PALACKÝ UNIVERSITY OLOMOUC
FACULTY OF SCIENCE
DEPARTMENT OF ALGEBRA AND GEOMETRY

DISSERTATION THESIS

# Some classes of basic algebras and related structures

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Jan Krňávek

I declare that the thesis has been elaborated by myself, under the supervision of Jan Kühr, using only the cited references.

Jan Krňávek

# Contents

# Preface

This thesis is primarily devoted to the so-called 'basic algebras' which were introduced in **[11]** as a by-product of an attempt to find a good common generalization of orthomodular lattices and MV-algebras. The original problem posed in **[11]** was to describe 'MV-like algebras' (i.e. algebras of the same signature as MV-algebras) that would stand to orthomodular lattices as MV-algebras to Boolean algebras, and to MV-algebras as orthomodular lattices to Boolean algebras. This means that the algebras in question should be algebras $(A, \oplus, \neg, 0, 1)$ such that (i) the rule $x \leq y$ iff $\neg x \oplus y = 1$ defines a bounded lattice; (ii) the elements $x \in A$ for which $\neg x$ is a complement in this lattice form a subalgebra which is an orthomodular lattice in its own right; and (iii) the algebra is the set-theoretical union of its blocks, where a block is a maximal subalgebra which is an MV-algebra. These requirements, however, are quite restrictive and lead to lattice effect algebras.

From another point of view, besides being lattice based algebras, both MV-algebras and orthomodular lattices have the following interesting property: all principal filters (as well as all principal ideals, and in fact all intervals) bear certain natural antitone involutions. Indeed, in any MV-algebra $(A, \oplus, \neg, 0, 1)$, for every $a \in A$, the map $\gamma_a \colon x \mapsto \neg x \oplus a$ is an antitone involution on $[a, 1]$, and dually, $\delta_a \colon x \mapsto \neg(x \oplus \neg a)$ is an antitone involution on $[0, a]$. The initial MV-algebra can be easily retrieved from its underlying lattice and the $\gamma_a$'s (or $\delta_a$'s), because $\neg x = \gamma_0(x) = \delta_1(x)$ and $x \oplus y = \gamma_y(\neg x \vee y) = \neg\delta_{\neg y}(x \wedge \neg y)$, for all $x, y \in A$. Likewise, in any orthomodular lattice $(A, \vee, \wedge, ^\perp, 0, 1)$, the map $\gamma_a \colon x \mapsto x^\perp \vee a$ is an antitone involution on $[a, 1]$, and $\delta_a \colon x \mapsto x^\perp \wedge a$ on $[0, a]$. In fact, for an ortholattice, orthomodularity amounts to saying that every $\gamma_a$ is an antitone involution on $[a, 1]$ (or equivalently, that every $\delta_a$ is an antitone involution on $[0, a]$). Now, using the $\gamma_a$'s (or the $\delta_a$'s), we can make $(A, \vee, \wedge, ^\perp, 0, 1)$ into an 'MV-like algebra' $(A, \oplus, \neg, 0, 1)$; namely, it suffices to put $\neg x = x^\perp$ and $x \oplus y = \gamma_y(x^\perp \vee y) = (x \wedge y^\perp) \vee y$.

Therefore, bounded lattices with antitone involutions (on principal filters or ideals) can serve as a common framework for MV-algebras and orthomodular lattices, and basic algebras are algebras $(A, \oplus, \neg, 0, 1)$ corresponding to such lattices in the way indicated above, i.e. $\neg x = \gamma_0(x)$ and $x \oplus y = \gamma_y(\neg x \vee y)$. The class of basic algebras is a variety which may be axiomatized by the identities

$$x \oplus 0 = x,$$
$$\neg\neg x = x,$$
$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x,$$
$$\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = 1.$$

The underlying order is defined by $x \leq y$ iff $\neg x \oplus y = 1$, and the associated lattice operations are given by $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$. Moreover, for every $a$, the map $\gamma_a \colon x \mapsto \neg x \oplus a$ is an antitone involution on $[a, 1]$ and $\delta_a \colon x \mapsto \neg(x \oplus \neg a)$ on $[0, a]$. MV-algebras are precisely the associative basic algebras, and orthomodular lattices can be identified with basic algebras satisfying the quasi-identity $x \leq y \Rightarrow y \oplus x = y$. The above mentioned lattice effect algebras are equivalent to basic algebras satisfying the quasi-identity $x \oplus y \leq \neg z \Rightarrow (x \oplus y) \oplus z = x \oplus (y \oplus z)$.

The name 'basic algebra' was used just as a makeshift and does not establish any connection with other 'basic' structures, such as Hájek's BL-algebras. In fact, a BL-algebra can be made into a basic algebra only if it satisfies the law of double negation in which case it is (equivalent to) an MV-algebra. We should mention that—though basic algebras as such were first defined in [**11**]—the idea of associating algebras $(A, \oplus, \neg, 0, 1)$ with bounded lattices with antitone involutions can be traced back to [**9**].

The thesis is based on the papers [**28**], [**27**] and [**29**] and is divided into four chapters. The first part of Chapter 1 is an introduction to basic algebras and lattices with antitone involutions. In the second part, we deal with algebras satisfying the identities

$$x \oplus (\neg x \wedge y) = x \oplus y, \tag{C}$$

$$x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z). \tag{M}$$

The motivation behind these identities is explained in Section 1.3. We generalize most of the result proved by Botur and Halaš for commutative basic algebras, which are in a sense too similar to MV-algebras. In particular, we prove that every finite basic algebra satisfying the identity (C) is an MV-algebra; cf. [**6**].

Chapter 2 is devoted to pre-ideals of basic algebras, i.e. downwards closed subsets which are also closed under $\oplus$. The name 'ideal' is reserved for the 0-classes of congruences. The concept of pre-ideal is quite general and pre-ideals may fail to have some desirable properties—for instance, a pre-ideal need not be a lattice ideal—and hence we focus mainly on basic algebras satisfying (M). We prove that the pre-ideal lattice $Pr(A)$ of such a basic algebra is an algebraic distributive lattice which contains the ideal lattice $Id(A)$ as a complete sublattice. We describe the pseudocomplements and meet-prime elements in $Pr(A)$ and we show that $Pr(A)$ belongs to the class IRN of ideal lattices of the so-called relatively normal lattices (see [**35**]).

In Chapter 3, generalizing the well-known equivalence between MV-algebras and unital lattice-ordered Abelian groups, we try to shed light on the connections between (commutative) basic algebras and lattice-ordered commutative loops. First, we prove that in any lattice-ordered commutative loop $L$, every interval $[0, u]$ can be made into a basic algebra which is monotone but need not be commutative. As in the case of MV-algebras, we denote this interval algebra by $\Gamma(L, u)$, although $x \oplus y$ cannot be defined simply as $(x + y) \wedge u$ (where $+$ is the addition in the loop $L$). On the other hand, we prove that every semilinear commutative basic algebra is isomorphic to $\Gamma(L, u)$ for a suitable lattice-ordered commutative loop $L$ and a positive element $u \in L$, where semilinearity means that the algebra is isomorphic to subdirect product of linearly ordered algebras. We generalize Chang's construction for linearly oredered algebras as well as Mundici's method of good sequences.

We also present a new example of a proper commutative basic algebra which is derived from linearly ordered commutative loops.

In Chapter 4, we deal with derivations on basic algebras, where by a derivation is meant an additive map satisfying $d(x \odot y) = (d(x) \odot y) \oplus (x \odot d(y))$ or, which turns out to be the same, $d(x \odot y) = (x \odot d(y)) \oplus (d(x) \odot y)$. We give a simple complete characterization of derivations; we prove that $d(x) = x \wedge d(1)$ and that $d$ is actually a homomorphism onto the interval algebra $[0, d(1)]$. In some particular cases, such as in MV-algebras or lattice effect algebras, the element $d(1)$ is even central, and hence derivations on such algebras correspond to certain direct product decompositions. We also prove some auxiliary result on sharp and central elements of basic algebras.

CHAPTER 1

# Basic algebras

In the first chapter, after some introductory material on basic algebras and lattices with antitone involutions, we focus on basic algebras satisfying the identities (C) or (M) for which we prove some technical results. Among others, we show that these basic algebras are distributive as lattices and satisfy a natural version of the Riesz decomposition property. The main result of Chapter 1 is Theorem 1.4.4 stating that all finite basic algebras satisfying (C) are MV-algebras.

## 1.1. Introduction

DEFINITION 1.1.1 (cf. [**11, 13**]). A *basic algebra* is an algebra $(A, \oplus, \neg, 0, 1)$ of type $(2, 1, 0, 0)$ satisfying the identities

$$x \oplus 0 = x, \tag{1.1}$$

$$\neg\neg x = x, \tag{1.2}$$

$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x, \tag{1.3}$$

$$\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = 1. \tag{1.4}$$

If the operation $\oplus$ is commutative or associative, then the basic algebra $(A, \oplus, \neg, 0, 1)$ is said to be *commutative* or *associative*, respectively.

