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DELAY DIFFERENCE EQUATIONS AND THEIR APPLICATIONS

DIFERENČNÍ ROVNICE SE ZPOŽDĚNÍM A JEJICH APLIKACE

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Abstrakt
Disertační práce se zabývá vyšetřováním kvalitativních vlastností diferenčních rovnic se zpožděním, které vznikly diskretizací příslušných diferenciálních rovnic se zpožděním pomocí tzv. Θ-metody. Cílem je analyzovat asymptotické vlastnosti numerického řešení těchto rovnic a formulovat jeho horní odhady. Studována je rovněž stabilita vybraných numerických diskretizací. Práce obsahuje také srovnání s dosud známými výsledky a několik příkladů ilustrujících hlavní dosažené výsledky.

Summary
This thesis discusses the qualitative properties of some delay difference equations. These equations originate from the Θ-method discretizations of the differential equations with a delayed argument. Our purpose is to analyse the asymptotic properties of these numerical solutions and formulate their upper bounds. We also discuss stability properties of the studied discretizations. Several illustrating examples and comparisons with the known results are presented as well.

Klíčová slova
Diferenční rovnice se zpožděním, diferenciální rovnice se zpožděním, asymptotické chování, stabilita, Θ-metoda.

Keywords
Delay difference equation, delay differential equation, asymptotic behaviour, stability, the Θ-method.

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Prohlašuji, že jsem disertační práci s názvem *Diferenční rovnice se zpožděním a jejich aplikace* vypracoval samostatně pod vedením Doc. RNDr. Jana Čermáka, CSc. s využitím materiálů uvedených v seznamu literatury.

Ing. Jiří Jánský

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Ing. Jiří Jánský

Contents

1. Introduction

Mathematical modeling of various problems via delay differential equations (DDEs) is a conventional and classical topic. It turns out to be useful especially in the situation, where the mathematical description of investigated systems depends not only on the position of a system in the current time, but also in a preceding time. In such a case the modeling via ordinary differential equations (ODEs) turns out to be insufficient. There are many interesting applications of these equations in various areas ranging from the control theory to industrial problems (see, e.g. [27]).

It is known that DDEs can be solved analytically only in some exceptional cases. There are no special types of DDEs and no computational methods (analogous to basic methods utilized for ODEs such as the variation of constants method, the separation of variables method and others) which can produce the exact solution. Therefore, the qualitative and numerical methods of solving of DDEs are of a fundamental importance even in the study of basic (linear) types of DDEs.

Roughly speaking, basic numerical methods for DDEs originates from the corresponding procedures for ODEs, where some additional operations (especially the interpolation of delayed terms) are necessary. The resulting formulae are then delay difference equations. Their previous qualitative investigation is rather rare because - contrary to DDEs - there do not exist many original significant applications for this type of difference equations. Therefore it is just a numerical discretization of DDEs which motivates the investigation of delay difference equations.

The aim of this thesis is to discuss some properties of the numerical solution of a special delay differential equation in the form

$$y'(t) = ay(t) + by(\lambda t), \quad 0 < \lambda < 1, \qquad t \geq 0\,, \tag{1.1}$$

where $a, b \in \mathbb{C}$, which appears as a mathematical model of several problems (see, e.g. [32]). Among these applications we mention a technical problem on railways (see [37]) which gave the name to (1.1) - namely the pantograph equation. This thesis discusses its Θ-method discretizations which lead to delay difference equations with some specific properties (in particular, all the discretizations discussed throughout this thesis are difference equations of a variable order).

The structure of this thesis is as follows: In Section 2 we consider equation involving a general delayed argument and present its Θ-method discretization. The form of this discretization is discussed in [21]. Section 3 deals with asymptotic analysis of discretized nonautonomous equation (1.1). We present the result formulating the upper bound of its solutions. The results stated in this section form a part of the paper [8].

Section 4 contains extensions of results derived in Section 3. The results presented in this section are the subject of the papers [8, 18, 20]. In Subsection 4.1 we consider the nonautonomous equation (1.1) with a general delayed argument and derive the extension of our results presented in Section 3. Then we consider two simple modifications of the equation (1.1) and investigate the asymptotic behaviour of solutions of their discretizations. In particular, we consider the equation (1.1) with several proportional delays and the equation (1.1) involving a forcing term.

Section 5 discusses stability analysis of the Euler formula for the equation (1.1). The results of this section will appear in [22]. In Section 6 we present some numerical consequences of our results and comparisons with the results of other authors. The considerations stated in this section form a part of the papers [8, 21].

2. The derivation of the Θ-methods for linear DDEs

We consider the differential equation with a delayed argument in the form

$$y'(t) = a(t)y(t) + b(t)y(\tau(t)), \quad t \geq t_0, \tag{2.1}$$

where $a(t), b(t), \tau(t)$ are continuous (possibly complex valued) functions on $[t_0, \infty)$ and $\tau(t)$ is a differentiable function which is strictly monotonically increasing and satisfies $\tau(t_0) = t_0$, $\tau(t) < t$ for all $t > t_0$.

The popular discretization of the equation (2.1) is the well-known Θ-method involving both Euler methods and the trapezoidal rule as particular cases. Some other types of discretization of (2.1) are described in [19].

Now we sketch derivation of the Θ-method: The integration of (2.1) yields

$$\int_0^t y'(u)du = \int_0^t a(u)y(u)du + \int_0^t b(u)y(\tau(u))du. \tag{2.2}$$

After the discretization we get

$$y_{n+1} - y_n = \int_{t_0+nh}^{t_0+(n+1)h} a(u)y(u)du + \int_{t_0+nh}^{t_0+(n+1)h} b(u)y(\tau(u))du, \tag{2.3}$$

where $y_n \approx y(t_0 + nh)$ and $h > 0$ is the stepsize. The integrals on the right-hand side of (2.3) can be approximated by use of the explicit rectangular formula as well as implicit rectangular formula. For the simplification we denote: $\tau_n := \tau(t_0 + nh)$, $\bar{\tau}_n := (\tau_n - t_0)/h$, $a_n := a(t_0 + nh)$ and $b_n := b(t_0 + nh)$.

First we approximate both integrals on the right-hand side of the equation (2.3) using the rectangular formula with the left grid point, i.e.

$$\int_{t_0+nh}^{t_0+(n+1)h} a(u)y(u)du \approx ha_n y_n,$$

Since the point τ_n is not usually a grid point, we approximate the second integral as

$$\int_{t_0+nh}^{t_0+(n+1)h} b(u)y(\tau(u))du \approx hb_n y^h(\tau_n),$$

where we define the value $y^h(\tau_n)$ as the linear interpolation utilizing the left and right neighbours of τ_n, namely

$$y^h(\tau_n) = (1 - r_n)\, y_{\lfloor \frac{\tau_n - t_0}{h} \rfloor} + r_n y_{\lfloor \frac{\tau_n - t_0}{h} \rfloor + 1}, \tag{2.4}$$

where $r_n := \frac{\tau_n - t_0}{h} - \lfloor \frac{\tau_n - t_0}{h} \rfloor$ and the symbol $\lfloor\ \rfloor$ means an integer part.

Then the equation (2.3) becomes

$$y_{n+1} = y_n + ha_n y_n + hb_n y^h(\tau_n). \tag{2.5}$$

Now we proceed to another way of discretization, which is based on the fact that integrals on the right-hand side of the equation (2.3) are approximated using the rectangular formulae with the right grid point. Since the substitution of the first integral is quite simple, it is omitted here. The substitution of the second integral has the form

$$\int\limits_{t_0+nh}^{t_0+(n+1)h} b(u)y(\tau(u))du \approx hb_{n+1}y^h(\tau_{n+1}).$$

Thus we get

$$y_{n+1} = y_n + ha_{n+1}y_{n+1} + hb_{n+1}y^h(\tau_{n+1}). \tag{2.6}$$

The linear combination of (2.5) and (2.6) yields the Θ-method in the form

$$y_{n+1} = y_n + h((1-\Theta)a_n y_n + \Theta a_{n+1}y_{n+1} + (1-\Theta)b_n y^h(\tau_n) + \Theta b_{n+1}y^h(\tau_{n+1})), \tag{2.7}$$

where $\Theta \in [0,1]$ and instead of $y^h(\tau_n), y^h(\tau_{n+1})$ we substitute the term from (2.4). Note that the equation (2.7) was derived using the procedure stated in [33].

Let $1 - \Theta h a_{n+1} \neq 0$. Then the equation (2.7) can be also rewritten as

$$y_{n+1} = R_n y_n + S_n \left(\beta_n y_{\lfloor \bar\tau_n \rfloor} + \alpha_n y_{\lfloor \bar\tau_n \rfloor +1} + \widehat{\beta}_n y_{\lfloor \bar\tau_{n+1} \rfloor} + \widehat{\alpha}_n y_{\lfloor \bar\tau_{n+1} \rfloor +1}\right), \quad n = 0, 1, \dots, \tag{2.8}$$

where

$$R_n := \frac{1 + (1-\Theta)ha_n}{1 - \Theta h a_{n+1}}, \qquad S_n := \frac{b_n h}{1 - \Theta h a_{n+1}} \tag{2.9}$$

and

$$\begin{aligned} \alpha_n &:= (1-\Theta)(\bar\tau_n - \lfloor \bar\tau_n \rfloor), & \beta_n &:= 1 - \Theta - \alpha_n\,, \\ \widehat{\alpha}_n &:= \frac{b_{n+1}}{b_n}\Theta(\bar\tau_{n+1} - \lfloor \bar\tau_{n+1} \rfloor), & \widehat{\beta}_n &:= \frac{b_{n+1}}{b_n}\Theta - \widehat{\alpha}_n\,. \end{aligned} \tag{2.10}$$

Now we present another way of discretization of (2.1). We introduce the substitution $v = \tau(u)$ in (2.2) and denote

$$\psi(v) := \tau^{-1}(v).$$

Then the equation (2.2) becomes

$$y(t) - y(0) = \int\limits_0^t a(u)y(u)du + \int\limits_0^{\tau(t)} b(\tau^{-1}(u))\psi'(u)y(u)du.$$

After the discretization we get

$$y_{n+1} - y_n = \int\limits_{t_0+nh}^{t_0+(n+1)h} a(u)y(u)du + \int\limits_{\tau(t_0+nh)}^{\tau(t_0+(n+1)h)} b(\tau^{-1}(u))\psi'(u)y(u)du. \tag{2.11}$$

Both integrals on the right-hand side of the equation (2.11) are replaced as follows: At first, they are approximated via the rectangular formulae with the left grid point. The approximation of the second integral has the form

$$\int_{\tau_n}^{\tau_{n+1}} b(\tau^{-1}(u))\psi'(u)y(u)du \approx (\tau_{n+1}-\tau_n)b_n\psi'(\tau_n)\, y^h(\tau_n).$$

Thus we get

$$y_{n+1} = y_n + ha_ny_n + b_n\left(\tau_{n+1}-\tau_n\right)\psi'\left(\tau_n\right)y^h(\tau_n), \tag{2.12}$$

where the value $y^h(\tau_n)$ is given by (2.4). Similarly we can arrive at

$$y_{n+1} = y_n + ha_{n+1}y_{n+1} + b_{n+1}\left(\tau_{n+1}-\tau_n\right)\psi'\left(\tau_{n+1}\right)y^h(\tau_{n+1}). \tag{2.13}$$

Contrary to the previous case the value $y^h(\tau_{n+1})$ now replace by

$$y^h\left(\tau_{n+1}\right) = \left(1-k_n\right)y_{\lfloor\frac{\tau_n-t_0}{h}\rfloor} + k_ny_{\lfloor\frac{\tau_n-t_0}{h}\rfloor+1}, \tag{2.14}$$

where $k_n := \frac{\tau_{n+1}-t_0}{h} - \lfloor\frac{\tau_n-t_0}{h}\rfloor$. We note that the value k_n can be greater then 1. In other words $y^h(\tau_{n+1})$ is calculated via the linear interpolation utilizing the left and right neighbours of τ_n. The linear combination of (2.12) and (2.13) yields the Θ-method in the form

$$\begin{aligned} y_{n+1} &= y_n + h((1-\Theta)a_ny_n + \Theta a_{n+1}y_{n+1}) \\ &\quad +(\tau_{n+1}-\tau_n)((1-\Theta)b_n\psi'\left(\tau_n\right)y^h(\tau_n) + \Theta b_{n+1}\psi'\left(\tau_{n+1}\right)y^h(\tau_{n+1})), \end{aligned} \tag{2.15}$$

where $y^h(\tau_n)$ and $y^h(\tau_{n+1})$ are given by (2.4), (2.14) respectively.

Considering $1-\Theta ha_{n+1} \neq 0$ the recurrence relation (2.15) can be rewritten as

$$y_{n+1} = R_ny_n + S_n\left(\tilde{\beta}_ny_{\lfloor\bar{\tau}_n\rfloor} + \tilde{\alpha}_ny_{\lfloor\bar{\tau}_n\rfloor+1}\right), \tag{2.16}$$

where R_n, S_n are given by (2.9) and

$$\tilde{\alpha}_n = \frac{1}{2h}(\tau_{n+1}-\tau_n)\left(\psi'(\tau_n)\left(\bar{\tau}_n-\lfloor\bar{\tau}_n\rfloor\right) + \frac{b_{n+1}}{b_n}\psi'(\tau_{n+1})\left(\bar{\tau}_{n+1}-\lfloor\bar{\tau}_n\rfloor\right)\right),$$

$$\tilde{\beta}_n = \frac{1}{2h}(\tau_{n+1}-\tau_n)\left(\psi'(\tau_n) + \frac{b_{n+1}}{b_n}\psi'(\tau_{n+1})\right) - \tilde{\alpha}_n.$$

Note that this equation can be found in the particular case $\Theta = 1/2$ in [6]. In the sequel we consider the formulae arising from (2.8).

3. The asymptotic behaviour of the Θ-method for the nonautonomous pantograph equation

We consider the nonautonomous pantograph equation as the particular case of (2.1) via the choice $\tau(t) = \lambda t$, $0 < \lambda < 1$, in the form

$$y'(t) = a(t)y(t) + b(t)y(\lambda t), \quad t \geq t_0 \,. \tag{3.1}$$

The procedure sketched in Section 2 yields the recurrence relation

$$y_{n+1} = R_n y_n + S_n \left(\beta_n y_{\lfloor \lambda n \rfloor} + \alpha_n y_{\lfloor \lambda n \rfloor + 1} + \widehat{\beta}_n y_{\lfloor \lambda(n+1) \rfloor} + \widehat{\alpha}_n y_{\lfloor \lambda(n+1) \rfloor + 1} \right), \tag{3.2}$$

n=0,1,..., where R_n, S_n are given by (2.9), i.e.

$$R_n := \frac{1 + (1 - \Theta)ha_n}{1 - \Theta h a_{n+1}}, \qquad S_n := \frac{hb_n}{1 - \Theta h a_{n+1}} \tag{3.3}$$

and the relations (2.10) become

$$\alpha_n := (1 - \Theta)(\lambda n - \lfloor \lambda n \rfloor), \qquad \beta_n := 1 - \Theta - \alpha_n,$$

$$\widehat{\alpha}_n := \frac{b_{n+1}}{b_n} \Theta(\lambda(n+1) - \lfloor \lambda(n+1) \rfloor), \qquad \widehat{\beta}_n := \frac{b_{n+1}}{b_n}\Theta - \widehat{\alpha}_n \,. \tag{3.4}$$

We emphasize that the relation (3.2) is a delay difference equation of a variable order. More precisely, the order of (3.2) becomes infinite as $n \to \infty$.

This section presents the result formulating the upper bound of the solutions y_n of (3.2). To describe this asymptotic estimate we introduce the inequality

$$|S_n|(|\beta_n|\varrho_{\lfloor \lambda n \rfloor} + |\alpha_n|\varrho_{\lfloor \lambda n \rfloor + 1} + |\widehat{\beta}_n|\varrho_{\lfloor \lambda(n+1) \rfloor} + |\widehat{\alpha}_n|\varrho_{\lfloor \lambda(n+1) \rfloor + 1}) \leq (1 - |R_n|)\, \varrho_n, \tag{3.5}$$

n=0,1,..., which plays the key role in our investigations. To simplify the analysis we further assume that

$$\tilde{S} := \sup_{n \in \mathbb{Z}^+} (|S_n|) < \infty,\ \tilde{\eta} := \sup_{n \in \mathbb{Z}^+} (|\beta_n| + |\alpha_n| + |\widehat{\beta}_n| + |\widehat{\alpha}_n|) < \infty,\ \tilde{R} := \sup_{n \in \mathbb{Z}^+} (|R_n|) < 1 \,. \tag{3.6}$$

If we set

$$\tilde{\gamma} := \frac{\tilde{S}\tilde{\eta}}{1 - \tilde{R}}, \tag{3.7}$$

then we can present the explicit form of a solution of (3.5).

Proposition 3.1. *Consider the inequality (3.5) and assume that (3.6) holds. Then the sequence*

$$\varrho_n = \begin{cases} \left(n - \frac{1+\lambda}{1-\lambda}\right)^{-\log_\lambda \tilde{\gamma}} & \text{for } \tilde{\gamma} \geq 1, \\ \left(n + \frac{1}{1-\lambda}\right)^{-\log_\lambda \tilde{\gamma}} & \text{for } 0 < \tilde{\gamma} < 1 \end{cases} \tag{3.8}$$

defines the positive solution of (3.5) for all $n \in \mathbb{Z}^+$, $n \geq (1+\lambda)/(1-\lambda)$.

Proof: First let $\tilde{\gamma} \geq 1$. Then ϱ_n is the nondecreasing sequence and we can write

$$|S_n| \left(|\beta_n| \varrho_{\lfloor \lambda n \rfloor} + |\alpha_n| \varrho_{\lfloor \lambda n \rfloor + 1} + |\widehat{\beta}_n| \varrho_{\lfloor \lambda(n+1) \rfloor} + |\widehat{\alpha}_n| \varrho_{\lfloor \lambda(n+1) \rfloor + 1} \right) \leq \tilde{S} \tilde{\eta} \varrho_{\lfloor \lambda(n+1) \rfloor + 1} .$$

Substituting the corresponding form of ϱ_n one gets

$$\begin{aligned} \tilde{S}\tilde{\eta} \left(\lfloor \lambda(n+1) \rfloor + 1 - \frac{1+\lambda}{1-\lambda} \right)^{-\log_\lambda \tilde{\gamma}} &\leq \tilde{S}\tilde{\eta} \left(\lambda n - \frac{\lambda + \lambda^2}{1-\lambda} \right)^{-\log_\lambda \tilde{\gamma}} \\ = \frac{\tilde{S}\tilde{\eta}}{\tilde{\gamma}} \left(n - \frac{1+\lambda}{1-\lambda} \right)^{-\log_\lambda \tilde{\gamma}} = (1-\tilde{R}) \left(n - \frac{1+\lambda}{1-\lambda} \right)^{-\log_\lambda \tilde{\gamma}} &= (1-\tilde{R}) \varrho_n \end{aligned}$$

by use of (3.7). The case $0 < \tilde{\gamma} < 1$ can be dealt with quite similarly. □

Now we can state the main assertion of this section formulating the asymptotic estimate of all solutions y_n of (3.2).

Theorem 3.2. *Let y_n be a solution of the delay difference equation (3.2), where we assume the validity of the hypothesis (3.6) and let $\tilde{\gamma}$ be given by (3.7). Then*

$$y_n = O\left(n^{-\log_\lambda \tilde{\gamma}}\right) \qquad \text{as } n \to \infty . \tag{3.9}$$

Proof: We introduce the substitution $z_n = y_n/\varrho_n$ in (3.2), where ϱ_n is given by (3.8). Then

$$\begin{aligned} \varrho_{n+1} z_{n+1} = \; & R_n \varrho_n z_n + S_n \Big(\beta_n \varrho_{\lfloor \lambda n \rfloor} z_{\lfloor \lambda n \rfloor} + \alpha_n \varrho_{\lfloor \lambda n \rfloor + 1} z_{\lfloor \lambda n \rfloor + 1} + \widehat{\beta}_n \varrho_{\lfloor \lambda(n+1) \rfloor} z_{\lfloor \lambda(n+1) \rfloor} \\ & + \widehat{\alpha}_n \varrho_{\lfloor \lambda(n+1) \rfloor + 1} z_{\lfloor \lambda(n+1) \rfloor + 1} \Big) . \end{aligned} \tag{3.10}$$

We aim at showing that every solution z_n of (3.10) is bounded as $n \to \infty$. Choose

$$\sigma_0 > \max \left(\frac{1+\lambda}{1-\lambda}, \frac{2-\lambda}{(1-\lambda)\lambda} \right), \qquad \sigma_0 \in \mathbb{Z}^+ \tag{3.11}$$

and define points $\sigma_{m+1} := \lfloor \frac{\sigma_m - 1}{\lambda} \rfloor$, where $m = 0, 1, \ldots$. The condition (3.11) guarantees that $\sigma_1 > \sigma_0$ and $\varrho_n > 0$ for $n = \lfloor \lambda \sigma_0 \rfloor, \lfloor \lambda \sigma_0 \rfloor + 1, \ldots$. Moreover, it holds

$$\lambda^{-m} \big(\sigma_0 - \frac{1+\lambda}{1-\lambda} \big) \leq \sigma_m \leq \lambda^{-1} \sigma_{m-1}, \qquad m = 1, 2, \ldots . \tag{3.12}$$

Further, we introduce intervals $I_0 := [\lfloor \lambda \sigma_0 \rfloor, \sigma_0] \cap \mathbb{Z}^+$, $I_{m+1} := [\sigma_m, \sigma_{m+1}] \cap \mathbb{Z}^+$ and denote

$$B_m := \sup(|z_k|, \, k \in \cup_{j=0}^m I_j), \qquad m = 0, 1, 2 \ldots . \tag{3.13}$$

Let $n^\star \in I_{m+1}$, $n^\star > \sigma_m$ be arbitrary. Using the inequality $\lfloor \lambda n^\star \rfloor + 1 \leq \sigma_m$ following from the definition of σ_{m+1} we wish to express and estimate $z_{n^\star}$ in terms of z_k, where $k \in \cup_{j=0}^m I_j$. On this account it is necessary to distinguish the following three cases:

(i) Let $R_{n^\star-1}=0$. Then

$$\begin{aligned} z_{n^\star} &= \frac{1}{\varrho_{n^\star}} S_{n^\star-1}\big(\beta_{n^\star-1}\varrho_{\lfloor\lambda(n^\star-1)\rfloor} z_{\lfloor\lambda(n^\star-1)\rfloor} + \alpha_{n^\star-1}\varrho_{\lfloor\lambda(n^\star-1)\rfloor+1} z_{\lfloor\lambda(n^\star-1)\rfloor+1} \\ &\quad + \widehat{\beta}_{n^\star-1}\varrho_{\lfloor\lambda n^\star\rfloor} z_{\lfloor\lambda n^\star\rfloor} + \widehat{\alpha}_{n^\star-1}\varrho_{\lfloor\lambda n^\star\rfloor+1} z_{\lfloor\lambda n^\star\rfloor+1}\big)\,. \end{aligned}$$

Taking absolute values we get

$$\begin{aligned} |z_{n^\star}| &\le B_m \frac{1}{\varrho_{n^\star}} |S_{n^\star-1}|(|\beta_{n^\star-1}|\varrho_{\lfloor\lambda(n^\star-1)\rfloor} + |\alpha_{n^\star-1}|\varrho_{\lfloor\lambda(n^\star-1)\rfloor+1} + |\widehat{\beta}_{n^\star-1}|\varrho_{\lfloor\lambda n^\star\rfloor} \\ &\quad + |\widehat{\alpha}_{n^\star-1}|\varrho_{\lfloor\lambda n^\star\rfloor+1}). \end{aligned}$$

Then using (3.5) we can estimate $|z_{n^\star}|$ as

$$|z_{n^\star}| \le \frac{\varrho_{n^\star-1}}{\varrho_{n^\star}} B_m.$$

If $\tilde{\gamma}\ge 1$, then ϱ_n is the nondecreasing sequence, hence $|z_{n^\star}|\le B_m$. If $0<\tilde{\gamma}<1$, then we can use (3.8), (3.12) and the binomial formula to derive the relation

$$|z_{n^\star}| \le B_m(1+K_1\lambda^m)\,, \tag{3.14}$$

where K_1 is a positive real constant.

