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ÚSTAV MATEMATIKY

NONLINEAR DIFFERENTIAL EQUATIONS IN THE FRAMEWORK OF THE KARAMATA THEORY
NELINEÁRNÍ DIFERENCIÁLNÍ ROVNICE A KARAMATOVA TEORIE

MASTER'S THESIS
DIPLOMOVÁ PRÁCE

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Specification Master's Thesis

Department:	Institute of Mathematics
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Pursuant to Act no. 111/1998 concerning universities and the BUT study and examination rules, you have been assigned the following topic by the institute director Master's Thesis:

Nonlinear differential equations in the framework of the Karamata theory

Concise characteristic of the task:

It will be studied asymptotic behavior of solutions to nonlinear (in particular, sublinear, superlinear, and nearly–linear) differential equations that are important in applications. An important role in the analysis will be played by the theory of regular variation, fixed point theorems, and inequalities.

Goals Master's Thesis:

1. An overview of basic concepts from the theory of regular variation.
2. A description and possibly comparison of selected important approaches (related to utilization of the Karamata theory in differential equations) that appear in the topical literature.
3. Application of regular variation (in combination with other tools) in the study of asymptotic behavior of solutions to differential equation of a particular type.

Recommended bibliography:

BINGHAM, N. H., GOLDIE, C. M., TEUGELS, J. L. Regular Variation. Encyclopedia of Mathematics and its Applications, Vol. 27, Cambridge Univ. Press, 1987.

MARIC, V. Regular Variation and Differential Equations, Lecture Notes in Mathematics 1726, Springer-Verlag, Berlin-Heidelberg-New York, 2000.

ŘEHÁK, P. Nonlinear Differential Equations in the Framework of Regular Variation, AMathNet 2014.

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Abstrakt

Cílem této diplomové práce je sjednotit a zobecnit známé výsledky z literatury, studovat asymptotické chování kladných regulárně se měnících řešení jisté třídy nelineárních diferenciálních rovnic (tzv. skoro pololineárních diferenciálních rovnic) pomocí dostupných nástrojů. Tato práce zahrnuje popis teorie regulární variace, některé informace o nelineárních diferenciálních rovnicích různých typů, detailní odvození výsledků týkajících se asymptotického chování řešení a příklady aplikace získaných výsledků.

Abstract

The goal of the thesis is to unify and generalize known results from literature, to study asymptotic behaviour of positive regularly varying solutions to the certain type of non-linear differential equations (known as nearly-half-linear differential equations) using available tools. This work includes description of theory of regular variation, some information on non-linear differential equations of various types, detailed derivations of results related to asymptotic behaviour of the solutions and examples of application of obtained results.

Klíčová slova

nelineární diferenciální rovnice druhého řádu, regulárně se měnící funkce, asymptotické chování

Keywords

non-linear second order differential equation, regularly varying function, asymptotic behaviour

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I declare that I have written the master's thesis *Nonlinear differential equations in the framework of the Karamata theory* independently and supervised by doc. Mgr. Pavel Řehák, Ph.D. using the sources listed in the list of references.

Denys Bukotin

Contents

Introduction

The concept of regular variation was first introduced in 1930 in the Jovan Karamata's paper [9]. Regular variation is factually a field in classical real variable theory, together with its applications in integral transforms – complex analysis, probability theory, analytic number theory and differential equations (see [2], [7], [8], [11], [17]). Among others, it was applied in Tauberian theorems, giving in fact asymptotic behavior of integrals and series, the Fourier ones in particular. The theory of regular variation has been shown as a very useful tool in some fields of qualitative theory of differential equations of various forms (see [11] and [14]). The most complete presentation of Karamata theory and its generalizations as well as the majority of the applications are contained in [2].

In this work we study asymptotic behaviour of solutions to non-linear second order differential equations of different types: half-linear, nearly-linear and in some sense "combination" of them – nearly-half-linear. They have not been studied a lot yet, but these types of equations are shown as an useful tool in applications, for example for modelling of fluid mechanics problems. Exploration of behaviour of solutions is made by means of regular variation and de Haan theory.

In the first chapter we introduce basic definitions from the Karamata theory. We provide important theorems and show properties of regularly varying functions. We also describe de Haan theory and present the definition and properties of a special class of functions called Π-class.

In Chapter 2 we discuss some types of non-linear second order differential equations. We introduce nearly-half-linear equations and explain how this type of equations is related to half-linear and nearly-linear equations. We briefly present some known results and applications of these equations from literature. Here we formulate our main goal: to unify and generalize asymptotic formulae for slowly varying solutions of the nearly-half-linear equations.

Chapter 3 deals with nearly-half-linear equations and we study behaviour of their slowly varying solutions. We investigate existence of such solutions, discuss required conditions for deducing the asymptotic formula for them using different approaches and summarise obtained results. This chapter is divided to two sections where in first we are interested in decreasing slowly varying solutions, and in the second we work with increasing ones. We also prove other important results, such as statements related to monotonicity of slowly varying solutions and asymptotic estimates of such solutions. Many of those results are new or can be taken as an improvement or extension of existing results for special cases.

Chapter 4 is devoted to presentation of a couple of examples of equations we discussed earlier in Chapter 3. We show applications of the results obtained in the previous chapters and other literature and discuss different modifications of such equations.

The last chapter describes possible directions of further exploration of asymptotic behaviour of solutions to nearly-half-linear equations, specifically solutions which are not slowly varying, asymptotic estimates for the general case of the equations and other methods, which can be useful for resolving additional problems.

1 Theory of Regular Variation

In this chapter we will provide main definitions, important properties and show some examples of regularly varying functions. Further we will introduce important facts from Karamata theory, which will play significant role in the latter chapters and will be used for proving theorems. The main aim of this chapter is to prepare all needed information for latter exploration of asymptotic behaviour of solutions of certain types of non-linear differential equations.

1.1 Regular and slow variation

In its basic form the theory of regular variation studies relations such that

$$\frac{f(\lambda t)}{f(t)} \to g(\lambda) \in (0, \infty) \text{ as } t \to \infty \text{ for every } \lambda > 0.$$

We start with two fundamental definitions of regular variation and slow variation.

Definition 1.1. A measurable function $f : [a, \infty) \to (0, \infty)$ is called *regularly varying (at infinity) of index* ϑ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\vartheta} \text{ for every } \lambda > 0; \tag{1.1}$$

we write $f \in \mathcal{RV}(\vartheta)$. The class of all regularly varying functions is denoted as

$$\mathcal{RV} = \bigcup_{\vartheta\in\mathbb{R}} \mathcal{RV}(\vartheta).$$

Definition 1.2. A measurable function $L : [a, \infty) \to (0, \infty)$ is called *slowly varying (at infinity)* if

$$\lim_{t\to\infty} \frac{L(\lambda t)}{L(t)} = 1; \tag{1.2}$$

we write $L \in \mathcal{SV}$.

The set of slowly varying functions is a proper subset of the set or regularly varying functions and in fact, $\mathcal{SV} = \mathcal{RV}(0)$. The condition in the definition of $\mathcal{RV}$ functions mentioned above can be weakened. The limit in the Definition 1.1 is sufficient to hold only for λ in a set of positive measure a then the regular variation follows. Moreover, if the limit

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = g(\lambda) \in (0, \infty)$$

exists for λ in a set of positive measure, then the function g is necessarily in the form $g(\lambda) = \lambda^{\vartheta}$, where ϑ is a real number.

A slowly varying function is customarily denoted by L because of the first letter of the French word "lentement" which means "slowly". Using Definitions 1.1 and 1.2, it is easy to show that $f \in \mathcal{RV}(\vartheta)$, where $\vartheta \in \mathbb{R}$, if and only if it is possible to write the function in the form

$$f(t) = t^{\vartheta} L(t), \text{ where } L \in \mathcal{SV}. \tag{1.3}$$

So for many purposes in the study of regular variation it is enough to explore the properties of slowly varying functions. Let us give some examples of such functions:

$$
\begin{aligned}
&L(t) = \prod_{i=1}^{n} (\ln_i t)^{\mu_i}, \text{ where } \ln_i t = \ln \ln_{i-1} t \text{ and } \mu_i \in \mathbb{R},\\
&L(t) = \exp\Big(\prod_{i=1}^{n} (\ln_i t)\Big)^{\nu_i}, \text{ where } 0 < \nu_i < 1,\\
&L(t) = 2 + \sin(\ln_2 t),\\
&L(t) = \frac{1}{t}\int_a^t \frac{1}{\ln s} \mathrm{d}s,\\
&L(t) = \exp\big((\ln t)^{\frac{1}{3}} \cos(\ln t)^{\frac{1}{3}}\big).
\end{aligned}
$$

Let us prove slow variation of selected functions. We start with the simplest one $L(t) = \ln t$, then applying l'Hospital's rule we have:

$$\lim_{t\to\infty} \frac{\ln(\lambda t)}{\ln t} = \lim_{t\to\infty} \frac{(\ln(\lambda t))'}{(\ln t)'} = \frac{\lambda/\lambda t}{1/t} = \frac{t}{t} = 1.$$

Let us add a power $L(t) = (\ln t)^{\mu}$ and we obtain again a slowly varying function by Proposition 1.1 presented below, which says that if $f \in \mathcal{RV}(\vartheta)$, then $f^a \in \mathcal{RV}(\alpha\vartheta)$ for every $\alpha \in \mathbb{R}$. Let us consider another function $L(t) = \ln(\ln t)$, then we have:

$$\lim_{t\to\infty} \frac{\ln(\ln(\lambda t))}{\ln(\ln t)} = \lim_{t\to\infty} \frac{(\ln(\ln(\lambda t)))'}{(\ln(\ln t))'} = \lim_{t\to\infty} \frac{t \ln t}{t \ln(\lambda t)} = 1.$$

Thus we get $L(t) = \ln(\ln t) \in \mathcal{SV}$. If we take $L(t) = (\ln t)^{\mu_1}(\ln(\ln t))^{\mu_2}$, then using Proposition 1.1, we can conclude that $L(t) = \prod_{i=1}^{n} (\ln_i t)^{\mu_i} \in \mathcal{SV}$.

Let us take a function $L(t) = 2 + \sin(\ln_2 t)$. To prove that this function is slowly varying we will use again properties of regularly varying functions from Proposition 1.1. We want to show that $tg'(t)/g(t) \to \vartheta$, $g \in C^1$, then $g \in \mathcal{RV}(\vartheta)$ such that $g(t) \sim f(t)$ as $t \to \infty$ and so $f \in \mathcal{RV}(\vartheta), \vartheta \neq 0$. If we deal with a slowly varying function, we assume $\vartheta = 0$. Let us prove that $tL'(t)/L(t) \to 0$ as $t \to \infty$. Compute

$$\frac{tL'(t)}{L(t)} = \frac{t \cos(\ln_2 t)}{t \ln t(2 + \sin(\ln_2 t))} \to 0$$

as $t \to \infty$, because a cosine/sine function is bounded and $\ln t \to \infty$. Now we can conclude that $L(t) = 2 + \sin(\ln_2 t) \in \mathcal{SV}$.

Let us prove slow variation of the function $L(t) = \frac{1}{t}\int_a^t \frac{1}{\ln s}\mathrm{d}s$. Recall that $\ln t \in \mathcal{SV}$ and use Karamata's theorem 1.3, which will be introduced in the next section we prove that

$$L(t) = \frac{1}{t}\int_a^t \frac{1}{\ln s}\mathrm{d}s \sim \frac{t}{t \ln t} \in \mathcal{SV}.$$

The class $\mathcal{RV}$ includes a wide variety of functions. In particular, slowly varying functions do not need to be monotone eventually. The exponential functions $\exp(t)$ or $\exp(-t)$ are not regularly varying, but $1 + \exp(-t)$ is slowly varying. The last example $L(t) = \exp\big((\ln t)^{\frac{1}{3}} \cos(\ln t)^{\frac{1}{3}}\big)$ provides a slowly varying function which exhibits "infinite oscillation", i.e. $\liminf_{t\to\infty} L(t) = 0$, $\limsup_{t\to\infty} L(t) = \infty$.

We have defined regular variation at infinity. Of course, this is not the only possibility. A measurable function $f : [a, \infty) \to (0, \infty)$ is called *regularly varying at zero of index ϑ* if $\lim_{t\to 0+} \frac{f(\lambda t)}{f(t)} = \lambda^{\vartheta}$ for every $\lambda > 0$ – we write $f \in \mathcal{RV}_0(\vartheta)$. Since regular variation of $f(\cdot)$ at zero of index ϑ means in fact regular variation of $f(1/t)$ at infinity of index $-\vartheta$, properties of $\mathcal{RV}_0$ functions can be easily deduced from theory of $\mathcal{RV}$ functions. Regular variation can now be defined at any finite point by shifting the origin of the function to this point. In the next remark we examine functions for which the limit in (1.1) attains the extreme values.

Remark 1.1. A measurable function $f : [a; 1) \to (0; 1)$ is called *rapidly varying of index 1*, we write $f \in \mathcal{RPV}(1)$, if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \begin{cases} 0 & \text{for } 0 < \lambda < 1, \\ \infty & \text{for } \lambda > 1. \end{cases}$$

and is called *rapidly varying of index $-\infty$*, we write $f \in \mathcal{RPV}(-\infty)$, if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \begin{cases} \infty & \text{for } 0 < \lambda < 1, \\ 0 & \text{for } \lambda > 1. \end{cases}$$

The class of all rapidly varying solutions is denoted as $\mathcal{RPV}$.

Let us introduce a couple of notations which will occur later in this thesis. For eventually positive f and g we denote:

- $f(t) \sim g(t)$ if $\lim_{t\to\infty} f(t)/g(t) = 1$;
- $f(t) = o(g(t))$ if $\lim_{t\to\infty} f(t)/g(t) = 0$;
- $f(t) = O(g(t))$ if $\exists c \in (0, \infty)$ such that $f(t) \le cg(t)$ for large t.

1.2 Karamata theory

In this section we will introduce basic information on Karamata theory, which will help us during the analysis of behaviour of solutions to differential equations. The following theorems are very important in the theory and properties obtained from them will be useful for exploring $\mathcal{RV}$ functions and investigation of solutions to differential equations. The first statement is the so-called Uniform Convergence Theorem.

Theorem 1.1. *If $f \in \mathcal{RV}(\vartheta)$, then the relation* (1.1) *(and so* (1.2)*) holds uniformly on each compact λ-set in* $(0, \infty)$.

The second fundamental result is the following Representation theorem. Its proof is based on Theorem 1.1.

Theorem 1.2. *(Representation theorem) A function L is slowly varying if and only if it has the form.*

$$L(t) = \phi(t) \exp\Big[\int_a^t \frac{\psi(s)}{s} \mathrm{d}s\Big], \tag{1.4}$$

$t \geq a$, for some $a > 0$, where ϕ, ψ are measurable with $\lim_{t\to\infty} \phi(t) = C \in (0, \infty)$ and $\lim_{t\to\infty} \psi(t) = 0$. A function $f \in \mathcal{RV}(\vartheta)$ if and only if

$$f(t) = \phi(t) t^{\vartheta} \exp\Big[\int_a^t \frac{\psi(s)}{s} \mathrm{d}s\Big], \tag{1.5}$$

$t \geq a$, for some $a > 0$, where ϕ, ψ are measurable with $\lim_{t\to\infty} \phi(t) = C \in (0, \infty)$ and $\lim_{t\to\infty} \psi(t) = 0$.

Since L, ϕ, ψ may get changed on finite intervals, the value of a is unimportant: if $a = 0$ one can take $\psi \equiv 0$ on a neighbourhood of 0 to avoid divergence of the integral at the origin. The Karamata's representation (1.4) is essentially non-unique: within limits, one may always adjust one of $\phi(t)$, $\psi(t)$ making a compensating adjustment to the other. From some points of view slowly varying functions are of our interest only to within an asymptotic equivalence. We would not lose anything by restricting attention to the case $\phi(t) \equiv 0$ in (1.4) or (1.5). The following definition is appropriate to this case.

Definition 1.3. The regularly varying function of index ϑ

$$f(t) = C t^{\vartheta} \exp\Big[\int_a^t \frac{\psi(s)}{s} \mathrm{d}s\Big], \tag{1.6}$$

$\lim_{t\to\infty} \psi(t) = 0$, $C \in (0, \infty)$, is called *normalized*. We write $f \in \mathcal{NRV}(\vartheta)$. The set of normalized slowly varying functions, i.e., $\mathcal{NRV}(0)$, is denoted as $\mathcal{NSV}$.

If f is a C^1 function and $\lim_{t\to\infty} t f'(t)/f(t) = \vartheta$, then $f \in \mathcal{NRV}(\vartheta)$. Conversely, if $f \in \mathcal{NRV}(\vartheta) \cap C^1$, then $\lim_{t\to\infty} t f'(t)/f(t) = \vartheta$.

The following results will be useful in applications to the theory of differential equations. This theorem is also called Karamata's theorem and will be used for proving formulae for asymptotic solutions of different types to differential equations. The proof of the theorem is provided in [2].

Theorem 1.3. *(Karamata's theorem) If $L \in \mathcal{SV}$, then*

$$\int_t^{\infty} s^{\vartheta} L(s) \mathrm{d}s \sim \frac{1}{-\vartheta - 1} t^{\vartheta+1} L(t) \text{ provided } \vartheta < -1, \tag{1.7}$$

$$\int_a^t s^{\vartheta} L(s) \mathrm{d}s \sim \frac{1}{\vartheta + 1} t^{\vartheta+1} L(t) \text{ provided } \vartheta > -1. \tag{1.8}$$

at $t \to \infty$. Moreover, if $\int_a^{\infty} L(s)/s\mathrm{d}s$ converges, then $\tilde{L} = \int_t^{\infty} L(s)/s\mathrm{d}s$ is a $\mathcal{SV}$ function; if $\int_a^{\infty} L(s)/s\mathrm{d}s$ diverges, then $\tilde{L} = \int_a^t L(s)/s\mathrm{d}s$ is a $\mathcal{SV}$ function. In both cases $L(t)/\tilde{L}(t) \to 0$ as $t \to \infty$.

