VYSOKÉ UČENÍ TECHNICKÉ V BRNĚ

BRNO UNIVERSITY OF TECHNOLOGY

FAKULTA STROJNÍHO INŽENÝRSTVÍ
ÚSTAV MATEMATIKY

FACULTY OF MECHANICAL ENGINEERING
INSTITUTE OF MATHEMATICS

FOUNDATIONS OF FRACTIONAL CALCULUS ON TIME SCALES

BAKALÁŘSKÁ PRÁCE
BACHELOR'S THESIS

AUTOR PRÁCE — MATEJ DOLNÍK
AUTHOR

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ZÁKLADY ZLOMKOVÉHO KALKULU NA ČASOVÝCH ŠKÁLÁCH

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AUTOR PRÁCE — MATEJ DOLNÍK
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SUPERVISOR

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Vysoké učení technické v Brně, Fakulta strojního inženýrství

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ZADÁNÍ BAKALÁŘSKÉ PRÁCE

student(ka): Matej Dolník

který/která studuje v **bakalářském studijním programu**

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Ředitel ústavu Vám v souladu se zákonem č.111/1998 o vysokých školách a se Studijním a zkušebním řádem VUT v Brně určuje následující téma bakalářské práce:

Základy zlomkového kalkulu na časových škálách

v anglickém jazyce:

Foundations of Fractional Calculus on Time Scales

Stručná charakteristika problematiky úkolu:

Zlomkový kalkulus a teorie časových škál jsou disciplíny směřující k zobecňování pojmů klasické analýzy. Jejich vzájemné propojení je v posledních letech velmi aktuálním tématem poskytujícím široké možnosti výzkumu. Hlavní náplní práce bude rozšíření výsledků publikovaných v [1] na obecné časové škály, tedy i na škály s nulovou funkcí zrnitosti.

Cíle bakalářské práce:

1. Diskutovat existenci a jednoznačnost zavedení zlomkových operátorů na obecné časové škále.
2. Odvodit základní vlastnosti těchto operátorů.

Seznam odborné literatury:

[1] T. Kisela: Power functions and essentials of fractional calculus on isolated time scales, Advances in Difference Equations, Vol.2013, (2013), No.8, pp.1-18.
[2] M. Bohner, A. Peterson: Advances in Dynamic Equations on Time Scales. ISBN 0-8176-4293-5.
[3] I. Podlubný: Fractional Differential Equations. Academic Press, USA, 1999.

Vedoucí bakalářské práce: Ing. Tomáš Kisela, Ph.D.

Termín odevzdání bakalářské práce je stanoven časovým plánem akademického roku 2014/2015.

V Brně, dne 19.11.2014

L.S.

prof. RNDr. Josef Šlapal, CSc.	doc. Ing. Jaroslav Katolický, Ph.D.
Ředitel ústavu	Děkan fakulty

ABSTRACT
The bachelor thesis concerns fractional calculus on time scales, more precisely, it introduces fractional calculus on time scales and also investigates the property of uniqueness of the axiomatic definition of the power functions. After introducing basic concepts, the subject of discussion is mostly generalized Laplace transform as well as proof of uniqueness of generalized Laplace transform, which is used as a tool to proving the uniqueness of fractional power functions on time scales.

KEYWORDS
time scales, fractional calculus, Laplace transform, power functions, monomials, uniqeness

ABSTRAKT
Bakalářská práce pojednává o zlomkovém kalkule na časových škálach, přesněji - zavádí zlomkový kalkulus na časových škálach a taktéž vyšetřuje jednoznačnost axiomatické definice zavádějící mocninné funkce. Po zavedení základních pojmů je předmětem diskuze hlavně zobecněná Laplaceova transformace a důkaz jednoznačnosti zobecněné Laplaceovy transformace, která je použita jako nástroj pro dokázání jednoznačnosti zlomkových mocniných funkcií na časových škálach.

KLÍČOVÁ SLOVA
časové škály, zlomkový kalkulus, Laplaceova transformáce, mocninné funkce, monomiály, jednoznačnost

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DECLARATION

I declare that I have elaborated my bachelor's thesis on the theme of "Foundations of Fractional Calculus on Time Scales" independently, under the supervision of the bachelor's thesis supervisor and with the use of technical literature and other sources of information which are all quoted in the thesis and detailed in the list of literature at the end of the thesis.

As the author of the bachelor's thesis I furthermore declare that, concerning the creation of this bachelor's thesis, I have not infringed any copyright. In particular, I have not unlawfully encroached on anyone's personal copyright and I am fully aware of the consequences in the case of breaking Regulation § 11 and the following of the Copyright Act No 121/2000 Vol., including the possible consequences of criminal law resulted from Regulation § 152 of Criminal Act No 140/1961 Vol.

Brno ....

(author's signature)

Ďakujem Ing. Tomášovi Kiselovi, Ph.D. za odbornú pomoc pri vedení bakalárskej práce, informácie, rady, trpezlivosť a ochotu.

Matej Dolník

CONTENTS

INTRODUCTION

A unification of the diverse fields of mathematics is a goal of many scientists throughout the history. Some of the greatest mathematicians have expressed views that whole theory should be fitted into one subject.

One of such attempts is a time scale calculus (or a measure chain calculus), which is a unification of the theory of continuous and discrete analysis and creating a new formalism to study discrete-continuous dynamical systems. This feature is enabled by the general definition of a time scale , which is a closed non-empty subset of real numbers. Time scale calculus originates from Ph.D. dissertation by the German mathematician Stefan Hilger, 1988 [1].

An important feature of time scale calculus is the possibility of an application of gained results to the more general sets than those in continuous and discrete analysis, such as the Cantor set. Recently, the subject of dynamic equations on time scales continues to be a rapidly growing area of research, with many applications in various fields such as neural networks, heat transfer, epidemic models, population dynamics and so on.

Another successful attempt of generalization is the continuous fractional calculus. The initial idea have been introduced almost at the time of developing the derivative. First record of such an idea leads to the end of the 17th century to Leibniz's letter to l'Hospital, where he discussed the meaning of the derivative. Many well-known authors contributed to fractional calculus throughout the history. However, only after the first international conference in 1974 specialized on this subject and the publication of a comprehensive survey of fractional calculus theory and its applications [2], this subject has changed to a fast-growing and a respected field of mathematics with many applications in electrical engineering, rheology, control theory etc.

Origins of discrete fractional calculus are papers [3], [4], where the first definitions of non-integer order differences and sums were proposed. Recently, discrete fractional calculus attracted attention and was unified and generalized.

The paper is organized as following: Firstly, we recall the basics of time scale theory and underlay them with a sufficient amount of examples to understand the matter. Secondly, we discuss properties of generalized time scale nabla Laplace transform. Finally, we employ our findings to extend some known results on time scale fractional operators.

1 TIME SCALE CALCULUS

1.1 Basic definitions

In this section we introduce the basics of the time scale calculus. The employed definition of a difference operator divides the time scale calculus into two categories: delta and nabla calculus. In this work we utilize the nabla calculus, which is built on the notion of a backward difference.

Following basic definitions are adapted from [5].

Definition 1.1.1. A time scale $\mathbb{T}$ is an arbitrary non-empty closed subset of real numbers.

Example 1.1.1. $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ are typical examples of time scales. Also

$$\mathbb{T} = \{5^n; n \in \mathbb{Z}\}$$

$$\mathbb{T} = \{\sqrt[k]{n}; k \in \mathbb{Z} n \in \mathbb{R}^+\}$$

$$\mathbb{T} = \overline{q^{\mathbb{Z}}} = \{q^k, k \in \mathbb{Z}, q \in \mathbb{R}^+\} \cup \{0\}$$

$$\mathbb{T} = h\mathbb{Z} = \{kh; n \in \mathbb{Z}; h \in \mathbb{R}^+\}$$

are time scales.

Note that $\mathbb{Q}, \mathbb{C}$ are not the time scales.

In the following definitions we put $sup\{\mathbb{T}\} = inf\{\emptyset\}$ and $inf\{\mathbb{T}\} = sup\{\emptyset\}$.

Definition 1.1.2. Let $\mathbb{T}$ be a time scale.

a) Forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ for $t \in \mathbb{T}$ is defined by $\sigma(t) := inf\{s \in \mathbb{T} : s > t\}$.
b) Let $\mathbb{T}$ be a time scale. Backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ for $t \in \mathbb{T}$ is defined by $\rho(t) := sup\{s \in \mathbb{T} : s < t\}$.
c) Let $\mathbb{T}$ be a time scale. Forward graininess function $\mu : \mathbb{T} \to [0, \infty)$ for $t \in \mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$.
d) Let $\mathbb{T}$ be a time scale. Backward graininess function $\nu : \mathbb{T} \to [0, \infty)$ for $t \in \mathbb{T}$ is defined by $\nu(t) := t - \rho(t)$.