(ii) Let $R_n\neq 0$ for any $n\in[\sigma_m, n^\star-1]\cap\mathbb{Z}^+$. Multiplying the equation (3.10) by $\prod_{l=\sigma_m}^{n}\frac{1}{R_l}$ we obtain

$$\begin{aligned} \Delta\left(\varrho_n z_n \prod_{l=\sigma_m}^{n-1}\frac{1}{R_l}\right) &= S_n\Big(\beta_n\varrho_{\lfloor\lambda n\rfloor} z_{\lfloor\lambda n\rfloor} + \alpha_n\varrho_{\lfloor\lambda n\rfloor+1} z_{\lfloor\lambda n\rfloor+1} + \widehat{\beta}_n\varrho_{\lfloor\lambda(n+1)\rfloor} z_{\lfloor\lambda(n+1)\rfloor} \\ &\quad + \widehat{\alpha}_n\varrho_{\lfloor\lambda(n+1)\rfloor+1} z_{\lfloor\lambda(n+1)\rfloor+1}\Big)\prod_{l=\sigma_m}^{n}\frac{1}{R_l}, \end{aligned}$$

where we put $\prod_{l=k}^{k-1}\frac{1}{R_l}=1$ for any $k\in\mathbb{Z}^+$. Summing this relation from σ_m to $n^\star-1$ we arrive at

$$\begin{aligned} \varrho_{n^\star} z_{n^\star}\prod_{l=\sigma_m}^{n^\star-1}\frac{1}{R_l} - \varrho_{\sigma_m} z_{\sigma_m} &= \sum_{p=\sigma_m}^{n^\star-1} S_p\big(\beta_p\varrho_{\lfloor\lambda p\rfloor} z_{\lfloor\lambda p\rfloor} + \alpha_p\varrho_{\lfloor\lambda p\rfloor+1} z_{\lfloor\lambda p\rfloor+1} \\ &\quad + \widehat{\beta}_p\varrho_{\lfloor\lambda(p+1)\rfloor} z_{\lfloor\lambda(p+1)\rfloor} + \widehat{\alpha}_p\varrho_{\lfloor\lambda(p+1)\rfloor+1} z_{\lfloor\lambda(p+1)\rfloor+1}\big)\prod_{l=\sigma_m}^{p}\frac{1}{R_l}\,, \end{aligned}$$

i.e.

$$\begin{aligned} z_{n^\star} &= \frac{\varrho_{\sigma_m}}{\varrho_{n^\star}} z_{\sigma_m}\prod_{l=\sigma_m}^{n^\star-1} R_l + \frac{1}{\varrho_{n^\star}}\sum_{p=\sigma_m}^{n^\star-1} S_p\big(\beta_p\varrho_{\lfloor\lambda p\rfloor} z_{\lfloor\lambda p\rfloor} + \alpha_p\varrho_{\lfloor\lambda p\rfloor+1} z_{\lfloor\lambda p\rfloor+1} \\ &\quad + \widehat{\beta}_p\varrho_{\lfloor\lambda(p+1)\rfloor} z_{\lfloor\lambda(p+1)\rfloor} + \widehat{\alpha}_p\varrho_{\lfloor\lambda(p+1)\rfloor+1} z_{\lfloor\lambda(p+1)\rfloor+1}\big)\prod_{l=p+1}^{n^\star-1} R_l\,. \end{aligned}$$

Then

$$|z_{n^\star}| \le B_m \left(\frac{\varrho_{\sigma_m}}{\varrho_{n^\star}} \prod_{l=\sigma_m}^{n^\star-1} |R_l| + \frac{1}{\varrho_{n^\star}} \sum_{p=\sigma_m}^{n^\star-1} (1-|R_p|)\, \varrho_p \prod_{l=p+1}^{n^\star-1} |R_l| \right) \tag{3.15}$$

by use of (3.5). Now we consider the obvious identity

$$(1-|R_p|) \prod_{l=p+1}^{n^\star-1} |R_l| = \Delta \prod_{l=p}^{n^\star-1} |R_l| \,.$$

Substituting this into (3.15) and summing by parts one gets

$$\begin{aligned}
|z_{n^\star}| &\le B_m \left(\frac{\varrho_{\sigma_m}}{\varrho_{n^\star}} \prod_{l=\sigma_m}^{n^\star-1} |R_l| + \frac{1}{\varrho_{n^\star}} \sum_{p=\sigma_m}^{n^\star-1} \varrho_p \Delta \prod_{l=p}^{n^\star-1} |R_l| \right) \\
&= B_m \left(\frac{\varrho_{\sigma_m}}{\varrho_{n^\star}} \prod_{l=\sigma_m}^{n^\star-1} |R_l| + 1 - \prod_{l=\sigma_m}^{n^\star-1} |R_l| \frac{\varrho_{\sigma_m}}{\varrho_{n^\star}} - \sum_{p=\sigma_m}^{n^\star-1} \prod_{l=p+1}^{n^\star-1} |R_l| \frac{\Delta \varrho_p}{\varrho_{n^\star}} \right) \\
&= B_m \left(1 - \frac{1}{\varrho_{n^\star}} \sum_{p=\sigma_m}^{n^\star-1} \Delta \varrho_p \prod_{l=p+1}^{n^\star-1} |R_l| \right) \\
&= B_m \left(1 - \frac{1}{\varrho_{n^\star}} \sum_{p=\sigma_m}^{n^\star-1} \frac{\Delta \varrho_p}{1-|R_p|} \Delta \prod_{l=p}^{n^\star-1} |R_l| \right) .
\end{aligned}$$

If $\tilde{\gamma} \ge 1$, then ϱ_p is nondecreasing, hence $\Delta \varrho_p \ge 0$ and $|z_{n^\star}| \le B_m$. If $0 < \tilde{\gamma} < 1$, then $\Delta \varrho_p$ is negative and nondecreasing, hence

$$\begin{aligned}
|z_{n^\star}| &\le B_m \left(1 - \frac{\Delta \varrho_{\sigma_m}}{\varrho_{n^\star}(1-\tilde{R})} \sum_{p=\sigma_m}^{n^\star-1} \Delta \prod_{l=p}^{n^\star-1} |R_l| \right) \\
&= B_m \left(1 - \frac{\Delta \varrho_{\sigma_m}}{\varrho_{n^\star}(1-\tilde{R})} \left(1 - \prod_{l=\sigma_m}^{n^\star-1} |R_l| \right) \right) \\
&\le B_m \left(1 - \frac{\Delta \varrho_{\sigma_m}}{\varrho_{\sigma_{m+1}}(1-\tilde{R})} \right) .
\end{aligned}$$

Substituting the corresponding form of ϱ_n we can derive

$$\frac{-\Delta \varrho_{\sigma_m}}{\varrho_{\sigma_{m+1}}(1-\tilde{R})} = \frac{(\sigma_m + \frac{1}{1-\lambda})^{-\log_\lambda \tilde{\gamma}} \left(1 - \left(1 + \frac{1}{\sigma_m + \frac{1}{1-\lambda}}\right)^{-\log_\lambda \tilde{\gamma}}\right)}{(1-\tilde{R})(\sigma_{m+1} + \frac{1}{1-\lambda})^{-\log_\lambda \tilde{\gamma}}} \,.$$

Considering (3.12) and using the binomial formula we arrive at

$$\frac{-\Delta \varrho_{\sigma_m}}{\varrho_{\sigma_{m+1}}(1-\tilde{R})} \le \frac{\log_\lambda \tilde{\gamma}}{\tilde{\gamma}(1-\tilde{R})\sigma_m} \le K_2 \lambda^m \,,$$

where

$$K_2 = \frac{\log_\lambda \tilde{\gamma}}{\tilde{\gamma}(1-\tilde{R})(\sigma_0 - \frac{1+\lambda}{1-\lambda})} > 0 \,. \tag{3.16}$$

Consequently,

$$|z_{n^\star}| \le B_m(1+K_2\lambda^m)\,. \tag{3.17}$$

(iii) Let $R_{n^\star-1} \neq 0$ and $R_k = 0$ for some $k \in [\sigma_m, n^\star-2] \cap \mathbb{Z}^+$. The proof technique applied in this case is a combination of procedures utilized in cases (i)-(ii) and therefore we present only the main idea.

First we denote $\sigma^\star := \sup(k,\ k \in [\sigma_m, n^\star-2] \cap \mathbb{Z}^+ \text{ and } R_k = 0)$. Then we multiply the equation (3.10) by $\prod_{l=\sigma^\star+1}^{n} \frac{1}{R_l}$ and sum from $\sigma^\star+1$ to $n^\star-1$ to obtain

$$\begin{aligned} z_{n^\star} \;&=\; \frac{\varrho_{\sigma^\star+1}}{\varrho_{n^\star}} z_{\sigma^\star+1} \prod_{l=\sigma^\star+1}^{n^\star-1} R_l + \frac{1}{\varrho_{n^\star}} \sum_{p=\sigma^\star+1}^{n^\star-1} S_p \left(\beta_p \varrho_{\lfloor\lambda p\rfloor} z_{\lfloor\lambda p\rfloor} + \alpha_p \varrho_{\lfloor\lambda p\rfloor+1} z_{\lfloor\lambda p\rfloor+1}\right. \\ &\left. + \widehat{\beta}_p \varrho_{\lfloor\lambda(p+1)\rfloor} z_{\lfloor\lambda(p+1)\rfloor} + \widehat{\alpha}_p \varrho_{\lfloor\lambda(p+1)\rfloor+1} z_{\lfloor\lambda(p+1)\rfloor+1}\right) \prod_{l=p+1}^{n^\star-1} R_l. \end{aligned}$$

The definition of $\sigma^\star$ implies $R_{\sigma^\star} = 0$, hence by the case (i) we can use the estimate

$$|z_{\sigma^\star+1}| \le B_m(1+K_1\lambda^m)\,.$$

Then the application of (3.5) yields

$$|z_{n^\star}| \le B_m(1+K_1\lambda^m)\Big(\frac{\varrho_{\sigma^\star+1}}{\varrho_{n^\star}} \prod_{l=\sigma^\star+1}^{n^\star-1} |R_l| + \frac{1}{\varrho_{n^\star}} \sum_{p=\sigma^\star+1}^{n^\star-1} (1-|R_p|)\varrho_p \prod_{l=p+1}^{n^\star-1} |R_l|\Big).$$

The right-hand side of this inequality is a modification of the corresponding term involved in (3.15) with σ_m replaced by $\sigma^* + 1$. Using the same line of arguments as given in the case (ii) we arrive at

$$|z_{n^\star}| \le B_m(1+K_1\lambda^m)(1+K_2\lambda^m) \le B_m(1+K_3\lambda^m)\,, \tag{3.18}$$

where K_3 is a positive real constant.

Summarizing cases (i)-(iii), the estimates (3.14), (3.17) and (3.18) imply that

$$|z_{n^\star}| \le B_m(1+K\lambda^m) \qquad \text{as } m \to \infty$$

for arbitrary $n^\star \in I_{m+1}$, $n^* > \sigma_m$ and a suitable $K > 0$. Consequently,

$$B_{m+1} \le B_m\left(1+K\lambda^m\right) \le B_0 \prod_{j=0}^{m} \left(1+K\lambda^j\right) \le B_0 \exp\{\frac{K}{1-\lambda}\} \tag{3.19}$$

and the sequence (B_m) is uniformly bounded. The estimate (3.9) is proved. □

Remark 3.3. The significance of the hypothesis (3.6) consists in the fact that it provides the explicit form of a solution ϱ_n of the inequality (3.5) and thus enables us to formulate the effective asymptotic criterion for the Θ-method (3.2). Let us emphasize that the Theorem 3.2 can be extended to particular cases of (3.2) not satisfying some of the assumptions involved in (3.6).

To outline this possible extension we first assume that $|S_n|$ is a nondecreasing and unbounded sequence, i.e. $\tilde{S} = \infty$ (the validity of other assumptions of (3.6) remains preserved). Then the inequality (3.5) always admits a positive and increasing solution ϱ_n. Indeed, e.g. the sequence

$$\varrho_n = (\eta|S_n|/(1-\tilde{R}))^n$$

satisfies (3.5) for all n large enough. Now it is easy to verify that the technique applied in the proof of the Theorem 3.2 is utilizable also provided such a solution ϱ_n is considered instead of (3.8). In particular, $\Delta\varrho_n > 0$ for all n large enough and we can omit the parts of the proof discussing the asymptotic stable case. Then the asymptotic bound (3.9) presented in the Theorem 3.2 can be slightly modified as

$$y_n = O(\varrho_n) \qquad \text{as } n \to \infty\,. \tag{3.20}$$

Of course, this asymptotic estimate (with the above specified ϱ_n) may be too rough in particular cases. Then, considering a concrete equation, we can try to find a more suitable (positive and increasing) solution ϱ_n of (3.5) representing the stronger upper bound sequence for the estimate (3.20). The illustration of this procedure is given in the Example 3.4.

Similarly we can discuss the case $\tilde{\eta} = \infty$ as well as the case $\tilde{S} = \tilde{\eta} = \infty$. The possible omission of the last condition of (3.6), namely $\tilde{R} < 1$, is the most interesting point. First note that if $|R_n| \geq 1$ for all n sufficiently large, then the inequality (3.5) does not admit any positive solution ϱ_n. In particular, if $a_n \equiv a$ is a constant, then the assumption $\tilde{R} < 1$ (which is satisfied if and only if $2\mathrm{Re}\, a < (2\Theta - 1)|a|^2 h$) cannot be omitted. If a_n is not a constant, then we can consider the case where $|R_n| < 1$ for all n sufficiently large and $\lim_{n\to\infty} |R_n| = 1$, i.e. $\tilde{R} = 1$. Under some particular choices of b_n the inequality (3.5) can admit a positive and nondecreasing solution ϱ_n, hence the estimate (3.20) remains valid. In particular, if we substitute $\varrho_n \equiv \text{const}$ into (3.5), then we obtain the inequality

$$|S_n|\left(|\beta_n| + |\alpha_n| + |\widehat{\beta}_n| + |\widehat{\alpha}_n|\right) \leq 1 - |R_n|\,, \qquad n = 0, 1, \ldots,$$

which is the condition guaranteing (without assuming (3.6)) the stability of the discretization (3.2). The case where ϱ_n decreases is much more complicated. Besides the determination of the form of ϱ_n we have to verify some additional nontrivial requirements on ϱ_n and a_n following from calculations performed in the corresponding part of the proof of the Theorem 3.2.

To summarize, in particular cases the omission of some assumptions involved in the hypothesis (3.6) is possible, but searching for a suitable solution ϱ_n of (3.5) without assuming (3.6) is, in general, a difficult task (especially in the asymptotic stable case).

The following example illustrates the extension of the Theorem 3.2 to the case where the assumption $\tilde{S} < \infty$ is not satisfied.

Example 3.4. We consider the differential equation

$$y'(t) = ay(t) + bty(t/2), \qquad t \geq 0\,,$$

where $a < 0$ and $b \neq 0$ are real scalars. The discretization of this equation based on the recurrence (3.2) with $\Theta = 1/2$ yields the relation

$$y_{n+1} = Ry_n + S_n(\beta_n y_{\lfloor n/2 \rfloor} + \alpha_n y_{\lfloor n/2 \rfloor+1} + \widehat{\beta}_n y_{\lfloor (n+1)/2 \rfloor} + \widehat{\alpha}_n y_{\lfloor (n+1)/2 \rfloor+1}) \tag{3.21}$$

with

$$R=\frac{2+ha}{2-ha},\qquad S_n=\frac{2bnh^2}{2-ha}$$

and

$$\alpha_n=\begin{cases}0, & n \text{ is even},\\ \frac{1}{4}, & n \text{ is odd},\end{cases}\qquad \beta_n=\begin{cases}\frac{1}{2}, & n \text{ is even},\\ \frac{1}{4}, & n \text{ is odd},\end{cases}$$

$$\widehat{\alpha}_n=\begin{cases}\frac{1}{4}+\frac{1}{4n}, & n \text{ is even},\\ 0, & n \text{ is odd},\end{cases}\qquad \widehat{\beta}_n=\begin{cases}\frac{1}{4}+\frac{1}{4n}, & n \text{ is even},\\ \frac{1}{2}+\frac{1}{2n}, & n \text{ is odd},\end{cases}$$

i.e. $\alpha_n+\beta_n+\widehat{\alpha}_n+\widehat{\beta}_n=1+\frac{1}{2n}$ for all $n=1,2,\dots$. Although the assumption $\tilde{S}<\infty$ involved in the hypothesis (3.6) is not satisfied, we outline the applicability of the Theorem 3.2 regardless of the invalidity of (3.6). It is enough to find an appropriate solution of the inequality

$$|S_n|(\beta_n\varrho_{\lfloor n/2\rfloor}+\alpha_n\varrho_{\lfloor n/2\rfloor+1}+\widehat{\beta}_n\varrho_{\lfloor (n+1)/2\rfloor}+\widehat{\alpha}_n\varrho_{\lfloor (n+1)/2\rfloor+1})\le(1-|R|)\varrho_n \tag{3.22}$$

resulting from (3.5). On this account we consider the auxiliary functional equation

$$pt\varphi(\frac{t}{2})=q\varphi(t),\qquad t>0\,,$$

p, $q>0$ are real scalars, which turns out to be of the key importance in this investigation. To our knowledge, one of the first papers discussing this equation was that of [12]. Utilizing the Mellin transform method the searched solution φ was derived in the form

$$\varphi(t)=t^{\log_2\frac{p}{q}+\frac{1}{2}(\log_2 t+1)}\,.$$

This relation (with $p=2|b|h^2/|2-ha|$ and $q=1-|2+ha|/|2-ha|$) can be only slightly modified to obtain the form

$$\varrho_n=\frac{n-13/2}{n-3}(n-3)^{\log_2\frac{p}{q}+\frac{1}{2}(\log_2(n-3)+1)} \tag{3.23}$$

defining the required solution of (3.22) for $n\ge 7$. Indeed, since ϱ_n is eventually increasing and $\alpha_n+\beta_n+\widehat{\alpha}_n+\widehat{\beta}_n=1+\frac{1}{2n}$ we can simplify the inequality (3.22) as

$$|S_n|\,(1+\frac{1}{2n})\varrho_{\lfloor (n+1)/2\rfloor+1}\le(1-|R|)\varrho_n.$$

Then substituting (3.23) into this relation and using some straightforward calculations one can check the validity of this inequality.

Then the Theorem 3.2 with respect to the Remark 3.3 implies that

$$y_n=O\left(n^{\log_2\frac{2|b|h^2}{|2-ha|-|2+ha|}+\frac{1}{2}(\log_2 n+1)}\right).$$

for any solution y_n of (3.21).

The previous example described the asymptotic bound for the solution of the delay difference equation (3.2) with $\Theta=1/2$ arising from the nonautonomous equation (3.1). The application of the Theorem 3.2 to the significant autonomous case $a(t)\equiv a$, $b(t)\equiv b$ will be discussed in Section 6. Here we also mention some relevant references and comparisons with the known results.

4. The asymptotic analysis of the Θ-method for the modified pantograph equation

In this section, we discuss possible extensions of the Theorem 3.2 and related proof technique. Some of these extensions are quite straightforward (e.g. the involvement of several proportional delays into our considerations), while others require some additional operations.

4.1. The equation (3.1) with a general delay

We focus on the asymptotic investigation of the Θ-method

$$y_{n+1} = R_n y_n + S_n \left(\beta_n y_{\lfloor \tau_n \rfloor} + \alpha_n y_{\lfloor \tau_n \rfloor + 1} + \widehat{\beta}_n y_{\lfloor \tau_{n+1} \rfloor} + \widehat{\alpha}_n y_{\lfloor \tau_{n+1} \rfloor + 1} \right) \tag{4.1}$$

with R_n, S_n given by (2.9) and $\alpha_n, \beta_n, \widehat{\alpha}_n, \widehat{\beta}_n$ given by (2.10), which originates from the discretization of the differential equation

$$y'(t) = a(t)y(t) + b(t)y(\tau(t)), \quad t \geq t_0\,, \tag{4.2}$$

involving a general delayed argument (see Section 2 and the equations (2.1) and (2.8)).

The asymptotic investigation of equations (4.2) and (4.1) is less developed than the study of their particular cases (3.1) and (3.2). Among papers related to our discussions on (4.2) we refer to papers [7, 9, 14, 36], where some asymptotic estimations for the equation (4.2) with infinite time lag (i.e. such that $\limsup(t - \tau(t)) = \infty$ as $t \to \infty$) have been performed. The derivation of the corresponding Θ-method discretization (4.1) as well as discussions on the stability analysis of (4.1) belong to the topics of papers [6, 15].