Let us provide some properties of regularly varying functions. More details on these proofs can be found in monographs [2], [7] and [17].

Proposition 1.1.

- *If $f \in \mathcal{RV}(\vartheta)$, then $\ln f(t)/\ln t \to \vartheta$ as $t \to \infty$. It then implies that $\lim_{t\to\infty} f(t) = 0$ provided $\vartheta < 0$ and $\lim_{t\to\infty} f(t) = \infty$ provided $\vartheta > 0$;*

- *If* $f \in \mathcal{RV}(\vartheta)$, *then* $f^a \in \mathcal{RV}(\alpha\vartheta)$ *for every* $\alpha \in \mathbb{R}$*;*
- *If* $f_i \in \mathcal{RV}(\vartheta_i),\ i = 1, 2,\ f_2(t) \to \infty$ *as* $t \to \infty$, *then* $f_1 \circ f_2 \in \mathcal{RV}(\vartheta_1\vartheta_2)$*;*
- *If* $f_i \in \mathcal{RV}(\vartheta_i),\ i = 1, 2$, *then* $f_1 + f_2 \in \mathcal{RV}(\max\{\vartheta_1, \vartheta_2\})$*;*
- *If* $f_i \in \mathcal{RV}(\vartheta_i),\ i = 1, 2$, *then* $f_1 f_2 \in \mathcal{RV}(\vartheta_1 + \vartheta_2)$*;*
- *If* $f_1, \dots, f_n \in \mathcal{RV}$, $n \in \mathbb{N}$ *and* $R(x_1, \dots, x_n)$ *is a rational function with non-negative coefficients, then* $R(f_1, \dots, f_n) \in \mathcal{RV}$*;*
- *If* $L \in \mathcal{SV}$ *and* $\vartheta > 0$, *then* $t^{\vartheta} L(t) \to \infty,\ t^{-\vartheta} L(t) \to 0$ *as* $t \to \infty$*;*
- *If* $f \in \mathcal{RV}(\vartheta), \vartheta \neq 0$, *then there exists* $g \in C^1$ *with* $g(t) \sim f(t)$ *as* $t \to \infty$ *and such that* $tg'(t)/g(t) \to \vartheta$, *hence* $g \in \mathcal{NRV}(\vartheta)$. *Moreover,* g *can be taken such that* $|g'| \in \mathcal{NRV}(\vartheta - 1)$*;*
- *If* $|f'| \in \mathcal{RV}(\vartheta),\ \vartheta \neq -1$ *with* f' *being eventually of one sign, then* $f \in \mathcal{NRV}(\vartheta + 1)$*;*
- *Let* $g \in \mathcal{RV}_0(\vartheta)$ *with* $\vartheta > 0$ *be increasing in a right neighbourhood of zero. Then* $g^{-1} \in \mathcal{RV}_0(1/\vartheta)$, *where* g^{-1} *stands for the inverse of* g.

1.3 De Haan theory

De Haan theory can be understood as a refinement of Karamata theory. The theory was studied by de Haan in his thesis of 1970 [8].

Definition 1.4. A measurable function $f \in [a, \infty) \to \mathbb{R}$ is said to belong to the class Π if there exists a function $\omega : (0, \infty) \to (0, \infty)$ such that for $\lambda > 0$

$$\lim_{t \to \infty} \frac{f(\lambda t) - f(t)}{\omega(t)} = \ln \lambda; \tag{1.9}$$

we write $f \in \Pi$ or $f \in \Pi(\omega)$. The function w is called an *auxiliary function* for f.

Let us give some examples of functions belonging to the class Π. The functions f defined by

$$\begin{aligned} f(t) &= \ln t + o(1), \\ f(t) &= (\ln t)^{\alpha} (\ln_2 t)^{\beta} + o((\ln t)^{\alpha - 1}),\ \alpha > 0,\ \beta \in \mathbb{R}, \\ f(t) &= \exp((\ln t)^{\gamma}) + o((\ln t)^{\gamma - 1}) \exp((\ln t)^{\gamma}),\ 0 < \gamma < 1, \end{aligned}$$

are in Π. For example, the function $f(t) = 2 \ln t + \sin(\ln t)$ is in $\mathcal{SV}$ but not in Π.

Now we will show selected properties of functions in the class Π. The proves of them are presented in [7] and [2].

Proposition 1.2.

- *If* $0 < c < d < \infty$ *relation* (1.9) *holds uniformly for* $\lambda \in [c, d]$*;*
- *Auxiliary function is unique up to asymptotic equivalence;*

- *The statements* $f \in \Pi$ *and there exists* $L \in \mathcal{SV}$ *such that*

$$f(t) = L(t) + \int_a^t \frac{L(s)}{s} ds \tag{1.10}$$

are equivalent;

- *If* f *satisfies* (1.10)*, then* $f \in \Pi(L)$ *and*

$$L(t) \sim f(t) - \frac{1}{t} \int_a^t f(s) ds \tag{1.11}$$

as $t \to \infty$*. If* $f \in \Pi(L)$ *is integrable on finite intervals of* $(0, \infty)$*, then* (1.11) *holds;*

- *If* $f \in \Pi$*, then* $\lim_{t\to\infty} f(t) =: f(\infty) \le \infty$ *exists. If the limit is infinite, then* $f \in \mathcal{SV}$*. If the limit is finite, then* $f(\infty) - f(t) \in \mathcal{SV}$*.*

Let us prove one more proposition, which is very useful and will be used in the next chapters.

Proposition 1.3. *If* $f' \in \mathcal{RV}(-1)$*, then* $f \in \Pi(tf'(t))$*.*

Proof. We have $f' \in \mathcal{RV}(-1)$. Let us check if (1.9) is satisfied for $w(t) = tf'(t)$. Integrating by substitution $u = st$, for every $\lambda > 0$ and using Uniform Convergence Theorem 1.1 we obtain

$$\frac{f(\lambda t) - f(t)}{tf'(t)} = \int_t^{\lambda t} \frac{f'(u)}{tf'(t)} \mathrm{d}u = \int_1^{\lambda} \frac{f'(st)}{f'(t)} \mathrm{d}s \to \int_1^{\lambda} \frac{1}{s} \mathrm{d}s = \ln \lambda, \tag{1.12}$$

as $t \to \infty$. □

Let us prove that the functions presented above before are indeed in the class Π. The proof that the first function $f(t) = \ln t + o(1)$ belongs to the class Π is easy. Indeed, if we take $w(t) = tf'(t) = t \cdot 1/t$, then

$$\frac{\ln(\lambda t) - \ln t}{t \cdot 1/t} = \ln(\lambda t) - \ln t = \ln\left(\frac{\lambda t}{t}\right) \to \ln \lambda$$

as $t \to \infty$. We continue with a function $f(t) = (\ln t)^2$. Let us use Proposition 1.3 and recall that $\ln t \in \mathcal{SV}$. Compute:

$$f'(t) = \frac{2 \ln t}{t} \in \mathcal{RV}(-1)$$

by Proposition 1.1, thus $f(t) = (\ln t)^2 \in \Pi(2 \ln t)$. In the next step we will prove that $f(t) = (\ln t)(\ln(\ln t))$ belongs to the class Π. Let us use again Proposition 1.3 and the fact that $f(t) = (\ln t)(\ln(\ln t)) \in \mathcal{SV}$. Compute a derivative of f:

$$f'(t) = \frac{\ln(\ln t)}{t} + \frac{1}{t} \in \mathcal{RV}(-1)$$

by Proposition 1.1, so due to the property we mentioned before we are able to conclude that $f \in \Pi(\ln(\ln t) + 1)$.

Let us generalize this result for general powers. Take a function $f(t) = (\ln t)^{\alpha}(\ln(\ln t))^{\beta}$. We will follow the same argumentation, so let us compute f':

$$f' = \frac{\beta(\ln(t))^{\alpha-1}(\ln(\ln t))^{\beta-1}}{t} + \frac{\alpha(\ln(t))^{\alpha-1}(\ln(\ln t))^{\beta}}{t} \in \mathcal{RV}(-1),$$

and from this fact we conclude that

$$f \in \Pi(\beta(\ln(t))^{\alpha-1}(\ln(\ln t))^{\beta-1} + \beta(\ln(t))^{\alpha-1}(\ln(\ln t))^{\beta-1}).$$

Remark 1.2. In [8] by de Haan was introduced and studied another class called Γ, which can be understood as an "inverse" of the class Π. This class is also useful for solving differential equations. A non-decreasing function $f : \mathbb{R} \to (0, \infty)$ is said to belong to the class Γ if there exists a function $v : (0, \infty) \to (0, \infty)$ such that for all $\lambda \in \mathbb{R}$

$$\lim_{t\to\infty} \frac{f(\lambda t) + \lambda v(t)}{f(t)} = e^{\lambda};$$

we write $f \in \Gamma$ or $f \in \Gamma(v)$.

2 Non-Linear Second Order Differential Equations

In the last decades a great attention has been paid to the differential equations with *p-Laplacian*. Let us recall that the p-Laplacian is a partial differential operator of the form:

$$\Delta_p u := \text{div } (||\nabla u||^{p-2}\nabla u),$$

where for $u = u(x) = u(x_1, \dots, x_N)$

$$\nabla u = \Big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_N}\Big)$$

is the Hamilton nabla operator and for $v(x) = (v_1(x), \dots, v_N(x))$

$$\text{div } v(x) = \sum_{i=1}^{N} \frac{\partial v}{\partial x_i}(x)$$

is the usual divergence operator. If u is a radially symmetric function, i.e. $u(x) = y(t)$, $t = ||x||$, $||\cdot||$ being the Euclidean norm in $\mathbb{R}^N$, the (partial) differential operator Δ_p can be reduced to the ordinary differential operator

$$\Delta_p u(x) = t^{1-N}\big(t^{N-1}\Phi(y'(t))\big)', \quad ' = \frac{\mathrm{d}}{\mathrm{d}t},$$

where $\Phi(u) := |u|^{p-1}\operatorname{sgn} u$, $p > 1$. The p-Laplacian operator is useful for studying and modelling of the flow of a liquid through a porous medium, that was one of the problems which big cities were dealing with in the 18$^{\text{th}}$ century. It was found as a useful tool for the the Darcy's law for the turbulent flow, so the velocity of the flow is higher and/or the aquifer is more coarse-grained. Also, p-Laplacian is useful for exploration of the de Prony's law for small velocities. More information to this topic can be found in [1]. One of the important prototypes of equations with p-Laplacian is a quasilinear differential equation in the form:

$$\big(r(t)\Phi_\alpha(u')\big)' = p(t)\Phi_\lambda(u), \tag{2.1}$$

where $\Phi_\gamma(w) := |w|^{\gamma-1}\operatorname{sgn} w$, $\gamma > 1$. If we take different α and λ, then this class of equations contains also Emden-Fowler equations. Equations and systems of Emden-Fowler type are investigated in the framework of regular variation e.g. in [5] and [10]. On the other hand, if $\alpha = \lambda$, then we get half-linear equations. Even though we are dealing with non-linear equations, on the contrary to (2.1) with different indices α and λ, the half-linear equations in lots of aspects are closer to the linear case and methods used for solving them are different from ones used for (2.1), $\alpha \neq \lambda$. In case of $\alpha = \lambda = 2$, then (2.1) reduces to the linear equation.

Non-linearity does not need to be purely in the form of power functions, but it can have a perturbation in the form of a slowly varying function, which enables us to include a wider set of equations. Half-linear equations can be naturally generalized by substituting $\Phi_\alpha(\cdot)$ with continuous functions $F(|\cdot|)$ and $G(|\cdot|)$ such that $F(u) = |u|^{\alpha-1}\operatorname{sgn}(u)L_F(|u|)$, where $L_F \in \mathcal{SV}$ or $L_F \in \mathcal{SV}_0$ and similarly G such that $G(u) = |u|^{\alpha-1}\operatorname{sgn}(u)L_G(|u|)$, where $L_G \in \mathcal{SV}$ or $L_G \in \mathcal{SV}_0$. The we obtain a differential equation in the form:

$$\big(r(t)G(y'(t))\big)' = p(t)F(y(t)), \tag{2.2}$$

where p and r are positive continuous functions on $[a, \infty)$ and $F(|\cdot|)$ and $G(|\cdot|)$ are continuous functions which are regularly varying (at infinity or at zero) of index $\alpha - 1$, where $\alpha > 1$ and $\Phi_\alpha(u) = |u|^{\alpha-1}\mathrm{sgn}(u)$. To simplify our considerations we suppose that F and G are increasing and odd functions. We use the convention that a slowly varying component of $f \in \mathcal{RV}(\vartheta)$ denoted as $L_f \in \mathcal{RV}$ is represented in the form $L_f = t^{-\vartheta} f(t)$. Examples of functions $F(u)$ and $G(u)$ such that the equation is non-linear and can be explored within the theory are $\Phi_\alpha(u)|\ln|u||$, $\Phi_\alpha(u)/|\ln|u||$ or $\frac{\Phi_\alpha(u)}{\sqrt{1 \pm u^2}}$ and many others. For $\alpha = 2$ the function $\frac{u}{\sqrt{1+u^2}}$ is called the Euclidean mean curvature operator and arises in the search for radial solutions of partial differential equations which model fluid mechanics problems, in particular capillarity-type phenomena for compressible and incompressible fluids. On the other hand, $\frac{u}{\sqrt{1-u^2}}$ is called Minkowski mean curvature operator (or relativity operator) and it is used for studying properties of the mean curvature of hyper-surfaces in the relativity theory.

We call the equation (2.2) "nearly-half-linear", because they can be understood as a combination or unification of two other types of differential equations: half-linear equation (2.3) and nearly-linear equation (2.6), which we will discuss later in the chapter where we will recall important properties of such equations and will present asymptotic formulae for positive solutions.

We define $\beta = \frac{\alpha}{\alpha-1}$ is a conjugate number of α. For function p we will be using index δ so if p is regularly varying then we write $p \in \mathcal{RV}(\delta)$. This index will play significant role in the future analysis of solutions of nearly-half-linear equations, because this index will influences character of slowly varying solutions of (2.2).

The space of solutions of (2.2) is neither homogeneous nor additive. Without loss of generality, we work only with positive solutions, i.e. with the set

$$\mathcal{PS} = \{y : \ y(t) \text{ is a positive solution of (2.2) for large } t\}.$$

Because of the sign condition on the coefficients, all positive solution $\mathcal{PS}$ of (2.2) are eventually monotone, therefore they belong to one of the following disjoint classes of decreasing and increasing solutions:

$$\mathcal{IS} = \{y \in \mathcal{PS} : y'(t) > 0 \text{ for large } t\};$$
$$\mathcal{DS} = \{y \in \mathcal{PS} : y'(t) < 0 \text{ for large } t\}.$$

These classes can be further divided to the disjoint subclasses:

$$\mathcal{IS}_\infty = \{y \in \mathcal{IS} : \lim_{t\to\infty} y(t) = \infty\};$$
$$\mathcal{IS}_B = \{y \in \mathcal{IS} : \lim_{t\to\infty} y(t) = l \in \mathbb{R}\};$$
$$\mathcal{DS}_B = \{y \in \mathcal{DS} : \lim_{t\to\infty} y(t) = l > 0\};$$
$$\mathcal{DS}_0 = \{y \in \mathcal{DS} : \lim_{t\to\infty} y(t) = 0\}.$$

As far as we know nearly-half-linear equations have almost not been studied in the literature. Our main goal is to obtain results, which will generalize known results and using available tools to show similarities of the nearly-half-linear type of equations compared to the half-linear and nearly-linear differential equations. Among others, we will generalize and unify Theorem 2.1 and Theorem 2.2 presented below. Furthermore, we will prove other types of results, which can also be shown as new: nearly-linear equation with general r, increasing solutions in case of nearly-linear equations and other properties of decreasing and increasing solutions of such equations.

2.1 Half-linear differential equations

We consider the equation

$$\big(r(t)\Phi_\alpha(y'(t))\big)' = p(t)\Phi_\alpha(y(t)), \tag{2.3}$$

where p and r are positive continuous functions on $[a,\infty)$ and $\Phi_\alpha(u) = |u|^{\alpha-1}\mathrm{sgn}(u)$ with $\alpha > 1$. This equation is non-oscillatory or in other words all its non-trivial solutions are eventually of constant sign, what explained in [4]. The terminology "half-linear" differential equation reflects the fact that the solution space of (2.3) is homogeneous, but not additive, what is the basic difference between linear and half-linear equations.

Further we will show one illustrative result on positive solutions of half-linear equations. The next theorem provides an asymptotic formula for decreasing solutions of a half-linear differential equation.

Theorem 2.1. (Theorem 5 in [16]) *Let $p \in \mathcal{RV}(\delta)$ and $r \in \mathcal{RV}(\delta+\alpha)$ with $\delta < -1$. Assume $\frac{L_p(t)}{L_r(t)} \to 0$ as $t \to \infty$, then $\mathcal{DS} \subset \mathcal{SV}$. If $y \in \mathcal{DS} \cap \mathcal{SV}$, then $y \in \Pi(-ty'(t))$. Moreover, for every $y \in \mathcal{DS}$,*

- *if $\int_a^\infty (\frac{sp(s)}{r(s)})^{\frac{1}{\alpha-1}} ds = \infty$, then there exists $\varepsilon(t)$ with $\varepsilon(t) \to 0$ as $t \to \infty$ such that*

$$y(t) = \exp\Big\{ -\int_a^t \big(1+\varepsilon(s)\big)\Big[-\frac{sp(s)}{(\delta+1)r(s)}\Big]^{\frac{1}{\alpha-1}} ds\Big\} \tag{2.4}$$

  and $y(t) \to 0$ as $t \to \infty$;

- *if $\int_a^\infty (\frac{sp(s)}{r(s)})^{\frac{1}{\alpha-1}} ds < \infty$, then there exists $\varepsilon(t)$ with $\varepsilon(t) \to 0$ as $t \to \infty$ such that*

$$y(t) = l\exp\Big\{ \int_t^\infty \big(1+\varepsilon(s)\big)\Big[-\frac{sp(s)}{(\delta+1)r(s)}\Big]^{\frac{1}{\alpha-1}} ds\Big\} \tag{2.5}$$

  and $y(t) \to l \in (0,\infty)$ as $t \to \infty$.