Example 1.1.2. We present some examples of backward graininess function.

$$\nu(t) = 0 \quad \text{for} \quad \mathbb{R}$$

$$\nu(t) = h \quad \text{for} \quad h\mathbb{Z}$$

$$\nu(t) = t(1 - \frac{1}{q}) \quad \text{for} \quad \overline{q^{\mathbb{Z}}}$$

Definition 1.1.3. a) If $\sigma(t) > t$ we say that t is a right-scattered point.
b) If $\rho(t) < t$ we say that t is a left-scattered point.
c) If $\sigma(t) > t$ and $\rho(t) < t$ we say that t is an isolated point.
d) If $t < sup\{\mathbb{T}\}$ and $\sigma(t) = t$ we say that t is a right-dense point.
e) If $t > inf\{\mathbb{T}\}$ and $\rho(t) = t$ we say that t is a left-dense point.
f) If $sup\{\mathbb{T}\} > t > inf\{\mathbb{T}\}$ and $\rho(t) = t = \sigma(t)$ we say that t is a dense point.

Definition 1.1.4. If $\mathbb{T}$ has a right scattered minimum m, then $\mathbb{T}_\kappa := \mathbb{T} - m$, otherwise $\mathbb{T}_\kappa = \mathbb{T}$. We call $\mathbb{T}_\kappa$ the truncated time scale.

Definition 1.1.5. The function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous (rd-continuous) if it is continuous at each right-dense point in $\mathbb{T}$ and if it has finite left sided limits at left dense points in $\mathbb{T}$.

Definition 1.1.6. The function $f : \mathbb{T} \to \mathbb{R}$ is left-dense continuous (ld-continuous) if it is continuous at each left-dense point in $\mathbb{T}$ and if it has finite right sided limits at right dense points in $\mathbb{T}$.

Theorem 1.1.1. Assume $f : \mathbb{T} \to \mathbb{R}$, then the following statements are true:
a) If f is continuous, then f is rd-continuous and ld-continuous.
b) Forward jump operator σ is rd-continuous.
c) Backward jump operator ρ is ld-continuous.

Following definitions are borrowed from [6].

Definition 1.1.7. The function $f : \mathbb{T} \to \mathbb{R}$ is ν-regressive if

$$1 - \nu(t)f(t) \neq 0 \quad \text{for all } t \in \mathbb{T}_\kappa.$$

Definition 1.1.8. The class of all scalar ld-continuous and ν-regressive functions on $\mathbb{T}$ is denoted by $\mathcal{R}_\nu$, i.e. $\mathcal{R}_\nu = \{f : \mathbb{T} \to \mathbb{R}; f(t)$is ld-continuous and ν-regressive $\}$.
Further, $\mathcal{R}_\nu^+ = \{f \in \mathcal{R}_\nu; 1 - f(t)\nu(t) > 0\}$ for all $t \in \mathbb{T}_\kappa$.

Definition 1.1.9. We define a circle plus addition for $p, v \in \mathcal{R}_\nu$ by

$$(p \oplus_\nu q)(t) := p(t) + q(t) - p(t)q(t)\nu(t)$$

for all $t \in \mathbb{T}_\kappa$.

Definition 1.1.10. We define a circle minus subtraction for $p \in \mathcal{R}_\nu$ by

$$\ominus_\nu p(t) := -\frac{p(t)}{1 - p(t)\nu(t)}$$

Thorough this work, we use $\oplus$,$\ominus$ instead of $\oplus_\nu$,$\ominus_\nu$.

Theorem 1.1.2. $(\mathcal{R}_\nu, \oplus)$ is an Abelian group.

Theorem 1.1.3. $(\mathcal{R}_\nu^+, \oplus)$ is a subgroup of $(\mathcal{R}_\nu, \oplus)$.

Definition 1.1.11. For $h > 0$, the Hilger complex numbers are defined as

$$\mathbb{C}_h = \{z \in \mathbb{C} : z \neq \frac{1}{h}\}.$$

Definition 1.1.12. For $h > 0$, the strip $\mathbb{Z}_h$ is defined as

$$\mathbb{Z}_h = \{z \in \mathbb{C} : -\frac{\pi}{h} < Im(z) \leq \frac{\pi}{h}\}.$$

Following theorems and definitions comprising nabla derivative can be found in [5].

Definition 1.1.13. Let $f : \mathbb{T} \to \mathbb{R}$ be a function and let $t \in \mathbb{T}_\kappa$. Then ∇-derivative $\nabla f(t)$ (or $f^\nabla(t)$), if it exists, is defined to be a number, with the property that for any given number $\epsilon > 0$, there is a delta neighbourhood U_δ of t (i.e., $U_\delta = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|f(s) - f(\rho(t)) - f^\nabla(t)(s - \rho(t))| \leq \epsilon|s - \rho(t)| \qquad \forall s \in U_\delta.$$

Theorem 1.1.4. Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}_\kappa$. Then following statements are true.

a) If f is differentiable at t, then f is continuous at t.

b) If f is continuous at t and t is left-scattered, then f is nabla differentiable at t with

$$f^\nabla(t) = \frac{f(t) - f(\rho(t))}{\nu(t)}. \tag{1.1}$$

c) If t is left-dense, then f is nabla differentiable at t iff a limit

$$\lim_{s \to t} \frac{f(s) - f(t)}{s - t}.$$

exists and is finite. Then

$$f^\nabla(t) = \frac{f(s) - f(t)}{s - t}.$$

d) If t is nabla differentiable at t, then

$$f(\rho(t)) = \nu(t) f^\nabla(t) + f(t).$$

Theorem 1.1.5. Assume that $f, g : \mathbb{T} \to \mathbb{R}$ are nabla differentiable at $t \in \mathbb{T}_\kappa$. Then following statements hold:

a) $(fg)^\nabla(t) = f^\nabla(t)g(t) + f(\rho(t))g^\nabla(t) = f(t)g^\nabla(t) + f^\nabla(t)g(\rho(t))$.

b) $(f+g)^\nabla(t) = f^\nabla(t) + g^\nabla(t)$.

c) $\left(\frac{f}{g}\right)^\nabla(t) = \frac{f^\nabla(t)g(t) - f(t)g^\nabla(t)}{g(t)g(\rho(t))}$.

Definition 1.1.14. Let $a, b \in \mathbb{T}$ be such that $a < b$. Let $f : [a,b]_\mathbb{T} \to \mathbb{R}$ and $F : [a,b]_\mathbb{T} \to \mathbb{R}$ be functions such that $F^\nabla(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$. Then the function $F(t)$ is called the antiderivative of function $f(t)$ over $[a,b]_\mathbb{T}$ as $\int_a^b f(t)\nabla t = F(b) - F(a)$. $\int_a^b f(t)\nabla t$ is called nabla integral.

We may also note that $\int_b^a f(t)\nabla t = -\int_a^b f(t)\nabla t$ and $\int_a^a f(t)\nabla t = 0$.

Theorem 1.1.6. Every ld-continuous function has the nabla antiderivative.

Theorem 1.1.7. Let $\mathbb{T}$ be an isolated time scale. Then the nabla integral can be calculated as

$$\int_a^b f(t)\nabla t = \sum_{t \in (a,b]_\mathbb{T}} \nu(t) f(t).$$

Definition 1.1.15. Let $a \in \mathbb{T}$, $sup\{\mathbb{T}\} = \infty$, $f : [a,\infty)_\mathbb{T} \to \mathbb{R}$ be ld-continuous. Then the improper integral of first kind over $f(t)$ over $[a,\infty)_\mathbb{T}$ is defined by:

$$\int_a^\infty f(t)\nabla t = lim_{b\to\infty} \int_a^b f(t)\nabla t.$$

Definition 1.1.16. Let $a, b, c \in \mathbb{T}$ be such that $a < b < c$ and let $f : (a,c]_\mathbb{T} \to \mathbb{R}$ be ld-continuous on any interval $[b,c]_\mathbb{T}$. Then the improper integral of second kind of $f(t)$ over $[a,c]_\mathbb{T}$ is defined by:

$$\int_a^c f(t)\nabla t = \begin{cases} lim_{b\to a+}(\int_b^c f(t)\nabla t), & \text{if } a \text{ is right-dense} \\ \int_a^c f(t)\nabla t, & \text{if } a \text{ is right-scattered.} \end{cases}$$

From 1.1.5 a) we obtain following integral by parts formula

Theorem 1.1.8. Assume that $f, g : \mathbb{T} \to \mathbb{R}$ are nabla differentiable at $t \in \mathbb{T}_\kappa$. Then

$$\int f^\nabla(t)g(t)\nabla t = f(t)g(t) - \int f(\rho(t))g^\nabla(t)\nabla t.$$

Proof of next theorem is modified version of delta version of this proof presented in [5], but first mentions of delta version of proof have been presented in [7].