To analyse the asymptotics of (4.1), we have to appropriately modify the key inequality (3.5). As it might be expected, the relation

$$|S_n| \left(|\beta_n| \varrho_{\lfloor \bar{\tau}_n \rfloor} + |\alpha_n| \varrho_{\lfloor \bar{\tau}_n \rfloor + 1} + |\widehat{\beta}_n| \varrho_{\lfloor \bar{\tau}_{n+1} \rfloor} + |\widehat{\alpha}_n| \varrho_{\lfloor \bar{\tau}_{n+1} \rfloor + 1} \right) \leq (1 - |R_n|)\, \varrho_n, \qquad n = 0, 1, \ldots \tag{4.3}$$

seems to be the natural replacement of (3.5). To confirm this conjecture we start with the searching for a suitable solution of (4.3). On this account we consider the auxiliary functional equation

$$\varphi(\tau(t)) = \kappa \varphi(t), \qquad \kappa = \tau'(t_0), \quad t \geq t_0 \tag{4.4}$$

which is usually referred to as the Schröder equation. It is known (see, e.g. [28]) that if $\tau \in C^2([t_0, \infty))$, $\tau(t_0) = t_0$, $\tau(t) < t$ for all $t > t_0$, τ' is positive on $[t_0, \infty)$ and $\tau'(t_0) < 1$, then there exists a unique strictly increasing and continuously differentiable solution φ of (4.4) satisfying $\varphi'(t_0) = 1$. This solution is given by the formula

$$\varphi(t) = \lim_{n \to \infty} \kappa^{-n} (\tau^n(t) - t_0), \qquad t \geq t_0\,, \tag{4.5}$$

where τ^n means the n-th iterate of τ. In the sequel we mention a slightly modified version of this result, where further condition on τ (namely τ' nonincreasing) is imposed to ensure some additional properties of φ. We utilize these properties in the proof of the main result of this section.

Proposition 4.1. *Let $\tau \in C^2([t_0,\infty))$ be such that $\tau(t_0) = t_0$, $\tau(t) < t$ for all $t > t_0$, τ' is positive and nonincreasing on $[t_0,\infty)$ and $\tau'(t_0) < 1$. Then the function φ defined by (4.5) is the solution of (4.4) such that φ' is positive, continuous and nonincreasing on $[t_0,\infty)$ and, furthermore, $\varphi'(t)/\varphi(t) \le 1/(t-t_0)$ for all $t > t_0$.*

Proof: Differentiating (4.4) one can obtain

$$\varphi'(\tau(t))\tau'(t) = \kappa\varphi'(t), \qquad t \ge t_0$$

which implies that φ' is positive and nonincreasing. Similarly,

$$\frac{\varphi'(t)}{\varphi(t)} = \frac{\varphi'(t)}{\varphi(t)-\varphi(t_0)} \le \frac{\varphi'(t)}{\varphi'(t)(t-t_0)} = \frac{1}{t-t_0}, \qquad t > t_0\,. \qquad \square$$

Throughout this section we shall assume that all the assumptions imposed on τ in the Proposition 4.1 are satisfied and φ is the function defined by (4.5) with the properties guaranteed by the Proposition 4.1. Then we consider the differential equation (4.2), its Θ - method discretization (4.1) and the inequality (4.3). To formulate the upper bound of the solutions of (4.1) it is necessary to present the exact form of the solutions of (4.3).

Proposition 4.2. *Consider the inequality (4.3) and assume that (3.6) holds. Further, let $t^* \ge t_0$ be a (unique) real root of the equation $t - \tau(t+h) = h$ and let $k^* = \lfloor (t^*-t_0)/h \rfloor + 1$. Then*

$$\varrho_n = \begin{cases} (\varphi(t_0+(n-k^*)h))^{-\log_\kappa \tilde\gamma} & \text{for } \tilde\gamma \ge 1, \\ (\varphi(t_0+(n+k^*)h))^{-\log_\kappa \tilde\gamma} & \text{for } 0 < \tilde\gamma < 1\,, \end{cases} \tag{4.6}$$

where $\tilde\gamma$, $\tilde\eta$ are given by (3.7) and (3.6) respectively, defines the solution of (4.3). Moreover, if $\tilde\gamma \ge 1$, then $\Delta\varrho_n$ is nonnegative and if $0 < \tilde\gamma < 1$, then $\Delta\varrho_n$ is negative and nondecreasing.

Proof: First let $\tilde\gamma \ge 1$. Then ϱ_n is nondecreasing and

$$|S_n|\left(|\beta_n|\varrho_{\lfloor\bar\tau_n\rfloor} + |\alpha_n|\varrho_{\lfloor\bar\tau_n\rfloor+1} + |\widehat\beta_n|\varrho_{\lfloor\bar\tau_{n+1}\rfloor} + |\widehat\alpha_n|\varrho_{\lfloor\bar\tau_{n+1}\rfloor+1}\right) \le \tilde S\tilde\eta\varrho_{\lfloor\bar\tau_{n+1}\rfloor+1}$$

Substituting the corresponding form of ϱ_n one gets

$$\begin{aligned} \tilde S\tilde\eta\varrho_{\lfloor\bar\tau_{n+1}\rfloor+1} &\le \tilde S\tilde\eta(\varphi(t_0+\bar\tau_{n+1}h+h-k^*h))^{-\log_\kappa\tilde\gamma} \\ &= \tilde S\tilde\eta\,(\varphi(\tau_{n+1}+h-k^*h))^{-\log_\kappa\tilde\gamma} \\ &\le \tilde S\tilde\eta(\varphi(\tau_{n-k^*}))^{-\log_\kappa\tilde\gamma} = (1-\tilde R)\varrho_n \end{aligned}$$

by use of (4.4).

The case $0 < \tilde\gamma < 1$ can be dealt with quite similarly. Moreover, the additional properties of $\Delta\varrho_n$ follow from the corresponding properties of φ. $\square$

Remark 4.3. The sequence (4.6) is defined for all $n \geq k^*$ provided $\tilde{\gamma} \geq 1$. If $0 < \tilde{\gamma} < 1$, then ϱ_n defines the solution of (4.3) for all $n \geq 0$.

Now we can formulate the following generalization of the Theorem 3.2.

Theorem 4.4. *Let y_n be a solution of (4.1), where we assume the validity of the hypothesis (3.6), let $\tilde{\gamma}$ be given by (3.7) and let $\kappa = \tau'(t_0)$. Then*

$$y_n = O\left((\varphi(n))^{-\log_\kappa \tilde{\gamma}}\right) \qquad \text{as } n \to \infty\,. \tag{4.7}$$

Proof: The proof method is a modification of the procedure utilized in the proof of the Theorem 3.2. First we introduce the substitution $z_n = y_n/\varrho_n$, where ϱ_n is given by (4.6). Then

$$\begin{aligned} \varrho_{n+1} z_{n+1} &= R_n \varrho_n z_n + S_n \Big(|\beta_n| \varrho_{\lfloor \bar{\tau}_n \rfloor} z_{\lfloor \bar{\tau}_n \rfloor} + |\alpha_n| \varrho_{\lfloor \bar{\tau}_n \rfloor + 1} z_{\lfloor \bar{\tau}_n \rfloor + 1} + |\widehat{\beta}_n| \varrho_{\lfloor \bar{\tau}_{n+1} \rfloor} z_{\lfloor \bar{\tau}_{n+1} \rfloor} \\ &\quad + |\widehat{\alpha}_n| \varrho_{\lfloor \bar{\tau}_{n+1} \rfloor + 1} z_{\lfloor \bar{\tau}_{n+1} \rfloor + 1} \Big)\,. \end{aligned}$$

Choose $\sigma_0 > \max\left(\frac{1+\kappa}{1-\kappa}, \frac{\tau^{-1}(t_0 + k^* h) - t_0}{h}\right)$, $\sigma_0 \in \mathbb{Z}^+$ and define $I_0 := [\lfloor \bar{\tau}_{\sigma_0} \rfloor, \sigma_0] \cap \mathbb{Z}^+$, $\sigma_{m+1} := \lfloor \frac{\tau^{-1}(t_0 + (\sigma_m - 1)h) - t_0}{h} \rfloor$, $I_{m+1} := [\sigma_m, \sigma_{m+1}] \cap \mathbb{Z}^+$, $B_m := \sup(|z_k|,\, k \in \cup_{j=0}^m I_j)$, $m = 0, 1, \ldots$. Now considering arbitrary $n^* \in I_{m+1}$, $n^* > \sigma_m$ we distinguish the following cases:

(i) Let $R_{n^\star - 1} = 0$. Using the same line of arguments as given in the proof of the Theorem 3.2 we arrive at the estimate

$$|z_{n^\star}| \leq \frac{\varrho_{n^\star - 1}}{\varrho_{n^\star}} B_m\,.$$

If $\tilde{\gamma} \geq 1$ then $|z_{n^\star}| \leq B_m$. If $0 < \tilde{\gamma} < 1$, then we utilize the mean value theorem, the binomial formula and properties of φ guaranteed by the Proposition 4.1 to rewrite the term $\varrho_{n^*-1}/\varrho_{n^*}$ as

$$\begin{aligned} \frac{\varrho_{n^\star - 1}}{\varrho_{n^\star}} &= \left(\frac{\varphi(t_0 + (n^* + k^*)h)}{\varphi(t_0 + (n^* - 1 + k^*)h)} \right)^{\log_\kappa \tilde{\gamma}} \\ &= \left(1 + \frac{\varphi(t_0 + (n^* + k^*)h) - \varphi(t_0 + (n^* - 1 + k^*)h)}{\varphi(t_0 + (n^* - 1 + k^*)h)} \right)^{\log_\kappa \tilde{\gamma}} \\ &\leq \left(1 + h \frac{\varphi'(t_0 + (n^* - 1 + k^*)h)}{\varphi(t_0 + (n^* - 1 + k^*)h)} \right)^{\log_\kappa \tilde{\gamma}} \leq \left(1 + \frac{1}{\sigma_m} \right)^{\log_\kappa \tilde{\gamma}} \leq 1 + \frac{K_1}{\sigma_m}\,, \end{aligned}$$

where K_1 is a positive real constant. Consequently,

$$|z_{n^\star}| \leq B_m (1 + \frac{K_1}{\sigma_m})\,.$$

(ii) Let $R_n \neq 0$ for any $n \in [\sigma_m, n^\star - 1] \cap \mathbb{Z}^+$. Applying the corresponding steps performed in the proof of the Theorem 3.2 we can derive the estimate

$$|z_{n^*}| \leq B_m \left(1 - \frac{1}{\varrho_{n^\star}} \sum_{p=\sigma_m}^{n^\star - 1} \frac{\Delta \varrho_p}{1 - |R_p|} \Delta \prod_{l=p}^{n^\star - 1} |R_l| \right)\,.$$

If $\tilde{\gamma} \geq 1$, then $|z_{n^\star}| \leq B_m$. If $0 < \tilde{\gamma} < 1$, then, by the Proposition 4.2, $\Delta \varrho_p$ is negative and nondecreasing, hence

$$|z_{n^\star}| \leq B_m \left(1 - \frac{\Delta \varrho_{\sigma_m}}{\varrho_{\sigma_{m+1}}(1-\tilde{R})}\right).$$

To estimate the ratio term we use the mean value theorem and the monotonicity of φ' to obtain

$$\begin{aligned} -\Delta \varrho_{\sigma_m} &= (\varphi(t_0 + (\sigma_m + k^*)h))^{-\log_\kappa \tilde{\gamma}} - (\varphi(t_0 + (\sigma_m + 1 + k^*)h))^{-\log_\kappa \tilde{\gamma}} \\ &\leq h \log_\kappa \tilde{\gamma} (\varphi(t_0 + (\sigma_m + k^*)h))^{-\log_\kappa \tilde{\gamma} - 1} \varphi'(t_0 + (\sigma_m + k^*)h). \end{aligned}$$

Similarly,

$$\begin{aligned} \varrho_{\sigma_{m+1}} &= (\varphi(t_0 + (\sigma_{m+1} + k^*)h))^{-\log_\kappa \tilde{\gamma}} \\ &\geq (\varphi(\tau^{-1}(t_0 + (\sigma_m - 1)h) + k^* h))^{-\log_\kappa \tilde{\gamma}} \\ &\geq (C\varphi(\tau^{-1}(t_0 + (\sigma_m - 1)h)))^{-\log_\kappa \tilde{\gamma}} \\ &\geq \left(\frac{C}{\kappa}\right)^{-\log_\kappa \tilde{\gamma}} (\varphi(t_0 + (\sigma_m + k^*)h))^{-\log_\kappa \tilde{\gamma}} \end{aligned}$$

by use of (4.4), C being a suitable positive real constant. Consequently,

$$\frac{-\Delta \varrho_{\sigma_m}}{\varrho_{\sigma_{m+1}}(1-\tilde{R})} \leq \left(\frac{C}{\kappa}\right)^{\log_\kappa \tilde{\gamma}} \frac{h \log_\kappa \tilde{\gamma}}{1-\tilde{R}} \frac{\varphi'(t_0 + (\sigma_m + k^*)h)}{\varphi(t_0 + (\sigma_m + k^*)h)} \leq \frac{K_2}{\sigma_m}$$

and

$$|z_{n^\star}| \leq B_m \left(1 + \frac{K_2}{\sigma_m}\right),$$

where K_2 is a positive real constant.

(iii) Let $R_{n^*-1} \neq 0$ and $R_k = 0$ for some $k \in [\sigma_m, n^\star - 2] \cap \mathbb{Z}^+$. This case is fully covered by the corresponding part of the proof of the Theorem 3.2.

The cases (i)-(iii) imply that

$$|z_{n^\star}| \leq B_m \left(1 + O(\frac{1}{\sigma_m})\right) \qquad \text{as } m \to \infty,$$

where $n^* \in I_{m+1}$, $n^* > \sigma_m$ is arbitrary. Hence $B_{m+1} \leq B_m(1 + O(1/\sigma_m))$ and it remains to show that the product $\prod_{j=1}^m (1 + 1/\sigma_j)$ converges as $m \to \infty$. Using the property $\delta\varphi(t + t_0) \geq \varphi(\delta t + t_0)$, $t \geq 0$, $\delta \geq 1$ following from the properties of φ stated in the Proposition 4.1 we can write

$$\begin{aligned} \sigma_{m+1} &\geq \frac{1}{h}\big(\tau^{-1}((\sigma_m - 1)h + t_0) - t_0 - h\big) \\ &= \frac{1}{h}\Big(\varphi^{-1}\big(\frac{1}{\kappa}\varphi((\sigma_m - 1)h + t_0)\big) - t_0 - h\Big) \\ &\geq \frac{1}{h}\Big(\varphi^{-1}\big(\varphi(\frac{1}{\kappa}(\sigma_m - 1)h + t_0)\big) - t_0 - h\Big) \\ &= \frac{1}{\kappa}\sigma_m - \frac{1}{\kappa} - 1, \end{aligned}$$

hence $\sigma_m \geq \kappa^{-m}(\sigma_0 - \frac{1+\kappa}{1-\kappa})$ and the corresponding infinite product converges. Now the validity of (4.7) follows from the boundedness of B_m as $m \to \infty$. $\square$

Remark 4.5. We can verify that the Theorem 4.4 actually represents the direct generalization of the Theorem 3.2. Indeed, if $\tau(t) = \lambda t$, $0 < \lambda < 1$, $t \geq 0$, then all the assumptions of the Proposition 4.1 are satisfied and the corresponding Schröder equation

$$\varphi(\lambda t) = \lambda \varphi(t), \qquad t \geq 0$$

admits the identity function as the required solution. Now obviously the asymptotic property (4.7) becomes (3.9).

To illustrate the applicability of the Theorem 4.4 also to other types of delays we consider the differential equation (4.2) with the power delayed argument in the form

$$y'(t) = a(t)y(t) + b(t)y(t^{\omega}), \quad t \geq 1\,, \tag{4.8}$$

where $0 < \omega < 1$ is a real scalar and a, b are nonzero continuous functions on $[1, \infty)$. The Θ-method formula (4.1) now yields the recurrence relation

$$\begin{aligned} y_{n+1} \quad = \quad & R_n y_n + S_n \Big(\beta_n y_{\lfloor \frac{(1+nh)^{\omega}-1}{h} \rfloor} + \alpha_n y_{\lfloor \frac{(1+nh)^{\omega}-1}{h} \rfloor +1} + \widehat{\beta}_n y_{\lfloor \frac{(1+(n+1)h)^{\omega}-1}{h} \rfloor} \\ & + \widehat{\alpha}_n y_{\lfloor \frac{(1+(n+1)h)^{\omega}-1}{h} \rfloor +1} \Big)\,, \end{aligned} \tag{4.9}$$

where R_n, S_n are given by (2.9) with $a_n = a(1+nh)$, $b_n = b(1+nh)$ and

$$\alpha_n := (1-\Theta)(\frac{(1+nh)^{\omega}-1}{h} - \lfloor \frac{(1+nh)^{\omega}-1}{h} \rfloor), \qquad \beta_n := 1 - \Theta - \alpha_n,$$

$$\widehat{\alpha}_n := \frac{b_{n+1}}{b_n}\Theta(\frac{(1+(n+1)h)^{\omega}-1}{h} - \lfloor \frac{(1+(n+1)h)^{\omega}-1}{h} \rfloor), \qquad \widehat{\beta}_n := \frac{b_{n+1}}{b_n}\Theta - \widehat{\alpha}_n\,.$$

To apply the conclusion of the Theorem 4.4 it is easy check that the assumptions imposed on τ in the Proposition 4.1 are satisfied. Then the asymptotic property (4.7) yields the effective result for the equation (4.9) provided we are able to solve explicitly the corresponding Schröder equation (4.4). This task is not difficult because considering $\tau(t) = t^{\omega}$ the relation (4.4) becomes the functional equation

$$\varphi(t^{\omega}) = \omega \varphi(t), \qquad t \geq 1$$

with the solution $\varphi(t) = \log t$. Hence, we can present the following consequence of the Theorem 4.4.

Corollary 4.6. *Let y_n be a solution of (4.9), where we assume the validity of the hypothesis (3.6) and let $\tilde{\gamma}$ be given by (3.7). Then*

$$y_n = O\left((\log n)^{-\log_{\omega}\tilde{\gamma}}\right) \qquad \text{as } n \to \infty\,. \tag{4.10}$$

Discussing some particular cases of (4.8) we can observe close similarities between the formula (4.10) and the asymptotics of the exact equation (4.8) investigated, e.g. in [7]. Indeed, it follows from the Theorem 3.1 and the Corollary 3.6 of [7] that under some additional assumptions on coefficients a and b the upper bound for the exact solution of (4.8) can be expressed via the function $(\log t)^{-\delta}$, $\delta = \log_{\omega} Q$, where $Q > 0$ is a majorant constant of the ratio $|b(t)/a(t)|$ which is assumed to be uniformly bounded on $[1, \infty)$. For other results discussing this type of asymptotics of the differential equations with a power deviating argument we refer to paper [36] (the delayed case) and [13] (the advanced case).

4.2. The equation (3.1) with several delays

In this subsection, we discuss the numerical properties of the equation (3.1) with several proportional delays. For the sake of simplicity we consider the corresponding equation with constant coefficients. The extension to the nonautonomous case can be easily done via the modified proof technique employed in Section 3.

We consider the DDE

$$y'(t) = ay(t) + \sum_{i=1}^{k} b_i y(\lambda_i t), \quad t \geq 0\,, \tag{4.11}$$

where a, $b_i \neq 0$ are complex scalars, $0 < \lambda_i < 1$ are real scalars, $i \in \{1, 2, \dots, k\}$. We focus on delay difference equations arising from (4.11) by use of the Θ-method discretization.

Using the procedure analogical with the procedure of derivation of (2.8) we arrive at

$$y_{n+1} = Ry_n + \sum_{i=1}^{k} S_i \left(\beta_{n,i} y_{\lfloor \lambda_i n \rfloor} + \alpha_{n,i} y_{\lfloor \lambda_i n \rfloor + 1} + \widehat{\beta}_{n,i} y_{\lfloor \lambda_i (n+1) \rfloor} + \widehat{\alpha}_{n,i} y_{\lfloor \lambda_i (n+1) \rfloor + 1} \right), \tag{4.12}$$

n=0,1,..., where $y_n \approx y(nh)$, h is the stepsize,

$$R := \frac{1 + (1-\Theta)ha}{1 - \Theta ha}, \qquad S_i := \frac{hb_i}{1 - \Theta ha}$$

and

$$\alpha_{n,i} := (1-\Theta)(\lambda_i n - \lfloor \lambda_i n \rfloor),\ \beta_{n,i} := 1 - \Theta - \alpha_{n,i},$$
$$\widehat{\alpha}_{n,i} := \Theta(\lambda_i(n+1) - \lfloor \lambda_i(n+1) \rfloor),\ \widehat{\beta}_{n,i} := \Theta - \widehat{\alpha}_{n,i}\,.$$

Now we present the inequality which is useful in our further calculations. It is analogous to (3.5) and has the form:

$$\sum_{i=1}^{k} |S_i| \left(|\beta_{n,i}| \varrho_{\lfloor \lambda_i n \rfloor} + |\alpha_{n,i}| \varrho_{\lfloor \lambda_i n \rfloor + 1} + |\widehat{\beta}_{n,i}| \varrho_{\lfloor \lambda_i (n+1) \rfloor} + |\widehat{\alpha}_{n,i}| \varrho_{\lfloor \lambda_i (n+1) \rfloor + 1} \right) \leq (1 - |R|) \varrho_n, \tag{4.13}$$

n=0,1,.... Assuming

$$|R| < 1 \tag{4.14}$$

we can formulate the following assertion.

Lemma 4.7. *Let (4.14) hold. Then the sequence*

$$\varrho_n := \begin{cases} \left(n - \frac{1+\lambda}{1-\lambda}\right)^{-\log_\lambda \tilde{\gamma}} & \text{for } \tilde{\gamma} \geq 1, \\ \left(n + \frac{1}{1-\lambda}\right)^{-\log_\lambda \tilde{\gamma}} & \text{for } 0 < \tilde{\gamma} < 1, \end{cases} \tag{4.15}$$

where

$$\lambda := \begin{cases} \max(\lambda_1, \lambda_2, \dots \lambda_k) & \text{for } \tilde{\gamma} \geq 1, \\ \min(\lambda_1, \lambda_2, \dots \lambda_k) & \text{for } 0 < \tilde{\gamma} < 1 \end{cases} \tag{4.16}$$

and

$$\tilde{\gamma} := \frac{\sum_{i=1}^{k} |S_i|}{1 - |R|}\,,$$

is a solution of the inequality (4.13).