The original definition in [**11**] included the redundant identities $x \oplus 1 = 1 = 1 \oplus x$, and the constant 1, which was not in the signature, was defined as $\neg 0$. The above axiomatization is independent, see [**13**]. The name 'basic algebra' was used just because these algebras were in a sense a common base for all other structures considered in [**11**].

In addition to the negation $\neg$ and the addition $\oplus$, it is useful to define multiplication and two subtractions by

$$x \odot y = \neg(\neg x \oplus \neg y), \quad x \ominus y = \neg(y \oplus \neg x), \quad \text{and} \quad x \oslash y = \neg(\neg x \oplus y).$$

Note that $\oslash$ and $\ominus$ coincide only in commutative basic algebras. The identities (1.3) and (1.4) can be rewritten as $(x \oslash y) \oplus y = (y \oslash x) \oplus x$ and $(((x \oplus y) \oslash y) \oplus z) \oslash (x \oplus z) = 0$, respectively. It is also possible to prove (see [**11**]) that basic algebras are term-equivalent to algebras $(A, \ominus, 0, 1)$ of type $(2, 0, 0)$ satisfying the identities

$$x \ominus 0 = x, \quad x \ominus 1 = 0, \quad x \ominus (x \ominus y) = y \ominus (y \ominus x),$$
$$(x \ominus y) \ominus (x \ominus (y \ominus (y \ominus z))) = 0.$$

The one-to-one correspondence between basic algebras and lattices with antitone involutions can be summarized as follows:

PROPOSITION 1.1.2 (cf. [**11**, **15**, **16**]).

(i) *Let* $(A, \oplus, \neg, 0, 1)$ *be a basic algebra. If we define*

$$x \vee y = (x \oslash y) \oplus y \quad \textit{and} \quad x \wedge y = x \ominus (x \ominus y), \tag{1.5}$$

*then* $(A, \vee, \wedge, 0, 1)$ *is a bounded lattice that we call the* underlying lattice *of* $A$ *and denote it by* $\ell(A)$. *The induced ordering of the lattice is given by*

$$x \leq y \quad \textit{iff} \quad \neg x \oplus y = 1 \quad \textit{iff} \quad x \oslash y = 0 \quad \textit{iff} \quad x \ominus y = 0 \quad \textit{iff} \quad x \odot \neg y = 0.$$

*Moreover, for every* $a \in A$, *the maps*

$$\gamma_a \colon x \mapsto \neg x \oplus a \quad \textit{and} \quad \delta_a \colon x \mapsto a \ominus x$$

*are antitone involutions on the principal filter* $[a, 1]$ *and on the principal ideal* $[0, a]$, *respectively. The initial algebra* $(A, \oplus, \neg, 0, 1)$ *is retrieved by* $\neg x = \gamma_0(x) = \delta_1(x)$ *and* $x \oplus y = \gamma_y(\neg x \vee y) = \neg(\delta_{\neg y}(x \wedge \neg y))$; *also* $x \ominus y = \delta_x(x \wedge y)$.

(ii) *Let* $(A, \vee, \wedge, 0, 1)$ *be a bounded lattice and let* $\gamma_a$ *be fixed antitone involutions on the principal filters* $[a, 1]$. *If we define*

$$\neg x = \gamma_0(x) \quad \textit{and} \quad x \oplus y = \gamma_y(\neg x \vee y), \tag{1.6}$$

*then* $(A, \oplus, \neg, 0, 1)$ *is a basic algebra; its induced lattice is just* $(A, \vee, \wedge, 0, 1)$ *and the induced antitone involutions* $x \mapsto \neg x \oplus a$ *on the principal filters* $[a, 1]$ *are just the* $\gamma_a$*'s.*

(iii) *Let* $(A, \vee, \wedge, 0, 1)$ *be a bounded lattice and let* $\delta_a$ *be fixed antitone involutions on the principal ideals* $[0, a]$. *If we define*

$$\neg x = \delta_1(x) \quad \textit{and} \quad x \oplus y = \neg(\delta_{\neg y}(x \wedge \neg y)),$$

*then* $(A, \oplus, \neg, 0, 1)$ *is a basic algebra in which* $x \ominus y = \delta_x(x \wedge y)$. *The induced lattice is just* $(A, \vee, \wedge, 0, 1)$ *and the induced antitone involutions* $x \mapsto a \ominus x$ *of the principal ideals* $[0, a]$ *are just the* $\delta_a$*'s.*

## 1.2. Special classes of basic algebras

**1.2.1. MV-algebras.** Let $(G, \vee, \wedge, +, -, 0)$ be a lattice-ordered Abelian group, $u \in G$ any positive element of $G$ and $A = [0, u]$. As is well-known, $\Gamma(G, u) = (A, \oplus, \neg, 0, u)$, where $\neg x = u - x$ and $x \oplus y = (x + y) \wedge u$, is an MV-algebra. In view of Proposition 1.1.2, $A$ can be made into a basic algebra: one readily sees that $\delta_a \colon x \mapsto a - x$ is an antitone involution on $[0, a]$, for every $a \in A$, hence if we define $\neg x = \delta_u(x) = u - x$ and $x \oplus y = \neg(\delta_{\neg y}(x \wedge \neg y)) = u - [(u - y) - (x \wedge (u - y))] = (x \wedge (u - y)) + y = (x + y) \wedge u$, then $(A, \oplus, \neg, 0, u)$ is a basic algebra, which is the same as the MV-algebra $\Gamma(G, u)$. Consequently, any MV-algebra is a basic algebra. A direct verification of the identities (1.1)–(1.4) is also easy. It is not hard to show that a basic algebra is an MV-algebra iff the addition $\oplus$ is associative (and hence also commutative). On the other hand, commutativity does not entails associativity; an example of a commutative basic algebra is given in Section 3.4.

Thus basic algebras, and especially commutative basic algebras, can be seen as a non-associative generalization of MV-algebras. Other possible non-associative generalizations are NMV-algebras [**14**] and WMV-algebras (also weak basic algebras) [**10**, **24**]. Both are algebras $(A, \oplus, \neg, 0, 1)$ such that the rule $x \leq y$ iff $\neg x \oplus y = 1$ defines a bounded poset in which the element $(x \oslash y) \oplus y = (y \oslash x) \oplus x$ is an upper bound of $\{x, y\}$, but not necessarily the supremum, and where the maps $\gamma_a \colon x \mapsto \neg x \oplus a$ are involutions on the principal filters $[a, 1]$; in case of WMV-algebras, the $\gamma_a$'s are antitone, in case of NMV-algebras, only the negation $\neg = \gamma_0$ is antitone.

**1.2.2. Orthomodular lattices,** see [**25**]. Let $(A, \vee, \wedge, ^\perp, 0, 1)$ be an orthomodular lattice. The orthomodular law $x \leq y \Rightarrow x \vee (x^\perp \wedge y) = y$ entails that for every $a \in A$, $\gamma_a \colon x \mapsto x^\perp \vee a$ is an antitone involution on $[a, 1]$, and $\delta_a \colon x \mapsto x^\perp \wedge a$ is an antitone involution on $[0, a]$. Thus the algebra $(A, \oplus, \neg, 0, 1)$, where $\neg x = x^\perp$ and $x \oplus y = (x \wedge y^\perp) \vee y$, is a basic algebra in which $x \ominus y = x \wedge (x \wedge y)^\perp$ and which satisfies the quasi-identity $x \leq y \Rightarrow y \oplus x = y$. In fact, orthomodular lattices are term-equivalent to basic algebras satisfying this quasi-identity. However, the equivalence between orthomodular lattices and basic algebras that fulfill $x \leq y \Rightarrow y \oplus x = y$ is based on the choice of the above natural antitone involutions $\gamma_a \colon x \mapsto x^\perp \vee a$ (or $\delta_a \colon x \mapsto x^\perp \wedge a$), so it does not mean that the induced lattice of a basic algebra is orthomodular iff the quasi-identity is satisfied; to be more concrete, the induced lattice of the basic algebra associated with an orthomodular lattice is always an orthomodular lattice, regardless of the choice of the antitone involutions in the intervals $[a, 1]$ for $a \neq 0$ (or $[0, a]$ for $a \neq 1$).