Theorem 1.1.9. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is nabla differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is nabla differentiable and the following formula holds

$$(f \circ g)^{\nabla}(t) = g^{\nabla}(t) \int_0^1 f'(g(t) - u\nu(t)g^{\nabla}(t))du.$$

Proof. We apply ordinary substitution from the calculus:

$$(f \circ g)(s) - (f \circ g)(\rho(t)) = f(g(s)) - f(g(\rho(t)) =$$
$$= \int_{g(\rho(t))}^{g(s)} f'(\tau)d\tau = (g(s) - g(\rho(t)) \int_0^1 f'(hg(s) + (1-h)g(\rho(t)))dh$$

Let $t \in \mathbb{T}_{\kappa}$ and $\epsilon > 0$ be given. Since g is nabla differentiable at t, there exists a neighbourhood U_1 of t such that

$$|g(s) - g(\rho(t)) - g^{\nabla}(t)(s - \rho(t))| \leq \epsilon^* |s - \rho(t)|$$

for all $s \in U_1$, where

$$\epsilon^* = \frac{\epsilon}{2\int_0^1 f'(hg(t) + (1-h)g(\rho(t)))dh}$$

We know that if f' is continuous on $\mathbb{R}$ and therefore uniformly continuous on closed subsets of $\mathbb{R}$, there exists a neighbourhood U_2 of t such that:

$$|f'(hg(s) + (1-h)g(\rho(t))) - f'(hg(t) + (1-h)g(\rho(t)))| \leq$$
$$\leq \frac{\epsilon}{2(\epsilon^* + |g^{\nabla}(t)|)}$$

Also

$$|(hg(s) + (1-h)g(\rho(t))) - (hg(t) +$$
$$+ (1-h)g(\rho(t)))| \leq (1-h)|g(s) - g(t)| \leq |g(s) - g(t)|$$

holds for all $0 \leq h \leq 1$. Then we define $U = U_1 \cap U_2$ and set $s \in U$ and we put

$$\alpha = (hg(s) + (1-h)g(\rho(t))$$

$$\beta = (hg(t) + (1-h)g(\rho(t))$$

Then we have

$$|(f \circ g)(s) - (f \circ g)(\rho(t)) - (s - \rho(t))g^{\nabla}(t) \int_0^1 f'(\beta)dh| =$$

$$
\begin{aligned}
=&|(g(s)-g(\rho(t))\int_0^1 f'(\alpha)dh-(s-\rho(t))g^{\nabla}(t)\int_0^1 f'(\beta)dh|=\\
=&|(g(s)-g(\rho(t)-(s-\rho(t))g^{\nabla}(t))\int_0^1 f'(\alpha)dh+\\
&+(s-\rho(t))g^{\nabla}(t)\int_0^1 f'(\alpha)-f'(\beta)dh|\leq\\
\leq&|(g(s)-g(\rho(t)-(s-\rho(t))g^{\nabla}(t))|\int_0^1 |f'(\alpha)|dh+\\
&+|(s-\rho(t))||g^{\nabla}(t)|\int_0^1 |f'(\alpha)-f'(\beta)|dh\leq\\
\leq&\epsilon^*|s-\rho(t)|\int_0^1 |f'(\alpha)|dh+\\
&+|(s-\rho(t))||g^{\nabla}(t)|\int_0^1 |f'(\alpha)-f'(\beta)|dh\leq\\
\leq&\epsilon^*|s-\rho(t)|\int_0^1 |f'(\beta)|dh+\\
&+|(s-\rho(t))||g^{\nabla}(t)+\epsilon^*|\int_0^1 |f'(\alpha)-f'(\beta)|dh\leq\\
\leq&\epsilon^*|s-\rho(t)|\int_0^1 |f'(\beta)|dh+\frac{\epsilon}{2}|s-\rho(t)|\leq\\
\leq&\frac{\epsilon}{2}|s-\rho(t)|+\frac{\epsilon}{2}|s-\rho(t)|=\\
=&\epsilon|s-\rho(t)|
\end{aligned}
$$

This implies that $f\circ g$ is nabla differentiable at t and its derivative is

$$g^{\nabla}(t)\int_0^1 f'(\beta)dh = g^{\nabla}(t)\int_0^1 f'(hg(t)+(1-h)g(\rho(t))dh$$

Utilizing simple substitution, we prove our theorem. □

Following results are dual to delta results presented in [6]. We utilized same methods in nabla calculus to determine them.

Using chain rule provided by the Theorem 1.1.9 we derive the following formula:

$$(x^{\alpha})^{\nabla}(t)=x^{\nabla}(t)\int_0^1 \alpha(x(t)-u\nu(t)x^{\nabla}(t))^{\alpha-1}dh$$

If $x(t)\neq 0$, then

$$(x^{\alpha})^{\nabla}(t)=x^{\alpha}(t)\frac{x^{\nabla}(t)}{x(t)}\alpha\int_0^1 (1-u\nu(t)\frac{x^{\nabla}(t)}{x(t)})^{\alpha-1}du \tag{1.2}$$

In order to have everything well defined, we want to assume for $\alpha\in\mathbb{R}|\mathbb{N}$ that

$$(1-\nu(t)\frac{x^{\nabla}(t)}{x(t)}u)^{\alpha-1}>0$$

for $h \in [0,1]$ and $t \in \mathbb{T}$. Sufficient condition for this is:

$$\mathcal{R}(\alpha) := \begin{cases} \mathcal{R}^+ & \text{for} \quad \alpha \in \mathbb{R}|\mathbb{N} \\ \mathcal{R} & \text{for} \quad \alpha \in \mathbb{N} \end{cases}$$

From this we get the following definition:

Definition 1.1.17. For $\alpha \in \mathbb{R}$ and for $p \in \mathcal{R}(\alpha)$ we define the dot multiplication

$$(\alpha \odot p)(t) = p(t)\alpha \int_0^1 (1 - u\nu(t)p(t))^{\alpha-1} du.$$

Theorem 1.1.10. Let $\alpha \in \mathbb{R}$. If $\alpha \in \mathbb{N}$, suppose that $\alpha \neq 0$ for all $t \in \mathbb{T}$. If $\alpha \notin \mathbb{N}$, suppose that $x(t)x(\rho(t)) > 0$ for all $t \in \mathbb{T}$. Then

$$\frac{(x^\alpha)^\nabla}{x^\alpha} = (\alpha \odot \frac{x^\nabla}{x}) \tag{1.3}$$

Proof.

$$1 - \nu \frac{(x(t))^\nabla}{x(t)} = \frac{x(\rho(t))}{x(t)}$$

Then the theorem follows directly from 1.2. □

Theorem 1.1.11. Let $\alpha \in \mathbb{R}$. If $p \in \mathcal{R}(\alpha)$, then

$$1 - \nu(\alpha \odot p) = (1 - \nu p)^\alpha.$$

Proof.

$$\begin{aligned} 1 - \nu(\alpha \odot p) &= 1 - \nu p\alpha \int_0^1 (1 - u\nu p))^{\alpha-1} du = \\ &= 1 - \int_0^1 \nu p\alpha(1 - u\nu p))^{\alpha-1} du = \\ &= 1 - \int_1^{1-\nu p} \alpha s^{\alpha-1} ds = \\ &= 1 + (1 - \nu p)^\alpha - 1^\alpha = (1 - \nu p)^\alpha. \end{aligned}$$

□

1.2 Time scale functions

In this section we introduce a generalized nabla time scale monomials (generalized polynomials), a generalized nabla time scale exponential function and present few demonstrative examples followed by brief introduction to their determination. The exponential function plays a crucial role in our next chapter - Laplace transform. Monomials are inevitable for series expansions, namely the Taylor's series expansions.

Following results concerning monomials are adapted from [8].

Definition 1.2.1. Monomials $\hat{h}_n : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ for $n \in \mathbb{N}$ are defined by recursion

$$\hat{h}_0(t,s) = 1$$
$$\hat{h}_n(t,s) = \int_s^t \hat{h}_{n-1}(\tau,s)\nabla\tau.$$

Theorem 1.2.1. Let $n \in \mathbb{Z}^+$ and $s,t \in \mathbb{T}$. Then

a) $\hat{h}_n(t,t) = 0$,

b) $\nabla\hat{h}_n(t,s) = \hat{h}_{n-1}(t,s) \quad \text{for } t \in \mathbb{T}_\kappa$,

c) $\nabla\hat{h}_1(t,s) = (t-s)$.

Theorem 1.2.2. Let $s,t \in \mathbb{T}$, $n \in \mathbb{Z}_0^+$. Then the following statements are true.

a) If $\mathbb{T} = \mathbb{R}$, then $\hat{h}_n(t,s) = \frac{(t-s)^m}{m!}$.

b) If $\mathbb{T} = h\mathbb{Z}$ and $\sigma^k(s) = t$, then

$$\hat{h}_n(t,s) = h^n \binom{n+k-1}{k-1} = (-1)^{k-1}h^n \binom{-n-1}{k-1}.$$

Theorems and definitions comprising generalized nabla time scale exponential function are borrowed from [5, 6].

Definition 1.2.2. For any $f \in \mathcal{R}_\nu$, the exponential function $\hat{e}_f : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ is defined as a unique solution of the initial value problem

$$\nabla y(t) = f(t)y(t) \qquad y(s) = 1$$

Definition 1.2.3. The h-cylinder transformation $\hat{\xi}_h : \mathbb{C}_h \to \mathbb{Z}_h$ is defined by:

$$\hat{\xi}_h(z) = -\frac{1}{h}Log(1-zh)$$

for $h > 0$ and where Log is the principal logarithm function.