Proof: We only deal with the case $\tilde\gamma < 1$ because the case $\tilde\gamma \geq 1$ is analogical. If $\tilde\gamma < 1$, then ϱ_n is a decreasing sequence. Hence we can write

$$\sum_{i=1}^{k} |S_i| \left(|\beta_{n,i}|\varrho_{\lfloor\lambda_i n\rfloor} + |\alpha_{n,i}|\varrho_{\lfloor\lambda_i n\rfloor+1} + |\widehat\beta_{n,i}|\varrho_{\lfloor\lambda_i(n+1)\rfloor} + |\widehat\alpha_{n,i}|\varrho_{\lfloor\lambda_i(n+1)\rfloor+1}\right) \leq \sum_{i=1}^{k} |S_i|\varrho_{\lfloor\lambda_i n\rfloor}.$$

Further

$$\begin{aligned}
\sum_{i=1}^{k} |S_i|\varrho_{\lfloor\lambda_i n\rfloor} &= \sum_{i=1}^{k} |S_i|(\lfloor\lambda_i n\rfloor + \frac{1}{1-\lambda})^{-\log_\lambda\tilde\gamma} \\
&\leq \sum_{i=1}^{k} |S_i|(\lambda_i n - 1 + \frac{1}{1-\lambda})^{-\log_\lambda\tilde\gamma} \\
&= \sum_{i=1}^{k} |S_i|(\lambda_i n + \frac{\lambda}{1-\lambda})^{-\log_\lambda\tilde\gamma} \\
&\leq \sum_{i=1}^{k} |S_i|(\lambda n + \frac{\lambda}{1-\lambda})^{-\log_\lambda\tilde\gamma} \\
&= \sum_{i=1}^{k} |S_i|\lambda^{-\log_\lambda\tilde\gamma}\varrho_n. \\
&= (1-|R|)\varrho_n. \qquad \square
\end{aligned}$$

The main result of this subsection is the following

Theorem 4.8. *Let y_n be a solution of (4.12), where $|R| < 1$, $S_i \neq 0$ and $0 < \lambda_i < 1$ for all $i \in \{1, 2, \ldots, k\}$. Further let λ be given by (4.16). Then*

$$y_n = O\left(n^{-\log_\lambda\tilde\gamma}\right) \qquad \text{as } n \to \infty, \qquad \tilde\gamma := \frac{\sum_{i=1}^{k}|S_i|}{1-|R|}. \tag{4.17}$$

Proof: We use the substitution $z_n = y_n/\varrho_n$ in (4.12), where ϱ_n is given by (4.15). Then

$$\begin{aligned}
\varrho_{n+1}z_{n+1} &= R\varrho_n z_n + \sum_{i=1}^{k} S_i \left(\beta_{n,i}\varrho_{\lfloor\lambda_i n\rfloor}z_{\lfloor\lambda_i n\rfloor} + \alpha_{n,i}\varrho_{\lfloor\lambda_i n\rfloor+1}z_{\lfloor\lambda_i n\rfloor+1}\right. \\
&\qquad \left. +\widehat\beta_{n,i}\varrho_{\lfloor\lambda_i(n+1)\rfloor}z_{\lfloor\lambda_i(n+1)\rfloor} + \widehat\alpha_{n,i}\varrho_{\lfloor\lambda_i(n+1)\rfloor+1}z_{\lfloor\lambda_i(n+1)\rfloor+1}\right).
\end{aligned} \tag{4.18}$$

Now we choose

$$\sigma_0 \geq \max\left(\frac{1+\lambda}{1-\lambda}, \frac{2-\lambda}{(1-\lambda)\lambda}, 2\log_\lambda\tilde\gamma\right),$$

$\sigma_0 \in \mathbb{Z}^+$ and define points $\sigma_{m+1} := \lfloor\frac{\sigma_m - 1}{\lambda}\rfloor$, where $m = 0, 1, \ldots$. After some calculations, we obtain

$$\lambda^{-m}\big(\sigma_0 - \frac{1+\lambda}{1-\lambda}\big) \leq \sigma_m \leq \lambda^{-1}\sigma_{m-1}, \qquad m \in \mathbb{Z}^+. \tag{4.19}$$

Next we introduce intervals $I_0 := [\lambda(\sigma_0-1),\sigma_0]\cap\mathbb{Z}^+$, $I_{m+1} := [\sigma_m,\sigma_{m+1}]\cap\mathbb{Z}^+$ and denote

$$B_m := \sup(|z_s|,\ s\in\cup_{j=0}^m I_j),\qquad m=0,1,2\dots. \tag{4.20}$$

Now we choose $n^\star\in I_{m+1}$, $n^\star>\sigma_m$ arbitrarily and distinguish two cases with respect to R.

(i) First, we deal with the case $R=0$. In this case

$$\begin{aligned} z_{n^\star} &= \frac{1}{\varrho_{n^\star}}\sum_{i=1}^k S_i\left(\beta_{n^\star-1,i}\varrho_{\lfloor\lambda_i(n^\star-1)\rfloor}z_{\lfloor\lambda_i(n^\star-1)\rfloor}+\alpha_{n^\star-1,i}\varrho_{\lfloor\lambda_i(n^\star-1)\rfloor+1}z_{\lfloor\lambda_i(n^\star-1)\rfloor+1}\right.\\ &\quad\left.+\widehat{\beta}_{n^\star-1,i}\varrho_{\lfloor\lambda_i n^\star\rfloor}z_{\lfloor\lambda_i n^\star\rfloor}+\widehat{\alpha}_{n^\star-1,i}\varrho_{\lfloor\lambda_i n^\star\rfloor+1}z_{\lfloor\lambda_i n^\star\rfloor+1}\right), \end{aligned}$$

hence

$$\begin{aligned} |z_{n^\star}| &\le B_m\frac{1}{\varrho_{n^\star}}\sum_{i=1}^k |S_i|\left(|\beta_{n^\star-1,i}|\varrho_{\lfloor\lambda_i(n^\star-1)\rfloor}+|\alpha_{n^\star-1,i}|\varrho_{\lfloor\lambda_i(n^\star-1)\rfloor+1}\right.\\ &\quad\left.+|\widehat{\beta}_{n^\star-1,i}|\varrho_{\lfloor\lambda_i n^\star\rfloor}+|\widehat{\alpha}_{n^\star-1,i}|\varrho_{\lfloor\lambda_i n^\star\rfloor+1}\right). \end{aligned}$$

Using (4.13), we arrive at

$$|z_{n^\star}|\le\frac{\varrho_{n^\star-1}}{\varrho_{n^\star}}B_m.$$

If $\tilde\gamma\ge1$ then ϱ_n is the nondecreasing sequence and we obtain $|z_{n^\star}|\le B_m$. If $0<\tilde\gamma<1$, we derive with respect to (4.15), (4.19) and the binomial formula the relation

$$\begin{aligned} \frac{\varrho_{n^\star-1}}{\varrho_{n^\star}} &= \left(\frac{n^\star+\frac{1}{1-\lambda}-1}{n^\star+\frac{1}{1-\lambda}}\right)^{-\log_\lambda\tilde\gamma}\le\frac{1}{\left(1+\frac{1}{\sigma_m}\right)^{-\log_\lambda\tilde\gamma}}\\ &\le \frac{1}{1+\frac{-\log_\lambda\tilde\gamma}{\sigma_m}}\le 1+\frac{2\log_\lambda\tilde\gamma}{\sigma_m}. \end{aligned}$$

This inequality implies

$$|z_{n^\star}|\le B_m\Big(1+\frac{2\log_\lambda\tilde\gamma}{\sigma_0-\frac{1+\lambda}{1-\lambda}}\lambda^m\Big). \tag{4.21}$$

(ii) Let $R\ne0$. Then we can multiply the equation (4.18) by $\frac{1}{R^{n+1}}$ and get

$$\begin{aligned} \Delta\left(\frac{\varrho_n z_n}{R^n}\right) &= \frac{1}{R^{n+1}}\sum_{i=1}^k S_i\left(\beta_{n,i}\varrho_{\lfloor\lambda_i n\rfloor}z_{\lfloor\lambda_i n\rfloor}+\alpha_{n,i}\varrho_{\lfloor\lambda_i n\rfloor+1}z_{\lfloor\lambda_i n\rfloor+1}\right.\\ &\quad\left.+\widehat{\beta}_{n,i}\varrho_{\lfloor\lambda_i(n+1)\rfloor}z_{\lfloor\lambda_i(n+1)\rfloor}+\widehat{\alpha}_{n,i}\varrho_{\lfloor\lambda_i(n+1)\rfloor+1}z_{\lfloor\lambda_i(n+1)\rfloor+1}\right). \end{aligned}$$

If we sum this relation from σ_m to $n^\star-1$, then we obtain

$$\begin{aligned} \frac{\varrho_{n^\star}z_{n^\star}}{R^{n^\star}}-\frac{\varrho_{\sigma_m}z_{\sigma_m}}{R^{\sigma_m}} &= \sum_{p=\sigma_m}^{n^\star-1}\frac{1}{R^{p+1}}\sum_{i=1}^k S_i\left(\beta_{p,i}\varrho_{\lfloor\lambda_i p\rfloor}z_{\lfloor\lambda_i p\rfloor}+\alpha_{p,i}\varrho_{\lfloor\lambda_i p\rfloor+1}z_{\lfloor\lambda_i p\rfloor+1}\right.\\ &\quad\left.+\widehat{\beta}_{p,i}\varrho_{\lfloor\lambda_i(p+1)\rfloor}z_{\lfloor\lambda_i(p+1)\rfloor}+\widehat{\alpha}_{p,i}\varrho_{\lfloor\lambda_i(p+1)\rfloor+1}z_{\lfloor\lambda_i(p+1)\rfloor+1}\right), \end{aligned}$$

i.e.

$$\begin{aligned} z_{n^\star} &= \frac{\varrho_{\sigma_m}}{\varrho_{n^\star}} R^{n^\star-\sigma_m} z_{\sigma_m} + \frac{R^{n^\star}}{\varrho_{n^\star}} \sum_{p=\sigma_m}^{n^\star-1} \frac{1}{R^{p+1}} \sum_{i=1}^{k} S_i \left(\beta_{p,i} \varrho_{\lfloor \lambda_i p \rfloor} z_{\lfloor \lambda_i p \rfloor} + \alpha_{p,i} \varrho_{\lfloor \lambda_i p \rfloor + 1} z_{\lfloor \lambda_i p \rfloor + 1} \right. \\ &\quad \left. + \widehat{\beta}_{p,i} \varrho_{\lfloor \lambda_i (p+1) \rfloor} z_{\lfloor \lambda_i (p+1) \rfloor} + \widehat{\alpha}_{p,i} \varrho_{\lfloor \lambda_i (p+1) \rfloor + 1} z_{\lfloor \lambda_i (p+1) \rfloor + 1} \right). \end{aligned}$$

Thus

$$\begin{aligned} |z_{n^\star}| &\leq B_m \left(\frac{\varrho_{\sigma_m}}{\varrho_{n^\star}} |R|^{n^\star-\sigma_m} + \frac{|R|^{n^\star}}{\varrho_{n^\star}} \sum_{p=\sigma_m}^{n^\star-1} \frac{1}{|R|^{p+1}} \sum_{i=1}^{k} |S_i| (|\beta_{p,i}| \varrho_{\lfloor \lambda_i p \rfloor} \right. \\ &\quad \left. + |\alpha_{p,i}| \varrho_{\lfloor \lambda_i p \rfloor + 1} + |\widehat{\beta}_{p,i}| \varrho_{\lfloor \lambda_i (p+1) \rfloor} + |\widehat{\alpha}_{p,i}| \varrho_{\lfloor \lambda_i (p+1) \rfloor + 1}) \right). \end{aligned}$$

Using (4.13), we get

$$|z_{n^\star}| \leq B_m \left(\frac{\varrho_{\sigma_m}}{\varrho_{n^\star}} |R|^{n^\star-\sigma_m} + \frac{|R|^{n^\star}}{\varrho_{n^\star}} \sum_{p=\sigma_m}^{n^\star-1} \frac{1-|R|}{|R|^{p+1}} \varrho_p \right). \tag{4.22}$$

Now using the relation

$$\frac{1-|R|}{|R|^{p+1}} = \Delta \left(\frac{1}{|R|} \right)^p \tag{4.23}$$

and summing by parts we get

$$\begin{aligned} |z_{n^\star}| &\leq B_m \left(\frac{\varrho_{\sigma_m}}{\varrho_{n^\star}} |R|^{n^\star-\sigma_m} + \frac{|R|^{n^\star}}{\varrho_{n^\star}} \sum_{p=\sigma_m}^{n^\star-1} \Delta \left(\frac{1}{|R|} \right)^p \varrho_p \right) \\ &= B_m \left(\frac{\varrho_{\sigma_m}}{\varrho_{n^\star}} |R|^{n^\star-\sigma_m} + 1 - \frac{\varrho_{\sigma_m}}{\varrho_{n^\star}} |R|^{n^\star-\sigma_m} - \frac{|R|^{n^\star}}{\varrho_{n^\star}} \sum_{p=\sigma_m}^{n^\star-1} \frac{1}{|R|^{p+1}} \Delta \varrho_p \right) \\ &= B_m \left(1 - \frac{|R|^{n^\star}}{\varrho_{n^\star}} \sum_{p=\sigma_m}^{n^\star-1} \frac{1}{|R|^{p+1}} \Delta \varrho_p \right). \end{aligned}$$

Now using (4.23), we get

$$|z_{n^\star}| \leq B_m \left(1 - \frac{|R|^{n^\star}}{\varrho_{n^\star}} \sum_{p=\sigma_m}^{n^\star-1} \frac{\Delta \varrho_p}{1-|R|} \Delta \left(\frac{1}{|R|} \right)^p \right).$$

If $\tilde{\gamma} \geq 1$ then ϱ_p is nondecreasing, therefore $\Delta \varrho_p \geq 0$ and $|z_{n^\star}| \leq B_m$. In the case $0 < \tilde{\gamma} < 1$, some simple calculations are necessary to derive that $\Delta \varrho_p$ is negative and nondecreasing. Hence, we can write

$$\begin{aligned} |z_{n^\star}| &\leq B_m \left(1 - \frac{|R|^{n^\star}}{1-|R|} \frac{\Delta \varrho_{\sigma_m}}{\varrho_{n^\star}} \sum_{p=\sigma_m}^{n^\star-1} \Delta \left(\frac{1}{|R|} \right)^p \right) \\ &= B_m \left(1 - \frac{|R|^{n^\star}}{1-|R|} \frac{\Delta \varrho_{\sigma_m}}{\varrho_{n^\star}} \left(\frac{1}{|R|^{n^\star}} - \frac{1}{|R|^{\sigma_m}} \right) \right) \\ &\leq B_m \left(1 + \frac{1}{1-|R|} \frac{-\Delta \varrho_{\sigma_m}}{\varrho_{n^\star}} \right) \\ &\leq B_m \left(1 + \frac{1}{1-|R|} \frac{-\Delta \varrho_{\sigma_m}}{\varrho_{\sigma_{m+1}}} \right). \end{aligned}$$

Substituting the corresponding form of ϱ_n (see (4.15)) and using the binomial formula, we can derive

$$\begin{aligned}
-\Delta\varrho_{\sigma_m} &= \varrho_{\sigma_m} - \varrho_{\sigma_m+1} \\
&= (\sigma_m + \frac{1}{1-\lambda})^{-\log_\lambda \tilde{\gamma}} - (\sigma_m + 1 + \frac{1}{1-\lambda})^{-\log_\lambda \tilde{\gamma}} \\
&= (\sigma_m + \frac{1}{1-\lambda})^{-\log_\lambda \tilde{\gamma}}(1 - (1 + \frac{1}{\sigma_m + \frac{1}{1-\lambda}})^{-\log_\lambda \tilde{\gamma}}) \\
&\leq (\sigma_m + \frac{1}{1-\lambda})^{-\log_\lambda \tilde{\gamma}}(1 - (1 + \frac{-\log_\lambda \tilde{\gamma}}{\sigma_m + \frac{1}{1-\lambda}})) \\
&\leq (\sigma_m + \frac{1}{1-\lambda})^{-\log_\lambda \tilde{\gamma}} \frac{\log_\lambda \tilde{\gamma}}{\sigma_m}
\end{aligned}$$

and analogically

$$\begin{aligned}
\varrho_{\sigma_{m+1}} &= (\sigma_{m+1} + \frac{1}{1-\lambda})^{-\log_\lambda \tilde{\gamma}} \\
&\geq (\frac{1}{\lambda}\sigma_m + \frac{1}{1-\lambda})^{-\log_\lambda \tilde{\gamma}} \\
&\geq (\frac{1}{\lambda}\sigma_m + \frac{1}{\lambda}\frac{1}{1-\lambda})^{-\log_\lambda \tilde{\gamma}} \\
&= \tilde{\gamma}(\sigma_m + \frac{1}{1-\lambda})^{-\log_\lambda \tilde{\gamma}}.
\end{aligned}$$

Considering (4.19) we arrive at

$$\frac{-\Delta\varrho_{\sigma_m}}{\varrho_{\sigma_{m+1}}(1-|R|)} \leq \frac{\log_\lambda \tilde{\gamma}}{\tilde{\gamma}(1-|R|)}\frac{1}{\sigma_m} \leq \frac{\log_\lambda \tilde{\gamma}}{\tilde{\gamma}(1-|R|)}\frac{1}{(\sigma_0 - \frac{1+\lambda}{1-\lambda})}\lambda^m .$$

Hence

$$|z_{n^\star}| \leq B_m(1 + \frac{\log_\lambda \tilde{\gamma}}{\tilde{\gamma}(1-|R|)(\sigma_0 - \frac{1+\lambda}{1-\lambda})}\lambda^m) . \tag{4.24}$$

Using the notation

$$L := \frac{2\log_\lambda \tilde{\gamma}}{\tilde{\gamma}(1-|R|)(\sigma_0 - \frac{1+\lambda}{1-\lambda})} , \tag{4.25}$$

summarizing cases (i)-(ii) and using the estimates (4.21) and (4.24) we get

$$|z_{n^\star}| \leq B_m(1 + L\lambda^m) \qquad \text{as } m \to \infty$$

for arbitrary $n^\star \in I_{m+1}$, $n^* > \sigma_m$. Thus

$$B_{m+1} \leq B_m(1 + L\lambda^m) \qquad \text{as } m \to \infty .$$

Now we can estimate B_{m+1} as

$$\begin{aligned}
B_{m+1} &\leq B_m\,(1 + L\lambda^m) \leq B_0 \prod_{j=0}^{m} \left(1 + L\lambda^j\right) \\
&\leq B_0 \prod_{j=0}^{\infty} \left(1 + L\lambda^j\right) \leq B_0 \exp(L\frac{1}{1-\lambda}).
\end{aligned}$$

Thus

$$B_m \leq B_0 \exp(L\frac{1}{1-\lambda}) \qquad \text{as } m \to \infty.$$

The estimate (4.17) is proved. □

Remark 4.9. We can specify the O-term in (4.17). Following some steps in the proof of the Theorem 4.8 we obtain the estimate (4.17) in the form

$$|y_n| \leq K n^{-\log_\lambda \tilde{\gamma}} \qquad \text{for } n = \sigma_0, \sigma_0+1, \sigma_0+2, \ldots,$$

where $K := B_0 \exp(L\frac{1}{1-\lambda})$. Note that the constant L is given by (4.25) and B_0 can be computed via (4.20) as

$$B_0 := \sup(|y_n|/\varrho_n,\ n \in [\lfloor\lambda(\sigma_0-1)\rfloor, \sigma_0] \cap \mathbb{Z}^+). \tag{4.26}$$

The constant σ_0 should be proposed with respect to a concrete equation. If we choose $\sigma_0 \geq \max(\frac{1+\lambda}{1-\lambda}, \frac{2-\lambda}{(1-\lambda)\lambda}, 2\log_\lambda \tilde{\gamma})$ too small, then the constant L can be too large and the estimate (4.17) becomes worse. If we choose σ_0 too large, then it will be necessary to calculate the constant B_0 in (4.26) in too large interval.

Example 4.10. In this example we illustrate the application of the Theorem 4.8. Let us consider the following initial value problem

$$y'(t) = -y(t) - 0.25y(t/4) - 0.2y(t/3), \quad t \geq 0\,, \quad y(0) = 1. \tag{4.27}$$

After a trapezoidal rule discretization of (4.27) with the stepsize $h = 0.05$ we obtain

$$\begin{aligned} y_0 &= 1, \\ y_{n+1} &= \frac{39}{41}y_n - \frac{1}{82}\left(\beta_{n,1}y_{\lfloor n/4\rfloor} + \alpha_{n,1}y_{\lfloor n/4\rfloor+1} + \widehat{\beta}_{n,1}y_{\lfloor (n+1)/4\rfloor} + \widehat{\alpha}_{n,1}y_{\lfloor (n+1)/4\rfloor+1}\right) \\ &\quad -\frac{2}{205}\left(\beta_{n,2}y_{\lfloor n/3\rfloor} + \alpha_{n,2}y_{\lfloor n/3\rfloor+1} + \widehat{\beta}_{n,2}y_{\lfloor (n+1)/3\rfloor} + \widehat{\alpha}_{n,2}y_{\lfloor (n+1)/3\rfloor+1}\right), \end{aligned}$$

where

$$\alpha_{n,1} := 1/2(n/4 - \lfloor n/4\rfloor), \qquad \beta_{n,1} := 1/2 - \alpha_{n,1},$$
$$\widehat{\alpha}_{n,1} := 1/2((n+1)/4 - \lfloor (n+1)/4\rfloor), \qquad \widehat{\beta}_{n,1} := 1/2 - \widehat{\alpha}_{n,2}$$

and

$$\alpha_{n,2} := 1/2(n/3 - \lfloor n/3\rfloor), \qquad \beta_{n,2} := 1/2 - \alpha_{n,1},$$
$$\widehat{\alpha}_{n,2} := 1/2((n+1)/3 - \lfloor (n+1)/3\rfloor), \qquad \widehat{\beta}_{n,2} := 1/2 - \widehat{\alpha}_{n,2}\,.$$

Now if we set $\sigma_0 = 488$, then using the Theorem 4.8 with respect to the Remark 4.9 we obtain the estimate

$$|y_n| \leq 2.138n^{-0.576}, \qquad \text{for } n = 488, 489, \ldots \tag{4.28}$$

For a better graphic illustration we denote $y^h(t)$ as the linear interpolation of $\{y_n\}_{n=0}^{n=\infty}$, i.e.

$$y^h(t) = \frac{(n+1)h - t}{h}y_n + \frac{t - nh}{h}y_{n+1}, \qquad t \in [nh, (n+1)h], \qquad n = 0, 1, 2, \ldots \tag{4.29}$$

and consider $t \in [25, 4000]$. The Fig. 4.1 displays the real numerical solution of the problem (4.27) and its upper bound $g(t) = 2.138h^{0.576}t^{-0.576} \approx 0.3807t^{-0.576}$.