2.2 Nearly-linear differential equations

We consider the equation

$$(G(y'(t))' = p(t)F(y(t)), \tag{2.6}$$

where p is positive (at infinity or at zero) continuous functions on $[a,\infty)$ and $F(|\cdot|)$ and $G(|\cdot|)$ are continuous functions on $\mathbb{R}$ which are regularly varying (at infinity or at zero) of index one with $uF(u) > 0$ and $uG(u) > 0$ for $u \neq 0$. This condition justifies the terminology a nearly linear equation. If we make the trivial choice of the functions $F = G = id$, then (2.6) reduces to a linear equation.

We know, that the solution space of (2.3) is neither homogeneous nor additive, but we still are allowed to use the same methods of exploration as for the linear case. In the following theorem we will show a general form of a decreasing solution of a nearly-linear differential equation. The proof of the theorem and following remarks are in [14]. Conditions for existence of decreasing solutions and $\mathcal{DS} \subset \mathcal{NSV}$ are

$$\lim_{t\to\infty} t\int_t^\infty p(s)\mathrm{d}s = 0, \quad \limsup_{u\to 0_+} L_F(u) < \infty, \quad \liminf_{u\to 0_+} L_G(u) > 0.$$

Before we get to the asymptotic formula for solutions we have to introduce function $\hat{F}$ in the way:

$$\hat{F}(x) = \int_1^x \frac{\mathrm{d}u}{F(u)}, \quad x > 0.$$

The constant 1 in the integral can be replaced by any positive constant. Also we denote $\hat{F}^{-1}$ as the inverse function.

Theorem 2.2. (Theorem 2 in [15]) *Let $p \in \mathcal{RV}(-2)$ and $\lim_{u \to 0_+} |\hat{F}(u)| = \infty$. Assume that $L_G(ug(u)) \sim L_G(u)$ as $u \to 0_+$ for all $g \in \mathcal{SV}_0$. If $y \in \mathcal{DS} \cap \mathcal{SV}$, then $y \in \Pi(-ty'(t))$. Moreover, for every $y \in \mathcal{DS}$,*

- *if $\int_a^\infty (\frac{sp(s)}{L_G(1/s)}) ds = \infty$, then there exists $\varepsilon(t)$ with $\varepsilon(t) \to 0$ as $t \to \infty$ such that*

$$y(t) = \hat{F}^{-1}\Big\{ - \int_a^t \big(1 + \varepsilon(s)\big) \frac{sp(s)}{L_G(1/s)} ds \Big\} \tag{2.7}$$

  and $y(t) \to 0$ as $t \to \infty$;

- *if $\int_a^\infty (\frac{sp(s)}{L_G(1/s)}) ds < \infty$, then there exists $\varepsilon(t)$ with $\varepsilon(t) \to 0$ as $t \to \infty$ such that*

$$y(t) = \hat{F}^{-1}\Big\{ \hat{F}\big(y(\infty)\big) + \int_t^\infty \big(1 + \varepsilon(s)\big) \frac{sp(s)}{L_G(1/s)} ds \Big\} \tag{2.8}$$

  and $y(t) \to y(\infty) \in (0, \infty)$ as $t \to \infty$.

3 Asymptotic Behaviour of Solutions to Nearly-Half-Linear Equations

In this chapter we will study asymptotic behaviour of positive solutions of nearly-half-linear differential equations, in particular positive slowly varying solutions. First we will describe the relation between the indices of regular variation of functions p and r. We will divide the results to two groups depending on the indices of regular variation of functions p and r. As we mentioned in the previous chapter, asymptotic behaviour of solutions is affected by the index of regular variation of p, so we will work with these cases separately. We will show different approaches to proving an asymptotic formula for solutions to such equations. Some of these results are new even in the special cases and some of them generalize already known results.

3.1 Decreasing slowly varying solutions

In this section we will be dealing with a nearly-half-linear equation and assume that the function $p(t)$ is regularly varying with index $\delta < -1$. At the beginning we will explore existence of slowly varying solutions. Then we will provide an asymptotic formula for solutions to nearly-half-linear equations. Further we will show a couple of special cases for certain functions p, r and L_F or L_G. At the end of this subsection we will briefly discuss possibility of an asymptotic estimate of the solution and restrictions which are required for proving such statement.

The following theorem proves non-emptiness of the set of positive decreasing solutions. Moreover, we will show that all of such solutions are normalized and slowly varying.

Theorem 3.1. *Consider the equation* (2.2). *Assume that*

$$\int_a^\infty p(s)\mathrm{d}s < \infty, \tag{3.1}$$

$$\lim_{t\to\infty} \frac{t^{\alpha-1}}{r(t)} \int_t^\infty p(s)\mathrm{d}s = 0, \tag{3.2}$$

$$\limsup_{u\to 0_+} L_F(u) < \infty, \tag{3.3}$$

$$\int_a^\infty G^{-1}\Big(\frac{M}{r(s)}\Big)\mathrm{d}s = \infty \quad \textit{for some} \quad M \in (0,\infty) \tag{3.4}$$

$$\textit{and} \quad \liminf_{u\to 0_+} L_G(u) > 0. \tag{3.5}$$

Then $\emptyset \neq \mathcal{DS} \subset \mathcal{NSV}$.

Proof. Firstly, we rewrite the equation (2.2) as an equivalent system of two equations

$$y' = -G^{-1}\Big(\frac{u}{r(t)}\Big), \qquad u' = -p(t)F(y),$$

where G^{-1} is the inverse of function G. For this system we apply the existence theorem from [3], which requires $-G^{-1}(u/r(t)) \le 0$ and $-p(t)F(y) \le 0$ for $y > 0$ and $u > 0$. Both of the conditions are satisfied, so we can conclude that $\mathcal{DS} \neq \emptyset$.

Take $y \in \mathcal{DS}$ so $y(t) > 0,\ y'(t) < 0$ for large t, say $t \ge a$. Then from (2.2) $r(t)G(y'(t))$ is negative increasing for $t \ge a$. Then there exists the limit

$$\lim_{t\to\infty} -r(t)G(-y'(t)) = -M \in (-\infty, 0].$$

If $-M < 0$, then $-r(t)G(-y'(t)) \le -M$ for large t because of the fact that $-r(t)G(-y'(t))$ is increasing. Thus

$$G(y'(t)) \le -\frac{M}{r(t)}.$$

Now use inverse function G^{-1} and we obtain

$$y'(t) \le -G^{-1}\Big(\frac{M}{r(t)}\Big).$$

Integrating the inequality from a to t we have

$$y(t) \le y(a) - \int_a^t G^{-1}\Big(\frac{M}{r(s)}\Big)\mathrm{d}s \to -\infty$$

as $t \to \infty$ due to (3.4), what contradicts with the fact that we are working with positive solutions. It means that we can conclude $M = 0$.

Now integrate the equation (2.2) from t to ∞ and obtain

$$-r(t)G(y'(t)) = \int_t^\infty p(s)F(y(s))\mathrm{d}s. \tag{3.6}$$

Using the definition of functions G and F, recalling that we care only of positive solutions and from the fact that $y \in \mathcal{DS}$ we get

$$\begin{aligned} r(t)(-y'(t))^{\alpha-1}L_G(|y'(t)|) &= \int_t^\infty p(s)(y(s))^{\alpha-1}L_F(y(s))\mathrm{d}s \\ &\le (y(t))^{\alpha-1}\int_t^\infty p(s)L_F(y(s))\mathrm{d}s. \end{aligned}$$

Divide this inequality by $(y(t))^{\alpha-1}$ and multiply by $t^{\alpha-1}$. Thus,

$$\Big(-\frac{ty'(t)}{y(t)}\Big)^{\alpha-1} \le \frac{t^{\alpha-1}L_F(y(t))}{L_G(|y'(t)|)r(t)}\int_t^\infty p(s)\mathrm{d}s \le \frac{t^{\alpha-1}K}{r(t)}\int_t^\infty p(s)\mathrm{d}s \tag{3.7}$$

for large t, where $K \in (0,\infty)$ is some constant reasoned by the conditions (3.3) and (3.5) such that $K = P/N$, where $P \ge L_F(y(t))$ and $1/N \ge 1/L_G(|y'(t)|)$ for large t. Since the right-hand side of the relation (3.7) tends to zero thanks to (3.2), we obtain that $\frac{-ty'(t)}{y(t)} \to 0$ as $t \to \infty$. From this fact $y \in \mathcal{NSV}$ follows. □

For establishing asymptotic formula we will assume that both of the functions p and r are regularly varying. It would be natural for us to assume these functions with general indices $p \in \mathcal{RV}(\delta)$ and $r \in \mathcal{RV}(\gamma)$. In the next remark we will show that when we work with $\mathcal{SV}$ solutions it is in fact necessary to have certain relation between the indices of regular variation of these functions.

Remark 3.1. Assume that $p \in \mathcal{RV}(\delta)$, $r \in \mathcal{RV}(\gamma)$ and $y \in \mathcal{DS}_0 \cap \mathcal{SV}$ of (2.2). Then recalling the definition of F and with the help of properties from Proposition 1.1 we have

$$\big(r(t)G(y'(t))\big)' = p(t)F(y(t)) \in \mathcal{RV}(\delta).$$

Integrate (2.2) from t to ∞ and we get the relation (3.6). We know that $y'(t) < 0$ for $t \geq a$, then we can rewrite the functions F and G as

$$-r(t)(y'(t))^{\alpha-1}L_G(|y'(t)|) = \int_t^\infty p(s)(y(s))^{\alpha-1}L_F(y(s))\mathrm{d}s \in \mathcal{RV}(\delta+1),$$

using properties of $\mathcal{RV}$ from Proposition 1.1 and Karamata's theorem 1.3, and so we get $(y'(t))^{\alpha-1} \in \mathcal{RV}(\delta+1-\gamma)$, because $r \in \mathcal{RV}(\gamma)$ and $L_G \in \mathcal{SV}$, which implies that $y'(t) \in \mathcal{RV}\big(\frac{\delta+1-\gamma}{\alpha-1}\big)$. If $\frac{\delta+1-\gamma}{\alpha-1} \neq -1$ we can easily prove that $y(t) \in \mathcal{RV}\big(\frac{\delta+1-\gamma}{\alpha-1}+1\big)$ using the Karamata's theorem 1.3. Therefore, since $y \in \mathcal{SV}$, i.e. $y \in \mathcal{RV}(0)$, we obtain $\frac{\delta+1-\gamma}{\alpha-1}+1 = 0$ and it is a contradiction with the assumption $\frac{\delta+1-\gamma}{\alpha-1} \neq -1$. Thus $\frac{\delta+1-\gamma}{\alpha-1} = -1$, so we get $\gamma = \delta + \alpha$.

In the following remark we will show how the conditions guaranteeing slow variation of any decreasing solution in Theorem 3.1 can be rewritten in the case when functions $p \in \mathcal{RV}(\delta)$ and $r \in \mathcal{RV}(\delta+\alpha)$ with $\delta < -1$.

Remark 3.2. Assume $p \in \mathcal{RV}(\delta)$ and $r \in \mathcal{RV}(\delta+\alpha)$, where $\delta < -1$. Then by Karamata's theorem 1.3 we get $\int_a^\infty p(s)\mathrm{d}s < \infty$.

If $G \in \mathcal{RV}_0(\alpha-1)$, then $G^{-1} \in \mathcal{RV}_0\big(\frac{1}{\alpha-1}\big)$. Applying Proposition 1.1 we obtain $G^{-1}(1/r(s)) \in \mathcal{RV}((-\delta-\alpha)(1/(\alpha-1))) = \mathcal{RV}((\delta+\alpha)(1-\beta))$ and so we get

$$\int_a^\infty G^{-1}\Big(\frac{M}{r(s)}\Big)\mathrm{d}s = \int_a^\infty r^{1-\beta}(s)\cdot h(s)\mathrm{d}s,$$

where $h \in \mathcal{SV}$. In fact, $\int_a^\infty r^{1-\beta}(s)\mathrm{d}s = \infty$, which follows from

$$(1-\beta)(\delta+\alpha) = (1-\beta)(\delta+1+\alpha-1) = (1-\beta)(\delta+1)-1$$

with $\delta < -1$, so $(1-\beta)(\delta+\alpha) > -1$ and Karamata's theorem 1.3.

Take the right-hand side of (3.7) and apply Karamata's theorem 1.3:

$$\frac{t^{\alpha-1}K}{r(t)}\int_t^\infty p(s)\mathrm{d}s \sim \frac{t^{\alpha-1}K}{t^{\delta+\alpha}L_r(t)}\cdot\frac{1}{-\delta-1}t^{\delta+1}L_p(t) = \frac{L_p(t)}{L_r(t)}\cdot\frac{1}{-\delta-1}. \tag{3.8}$$

Thus,

$$\frac{t^{\alpha-1}}{r(t)}\int_t^\infty p(s)\mathrm{d}s \sim \frac{L_p(t)}{L_r(t)}.$$

In the following proposition we will prove that the condition (3.2) is necessary for existing of $\mathcal{NSV}$ solutions.

Proposition 3.1. *Assume that $r \in \mathcal{RV}(\delta+\alpha)$ where $\delta < -1$, $\liminf_{u\to 0_+} L_F(u) > 0$ and $\limsup_{u\to 0_+} L_G(u) < \infty$. The condition (3.2) is necessary for existence of a decreasing normalized slowly varying solution.*

Proof. In fact, we will prove that if there exists such $y \in \mathcal{DS} \cap \mathcal{NSV}$, then (3.2) is satisfied. Later we will show that $\mathcal{SV}$ solutions necessarily decrease.

Let $y \in \mathcal{PS}$. Consider

$$w(t) = \frac{r(t)G(y'(t))}{\Phi(y(t))}$$

and $w(t)$ satisfies the generalized Riccati type equation in the form:

$$w' - p\frac{F(y)}{\Phi(y)} + (\alpha - 1)|w|^{\beta}\big(rL_G(y')\big)^{1-\beta} = 0. \tag{3.9}$$

The derivative of w:

$$\begin{aligned} w' &= \frac{(rG(y'))'\Phi(y) - rG(y')(\Phi(y))'}{\Phi^2(y)} = \frac{(rG(y'))'}{\Phi(y)} - \frac{(\alpha-1)rG(y')|y|^{\alpha-2}y'}{|y|^{2\alpha-2}(\operatorname{sgn}(y))^2} \\ &= \frac{pF(y)}{\Phi(y)} - \frac{(\alpha-1)r^{\beta}r^{1-\beta}|y'|^{\alpha}(L_G(|y'|))^{\beta}(L_G(|y'|))^{1-\beta}}{|y|^{\alpha}} \\ &= p\frac{F(y)}{\Phi(y)} - (\alpha-1)|w|^{\beta}\big(rL_G(|y'|)\big)^{1-\beta}. \end{aligned}$$

Notice, that the Riccati equation (3.9) is dependent not only on the function w, but also on the function y, because of the general form of non-linear functions L_F and L_G. Remember that we work only with positive solutions y, so instead of working with the function $\Phi(y) = |y|^{\alpha-1}\operatorname{sgn}(y)$ we continue to work with the function $y^{\alpha-1}$. There exists a constant $N \in (0, \infty)$ because of the boundedness condition on L_G such that

$$0 < -\frac{t^{\alpha-1}}{r(t)}w(t) = -\frac{t^{\alpha-1}G(y'(t))}{(y(t))^{\alpha-1}} \leq -N\Big(\frac{ty'(t)}{y(t)}\Big)^{\alpha-1} \to 0 \tag{3.10}$$

as $t \to \infty$, because $y \in \mathcal{NSV}$.