Theorem 1.2.3. Solution of the initial value problem $\nabla y(t) = f(t)y(t)$, $y(s) = 1$ can be written as:

$$\hat{e}_f(t,s) = exp\{\int_s^t \hat{\xi}_{\nu(\tau)}(f(\tau))\nabla\tau\} \qquad s,t \in \mathbb{T} \tag{1.4}$$

where $\hat{\xi}_h(z)$ is the ν-cylinder transformation.

Theorem 1.2.4. If $p \in \mathcal{R}_\nu$, then the semi-group property for $s, t, r \in \mathbb{T}$ is satisfied.

$$\hat{e}_f(t,r)\,\hat{e}_f(r,s) = \hat{e}_f(t,s).$$

Theorem 1.2.5. Let $p, q \in \mathcal{R}_\nu$ and $s, t \in \mathbb{T}$. Then the following statements are valid

a) $\hat{e}_0(t,s) = 1$
b) $\hat{e}_p(t,t) = 1$
c) $\hat{e}_p(t,s)\,\hat{e}_q(t,s) = \hat{e}_{p\oplus q}(t,s)$
d) $\frac{1}{\hat{e}_p(t,s)} = \hat{e}_p(s,t) = \hat{e}_{\ominus p}(t,s)$
e) $\hat{e}_p(\rho(t),s) = (1-\nu(t)p(t))\,\hat{e}_p(t,s)$

Theorem 1.2.6. Let $p \in \mathcal{R}_\nu$ and $t_0 \in \mathbb{T}$. Then the following statements are true:

a) if $p \in \mathcal{R}_\nu^+$, then $\hat{e}_p(t,t_0) > 0$ for all $t \in \mathbb{T}$.
b) if $1-\nu(t)p(t) < 0$ for some $t \in \mathbb{T}_\kappa$, then $\hat{e}_p(\rho(t),t_0)\,\hat{e}_p(t,t_0) < 0$.
c) if $1-\nu(t)p(t) < 0$ for all $t \in \mathbb{T}_\kappa$, then $\hat{e}_p(t,t_0)$ changes sign at every point $t \in \mathbb{T}$.

Theorem 1.2.7. Let $z \in \mathcal{R}_\nu$. Then

$$\hat{e}_{\ominus z}(\rho(t),s) = \frac{-\ominus z}{z}\,\hat{e}_{\ominus z}(t,s).$$

Proof. Utilizing the Theorem 1.2.5 e)

$$\hat{e}_{\ominus z}(\rho(t),s) = (1-\nu(t)\ominus z)\,\hat{e}_{\ominus z}(t,s) = \frac{\hat{e}_{\ominus z}(t,s)}{1-\nu(t)z} = \frac{-\ominus z}{z}\,\hat{e}_{\ominus z}(t,s).$$

□

Here we present some examples of the time scale exponential functions as well as the basic method of their determination.

Example 1.2.1. Let $\mathbb{T} = h\mathbb{Z}$ and $f(t) = c \in \mathbb{R}$,$s \in \mathbb{T}$ and the initial value problem

$$\nabla y(t) = f(t)y(t) \quad \text{and} \quad y(s) = 1$$

$$\nabla y(t) = cy(t)$$

Using Theorem 1.1.4, we obtain

$$\frac{y(t) - y(\rho(t))}{h} = cy(t).$$

After few calculations and utilizing boundary conditions we get

$$y(h+s) = \frac{1}{1-ch}.$$

Finally, we may determine that

$$y(t) = \hat{e}_c(t,s) = \frac{1}{(1-ch)^{\frac{t-s}{h}}}.$$

Example 1.2.2. Let $\mathbb{T} = q^{\mathbb{Z}} = \{q^k, k \in \mathbb{Z}, q \in \mathbb{R}_0^+\}$ and the initial value problem

$$\nabla y(t) = f(t)y(t) \quad \text{and} \quad y(s) = 1.$$

Then

$$\frac{y(t) - y(\rho(t))}{t(1 - \frac{1}{q})} = f(t)y(t)$$

$$y(t) = \frac{y(\rho(t))}{1 - (1 - \frac{1}{q})tf(t)}$$

Using the boundary conditions

$$y(sq) = \frac{1}{1 - (1 - \frac{1}{q})sq(f(sq))}$$

$$y(t) = y(\sigma^a(s)) = \prod_{k=1}^{a} \frac{1}{1 - (1 - \frac{1}{q})\sigma^k(s)(f(\sigma^k(s))}$$

$$y(t) = y(\rho^a(s)) = \prod_{k=1}^{a} 1 - (1 - \frac{1}{q})\rho^k(s)(f(\rho^k(s)).$$

Example 1.2.3. Let $\mathbb{T}$ be an arbitrary isolated time scale, $s, t \in \mathbb{T}$,$f : \mathbb{T} \to \mathbb{R}$ such that $f \in \mathcal{R}_\nu$ and the initial value problem

$$\nabla y(t) = f(t)y(t) \quad \text{and} \quad y(s) = 1.$$

Then

$$y(t) = \frac{y(\rho(t))}{\nu(t) - \nu(t)f(t)}$$

From the boundary condition we know that

$$y(\sigma(s)) = \frac{1}{\nu(t) - \nu(t)f(t)}.$$

Let $t = \sigma^n(s)$

$$y(\sigma^n(s)) = \prod_{k=1}^{n} \frac{1}{\nu(\sigma^k(s)) - \nu(\sigma^k(s))f(\sigma^k(s))}.$$

Example 1.2.4. Following table contains other examples of exponential functions.

$\mathbb{T}$	$\hat{e}_\alpha(t,s)$
$\mathbb{R}$	$e^{\alpha(t-s)}$
$\mathbb{Z}$	$\left(\frac{1}{1-\alpha}\right)^{t-s}$
$h\mathbb{Z}$	$\left(\frac{1}{1-h\alpha}\right)^{(t-s)/h}$
$\frac{1}{n}\mathbb{Z}$	$\left(\frac{n}{n-\alpha}\right)^{n(t-s)}$
$q^{\mathbb{N}_0}$	$\prod_{s\in[s,t)} \frac{1}{1-(q-1)\alpha s}$ if $t \geq s$

Following theorems are dual to delta versions presented in [6].

Theorem 1.2.8. If $\alpha \in \mathbb{R}$ and $p \in \mathcal{R}(\alpha)$, then

$$\hat{e}_{\alpha \odot p} = \hat{e}_p^{\alpha}$$

Proof. Let $t_0 \in \mathbb{T}$ and put:

$$y = \hat{e}_p^{\alpha}(\cdot, t_0)$$

Then $y(t_0) = 0$ and by the Theorem 1.3

$$y^{\nabla} = (\hat{e}_p^{\alpha})^{\nabla} = (\alpha \odot \frac{\hat{e}_p^{\nabla}}{\hat{e}_p})\, \hat{e}_p^{\alpha} = (\alpha \odot p) y$$

We know that y solves the initial value problem

$$\nabla y(t) = (\alpha \odot p)(t) y \qquad y(t_0) = 1 \tag{1.5}$$

Therefore,

$$\hat{e}_p^{\alpha} = y = \hat{e}_{\alpha \odot p}\,.$$

□

2 LAPLACE TRANSFORM

The generalized time scale nabla Laplace transform plays the key role in the following investigation of the uniqueness of the fractional operators on the time scales. We employ these definitions and the theorems from the following papers: [9], [8].

2.1 Basic definitions

Definition 2.1.1. Let $\mathbb{T}$ be such that $sup\{\mathbb{T}\} = \infty$ and fix $t_0 \in \mathbb{T}$. For any given function $f : [t_0, \infty)_{\mathbb{T}} \to \mathbb{C}$, the solution of the shifting problem

$$\nabla u(t, \rho(s)) = -\tilde{\nabla} u(t, s) \qquad t, s \in \mathbb{T}, t > s > t_0$$

is called shift of $f(t)$. The symbol $\tilde{\nabla}$ denotes the derivative with respect to the second variable.

Definition 2.1.2. For any given function $f, g : [s, \infty)_{\mathbb{T}} \to \mathbb{R}$, their convolution is defined by:

$$(f * g)(t) = \int_s^t \hat{f}(t, \rho(t)) g(\tau) \nabla \tau$$

where $\hat{f}$ is the shift of f.

Theorem 2.1.1. Convolution defined by previous definition is associative e.g. $(f * g) * u = f * (g * u)$.

Proof. The assertion can be proven utilizing the nabla analogy of the technique performed in the proof presented in [6]. □

Definition 2.1.3. Let $sup\{\ \mathbb{T}\} = \infty$, $s \in T$ and let $f(t)$ be a real function defined at least on $(s, \infty)_{\mathbb{T}}$. The generalized time scale nabla Laplace transform of $f(t)$ is defined by

$$\mathcal{L}_s\{f\}(z) = \int_s^\infty f(t) \hat{e}_{\ominus z}(\rho(t), s) \nabla t \qquad for \quad z \in \mathcal{D}(f)$$

where $\mathcal{D}(f)$ consist of all complex numbers z such that $Re(z) \in \mathbb{R}_\nu$ for which the improper integral exists.