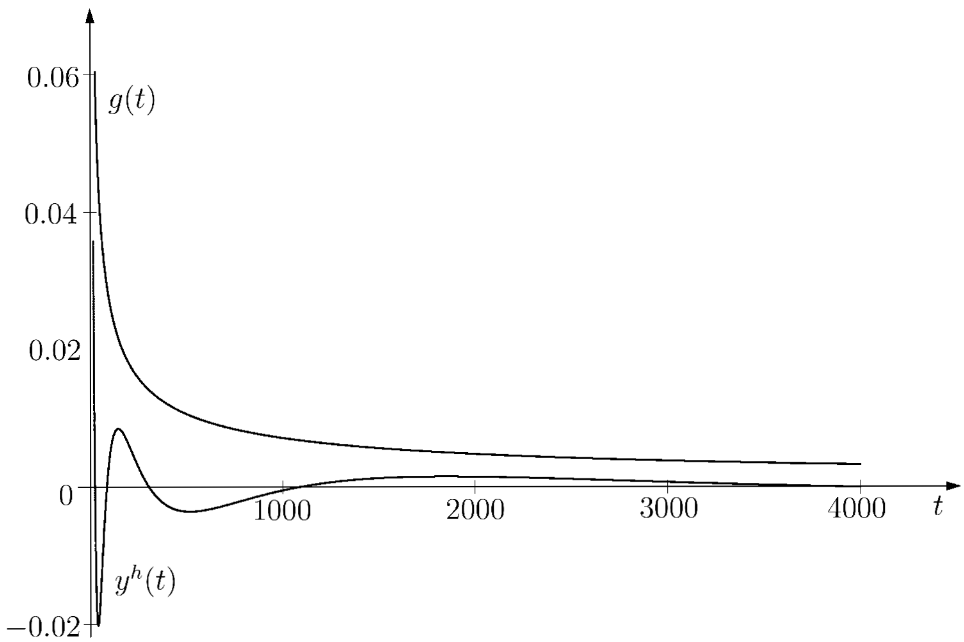

Fig. 4.1: The solution $y^h(t)$ and its estimate.

4.3. The equation (3.1) with a forcing term

We extend the problem of the study of numerical discretization of (3.1) to the nonhomogenous case. We consider the equation

$$y'(t) = ay(t) + by(\lambda t) + f(t), \quad t \geq 0\,, \tag{4.30}$$

where a, $b \neq 0$ are complex numbers, $0 < \lambda < 1$ is a real number and f is a complex-valued function. The generalization to the case $a = a(t)$, $b = b(t)$ is analogous to procedures from the Theorem 3.2.

The corresponding Θ-method discretization of (4.30) arises as a simple modification of (3.2) in the form

$$\begin{aligned} y_{n+1} &= Ry_n + S\left(\beta_n y_{\lfloor \lambda n \rfloor} + \alpha_n y_{\lfloor \lambda n \rfloor + 1} + \widehat{\beta}_n y_{\lfloor \lambda(n+1) \rfloor} + \widehat{\alpha}_n y_{\lfloor \lambda(n+1) \rfloor + 1}\right) \\ &\quad + \frac{(1-\Theta)hf_n + \Theta h f_{n+1}}{1-\Theta a h}, \quad n = 0, 1, \dots\,, \end{aligned} \tag{4.31}$$

where

$$R := \frac{1 + (1-\Theta)ha}{1-\Theta ha}, \qquad S := \frac{bh}{1-\Theta ha}$$

and

$$\alpha_n := (1-\Theta)(\lambda n - \lfloor \lambda n \rfloor), \qquad \beta_n := 1 - \Theta - \alpha_n,$$
$$\widehat{\alpha}_n := \Theta(\lambda(n+1) - \lfloor \lambda(n+1) \rfloor), \qquad \widehat{\beta}_n := \Theta - \widehat{\alpha}_n\,.$$

The useful inequality (3.5) becomes

$$|S|\left(|\beta_n|\varrho_{\lfloor\lambda n\rfloor}+|\alpha_n|\varrho_{\lfloor\lambda n\rfloor+1}+|\widehat{\beta}_n|\varrho_{\lfloor\lambda(n+1)\rfloor}+|\widehat{\alpha}_n|\varrho_{\lfloor\lambda(n+1)\rfloor+1}\right)\le(1-|R|)\varrho_n, \qquad (4.32)$$

n=0,1,... To find its solution sequence ϱ_n we have to specify the meaning of the symbols $\tilde{S}$, $\tilde{\eta}$ and $\tilde{R}$ occurring in the hypothesis (3.6). Obviously $\tilde{S}=|S|$, $\tilde{\eta}=1$ and $\tilde{R}=|R|$, where we assume $|R|<1$. Thus the solution of (4.32) is given by (3.8), i.e.

$$\varrho_n=\begin{cases}\left(n-\frac{1+\lambda}{1-\lambda}\right)^{-\log_\lambda\tilde{\gamma}} & \text{for } \tilde{\gamma}\ge 1,\\ \left(n+\frac{1}{1-\lambda}\right)^{-\log_\lambda\tilde{\gamma}} & \text{for } 0<\tilde{\gamma}<1,\end{cases} \qquad (4.33)$$

where $\tilde{\gamma}$ in (3.7) becomes

$$\tilde{\gamma}:=\frac{|S|}{1-|R|}.$$

Now we present the main result of this section. We introduce the following assumption

$$f_n=O(n^\nu) \qquad \text{as } n\to\infty \qquad (4.34)$$

for a suitable real scalar $\nu<-\log_\lambda\tilde{\gamma}$. Then we can formulate the following

Theorem 4.11. *Let y_n be a solution of (4.31), where $|R|<1$, $S\neq 0$ and $0<\lambda<1$. Further let (4.34) hold. Then*

$$y_n=O\left(n^{-\log_\lambda\tilde{\gamma}}\right) \qquad \text{as } n\to\infty, \qquad \tilde{\gamma}=\frac{|S|}{1-|R|}. \qquad (4.35)$$

Proof: We use the substitution $z_n=y_n/\varrho_n$ in (4.31), where ϱ_n is given by (4.33). Then

$$\begin{aligned}\varrho_{n+1}z_{n+1} &= R\varrho_n z_n+S\left(\beta_n\varrho_{\lfloor\lambda n\rfloor}z_{\lfloor\lambda n\rfloor}+\alpha_n\varrho_{\lfloor\lambda n\rfloor+1}z_{\lfloor\lambda n\rfloor+1}\right. \qquad (4.36)\\ &\quad \left.+\widehat{\beta}_n\varrho_{\lfloor\lambda(n+1)\rfloor}z_{\lfloor\lambda(n+1)\rfloor}+\widehat{\alpha}_n\varrho_{\lfloor\lambda(n+1)\rfloor+1}z_{\lfloor\lambda(n+1)\rfloor+1}\right)+\frac{(1-\Theta)hf_n+\Theta hf_{n+1}}{1-\Theta ah}.\end{aligned}$$

Now we choose $\sigma_0\ge\max(\frac{1+\lambda}{1-\lambda},\frac{2-\lambda}{(1-\lambda)\lambda},2\log_\lambda\tilde{\gamma})$, $\sigma_0\in\mathbb{Z}^+$ and define points $\sigma_{m+1}:=\lfloor\frac{\sigma_m-1}{\lambda}\rfloor$, where $m=0,1,\ldots$. After some calculations, we obtain

$$\lambda^{-m}\left(\sigma_0-\frac{1+\lambda}{1-\lambda}\right)\le\sigma_m\le\lambda^{-1}\sigma_{m-1}, \qquad m=1,2,\ldots.$$

Next we introduce intervals $I_0:=[\lambda(\sigma_0-1),\sigma_0]\cap\mathbb{Z}^+$, $I_{m+1}:=[\sigma_m,\sigma_{m+1}]\cap\mathbb{Z}^+$ and denote $B_m:=\sup(|z_s|,\ s\in\cup_{j=0}^m I_j)$, $m=0,1,2\ldots$.

In the sequel we use the estimate

$$\left|\frac{(1-\Theta)hf_n+\Theta hf_{n+1}}{1-\Theta ah}\right|\le K_1\varrho_n n^{\nu+\log_\lambda\tilde{\gamma}}, \qquad (4.37)$$

where K_1 is a positive real constant, we choose $n^\star\in I_{m+1}$, $n^\star>\sigma_m$ arbitrarily and distinguish two cases with respect to R.

(i) First, we deal with the case $R=0$. In this case

$$
\begin{aligned}
z_{n^\star} &= \frac{1}{\varrho_{n^\star}} S\left(\beta_{n^\star-1}\varrho_{\lfloor\lambda(n^\star-1)\rfloor} z_{\lfloor\lambda(n^\star-1)\rfloor} + \alpha_{n^\star-1}\varrho_{\lfloor\lambda(n^\star-1)\rfloor+1} z_{\lfloor\lambda(n^\star-1)\rfloor+1}\right.\\
&\quad \left. + \widehat{\beta}_{n^\star-1}\varrho_{\lfloor\lambda n^\star\rfloor} z_{\lfloor\lambda n^\star\rfloor} + \widehat{\alpha}_{n^\star-1}\varrho_{\lfloor\lambda n^\star\rfloor+1} z_{\lfloor\lambda n^\star\rfloor+1}\right) + \frac{(1-\Theta)hf_{n^\star-1} + \Theta h f_{n^\star}}{\varrho_{n^\star}(1-\Theta a h)}\,.
\end{aligned}
$$

Then

$$
\begin{aligned}
|z_{n^\star}| &\le B_m \frac{1}{\varrho_{n^\star}} |S| \left(|\beta_{n^\star-1}|\varrho_{\lfloor\lambda(n^\star-1)\rfloor} + |\alpha_{n^\star-1}|\varrho_{\lfloor\lambda(n^\star-1)\rfloor+1}\right.\\
&\quad \left. + |\widehat{\beta}_{n^\star-1}|\varrho_{\lfloor\lambda n^\star\rfloor} + |\widehat{\alpha}_{n^\star-1}|\varrho_{\lfloor\lambda n^\star\rfloor+1}\right) + \left|\frac{(1-\Theta)hf_{n^\star-1} + \Theta h f_{n^\star}}{\varrho_{n^\star}(1-\Theta a h)}\right|\,.
\end{aligned}
$$

Using (4.32) and (4.37) we get

$$
|z_{n^\star}| \le \frac{\varrho_{n^\star-1}}{\varrho_{n^\star}} B_m + \frac{\varrho_{n^\star-1}}{\varrho_{n^\star}} K_1 (n^\star-1)^{\nu+\log_\lambda\tilde{\gamma}}.
$$

Hence

$$
|z_{n^\star}| \le \frac{\varrho_{n^\star-1}}{\varrho_{n^\star}} \left(B_m + K_1 {\sigma_m}^{\nu+\log_\lambda\tilde{\gamma}}\right).
$$

Applying the same procedure as in Subsection 4.2 we get the relation

$$
|z_{n^\star}| \le (B_m + K_1 {\sigma_m}^{\nu+\log_\lambda\tilde{\gamma}})\Big(1 + \frac{2\log_\lambda\tilde{\gamma}}{\sigma_0 - \frac{1+\lambda}{1-\lambda}}\lambda^m\Big)\,. \tag{4.38}
$$

(ii) Let $R \neq 0$. Then we can multiply the equation (4.36) by $\frac{1}{R^{n+1}}$ and get

$$
\begin{aligned}
\Delta\left(\frac{\varrho_n z_n}{R^n}\right) &= \frac{1}{R^{n+1}} S\left(\beta_n\varrho_{\lfloor\lambda n\rfloor} z_{\lfloor\lambda n\rfloor} + \alpha_n\varrho_{\lfloor\lambda n\rfloor+1} z_{\lfloor\lambda n\rfloor+1}\right.\\
&\quad \left. + \widehat{\beta}_n\varrho_{\lfloor\lambda(n+1)\rfloor} z_{\lfloor\lambda(n+1)\rfloor} + \widehat{\alpha}_n\varrho_{\lfloor\lambda(n+1)\rfloor+1} z_{\lfloor\lambda(n+1)\rfloor+1}\right)\\
&\quad + \frac{1}{R^{n+1}} \frac{(1-\Theta)hf_n + \Theta h f_{n+1}}{1-\Theta a h}.
\end{aligned}
$$

If we sum this relation from σ_m to $n^\star - 1$, then we obtain

$$
\begin{aligned}
\frac{\varrho_{n^\star} z_{n^\star}}{R^{n^\star}} - \frac{\varrho_{\sigma_m} z_{\sigma_m}}{R^{\sigma_m}} &= \sum_{p=\sigma_m}^{n^\star-1} \frac{1}{R^{p+1}} S\left(\beta_p\varrho_{\lfloor\lambda p\rfloor} z_{\lfloor\lambda p\rfloor} + \alpha_p\varrho_{\lfloor\lambda p\rfloor+1} z_{\lfloor\lambda p\rfloor+1}\right.\\
&\quad \left. + \widehat{\beta}_p\varrho_{\lfloor\lambda(p+1)\rfloor} z_{\lfloor\lambda(p+1)\rfloor} + \widehat{\alpha}_p\varrho_{\lfloor\lambda(p+1)\rfloor+1} z_{\lfloor\lambda(p+1)\rfloor+1}\right)\\
&\quad + \sum_{p=\sigma_m}^{n^\star-1} \frac{1}{R^{p+1}} \frac{(1-\Theta)hf_p + \Theta h f_{p+1}}{1-\Theta a h},
\end{aligned}
$$

i.e.

$$
\begin{aligned}
z_{n^\star} &= \frac{\varrho_{\sigma_m}}{\varrho_{n^\star}} R^{n^\star-\sigma_m} z_{\sigma_m} + \frac{R^{n^\star}}{\varrho_{n^\star}} \sum_{p=\sigma_m}^{n^\star-1} \frac{1}{R^{p+1}} S\left(\beta_p\varrho_{\lfloor\lambda p\rfloor} z_{\lfloor\lambda p\rfloor} + \alpha_p\varrho_{\lfloor\lambda p\rfloor+1} z_{\lfloor\lambda p\rfloor+1}\right.\\
&\quad \left. + \widehat{\beta}_p\varrho_{\lfloor\lambda(p+1)\rfloor} z_{\lfloor\lambda(p+1)\rfloor} + \widehat{\alpha}_p\varrho_{\lfloor\lambda(p+1)\rfloor+1} z_{\lfloor\lambda(p+1)\rfloor+1}\right)\\
&\quad + \frac{R^{n^\star}}{\varrho_{n^\star}} \sum_{p=\sigma_m}^{n^\star-1} \frac{1}{R^{p+1}} \frac{(1-\Theta)hf_p + \Theta h f_{p+1}}{1-\Theta a h}.
\end{aligned}
$$

Taking absolute values we get

$$
\begin{aligned}
|z_{n^\star}| \quad \leq \quad & B_m \left(\frac{\varrho_{\sigma_m}}{\varrho_{n^\star}} |R|^{n^\star - \sigma_m} + \frac{|R|^{n^\star}}{\varrho_{n^\star}} \sum_{p=\sigma_m}^{n^\star - 1} \frac{1}{|R|^{p+1}} |S| (|\beta_p| \varrho_{\lfloor \lambda p \rfloor} \right. \\
& \left. + |\alpha_p| \varrho_{\lfloor \lambda p \rfloor + 1} + |\widehat{\beta}_p| \varrho_{\lfloor \lambda (p+1) \rfloor} + |\widehat{\alpha}_p| \varrho_{\lfloor \lambda (p+1) \rfloor + 1}) \right) \\
& + \frac{|R|^{n^\star}}{\varrho_{n^\star}} \sum_{p=\sigma_m}^{n^\star - 1} \frac{1}{|R|^{p+1}} \left| \frac{(1-\Theta) h f_p + \Theta h f_{p+1}}{1 - \Theta a h} \right| .
\end{aligned}
$$

Thus

$$
\begin{aligned}
|z_{n^\star}| \quad \leq \quad & B_m \left(\frac{\varrho_{\sigma_m}}{\varrho_{n^\star}} |R|^{n^\star - \sigma_m} + \frac{|R|^{n^\star}}{\varrho_{n^\star}} \sum_{p=\sigma_m}^{n^\star - 1} \frac{1}{|R|^{p+1}} |S| (|\beta_p| \varrho_{\lfloor \lambda p \rfloor} \right. \\
& \left. + |\alpha_p| \varrho_{\lfloor \lambda p \rfloor + 1} + |\widehat{\beta}_p| \varrho_{\lfloor \lambda (p+1) \rfloor} + |\widehat{\alpha}_p| \varrho_{\lfloor \lambda (p+1) \rfloor + 1}) \right) \\
& + \frac{|R|^{n^\star}}{\varrho_{n^\star}} \sum_{p=\sigma_m}^{n^\star - 1} \frac{1-|R|}{|R|^{p+1}} \varrho_p \frac{K_1 p^{\nu + \log_\lambda \tilde{\gamma}}}{1 - |R|} .
\end{aligned}
$$

Using (4.32), we get

$$
|z_{n^\star}| \leq B_m \frac{\varrho_{\sigma_m}}{\varrho_{n^\star}} |R|^{n^\star - \sigma_m} + \frac{|R|^{n^\star}}{\varrho_{n^\star}} (B_m + \frac{K_1 \sigma_m^{\nu + \log_\lambda \tilde{\gamma}}}{1 - |R|}) \sum_{p=\sigma_m}^{n^\star - 1} \frac{1-|R|}{|R|^{p+1}} \varrho_p
$$

and

$$
|z_{n^\star}| \leq (B_m + \frac{K_1 \sigma_m^{\nu + \log_\lambda \tilde{\gamma}}}{1 - |R|}) \left(\frac{\varrho_{\sigma_m}}{\varrho_{n^\star}} |R|^{n^\star - \sigma_m} + \frac{|R|^{n^\star}}{\varrho_{n^\star}} \sum_{p=\sigma_m}^{n^\star - 1} \frac{1-|R|}{|R|^{p+1}} \varrho_p \right) .
$$

This equation is analogous to (4.22). Hence we can apply the same procedure as in Subsection 4.2 and obtain the analogy of (4.24) in the form

$$
|z_{n^\star}| \leq (B_m + \frac{K_1}{1-|R|} \sigma_m^{\nu + \log_\lambda \tilde{\gamma}})(1 + \frac{\log_\lambda \tilde{\gamma}}{\tilde{\gamma}(1-|R|)(\sigma_0 - \frac{1+\lambda}{1-\lambda})} \lambda^m). \tag{4.39}
$$

Now using the definition of L by (4.25), summarizing cases (i)-(ii) and using estimates (4.38) and (4.39) we get

$$
|z_{n^\star}| \leq (B_m + \frac{K_1 \sigma_m^{\nu + \log_\lambda \tilde{\gamma}}}{1 - |R|})(1 + L\lambda^m) \qquad \text{as } m \to \infty
$$

for arbitrary $n^\star \in I_{m+1}$, $n^* > \sigma_m$. Hence

$$
B_{m+1} \leq (B_m + K_2 \lambda^{-m(\nu + \log_\lambda \tilde{\gamma})})(1 + L\lambda^m) \qquad \text{as } m \to \infty
$$

where

$$
K_2 := \frac{K_1}{(1-|R|)} (\sigma_0 - \frac{1+\lambda}{1-\lambda})^{\nu + \log_\lambda \tilde{\gamma}}. \tag{4.40}
$$

Now we can estimate B_m in this way:

$$\begin{aligned} B_{m+1} &\le (B_m + K_2\lambda^{-m(\nu+\log_\lambda\tilde\gamma)})\,(1+L\lambda^m) \le (B_0 + K_2\sum_{m=0}^{\infty}\lambda^{-m(\nu+\log_\lambda\tilde\gamma)})\prod_{m=0}^{\infty}(1+L\lambda^m) \\ &\le (B_0 + K_2\frac{1}{1-\lambda^{-(\nu+\log_\lambda\tilde\gamma)}})\exp(L\frac{1}{1-\lambda}). \end{aligned}$$

Thus

$$B_m \le (B_0 + K_2\frac{1}{1-\lambda^{-(\nu+\log_\lambda\tilde\gamma)}})\exp(L\frac{1}{1-\lambda}) \qquad \text{as } m\to\infty.$$

The estimate (4.35) is proved. □

Remark 4.12. The specification of the O-term in (4.35) is easy. It requires to follow some steps performed in the proof of the Theorem 4.11. This problem is discussed in the following example.

Example 4.13. In this example we show the application of the Theorem 4.11. Let us consider the following initial value problem

$$y'(t) = -2y(t) + y(t/2) + \frac{t}{(t+1)^2(t+2)}, \quad t\ge 0\,, \quad y(0)=1 \tag{4.41}$$

with the exact solution

$$y(t) = \frac{1}{t+1}. \tag{4.42}$$

The discretization of (4.41) via the formula (4.31) with $\Theta = 2/3$ and the stepsize $h = 0.1$ becomes

$$\begin{aligned} y_0 &= 1, \\ y_{n+1} &= \frac{14}{17}y_n + \frac{3}{34}\left(\beta_n y_{\lfloor\lambda n\rfloor} + \alpha_n y_{\lfloor\lambda n\rfloor+1} + \widehat\beta_n y_{\lfloor\lambda(n+1)\rfloor} + \widehat\alpha_n y_{\lfloor\lambda(n+1)\rfloor+1}\right) \\ &\quad + \frac{h^2}{3-2ah}\left(\frac{n}{(nh+1)^2(nh+2)} + \frac{2(n+1)}{(nh+h+1)^2(nh+h+2)}\right), \end{aligned}$$

where

$$\alpha_n := \frac{1}{3}(\frac{n}{2} - \lfloor\frac{n}{2}\rfloor), \qquad \beta_n := \frac{1}{3} - \alpha_n,$$

$$\widehat\alpha_n := \frac{2}{3}(\frac{n+1}{2} - \lfloor\frac{n+1}{2}\rfloor), \qquad \widehat\beta_n := \frac{2}{3} - \widehat\alpha_n\,.$$

Now we specify the O-term in (4.35). We determine the constant K such that

$$|y_n| \le Kn^{-\log_\lambda\tilde\gamma} \qquad \text{for } n = \sigma_0, \sigma_0+1, \sigma_0+2, \ldots.$$

On this account we use the estimate $\frac{t}{(t+1)^2(t+2)} \le t^{-2}$ for all $t>0$ and put $\nu = -2$ in (4.34). Further we set $\sigma_0 = 50$ and compute constants K_1 and K_2 by use (4.37) and (4.40), respectively. Then we put

$$K = (B_0 + K_2\frac{1}{1-\lambda^{-(\nu+\log_\lambda\tilde\gamma)}})\exp(L\frac{1}{1-\lambda}).$$

Then it requires only some simple calculations to obtain the estimate (4.35) in the form

$$|y_n| \le 23.4055 n^{-1}, \qquad \text{for } n = 50, 51, \ldots \tag{4.43}$$

The decay rate of the numerical solution of (4.41) corresponds to the decay rate of the exact solution (4.42). Using (4.29), the Fig. 4.2 displays the real numerical solution of the problem (4.41) and its upper bound $g(t) = 23.4055 h t^{-1} \approx 2.3406 t^{-1}$.