**1.2.3. Lattice effect algebras,** see [**19**]. Let $(E, +, 0, 1)$ be a lattice effect algebra (we use $\vee$, $\wedge$ and $'$ to denote the lattice operations and the operation of taking supplements, respectively). Since, for every $e \in E$, the map $\gamma_e \colon x \mapsto x' + e$ is an antitone involution on $[e, 1]$ (this is true in any effect algebra, not necessarily a lattice one), we can make $(E, +, 0, 1)$ into a basic algebra by setting $\neg x = x'$ and $x \oplus y = (x' \vee y)' + y = (x \wedge y') + y$. The basic algebra $(E, \oplus, \neg, 0, 1)$ thus defined satisfies the quasi-identity

$$x \leq \neg y \quad \& \quad x \oplus y \leq \neg z \quad \Rightarrow \quad (x \oplus y) \oplus z = x \oplus (z \oplus y), \tag{1.7}$$

which says that the partial addition $+$, which is the restriction of $\oplus$ to the pairs $(a, b) \in E \times E$ with $a \leq \neg b$, is both commutative and associative. It is worth noticing that the partial subtraction $-$ is the restriction of $\oslash$ as well as of $\ominus$ to the pairs $(a, b) \in E \times E$ with $a \geq b$. Indeed, by definition, $a - b$ is the only element of $E$ such that $(a - b) + b = a$ provided $a \geq b$. Hence if $a \geq b$, then $a = a \vee b = (a \oslash b) \oplus b = (a \oslash b) + b$ as $a \oslash b \leq \neg b$, thus $a \oslash b = a - b$. Moreover, $\neg a + b = b + \neg a$ is defined when $a \geq b$, and so $a \oslash b = \neg(\neg a \oplus b) = \neg(b \oplus \neg a) = a \ominus b$.

Conversely, if a basic algebra $(A, \oplus, \neg, 0, 1)$ satisfies (1.7), then the corresponding lattice effect algebra $(A, +, 0, 1)$ is obtained as follows: $a + b$ is defined iff $a \leq \neg b$ in which case $a + b = a \oplus b$.

Basic algebras satisfying (1.7), sometimes called *effect basic algebras*, are therefore equivalent to lattice effect algebras. Since (1.7) can be rewritten as an identity, effect basic algebras form a variety which contains both the variety of MV-algebras and the variety which is termwise equivalent to orthomodular lattices (but it is not their join in the lattice of varieties of basic algebras).

## 1.3. Basic properties

In the first lemma we record some properties that will be used in doing calculations and that can easily be derived from the correspondence between the operations $\oplus, \neg, \oslash, \ominus, \odot$ and the antitone involutions $\Gamma = \{\gamma_e \mid e \in A\}$ and $\Delta = \{\delta_e \mid e \in A\}$:

Lemma 1.3.1. *Let $(A, \oplus, \neg, 0, 1)$ be a basic algebra. For all $x, y, z \in A$ we have:*

(a) *if $x \leq y$, then $x \oplus z \leq y \oplus z$, $x \oslash z \leq y \oslash z$, $x \odot z \leq y \odot z$ and $z \ominus y \leq z \ominus x$;*
(b) *$z \geq x \oslash y$ iff $z \oplus y \geq x$;*
(c) *$(x \oplus y) \oslash y = x \wedge \neg y$;*
(d) *$(x \wedge y) \oplus z = (x \oplus z) \wedge (y \oplus z)$, $(x \vee y) \oslash z = (x \oslash z) \vee (y \oslash z)$, $(x \vee y) \odot z = (x \odot z) \vee (y \odot z)$ and $z \ominus (x \wedge y) = (z \ominus x) \vee (z \ominus y)$;*
(e) *$x \oslash y = (x \vee y) \oslash y$ and $x \ominus y = x \ominus (x \wedge y)$.*

Lemma 1.3.2. *Let $(A, \oplus, \neg, 0, 1)$ be a basic algebra. For every $a \in A$, the algebra $([0, a], \oplus_a, \neg_a, 0, a)$, where*

$$x \oplus_a y = a \ominus ((a \ominus y) \ominus x) \quad \text{and} \quad \neg_a x = a \ominus x,$$

*is a basic algebra. The operation $\ominus_a$ is the restriction of $\ominus$ to $[0, a]$.*

Proof. We know that for every $e \in A$, the map $\delta_e \colon x \mapsto e \ominus x$ is an antitone involution on the interval $[0, e]$. Hence $([0, a], \vee, \wedge, 0, a)$ is a bounded lattice with the set of antitone involutions $\Delta_a = \{\delta_e \mid e \in [0, a]\}$, and so in the associated basic algebra $[0, a]$ we have $\neg_a x = a \ominus_a x$ and $x \oplus_a y = \neg_a(\neg_a y \ominus_a x)$, where $\ominus_a$ is given by $x \ominus_a y = \delta_x(x \wedge y) = x \ominus y$. □

Lemma 1.3.3. *Let $A$ be a basic algebra and $[a, b]$ an interval in $\ell(A)$. Then $[a, b]$—as a lattice—is isomorphic to $[0, b \oslash a]$ and anti-isomorphic to $[0, b \ominus a]$.*

Proof. We show that $\phi_1 \colon x \mapsto x \oslash a$ is an isomorphism of $[a, b]$ onto $[0, b \oslash a]$, and that $\phi_2 \colon x \mapsto b \ominus x$ is an anti-isomorphism of $[a, b]$ onto $[0, b \ominus a]$.

The definition of $\phi_1$ is correct since $x \leq b$ implies $x \oslash a \leq b \oslash a$. Let $z \in [0, b \oslash a]$. Then $a \leq z \oplus a \leq (b \oslash a) \oplus a = b \vee a = b$ and $\phi_1(z \oplus a) = (z \oplus a) \oslash a = z \wedge \neg a = z$ since $z \leq b \oslash a \leq 1 \oslash a = \neg a$. Thus $\phi_1$ is onto. Let $x, y \in [a, b]$. If $x \leq y$, then $x \oslash a \leq y \oslash a$, which implies $x = x \vee a = (x \oslash a) \oplus a \leq (y \oslash a) \oplus a = y \vee a = y$. Hence $x \leq y$ iff $\phi_1(x) \leq \phi_1(y)$, so that $\phi_1$ is an isomorphism.

Analogously, if $a \leq x$, then $b \ominus x \leq b \ominus a$, so $\phi_2$ is well-defined. For every $z \in [0, b \ominus a]$ we have $z \oplus \neg b \leq (b \ominus a) \oplus \neg b = \neg(a \oplus \neg b) \oplus \neg b = \neg a \vee \neg b = \neg a$, whence $b \ominus z = \neg(z \oplus \neg b) \geq a$. Thus $b \ominus z \in [a, b]$ and $\phi_2(b \ominus z) = b \ominus (b \ominus z) = b \wedge z = z$. Finally, if $x, y \in [a, b]$, then $x \leq y$ implies $b \ominus x \geq b \ominus y$ whence $x = x \wedge b = b \ominus (b \ominus x) \leq b \ominus (b \ominus y) = y \wedge b = y$. Thus $x \leq y$ iff $\phi_2(x) \geq \phi_2(y)$, proving that $\phi_2$ is an anti-isomorphism. □

In all basic algebras, the operations $\oplus$, $\oslash$, $\odot$ and $\ominus$ are monotone (isotone) in one argument (see Lemma 1.3.1 (a)), while monotonicity in the other argument

$$x \leq y \quad \Rightarrow \quad z \oplus x \leq z \oplus y, \quad z \oslash y \leq z \oslash x, \quad z \odot x \leq z \odot y \quad \text{and} \quad x \ominus z \leq y \ominus z \tag{1.8}$$

does not hold in general.

Definition 1.3.4 (cf. [**8**,**16**]). A basic algebra is said to be *monotone* if it satisfies (1.8).

Obviously, every commutative basic algebra is monotone. In view of [**16**, Thm. 6.4], an orthomodular lattice regarded as a basic algebra is monotone iff it is a Boolean algebra, and a lattice effect algebra is monotone iff it is an MV-algebra.

In what follows, we focus on basic algebras which fulfill the identities

$$x \oplus (\neg x \wedge y) = x \oplus y, \tag{C}$$

$$x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z). \tag{M}$$

Since $x \oplus \neg x = 1$, it is readily seen that (C) follows from (M). Also, in the language of the subtractions $\oslash$ and $\ominus$, the two identities translate into

$$x \oslash (x \wedge y) = x \oslash y, \quad x \odot (\neg x \vee y) = x \odot y, \quad (x \vee y) \ominus y = x \ominus y, \tag{C'}$$

$$x \oslash (y \wedge z) = (x \oslash y) \vee (x \oslash z), \quad x \odot (y \vee z) = (x \odot y) \vee (x \odot z), \tag{M'}$$

$$(x \vee y) \ominus z = (x \ominus z) \vee (y \ominus z).$$

The motivation for (C) comes from lattice effect algebras where the identity captures compatibility of elements. To be more precise, we recall that two elements $a, b$ in an effect algebra $(E, +, 0, 1)$ are said to be *compatible* if there exist $a_1, b_1, c \in E$ such that $a = a_1 + c$, $b = b_1 + c$ and $a_1 + b_1 + c$ is defined. If $(E, +, 0, 1)$ is a lattice effect algebra, then $a, b$ are compatible iff $(a \vee b) - b = a - (a \wedge b)$. Hence, using Subsection 1.2.3 and Lemma 1.3.1 (e), if $(E, \oplus, \neg, 0, 1)$ is the associated effect basic algebra, then for all $a, b \in E$, the following are equivalent:

(a) the elements $a, b$ are compatible;
(b) $a \oslash b = a \oslash (a \wedge b)$;
(c) $(a \vee b) \ominus b = a \ominus b$;
(d) $a \oplus (\neg a \wedge b) = a \oplus b$;
(e) $a \odot (\neg a \vee b) = a \odot b$.