Integrate (3.9) from t to ∞ and multiply by $\frac{t^{\alpha-1}}{r(t)}$, so we obtain:

$$-\frac{t^{\alpha-1}}{r(t)}w(t) = \frac{t^{\alpha-1}}{r(t)}\int_t^{\infty} p(s)L_F(y(s))\mathrm{d}s - (\alpha-1)\frac{t^{\alpha-1}}{r(t)}\int_t^{\infty} |w(s)|^{\beta}\big(r(s)L_G(|y'(s)|)\big)^{1-\beta}\mathrm{d}s. \tag{3.11}$$

The left-hand side of the equation (3.11) tends to zero due to the observation in (3.10). Let us work with the second term on the right side of the equation and find a limit using l'Hospital's rule:

$$\begin{aligned} \lim_{t\to\infty}\frac{t^{\alpha-1}}{r(t)}\int_t^{\infty} |w(s)|^{\beta}\big(r(s)L_G(|y'(s)|)\big)^{1-\beta}\mathrm{d}s &= \lim_{t\to\infty}\frac{|w(t)|^{\beta}\big(r(t)L_G(|y'(t)|)\big)^{1-\beta}}{r'(t)t^{1-\alpha} + (1-\alpha)r(t)t^{-\alpha}} \\ &= \lim_{t\to\infty}\frac{(y'(t))^{\alpha}L_G(|y'(t)|)}{(y(t))^{\alpha}(r'(t)/r(t)t^{1-\alpha} + (1-\alpha)t^{-\alpha})} \\ &= \lim_{t\to\infty}\frac{t^{\alpha}(y'(t))^{\alpha}L_G(|y'(t)|)}{(y(t))^{\alpha}(tr'(t)/r(t) + (1-\alpha))} \\ &= \frac{0\cdot M}{-\delta-\alpha+\alpha-1} = 0, \end{aligned}$$

where $M \in (0,\infty)$ is a constant, which follows from $\limsup_{u\to u_0} L_G(u) < \infty$. From (3.11) we get the limit

$$\lim_{t\to\infty} \frac{t^{\alpha-1}}{r(t)} \int_t^\infty p(s) L_F(y(s)) \mathrm{d}s = 0,$$

and if we define $K \in (0,\infty)$ such that $L_F(y(t)) \geq K$ for large t due to the assumptions above, then we get the condition

$$\lim_{t\to\infty} \frac{t^{\alpha-1}}{r(t)} \int_t^\infty p(s) \mathrm{d}s = 0.$$

□

If, in addition $p \in \mathcal{RV}(\delta)$, then the necessary condition is $L_p(t)/L_r(t) \to 0$ as $t \to \infty$ as we showed it earlier. A closer and more detailed examination of the proofs actually shows that the condition $r \in \mathcal{RV}(\delta+\alpha)$ can also be relaxed to the existence of $r_i \in \mathcal{RV}(\delta_i + \alpha)$, $i = 1,\ 2$, with $r_1(t) \leq r(t) \leq r_2(t)$ for large t, and $\delta_1,\ \delta_2 < -1$.

Further we will show that slowly varying solutions necessarily decrease. The proof is made by contradiction.

Proposition 3.2. *Assume $p \in \mathcal{RV}(\delta)$, $r \in \mathcal{RV}(\delta+\alpha)$, with $\delta < -1$ and $y \in \mathcal{PS} \cap \mathcal{SV}$, then $y \in \mathcal{DS}$.*

Proof. Take $y \in \mathcal{IS} \cap \mathcal{SV}$. Since y is positive, then $r(t)G(y'(t))$ is positive increasing. Hence, there exists some positive constant M such that $r(t)(y'(t))^{\alpha-1} L_G(y'(t)) \geq M$ for t sufficient large. Dividing by $r(t)$ and raising by $\frac{1}{\alpha-1}$, it follows that

$$y'(t) \geq \Big(\frac{M}{r(t) L_G(y'(t))}\Big)^{\frac{1}{\alpha-1}},$$

which after integration both of the sides from a to t implies

$$y(t) \geq y(a) + M^{\frac{1}{\alpha-1}} \int_a^t \Big(\frac{1}{r(s) L_G(y'(s))}\Big)^{\frac{1}{\alpha-1}} \mathrm{d}s. \tag{3.12}$$

Since $r \in \mathcal{RV}(\delta+\alpha)$ and $L_G \in \mathcal{SV}$, it holds

$$\Big(\frac{1}{r(t) L_G(y'(t))}\Big)^{\frac{1}{\alpha-1}} \in \mathcal{RV}\Big(\frac{-\delta-\alpha}{\alpha-1}\Big).$$

From hypothesis $\delta < -1$ we can conclude that $-\delta-\alpha > 1-\alpha$, so $\frac{-\delta-\alpha}{\alpha-1} > -1$. Applying Karamata's theorem 1.3 and Proposition 1.1 we then obtain that

$$\int_a^t \Big(\frac{1}{r(s) L_G(y'(s))}\Big)^{\frac{1}{\alpha-1}} \mathrm{d}s \in \mathcal{RV}\Big(\frac{-\delta-\alpha}{\alpha-1} + 1\Big) = \mathcal{RV}\Big(\frac{-\delta-1}{\alpha-1}\Big).$$

Since $\delta < -1$, it follows that $\frac{-\delta-1}{\alpha-1} > 0$, therefore (3.12) implies that y is greater than or equal to a regular varying function with positive index $\Big(\frac{-\delta-1}{\alpha-1}\Big)$ and therefore cannot be slowly varying, what is a contradiction. □

Remark 3.3. In the previous proposition we proved that slowly varying solutions are necessarily decreasing and in Theorem 3.1 we showed that any decreasing solution is normalized slowly varying. Notice, that from these two facts it is clear that

$$\mathcal{PS} \cap \mathcal{NSV} = \mathcal{DS}.$$

Before we prove the following theorem we need to define a function $\hat{F}$ such that

$$\hat{F}(x) = \int_1^x \frac{\mathrm{d}u}{(F(u))^{\frac{1}{\alpha-1}}}, \quad x > 0. \tag{3.13}$$

The constant 1 in the integral can be replaced by any positive constant. Denote the inverse of $\hat{F}(x)$ by $\hat{F}^{-1}(x)$. We have $|\hat{F}| \in \mathcal{SV}_0$ and in general $\lim_{u\to 0_+} |\hat{F}(u)|$ can be finite or infinite. Slow variation of $|\hat{F}|$ follows from the fact that $F \in \mathcal{RV}_0(\alpha - 1)$, which implies $F^{\frac{1}{\alpha-1}} \in \mathcal{RV}_0(1)$ due to Proposition 1.1. Applying Karamata's theorem 1.3 to $1/F^{\frac{1}{\alpha-1}} \in \mathcal{RV}_0(-1)$ we obtain

$$\Big| \int_a^x \frac{1}{(F(u))^{\frac{1}{\alpha-1}}} \mathrm{d}u \Big| \in \mathcal{SV}_0$$

Now we are ready to derive an asymptotic formula for a solution of the nearly-half-linear differential equation (2.2), what we will do in the following theorem. Assumption of $r \in \mathcal{RV}(\delta + \alpha)$ thanks to Remark 3.1 is not restrictive, but natural.

Theorem 3.2. *Let $p \in \mathcal{RV}(\delta)$ and $r \in \mathcal{RV}(\delta+\alpha)$ with $\delta < -1$. Assume $\lim_{u\to 0_+} |\hat{F}(u)| = \infty$ and $L_G(ug(u)) \sim L_G(u)$ as $u \to 0_+$ for all $g \in \mathcal{SV}_0$. If $y \in \mathcal{DS} \cap \mathcal{NSV}$ to (2.2), then $-y(t) \in \Pi(-ty'(t))$. Moreover, for every $y \in \mathcal{DS} \cap \mathcal{NSV}$,*

- *if*

$$\int_a^\infty \Big(\frac{sp(s)}{r(s)L_G(1/s)}\Big)^{\frac{1}{\alpha-1}} \mathrm{d}s = \infty,$$

then there exists $\varepsilon(t)$ with $\varepsilon(t) \to 0$ as $t \to \infty$ such that

$$y(t) = \hat{F}^{-1}\Big\{ - \int_a^t \big(1 + \varepsilon(s)\big)\Big[- \frac{sp(s)}{(\delta+1)r(s)L_G(1/s)}\Big]^{\frac{1}{\alpha-1}} \mathrm{d}s \Big\} \tag{3.14}$$

and $y(t) \to 0$ as $t \to \infty$;

- *if*

$$\int_a^\infty \Big(\frac{sp(s)}{r(s)L_G(1/s)}\Big)^{\frac{1}{\alpha-1}} \mathrm{d}s < \infty,$$

then there exists $\varepsilon(t)$ with $\varepsilon(t) \to 0$ as $t \to \infty$ such that

$$y(t) = \hat{F}^{-1}\Big\{ \hat{F}\big(y(\infty)\big) + \int_t^\infty \big(1 + \varepsilon(s)\big)\Big[- \frac{sp(s)}{(\delta+1)r(s)L_G(1/s)}\Big]^{\frac{1}{\alpha-1}} \mathrm{d}s \Big\} \tag{3.15}$$

and $y(t) \to y(\infty) \in (0, \infty)$ as $t \to \infty$.

Moreover, $|y(\infty) - y(t)| \in \mathcal{SV}$ *and*

$$\frac{L_p^{\beta-1}(t) L_F^{\beta-1}(y(t))}{L_r^{\beta-1}(t) L_G^{\beta-1}(1/t)(y(\infty) - y(t))} = o(1) \tag{3.16}$$

as $t \to \infty$.

Proof. Take $y \in \mathcal{DS} \cap \mathcal{NSV}$ and let a be such that $y(t) > 0$ and $y'(t) < 0$ for $t \geq a$. Then we claim that

$$\big(r(t)G(y'(t))\big)' = p(t)F(y(t)) \in \mathcal{RV}(\delta), \tag{3.17}$$

provided by $y(t) \to 0$ as $t \to \infty$ by Proposition 1.1. If $y(t) \to C \in (0, \infty)$, then the conclusion is the same since $F(y(t)) \to F(C) \in (0, \infty)$ and so $p(t)F(y(t)) \in \mathcal{RV}(\delta)$. Thus,

$$r(t)G(-y'(t)) = \int_t^\infty \big(r(s)G(-y'(s))\big)' \mathrm{d}s \in \mathcal{RV}(\delta + 1), \tag{3.18}$$

what follows from the property of regularly varying functions from Proposition 1.1 and Karamata's theorem 1.3. Since $r \in \mathcal{RV}(\delta + \alpha)$ we conclude that $G(-y'(t)) \in \mathcal{RV}_0(1 - \alpha)$ by Proposition 1.1. From the definition of the inverse function G^{-1} it is clear that we can write $-y'(t) = G^{-1}\big(G(-y'(t))\big)$, so we get $-y'(t) \in \mathcal{RV}_0(-1)$. By Proposition 1.3 we are able to conclude that $-y(t) \in \Pi(-ty'(t))$. Set the function

$$h(t) = -t^{-\delta-1} r(t)G(-y'(t)) - (\delta + 1) \int_a^t s^{-\delta-2} r(s)G(-y'(s))\mathrm{d}s.$$

We will show that $h \in \Pi(-ty'(t))$ and also $h \in \Pi(-(\delta + 1)t^{-\delta-1} r(t)G(-y'(t)))$, what will help us to get an asymptotic solution in the next few steps. Let us recall that $F(y(t)) \in \mathcal{SV}$, what follows from Proposition 1.1. Compute the derivative of h:

$$\begin{aligned} h'(t) &= -(-\delta - 1)t^{-\delta-2} r(t)G(-y'(t)) + t^{-\delta-1} p(t)F(y(t)) + (-\delta - 1)t^{-\delta-2} r(t)G(-y'(t)) \\ &= t^{-\delta-1} p(t)F(y(t)) \ \in \mathcal{RV}(-\delta - 1 + \delta + 0). \end{aligned}$$

Thus $h'(t) \in \mathcal{RV}(-1)$ and using Proposition 1.3 we conclude that $h \in \Pi(th'(t))$. Moreover, fix $\lambda > 0$ and let us prove that the function h belongs to the class Π using again the same proposition, so we get

$$\begin{aligned} \frac{h(\lambda t) - h(t)}{-(\delta + 1)t^{-\delta-1} r(t)G(-y'(t))} &= -\frac{\lambda^{-\delta-1} t^{-\delta-1} r(\lambda t)G(-y'(\lambda t))}{-(\delta + 1)t^{-\delta-1} r(t)G(-y'(t))} \\ &+ \frac{t^{-\delta-1} r(t)G(-y'(t))}{-(\delta + 1)t^{-\delta-1} r(t)G(-y'(t))} - \frac{(\delta + 1)\int_a^{\lambda t} s^{-\delta-2} r(s)G(-y'(s))\mathrm{d}s}{-(\delta + 1)t^{-\delta-1} r(t)G(-y'(t))} \\ &+ \frac{(\delta + 1)\int_a^{t} s^{-\delta-2} r(s)G(-y'(s))\mathrm{d}s}{-(\delta + 1)t^{-\delta-1} r(t)G(-y'(t))} = -\frac{\lambda^{-\delta-1} t^{-\delta-1} r(\lambda t)G(-y'(\lambda t))}{-(\delta + 1)t^{-\delta-1} r(t)G(-y'(t))} \\ &+ \frac{t^{-\delta-1} r(t)G(-y'(t))}{-(\delta + 1)t^{-\delta-1} r(t)G(-y'(t))} - \frac{(\delta + 1)\int_t^{\lambda t} s^{-\delta-2} r(s)G(-y'(s))\mathrm{d}s}{-(\delta + 1)t^{-\delta-1} r(t)G(-y'(t))}. \end{aligned}$$

Substitute $s = tu$ in the integral in the last term. Thus, $\mathrm{d}s = t\mathrm{d}u$ and

$$\begin{aligned}\frac{h(\lambda t)-h(t)}{-(\delta+1)t^{-\delta-1}r(t)G(-y'(t))} &= -\frac{\lambda^{-\delta-1}r(\lambda t)G(-y'(\lambda t))}{-(\delta+1)r(t)G(-y'(t))} - \frac{1}{\delta+1} \\ &\quad - \int_1^\lambda \frac{t^{-\delta-2}u^{-\delta-2}r(tu)G(-y'(tu))}{-t^{-\delta-1}r(t)G(-y'(t))} t\mathrm{d}u \\ &= -\frac{\lambda^{-\delta-1}r(\lambda t)G(-y'(\lambda t))}{-(\delta+1)r(t)G(-y'(t))} - \frac{1}{\delta+1} \\ &\quad - \int_1^\lambda \frac{u^{-\delta-2}r(tu)G(-y'(tu))}{-r(t)G(-y'(t))} \mathrm{d}u.\end{aligned}$$

Let us work with separate terms of the right-hand side of the relation showed above. We start with the first fraction. Since $r(t)G(-y'(t)) \in \mathcal{RV}(\delta+1)$ and using Definition 1.1 of regular variation we obtain

$$\lim_{t\to\infty} -\frac{\lambda^{-\delta-1}r(\lambda t)G(-y'(\lambda t))}{-(\delta+1)r(t)G(-y'(t))} = \frac{\lambda^{-\delta-1}}{\delta+1}\lambda^{\delta+1} = \frac{1}{\delta+1}.$$

Recall that $r(t)G(-y'(t)) \in \mathcal{RV}(\delta+1)$ and so the uniform convergence of

$$\frac{r(tu)G(-y'(tu))}{r(t)G(-y'(t))} \longrightarrow u^{\delta+1}$$

implies

$$\lim_{t\to\infty}\Big[-\int_1^\lambda \frac{u^{-\delta-2}r(tu)G(-y'(tu))}{-r(t)G(-y'(t))}\mathrm{d}u\Big] = \int_1^\lambda u^{-\delta-2}u^{\delta+1}\mathrm{d}u = \ln\lambda,$$

so we conclude $h \in \Pi(-(\delta+1)t^{-\delta-1}r(t)G(-y'(t)))$. Because of the uniqueness of the auxiliary function up to asymptotic equivalence we obtain the following relation

$$-(\delta+1)t^{-\delta-1}r(t)G(-y'(t)) \sim th'(t) = t^{-\delta}p(t)F(y(t)),$$

which implies

$$\frac{G(-y'(t))}{F(y(t))} \sim -\frac{tp(t)}{(\delta+1)r(t)}$$

as $t \to \infty$ and using the condition $L_G(ug(u)) \sim L_G(u)$ as $u \to 0_+$ for all $g \in \mathcal{SV}_0$, rewrite it equivalently as $L_G(v(t)/t) \sim L_G(1/t)$ as $t \to \infty$ for all $v \in \mathcal{SV}$. Let us remind again that $-y'(t) \in \mathcal{RV}_0(-1)$, then we can rewrite it in the form $y'(t) = \dfrac{L_{|y'|}(t)}{t}$, where $L_{|y'|}(t) \in \mathcal{SV}$. Hence,

$$G(-y'(t)) = (-y'(t))^{\alpha-1}L_G(L_{|y'|}(t)/t) \sim (-y'(t))^{\alpha-1}L_G(1/t) \tag{3.19}$$

as $t \to \infty$. Combining all these relations we obtain

$$\frac{(-y'(t))^{\alpha-1}}{F(y(t))} \sim -\frac{tp(t)}{(\delta+1)r(t)L_G(1/t)} \Rightarrow \frac{-y'(t)}{(F(y(t)))^{\frac{1}{\alpha-1}}} \sim \Big[-\frac{tp(t)}{(\delta+1)r(t)L_G(1/t)}\Big]^{\frac{1}{\alpha-1}}.$$

Therefore, there exists a function $\varepsilon(t)$ satisfying $\lim_{t\to\infty}\varepsilon(t) = 0$ such that

$$\frac{y'(t)}{\big(F(y(t))\big)^{\frac{1}{\alpha-1}}} = -(1+\varepsilon(t))\Big[-\frac{tp(t)}{(\delta+1)r(t)L_G(1/t)}\Big]^{\frac{1}{\alpha-1}} \tag{3.20}$$

as $t \to \infty$. Assume now that

$$\int_a^\infty \Big(\frac{sp(s)}{r(s)L_G(1/s)}\Big)^{\frac{1}{\alpha-1}} \mathrm{d}s = \infty.$$

Integrating (3.20) from a to t we obtain

$$\int_a^t \frac{y'(s)}{\big(F(y(s))\big)^{\frac{1}{\alpha-1}}} \mathrm{d}s = -\int_a^t \big(1+\varepsilon(s)\big)\Big[-\frac{sp(s)}{(\delta+1)r(s)L_G(1/s)}\Big]^{\frac{1}{\alpha-1}} \mathrm{d}s.$$

Substituting $u = y(s)$ in the integral on the left side of the relation, then we get $\mathrm{d}u = y'(s)\mathrm{d}s$, what brings us to the new interval of integration from $y(a)$ to $y(t)$

$$\int_a^t \frac{y'(s)}{\big(F(y(s))\big)^{\frac{1}{\alpha-1}}} \mathrm{d}s = \int_{y(a)}^{y(t)} \frac{\mathrm{d}u}{\big(F(u)\big)^{\frac{1}{\alpha-1}}}$$

and using the definition (3.13) of the function $\hat{F}$ we have

$$\hat{F}(y(t)) = \hat{F}(y(a)) - \int_a^t \big(1+\varepsilon(s)\big)\Big[-\frac{sp(s)}{(\delta+1)r(s)L_G(1/s)}\Big]^{\frac{1}{\alpha-1}} \mathrm{d}s. \tag{3.21}$$

There exists $\hat{\varepsilon}(t) \to 0$ such that

$$\hat{F}(y(a)) - \int_a^t \big(1+\varepsilon(s)\big)\Big[H(s)\Big]^{\frac{1}{\alpha-1}} \mathrm{d}s = -\int_a^t \big(1+\hat{\varepsilon}(s)\big)\Big[H(s)\Big]^{\frac{1}{\alpha-1}} \mathrm{d}s,$$

where $H(s) = -\dfrac{sp(s)}{(\delta+1)r(s)L_G(1/s)}$, which implies for $\int_a^\infty \Big(\dfrac{sp(s)}{r(s)L_G(1/s)}\Big)^{\frac{1}{\alpha-1}} \mathrm{d}s = \infty$

$$y(t) = \hat{F}^{-1}\Big\{-\int_a^t \big(1+\varepsilon(s)\big)\Big[-\frac{sp(s)}{(\delta+1)r(s)L_G(1/s)}\Big]^{\frac{1}{\alpha-1}} \mathrm{d}s\Big\}.$$

Clearly $y(t) \to 0$ as $t \to \infty$, otherwise we get a contradiction with the divergence of the integral (3.21). On the contrary, if

$$\int_a^\infty \Big(\frac{sp(s)}{r(s)L_G(1/s)}\Big)^{\frac{1}{\alpha-1}} \mathrm{d}s < \infty$$

we integrate (3.20) over the interval (t, ∞) and repeating the same steps as we did above, we obtain

$$y(t) = \hat{F}^{-1}\Big\{\hat{F}\big(y(\infty)\big) + \int_t^\infty \big(1+\varepsilon(s)\big)\Big[-\frac{sp(s)}{(\delta+1)r(s)L_G(1/s)}\Big]^{\frac{1}{\alpha-1}} \mathrm{d}s\Big\}.$$

In this case $y(t)$ must tend to a positive constant $y(\infty)$ as $t \to \infty$, otherwise the left side of the relation (3.21) becomes unbounded, which would contradict the assumption of boundedness of the integral.