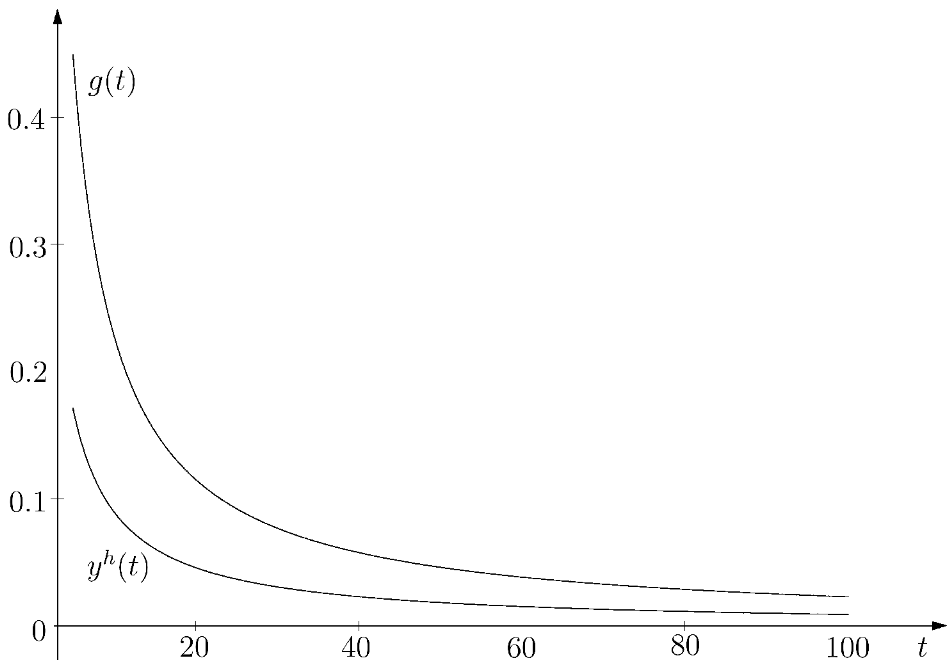

Fig. 4.2: The solution $y^h(t)$ and its estimate.

5. Stability analysis of the Euler formula for the pantograph equation

In this section, we analyse a change of the qualitative behaviour of the numerical solution of the scalar pantograph equation

$$y'(t) = ay(t) + by(\lambda t), \quad 0 < \lambda < 1 \tag{5.1}$$

which is based on the implicit Euler discretization in the form

$$y_{n+1} = \mathcal{R}y_n + \mathcal{S}y_{\lfloor \lambda(n+1) \rfloor}, \qquad n = 0, 1, 2, \ldots, \tag{5.2}$$

where

$$\mathcal{R} := \frac{1}{1 - ah}, \quad \mathcal{S} := \frac{bh}{1 - ah}, \tag{5.3}$$

$h > 0$ is the stepsize. The derivation of this equation is sketched in Section 2. Indeed, the formula (5.2) originates from (2.6) by use of $\tau_n = \lambda nh$ and $y^h(\tau_{n+1}) = y_{\lfloor \lambda(n+1) \rfloor}$. Note also that there is some analogy between (5.3) and (2.9). The relation (5.3) corresponds to the case of the constant coefficients and the choice $\Theta = 1$ in (2.9).

Assume that a, b are real scalars, $|a| + b < 0$ and $0 < 1 - \lambda << 1$. Then the numerical solution of (5.1) has a tendency to tend to zero solution, but after reaching a certain critical index this tendency vanishes and the solution is "blowing up". Our next investigation is inspired by the paper [34], where this phenomenon (familiarly referred to as the numerical nightmare) has been investigated using the explicit Euler method. In the connection with the studied problem we can mention the other useful sources [1-3,10,11,17,30].

The difference equation (5.2) is of an increasing order, but for

$$n \in I_m := \left(\frac{m + \lambda - 1}{1 - \lambda}, \frac{m + \lambda}{1 - \lambda} \right], \quad m \in \mathbb{Z}^+$$

the order is fixed to the value m. Then we can rewrite the equation (5.2) as a three-term difference equation

$$y_{n+1} - \mathcal{R}y_n - \mathcal{S}y_{n-m} = 0, \qquad n \in I_m, \tag{5.4}$$

where $\mathcal{R}$, $\mathcal{S}$ are given by (5.3).

Our aim is to estimate the maximal order $m^\star$ of the difference equation (5.4), where the condition for the asymptotic stability of its solutions is still guaranteed, but starting from $m = m^\star + 1$ is no more valid.

It is well-known that the solution of linear difference equation (5.4) is asymptotically stable if and only if all the zeros of the corresponding characteristic polynomial lie inside a unit disk. Therefore we recall the Schur-Cohn criterion (see, e.g. [10, p. 164]) which plays a key role in our investigation. For our purposes it is sufficient to reformulate this criterion directly to the three-term difference equation (5.4).

Theorem 5.1. *The zeros of the characteristic polynomial*

$$P(\mu) = \mu^{m+1} - \mathcal{R}\mu^m - \mathcal{S} \tag{5.5}$$

of the difference equation (5.4) lie inside a unit disk if and only if the following holds:

(i) $P(1) > 0$,

(ii) $(-1)^{m+1}P(-1) > 0$,

(iii) the $m \times m$ *matrices*

$$M_m^{\pm} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -\mathcal{R} & 1 & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \cdots & 0 & -\mathcal{R} & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 & \cdots & 0 & -\mathcal{S} \\ 0 & & & -\mathcal{S} & 0 \\ \vdots & & & & 0 \\ 0 & -\mathcal{S} & & & \vdots \\ -\mathcal{S} & 0 & 0 & \cdots & 0 \end{pmatrix}$$

are positive innerwise (i.e. determinants of all of its inners are positive).

In the sequel, we derive an auxiliary difference equation arising from the application of the Schur-Cohn criterion to the equation (5.4). The analysis of this auxiliary equation (in particular, the discussion of the sign of its solutions with respect to the assumptions (i)–(iii) of the Theorem 5.1) enables us to investigate the problem when the discretization (5.2) admits a sudden change of its behaviour.

We start our analysis with discussions of the assumptions of the Theorem 5.1 in the connection with our problem. Under the assumption $|a| + b < 0$ we can rewrite the condition (i) as $1 - \mathcal{R} - \mathcal{S} > 0$ which is equivalent to

$$-(a+b)\frac{h}{1-ah} > 0,$$

i.e.

$$1/h > a. \tag{5.6}$$

Condition (ii) of the Theorem 5.1 implies that we have to assume

$$1 + \mathcal{R} - \mathcal{S} > 0 \quad \text{and} \quad 1 + \mathcal{R} + \mathcal{S} > 0.$$

These inequalities are satisfied if and only if

$$h < 2/(a + |b|). \tag{5.7}$$

Note that relation (5.7) implies the previous condition (5.6).

Now let $|a| + b < 0$ and $h < 2/(a + |b|)$ (ensuring that (i) and (ii) are valid). We show that there exists $m^\star \in \mathbb{Z}^+$ such that the third condition (iii) holds provided $m = 1, \dots, m^\star$ and it is not valid for all integers $m > m^\star$. On this account we derive a three-term difference equation for determinants $D_m := \det(M_m^{\pm})$, $m = 1, 2, \dots$ (see the Theorem 5.1). We introduce here $\tilde{\mathcal{S}} := \pm\mathcal{S}$ to cover both sign cases in next computations. Then we can express D_{m+2} as

$$D_{m+2} = \begin{vmatrix} 1 & 0 & \dots & 0 & -\tilde{\mathcal{S}} \\ -\mathcal{R} & & & & 0 \\ \vdots & & \boxed{M_m^{\pm}} & & \vdots \\ 0 & & & & 0 \\ -\tilde{\mathcal{S}} & 0 & \dots & 0 \; -\mathcal{R} & 1 \end{vmatrix}$$

$$= \left|\begin{array}{cc} \boxed{\quad M_m^{\pm} \quad} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ 0 \ \dots \ 0 \ -\mathcal{R} & 1 \end{array}\right| - (-1)^{m+3}\tilde{\mathcal{S}} \left|\begin{array}{cc} \begin{matrix}-\mathcal{R}\\ \vdots\\ 0\end{matrix} & \boxed{\quad M_m^{\pm} \quad} \\ -\tilde{\mathcal{S}} & 0 \ \dots \ 0 \ -\mathcal{R} \end{array}\right|.$$

Now we apply the Laplace expansion along the last column in the first matrix and along the first column in the second one. Then we get

$$D_{m+2} = (1-\tilde{\mathcal{S}}^2)D_m - (-1)^m \mathcal{R}\tilde{\mathcal{S}} \left|\begin{array}{ccc} \begin{matrix}-\mathcal{R}\\ \vdots\\ 0\end{matrix} & \boxed{\quad M_{m-2}^{\pm} \quad} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ -\tilde{\mathcal{S}} & 0 \ \dots \ 0 \ -\mathcal{R} & 1 \\ 0 & 0 \quad \dots \quad 0 & -\mathcal{R} \end{array}\right|. \tag{5.8}$$

Analogously we can write

$$D_{m+4} = (1-\tilde{\mathcal{S}}^2)D_{m+2} - (-1)^m \mathcal{R}\tilde{\mathcal{S}} \left|\begin{array}{ccc} \begin{matrix}-\mathcal{R}\\ \vdots\\ 0\end{matrix} & \boxed{\quad M_{m}^{\pm} \quad} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ -\tilde{\mathcal{S}} & 0 \ \dots \ 0 \ -\mathcal{R} & 1 \\ 0 & 0 \quad \dots \quad 0 & -\mathcal{R} \end{array}\right|.$$

Using the Laplace expansion along the last row we obtain

$$D_{m+4} = (1-\tilde{\mathcal{S}}^2)D_{m+2} + (-1)^m \mathcal{R}^2\tilde{\mathcal{S}} \left|\begin{array}{cc} \begin{matrix}-\mathcal{R}\\ \vdots\\ 0\end{matrix} & \boxed{\quad M_{m}^{\pm} \quad} \\ -\tilde{\mathcal{S}} & 0 \ \dots \ 0 \ -\mathcal{R} \end{array}\right|.$$

Now using the Laplace expansion along the first column we arrive at

$$D_{m+4} = (1-\tilde{\mathcal{S}}^2)D_{m+2} - \mathcal{R}^3(-1)^m \tilde{\mathcal{S}} \left|\begin{array}{ccc} \begin{matrix}-\mathcal{R}\\ \vdots\\ 0\end{matrix} & \boxed{\quad M_{m-2}^{\pm} \quad} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ -\tilde{\mathcal{S}} & 0 \ \dots \ 0 \ -\mathcal{R} & 1 \\ 0 & 0 \quad \dots \quad 0 & -\mathcal{R} \end{array}\right| - \mathcal{R}^2\tilde{\mathcal{S}}^2 D_m.$$

Substituting from (5.8), we can rewrite this relation as the linear difference equation of the fourth order

$$D_{m+4} - (1+\mathcal{R}^2-\tilde{\mathcal{S}}^2)D_{m+2} + \mathcal{R}^2 D_m = 0 \tag{5.9}$$

subject to initial conditions

$$\begin{aligned} D_1 &= 1-\tilde{\mathcal{S}}, \\ D_2 &= 1-\tilde{\mathcal{S}}\mathcal{R}-\tilde{\mathcal{S}}^2, \\ D_3 &= 1-\tilde{\mathcal{S}}-\mathcal{R}^2\tilde{\mathcal{S}}-\tilde{\mathcal{S}}^2+\tilde{\mathcal{S}}^3, \\ D_4 &= 1-\mathcal{R}^3\tilde{\mathcal{S}}-\mathcal{R}^2\tilde{\mathcal{S}}^2+\mathcal{R}\tilde{\mathcal{S}}^3-\mathcal{R}\tilde{\mathcal{S}}+\tilde{\mathcal{S}}^4-2\tilde{\mathcal{S}}^2. \end{aligned} \tag{5.10}$$

Let us emphasize that the difference equation (5.9) is the same for both cases $\tilde{\mathcal{S}} = -\mathcal{S}$ and $\tilde{\mathcal{S}} = \mathcal{S}$, but the sign of $\tilde{\mathcal{S}}$ influences the initial conditions.

In the sequel we find the general solution of the difference equation (5.9). The characteristic polynomial of (5.9) is

$$\zeta^4 - (1 + \mathcal{R}^2 - \tilde{\mathcal{S}}^2)\zeta^2 + \mathcal{R}^2 \tag{5.11}$$

and has the roots in the form

$$\zeta_{1,2}^2 = \frac{1}{2}\left(1 + \mathcal{R}^2 - \tilde{\mathcal{S}}^2 \pm \sqrt{(1 + \mathcal{R}^2 - \tilde{\mathcal{S}}^2)^2 - 4\mathcal{R}^2}\right),$$

where $(1 + \mathcal{R}^2 - \tilde{\mathcal{S}}^2)^2 - 4\mathcal{R}^2 < 0$. Indeed, since

$$a^2 - b^2 < 0 \quad \text{and} \quad 4 - 4ah + (a^2 - b^2)h^2 > 0 \quad \text{for} \quad 0 < h < \frac{2}{a + |b|},$$

we have

$$\frac{h^2(a^2 - b^2)}{(1 - ah)^2} \cdot \frac{(4 - 4ah + (a^2 - b^2)h^2)}{(1 - ah)^2} < 0,$$

i.e.

$$(1 + \mathcal{R}^2 - \tilde{\mathcal{S}}^2 - 2\mathcal{R}) \cdot (1 + \mathcal{R}^2 - \tilde{\mathcal{S}}^2 + 2\mathcal{R}) < 0.$$

Using the notation

$$A = \frac{1}{2}(1 + \mathcal{R}^2 - \tilde{\mathcal{S}}^2), \qquad B = \frac{1}{2}\sqrt{4\mathcal{R}^2 - (1 + \mathcal{R}^2 - \tilde{\mathcal{S}}^2)^2},$$

the roots of (5.11) can be expressed as

$$\zeta_{1,2,3,4} = (A \pm Bi)^{1/2} = \left[\sqrt{A^2 + B^2}\left(\frac{A}{\sqrt{A^2 + B^2}} \pm i\frac{B}{\sqrt{A^2 + B^2}}\right)\right]^{1/2}$$

which implies

$$\begin{aligned} \zeta_{1,2} &= (A^2 + B^2)^{1/4}(\cos(\varphi/2) \pm i\sin(\varphi/2)), \\ \zeta_{3,4} &= (A^2 + B^2)^{1/4}(\cos(\varphi/2 + \pi) \pm i\sin(\varphi/2 + \pi)), \end{aligned}$$

where φ is given by

$$\varphi = \arcsin\frac{B}{\sqrt{A^2 + B^2}}.$$

To summarize this, the solution of (5.9) can be written in the form

$$D_m = (A^2 + B^2)^{m/4}\left[(C_1 + (-1)^m C_3)\cos(m\varphi/2) + (C_2 + (-1)^m C_4)\sin(m\varphi/2)\right],$$

where $C_1, \ldots, C_4$ are general constants. In the sequel we specify a certain relation among them. We emphasize that the next calculations are analogous for both cases $\tilde{\mathcal{S}} = \pm\mathcal{S}$. Utilizing initial conditions (5.10) we arrive at

$$\begin{aligned} D_2 &= (C_1 + C_3)A + (C_2 + C_4)B, \\ D_4 &= (C_1 + C_3)(A^2 - B^2) + (C_2 + C_4)2AB, \end{aligned}$$

hence

$$\begin{aligned} C_1 + C_3 &= \tfrac{2AD_2 - D_4}{A^2+B^2} = 1, \\ C_2 + C_4 &= \tfrac{D_2 - A}{B} = \tfrac{1-\tilde{\mathcal{S}}^2 - \mathcal{R}^2 - 2\mathcal{R}\tilde{\mathcal{S}}}{\sqrt{4\mathcal{R}^2-(1+\mathcal{R}^2-\tilde{\mathcal{S}}^2)^2}}. \end{aligned}$$

Analogously we can write

$$\begin{aligned} D_1 &= (C_1 - C_3)\tfrac{1}{\sqrt{2}}\sqrt{\mathcal{R}+A} + (C_2 - C_4)\tfrac{1}{\sqrt{2}}\sqrt{\mathcal{R}-A}, \\ D_3 &= (C_1 - C_3)\tfrac{1}{\sqrt{2}}\sqrt{\mathcal{R}+A}(2A-\mathcal{R}) + (C_2 - C_4)\tfrac{1}{\sqrt{2}}\sqrt{\mathcal{R}-A}(2A+\mathcal{R}), \end{aligned}$$

hence

$$\begin{aligned} C_1 - C_3 &= \tfrac{D_1\sqrt{2}(2A+\mathcal{R}) - D_3\sqrt{2}}{2\mathcal{R}\sqrt{\mathcal{R}+A}} = \tfrac{1+\mathcal{R}-\tilde{\mathcal{S}}}{\sqrt{2}\sqrt{\mathcal{R}+(1+\mathcal{R}^2-\tilde{\mathcal{S}}^2)/2}}, \\ C_2 - C_4 &= \tfrac{D_1\sqrt{2}(2A-\mathcal{R}) - D_3\sqrt{2}}{-2\mathcal{R}\sqrt{\mathcal{R}-A}} = \tfrac{1-\mathcal{R}-\tilde{\mathcal{S}}}{\sqrt{2}\sqrt{\mathcal{R}-(1+\mathcal{R}^2-\tilde{\mathcal{S}}^2)/2}}. \end{aligned}$$

Now we can observe that

$$\frac{C_1 + C_3}{C_2 + C_4} = \frac{\sqrt{4\mathcal{R}^2 - (1+\mathcal{R}^2-\tilde{\mathcal{S}}^2)^2}}{1-\tilde{\mathcal{S}}^2-\mathcal{R}^2-2\mathcal{R}\tilde{\mathcal{S}}},$$

i.e.

$$\frac{C_1 - C_3}{C_2 - C_4} = \frac{C_1 + C_3}{C_2 + C_4}. \tag{5.12}$$

Using the property (5.12) we are going to analyse the sign of D_m. It follows from (5.10) that the condition $h < 1/(a + |b|)$ implies $D_1 > 0$. To find whether

$$D_{m^\star} D_{m^\star+1} \le 0 \tag{5.13}$$

for a suitable $m^\star \in \mathbb{Z}^+$ we note that by previous calculations, the condition (5.13) is equivalent to

$$\tilde{D}_{m^\star} \tilde{D}_{m^\star+1} \le 0,$$

where

$$\tilde{D}_{m^\star} = \frac{C_1 + C_3}{C_2 + C_4}\cos(m^\star\varphi/2) + \sin(m^\star\varphi/2).$$

Considering $\tilde{D}_{m^\star}$ as a function $\tilde{D} = \tilde{D}(u)$ of a continuous argument u (instead of index $m^\star$), we need to solve the equation $\tilde{D}(u) = 0$, i.e.

$$-\frac{C_1 + C_3}{C_2 + C_4} = \tan(u\varphi/2). \tag{5.14}$$

One can easily verify that the left-hand side of the previous equation is negative and positive for $\tilde{\mathcal{S}} = \mathcal{S} < 0$ and $\tilde{\mathcal{S}} = -\mathcal{S} > 0$, respectively. Then the smallest positive root of (5.14) is given by

$$u = \begin{cases} \frac{2}{\varphi}\left[\pi + \arctan\left(-\frac{C_1+C_3}{C_2+C_4}\right)\right] & \text{for } \tilde{\mathcal{S}} = \mathcal{S} < 0, \\ \frac{2}{\varphi}\left[\arctan\left(-\frac{C_1+C_3}{C_2+C_4}\right)\right] & \text{for } \tilde{\mathcal{S}} = -\mathcal{S} > 0. \end{cases} \tag{5.15}$$

We recall that the condition (iii) has to be fulfilled for $\tilde{\mathcal{S}} = \mathcal{S}$ and $\tilde{\mathcal{S}} = -\mathcal{S}$ simultaneously, hence we put

$$m_0 := 2\arctan\left(-\frac{\sqrt{4\mathcal{R}^2 - (1+\mathcal{R}^2-\mathcal{S}^2)^2}}{1-\mathcal{S}^2-\mathcal{R}^2+2\mathcal{R}\mathcal{S}}\right) \Big/ \arcsin\frac{B}{\sqrt{A^2+B^2}},$$

i.e.

$$m_0 = 2\arctan\left(-\frac{\sqrt{4\mathcal{R}^2-(1+\mathcal{R}^2-\mathcal{S}^2)^2}}{1-\mathcal{S}^2-\mathcal{R}^2+2\mathcal{R}\mathcal{S}}\right)\Big/\arcsin\frac{\sqrt{4\mathcal{R}^2-(1+\mathcal{R}^2-\mathcal{S}^2)^2}}{2\mathcal{R}}.$$

Now we can express the discussed critical order $m^\star$ as

$$m^\star := \begin{cases} \lfloor m_0 \rfloor, & m_0 \notin \mathbb{Z}^+, \\ m_0 - 1, & m_0 \in \mathbb{Z}^+. \end{cases}$$

To summarize all previous calculations we can observe that considering any positive integer $m \le m^\star$, the polynomial (5.5) has all its roots in the unit disk, hence the difference equation (5.4) is asymptotically stable. On the contrary, for any $m > m^\star$ the polynomial (5.5) does not have this property. Indeed, it is obvious from (5.15) that $D_{m^\star+1} \le 0$ and $D_{m^\star+2} < 0$ provided $\tilde{\mathcal{S}} = -\mathcal{S}$. Since either $M^{\pm}_{m^\star+1}$ or $M^{\pm}_{m^\star+2}$ always appears as an inner in every $M^{\pm}_m$, $m > m^\star + 2$, then the property (iii) of the Theorem 5.1 is not fulfilled for any $m > m^\star$.