Thorough this work, we denote $\mathcal{D}(f)$ region of convergence of generalized time scale nabla Laplace transform.

Theorem 2.1.2. The necessary condition for the existence of the a Laplace transform $\mathcal{L}_s\{f\}(z)$ to exist is

$$lim_{t \to \infty} f(t) \hat{e}_{\ominus z}(\rho(t), s) = 0.$$

Theorem 2.1.3. Laplace transform of convolution is the product of images of Laplace transform, i.e.:

$$\mathcal{L}_s\{(f*g)(\cdot,s)\}(z) = \mathcal{L}_s\{f\}(z) * \mathcal{L}_s\{g\}(z).$$

Theorem 2.1.4. For $n \in \mathbb{Z}$,

$$\mathcal{L}_s\{\hat{h}_n(\cdot,0)\}(z) = z^{-n-1}.$$

2.2 Regions of convergence

In this section we investigate the regions of convergence of the time scale Laplace transform.

We know that $|1-z| \leq 1$ is a circle with center in $z=1$ and radius 1.

Example 2.2.1. Let us consider $\mathbb{T} = h\mathbb{Z}$

$$\mathcal{L}_0\{1\} = \int_0^\infty \hat{e}_{\ominus z}(\rho(t),0)\nabla t =$$
$$= \int_0^\infty (1-hz)^t \nabla t = \sum_{t=1}^\infty (1-hz)^t = \frac{1-hz}{hz}$$

Sum coverges, if $|1-hz| \leq 1$.

We also got circle, with the center in $z = \frac{1}{h}$ and the radius $\frac{1}{h}$.

Example 2.2.2. Let us consider the periodic time scale with backward graininess function $\nu(t) = h$ and $\nu(\sigma(t)) = H$.

$$\mathcal{L}_0\{1\} = \int_0^\infty \hat{e}_{\ominus z}(\rho(t),0)\nabla t =$$
$$= \int_0^\infty (1-hz)^{\frac{t}{2}}(1-Hz)^{\frac{t}{2}} \nabla t = \sum_{t=1}^\infty \nu_t (1-hz)^{\frac{t}{2}}(1-Hz)^{\frac{t}{2}}$$

Sum converges, if $|(1-hz)(1-Hz)| \leq 1$.

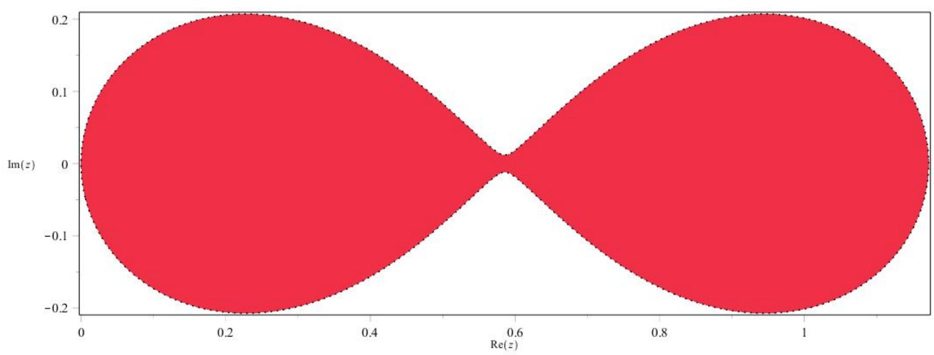

Fig. 2.1: An example of region of convergence for the periodic time scale $H = 5,825$, $h = 1$, which is inside of the shape.

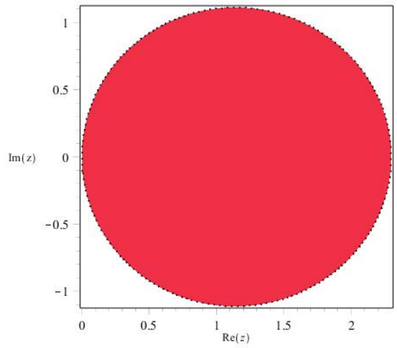

Fig. 2.2: An example of region of convergence for the periodic time scale $H = 0, 1$, $h = 1$, which is inside of the shape.

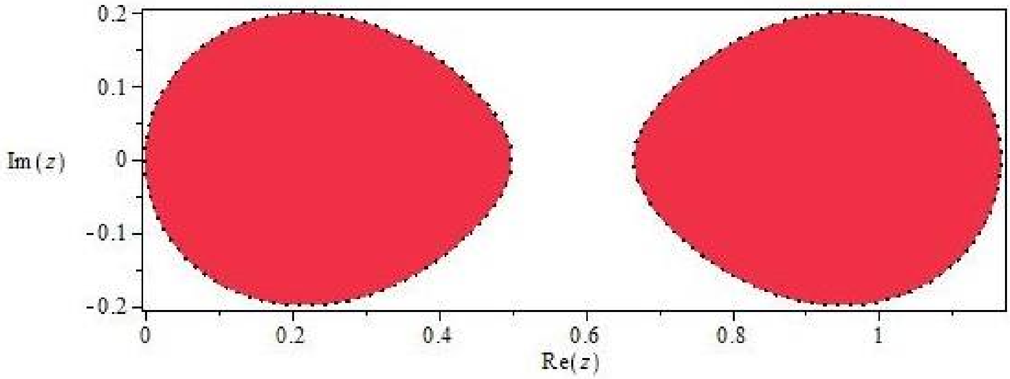

Fig. 2.3: An example of region of convergence for the periodic time scale $H = 6$, $h = 1$, which is inside of the shape.

Last figure shows us that the regions of convergence do not have to have the property of being connected, which is a very important feature for the next section.

2.3 Proof of Lerch's theorem

As we mentioned before, generalized time scale nabla Laplace transform is widely used as a tool to analyse properties of various functions, as well as proving their uniqueness. We know from the real analysis, that the Laplace transform on real numbers has uniquely determined image except for a null function. It also implies, that we may calculate the inversion and the various methods exist for such calculations on real numbers. Lerch's theorem proves the uniqueness of the Laplace transform. However, in our knowledge there is no valid proof of uniqueness of the generalized time scale nabla Laplace transform, nor general formula for an arbitrary time scale using nabla calculus, even though many authors assume it. This chapter is devoted to problems, concerning the proof of uniqueness of generalized time scale nabla Laplace transform on arbitrary time scales constructively, in fashion of proof presented in [10],[11]. Later on, we proved this chapter generalized Lerch's theorem for some specific types of time scales.

Definition 2.3.1. A function $f : [s, \infty)_{\mathbb{T}} \to \mathbb{R}$ is called a *null function* if

$$\int_s^t f(\eta)\nabla\eta = 0 \quad \text{for all } t \in [s, \infty)_{\mathbb{T}}.$$

The set of null functions from now on will be denoted by $\mathcal{N}[s, \infty)_{\mathbb{T}}$.

For $\mathbb{T} = \mathbb{R}$ null function is a function which is zero almost everywhere. The criterion for the null function was chosen, since it is also applicable to Riemann integration, while the function which is zero almost everywhere need not to be Riemann integrable.

Definition 2.3.2. The heaviside step function, or the unit step function is defined by

$$\mathrm{H}(k) = \begin{cases} 0 & k \in (-\infty, 0), \\ 1 & k \in [0, \infty). \end{cases}$$

Theorem 2.3.1. Let $\mathbb{T} = \mathbb{R}$. If $\mathcal{L}_0\{f(t)\} = \mathcal{L}_0\{g(t)\}$, then $f - g \in \mathcal{N}[0, \infty)_{\mathbb{T}}$.

The following theorem implies the previous one and it offers opportunity to prove theorem also for the functions non-integrable by Riemann's integral.

Theorem 2.3.2. Let $\mathcal{L}_0\{f(t)\} = F(z)$ vanishes on an infinite sequence of points that are located on equal intervals along a line parallel to the real axis:

$$F(z_0 + n\sigma) = 0 \quad (\sigma > 0, n \in \mathbb{N}).$$

Proof. Whole proof is presented in [10] or [11]. We present sketch of the proof in [10] to emphasize features, which we consider important for our further investigation.