To prove the last statement we use (3.20) so we get the asymptotic equivalence

$$y'(t) \sim \frac{y(\infty)}{t}\Big(\frac{1}{\delta+1}\cdot\frac{L_p(t)L_F(y(t))}{L_r(t)L_G(1/t)}\Big)^{\beta-1}$$

as $t \to \infty$. Integrate from t to ∞, so we obtain

$$y(\infty) - y(t) \sim \frac{y(\infty)}{(\delta+1)^{\beta-1}} \int_t^\infty \frac{1}{s}\Big(\frac{L_p(t)L_F(y(s))}{L_r(t)L_G(1/t)}\Big)^{\beta-1} \mathrm{d}s$$

as $t \to \infty$. Apply Karamata's theorem 1.3 we get to the conclusion that the integral is a slowly varying function and then we obtain (3.16). □

Remark 3.4. The proof represented above is based on the theory of the Π-class and uses properties of functions from this class of functions. Next we show another approach using Karamata's theorem. First part of the proof remains the same. We take $y \in \mathcal{DS} \cap \mathcal{NSV}$. Since $p(t)F(y(t)) \in \mathcal{RV}(\delta)$ with $\delta < -1$, from (3.18) and applying Karamata's theorem 1.3 we obtain

$$\begin{aligned} r(t)G(-y'(t)) &= \int_a^t p(s)F(y(s))\mathrm{d}s \sim \int_a^t s^\delta L_p(s)F(y(s))\mathrm{d}s \\ &\sim \frac{1}{\delta+1}t^{\delta+1}L_p(t)F(y(t)) \sim \frac{1}{\delta+1}tp(t)F(y(t)). \end{aligned}$$

Thus, dividing by $r(t)F(y(t))$, we get

$$\frac{G(-y'(t))}{F(y(t))} \sim \frac{tp(t)}{(\delta+1)r(t)},$$

which implies (3.19) and the rest of the proof remains the same.

Let us formulate the following corollary based on the results obtained in Theorem 3.1, Remark 3.2 and Theorem 3.2.

Corollary 3.1. *Consider the equation* (2.2). *Let* $p \in \mathcal{RV}(\delta)$ *and* $r \in \mathcal{RV}(\delta+\alpha)$ *with* $\delta < -1$. *Assume* $\frac{L_p(t)}{L_r(t)} \to 0$ *as* $t \to \infty$, $\limsup_{u\to 0_+} L_F(u) < \infty$ *and* $\liminf_{u\to 0_+} L_G(u) > 0$, *then there exists* $y \in \mathcal{DS} \cap \mathcal{NSV}$ *and* $-y(t) \in \Pi(-ty'(t))$. *Assume* $\lim_{u\to 0_+} |\hat{F}(u)| = \infty$ *and* $L_G(ug(u)) \sim L_G(u)$ *as* $u \to 0_+$ *for all* $g \in \mathcal{SV}_0$, *then for every* $y \in \mathcal{DS} \cap \mathcal{NSV}$,

- *if* $\int_a^\infty (\frac{sp(s)}{r(s)L_G(1/s)})^{\frac{1}{\alpha-1}}\mathrm{d}s = \infty$, *then there exists* $\varepsilon(t)$ *with* $\varepsilon(t) \to 0$ *as* $t \to \infty$ *such that* (3.14) *is an asymptotic formula and* $y(t) \to 0$ *as* $t \to \infty$;
- *if* $\int_a^\infty (\frac{sp(s)}{r(s)L_G(1/s)})^{\frac{1}{\alpha-1}}\mathrm{d}s < \infty$, *then there exists* $\varepsilon(t)$ *with* $\varepsilon(t) \to 0$ *as* $t \to \infty$ *such that* (3.15) *is an asymptotic formula and* $y(t) \to y(\infty) \in (0,\infty)$ *as* $t \to \infty$.

Moreover, $|y(\infty) - y(t)| \in \mathcal{SV}$ *and* (3.16) *is true as* $t \to \infty$.

If we take $L_F = 1$ and $L_G = 1$, then the equation (2.2) reduces to the half-linear equation (2.3) and notice, that the asymptotic formula of the solution is the same as one presented in Theorem 2.1. In case of $\alpha = 2$ and $r = 1$, the equation (2.2) reduces to the nearly-linear equation (2.6) and the formula is equivalent to one shown in Theorem 2.2. The formula for the case of a generalized function $r \in \mathcal{SV}$ is new for nearly-linear equations. Formula (3.16) is new even for the case $\alpha = 2$ and $r = 1$.

Let us show a simple example of computing $\hat{F}^{-1}$. Assume $F \in \mathcal{RV}_0(\alpha-1)$ with $L_F = 1$. Compute

$$\hat{F}(x) = \int_1^x \frac{\mathrm{d}u}{(F(u))^{\frac{1}{\alpha-1}}} = \int_1^x \frac{\mathrm{d}u}{(u^{\alpha-1})^{\frac{1}{\alpha-1}}} = \ln x.$$

Then we compute the inverse $\hat{F}^{-1}(x) = e^x$.

Notice, that to prove the asymptotic formulae for decreasing slowly varying solutions of the nearly-half-linear equation (2.2) we do not require (even one-side) boundedness conditions on L_F and L_G such as in Theorem 3.1. As for condition $L_G(ug(u)) \sim L_G(u)$ as $u \to 0_+$ for all $g \in \mathcal{SV}_0$ from Theorem 3.2 it is not too restrictive. Observe that, in fact many functions satisfy it, for example, $L_G(u) \to C \in (0, \infty)$ as $u \to 0_+$ or $L_G(u) = |\ln|u||^{\alpha_1} |\ln|\ln|u|||^{\alpha_2}$, $\alpha_1, \alpha_2 \in \mathbb{R}$. Take a look at the function $L_G(u) = |\ln|u||$. For simplicity let us choose a function $g(u) = |\ln|u||$. Here it is clear that the condition $L_G(ug(u)) \sim L_G(u)$ is satisfied:

$$\lim_{u\to 0_+} \frac{|\ln|u|\ln|u|||}{|\ln|u||} = \lim_{u\to 0_+} \frac{(|\ln|u|| + 1)|u|}{|u||\ln|u||} = \lim_{u\to 0_+} \Big(\frac{|\ln|u||}{|\ln|u||} + \frac{1}{|\ln|u||}\Big) = 1.$$

From the condition of Theorem 3.2 $\int_a^\infty (\frac{sp(s)}{r(s)L_G(1/s)})^{\frac{1}{\alpha-1}} \mathrm{d}s = \infty$, it does not follow that

$$y(t) \sim \hat{F}^{-1}\Big\{ -\int_a^t \Big[-\frac{sp(s)}{(\delta+1)r(s)L_G(1/s)}\Big]^{\frac{1}{\alpha-1}} \mathrm{d}s\Big\}$$

as $t \to \infty$ (see [6] for the linear case $y''(t) = p(t)y(t)$). However, we are able to deduce a lower estimate, but we will take stricter conditions. Due to technical reasons we will assume that solutions under our investigation are positive and decreasing in the interval $[0, \infty)$.

Theorem 3.3. *Consider the equation* (2.2)*, where $p \in \mathcal{RV}(-\alpha)$ and $r = 1$ are positive continuous functions. Take $\alpha \in (1, 2]$. Let*

$$\liminf_{u\to 0_+} L_G(u) > 0 \quad \text{and} \quad \lim_{t\to\infty} t \int_t^\infty p(s)\,ds = 0$$

holds. Denote $\tilde{F}(t) = \int_1^t \frac{du}{F(u)}$, $t > 0$. Then $y \in \mathcal{DS} \cap \mathcal{SV}$, such that $-y'(0) \le 1$, satisfies the estimate

$$\liminf_{t\to\infty} \frac{y(t)}{\tilde{F}^{-1}\Big\{\tilde{F}(y(0)) - \frac{1}{M}\int_0^t sp(s)\,ds\Big\}} \ge 1, \tag{3.22}$$

where M is some positive constant. The constant M can be taken as

$$M = \inf_{u\in[0,|y'(0)|]} L_G(u).$$

Proof. Take $y(t) \in \mathcal{DS} \cap \mathcal{SV}$, $t \ge 0$. Integrate equation (2.2) over the interval $(\lambda t, t)$, where $\lambda \in (0, 1)$, so we have

$$-G(y'(\lambda t)) + G(y(t)) = \int_{\lambda t}^t p(s)F(y(s))\mathrm{d}s.$$

Let us multiple both sides by $\frac{1}{F(y(\lambda t))}$ and recalling that $y \in \mathcal{DS}$ and so we get

$$\frac{-G(y'(\lambda t)) + G(y'(t))}{F(y(\lambda t))} = \frac{1}{F(y(\lambda t))}\int_{\lambda t}^t p(s)F(y(s))\mathrm{d}s \le \int_{\lambda t}^t p(s)\mathrm{d}s \tag{3.23}$$

for $t > 0$. Thanks to the fact that $\liminf_{u\to 0_+} L_G(u) > 0$, there exists $M > 0$ such that

$$M\frac{(-y'(\lambda t))^{\alpha-1}}{F(y(\lambda t))} - \frac{G(-y'(t))}{F(y(\lambda t))} \le \frac{(-y'(\lambda t))^{\alpha-1} L_G(|y'(\lambda t)|)}{F(y(\lambda t))} - \frac{G(-y'(t))}{F(y(\lambda t))} \le \int_{\lambda t}^{t} p(s)\mathrm{d}s, \tag{3.24}$$

$t > 0$, where the last estimate follows from (3.23). As we are assuming $\alpha \in (1, 2]$, $y' < 0$, y' is increasing, $-y'(0) \le 1$, then $|y'(t)| \le 1$ for $t \in [0, \infty)$ and we conclude that $(-y'(\lambda t))^{\alpha-1} \ge -y'(\lambda t)$. Then rewrite (3.24) as

$$M\frac{-y'(\lambda t)}{F(y(\lambda t))} - \frac{G(-y'(t))}{F(y(\lambda t))} \le M\frac{(-y'(\lambda t))^{\alpha-1}}{F(y(\lambda t))} - \frac{G(-y'(t))}{F(y(\lambda t))} \le \int_{\lambda t}^{t} p(s)\mathrm{d}s. \tag{3.25}$$

Integration over $\lambda \in (0, 1)$ yields

$$-\frac{M}{t}[\tilde{F}(y(t)) - \tilde{F}(y(0))] - \frac{G(-y'(t))}{t}\int_0^t \frac{\mathrm{d}s}{F(y(s))} \le \frac{1}{t}\int_0^t sp(s)\mathrm{d}s, \tag{3.26}$$

where we substituted $s = \lambda t$ in the first fraction of the left side of (3.25), so we get

$$\int_0^1 \frac{y'(\lambda t)}{F(y(\lambda t))}\mathrm{d}\lambda = \frac{1}{t}\int_0^t \frac{y'(s)}{F(y(s))}\mathrm{d}s$$

and further substituting $u = y(s)$ we obtain

$$\int_0^1 \frac{y'(\lambda t)}{F(y(\lambda t))}\mathrm{d}\lambda = \frac{1}{t}\int_{y(0)}^{y(t)} \frac{\mathrm{d}u}{F(u)} = \frac{1}{t}[\tilde{F}(y(t)) - \tilde{F}(y(0))].$$

In the second term we substitute similarly $s = \lambda t$ and we derive

$$\int_0^1 \frac{\mathrm{d}\lambda}{F(y(\lambda t))} = \frac{1}{t}\int_0^t \frac{\mathrm{d}s}{F(y(s))}.$$

On the right side of the inequality (3.25) we apply the Fubini theorem in

$$\int_0^1 \int_{\lambda t}^t p(s)\mathrm{d}s\mathrm{d}\lambda,$$

where we change the order of integration and the intervals: $0 \le s \le t$ and $0 \le \lambda \le s/t$ and we have

$$\int_0^1 \int_{\lambda t}^t p(s)\mathrm{d}s\mathrm{d}\lambda = \int_0^t p(s)\int_0^{s/t} \mathrm{d}\lambda\mathrm{d}s = \frac{1}{t}\int_0^t sp(s)\mathrm{d}s.$$

From the relation (3.26) applying the inverse function $\tilde{F}^{-1}$ we get

$$y(t) \ge \tilde{F}^{-1}\Big[\tilde{F}(y(0)) - \frac{G(-y'(t))}{MN}\int_0^t \frac{\mathrm{d}s}{F(y(s))} - \frac{1}{M}\int_0^t sp(s)\mathrm{d}s\Big]. \tag{3.27}$$

Since $F(y) \in \mathcal{SV}$ and so $1/F(y) \in \mathcal{SV}$ and recalling that y is a decreasing solution, the Karamata's theorem 1.3 yields

$$0 < G(-y'(t))\int_0^t \frac{\mathrm{d}s}{F(y(s))} \sim \frac{tG(-y'(t))}{F(y(t))} = \frac{t}{F(y(t))}\int_t^\infty p(s)F(y(s))\mathrm{d}s$$
$$\le t\int_t^\infty p(s)\mathrm{d}s \to 0$$

as $t \to \infty$ due to the assumption. Hence,

$$G(-y'(t)) \int_0^t \frac{\mathrm{d}s}{F(y(s))} = o(1)$$

as $t \to \infty$. Then from (3.27) it follows that

$$\liminf_{t\to\infty} \frac{y(t)}{\tilde{F}^{-1}\Big\{\tilde{F}(y(0)) - \frac{1}{M}\int_0^t sp(s)\mathrm{d}s\Big\}} \geq 1.$$

□

Notice, that for proving the previous theorem we did not use the fact that $p \in \mathcal{RV}(\delta)$ what is one of the differences with the approach used in the linear case (see [6]). It is reasonable to require the conditions $\lim_{u\to 0_+} |\tilde{F}| = \infty$ and $\int_0^\infty sp(s)\mathrm{d}s = \infty$ when we apply Theorem 3.3. If we take the theorem above as an improvement of information related to the solution we were dealing with in Theorem 3.2, even though with some restrictions, it is reasonable to consider $p \in \mathcal{RV}(\delta)$.