The previous analysis enables us to formulate the next result:

Theorem 5.2. *Let $|a| + b < 0$, $h < 1/(a + |b|)$ and let the values $\mathcal{R}, \mathcal{S}$ be given by (5.3). Then all the roots of the polynomial (5.5) lie inside the unit disk if and only if*

$$m \le m^\star := \begin{cases} \lfloor m_0 \rfloor, & m_0 \notin \mathbb{Z}^+, \\ m_0 - 1, & m_0 \in \mathbb{Z}^+, \end{cases}$$

where

$$m_0 = 2\arctan\left(-\frac{\sqrt{4\mathcal{R}^2-(1+\mathcal{R}^2-\mathcal{S}^2)^2}}{1-\mathcal{S}^2-\mathcal{R}^2+2\mathcal{R}\mathcal{S}}\right)\Big/\arcsin\frac{\sqrt{4\mathcal{R}^2-(1+\mathcal{R}^2-\mathcal{S}^2)^2}}{2\mathcal{R}}.$$

Furthermore,

$$\lim_{h\to 0} m^* h = \frac{2}{(b^2-a^2)^{1/2}}\arctan\frac{(b^2-a^2)^{1/2}}{a-b}\,.$$

Proof: The proof follows from our previous analysis. The relation $\lim_{h\to 0} m^* h$ follows from the L'Hôpital's rule. □

Hence, under the assumptions introduced in the Theorem 5.2, the solution of (5.2) has a tendency to reach the zero solution for $n \le n^\star = \left\lfloor \frac{m^\star+\lambda}{1-\lambda} \right\rfloor$. For $n > n^\star$ this tendency vanishes.

We can summarize that considering the numerical methods of the Euler type, our technique for the determination of $m^\star$ leads to the investigation of the asymptotic stability of the three-term difference equation (5.4). The stability analysis of (5.4) leads to another auxiliary difference equation (5.9) for the determinants D_m occurring in the assumptions of the Schur-Cohn criterion.

Now we present the example illustrating the contribution of the Theorem 5.2.

Example 5.3. We consider the initial value problem

$$y'(t) = -0.1y(t) - y(0.99t), \qquad t \ge 0, \qquad y(0) = 1 \tag{5.16}$$

and its implicit Euler discretization

$$y_{n+1} = \frac{100}{101} y_n - \frac{10}{101} y_{\lfloor 0.99(n+1) \rfloor}, \qquad n = 0, 1, 2, \ldots, \qquad y_0 = 1 \tag{5.17}$$

with the stepsize $h = 0.1$. Using (4.29), the Fig. 5.1 illustrates the behaviour of the solution of (5.17). For a better representation of the character of this solution we present the Fig. 5.2. This figure plots the values (t,$\log_{10}(|y^h(t)| + \epsilon)$), where $\varepsilon = 2.23 \times 10^{-308}$. It follows from the Fig. 5.2 that for $nh \in (100, 300)$ the values of $y^h(t)$ are already less then 10^{-40}. Considering such small values the solution of the problem (5.17) approximates the zero solution and it seems that the calculation could be finished. However, if $nh > 300$ then the solution increases quickly (in absolute values).

Using the Theorem 5.2 we are able to find the change point

$$n^* h = \left\lfloor \frac{m^* + \lambda}{1 - \lambda} \right\rfloor h = 1699h = 169.9,$$

where the character of this solution changes. For a better illustration the solution y_n of (5.17) close to this point is displayed on the Fig. 5.3. Moreover, we can find the point

$$t^* = \lim_{h \to 0} \left\lfloor \frac{m^* + \lambda}{1 - \lambda} \right\rfloor h = 167.9,$$

where the character of the exact solution of the problem (5.16) changes. It remains to note that this value equals to the value presented in [34], where the equation (5.16) was studied by use of the explicit Euler discretization.

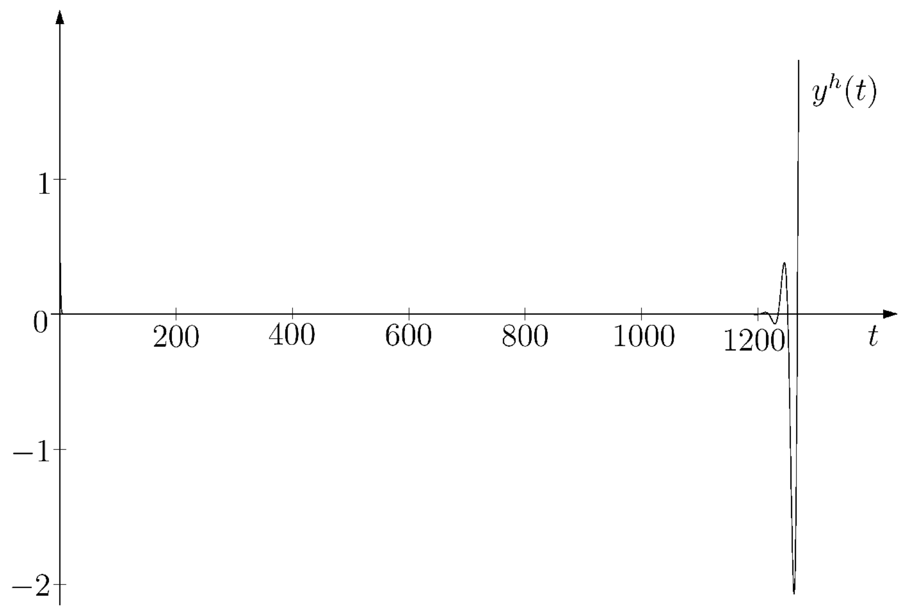

Fig. 5.1: The solution $y^h(t)$.

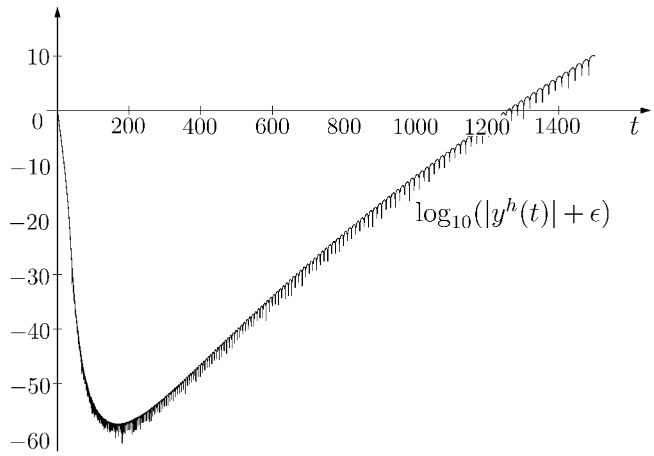

Fig. 5.2: The solution $\log_{10}(|y^h(t)| + \epsilon)$.

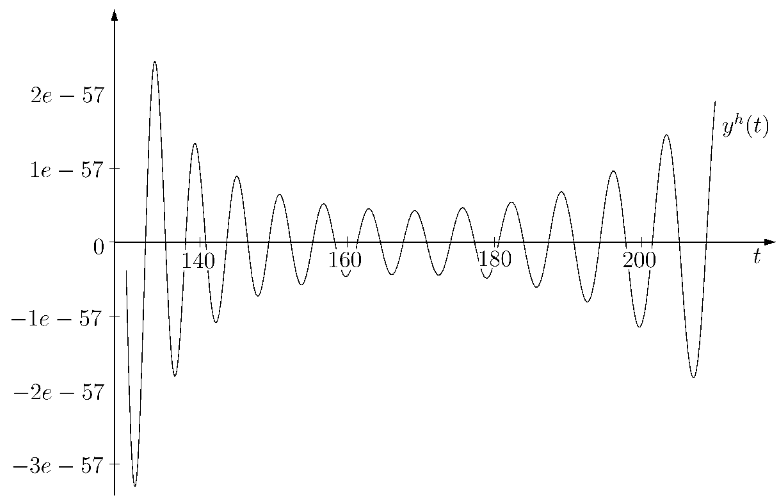

Fig. 5.3: The solution $y^h(t)$.

6. Some comparisons and examples

In this section, we mention several comparisons and numerical consequences concerning the asymptotic estimates of the exact pantograph equation and its Θ-method discretization. We consider the scalar pantograph equation

$$y'(t) = ay(t) + by(\lambda t), \quad t \geq 0\,, \tag{6.1}$$

where $0 < \lambda < 1$ and assume that $\mathrm{Re}\, a < 0, b \neq 0$. We note that this equation is a particular case of each of the equations (3.1), (4.2), (4.11) and (4.30) considered in previous sections.

In Subsection 6.1 we mention some basic numerical notions and characteristics associated with the stability of the equation (6.1). The aim of the following subsections is to illustrate the contribution of the asymptotic results mentioned in previous sections to the numerical investigation of the equation (6.1) and present various comparisons with the known results.

6.1. Some numerical preliminaries

It is well known (see [15]) that the necessary and sufficient asymptotic stability condition for (6.1) is

$$\mathrm{Re}\, a < 0, \qquad |b| < |a|.$$

This implies the analytical asymptotic stability region for the equation (6.1) in the form

$$S := \{(a, b) \in \mathbb{C}^2 : \mathrm{Re}\, a < 0, |b| + a < 0\}.$$

The discrete analogy of the asymptotic stability region S for the exact equation (6.1) is the corresponding numerical stability region. Considering the Θ-method

$$y_{n+1} = Ry_n + S\left(\beta_n y_{\lfloor \lambda n \rfloor} + \alpha_n y_{\lfloor \lambda n \rfloor + 1} + \widehat{\beta}_n y_{\lfloor \lambda (n+1) \rfloor} + \widehat{\alpha}_n y_{\lfloor \lambda (n+1) \rfloor + 1}\right), \; n = 0, 1, \ldots \tag{6.2}$$

where R, S are given by

$$R := \frac{1 + (1 - \Theta)ah}{1 - \Theta ah}, \qquad S := \frac{bh}{1 - \Theta ah} \tag{6.3}$$

and

$$\begin{aligned} \alpha_n &:= (1 - \Theta)(\lambda n - \lfloor \lambda n \rfloor), & \beta_n &:= 1 - \Theta - \alpha_n, \\ \widehat{\alpha}_n &:= \Theta(\lambda(n+1) - \lfloor \lambda(n+1) \rfloor), & \widehat{\beta}_n &:= \Theta - \widehat{\alpha}_n\,, \end{aligned} \tag{6.4}$$

we have the following definition.

Theorem 6.1. *The numerical stability region for the Θ-method (6.2) is defined as the set S_Θ of all couples of complex numbers (a, b) such that any solution y_n of (6.2) is tending to zero as $n \to \infty$ whenever $0 < \lambda < 1$. We say that the Θ-method (6.2) is asymptotically stable if*

$$S \subset S_\theta$$

for any $h \in \mathbb{R}^+$.

The asymptotic stability of the recurrence (6.2) is the subject matter of many papers. We mention at least the following results which are closely related to our investigations. If $\Theta = 1$ (the case of the implicit Euler method), then the method (6.2) is asymptotically stable (see [23, p. 266]). Further, if a, b are real scalars such that

$$a < 0, \qquad |b| < -a, \qquad \frac{1}{2} - \frac{1}{2}\frac{|b|}{a} \leq \Theta \leq 1, \tag{6.5}$$

then the solution y_n of the Θ-method (6.2) is tending to zero for all $\lambda \in (0, 1)$ (see [35]). A stronger result is proved in [15, 24] provided $\lambda = 1/L$, where $L \geq 2$ is an integer. It has been shown that the third condition in (6.5) can be weakened as

$$\frac{1}{2} \leq \Theta \leq 1.$$

In the other words, assuming $\lambda = 1/L$, the Θ-method (6.2) is asymptotically stable if and only if $1/2 \leq \Theta \leq 1$.

6.2. The asymptotic estimate for the exact and discretized pantograph equation.

The important theoretical question about numerical approximations is the problem whether the numerical and exact solution admit a related asymptotic behaviour on the unbounded domain. Recall that the qualitative behaviour of the solutions of the exact equation (6.1) is well known (see, e.g. [16, 25, 26]) and can be described as follows:

Theorem 6.2. *Let y be a solution of the equation (6.1), where* Re $a < 0$, $b \neq 0$ *and* $0 < \lambda < 1$. *Then*

$$y(t) = O\left(t^{-\log_\lambda |b/a|}\right) \qquad \text{as } t \to \infty\,. \tag{6.6}$$

Moreover, if $y(t) = o(t^{-\log_\lambda |b/a|})$ as $t \to \infty$, then y is the zero solution.

In other words, the estimate (6.6) is nonimprovable.

Now we are going to formulate the corresponding discrete estimate following from the Theorem 3.2 (as well as from the Theorem 4.4, the Theorem 4.8 or the Theorem 4.11).

Corollary 6.3. *Let y_n be a solution of the discretization (6.2) with R, S given by (6.3), where*

$$2\mathrm{Re}\; a < (2\Theta - 1)|a|^2 h, \tag{6.7}$$

$b \neq 0$ *and* $0 < \lambda < 1$. *Then*

$$y_n = O\left(n^{-\log_\lambda \tilde{\gamma}}\right) \qquad \text{as } n \to \infty, \qquad \tilde{\gamma} = \frac{|b|h}{|1 - \Theta a h| - |1 + (1 - \Theta) a h|}\,. \tag{6.8}$$

Remark 6.4. The condition (6.7) seems to be analogical with the condition Re $a < 0$ in the Theorem 6.2. Let Re $a < 0$. Then (6.7) is fulfilled for any $h > 0$ if and only if

$1/2 \le \Theta \le 1$. Assuming $0 \le \Theta < 1/2$, the condition (6.7) represents the restriction on the stepsize h and has the form

$$h < \frac{2\mathrm{Re}\, a}{(2\Theta - 1)|a|^2}.$$

Comparing (6.6) and (6.8), the natural question arises, namely what is the relation between the upper bound (6.6) derived in [16, 25, 26] for the exact solution of (6.1) and our upper bound (6.8) derived for its numerical solution.

Answering our question we first consider the case where a is a real constant (b can be complex). Then we can observe that $\tilde{\gamma}$ occurring in (6.8) becomes

$$\tilde{\gamma} = \begin{cases} |b/a| & \text{for } (1-\Theta)h|a| \le 1, \\ h|b|/(2 + h|a|(2\Theta - 1)) & \text{for } (1-\Theta)h|a| > 1. \end{cases}$$

Hence the value $|b/a|$ known from the asymptotic description of the exact scalar pantograph holds for discretization (6.2) with the modest restriction on the stepsize h. In other words both the exact solution and the numerical solution have exactly the same decay rate.

We note that in the case $\Theta = 1$ (the implicit Euler method) we get the equality $\tilde{\gamma} = |b/a|$ without any restriction to the stepsize h. We emphasize that this result is a significant strengthening of the asymptotic stability property of this method mentioned in the previous subsection. In particular, assuming $|b| < |a|$, we can guarantee that the discretization (6.2) with $\Theta = 1$ preserves not only the convergency to zero, but also the same decay rate of this convergency regardless of the stepsize h. In the remaining cases $0 \le \Theta < 1$ the condition $(1-\Theta)h|a| \le 1$ (ensuring the same decay rate of the exact and numerical solution) means the stepsize restriction.

We can also discuss some stability consequences following from the previous considerations. It follows from (6.8) that all complex couples (a, b), satisfying the condition

$$|b|h < |1 - \mathit{\Theta} ah| - |1 + (1 - \mathit{\Theta})ah| \tag{6.9}$$

belong to the stability region S_Θ. Let $a \in \mathbb{R}$, $b \in \mathbb{C}$ and $|b| < -a$ (we emphasize that this is just the asymptotic stability condition for the exact equation (6.1) with a real parameter a). If $(1-\Theta)h|a| \le 1$ holds, then (6.9) is satisfied trivially. If $(1-\Theta)h|a| \le 1$ does not hold, then (6.9) becomes

$$|b|h < 2 + (1 - 2\mathit{\Theta})ah.$$

To obtain the stability condition independent of the stepsize h, we can rewrite (6.9) into the form

$$\mathit{\Theta} > \frac{1}{2} + \frac{|b|}{2|a|} + \frac{1}{ah}.$$

From here we deduce that if $|b| < -a$ and

$$\frac{1}{2} - \frac{1}{2}\frac{|b|}{a} \le \Theta \le 1,$$

then any solution y_n of (6.2) tends to zero for any stepsize $h > 0$ and any $0 < \lambda < 1$. Note that we obtain the condition (6.5).

Finally, we consider the case where both parameters a, b are complex. If $\operatorname{Im} a \neq 0$, then the previous relation for $\tilde{\gamma}$ is no longer valid and it turns out that $\tilde{\gamma}$ is always greater then $|b/a|$ and $h|b|/(2 + h|a|(2\Theta - 1))$ provided $(1 - \Theta)h|a| \leq 1$ and $(1 - \Theta)h|a| > 1$, respectively.

6.3. Illustrating examples

In the next examples we specify the parameters a, b in (6.1) and calculate the upper bound for its Θ-method discretization (6.2).

Example 6.5. We illustrate the unstable case of the equation (6.1) via the choice $a = -0.5$, $b = -2$ and $\lambda = 1/2$. Consequently, we investigate the initial value problem

$$y'(t) = -0.5y(t) - 2y(t/2), \qquad t \geq 0, \qquad y(0) = 1\,. \tag{6.10}$$

The asymptotic estimate of the solution of (6.10) is given by (6.6) as

$$y(t) = O(t^2) \qquad \text{as } t \to \infty\,.$$

Now we compute the asymptotic estimate of the solution of the Θ-method discretization of (6.10). We choose the stepsize $h = 0.05$ and consider the trapezoidal rule discretization obtained from (6.2) via the choice $\Theta = 1/2$. Then $\tilde{\gamma} = \frac{|S|}{1-|R|} = \left|\frac{b}{a}\right| = 4$ and we can rewrite (6.8) as

$$|y_n| \leq L_1 n^2 \qquad \text{for all } n \text{ large enough}\,, \tag{6.11}$$

where $L_1 > 0$ is a suitable real constant. Following some steps in the proof of the Theorem 3.2 and noticing that for $\tilde{\gamma} > 1$ it holds $L_1 = B_0$, where

$$B_0 = \sup(|y_n(n-3)^{-2}|,\ n \in [\lfloor\frac{\sigma_0}{2}\rfloor, \sigma_0] \cap \mathbb{Z}^+)\,.$$

Now we choose σ_0 representing the starting point for the asymptotic estimation performed in the proof of the Theorem 3.2. By (3.11), it is enough to put $\sigma_0 = 7$. However, to obtain a reasonable computational and especially graphic illustration of (6.11), we suggest the choice of a larger value of σ_0, say $\sigma_0 = 150$. Then $B_0 \approx 0.000055$ and the asymptotic estimate (6.11) becomes

$$|y_n| \leq 0.000055\, n^2 \qquad \text{for all } n \text{ large enough}\,.$$

Now we consider the numerical solution $y^h(t)$ (see (4.29)) of the equation (6.10), where $t \in [7.5, 400]$ and its upper bound $g(t) = 0.000055\, t^2/h^2 \approx 0.022008\, t^2$. The following Fig. 6.1 plots $(t, \log_{10}(|y^h(t)| + \epsilon))$ and $(t, \log_{10} g(t))$, where $\varepsilon = 2.23 \times 10^{-308}$.

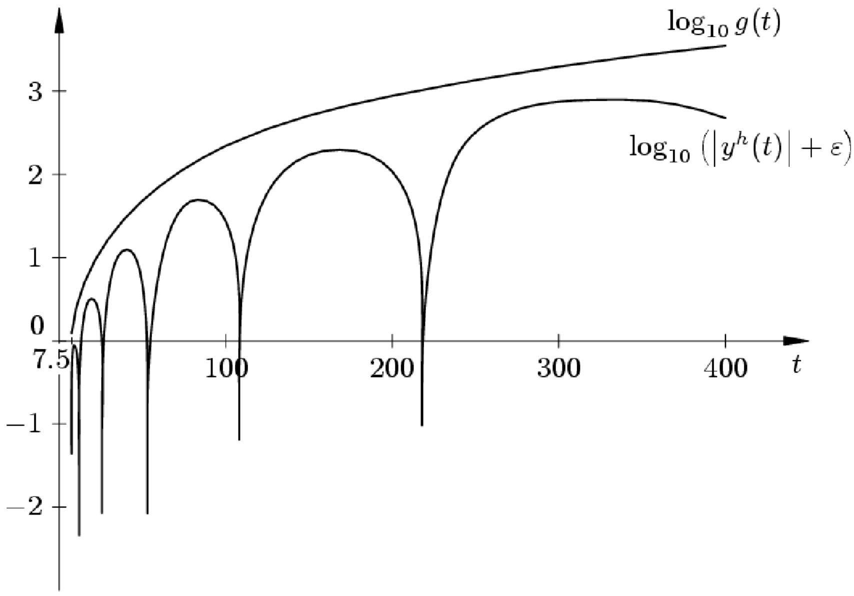

Fig. 6.1: The solution $\log_{10}(|y^h(t)| + \epsilon)$ and its estimate.

Example 6.6. In this example we illustrate the asymptotic stable case. Since we wish to perform this illustration via this example with the known exact solution, we have to consider nonhomogeneous scalar pantograph equation (see the form (4.30)).

We investigate the initial value problem

$$y'(t) = -160y(t) + 80y(t/2) + \frac{159t + 158}{(t+1)^2(t+1)}, \quad t \geq 0, \quad y(0) = 1 \tag{6.12}$$

with the exact solution $y(t) = \frac{1}{t+1}$. The corresponding Θ-method discretization (4.31) yields the sequence y_n which represents the approximate values of $y(t) = \frac{1}{t+1}$ at grid points $t = nh$. Applying the Theorem 4.11, we get the estimate

$$y_n = O(n^{-1}) \qquad \text{as } n \to \infty \tag{6.13}$$

provided $(1-\theta)h|a| \leq 1$. Note that this decay rate corresponds to the decay rate of the exact solution.

Now let $h = 0.05$. Then we can calculate to special values of Θ corresponding to the assumption $|R| < 1$ and $(1-\theta)h|a| \leq 1$. The first condition implies $\Theta > \frac{1}{2} - \frac{1}{h|a|} = 0.375$ and second one implies $\Theta \geq 1 - \frac{1}{h|a|} = 0.875$. Note that assuming $\Theta > 0.875$, the boundedness of the relative errors is guaranteed because of the property (6.13).