Let $0 < \tau < T$, and put $v(t) = f(t)e^{-zt}$. Then for any positive integer n

$$\int_0^T v(t)e^{n\sigma(T-t)}dt = -\int_T^\infty \int_T^t v(s)ds\; nke^{n\sigma(T-t)}dt$$

since $\int_T^\infty v(s)ds$ converges. Also

$$\left|\int_T^t v(s)ds\right| \leq M \quad \text{for all } t \geq T,$$

so that

$$\left|\int_0^T v(t)e^{n\sigma(T-t)}dt\right| \leq M.$$

The series converges uniformly for $0 \leq t \leq T$, e.g.

$$E_n(\tau - t) = \sum_1^\infty \frac{(-1)^{m-1}}{m!}e^{mn\sigma(\tau-t)} = 1 - exp(-e^{n\sigma(\tau-t)}).$$

Thus

$$\left|\int_0^T v(t)e^{n\sigma(T-t)}dt\right| \leq \sum_1^\infty \frac{1}{m!}e^{mn\sigma(\tau-t)}\left|\int_0^T v(t)e^{mn\sigma(T-t)}dt\right| \leq$$
$$\leq M(exp(-e^{n\sigma(T-\tau)}) - 1)$$

and so tends to 0 as $n \to \infty$. Also $E_n(\tau - t) \to H(\tau - t)$ as $n \to \infty$, and we can write

$$\int_0^T v(t)(E_n(\tau - t) - H(\tau - t))dt = I_1 + I_2 + I_3 + I_4.$$

Then by bounding I_1,I_2,I_3,I_4 author shows that $\int_0^\tau v(t)dt = 0$ for any $\tau > 0$. □

Now we shall discuss the properties of methods used to prove Theorem 2.3.2. First of all we have to consider, what is the region of convergence of the arbitrary generalized time scale nabla Laplace transform. From previous section we know, that the regions of convergence does not have to have the property of being connected. We also know, that the regions of convergence are located around limit points of backward graininess function. For example in $h\mathbb{Z}$, the regions of convergence are some neighbourhood of $\frac{1}{h}$. If we consider the periodic time scale with graininess function values $h_1, h_2, ...h_n$, the region of convergence are some neighbourhoods of points $\frac{1}{h_1}, \frac{1}{h_2}, ...\frac{1}{h_n}$. An arbitrary time scale may contain graininess function values without limit points and that would imply point-wise convergence in such a points. We also may note, that such a points are not ν-regressive $1 - \frac{1}{\nu(t)}\nu(t) = 0$.

Secondly, we have to consider the differences between the circle multiplication and the multiplication in sense of real numbers. We want the exponent of our exponential function to be the points of region of convergence of Laplace transform. Utilizing 1.1.17 we may write

$$n \odot \sigma = \frac{1 - \nu(t)\sigma}{-\nu(t)} + \frac{1}{\nu(t)}$$
$$(z_0 \oplus (n \odot \sigma)) = (z_0 + (n \odot \sigma) - z_0(n \odot \sigma)\nu(t)).$$

Therefore, for $\nu(t) \neq 0$

$$(z_0 \oplus (n \odot \sigma)) = (1 - \nu(t)\sigma)^n(-\frac{1}{\nu(t)} + z_0) + \frac{1}{\nu(t)}.$$

For $\nu(t) = 0$

$$(z_0 \oplus (n \odot \sigma)) = z_0 + n.\sigma.$$

This clearly means that $(z_0 \oplus (n \odot \sigma))$ is t dependant or more precisely $\nu(t)$ dependant. $\nu(t)$ dependence also concludes, that if we use exponential function $\hat{e}_{\ominus(z_0+n\odot\sigma)}$ for our Laplace transform, and backward graininess function is not constant, exponent of used exponential function is time dependent, ergo we do not get Laplace transform, which is defined only for constant parameters.

We may also conclude that on an arbitrary time scale, $z_0 \oplus (n \odot \sigma)$ lies in the region of convergence, except the possibility of non-limit points (note that by the limit points of $\nu(t)$ we mean the values of $\nu(t)$ for $t \to \infty$) of $\nu(t)$. Let us consider the following example:

Example 2.3.1. Let $\mathbb{T}$ be the time scale such as $t \in [0, T] \subset \mathbb{T}$, $\nu(t) = 0$ and $s \in (T, \infty)$, $\nu(s) = 1$. Thus "Laplace" integral of 1 in $z_0 \oplus (n \odot \sigma)$ is:

$$\int_0^\infty \hat{e}_{\ominus(z_0 \oplus n\odot\sigma)}(\rho(t), 0)\nabla t = \int_0^T e^{-(z_0+n\sigma)t}dt + \sum_{t=\sigma(T)}^{\infty}((1-z_0)(1-\sigma)^n)^t$$

We may notice, that our region of convergence is the circle with the radius 1 around point 1. Now let $\sigma > 0$, then our points $z_0 + n\sigma$ for $t \in [0, T]$ will clearly not stay in the region of convergence.

We also may note that

$$lim_{n\to\infty} = (z_0 \oplus (n \odot \sigma)) = \frac{1}{\nu}.$$

Lemma 2.3.1. Let $\mathbb{T}$ be a time scale, let $\max\{\nu(\tau)\}$ be the maximal graininess function for all $\tau \in \mathbb{T}$. Then if exists at least one point $p \in [s,t]$ such that $\nu(p) = \max\{\nu(\tau)\}$ the following statement holds.

$$lim_{x \to \frac{1}{\max\{\nu(t)\}}} \hat{e}_{\ominus x}(t,s) = 0$$

Proof. Firstly $1 - \frac{\nu(\tau)}{\max\{\nu(\tau)\}} \geq 0$.
Secondly, we may assume, that p is an isolated point.

$$lim_{x \to \frac{1}{\max\{\nu(t)\}}} \hat{e}_{\ominus x}(t,s) =$$
$$lim_{x \to \frac{1}{\max\{\nu(t)\}}} exp\{\int_s^t -\frac{1}{\nu(\tau)} Log(\frac{1}{1-\nu(\tau)x})\nabla\tau\} =$$
$$= lim_{x \to \frac{1}{\max\{\nu(t)\}}} exp\{\int_s^{\rho(p)} -\frac{1}{\nu(\tau)} Log(\frac{1}{1-\nu(\tau)x})\nabla\tau -$$
$$-\frac{1}{\nu(p)} Log(\frac{1}{1-\nu(p)x}) + \int_{\sigma(p)}^t -\frac{1}{\nu(\tau)} Log(\frac{1}{1-\nu(\tau)x})\nabla\tau\} = 0$$

For $\max\{\nu(\tau)\} = 0$ the case is obvious. □

Lemma 2.3.2. Let $f(x)$ be a continuous function, and suppose that the moments of every order of $f(x)$ on the finite interval (a,b) vanish, that is:

$$\int_a^b f(x)x^n dx = 0 \quad \text{n=0,1,...}$$

Then $f(x) = 0$ on (a,b).

Proof. Proof is located in [11]. □

Theorem 2.3.3. Let us consider a time scale such that for $t \in [s,a]$ we have graininess function $\nu(t) = 0$. Let $\tau \in [a,\infty]$ let $h = \nu(\tau) > max\{\nu(t)\}$, $h \in \mathbb{R}$. Let $c_k = z_0 \oplus (k \odot \sigma) = (1-h\sigma)^k(-\frac{1}{h} + z_0) + \frac{1}{h}$. Let $f : \mathbb{T} \to \mathbb{R}$. Suppose that

$$\mathcal{L}_s\{f(t)\}(c_k) = 0.$$

Then $f(t) \in \mathcal{N}[s,\infty)_{\mathbb{T}}$.

Proof. We have chosen condition $h = \nu(\tau) > max\{\nu(t)\}$ to secure that the exponents of the exponential functions at t will be from $\mathcal{R}_\nu^+$.

We know, that for $k = 0$, $c_0 = z_0$. From this may write, that $\int_s^\infty f(t)\, \hat{e}_{z_0}(\rho(t),s)\nabla t$ converges.

For any c_k we may write for every $T > a$:

$$\int_s^T f(t)\, \hat{e}_{\ominus(c_k)}(\rho(t),s)\nabla t + \int_T^\infty f(t)\, \hat{e}_{\ominus(c_k)}(\rho(t),s)\nabla t = 0$$

By Theorems 1.2.8,1.2.4

$$\int_s^T f(t)\,\hat{e}_{\ominus(c_k)}(\rho(t),s)\nabla t =$$
$$= -\int_T^\infty f(t)\,\hat{e}_{\ominus(c_k)}(\rho(t),s)\nabla t =$$
$$= -\int_T^\infty f(t)\,\hat{e}_{\ominus(z_0)}(\rho(t),s)\,\hat{e}^k_{\ominus(\sigma)}(\rho(t),s)\nabla t =$$
$$= -\int_T^\infty f(t)\,\hat{e}_{\ominus(z_0)}(\rho(t),T)\,\hat{e}^k_{\ominus(\sigma)}(\rho(t),T)\,\hat{e}_{\ominus(c_k)}(T,s)\nabla t$$

Now pushing $k \to \infty$, $c_k \to \frac{1}{h}$ we obtain by Lemma 2.3.1

$$\int_s^T f(t)\,\hat{e}_{\ominus(\frac{1}{h})}(\rho(t),s)\nabla t = 0.$$

This concludes that

$$\int_a^T f(t)\,\hat{e}_{\ominus\frac{1}{h}}(\rho(t),s)\nabla t = 0 \text{ for every } T > a.$$

Utilizing Theorem 1.1.8:

$$\int_a^T f(t)\nabla t\,\hat{e}_{\ominus\frac{1}{h}}(T,s) - \int_a^T \int_a^t f(x)\nabla x\,\hat{e}_{\ominus\frac{1}{h}}(t,s)\nabla t = 0.$$

Nabla differentiating at T we get

$$(\nabla_T \int_a^T f(t)\nabla t)\,\hat{e}_{\ominus\frac{1}{h}}(T,s) - \frac{1}{h}\int_a^T f(t)\nabla t\,\hat{e}_{\ominus\frac{1}{h}}(\rho(T),s) +$$
$$+\frac{1}{h}\int_a^T f(x)\nabla x\,\hat{e}_{\ominus\frac{1}{h}}(\rho(T),s)\nabla t = 0.$$

Therefore

$$(\nabla_T \int_a^T f(t)\nabla t)\,\hat{e}_{\ominus\frac{1}{h}}(T,s) = 0$$
$$\int_a^T f(t)\nabla t = 0$$

for all $T > a$. This implies

$$\int_a^T f(t)\,\hat{e}_{\ominus c_k}(\rho(t),a)\nabla t = 0.$$

Also

$$\int_s^a f(t)\,\hat{e}_{\ominus c_k}(\rho(t),s)\nabla t = 0.$$

From ν - regressivity of $\frac{1}{h}$ we know by 1.2.6 that $\hat{e}_{\ominus(\frac{1}{h})}(\rho(t), s)$ is positive at all $t \in [s, a]$. By assumption of the theorem, we may conclude that the integral converges.