Further, let us deduce an asymptotic estimate for a generalized r satisfying

$$\int_a^\infty r^{1-\beta}(s)\mathrm{d}s = \infty \quad \text{and} \quad L_G = 1.$$

Remark 3.5. Consider the equation

$$\big(r(t)\Phi(y'(t))\big)' = p(t)F(y(t)), \tag{3.28}$$

where $\lim_{t\to\infty} t\int_t^\infty p(s)\mathrm{d}s = 0$ and $\int_a^\infty r^{1-\beta}(s)\mathrm{d}s = \infty$. We will use the following transformation. Denote

$$R(t) = \int_a^t r^{1-\beta}(s)\mathrm{d}s$$

and R^{-1} is defined as the inverse function of R. We take new variable $s = \varphi(t)$ and new function $x(s) = y(t)$, such that φ is a differentiable function with $\varphi'(t) \neq 0$. Then

$$\varphi(t) = s \quad \Longrightarrow \quad \varphi'(t) = \frac{\mathrm{d}s}{\mathrm{d}t} \quad \Longrightarrow \quad \varphi'(t)\frac{\mathrm{d}}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}t}$$

and the equation (3.28) is transformed into the equation:

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big[\tilde{r}(s)\Phi\Big(\frac{\mathrm{d}x}{\mathrm{d}s}\Big)\Big] = \tilde{p}(s)F(x), \tag{3.29}$$

where

$$\tilde{r}(s) = (r \circ \varphi^{-1})(s)\Phi\big((\varphi' \circ \varphi^{-1})(s)\big) \quad \text{and} \quad \tilde{p} = \frac{(p \circ \varphi^{-1})(s)}{(\varphi' \circ \varphi^{-1})(s)}.$$

By suitable choice of $\varphi(t)$ we can transform (3.28) into (3.29) with $\tilde{r} = 1$. Indeed, set $\varphi(t) = R(t)$, then compute the derivative $\varphi'(t) = R'(t) = r^{1-\beta}(t)$. Notice that $\varphi^{-1}(s) = \varphi^{-1}(\varphi(t)) = t$. Then rewrite

$$\tilde{r}(s) = r(t)\Phi(\varphi'(t)) = r(t)\Phi(r^{1-\beta}(t)) = r(t)\cdot r^{(1-\beta)(\alpha-1)}(t) = r(t)\cdot r^{-1}(t) = 1.$$

Similarly rewrite

$$\tilde{p}(s) = \frac{(p \circ \varphi^{-1})(s)}{(\varphi' \circ \varphi^{-1})(s)} = \frac{p(t)}{(\varphi' \circ \varphi^{-1})(s)} = ((pr^{\beta-1}) \circ R^{-1})(s).$$

Now assume that $\tilde{p} \in \mathcal{RV}(\alpha)$. Since $\tilde{r} \in \mathcal{SV}$ and $\tilde{p} \in \mathcal{RV}(-\alpha)$, we can apply Theorem 3.3, so we obtain

$$\liminf_{s\to\infty} \frac{x(s)}{\tilde{F}^{-1}\Big\{\tilde{F}(x(0)) - \int_0^s u\tilde{p}(u)\mathrm{d}u\Big\}} \geq 1. \tag{3.30}$$

From the definition we know $s = \varphi(t)$, then $x(R^{-1}(s)) = y(t)$. Take the substitution $\tau = R^{-1}(u)$ in the integrand, then $u = R(\tau)$. We obtain $\mathrm{d}u = r^{1-\beta}\mathrm{d}\tau$. Rewrite (3.30) as

$$\begin{aligned}&\liminf_{t\to\infty} \frac{y(t)}{\tilde{F}^{-1}\Big\{\tilde{F}(y(0)) - \int_0^t R(\tau)p(\tau)\mathrm{d}\tau\Big\}}\\ =&\liminf_{t\to\infty} \frac{y(t)}{\tilde{F}^{-1}\Big\{\tilde{F}(y(0)) - \int_0^t R(\tau)p(\tau)r^{\beta-1}(\tau)r^{1-\beta}(\tau)\mathrm{d}\tau\Big\}} \geq 1,\end{aligned}$$

where $\tilde{F}(t) = \int_1^t \frac{\mathrm{d}u}{F(u)}$, $t > 0$.

3.2 Increasing slowly varying solutions

In this subsection we will be working with the same equation, but for the index of regular variation of the function p we require $\delta > -1$. Again we will study slowly varying solutions. As we will see later, we have to look for such solutions in the set $\mathcal{IS}$. The structure of the subsection is similar to the structure of one where we were dealing with the case $\delta < -1$. A lot of the following results are new (and even in special cases): asymptotic formula for the increasing solution of a nearly-linear equation with $r = 1$ and with generalized r.

The following theorem proves that if an increasing solution exists, then it is a normalized slowly varying function.

Theorem 3.4. *Assume that*

$$\int_a^\infty p(s)\mathrm{d}s = \infty, \tag{3.31}$$

$$\lim_{t\to\infty} \frac{t^{\alpha-1}}{r(t)} \int_a^t p(s)\mathrm{d}s = 0, \tag{3.32}$$

$$\limsup_{u\to\infty} L_F(u) < \infty \tag{3.33}$$

$$\textit{and} \quad \liminf_{u\to\infty} L_G(u) > 0. \tag{3.34}$$

Then $\mathcal{IS} \subset \mathcal{NSV}$.

Proof. Take $y \in \mathcal{IS}$ so $y(t) > 0$, $y'(t) > 0$ for $t \geq a$. Integrate equation (2.2) from a to t and recalling that y is increasing and so is $F(y)$, then

$$r(t)G(y'(t)) = r(a)G(y'(a)) + \int_a^t p(s)F(y(s))\mathrm{d}s \geq r(a)G(y'(a)) + F(y(a)) \int_a^t p(s)\mathrm{d}s$$

tends to ∞ as $t \to \infty$. As we mentioned earlier $y'(t) > 0$ for $t \geq a$, and because of the fact that $r(t)G(y'(t))$ tends to infinity, it is possible to find some positive constant K

$$r(t)G(y'(t)) \leq K \int_a^t p(s)F(y(s))\mathrm{d}s \tag{3.35}$$

for large t. Divide the inequality (3.35) by $r(t)(y(t))^{\alpha-1}L_G(y'(t))$ and multiply it by $t^{\alpha-1}$. Again use the fact that $F(y)$ is increasing. Thus,

$$0 < \Big(\frac{ty'(t)}{y(t)}\Big)^{\alpha-1} \leq \frac{t^{\alpha-1}KL_F(y(t))}{L_G(y'(t))r(t)} \int_a^t p(s)\mathrm{d}s \leq \frac{t^{\alpha-1}M}{r(t)} \int_a^t p(s)\mathrm{d}s \tag{3.36}$$

for large t, where $M \in (0, \infty)$ is some constant, which follows from conditions (3.33) and (3.34). Since the right-hand side of (3.36) tends to zero due to (3.32), we obtain $\frac{ty'(t)}{y(t)} \to 0$ as $t \to \infty$ and from this fact $y \in \mathcal{NSV}$ follows. □

Here, similarly as in the previous subsection, we want to justify the choice of $p \in \mathcal{RV}(\delta)$ and $r \in \mathcal{RV}(\delta + \alpha)$ with $\delta > -1$.

Remark 3.6. Assume that $p \in \mathcal{RV}(\delta)$ and $r \in \mathcal{RV}(\gamma)$ with $\delta > -1$. Take $y \in \mathcal{NSV} \cap \mathcal{IS}_\infty$ of (2.2). Then recalling the definition of F we have

$$\big(r(t)G(y'(t))\big)' = p(t)F(y(t)) \in \mathcal{RV}(\delta).$$

We work with the case $\delta > -1$, so integrate (2.2) from a to t:

$$r(t)G(y'(t)) - r(a)G(y'(a)) = \int_a^t p(s)F(y(s))\mathrm{d}s \to \infty$$

as $t \to \infty$ by Karamata's theorem 1.3. As we took $y \in \mathcal{SV} \cap \mathcal{IS}$, then we can rewrite the functions F and G in the form:

$$\begin{aligned} r(t)(y'(t))^{\alpha-1}L_G(y'(t)) - r(a)(y'(a))^{\alpha-1}L_G(y'(a)) &= \int_a^t p(s)(y(s))^{\alpha-1}L_F(y(s))\mathrm{d}s \\ &\in \mathcal{RV}(\delta+1), \end{aligned}$$

using properties of $\mathcal{RV}$ from Proposition 1.1 and Karamata's theorem 1.3, and so we have $(y'(t))^{\alpha-1} \in \mathcal{RV}(\delta + 1 - \gamma)$, because $r \in \mathcal{RV}(\gamma)$ and $L_G \in \mathcal{SV}$, which brings us to $y'(t) \in \mathcal{RV}\big(\frac{\delta+1-\gamma}{\alpha-1}\big)$ due to Proposition 1.1. From Remark 3.1, notice, that $\delta < -1$ is equivalent to $\gamma < \alpha - 1$, so if we assume $\delta > -1$ or equivalently $\gamma > \alpha - 1$, then the following steps are similar to ones in Remark 3.1, so we obtain $\gamma = \delta + \alpha$.

Remark 3.7. Assume that $p \in \mathcal{RV}(\delta)$ and $r \in \mathcal{RV}(\delta + \alpha)$ with $\delta > -1$, then by Karamata's theorem 1.3: $\int_a^\infty p(s)\mathrm{d}s = \infty$.

Earlier in Remark 3.2 we showed

$$\int_a^\infty G^{-1}\Big(\frac{M}{r(s)}\Big)\mathrm{d}s = \int_a^\infty r^{1-\beta}(s) \cdot h(s)\mathrm{d}s,$$

where $h \in \mathcal{SV}$. We know $r^{1-\beta} \in \mathcal{RV}((1-\beta)(\delta+\alpha))$. Further

$$(1-\beta)(\delta+\alpha) = (1-\beta)(\delta+1) - 1$$

and with the fact that $\delta > -1$ we get $(1-\beta)(\delta+\alpha) < -1$, and because r is positive we conclude that $\int_a^\infty r^{1-\beta}(s)\mathrm{d}s < \infty$. It means that integral $\int_a^\infty G^{-1}\Big(\frac{M}{r(s)}\Big)\mathrm{d}s$ is also convergent for some $M \in (0,\infty)$.

Applying Karamata's theorem 1.3 for the case $\delta > -1$ to the condition (3.32) we obtain:

$$\frac{t^{\alpha-1}}{r(t)}\int_a^t p(s)\mathrm{d}s \sim \frac{t^{\alpha-1}}{t^{\delta+\alpha}L_r(t)} \cdot \frac{1}{\delta+1}t^{\delta+1}L_p(t) = \frac{L_p(t)}{L_r(t)} \cdot \frac{1}{\delta+1}. \tag{3.37}$$

In this case, the condition (3.32) leads to $L_p(t)/L_r(t) \to 0$ as $t \to \infty$.

In the following remark we want to prove that the condition guaranteeing normalized slow variation of an increasing solution (3.32) is necessary for the existence of an increasing slowly varying solution of the equation. The proof is similar to one in the section dealing with the case $\delta < -1$. Let us show the main steps skipping similar computations.

Proposition 3.3. *Assume that $r \in \mathcal{RV}(\gamma)$ with $\delta > -1$. Take $\liminf_{u\to\infty} L_F(u) > 0$ and $\limsup_{u\to\infty} L_G(u) < \infty$. The condition (3.32) is necessary for existence of an increasing slowly varying solution.*

Proof. We want prove that if there exists $y \in \mathcal{IS} \cap \mathcal{NSV}$, then (3.32) is satisfied. Now we integrate the Riccati equation (3.11) from a to t and multiply it by $t^{\alpha-1}/r(t)$:

$$\begin{aligned}\frac{t^{\alpha-1}}{r(t)}w(t) - \frac{t^{\alpha-1}}{r(t)}w(a) &= \frac{t^{\alpha-1}}{r(t)}\int_a^t p(s)L_F(y(s))\mathrm{d}s \\ &\quad - (\alpha-1)\frac{t^{\alpha-1}}{r(t)}\int_a^t |w(s)|^\beta\big(r(s)L_G(y'(s))\big)^{1-\beta}\mathrm{d}s,\end{aligned}$$

where $\frac{t^{\alpha-1}}{r(t)}w(a) \to 0$ and $\frac{t^{\alpha-1}}{r(t)}w(t) \to 0$ as $t \to \infty$, which follows from the fact that $r \in \mathcal{RV}(\delta+\alpha)$, so $\frac{t^{\alpha-1}}{r(t)} \in \mathcal{RV}(\alpha-1-\delta-\alpha) = \mathcal{RV}(-1-\delta)$ with $\delta > -1$, thus $\frac{t^{\alpha-1}}{r(t)} \to 0$. Similarly as in Proposition 3.1 for the case $\delta < -1$ we can show that

$$\lim_{t\to\infty}\frac{t^{\alpha-1}}{r(t)}\int_a^t p(s)\mathrm{d}s = 0.$$

□

A closer examination of the proof shows that the condition $r \in \mathcal{RV}(\delta+\alpha)$ can be relaxed to the existence of $r_i \in \mathcal{RV}(\delta_i+\alpha)$, $i = 1,\ 2$, with $r_1(t) \le r(t) \le r_2(t)$ for large t, and $\delta_1,\ \delta_2 > -1$.

Further we will show that slowly varying solutions necessarily increase. We will prove this statement by contradiction.

Proposition 3.4. *Assume $p \in \mathcal{RV}(\delta)$, $r \in \mathcal{RV}(\delta+\alpha)$, with $\delta > -1$. If $y \in \mathcal{PS} \cap \mathcal{SV}$, then $y \in \mathcal{IS}$.*

Proof. Take $y \in \mathcal{DS} \cap \mathcal{SV}$. Then $G(y') = -(y')^{\alpha-1} L_G(|y'|)$ and $F(y) = y^{\alpha-1} L_F(y)$. Integrate (2.2) from a to t and get

$$r(t)G(y'(t)) = r(a)G(y'(a)) + \int_a^t p(s)F(y(s))\mathrm{d}s. \tag{3.38}$$

Since y is positive decreasing and r is a positive function, then $r(t)G(y'(t))$ is negative increasing, hence there exists a negative constant M such that

$$\lim_{t\to\infty} r(t)G(y'(t)) = M \in (-\infty, 0]\ .$$

Suppose that $y \in \mathcal{SV}$ and recall that $L_F \in \mathcal{SV}$, then $p(t)F(y(t)) \in \mathcal{RV}(\delta)$, thus $\int_a^t p(s)F(y(s))\mathrm{d}s \to \infty$, because of $\delta > -1$, what is a contradiction with (3.38). □

Remark 3.8. In the proposition above we proved that slowly varying solutions necessarily increase and in Theorem 3.4 we got to the conclusion that any increasing solution is normalized slowly varying. Combining these two observations we get

$$\mathcal{PS} \cap \mathcal{NSV} = \mathcal{IS}.$$

We have proved that we should look for slowly varying solutions in the set $\mathcal{IS}$, so our next goal is to deduce an asymptotic formula for solutions of the nearly-half-linear equation (2.2). We define $\hat{F}$ in the same way as in the previous section for the case $\delta < -1$, but now we have $|\hat{F}| \in \mathcal{SV}$.

Theorem 3.5. *Let $p \in \mathcal{RV}(\delta)$ and $r \in \mathcal{RV}(\delta+\alpha)$ with $\delta > -1$. Assume $\lim_{u\to\infty} |\hat{F}(u)| = \infty$ and $L_G(ug(u)) \sim L_G(u)$ as $u \to 0_+$ for all $g \in \mathcal{SV}_0$. If $y \in \mathcal{IS}\cap\mathcal{NSV}$, then $y \in \Pi(ty'(t))$. Moreover, for every $y \in \mathcal{IS} \cap \mathcal{NSV}$,*

- *if*
$$\int_a^\infty \Big(\frac{sp(s)}{r(s)L_G(1/s)}\Big)^{\frac{1}{\alpha-1}}\mathrm{d}s = \infty,$$
then there exists $\varepsilon(t)$ with $\varepsilon(t) \to 0$ as $t \to \infty$ such that
$$y(t) = \hat{F}^{-1}\Big\{\int_a^t \big(1+\varepsilon(s)\big)\Big[\frac{sp(s)}{(\delta+1)r(s)L_G(1/s)}\Big]^{\frac{1}{\alpha-1}}\mathrm{d}s\Big\} \tag{3.39}$$
and $y(t) \to \infty$ as $t \to \infty$;

- *if*
$$\int_a^\infty \Big(\frac{sp(s)}{r(s)L_G(1/s)}\Big)^{\frac{1}{\alpha-1}}\mathrm{d}s < \infty,$$
then there exists $\varepsilon(t)$ with $\varepsilon(t) \to 0$ as $t \to \infty$ such that
$$y(t) = \hat{F}^{-1}\Big\{\hat{F}\big(y(\infty)\big) - \int_t^\infty \big(1+\varepsilon(s)\big)\Big[\frac{sp(s)}{(\delta+1)r(s)L_G(1/s)}\Big]^{\frac{1}{\alpha-1}}\mathrm{d}s\Big\} \tag{3.40}$$
and $y(t) \to y(\infty) \in (0,\infty)$ as $t \to \infty$.