Thus, roughly speaking, basic algebras satisfying (C) are algebras with a single 'block' of compatible elements. This means that an effect basic algebra which satisfies (C) must be an MV-algebra since if all the elements in a lattice effect algebra are compatible, then the corresponding basic algebra is an MV-algebra.

We could also say that $a, b$ are compatible iff $a \oslash b = a \ominus b$, but the identity $x \oslash y = x \ominus y$ is equivalent to commutativity of $\oplus$, which is much stronger than (C). Indeed, in view of Lemma 1.3.1 (d), the identity $x \oplus y = y \oplus x$ would imply (M), which implies (C), but a basic algebra satisfying (C) need not satisfy (M) (for instance, unlike (M), the identity (C) is valid in all linearly ordered basic algebras) and there exist many non-commutative basic algebras that satisfy (M). Here, we give two examples:

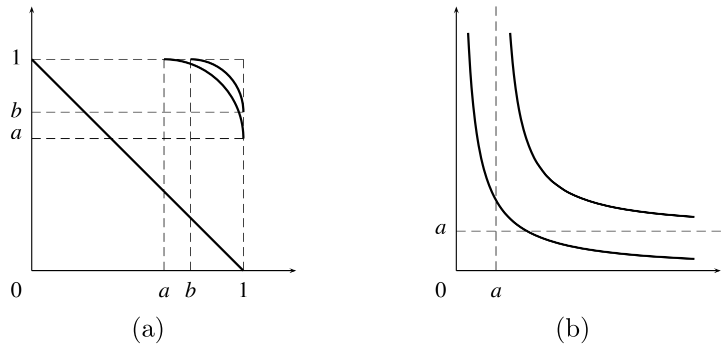


FIGURE 1. Examples of non-commutative basic algebras satisfying (M)

EXAMPLE 1.3.5. Let $L = ([0,1], \vee, \wedge, 0, 1)$ be the real interval $[0,1]$ with the usual linear order and let the intervals $[a,1]$ of $L$ be equipped with the antitone involutions $\gamma_a$ as follows (see Fig. 1 (a)):

$$\gamma_a(x) = \begin{cases} 1 - x & \text{for } a = 0, \\ a + \sqrt{(1-a)^2 - (x-a)^2} & \text{otherwise.} \end{cases}$$

Then the basic algebra $A = ([0,1], \oplus, \neg, 0, 1)$ associated with $L$ and $\Gamma = \{\gamma_a \mid a \in [0,1]\}$ by (1.6) satisfies (M), though it is not commutative.

Since the underlying lattice $\ell(A) = L$ is a chain, in order to show that $A$ satisfies (M), it suffices to verify that $y \leq z$ implies $x \oplus y \leq x \oplus z$. To see this, skipping the trivial case $y = z$, let $y < z$. Then there are three possible cases: If $\neg x \leq y$, then $x \oplus y = \gamma_y(\neg x \vee y) = \gamma_y(y) = 1$ and $x \oplus z = \gamma_z(\neg x \vee z) = \gamma_z(z) = 1$. If $y < \neg x \leq z$, then $x \oplus y = \gamma_y(\neg x \vee y) = \gamma_y(\neg x) < 1$ and $x \oplus z = \gamma_z(\neg x \vee z) = \gamma_z(z) = 1$. If $z < \neg x$, then $x \oplus y = \gamma_y(\neg x \vee y) = \gamma_y(\neg x)$ and $x \oplus z = \gamma_z(\neg x \vee z) = \gamma_z(\neg x)$ where $\gamma_y(\neg x) \leq \gamma_z(\neg x)$ (see Fig. 1 (a)).

Finally, it is not hard to find a pair of reals in $[0,1]$ witnessing non-commutativity of $A$. For instance, $\frac{1}{2} \oplus \frac{1}{4} = \frac{1+2\sqrt{2}}{4}$ and $\frac{1}{4} \oplus \frac{1}{2} = \frac{2+\sqrt{3}}{4}$.

REMARK 1.3.6. Botur **[4]** proved that (up to isomorphism) in all basic algebras on the real interval $[0,1]$ the negation $\neg x$ is given as $1 - x$.

EXAMPLE 1.3.7. Let $L = (\mathbb{R}_0^+ \cup \{\infty\}, \vee, \wedge, 0, \infty)$ where the set $\mathbb{R}_0^+$ of non-negative reals is linearly ordered in the usual way and $\infty$ is a new top element. For every $a \in \mathbb{R}_0^+$, let $\gamma_a$ be defined by $\gamma_a(x) = \frac{1}{x-a} + a$ for $a < x \in \mathbb{R}_0^+$ (see Fig. 1 (b)), $\gamma_a(a) = \infty$ and $\gamma_a(\infty) = a$. For completeness, $\gamma_\infty(\infty) = \infty$. Then the basic algebra $A = (\mathbb{R}_0^+ \cup \{\infty\}, \oplus, \neg, 0, \infty)$ corresponding to $L$ and $\Gamma = \{\gamma_a \mid a \in \mathbb{R}_0^+\} \cup \{\gamma_\infty\}$ fulfills (M), which can be verified by considering the three cases as in the previous example, but $A$ is not commutative because, e.g., $\frac{1}{2} \oplus \frac{1}{4} = \frac{23}{28}$ and $\frac{1}{4} \oplus \frac{1}{2} = \frac{11}{14}$.

LEMMA 1.3.8. *The underlying lattice of a basic algebra satisfying* (C) *is distributive.*

PROOF. We notice that $(x \vee y) \oslash y = x \oslash y = x \oslash (x \wedge y)$ owing to Lemma 1.3.1 (e) and (C'). Hence, assuming $a \vee c = b \vee c$ and $a \wedge c = b \wedge c$, we have $a = (a \oslash (a \wedge c)) \oplus (a \wedge c) = ((a \vee c) \oslash c) \oplus (a \wedge c) = ((b \vee c) \oslash c) \oplus (b \wedge c) = (b \oslash (b \wedge c)) \oplus (b \wedge c) = b$ by (1.5). $\square$

REMARK 1.3.9. The converse of Lemma 1.3.8 fails to be true. For, consider the basic algebra $A = (\{0, a, b, 1\}, \oplus, \neg, 0, 1)$ where $a \oplus a = b \oplus b = 1$, $a \oplus b = b$, $b \oplus a = a$, $\neg a = a$ and $\neg b = b$ (thus $A$ is the horizontal sum of two three-element MV-chains). Then $\ell(A)$ is a Boolean lattice, but $A$ does not satisfy (C): $a \oplus (\neg a \wedge b) = a \oplus 0 = a$, while $a \oplus b = b$.

LEMMA 1.3.10. *Let $A$ be a basic algebra. The lattice $\ell(A)$ is distributive if and only if $A$ satisfies any of the following equivalent identities:*

$$\begin{aligned} x \ominus (y \vee z) &= (x \ominus y) \wedge (x \ominus z), \quad (x \wedge y) \oslash z = (x \oslash z) \wedge (y \oslash z), \\ (x \vee y) \oplus z &= (x \oplus z) \vee (y \oplus z), \quad (x \wedge y) \odot z = (x \odot z) \wedge (y \odot z). \end{aligned} \tag{D}$$

PROOF. It is straightforward to check that the identities are equivalent to one another, so we will work with the first one. If $\ell(A)$ is a distributive lattice, then $x \ominus (y \vee z) = \delta_x(x \wedge (y \vee z)) = \delta_x((x \wedge y) \vee (x \wedge z)) = \delta_x(x \wedge y) \wedge \delta_x(x \wedge z) = (x \ominus y) \wedge (x \ominus z)$. Conversely, if $A$ satisfies the identity, then $x \wedge (y \vee z) = x \ominus (x \ominus (y \vee z)) = x \ominus ((x \ominus y) \wedge (x \ominus z)) = (x \ominus (x \ominus y)) \vee (x \ominus (x \ominus z)) = (x \wedge y) \vee (x \wedge z)$ by Lemma 1.3.1 (d). $\square$

DEFINITION 1.3.11. By an *additive term* we mean a term which is built from variables using $\oplus$ only; we also include 0 as an additive term.