Using series of expansions of exponential function, we may rewrite the integral to

$$\int_s^a e^{-c_k(t-s)} f(t) dt =$$
$$\sum_{j=1}^{\infty} \frac{(-c_k)^j}{j!} \int_s^a (t-s)^j f(t) dt =$$
$$= \sum_{j=1}^{\infty} \frac{(-c_k)^j}{j!} \int_s^a (t-s)^j f(t) dt.$$

Utilizing Lemma 2.3.2, we may conclude, that $f(t) = 0$ on $[s, a]$. Thus we may conclude, that $f(x) \in \mathcal{N}[s, \infty]_{\mathbb{T}}$.

□

3 FRACTIONAL CALCULUS ON TIME SCALES

Continuous fractional calculus is mathematical discipline, developed to generalize theory of continuous differential and integral calculus to non-integer or even complex orders. Following formula for the fractional integral of the real function, where γ is order of integration is fundamental to such studies:

$$_aD^{-\gamma}f(t) = \int_a^t \frac{(t-\tau)^{\gamma-1}}{\Gamma(\gamma)} f(\tau)d\tau \quad t > a, \gamma > 0$$

3.1 Power functions

This section employs some basic definitions of fractional calculus on time scales and summarizes results concerning the power functions. Generally, the explicit formulas of the power functions are not known, because the method of generalizing formulas from monomials works only for special cases of functions. That is the reason why following definition as well as a few other axiomatic definitions were independently proposed. In this work, we develop the definition presented in [8] and try widen the property of uniqueness of this definition on time scales with arbitrary constant backward graininess function.

Definition 3.1.1. Let $s, t \in \mathbb{T}$ and let $\beta, \alpha \in (-1, \infty)$. The time scale power functions $\hat{h}_\beta(t,s)$ are defined by a family of non-negative functions satisfying
i) $\int_s^t \hat{h}_\beta(t,\rho(\tau))\hat{h}_\gamma(\tau,s)\nabla\tau = \hat{h}_{\beta+\gamma+1}(t,s)$ for $t \geq s$
ii) $\hat{h}_0(t,s) = 1$ for $t \geq s$
iii) $\hat{h}_\beta(t,t) = 1$ for $\beta \in (0,1)$.
Further, the parameter β in $\hat{h}_\beta(t,s)$ is called the order of the function $\hat{h}_\beta(t,s)$.

Theorem 3.1.1. Let $m \in \mathbb{N}_0$, $\beta \in (-1,\infty)$, $s, t \in \mathbb{T}$ be such that $t > s$. Then

$$\nabla^m \hat{h}_\beta(t,s) = \begin{cases} \hat{h}_{\beta-m}(t,s) & \beta > m-1 \\ 0 & \beta \in 0,1,...,m-1. \end{cases}$$

Theorem 3.1.1 does not discuss the case $\beta \in (-1, m-1]|_{\{0,1,...m-1\}}$ due to an occurrence of a power function of order less than -1. Since such functions cannot be included in the Definition 3.1.1, we define by the Theorem 3.1.1.

Definition 3.1.2.

$$\hat{h}_\beta(t,s) = \nabla^{-\lceil\beta\rceil}\hat{h}_{\beta-\lceil\beta\rceil}(t,s)$$

for $\beta \in (-\infty,-1)|\mathbb{Z}$, $s,t \in \mathbb{T}$, $t \geq \sigma^{-\beta}(s)$, where $\lceil\beta\rceil$ is the ceiling function $\lceil\beta\rceil = min\{m \in \mathbb{Z}; m \geq \beta\}$.

Next results of investigation questions of existence and uniqueness of axiomatic definition of power functions on isolated time scales are provided by [12].

Theorem 3.1.2. Let $\beta \in (-1, \infty)$, $s, t \in \mathbb{T}_\kappa$ be such that $t > s$. Then $\hat{h}_\beta$ solves the shifting problem, i.e.:

$$\nabla_t \hat{h}(t, \rho(s)) = -\nabla_s \hat{h}_\beta(t, s)$$

Theorem 3.1.2 enables us to rewrite Definition 3.1.1 i) via the convolution:

$$(\hat{h}_\beta * \hat{h}_\gamma)(t, s) = \hat{h}_{\beta+\gamma+1}(t, s) \quad t \geq s, \beta, \gamma > -1.$$

Theorem 3.1.3. Let $\mathbb{T}$ be an isolated time scale, and let $r \in (-1, \infty)_Q$. Then Definition 3.1.1 determines uniquely the power function $\hat{h}_r(t, s)$ for all $s, t \in T$ such that $t > s$.

Theorem 3.1.4. Let $\mathbb{T}$ be an isolated time scale, and let $r \in (-1, \infty)_Q$, $s, t \in \mathbb{T}$, $t > s$, $\nu(t) \neq \nu(s)$ for all $t, s \in \mathbb{T}$. Then following formula holds

$$\hat{h}_r(t, s) = \frac{\nu(t)\hat{h}_r(t, \sigma(s)) - \nu(\sigma(s))\hat{h}_r(\rho(t), s)}{\nu(t) - \nu(\sigma(s))}.$$

Theorem 3.1.5. Let $\mathbb{T}$ be an isolated time scale, $t \in \mathbb{T}$ and let $r \in (-1, \infty)_\mathbb{Q}$. Then following statements are true

a) $\hat{h}_r(t, t) = 0$ for $r > 0$

b) $\hat{h}_r(t, t) = 0$ for $r = 0$

c) the value of $\hat{h}_r(t, t)$ for $-1 < r < 0$ is unbounded.

Definition 3.1.3. Let $\mathbb{T}_\mathcal{L}$ be an isolated time scale such that $sup\{\mathbb{T}_\mathcal{L}\} = \infty$ and $sup\{\nu(t), t \in \mathbb{T}_\mathcal{L}\} < \infty$.

Theorem 3.1.6. Let $a \in \mathbb{T}_\mathcal{L}$, and let $r \in (-1, \infty)_\mathbb{Q}$. Then it holds

$$\mathcal{L}_a\{\hat{h}_r(., a)\}(z) = z^{-r-1}.$$

Theorem 3.1.7. Let $\mathbb{T}$ be an arbitrary time scale. Then

$$\mathcal{L}_s\{1\}(z) = \frac{1}{z}.$$

Proof.

$$\mathcal{L}_s\{1\}(z) = \int_s^\infty \hat{e}_{\ominus z}(\rho(\eta), s)\nabla\eta = -\frac{1}{z}\,\hat{e}_{\ominus z}(\eta, s)|_{\eta\to s}^{\eta\to\infty} = \frac{1}{z}.$$

□

Theorem 3.1.8. Let $\mathbb{T}$ be an arbitrary time scale such that Laplace transform exists, $\alpha \in \mathbb{Q}$. Then Laplace transform of $\hat{h}_\alpha$ is $\frac{1}{z^{\alpha+1}}$.

Proof. We know that $\mathcal{L}_s\{\hat{h}_k(t,s)\}(z) = \frac{1}{z^{k+1}}$ for $n \in \mathbb{N}$. Now let Laplace transform of the convolution of m times $\hat{h}_\beta$ be

$$\mathcal{L}\{\hat{h}_\beta * \hat{h}_\beta ... * \hat{h}_\beta * \hat{h}_\beta\}(z) = \frac{1}{z^k}.$$

Via 3.1.2

$$\mathcal{L}\{\hat{h}_{m\beta+m-1}\}(z) = \frac{1}{z^k}$$
$$\mathcal{L}\{\hat{h}_{m\beta}\}(z) = \frac{1}{z^{k+1-m}}.$$

We may assume that $k = m\beta + m$, so $\beta = \frac{k}{m} - 1$. Then

$$\mathcal{L}_s\{\hat{h}_\beta\}(z) = \frac{1}{z^{\frac{k}{m}}} = \frac{1}{z^{\beta+1}}.$$

□

Theorem 3.1.9. Power functions on $\mathbb{T}$, where Laplace transform exists and is unique, are defined correctly by the Definition 3.1.1.

Proof. Proof is the direct consequence of Theorem 3.1.8. If we calculate Laplace transform using properties of Definition 3.1.1, and we are able to get only one result, the uniqueness of generalized time scale nabla Laplace transform provides the uniqueness of the Definition 3.1.1, except of the null function. □

3.2 Fractional operators on time scales

In this section we present fractional operators as well as some of their properties.