Moreover, $|y(\infty) - y(t)| \in \mathcal{SV}$ *and*

$$\frac{L_p^{\beta-1}(t)L_F^{\beta-1}(y(t))}{L_r^{\beta-1}(t)L_G^{\beta-1}(1/t)(y(\infty)-y(t))} = o(1) \tag{3.41}$$

as $t \to \infty$*.*

Proof. Take $y \in \mathcal{IS} \cap \mathcal{SV}$ and let a be such that $y(t) > 0,\ y'(t) > 0$ for $t \geq a$. Then

$$\big(r(t)G(y'(t))\big)' = p(t)F(y(t)) \in \mathcal{RV}(\delta),$$

provided $y(t) \to \infty$ as $t \to \infty$. If $y(t) \to C \in (0, \infty)$, then the conclusion is the same since $F(y(t)) \to F(C) \in (0, \infty)$ and so $p(t)F(y(t)) \in \mathcal{RV}(\delta)$. Thus, by Karamata's theorem 1.3:

$$r(t)G(y'(t)) - r(a)G(y'(a)) = \int_a^t \big(r(s)G(y'(s))\big)'\mathrm{d}s \in \mathcal{RV}(\delta+1).$$

Since $r \in \mathcal{RV}(\delta+\alpha)$ we obtain $G(y'(t)) \in \mathcal{RV}(1-\alpha)$ due to Proposition 1.1. In view of $y'(t) = G^{-1}(G(y'(t)))$, we get $y'(t) \in \mathcal{RV}(-1)$ and following Proposition 1.3 we conclude that $y(t) \in \Pi(ty'(t))$. Set

$$h(t) = t^{-\delta-1}r(t)G(y'(t)) + (\delta+1)\int_a^t s^{-\delta-2}r(s)G(y'(s))\mathrm{d}s.$$

Let us show that $h \in \Pi(ty'(t))$ and $h \in \Pi((\delta+1)t^{-\delta-1}r(t)G(y'(t)))$. Indeed,

$$\begin{aligned} h'(t) &= (-\delta-1)t^{-\delta-2}r(t)G(y'(t)) - t^{-\delta-1}p(t)F(y(t)) - (-\delta-1)t^{-\delta-2}r(t)G(y'(t)) \\ &= t^{-\delta-1}p(t)F(y(t)) \ \in \mathcal{RV}(-\delta-1+\delta+0). \end{aligned}$$

Thus $h'(t) \in \mathcal{RV}(-1)$ and applying results from Proposition 1.3 we can conclude that $h \in \Pi(th'(t))$. Moreover, fix $\lambda > 0$ and then integrate by substitution, so we get

$$\begin{aligned} \frac{h(\lambda t) - h(t)}{(\delta+1)t^{-\delta-1}r(t)G(y'(t))} &= \frac{\lambda^{-\delta-1}t^{-\delta-1}r(\lambda t)G(y'(\lambda t))}{(\delta+1)t^{-\delta-1}r(t)G(y'(t))} - \frac{t^{-\delta-1}r(t)G(y'(t))}{(\delta+1)t^{-\delta-1}r(t)G(y'(t))} \\ &+ \frac{(\delta+1)\int_t^{\lambda t} s^{-\delta-2}r(s)G(y'(s))\mathrm{d}s}{(\delta+1)t^{-\delta-1}r(t)G(y'(t))} \\ &= \frac{\lambda^{-\delta-1}r(\lambda t)G(y'(\lambda t))}{(\delta+1)r(t)G(y'(t))} + \frac{1}{\delta+1} + \int_1^\lambda \frac{u^{-\delta-2}r(tu)G(y'(tu))}{r(t)G(y'(t))}\mathrm{d}u. \end{aligned}$$

These calculations are similar to the previous case when we had $\delta < -1$. The most significant difference is a sign. Since $r(t)G(y'(t)) \in \mathcal{RV}(\delta+1)$ we obtain

$$\lim_{t\to\infty} -\frac{\lambda^{-\delta-1}r(\lambda t)G(y'(\lambda t))}{(\delta+1)r(t)G(y'(t))} = \frac{\lambda^{-\delta-1}}{\delta+1}\lambda^{\delta+1} = \frac{1}{\delta+1}$$

and the uniform convergence of $\frac{r(tu)G(y'(tu))}{r(t)G(y'(t))}$ to $u^{\delta+1}$ implies

$$\lim_{t\to\infty}\Big[\int_1^\lambda \frac{u^{-\delta-2}r(tu)G(y'(tu))}{r(t)G(y'(t))}\mathrm{d}u\Big] = \int_1^\lambda u^{-\delta-2}u^{\delta+1}\mathrm{d}u = \ln\lambda,$$

so $h \in \Pi((\delta+1)t^{-\delta-1}r(t)G(y'(t)))$. Because of the uniqueness of the auxiliary function up to asymptotic equivalence we obtain

$$(\delta+1)t^{-\delta-1}r(t)G(y'(t)) \sim th'(t) = t^{-\delta}p(t)F(y(t)),$$

which implies

$$\frac{G(y'(t))}{F(y(t))} \sim \frac{tp(t)}{(\delta+1)r(t)},$$

and using the condition $L_G(ug(u)) \sim L_G(u)$ as $u \to 0_+$ for all $g \in \mathcal{SV}_0$, rewrite it equivalently as $L_G(v(t)/t) \sim L_G(1/t)$ as $t \to \infty$ for all $v \in \mathcal{SV}_0$. Recall $y'(t) \in \mathcal{RV}(-1)$. Hence,

$$G(y'(t)) = (y'(t))^{\alpha-1}L_G(L_{y'}(t)/t) \sim (y'(t))^{\alpha-1}L_G(1/t) \tag{3.42}$$

as $t \to \infty$, where $L_{y'} \in \mathcal{SV}$. Putting all of these relations together we get

$$\frac{(y'(t))^{\alpha-1}}{F(y(t))} \sim \frac{tp(t)}{(\delta+1)r(t)L_G(1/t)} \Rightarrow \frac{y'(t)}{\big(F(y(t))\big)^{\frac{1}{\alpha-1}}} \sim \Big[\frac{tp(t)}{(\delta+1)r(t)L_G(1/t)}\Big]^{\frac{1}{\alpha-1}}$$

as $t \to \infty$. Therefore, there exists a function $\varepsilon(t)$ satisfying $\lim\limits_{t\to\infty} \varepsilon(t) = 0$, such that

$$\frac{y'(t)}{\big(F(y(t))\big)^{\frac{1}{\alpha-1}}} = (1+\varepsilon(t))\Big[\frac{tp(t)}{(\delta+1)r(t)L_G(1/t)}\Big]^{\frac{1}{\alpha-1}}. \tag{3.43}$$

Repeating the same procedures and justifying the steps in the same way as in the case $\delta < -1$, for

$$\int_a^\infty \Big(\frac{sp(s)}{(\delta+1)r(s)L_G(1/s)}\Big)^{\frac{1}{\alpha-1}} \mathrm{d}s = \infty$$

we get

$$y(t) = \hat{F}^{-1}\Big\{\int_a^t (1+\varepsilon(s))\Big[\frac{sp(s)}{(\delta+1)r(s)L_G(1/s)}\Big]^{\frac{1}{\alpha-1}} \mathrm{d}s\Big\}.$$

In this case $y(t) \to \infty$ as $t \to \infty$. On the other hand, for

$$\int_a^\infty \Big(\frac{sp(s)}{(\delta+1)r(s)L_G(1/s)}\Big)^{\frac{1}{\alpha-1}} \mathrm{d}s < \infty$$

we obtain

$$y(t) = \hat{F}^{-1}\Big\{\hat{F}\big(y(\infty)\big) - \int_t^\infty (1+\varepsilon(s))\Big[\frac{sp(s)}{(\delta+1)r(s)L_G(1/s)}\Big]^{\frac{1}{\alpha-1}} \mathrm{d}s\Big\}$$

and so $y(t)$ tends to a positive constant as $t \to \infty$.

Last part of the proof is essentially the same as for the case $\delta < -1$, we use (3.43), so we can conclude that

$$y'(t) \sim \frac{y(\infty)}{t}\Big(\frac{1}{\delta+1} \cdot \frac{L_p(t)L_F(y(t))}{L_r(t)L_G(1/t)}\Big)^{\beta-1}$$

as $t \to \infty$. Integrate it over the interval from t to ∞ and apply Karamata's theorem 1.3 and so we obtain (3.41). □

As well as in Theorem 3.2 we can prove Theorem 3.5 with help of Karamata theorem. Let us formulate the following corollary based on the results obtained in Theorem 3.4, Remark 3.7 and Theorem 3.5.

Corollary 3.2. *Consider the equation* (2.2)*. Let* $p \in \mathcal{RV}(\delta)$*,* $r \in \mathcal{RV}(\delta+\alpha)$ *where* $\delta > -1$*. Assume* $\frac{L_p(t)}{L_r(t)} \to 0$ *as* $t \to \infty$*,* $\limsup_{u\to\infty} L_F(u) < \infty$ *and* $\liminf_{u\to\infty} L_G(u) > 0$*, so if there exists* $y \in \mathcal{IS}$*, then* $y \in \mathcal{IS} \cap \mathcal{NSV}$ *and* $-y(t) \in \Pi(ty'(t))$*. Assume* $\lim\limits_{u\to\infty} |\hat{F}(u)| = \infty$ *and* $L_G(ug(u)) \sim L_G(u)$ *as* $u \to 0_+$ *for all* $g \in \mathcal{SV}_0$*, then for every* $y \in \mathcal{IS} \cap \mathcal{NSV}$*,*

- *if* $\int_a^\infty \left(\frac{sp(s)}{r(s)L_G(1/s)}\right)^{\frac{1}{\alpha-1}} \mathrm{d}s = \infty$*, then there exists* $\varepsilon(t)$ *with* $\varepsilon(t) \to 0$ *as* $t \to \infty$ *such that* (3.39) *is an asymptotic formula and* $y(t) \to \infty$ *as* $t \to \infty$*;*
- *if* $\int_a^\infty \left(\frac{sp(s)}{r(s)L_G(1/s)}\right)^{\frac{1}{\alpha-1}} \mathrm{d}s < \infty$*, then there exists* $\varepsilon(t)$ *with* $\varepsilon(t) \to 0$ *as* $t \to \infty$ *such that* (3.40) *is an asymptotic formula and* $y(t) \to y(\infty) \in (0,\infty)$ *as* $t \to \infty$*.*

Moreover, $|y(\infty) - y(t)| \in \mathcal{SV}$ *and* (3.41) *as* $t \to \infty$*.*

If we take $L_F = 1$ and $L_G = 1$, then the equation (2.2) reduces to the half-linear equation (2.3) the asymptotic formula for the solution is in the same form as it is shown in [16]. In case of $\alpha = 2$ and $r = 1$, the equation (2.2) reduces to the nearly-linear equation (2.6) and the analysis of the solutions as well as the asymptotic formula presented above are new. Moreover, here we have shown the case with generalized r. Formula (3.41) is also new even for the simpler case $\alpha = 2$ and $r = 1$.

4 Examples

In this chapter we will present applications of results obtained in the previous chapters to non-linear differential equations. We will start with an example of a half-linear linear equation.

Example 4.1. Consider the equation

$$\big(t^{\delta+\alpha}L_r(t)\Phi(y'(t))\big)' = t^{\delta}L_p(t)\Phi(y(t)), \tag{4.1}$$

where $L_r = (\ln t)^{\gamma_2} + h_1(t)$ and $L_p = (\ln t)^{\gamma_1} + h_2(t)$ with $|h_i(t)| = o((\ln t)^{\gamma_i})$ for $i = 1, 2$ for some $\gamma_1 < \gamma_2$. Examples of such functions h_i are $h_i(t) = \cos t$ or $h_i(t) = \ln(\ln t)$. For every $\lambda > 0$ we get:

$$\lim_{t\to\infty}\frac{(\ln\lambda t)^{\gamma_i} + h_i(\lambda t)}{(\ln t)^{\gamma_i} + h_i(t)} = \lim_{t\to\infty}\frac{\big(\frac{\ln\lambda t}{\ln t}\big)^{\gamma_i} + \frac{h_i(\lambda t)}{(\ln t)^{\gamma_i}}}{1 + \frac{h_i(t)}{(\ln t)^{\gamma_i}}} = \lim_{t\to\infty}\frac{\big(\frac{\ln\lambda t}{\ln t}\big)^{\gamma_i} + \frac{h_i(\lambda t)}{(\ln(\lambda t))^{\gamma_i}}\big(\frac{\ln\lambda t}{\ln t}\big)^{\gamma_i}}{1 + \frac{h_i(t)}{(\ln t)^{\gamma_i}}} = 1,$$

so we can conclude that L_p, $L_r \in \mathcal{SV}$. We have

$$\begin{aligned}\lim_{t\to\infty}\frac{L_p(t)}{L_r(t)} = \lim_{t\to\infty}\frac{(\ln t)^{\gamma_1} + h_1(t)}{(\ln t)^{\gamma_2} + h_2(t)} &= \lim_{t\to\infty}\frac{1 + \frac{h_1(t)}{(\ln t)^{\gamma_1}}}{(\ln t)^{\gamma_2-\gamma_1} + \frac{h_2(t)}{(\ln t)^{\gamma_1}}} \\ &= \lim_{t\to\infty}\frac{1 + \frac{h_1(t)}{(\ln t)^{\gamma_1}}}{(\ln t)^{\gamma_2-\gamma_1}[1 + \frac{h_2(t)}{(\ln t)^{\gamma_2}}]} = 0,\end{aligned} \tag{4.2}$$

because we set $\gamma_2 > \gamma_1$. Indeed the condition from Corollary 3.1 is satisfied. From (4.2) we get

$$\Big(\frac{tp(t)}{r(t)}\Big)^{\frac{1}{\alpha-1}} = \Big(\frac{t^{\delta+1}L_p(t)}{t^{\delta+\alpha}L_r(t)}\Big)^{\frac{1}{\alpha-1}} = \frac{1}{t}\Big[\frac{1 + \frac{h_1(t)}{(\ln t)^{\gamma_1}}}{(\ln t)^{\gamma_2-\gamma_1}[1 + \frac{h_2(t)}{(\ln t)^{\gamma_2}}]}\Big]^{\frac{1}{\alpha-1}} \sim \frac{1}{t}\Big(\frac{1}{\delta+1}\Big)^{\frac{1}{\alpha-1}}(\ln t)^{\frac{\gamma_1-\gamma_2}{\alpha-1}}$$

as $t \to \infty$. Since $\int_a^t \frac{1}{s}(\ln s)^{\lambda}\mathrm{d}s < \infty$ if and only if $\lambda < -1$, we have

$$\int_a^t \Big(\frac{sp(s)}{r(s)}\Big)^{\frac{1}{\alpha-1}}\mathrm{d}s < \infty$$

if and only if $\frac{\gamma_1-\gamma_2}{\alpha-1} < -1$. Notice that if $\eta \neq 1$, then

$$\int \frac{1}{s}(\ln s)^{\eta}\mathrm{d}s = \frac{(\ln t)^{\eta+1}}{\eta+1} + const.$$

On the other hand, if $\eta = 1$, then

$$\int \frac{1}{s\ln s}\mathrm{d}s = \ln(\ln t) + const.$$

Using results showed in Corollary 3.1 for $\delta < -1$ and Corollary 3.2 for $\delta > -1$ in the previous chapter for $\frac{\gamma_1-\gamma_2}{\alpha-1} < -1$ we have that every slowly varying solution has a finite non-zero limit l and

$$y(t) = l\exp\Big\{\operatorname{sgn}(\delta+1)\big(1+o(1)\big)(\ln t)^{\frac{\gamma_1-\gamma_2}{\alpha-1}+1}\frac{1-\alpha}{\gamma_1-\gamma_2+\alpha-1}\cdot\frac{1}{|\delta+1|^{\frac{1}{\alpha-1}}}\Big\}$$

as $t \to \infty$. On the other hand, if $\frac{\gamma_1-\gamma_2}{\alpha-1} > -1$, then in the case $\delta > -1$ increasing solutions are unbounded and for the case $\delta < -1$ decreasing solutions have zero limit and we have

$$y(t) = \exp\Big\{\operatorname{sgn}(\delta+1)\big(1+o(1)\big)(\ln t)^{\frac{\gamma_1-\gamma_2}{\alpha-1}+1}\frac{\alpha-1}{\gamma_1-\gamma_2+\alpha-1}\cdot\frac{1}{|\delta+1|^{\frac{1}{\alpha-1}}}\Big\}$$

as $t \to \infty$. Also, we can make a conclusion for the case $\frac{\gamma_1-\gamma_2}{\alpha-1} = -1$, so we have

$$y(t) = (\ln t)^{\operatorname{sgn}(\delta+1)(1+o(1))|\delta+1|^{-\frac{1}{\alpha-1}}}$$

as $t \to \infty$.

In the next example we will work with a nearly-linear differential equation, which we will later modify to a nearly-half linear one.

Example 4.2. Consider the equation

$$\big(y' L_G(|y'|)\big)' = \frac{L_p(t)y}{t^2|\ln|y||}, \tag{4.3}$$

where $L_G \in \mathcal{SV}_0$ and $L_p \in \mathcal{SV}$. Then

$$\hat{F}(u) = -\frac{(\ln(u))^2}{2}, \quad u \in (0,1)$$

so $\hat{F}(u) \to -\infty$ as $u \to 0_+$. Now we compute the inverse to this function

$$\hat{F}^{-1}(u) = \exp(-\sqrt{-2u}), \quad u < 0.$$

We deal only with positive solutions such that $y(t) < 1$ for $t \geq a$. It is a required condition because we need $F(u)$ to be increasing at least in a certain neighbourhood of zero (here it is $(0,1)$). A slight modification of F as $F(u) = \frac{u}{|\ln|u/k||}$, $k \in (0,\infty)$, ensures the required monotonicity of F on the (possibly bigger) interval $(0,k)$. Notice, that in our case $\delta = -2 < -1$.

Let us take the function $G(u) = u|\ln|u||$ and $L_p(t) = \frac{1}{\ln t + h(t)}$, where h is continuous function on $[a,\infty)$ with $|h(t)| = o(\ln t)$ as $t \to \infty$, and such that $\ln t + h(t) > 0$ for $t \in [a,\infty)$. Some examples of such functions are provided in the previous example. All of the conditions required in Corollary 3.1 are satisfied, so we can apply it and find an asymptotic formula for decreasing slowly varying solutions. Now, we analyse

$$\frac{tp(t)}{L_G(1/t)} = \frac{1}{t(\ln t + h(t))|\ln(1/t)|} = \frac{1}{t(\ln t + h(t))\ln t} \sim \frac{1}{t(\ln t)^2}$$

as $t \to \infty$. Thus,

$$\int^{\infty} \frac{sp(s)}{L_G(1/s)}\mathrm{d}s < \infty \quad \text{and we have} \quad \int_t^{\infty} \frac{sp(s)}{L_G(1/s)}\mathrm{d}s \sim \frac{1}{\ln t}$$

as $t \to \infty$. Due to the results presented earlier we are able to conclude that the decreasing slowly varying solution is in the form:

$$y(t) = \exp\Big[-\sqrt{(\ln y(\infty))^2 - \frac{2(1+o(1))}{\ln t}}\Big]$$

as $t \to \infty$.