The Table 6.6 involves the list of the exact values $y(nh) = \frac{1}{nh+1}$, the numerical values y_n and their relative errors (RE) at some grid points nh. It confirms our theoretical knowledge about the role of the "critical" parameter $\Theta = 0.375$.

nh		50	100	250	500
$y(nh)$		0.1961E-1	0.9901E-2	0.3984E-2	0.1996E-2
$\Theta = 0.374$	y_n	-0.2935E2	-0.1505E4	-0.2426E9	-0.1177E18
	RE	0.1498E4	0.1520E6	0.6088E11	0.5810E20
$\Theta = 0.375$	y_n	-0.7826E-1	-0.8910E-1	-0.8282E-1	-0.9139E-1
	RE	0.4091E2	0.9099E2	0.2089E3	0.4589E3
$\Theta = 0.376$	y_n	-0.2102E-1	-0.1057E-1	0.2567E-3	0.1338E-3
	RE	0.2072E1	0.2067E1	0.9356	0.9330
$\Theta = 0.4$	y_n	0.2751E-1	0.1386E-1	0.5937E-2	0.2973E-2
	RE	0.4031	0.3996	0.4902	0.4893
$\Theta = 0.5$	y_n	0.2811E-1	0.1415E-1	0.5732E-2	0.2870E-2
	RE	0.4335	0.4294	0.4387	0.4378
$\Theta = 0.8$	y_n	0.2817E-1	0.1418E-1	0.5697E-2	0.2853E-2
	RE	0.4367	0.4326	0.4299	0.4290
$\Theta = 1$	y_n	0.2804E-1	0.1412E-1	0.5669E-2	0.2839E-2
	RE	0.4299	0.4258	0.4210	0.4222

Table 6.6

6.4. The comparison with other asymptotic estimates for the Θ-method discretization of (6.1)

The asymptotic investigation of the discretized pantograph equation is rare. To our knowledge, the only paper dealing with the asymptotics of the Θ-method discretization is [33]. However, this paper discusses the Θ-method discretization on the quasigeometric mesh (characterized by the property $\lim_{n\to\infty} h_n = \infty$). Considering the asymptotics of the Θ-method on the uniform mesh, we can mention papers [6] and [29] dealing with the trapezoidal rule and Euler discretization of (6.1), respectively.

To compare the estimate (6.8) with the relevant estimate presented in [6] we need to make some minor modifications. The reason is that the discretization of (6.1) utilized in [6] is slightly different from the formula (6.2). The mentioned discretization is in the general case given by (2.16) and for the equation (6.1) has the form

$$y_{n+1} = Ry_n + S\left(\tilde{\beta}_n y_{\lfloor \lambda n \rfloor} + \tilde{\alpha}_n y_{\lfloor \lambda n \rfloor + 1}\right), \tag{6.14}$$

where R, S are given by (6.3) and

$$\tilde{\beta}_n := 1 - \tilde{\alpha}_n, \qquad \tilde{\alpha}_n := \lambda n - \lfloor \lambda n \rfloor + \Theta\lambda.$$

Since the discretization studied in [6] originates from the formula (6.14), we first reformulate the Corollary 6.3 for such a discretization. To perform this, we denote

$$\eta = \eta(\Theta, \lambda) := \sup_{n\in\mathbb{Z}^+} (|\tilde{\beta}_n| + |\tilde{\alpha}_n|) < \infty.$$

The next lemma yields the explicit form of η and can be found in the particular case $\Theta = 1/2$ in [6, Theorem 6].

Lemma 6.7. *Let $0 < \lambda < 1, 0 \le \Theta \le 1$. Then the function $\eta(\Theta, \lambda)$ has the following values:*

$$\eta(\Theta,\lambda) = \begin{cases} 1, & \lambda = K/L,\ \Theta K \le 1,\ K, L \in \{1, 2, \dots\} \text{ and relatively prime,} \\ 1 + 2\Theta\lambda - \frac{2}{L}, & \lambda = K/L,\ \Theta K \ge 1,\ K, L \in \{2, 3, \dots\} \text{ and relatively prime,} \\ 1 + 2\Theta\lambda, & \lambda \text{ irrational.} \end{cases} \tag{6.15}$$

Proof: First note that $1 \le \eta(\Theta, \lambda) \le 1 + 2\Theta\lambda$. Now assume $\lambda = \frac{K}{L}$, where $1 \le K < L$ and (K, L) are relatively prime. It is known that

$$\frac{nK}{L} - \lfloor \frac{nK}{L} \rfloor = \frac{nK \ mod \ L}{L}.$$

Then

$$\sup_{n\in\mathbb{Z}^+} \alpha_n = \Theta\lambda + \sup_{n\in\mathbb{Z}^+} (\lambda n - \lfloor \lambda n \rfloor) = \Theta\lambda + \frac{L-1}{L} = 1 + \Theta\lambda - \frac{1}{L}.$$

Thus the first two cases of (6.15) are true.

Let λ be an irrational number. The case $\Theta = 0$ is trivial, hence we deal only with $\Theta \ne 0$. In this case, for every $\epsilon > 0$, $\epsilon < \Theta\lambda$ there exists an n_ϵ such that

$$1 - \epsilon < \lambda n_\epsilon - \lfloor \lambda n_\epsilon \rfloor.$$

Furthermore,

$$\alpha_{n_\epsilon} > 1 + \Theta\lambda - \epsilon > 1$$

and we arrive at

$$\eta(\Theta, \lambda) \ge \alpha_{n_\epsilon} + |1 - \alpha_{n_\epsilon}| = 1 + 2\Theta\lambda - 2\epsilon.$$

Now we get $\eta(\Theta, \lambda) \ge 1 + 2\Theta\lambda$, because $\epsilon > 0$ can be made arbitrary small. □

Using this we can reformulate the Corollary 6.3 for the discretization (6.14) as follows:

Corollary 6.8. *Let y_n be a solution of the discretization (6.14) with R, S given by (6.3), where (6.7) holds, $b \ne 0$ and $0 < \lambda < 1$. Then*

$$y_n = O\left(n^{-\log_\lambda \gamma^*}\right) \qquad \text{as } n \to \infty, \qquad \gamma^* = \frac{|S|\eta}{1 - |R|}, \tag{6.16}$$

where $\eta = \eta(\Theta, \lambda)$ is given by (6.15).

Note that the estimate (6.16) can be weaker than the estimate (6.8) because the value of η can be grater then one. It follows from the Lemma 6.7 that if $\Theta K \le 1$ and $\lambda = K/L$ where $K, L \in \mathbb{Z}^+$ are relatively prime, then $\eta(\Theta, \lambda) = 1$. In this case the asymptotic estimates (6.16) and (6.8) coincide.

Now we can discuss the main goal of this subsection, namely the comparison of our estimate (6.16) with the relevant result from [6] describing the asymptotics of (6.14) for $\Theta = 1/2$. On this account, we introduce the notation.

$$\gamma := |R| + \eta|S|,$$

where $\eta = \eta(1/2, \lambda)$ is given by (6.15). Now we can read [6, Theorem 5] as follows:

Corollary 6.9. *Let y_n be a solution of the discretization (6.14), where* $\operatorname{Re} a < 0$, $b \neq 0$ *and* $0 < \lambda < 1$. *Further let* $\gamma \leq 1$. *Then*

$$y_n = O\left(n^{-\log_\lambda \gamma}\right) \qquad \text{as } n \to \infty, \qquad \gamma = |R| + \eta|S|\,. \tag{6.17}$$

Let us emphasize that this result have been derived in a more general case when the equation (6.1) and its discretization (6.14) involve the neutral term. On the other hand, the Corollary 6.9 discusses only the case $\gamma \leq 1$ and $\Theta = 1/2$.

Now we can easily compare our relation (6.16) with the asymptotic estimate (6.17) derived in [6, Theorem 5] under the assumption $\gamma \leq 1$. Considering this assumption we get

$$\gamma^* = \frac{|S|\eta}{1 - |R|} \leq |R| + \eta|S| = \gamma\,,$$

where the equality sign between γ^* and γ occurs if and only if $\gamma = 1$. In particular, substituting the values R and S from (6.3) into the inequality $\gamma^* \leq 1$ we can easily check that the solution of (6.14) is bounded if

$$\operatorname{Re} a < 0, \qquad \eta|b| + \frac{4 \operatorname{Re} a}{|2 + ha| + |2 - ha|} \leq 0\,,$$

which is the same stability condition as the one derived in [6] and [15] by use of the inequality $\gamma \leq 1$. However, considering the asymptotic stable case ($\gamma < 1$), the formula (6.16) provides a stronger asymptotic estimate than the formula (6.17) yields. More precisely, both formulae affirm the algebraic decay of y_n, but the asymptotic property (6.16) guarantees a stronger decay rate.

The next example illustrates the previous comparison. We specify the parameters a, b in (6.1) and discuss the upper bounds for (6.14) with the stepsize $h = 0.05$ with $\Theta = 1/2$.

Example 6.10. We choose $a = -1$ and $b = -0.5$ in (6.1), i.e. we consider the initial value problem

$$y'(t) = -y(t) - 0.5\, y(t/2), \qquad t \geq 0, \qquad y(0) = 1\,. \tag{6.18}$$

Then the corresponding discretization (6.14) becomes

$$y_{n+1} = Ry_n + S\left((3/4 - n/2 + \lfloor n/2 \rfloor)y_{\lfloor n/2 \rfloor} + (n/2 - \lfloor n/2 \rfloor + 1/4)y_{\lfloor n/2 \rfloor + 1}\right), \tag{6.19}$$

$y_0 = 1$, where the symbols R and S have been introduced in (6.3).

Then

$$\gamma = |R| + \eta(\lambda)|S| \approx 0.9756, \qquad \gamma^* = \frac{|S|}{1 - |R|} = \left|\frac{b}{a}\right| = 0.5$$

and the asymptotic estimates (6.17) and (6.16) become

$$y_n = O\left(n^{-0.0356}\right) \qquad \text{as } n \to \infty \tag{6.20}$$

and

$$y_n = O\left(n^{-1}\right) \qquad \text{as } n \to \infty\,, \tag{6.21}$$

respectively. Our next intention is the computational presentation of the estimate (6.21) and its graphic comparisons with the estimate (6.20) as well as with the real behaviour of the discretization (6.19). To make the estimate (6.21) more applicable from the computational viewpoint it is necessary to specify the O-term in (6.21), i.e. determine a constant $L_1 > 0$ such that

$$|y_n| \le L_1 n^{-1} \qquad \text{for all } n \text{ large enough.}$$

It follows from the proof of the Theorem 3.2 (the part (ii) and the relation (3.19) with respect to $K = K_2$) that $L_1 = B_0 \exp\{\frac{K_2}{1-\lambda}\}$, where the constants B_0 and K_2 can be calculated via (3.13) and (3.16) as

$$B_0 = \sup(|y_n(n+2)|,\ n \in [\lfloor \frac{\sigma_0}{2} \rfloor, \sigma_0] \cap \mathbb{Z}^+), \qquad K_2 = \frac{41}{\sigma_0 - 3}\,.$$

To obtain a satisfactory graphic illustration of our estimate we can choose, e.g. the same value of σ_0 as in the Example 6.5, i.e. $\sigma_0 = 150$. Then $K_2 \approx 0.279$ and for the specification of B_0 it remains to determine (or at least estimate) the values of y_n for $n = 1, 2, \ldots, 150$. By [33, Theorem 2], these values are uniformly bounded by $|y(0)| = 1$. However, to obtain a stronger majorant constant L_1, we prefer their direct calculation via (6.19). Then $B_0 \approx 1.9369$, hence $L_1 \approx 3.3834$ and we can precise the upper bound (6.21) for the solution y_n of (6.19) in the form

$$|y_n| \le 3.3834\, n^{-1} \qquad \text{for all } n \text{ large enough}$$

(more precisely, for $n = 150, 151, \ldots$).

Now we consider the estimate (6.20). Since $y_{\sigma_0} = y_{150} \approx 0.012$, we can choose the corresponding majorant constant L_2 specifying the O-term in (6.20) as $L_2 = 0.012 \times 150^{0.0356} \approx 0.0144$ (in other words, to obtain a sharp majorant constant we choose such L_2 that the values of y_n and its estimate $L_2 n^{-0.0356}$ coincide for $n = 150$). This implies

$$|y_n| \le 0.0144\, n^{-0.0356} \qquad \text{for all } n \text{ large enough}$$

(in the sequel we can see that this estimate holds for $n = 150, 151, \ldots$).

Now the gap between both asymptotic results can be simply illustrated by the following figure. We use here (4.29) and consider $t \in [7.5, 400]$ (note that the left-end point $t = 7.5$ corresponds to the starting index $\sigma_0 = 150$ of the asymptotic estimation via the relation $t = 150\,h$). The Fig. 6.2 plots the numerical solution y^h of (6.18) as well as its upper bounds $g(t) = 3.3834\,ht^{-1} \approx 0.1692\,t^{-1}$ and $f(t) = 0.0143\,h^{0.0356}t^{-0.0356} \approx 0.0129\,t^{-0.0356}$.

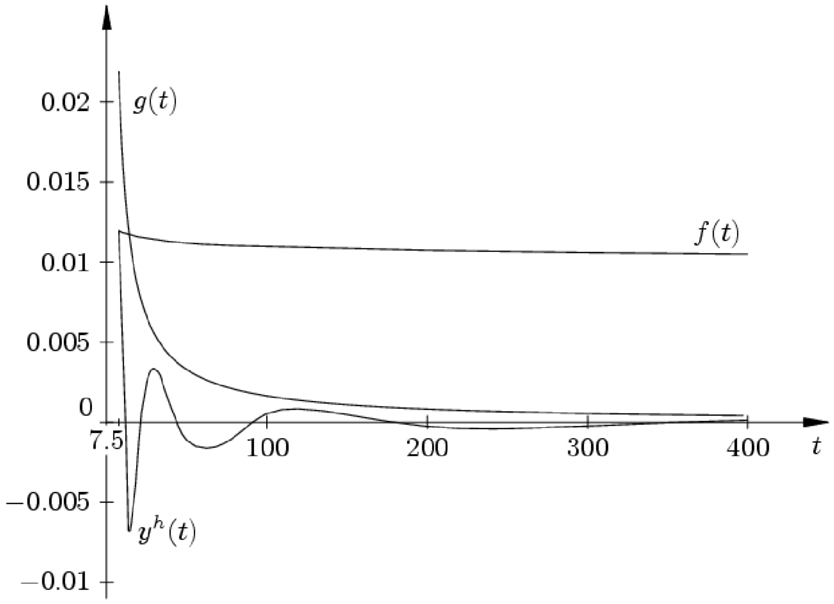

Fig. 6.2: The solution y^h and its upper bounds

6.5. The comparison of results describing the stability behaviour of numerical solution of (6.1)

In this subsection we consider the equation (6.1) and its simplest (Euler) discretization. Our intention is to discuss the relation between the result presented in Section 5 and result stated in [34]. We show that the result presented in Section 5 for the implicit Euler method can be modified also for the explicit Euler discretization and extend the result from [34]. On this account we consider explicit Euler discretization of (6.1) in the form

$$y_{n+1} = \mathcal{R}y_n + \mathcal{S}y_{\lfloor \lambda n \rfloor}, \tag{6.22}$$

where

$$\mathcal{R} := 1 + ah, \quad \mathcal{S} := bh. \tag{6.23}$$

This equation arises from (2.5), where we put $\tau_n = \lambda nh$ and use the piecewise constant interpolation $y^h(\tau_n) = y_{\lfloor \lambda n \rfloor}$. Let

$$n \in \left(\frac{m-1}{1-\lambda}, \frac{m}{1-\lambda} \right] \quad \text{for a suitable } m \in \mathbb{Z}^+.$$

Then we can convert the problem of finding the critical index (see the discussion performed in Section 5) to the analysis of the characteristic polynomial

$$P(\mu) = \mu^{m+1} - \mathcal{R}\mu^m - \mathcal{S}. \tag{6.24}$$

In particular, we focus on the calculation of the maximal value of order m^* for which (6.24) is of a Schur type, and starting from $m = m^* + 1$ the polynomial looses this property. On this account we mention the main result of [34], where the polynomial (6.24) is studied by use of the Kuruklis' result on the asymptotic stability of

$$y_{n+1} - \mathcal{R} y_n - \mathcal{S} y_{n-m} = 0$$

(see [31]).

Theorem 6.11. *Let $|a|+b<0$, $h<1/(a+|b|)$ and let the values $\mathcal{R},\mathcal{S}$ be given by (6.23). Then all the roots of the polynomial (6.24) lie inside the unit disk if and only if*

$$m \le m^\star := \begin{cases} \lfloor m_0 \rfloor, & m_0 \notin \mathbb{Z}^+, \\ m_0 - 1, & m_0 \in \mathbb{Z}^+, \end{cases}$$

where

$$m_0 = \begin{cases} \min\{\frac{1}{h(b^2-a^2)^{1/2}+O(h)} \left(\arctan \frac{(b^2-a^2)^{1/2}}{a} + O(h)\right), \frac{1}{ah}\}, & \text{for } a>0, \\ \frac{\pi}{4\arcsin(|b|h/2)} - \frac{1}{2}, & \text{for } a=0, \\ \frac{1}{h(b^2-a^2)^{1/2}+O(h)} \left(\pi + \arctan \frac{(b^2-a^2)^{1/2}}{a} + O(h)\right), & \text{for } a<0. \end{cases} \tag{6.25}$$

Furthermore,

$$\lim_{h\to 0} m^* h = \begin{cases} \min\{\frac{1}{(b^2-a^2)^{1/2}} \arctan \frac{(b^2-a^2)^{1/2}}{a}, \frac{1}{a}\}, & \text{for } a>0, \\ \frac{\pi}{2|b|}, & \text{for } a=0, \\ \frac{1}{(b^2-a^2)^{1/2}} \left(\pi + \arctan \frac{(b^2-a^2)^{1/2}}{a}\right), & \text{for } a<0. \end{cases}$$

The main utility of this theorem is following: There exists (see [34]) the critical point $t^* = \frac{1}{1-\lambda} \lim_{h\to 0} m^* h$ in the sense that the solution of (6.1) displays a tendency to decrease (in modulus) before t^* and to increase soon after t^*. The Theorem 6.11 makes the computation of t^* effective.

Now we mention our contribution to this discussion which follows from the results mentioned in Section 5. The equation (6.22) can be also analysed by use of the procedure performed in Section 5. Indeed, the characteristic polynomial (6.24) is identical with (5.5), where it is enough to consider $\mathcal{R} = 1 + ah$ and $\mathcal{S} = bh$. Therefore we can reformulate the Theorem 5.2 as follows.

Theorem 6.12. *Let $|a|+b<0$, $h<1/(a+|b|)$ and let the values $\mathcal{R},\mathcal{S}$ be given by (6.23). Then all the roots of the polynomial (6.24) lie inside the unit disk if and only if*

$$m \le m^\star := \begin{cases} \lfloor m_0 \rfloor, & m_0 \notin \mathbb{Z}^+, \\ m_0 - 1, & m_0 \in \mathbb{Z}^+, \end{cases}$$

where

$$m_0 = 2\arctan\left(-\frac{\sqrt{4\mathcal{R}^2 - (1+\mathcal{R}^2-\mathcal{S}^2)^2}}{1-\mathcal{S}^2-\mathcal{R}^2+2\mathcal{R}\mathcal{S}}\right) \Big/ \arcsin \frac{\sqrt{4\mathcal{R}^2 - (1+\mathcal{R}^2-\mathcal{S}^2)^2}}{2\mathcal{R}}.$$

Furthermore,

$$\lim_{h\to 0} m^* h = \frac{2}{(b^2-a^2)^{1/2}} \arctan \frac{(b^2-a^2)^{1/2}}{a-b}.$$

We note that this expression for m^* in the Theorem 6.12 does not depend on the sign of a. Moreover, contrary to the corresponding result in [34] we derive the exact expression for m_0 (in particular, the term $O(h)$ involved in (6.25) is specified). Consequently, we can compute the critical index $n^* = \left\lfloor \frac{m^*}{1-\lambda} \right\rfloor$ exactly.

We emphasize that the procedure utilized in Section 5 is applicable also in a more general situation. In particular, we can consider difference equations arising from (5.1) via more advanced discretizations. E.g. the Θ-method discretization with a piecewise constant interpolation leads to the recurrence in the form

$$y_{n+1} = Ry_n + S(\Theta y_{\lfloor \lambda(n+1) \rfloor} + (1-\Theta) y_{\lfloor \lambda n \rfloor}), \tag{6.26}$$

where R, S are given by (6.3). Of course, then we have to analyse the four-term difference equation (6.26) instead of the previously considered three-term equation (6.22). However, the advantage of our approach consists in the fact that the previous analysis utilizes the Schur-Cohn criterion which can be applied for any linear autonomous difference equation instead of Kuruklis' result [31] for three-term linear equations which is applied in [34]. This extension of our previous results to more general discretizations of (6.1) is the subject of further considerations.

7. Conclusion

The aim of this thesis was to present some qualitative properties of delay difference equations and their applications to the numerical analysis of given DDEs. A special attention was paid to the scalar pantograph equation

$$y'(t) = ay(t) + by(\lambda t), \quad 0 < \lambda < 1, \qquad t \geq 0$$

and its various modifications. We described the qualitative (mostly asymptotic and stability) properties of its Θ-method discretization

$$y_{n+1} = Ry_n + S\left(\beta_n y_{\lfloor \lambda n \rfloor} + \alpha_n y_{\lfloor \lambda n \rfloor + 1} + \widehat{\beta}_n y_{\lfloor \lambda(n+1) \rfloor} + \widehat{\alpha}_n y_{\lfloor \lambda(n+1) \rfloor + 1}\right), \; n = 0, 1, \ldots$$

where R,S are given by (6.3), α_n, β_n and $\widehat{\alpha}_n$, $\widehat{\beta}_n$ by (6.4). We compared these properties with the behaviour of the exact (differential) pantograph equation, which enabled us to formulate some numerical consequences of these qualitative results. Some comparisons with the known relevant results have been done and some illustrating examples have been involved as well. Furthermore, using the Schur-Cohn criterion on the asymptotic stability of the solutions we analysed a family of three-term difference equations and discussed a specific stability phenomenon for the Euler discretization of the pantograph equation.

Finally, we mention some open problems and general remarks. We recall that the problem of the asymptotic stability of the Θ-method discretization of the pantograph equation (in particular, the necessary and sufficient condition of the asymptotic stability in the form $1/2 \leq \Theta \leq 1$) is solved only for those λ which are reciprocal to positive integers. More generally, we pose a conjecture that the asymptotic estimate (6.8) holds without any restriction on the stepsize h provided $1/2 \leq \Theta \leq 1$. For the time being, this conjecture is confirmed for $\Theta = 1$, but numerical calculations and experiments indicate its validity for $1/2 \leq \Theta \leq 1$. Another natural extension of our results concerns the stability analysis performed in Section 5 for the Euler discretization of the pantograph equation. We hope that our proof technique can be extended to the analysis of the Θ-method (6.26).

The common investigation of the properties of the studied differential equations and its difference analogies (obtained via a suitable numerical discretization) was the unified viewpoint of the considerations and results mentioned in this thesis. This is an important aspect of the modern theory of dynamic equations on time scales (see [4] and [5]). Therefore, this theory can motivate us to other extensions of our previous results.

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