Definition 3.2.1. Let $\tilde{\mathbb{T}}$ be the time scale, such that the time scale monomials of any rational order are defined uniquely.

Example 3.2.1. All isolated time scales satisfy the Definition 3.2.1. Also $\mathbb{R}$. As we showed before, also time scales such that on $t \in [s,a]$ is backward graininess function zero and on $\tau \in (s,\infty)$ is backward graininess function constant.

Definition 3.2.2. Let $\gamma \geq 0$, $\alpha > 0$, $\tilde{a}, a, b \in \tilde{\mathbb{T}}$ be such that $\tilde{a} \leq a < b$. Then for the function $f : (\tilde{a}, b]_{\mathbb{T}} \to \mathbb{R}$ we define

a) the fractional integral of order $\gamma > 0$ with the lower limit a as

$$_a\nabla^{-\gamma} f(t) = \int_a^t \hat{h}_{\gamma-1}(t, \rho(\tau)) f(\tau) \nabla\tau, \quad t \in [a, b]_{\mathbb{T}} \cap (\tilde{a}, b]_{\mathbb{T}}$$

and for $\gamma = 0$, we put $_a\nabla^0 f(t) = f(t)$,

b) the Riemann-Liouville fractional derivative of order α with the lower limit a as

$$_a\nabla^{\alpha} f(t) = \nabla^{\lceil \alpha \rceil} {}_a\nabla^{\alpha - \lceil \alpha \rceil} f(t), \quad t \in [\sigma(a), b]_{\mathbb{T}} \cap (\sigma(\tilde{a}), b]_{\mathbb{T}}$$

c) the Caputo fractional derivative of order α with the lower limit a $(a > \tilde{a})$ as

$$^C_a\nabla^{\alpha} f(t) = {}_a\nabla^{\alpha - \lceil \alpha \rceil} \nabla^{\lceil \alpha \rceil} f(t), \quad t \in [\sigma(a), b]_{\mathbb{T}}$$

Theorem 3.2.1. Let $\gamma \geq 0$, $\alpha > 0$, $\tilde{a}, a, b \in \tilde{\mathbb{T}}$ be such that $\tilde{a} \leq a < b$. Then for any function $f : (\tilde{a}, b]_{\mathbb{T}} \to \mathbb{R}$ the fractional integral with the lower limit a is linear.

Proof. Linearity of fractional integral follows directly from the linearity of the time scale integral. □

For $\gamma = 1$, Definition 3.2.2 a) is reduced to a formula for the anti-derivative

$$_a\nabla^{-1} f(t) = \int_a^t f(\tau) \nabla\tau$$

known from the time scales theory. We also may note that in Definition 3.2.2 a), if $a > \tilde{a}$, we get usual definitions of difference calculus.

Theorem 3.2.2. Let $\alpha \in \mathbb{R}$, $\beta \in (-1, \infty)$ and $s, t \in \tilde{\mathbb{T}}$ such that $s > t$. Then it holds

$$_a\nabla^{\alpha} \hat{h}_{\beta}(t, a) = \begin{cases} \hat{h}_{\beta-\alpha}(t, s) & \text{for} \quad \beta > \alpha - 1 \\ 0 & \text{for} \quad \beta \in \{\alpha - \lceil \alpha \rceil, \alpha - \lceil \alpha \rceil + 1, ..., \alpha - 1\}. \end{cases}$$

Theorem 3.2.3. Let $\alpha > 0$, $\beta \in (-1, \infty)$ and $s, t \in \tilde{\mathbb{T}}$ be such that $t > s$. Then it holds

$$^C_a\nabla^{\alpha} \hat{h}_{\beta}(t, a) = \begin{cases} \hat{h}_{\beta-\alpha}(t, s) & \text{for} \quad \beta > \lceil \alpha \rceil - 1 \\ 0 & \text{for} \quad \beta \in \{\alpha - \lceil \alpha \rceil, \alpha - \lceil \alpha \rceil + 1, ..., \alpha - 1\}. \end{cases}$$

4 CONCLUSIONS

In the first few chapters the bachelor thesis covers basics of the time scale calculus (including the generalized time scale exponential function, the generalized time scale nabla Laplace transform) including demonstrative examples to fully understand the matter and deal with theory needed to pursue the later results.

The bachelor thesis contains results comprising investigation of axiomatic definition of power functions on time scales primarily using Laplace transform as a tool to provide the property of the uniqueness of definition used to define mentioned power functions. It shows relations of investigating the uniqueness of generalized time scale nabla Laplace transform and investigating uniqueness of the definition and concludes, that it is basically the same problem ergo by proving the uniqueness of generalized time scale Laplace transform we also show uniqueness of the definition of time scale monomials, as well as some other functions.

It also comprises in depth investigation of Lerch's theorem as well as method for proving such theorem. We also extended some known results about Lerch's theorem as well as about time scale power functions. We believe we contributed to the development of the fractional calculus on time scales and our results will be extended and summarized into a scientific paper later on.

We also developed properties of fractional operators, using uniqueness property obtained by extending Lerch's theorem.

Further investigation of the uniqueness of Laplace transform and possible generalization of Lerch's theorem to an arbitrary time scale is still open question for the future research and we proposed some facts we believe will be needed for obtaining general proof of Lerch's theorem, and thus we enhanced possible future development of fractional calculus on time scales.

Our work in the field of the fractional calculus will continue and hopefully, the proof of Lerch's theorem valid for every time scale, where Laplace transform may be defined, will be discovered.

BIBLIOGRAPHY

[1] S.Hilger. *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten, Ph. D. thesis*. Universität Würzburg, 1988.

[2] Keith B. Oldham and Jerome Spanier. *Fractional Calculus (Mathematics in Science and Engineering, Vol. 111)*. Elsevier Science, 1974. ISBN 0125255500.

[3] R. P. Agarwal. Certain fractional q-integrals and q-derivatives. 66(02):365, sep 1969. doi: 10.1017/s0305004100045060. URL `http://dx.doi.org/10.1017/S0305004100045060`.

[4] J. B. Díaz and T. J. Osler. Differences of fractional order. *Math. Comp.*, 28(125): 185–185, jan 1974. doi: 10.1090/s0025-5718-1974-0346352-5. URL `http://dx.doi.org/10.1090/S0025-5718-1974-0346352-5`.

[5] Martin Bohner and Allan Peterson. *Dynamic Equations on Time Scales: An Introduction with Applications*. Birkhäuser, 2001. ISBN 0817642250.

[6] Martin Bohner and Allan Peterson. *Advances in Dynamic Equations on Time Scales*. Birkhäuser, 2002. ISBN 978-0-8176-8230-9.

[7] C. Pötzsche. *Chain rule and invariance principle on measure chains*. J. Comput. Appl. Math., 2001.

[8] T. Kisela. *Basics of qualitative theory of linear fractional difference equations, Ph.D thesis*. Brno University of Technology, Faculty of Mechanical Engineering, 2012.

[9] C.R. Andreht. *Properties of the Generalized Laplace Transform and Transport Partial Dynamic Equation on Time Scales*. University of Nebraska, 2010.

[10] E.J. Watson. *Laplace Transformations and Applications (VNR new mathematics library ; 10)*. Van Nostrand Reinhold Company, 1981. ISBN 0442301766.

[11] G. Doetsch. *Introduction to the Theory and Application of the Laplace Transformation*. Springer, 1974. ISBN 3540064079.

[12] Tomáš Kisela. Power functions and essentials of fractional calculus on isolated time scales. *Advances in Difference Equations*, 2013(1):259, 2013. doi: 10.1186/1687-1847-2013-259.

LIST OF SYMBOLS

$\hat{\xi}_h$	h cylinder transformation
$\hat{e}_f(t,s)$	Nabla exponential function with exponent f and arguments t and s
$\int f(t)\nabla t$	Nabla integral of function f
$\mathbb{C}_h$	Hilger complex numbers
$\mathbb{T}^\kappa$	Truncated time scale
$\mathcal{L}_s f(z)$	Laplace transform of f on $[s,\infty]$ to variable z
$\mathcal{R}_\nu$	Class of ν regressive functions
$\mathcal{R}_\nu^+$	Class of positively ν regressive functions
μ	Forward graininess function
ν	Backward graininess function
$\odot$	Circle dot multiplication
$\ominus, \ominus_\nu$	Circle minus subtraction
$\oplus, \oplus_\nu$	Circle plus addition
ρ	Backward jump operator
σ	Forward jump operator
$expf$	Real exponential function with exponent f
f'	Real derivative of function f
$f * g$	Convolution operation of f and g
f^∇ or ∇f	Nabla derivative of function f
$inf\{M\}$	Infimum of M
$sup\{M\}$	Supremum of M
$\mathbb{C}$	Complex numbers
$\mathbb{N}$	Positive integers

$\mathbb{Q}$	Rational numbers
$\mathbb{R}$	Real numbers
$\mathbb{T}$	Time scale
$\mathbb{Z}$	Integers