LEMMA 1.3.12. *Every basic algebra $A$ satisfying* (C) *has the following Riesz decomposition property: For all $a, b_1, \dots, b_n \in A$, if $a \leq \tau(b_1, \dots, b_n)$ where $\tau$ is an $n$-ary additive term, then $a = \tau(c_1, \dots, c_n)$ for some $c_1, \dots, c_n \in A$ with $c_i \leq b_i$ $(i = 1, \dots, n)$.*

PROOF. We may assume that $\tau$ is binary, the rest is an easy induction. Let $a \leq b_1 \oplus b_2$. If we put $c_2 = a \wedge b_2$ and $c_1 = a \oslash c_2$, then $c_1 \oplus c_2 = (a \oslash c_2) \oplus c_2 = a \vee c_2 = a$. Trivially, $c_2 \leq b_2$, and by (C') and Lemma 1.3.1 (b) we also have $c_1 = a \oslash (a \wedge b_2) = a \oslash b_2 \leq b_1$. $\square$

LEMMA 1.3.13. *Let $A$ be a basic algebra satisfying* (M)*. Then*

(a) *the addition $\oplus$ is monotone, i.e., given any $n$-ary additive term $\tau$ and $a_1, b_1, \dots, a_n, b_n \in A$, if $a_i \leq b_i$ for all $i = 1, \dots, n$, then $\tau(a_1, \dots, a_n) \leq \tau(b_1, \dots, b_n)$;*
(b) *for every $n$-ary additive term $\tau$ we have $a \wedge \tau(b_1, \dots, b_n) \leq \tau(a \wedge b_1, \dots, a \wedge b_n)$ for all $a, b_1, \dots, b_n \in A$.*

PROOF. We prove the statements for $\tau(x_1, x_2) = x_1 \oplus x_2$; an induction then gives the result for an arbitrary additive term.

(a) By Lemma 1.3.1 (a), $a_1 \leq b_1$ implies $a_1 \oplus a_2 \leq b_1 \oplus a_2$, and by (M), $a_2 \leq b_2$ implies $b_1 \oplus a_2 \leq b_1 \oplus b_2$, thus $a_1 \oplus a_2 \leq b_1 \oplus b_2$.

(b) Using (M) and Lemma 1.3.1 (d), we have $(a \wedge b_1) \oplus (a \wedge b_2) = (a \oplus a) \wedge (a \oplus b_2) \wedge (b_1 \oplus a) \wedge (b_1 \oplus b_2) \geq a \wedge (b_1 \oplus b_2)$ since $(a \oplus a) \wedge (a \oplus b_2) \wedge (b_1 \oplus a) \geq a$. $\square$

We need one more concept:

DEFINITION 1.3.14. A basic algebra which is isomorphic to a subdirect product of linearly ordered basic algebras is called *semilinear* (or *representable*).

Semilinear (commutative) basic algebras form an equational class. This was proved in **[7]** for commutative basic algebras, and in **[16]** for general basic algebras. Here, we recall only the axiomatization in the commutative case.

PROPOSITION 1.3.15 (cf. **[7]**, Thm. 2.9). *A commutative basic algebra is semilinear if and only if it satisfies the identity*

$$[(x \oplus (y \oplus (z \ominus u))) \ominus (x \oplus y)] \wedge (u \ominus z) = 0.$$

We will study semilinear commutative basic algebras in Chapter 3.

## 1.4. Finite basic algebras satisfying (C) are MV-algebras

The aim of this section is clear from the title; as a matter of fact, we prove that every finite basic algebra that satisfies (C) is a direct product of linearly ordered ones, which are necessarily MV-algebras. Indeed, in a finite chain, say $0 = a_0 < a_1 < \cdots < a_n = 1$, there is only one way of defining the antitone involutions $\gamma_{a_i}$ on the intervals $[a_i, 1]$, namely, $\gamma_{a_i}(a_j) = a_{n+i-j}$ for $i \leq j$, this being exactly the way in which the antitone involutions are defined in the standard $(n+1)$-element MV-algebra $C_{n+1} = (\{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}, \oplus, \neg, 0, 1)$ where $\gamma_{\frac{i}{n}}(\frac{j}{n}) = \neg\frac{j}{n} \oplus \frac{i}{n} = \min\{1 - \frac{j}{n} + \frac{i}{n}, 1\} = \frac{n-j+i}{n}$ if $i \leq j$.

Let $(A, \oplus, \neg, 0, 1)$ be a basic algebra. For every $a \in A$ and $n \in \mathbb{N}_0$, the 'multiple' $n \otimes a$ is defined inductively:

$$\begin{aligned} &0 \otimes a = 0, \\ &n \otimes a = a \oplus ((n-1) \otimes a) \quad \text{for } n > 0. \end{aligned}$$

Thus, $n \otimes a = a \oplus (\cdots \oplus (a \oplus (a \oplus a)) \dots)$.

LEMMA 1.4.1. *Let $A$ be a basic algebra satisfying* (C). *For all $a, b \in A$, if $a \wedge b = 0$, then $a \oplus b = a \vee b$, and $(m \otimes a) \wedge (n \otimes b) = 0$ for all $m, n \in \mathbb{N}_0$.*

PROOF. Since $a \wedge b = 0$, we have $a \oslash b = a \oslash (a \wedge b) = a$ by (C'), and so $a \vee b = (a \oslash b) \oplus b = a \oplus b$. Further, it suffices to show that $a \wedge (m \otimes b) = 0$ for all $m \in \mathbb{N}_0$, which is an easy induction. If $m > 0$ and the assertion is true for all $k < m$, then $a \wedge ((m-1) \otimes b) = 0$ entails $a \vee ((m-1) \otimes b) = a \oplus ((m-1) \otimes b)$ whence $a = a \wedge [a \oplus ((m-1) \otimes b)]$ and

$$\begin{aligned} a \wedge (m \otimes b) &= a \wedge [b \oplus ((m-1) \otimes b)] \\ &= a \wedge [a \oplus ((m-1) \otimes b)] \wedge [b \oplus ((m-1) \otimes b)] \\ &= a \wedge [(a \wedge b) \oplus ((m-1) \otimes b)] \\ &= a \wedge ((m-1) \otimes b) \\ &= 0. \end{aligned}$$

□

Now, let $A$ be a *finite* basic algebra that fulfills (C). Let us denote by $\Omega$ the set of the atoms of (the underlying lattice of) $A$. Then for every $a \in \Omega$, the set

$$N(a) = \{n \otimes a \mid n \in \mathbb{N}_0\}$$

is a finite chain $0 < a < \cdots < \hat{a}$ where $\hat{a}$ is the greatest multiple of $a$ such that $n \otimes a > (n-1) \otimes a$. Owing to the Riesz decomposition property, $N(a)$ is the whole interval $[0, \hat{a}]$. Indeed, if $b \leq \hat{a}$ and $\hat{a} = n \otimes a$, then $b = c_1 \oplus (\cdots \oplus (c_{n-1} \oplus c_n) \dots)$ for some $c_i \in A$ with $c_i \leq a$. But $a \in \Omega$, and hence $c_i \in \{0, a\}$, which means that $b = k \otimes a$ for some $k \leq n$.

LEMMA 1.4.2. *Let $A$ be a finite basic algebra satisfying* (C). *With the above notation,*

$$\bigvee_{a \in \Omega} \hat{a} = 1.$$

PROOF. Let $c = \bigvee_{a \in \Omega} \hat{a}$ and suppose that $c < 1$. Then $\neg c > 0$ and there exists an atom $b \in \Omega$ with $b \leq \neg c$. Then $\neg b$ is a coatom and $\neg b \geq c \geq \hat{b}$. By the definition of $\hat{b}$ we have $b \oplus \hat{b} = \hat{b}$, and hence $0 = (b \oplus \hat{b}) \oslash \hat{b} = b \wedge \neg \hat{b} = b$, a contradiction. $\square$

By Lemma 1.3.8, the underlying lattice $\ell(A)$ of a basic algebra $A$ that satisfies (C) is distributive. By Lemma 1.4.1 we have $\hat{a} \wedge \hat{b} = 0$ for all $a, b \in \Omega$ with $a \neq b$, and hence Lemma 1.4.2 entails that the map

$$\eta \colon (x_a \mid a \in \Omega) \mapsto \bigvee_{a \in \Omega} x_a$$

is an isomorphism between the lattices $\prod_{a \in \Omega} [0, \hat{a}]$ and $\ell(A)$. We are going to prove that $\eta$ is an isomorphism between the basic algebras $\prod_{a \in \Omega} [0, \hat{a}]$ and $A$.

We recall from Lemma 1.3.2 that in the basic algebra $([0, \hat{a}], \oplus_{\hat{a}}, \neg_{\hat{a}}, 0, \hat{a})$, the operations are given by $\neg_{\hat{a}} x = \hat{a} \ominus x$, $x \oplus_{\hat{a}} y = \hat{a} \ominus ((\hat{a} \ominus y) \ominus x)$ and $x \ominus_{\hat{a}} y = x \ominus y$. In fact, $[0, \hat{a}]$ is an MV-algebra because the interval $[0, \hat{a}]$ is a finite chain.

LEMMA 1.4.3. *Let $A$ be a finite basic algebra satisfying* (C) *and let $a \in \Omega$. Then $x \oslash y = x \ominus y = x \oslash_{\hat{a}} y$ for all $x, y \in [0, \hat{a}]$.*

PROOF. The equalities obviously hold if $x \leq y$, so assume $x > y$. We can write $x \oslash y$ as $\bigvee_{b \in \Omega} z_b$ where $z_b \in [0, \hat{b}]$. For $b \neq a$ we have $x \wedge z_b = 0 = y \wedge z_b$ (since $y < x \leq \hat{a}$ and $z_b \leq \hat{b}$), which implies $z_b \oplus y = z_b \vee y$ and $x = x \vee y = (x \oslash y) \oplus y = (\bigvee_{b \in \Omega} z_b) \oplus y = \bigvee_{b \in \Omega} (z_b \oplus y)$ by Lemma 1.3.10, thus $x \geq z_b \oplus y = z_b \vee y \geq z_b$ for all $b \in \Omega \setminus \{a\}$. Hence $z_b = 0$ and so $x \oslash y = z_a \in [0, \hat{a}]$.