On the other hand if we take the function $G(u) = \frac{u}{\sqrt{1 \pm u^2}}$, then, similarly as in the previous case

$$\frac{tp(t)}{L_G(1/t)} = \frac{\sqrt{1 \pm \frac{1}{t^2}}}{t(\ln t + h(t))} \sim \frac{1}{t \ln t}$$

as $t \to \infty$. Note that $(\ln(\ln t))' = \frac{1}{t \ln t}$, and so

$$\int^{\infty} \frac{sp(s)}{L_G(1/s)} \mathrm{d}s = \infty.$$

In this case we get the result

$$y(t) = \exp\big[\sqrt{\big(-(1 + o(1)) \ln(\ln t)\big)}\big]$$

as $t \to \infty$ and y tends to zero. This kind of operators G we discussed earlier in Chapter 2 and it is called mean curvature operator and it is used in study of partial differential equations which model fluid mechanics problems.

Further, if we consider the equation

$$\big(t^{\delta+\alpha} L_r(t)(y')^{\alpha-1} L_G(|y'|)\big)' = \frac{t^{\delta} L_p(t) y^{\alpha-1}}{|\ln|y||}, \tag{4.4}$$

which is similar to (4.3), but we take $F \in \mathcal{RV}(\alpha - 1)$ and generalized $r \in \mathcal{RV}(\delta + \alpha)$, then the analysis is similar too. Notice that in this case we work with the nearly-half-linear equation. The previous example is a particular case of the generalized equation (4.4). Compute

$$\hat{F}(u) = -\frac{\alpha-1}{\alpha} |\ln u|^{\frac{\alpha}{\alpha-1}} \quad \text{or} \quad \hat{F}(u) = -\frac{|\ln u|^{\beta}}{\beta},$$

$u \in (0, 1)$ satisfies the condition $\hat{F}(u) \to -\infty$ as $u \to 0_+$. The inverse of $\hat{F}$ is

$$\hat{F}^{-1}(u) = \exp\big[-(-\beta u)^{1/\beta}\big].$$

Again, similarly we can choose different forms of function G such that all needed conditions from Corollary 3.1 for $\delta < -1$ and from Corollary 3.2 for $\delta > -1$ are satisfied, so then repeating all steps as before we can deduce an asymptotic formula for increasing slowly varying solutions. Notice, that we assume $L_r = (\ln t)^{\gamma_2} + h_1(t)$ and $L_p = (\ln t)^{\gamma_1} + h_2(t)$ with $|h_i(t)| = o((\ln t)^{\gamma_i})$ for $i = 1, 2$ for some $\gamma_1 < \gamma_2$.

Take $\delta < -1$, then similarly as in Example 4.1, we get two cases. For $\frac{\gamma_1 - \gamma_2}{\alpha - 1} < -1$ we have

$$\int_a^t \Big(\frac{sp(s)}{(\delta+1) r(s) L_G(1/s)} \Big)^{\frac{1}{\alpha-1}} \mathrm{d}s < \infty.$$

Using Corollary 3.1 we get decreasing $\mathcal{SV}$-solutions in the form (3.15) and they have a non-zero limit. For the case $\frac{\gamma_1 - \gamma_2}{\alpha - 1} > -1$ we have

$$\int_a^t \Big(\frac{sp(s)}{(\delta+1) r(s) L_G(1/s)} \Big)^{\frac{1}{\alpha-1}} \mathrm{d}s = \infty$$

and applying the same corollary, the asymptotic formula for decreasing $\mathcal{SV}$-solutions is in the form (3.14) and they have zero limit.

On the other hand, if we take $\delta > -1$, then the condition $\hat{F}(u) \to -\infty$ as $u \to \infty$ is also satisfied. Using Corollary 3.2 if there exist increasing solutions, then for $\frac{\gamma_1-\gamma_2}{\alpha-1} < -1$ we get increasing $\mathcal{SV}$-solutions in the form (3.40) and they have a non-zero limit. For the case $\frac{\gamma_1-\gamma_2}{\alpha-1} > -1$ we apply the same corollary and the asymptotic formula for increasing $\mathcal{SV}$-solutions is in the form (3.39) and they are unbounded.

Now, we will move to another example of nearly-half-linear equations.

Example 4.3. Consider the equation

$$\big(t^{\delta+\alpha}L_r(t)(y')^{\alpha-1}L_G(|y'|)\big)' = t^{\delta}L_p(t)y^{\alpha-1}, \tag{4.5}$$

where $L_r \in \mathcal{SV}$, $L_G \in \mathcal{SV}$, $L_p \in \mathcal{SV}$ and $L_F(u) = 1$. We have $F(u) = u^{\alpha-1}$, then

$$\hat{F}(u) = \ln u,$$

so $|\hat{F}(u)| \to \infty$ as $u \to 0_+$ and

$$\hat{F}^{-1}(u) = e^u, \quad u > 0.$$

Assume that $G(u) = u^{\alpha-1}|\ln|u||$, where $L_G(u) = |\ln|u||$, which satisfies condition $L_G(ug(u)) \sim L_G(u)$ as $u \to \infty$ for all $g \in \mathcal{SV}$. First take $\delta < -1$, so we will work with decreasing slowly varying solutions. Conditions for existence of the solution from Corollary 3.1 hold, so any decreasing solution is slowly varying and we are still able to find the asymptotic formula for the slowly varying solution.

Take $L_p(t) = \frac{1}{(\ln t)^\gamma + h(t)}$, where h is a continuous function on $[a,\infty)$ with $|h(t)| = o(\ln t)$ as $t \to \infty$ and such that $(\ln t)^\gamma + h(t) > 0$ for $t \in [a,\infty)$ and $L_r(t) = \frac{1}{(\ln t)^\mu + g(t)}$, where g is a continuous function on $[a,\infty)$ with $|g(t)| = o(\ln t)$ as $t \to \infty$ and such that $(\ln t)^\mu + g(t) > 0$ for $t \in [a,\infty)$ and $\mu < \gamma$. Examples of such functions are $h(t) = \cos t$ or $h(t) = \ln(\ln t)$. Note that the required monotonicity of G is ensured in a small neighbourhood of zero. Compute

$$\begin{aligned}\Big(\frac{tp(t)}{(\delta+1)r(t)L_G(1/t)}\Big)^{\frac{1}{\alpha-1}} &= \Big(\frac{t^{\delta+1}L_p(t)}{(\delta+1)t^{\delta+\alpha}L_r(t)|\ln(1/t)|}\Big)^{\frac{1}{\alpha-1}} \\ &= \Big(\frac{((\ln t)^\mu + g(t))}{(\delta+1)t^{\alpha-1}((\ln t)^\gamma + h(t))\ln t}\Big)^{\frac{1}{\alpha-1}} \sim \frac{1}{t}\Big(\frac{1}{\delta+1}\Big)^{\frac{1}{\alpha-1}}(\ln t)^{\frac{\mu-\gamma-1}{\alpha-1}}\end{aligned}$$

as $t \to \infty$. Notice, that

$$\int_a^\infty \frac{1}{s}(\ln s)^{\frac{\mu-\gamma-1}{\alpha-1}}\mathrm{d}s$$

converges if and only is $\frac{\mu-\gamma-1}{\alpha-1} < -1$. Then for $\frac{\mu-\gamma-1}{\alpha-1} < -1$ we have

$$\int_a^\infty \Big(\frac{sp(s)}{(\delta+1)r(s)L_G(1/s)}\Big)^{\frac{1}{\alpha-1}}\mathrm{d}s < \infty.$$

From Corollary 3.1 $y \in \mathcal{NSV} \cap \mathcal{DS}$ tends to $y(\infty) > 0$ and satisfies the formula

$$y(t) = \exp\Big[\ln(y(\infty)) + (1+o(1))\frac{\alpha-1}{\mu-\gamma+\alpha-2}(\ln t)^{(\frac{\mu-\gamma-1}{\alpha-1}+1)}\Big(\frac{1}{|\delta+1|}\Big)^{\frac{1}{\alpha-1}}\Big]$$

as $t \to \infty$. Let us assume $\frac{\mu-\gamma-1}{\alpha-1} > -1$, then

$$\int_t^\infty \Big(\frac{sp(s)}{(\delta+1)r(s)L_G(1/s)}\Big)^{\frac{1}{\alpha-1}} \mathrm{d}s = \infty$$

and applying results from Corollary 3.1, a decreasing slowly varying solution has zero limit and so we obtain:

$$y(t) = \exp\Big[-(1+o(1))\frac{\alpha-1}{\mu-\gamma+\alpha-2}(\ln t)^{(\frac{\mu-\gamma-1}{\alpha-1}+1)}\Big(\frac{1}{|\delta+1|}\Big)^{\frac{1}{\alpha-1}}\Big]$$

as $t \to \infty$.

Furthermore, if we choose $\delta > -1$, then we are investigating increasing slowly varying solutions. Similarly as in (4.2) we obtain

$$\lim_{t\to\infty} \frac{L_p(t)}{L_r(t)} = 0,$$

because $\gamma > \mu$. All of the required conditions from Corollary 3.1 are satisfied. Now we know that if there exists an increasing solution then it is also slowly varying and the asymptotic formula can be written. In the case $\frac{\mu-\gamma-1}{\alpha-1} < -1$ the integral

$$\int_a^\infty \Big(\frac{sp(s)}{(\delta+1)r(s)L_G(1/s)}\Big)^{\frac{1}{\alpha-1}} \mathrm{d}s < \infty$$

converges, then any increasing solution is in the form:

$$y(t) = \exp\Big[\ln(y(\infty)) - (1+o(1))\frac{\alpha-1}{\mu-\gamma+\alpha-2}(\ln t)^{(\frac{\mu-\gamma-1}{\alpha-1}+1)}\Big(\frac{1}{\delta+1}\Big)^{\frac{1}{\alpha-1}}\Big]$$

as $t \to \infty$. And finally, if we have $\frac{\mu-\gamma-1}{\alpha-1} > -1$, then

$$\int_t^\infty \Big(\frac{sp(s)}{(\delta+1)r(s)L_G(1/s)}\Big)^{\frac{1}{\alpha-1}} \mathrm{d}s = \infty.$$

In this case an increasing slowly varying solution tends to infinity and we have it in the form:

$$y(t) = \exp\Big[(1+o(1))\frac{\alpha-1}{\mu-\gamma+\alpha-2}(\ln t)^{(\frac{\mu-\gamma-1}{\alpha-1}+1)}\Big(\frac{1}{\delta+1}\Big)^{\frac{1}{\alpha-1}}\Big]$$

as $t \to \infty$.

5 Remarks

In this chapter we will briefly discuss some open problems and we will indicate possible directions of resolve for these problems.

Non-$\mathcal{SV}$ solutions. In the previous chapters we were discussing only decreasing solutions of (2.2) for $\delta < -1$ and increasing solutions in the case of $\delta > -1$. As we showed in Theorem 3.1 decreasing solutions are slowly varying functions and any slowly varying solution cannot increase. However increasing solutions exist and in some cases we are able to show that they belong to the class of regularly varying functions with non-zero index. First, take a look at the half-linear equation (2.3). For establishing an asymptotic formula we can use as one of the possible approaches the reciprocity principle, which is based on the following relation. If y is a solution of (2.3), then u is defined $u = Cr\Phi(y')$, where $C \in \mathbb{R}$, is a solution of the reciprocal equation:

$$\big(\hat{r}(t)(\hat{\Phi}(u'))\big)' = \hat{p}(t)\hat{\Phi}(u), \tag{5.1}$$

where $\hat{r} = p^{1-\beta}$, $\hat{p} = r^{1-\beta}$ and $\hat{\Phi}(u) = |u|^{\hat{\alpha}-1} \operatorname{sgn} u$ with $\hat{\alpha} = \beta$, $\hat{\Phi} = \Phi^{-1}$. Then assuming $p \in \mathcal{RV}(\delta)$ and $r \in \mathcal{RV}(\delta + \alpha)$ and $\lim_{t\to\infty} \frac{L_p(t)}{L_r(t)} = 0$, if $\delta < -1$, then $\mathcal{IS} \subset \mathcal{NSV}(\rho)$, if $\delta > -1$, then $\mathcal{DS} \subset \mathcal{NSV}(\rho)$, where

$$\rho = \frac{\delta + 1}{1 - \alpha}.$$

And we can deduce an asymptotic formula for non-$\mathcal{SV}$ solutions (Theorem 5.1. in [13]).

On the other hand, if we consider a nearly-half-linear equation (2.2), recalling that functions G^{-1} and F^{-1} are inverse functions of G and F and defining u such that

$$u(t) = r(t)G(y'(t)) \Rightarrow y'(t) = G^{-1}\Big(\frac{1}{r(t)}u(t)\Big),$$

the derivative u':

$$u'(t) = p(t)F(y(t)) \Rightarrow y(t) = F^{-1}\Big(\frac{1}{p(t)}u'(t)\Big),$$

so we can the following equation:

$$\Big(F^{-1}\Big(\frac{1}{p(t)}u'(t)\Big)\Big)' = G^{-1}\Big(\frac{1}{r(t)}u(t)\Big).$$

Notice, that this equation is of similar form as (2.2) and therefore it can be treated as a nearly-half-linear equation. Unfortunately, there are no asymptotic results for such equations non-linear components L_G and L_F at our disposal.

Non-emptiness of the set of eventually positive increasing solutions in the case $\delta > -1$. In Theorem 3.4 we proved that $\mathcal{IS} \subset \mathcal{NSV}$, so we got that any increasing solution is slowly varying. For the case $\delta < -1$ in Theorem 3.1 we proved in addition existence of decreasing solutions and all of them are slowly varying, i.e. $\emptyset \neq \mathcal{DS} \subset \mathcal{NSV}$. Moreover, we showed that $\mathcal{PS} \cap \mathcal{NSV} = \mathcal{DS}$ for $\delta < -1$ and $\mathcal{PS} \cap \mathcal{NSV} = \mathcal{IS}$ for $\delta > -1$. So the natural question would be whether the set of increasing solutions in the case $\delta > -1$ is non-empty. In Theorem 3.1 we applied the existence theorem from [3], but it is applicable only for decreasing solutions, so we could not prove a similar statement in Theorem 3.4.

Borderline case $\delta = -1$. We have explored behaviour of solutions of nearly-half-linear equations for two cases of the index δ, but we did not say anything about the borderline case $\delta = -1$. This case is the most complicated one for working with solutions of equations and we cannot apply Karamata's theorem 1.3 directly. In this particular case we have $p \in \mathcal{RV}(-1)$ and so we are not able to conclude whether $\int_a^\infty p(s)\mathrm{d}s$ converges or diverges. Moreover, analysis of this case gets more difficult because of the bigger effect of non-linear components L_G and L_F of functions G and F.

Asymptotic estimate. In Theorem 3.3 we have discussed the problem that

$$y(t) \sim \hat{F}^{-1}\Big\{ -\int_a^t \Big[-\frac{sp(s)}{(\delta+1)r(s)L_G(1/s)}\Big]^{\frac{1}{\alpha-1}}\mathrm{d}s\Big\}$$

as $t \to \infty$ does not follow from the condition $\int_a^\infty (\frac{sp(s)}{r(s)L_G(1/s)})^{\frac{1}{\alpha-1}}\mathrm{d}s = \infty$. We added a restriction on $r = 1$ and we also work with α on the shorter interval from 1 to 2. We also showed in Remark 3.5 that we could choose suitable transformation such that we obtain an asymptotic estimate for the equation with generalized r and $L_G = 1$. There are things which can be improved, for example, assuming $\alpha \in (2, \infty)$ and r in general form. In the case $\delta > -1$, we cannot take $r(t) = 1$ as we showed it in the end of Chapter 3. If we try to take a generalized $r \in \mathcal{RV}(\delta + \alpha)$, then it brings us to the fact that r is a function of λ. Moreover, we are not allowed to apply the method used in Remark 3.5.

Other methods. There is a number of methods which could be very useful for investigation half-linear and nearly-half-linear differential equations.

- Transformation of independent variable was used in Remark 3.5 and might be useful for exploring the borderline case $\delta = -1$. This method allows us by using suitable transformation to obtain an equation what we are able to work with.

- A modified Riccati technique is an asymptotic linearisation by means of suitable transformation of generalized Riccati equation into a Riccati equation for linear differential equations, so then we are able to explore the behaviour of positive solutions of the linearised problem.

These methods are presented in details for some cases in [12] and we conjecture that their modification could be useful also for nearly-half-linear equations.

Conclusion

In the beginning we presented important statements from theory of regular variation, specifically the Karamata theory and de Haan theory and properties useful in particular for an asymptotic analysis. In Chapter 2 we introduced some types of non-linear differential equations, which are the objects of our interest, and we showed some results from literature related to asymptotic behaviour of solutions to such equations.

In Chapter 3 we have presented results of study of asymptotic behaviour of slowly varying solutions to some types of non-linear differential equations in the framework of theory of regular variation. We have discussed positive decreasing and increasing solutions. If the nearly-half-linear equation satisfies certain conditions, then for slowly varying solutions which decrease we proved that the set of such solutions is non-empty and all of them are normalized slowly varying. Moreover, we showed that the set of normalized slowly varying solutions is eventually equal to the set of positive decreasing solutions. Similarly, we proved that positive increasing solutions are normalized slowly varying. Again we concluded that the set of positive normalized slowly varying solutions is eventually equal to the set of decreasing solutions. Sequentially we presented asymptotic formulae for nearly-half-linear equations for both cases. We proved them for the general case where we assumed that p and r are positive regularly varying functions. This can be understood as an improvement of results already obtained for half-linear and nearly-linear equations. Moreover, the case of study of asymptotic formulae for increasing slowly varying solutions is new for the nearly-linear case. Further, we showed some additional results related to such solutions: we discussed necessary conditions for existence of normalized slowly varying solutions and we found an asymptotic estimate for decreasing slowly varying solutions.

In the last part we illustrated some equations as examples for application of obtained results. We discussed different types of non-linear differential equations, existence of their slowly varying solutions and using theorems we proved in this work we deduced asymptotic formulae for their slowly varying solutions. In Chapter 5 we mentioned more methods for analysis of asymptotic behaviour of solutions to non-linear differential equations and indicated possible directions for improvement of results we got in this thesis.

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