By Lemma 1.3.3, the intervals $[0, x \oslash y]$ and $[0, x \ominus y]$ are finite chains of the same length. But we have just proved that $x \oslash y \in [0, \hat{a}]$, so that both $[0, x \oslash y]$ and $[0, x \ominus y]$ are subsets of the chain $[0, \hat{a}]$, and hence $x \oslash y = x \ominus y$. Finally, since $[0, \hat{a}]$ is an MV-algebra, we have $x \oslash_{\hat{a}} y = \neg_{\hat{a}} (\neg_{\hat{a}} x \oplus_{\hat{a}} y) = \neg_{\hat{a}} (y \oplus_{\hat{a}} \neg_{\hat{a}} x) = x \ominus_{\hat{a}} y = x \ominus y$. $\square$

Therefore, in order to prove that $\eta$ is an isomorphism between the algebras $\prod_{a \in \Omega} [0, \hat{a}]$ and $A$, it suffices to show that $\eta$ preserves the operation $\oslash$. To this end, let $(x_a \mid a \in \Omega)$,

$(y_a \mid a \in \Omega) \in \prod_{a\in\Omega}[0, \hat{a}]$. For every atom $a \in M$ we have

$$x_a \oslash \bigvee_{b\in\Omega} y_b = x_a \oslash (x_a \wedge \bigvee_{b\in\Omega} y_b) = x_a \oslash \bigvee_{b\in\Omega}(x_a \wedge y_b) = x_a \oslash y_a$$

since $x_a \wedge y_b = 0$ if $b \neq a$, and hence

$$\begin{aligned}\eta(x_a \mid a \in \Omega) \oslash \eta(y_a \mid a \in \Omega) &= \bigvee_{a\in\Omega} x_a \oslash \bigvee_{a\in\Omega} y_a = \bigvee_{a\in\Omega}(x_a \oslash \bigvee_{b\in\Omega} y_b) = \bigvee_{a\in\Omega}(x_a \oslash y_a)\\ &= \eta((x_a \mid a \in \Omega) \oslash (y_a \mid a \in \Omega)).\end{aligned}$$

We have proved the following generalization of [**6**]:

THEOREM 1.4.4. *Let $A$ be a finite basic algebra satisfying* (C). *Then $A$ is isomorphic to the MV-algebra $\prod_{a\in\Omega}[0, \hat{a}]$, hence $A$ is an MV-algebra.*

In [**8**] the same results were proved for basic algebras satisfying the identity $x \leq x \oplus y$. So, every basic algebra satisfying $x \leq x \oplus y$ is distributive and every finite basic algebra satisfying $x \leq x \oplus y$ is an MV-algebra.

CHAPTER 2

# Pre-ideals of basic algebras

The 0-classes of congruences of MV-algebras are characterized as ideals, i.e., non-empty subsets $J$ such that (i) $a \oplus b \in J$ for all $a, b \in J$, and (ii) if $a \in J$ and $b \leq a$, then $b \in J$. In basic algebras, this is not enough for a subset to be the 0-class of a congruence, and hence we will refer to the subsets satisfying the conditions (i) and (ii) as *pre-ideals*, and to the 0-classes of congruences as *ideals*. Unfortunately, the concept of pre-ideal is quite general and so pre-ideals can fail to have some natural properties we want them to have (for example, a pre-ideal of a basic algebra need not be an ideal of its underlying lattice). Therefore, we restrict ourselves to two subvarieties of basic algebras that are much closer to MV-algebras than basic algebras in general. Namely, we focus on basic algebras satisfying the identities (C) and (M).

## 2.1. Introduction and the pre-ideal lattice

Definition 2.1.1. Let $(A, \oplus, \neg, 0, 1)$ be a basic algebra. We call $\emptyset \neq P \subseteq A$ a *pre-ideal* of $A$ if

(P1) $a \oplus b \in P$ for all $a, b \in P$;
(P2) for every $a \in A$ and $b \in P$, if $a \leq b$, then $a \in P$.

An *ideal* of $A$ is a subset $J \subseteq A$ such that $J = [0]_\Theta$ for some congruence $\Theta$ of $A$.

It is proved in **[11]** that the variety of basic algebras is congruence regular, i.e., every congruence is specified by an arbitrary class, in particular, by the 0-class. Hence if $J$ is an ideal, then the congruence $\Theta = \Theta(J)$ the kernel of which is $J$ (i.e. $J = [0]_\Theta$) is uniquely determined as follows (cf. **[16]**):

$$(a, b) \in \Theta(J) \quad \text{iff} \quad a \oslash b, b \oslash a \in J \quad \text{iff} \quad a \ominus b, b \ominus a \in J.$$

Consequently, the assignments $\Theta \mapsto [0]_\Theta$ and $J \mapsto \Theta(J)$ are mutually inverse isomorphisms between the congruence lattice of $A$, $Con(A)$, and the ideal lattice of $A$, $Id(A)$.

Besides MV-algebras, in some classes of basic algebras, the description of ideals is quite simple. For instance, Pulmannová and Vinceková **[31]** proved that if $A$ is an effect basic algebra, then $\emptyset \neq J \subseteq A$ is an ideal of $A$ iff (i) $a \oplus b \in P$ for all $a, b \in P$ and (ii) $b \oslash a \in J$ for all $a \in A$, $b \in J$.

Remark 2.1.2. It is worth noticing that a pre-ideal of $A$ need not be an ideal of the lattice $\ell(A)$. As a counterexample we can consider the 6-element orthomodular lattice $(\{0, a, b, a', b', 1\}, \vee, \wedge, ', 0, 1)$ which is usually called $\mathrm{OM}_6$. Then $\{0, a, b\}$ is a pre-ideal of the corresponding basic algebra $(\{0, a, b, a', b', 1\}, \oplus, \neg, 0, 1)$ since $a \oplus a = a$, $b \oplus b = b$, $a \oplus b = b$ and $b \oplus a = a$, while $a \vee b = 1$, so $\{0, a, b\}$ is not a lattice ideal.

LEMMA 2.1.3. *Let $A$ be a basic algebra. For every $P \subseteq A$ such that $0 \in P$, the following are equivalent:*

(a) *$P$ is a pre-ideal;*

(b) *for all $a \in A$ and $b \in P$, if $a \oslash b \in P$, then $a \in P$;*

(c) *$P$ is closed under the term $\rho(x, y_1, y_2) = x \wedge (y_1 \oplus y_2)$, in the sense that $\rho(a, b_1, b_2) \in P$ for all $a \in A$ and $b_1, b_2 \in P$.*

PROOF. $(a) \Rightarrow (b)$ Let $a \oslash b \in P$ and $b \in P$. Then $a \vee b = (a \oslash b) \oplus b \in P$ which yields $a \in P$.

$(b) \Rightarrow (a)$ If $b \in P$ and $a \leq b$, then $a \oslash b = 0 \in P$, and so $a \in P$. Thus $P$ satisfies (P2). Hence, if $a, b \in P$, then $(a \oplus b) \oslash b = a \wedge \neg b \in P$, which implies $a \oplus b \in P$. Thus $P$ satisfies (P1), too.

The equivalence of $(a)$ and $(c)$ is evident. $\square$

For every $B \subseteq A$, let us denote by $Pg(B)$ the pre-ideal generated by $B$, i.e. the intersection of all pre-ideals that contain $B$. It can be described as follows:

LEMMA 2.1.4. *Let $A$ be a basic algebra and $\emptyset \neq B \subseteq A$. Then*

$$Pg(B) = \bigcup_{n \in \mathbb{N}_0} B^{(n)}$$

*where $B^{(0)} = B$ and, for $n > 0$, $B^{(n)} = \{\rho(a, b_1, b_2) \mid a \in A, b_1, b_2 \in B^{(n-1)}\}$. Moreover, if $A$ satisfies* (M)*, then $Pg(B)$ consists of those elements $a \in A$ that are less than or equal to finite sums of elements of $B$.*

PROOF. The former assertion easily follows from Lemma 2.1.3 (c). The latter one is proved by observing that if $A$ satisfies (M), then by Lemma 1.3.13 (a), the addition is monotone, and hence the elements $a \in A$ such that $a \leq \tau(b_1, \ldots, b_n)$, where $\tau$ is an $n$-ary additive term and $b_1, \ldots, b_n \in B$, form a pre-ideal of $A$. $\square$

The set of all pre-ideals of a basic algebra $A$ will be denoted by $Pr(A)$. In view of Lemma 2.1.4, $Pr(A)$ partially ordered by set-inclusion is an algebraic lattice.

PROPOSITION 2.1.5. *For every basic algebra $A$, the ideal lattice $Id(A)$ is a complete sublattice of the pre-ideal lattice $Pr(A)$.*

PROOF. Since the variety of basic algebras is congruence regular and permutable (see [**11**]), the map $\Theta \mapsto [0]_\Theta$ is an isomorphism between the congruence lattice $Con(A)$ and the ideal lattice $Id(A)$, and the join of $\Theta$ and $\Phi$ in $Con(A)$ is $\Theta \circ \Phi$.

For the moment we denote the join operations in $Pr(A)$ and $Id(A)$ by $\vee$ and $\sqcup$, respectively. It is clear that $[0]_\Theta \vee [0]_\Phi \subseteq [0]_\Theta \sqcup [0]_\Phi = [0]_{\Theta \circ \Phi}$ for all $\Theta, \Phi \in Con(A)$. Conversely, let $a \in [0]_\Theta \sqcup [0]_\Phi$, i.e. $(a, 0) \in \Theta \circ \Phi$. Then $(a, b) \in \Theta$ and $(b, 0) \in \Phi$ for some $b \in A$, so $a \oslash b \in [0]_\Theta$ and $b \in [0]_\Phi$ whence $a \vee b = (a \oslash b) \oplus b \in Pg([0]_\Theta \cup [0]_\Phi) = [0]_\Theta \vee [0]_\Phi$ and thus $a \in [0]_\Theta \vee [0]_\Phi$.

Let $\{\Theta_i \mid i \in I\}$ be a non-empty collection of congruences of $A$. Obviously, $\bigvee_{i\in I}[0]_{\Theta_i} \subseteq \bigsqcup_{i\in I}[0]_{\Theta_i}$. If $a \in \bigsqcup_{i\in I}[0]_{\Theta_i}$, then $a \in [0]_{\Theta_{i_1}\circ\cdots\circ\Theta_{i_n}} = [0]_{\Theta_{i_1}} \sqcup \cdots \sqcup [0]_{\Theta_{i_n}} = [0]_{\Theta_{i_1}} \vee \cdots \vee [0]_{\Theta_{i_n}}$ for some $i_1, \ldots, i_n \in I$, thus $a \in \bigvee_{i\in I}[0]_{\Theta_i}$, which proves $\bigsqcup_{i\in I}[0]_{\Theta_i} = \bigvee_{i\in I}[0]_{\Theta_i}$. $\square$

PROPOSITION 2.1.6. *Let $A$ be a basic algebra satisfying* (M). *Then $Pr(A)$ is a distributive lattice.*

PROOF. Let $P, Q_i \in Pr(A)$ with $i \in I$. If $a \in P \cap \bigvee_{i\in I} Q_i$, then $a \in P$ and $a \leq \tau(b_1, \ldots, b_n)$ for some $n$-ary additive term $\tau$ and $b_1, \ldots, b_n \in \bigcup_{i\in I} Q_i$. By Lemma 1.3.13 (b) we have $a = a \wedge \tau(b_1, \ldots, b_n) \leq \tau(a \wedge b_1, \ldots, a \wedge b_n)$ where $a \wedge b_j \in P \cap \bigcup_{i\in I} Q_i = \bigcup_{i\in I}(P \cap Q_i)$ for each $j = 1, \ldots, n$, and hence $a \in \bigvee_{i\in I}(P \cap Q_i) = Pg(\bigcup_{i\in I}(P \cap Q_i))$. Thus we get $P \cap \bigvee_{i\in I} Q_i \subseteq \bigvee_{i\in I}(P \cap Q_i)$. $\square$

Consequently, if $A$ satisfies (M), then the pre-ideal lattice $Pr(A)$ is relatively pseudocomplemented. In order to describe the relative pseudocomplements, we introduce the following notation: Given $\emptyset \neq B \subseteq A$ and a pre-ideal $Q \in Pr(A)$, we put

$$B \to Q = \{a \in A \mid a \wedge b \in Q \text{ for all } b \in B\}.$$

PROPOSITION 2.1.7. *Let $A$ be a basic algebra satisfying* (M). *For every $\emptyset \neq B \subseteq A$ and $Q \in Pr(A)$, $B \to Q$ is a pre-ideal, $B \to Q = Pg(B) \to Q$ and it is the relative pseudocomplement in $Pr(A)$ of $Pg(B)$ with respect to $Q$.*

PROOF. Clearly, $0 \in B \to Q$. If $a, b \in B \to Q$, then by Lemma 1.3.13 (b) we have $c \wedge (a \oplus b) \leq (c \wedge a) \oplus (c \wedge b) \in Q$ for every $c \in B$ since both $c \wedge a$ and $c \wedge b$ belong to $Q$. Hence $c \wedge (a \oplus b) \in Q$, proving $a \oplus b \in B \to Q$. Also, if $a \in B \to Q$ and $b \leq a$, then $b \wedge c \leq a \wedge c \in Q$ for each $c \in B$, and so $b \in B \to Q$. Thus $B \to Q$ is a pre-ideal.

Let $a \in B \to Q$ and $c \in Pg(B)$, i.e. $c \leq \tau(b_1, \ldots, b_n)$ for some additive term $\tau$ and $b_i \in B$. By Lemma 1.3.13 (b) we have $a \wedge c \leq \tau(a \wedge b_1, \ldots, a \wedge b_n) \in Q$ as each $a \wedge b_i$ is in $Q$. Hence $a \wedge c \in Q$, which shows that $B \to Q \subseteq Pg(B) \to Q$; the converse inclusion is obviously true, so $B \to Q = Pg(B) \to Q$.

Finally, it can easily be seen that, for every $P \in Pr(A)$, $P \subseteq Pg(B) \to Q$ iff $P \cap Pg(B) \subseteq Q$, and hence $Pg(B) \to Q$ is the relative pseudocomplement of $Pg(B)$ with respect to $Q$. $\square$

In particular, if we denote $B \to \{0\}$ by $B^\perp$, then $B^\perp = Pg(B)^\perp$ is the pseudocomplement of $Pg(B)$ in the lattice $Pr(A)$. We call $B^\perp$ the *polar* of $B$.

PROPOSITION 2.1.8. *Let $A$ be a basic algebra that satisfies* (M). *Then in the pre-ideal lattice $Pr(A)$ we have $Pg(a) \cap Pg(b) = Pg(a \wedge b)$ and $Pg(a) \vee Pg(b) = Pg(a \vee b) = Pg(a \oplus b)$ for all $a, b \in A$.*

PROOF. Let $c \in Pg(a) \cap Pg(b)$, i.e., for some additive terms $\tau_1$ and $\tau_2$, we have $c \leq \tau_1(a, \ldots, a)$ and $c \leq \tau_2(b, \ldots, b)$. Then, using Lemma 1.3.13 (b), we get

$$\begin{aligned} c \leq \tau_1(a, \ldots, a) \wedge \tau_2(b, \ldots, b) &\leq \tau_1(a \wedge \tau_2(b, \ldots, b), \ldots, a \wedge \tau_2(b, \ldots, b)) \\ &\leq \tau_1(\tau_2(a \wedge b, \ldots, a \wedge b), \ldots, \tau_2(a \wedge b, \ldots, a \wedge b)), \end{aligned}$$

and hence $c \in Pg(a \wedge b)$. Thus $Pg(a) \cap Pg(b) \subseteq Pg(a \wedge b)$. The other inclusion is obvious.

It is clear that $Pg(a) \vee Pg(b) \subseteq Pg(a \vee b) \subseteq Pg(a \oplus b)$ since $a \vee b \leq a \oplus b$. Conversely, if $c \in Pg(a \oplus b)$, then $c \leq \tau(a \oplus b, \dots, a \oplus b)$ for some additive term $\tau$. But $a, b \in Pg(a) \vee Pg(b)$ entails $\tau(a \oplus b, \dots, a \oplus b) \in Pg(a) \vee Pg(b)$, so $c \in Pg(a) \vee Pg(b)$ and hence $Pg(a \oplus b) \subseteq Pg(a) \vee Pg(b)$. $\square$

COROLLARY 2.1.9. *If $A$ satisfies* (M), *then the compact elements of the pre-ideal lattice $Pr(A)$ are precisely the pre-ideals $Pg(a)$, $a \in A$.*

PROOF. A pre-ideal $P$ is compact in $Pr(A)$ iff $P = Pg(B)$ for some finite $B \subseteq A$. Letting $a$ be the supremum of $B$, we have $Pg(B) = Pg(a)$, so that $P$ is compact iff $P = Pg(a)$ for some $a \in A$. $\square$

Let $(A, \oplus, \neg, 0, 1)$ be a basic algebra. Given a pre-ideal $P \in Pr(A)$, we define the relation $\Theta_l(P)$ on $A$ as follows:

$$\begin{aligned}(a,b) \in \Theta_l(P) \quad \text{iff} \quad & a = x_1 \oplus (\cdots \oplus (x_m \oplus b') \dots) \text{ and } b = y_1 \oplus (\cdots \oplus (y_n \oplus a') \dots) \\ & \text{for some } x_i, y_j \in P \text{ and } a', b' \in A \text{ with } a' \leq a \text{ and } b' \leq b.\end{aligned} \tag{2.1}$$

If $A$ satisfies (M), then in view of Lemmata 1.3.12 and 1.3.13 (a) we get

$$\begin{aligned}(a,b) \in \Theta_l(P) \quad \text{iff} \quad & a \leq x_1 \oplus (\cdots \oplus (x_m \oplus b) \dots) \text{ and } b \leq y_1 \oplus (\cdots \oplus (y_n \oplus a) \dots) \\ & \text{for some } x_i, y_j \in P.\end{aligned} \tag{2.2}$$

PROPOSITION 2.1.10. *Let $A$ be a basic algebra that satisfies* (C). *If $J$ is an ideal of $A$, then $\Theta_l(J)$ is the only congruence relation whose 0-class is $J$, i.e., $\Theta_l(J) = \Theta(J)$.*

PROOF. By (C') and (1.5) we have $(x \oslash y) \oplus (x \wedge y) = (x \oslash (x \wedge y)) \oplus (x \wedge y) = x$. Hence if $(a,b) \in \Theta(J)$, then $a \oslash b, b \oslash a \in [0]_{\Theta(J)} = J$ together with $a = (a \oslash b) \oplus (a \wedge b)$ and $b = (b \oslash a) \oplus (a \wedge b)$ imply $(a,b) \in \Theta_l(J)$.

Conversely, let $(a,b) \in \Theta_l(J)$, i.e., $a = x_1 \oplus (\cdots \oplus (x_m \oplus b') \dots)$ and $b = y_1 \oplus (\cdots \oplus (y_n \oplus a') \dots)$ for some $x_i, y_j \in J = [0]_{\Theta(J)}$ and $a' \leq a$, $b' \leq b$. Then $(a, b') \in \Theta(J)$ and $(b, a') \in \Theta(J)$, hence $b' = b' \wedge b \equiv_{\Theta(J)} b' \wedge a' \equiv_{\Theta(J)} a \wedge a' = a'$, so $(a,b) \in \Theta(J)$.

Thus, $\Theta(J) = \Theta_l(J)$. $\square$

PROPOSITION 2.1.11. *Let $A$ be a basic algebra satisfying* (C). *For every $P \in Pr(A)$, $\Theta_l(P)$ is an equivalence relation which is compatible with the meet operation $\wedge$, and the 0-class of $\Theta_l(P)$ is $P$. Moreover, if $A$ satisfies* (M), *then $\Theta_l(P)$ is a congruence of the lattice $\ell(A)$.*

PROOF. Obviously, $\Theta_l(P)$ is reflexive and symmetric. To prove transitivity, assume that $(a,b) \in \Theta_l(P)$ and $(b,c) \in \Theta_l(P)$. Then, by (2.1), $a = x_1 \oplus (\cdots \oplus (x_m \oplus b') \dots)$ and $b = y_1 \oplus (\cdots \oplus (y_n \oplus a') \dots)$ for some $x_i, y_j \in P$ and $a' \leq a$, $b' \leq b$, and at the same time, $b = u_1 \oplus (\cdots \oplus (u_r \oplus c') \dots)$ and $c = v_1 \oplus (\cdots \oplus (v_s \oplus b'') \dots)$ for some $u_k, v_l \in P$ and $b'' \leq b$, $c' \leq c$. By the Riesz decomposition property (Lemma 1.3.12) we can write $b' = u'_1 \oplus (\cdots \oplus (u'_r \oplus c'') \dots)$ for some $u'_k \leq u_k$ and $c'' \leq c'$, and likewise, $b'' = y'_1 \oplus (\cdots \oplus (y'_n \oplus a'') \dots)$ for some $y'_j \leq y_j$ and $a'' \leq a'$. Then we have $a = x_1 \oplus (\cdots \oplus (x_m \oplus (u'_1 \oplus (\cdots \oplus (u'_r \oplus c'') \dots))) \dots)$ where each $u'_k$ belongs to $P$ and $c'' \leq c$, and

$c = v_1 \oplus (\dots \oplus (v_s \oplus (y'_1 \oplus (\dots \oplus (y'_n \oplus a'') \dots))) \dots)$ where each $y'_j$ belongs to $P$ and $a'' \leq a$. Thus $(a,c) \in \Theta_l(P)$, and so $\Theta_l(P)$ is an equivalence relation. It is also clear that $(a,0) \in \Theta_l(P)$ iff $a \in P$, so $[0]_{\Theta_l(P)} = P$.

For compatibility with $\wedge$ we need the following

Claim. *For all $a, b \in A$ and $x \in P$ there exists $y \in P$ such that $(x \oplus a) \wedge b = y \oplus (a \wedge b)$.*

Indeed, if we put $c = (x \oplus a) \wedge b$ and $y = c \oslash a$, then we have $a \wedge c = a \wedge b$, which yields $y \oplus (a \wedge b) = (c \oslash a) \oplus (a \wedge c) = (c \oslash (a \wedge c)) \oplus (a \wedge c) = c$, and $y \in P$ because $x \oplus a \geq c$ implies $x \geq c \oslash a = y$. The claim is settled.

Now, let $(a,b) \in \Theta_l(P)$, i.e. $a = x_1 \oplus (\dots \oplus (x_m \oplus b') \dots)$ and $b = y_1 \oplus (\dots \oplus (y_n \oplus a') \dots)$ where $x_i, y_j \in P$ and $a' \leq a$, $b' \leq b$. Given any $c \in A$, by the above claim there exist $u_i, v_j \in P$ such that $a \wedge c = u_1 \oplus (\dots \oplus (u_m \oplus (b' \wedge c)) \dots)$ and $b \wedge c = v_1 \oplus (\dots \oplus (v_n \oplus (a' \wedge c)) \dots)$. Hence $(a \wedge c, b \wedge c) \in \Theta_l(P)$, proving that $\Theta_l(P)$ is compatible with $\wedge$.

Let $A$ satisfy (M). Let $(a,b) \in \Theta_l(P)$, i.e. $a \leq x_1 \oplus (\dots \oplus (x_m \oplus b) \dots)$ and $b \leq y_1 \oplus (\dots \oplus (y_n \oplus a) \dots)$ for some $x_i, y_j \in P$. Then, recalling Lemma 1.3.13 (a), for every $c \in A$ we have $x_1 \oplus (\dots \oplus (x_m \oplus (b \vee c)) \dots) \geq [x_1 \oplus (\dots \oplus (x_m \oplus b) \dots)] \vee [x_1 \oplus (\dots \oplus (x_m \oplus c) \dots)] \geq a \vee c$, and similarly, $y_1 \oplus (\dots \oplus (y_n \oplus (a \vee c)) \dots) \geq b \vee c$, so that $(a \vee c, b \vee c) \in \Theta_l(P)$. Thus $\Theta_l(P)$ is a lattice congruence. $\square$

Consequently, if $A$ is a basic algebra that fulfills (C), then for every $P \in Pr(A)$, the quotient set $A/\Theta_l(P)$ is a meet-semilattice when ordered as follows:

$$[a]_{\Theta_l(P)} \leq [b]_{\Theta_l(P)} \quad \text{iff} \quad a = c_1 \oplus (\dots \oplus (c_m \oplus b') \dots) \text{ for some } c_i \in P \text{ and } b' \leq b,$$

and if $A$ satisfies (M), then

$$[a]_{\Theta_l(P)} \leq [b]_{\Theta_l(P)} \quad \text{iff} \quad a \leq c_1 \oplus (\dots \oplus (c_m \oplus b) \dots) \text{ for some } c_i \in P.$$

Symmetrically to (2.2) we could also define the relation $\Theta_r(P)$ by saying: $(a,b) \in \Theta_r(P)$ iff $a \leq (\dots(b \oplus x_1) \oplus \dots) \oplus x_m$ and $b \leq (\dots(a \oplus y_1) \oplus \dots) \oplus y_n$ for some $x_i, y_j \in P$. It is not hard to show that $\Theta_r(P)$ is always an equivalence relation compatible with $\neg$, and if $A$ satisfies (M), then $\Theta_r(P)$ is a lattice congruence. However, the relation $\Theta_l(P)$ seems to behave better with respect to what we are interested in.

Another generalization of ideals corresponding to certain equivalence relations was considered in [**16**]. We now briefly discuss the connections between these 'one-sided ideals' and our pre-ideals.

Definition 2.1.12. Let $A$ be a basic algebra. We say that $\emptyset \neq J \subseteq A$ is a *weak ideal* of $A$ if, for all $a, b \in A$,

(i) if $a \ominus b \in J$ and $b \in J$, then also $a \in J$;
(ii) if $a \ominus b \in J$ and $a \geq b$, then $(c \ominus b) \ominus (c \ominus a) \in J$ for every $c \in A$.

For every weak ideal $J$, the relation $\Phi(J)$ defined by

$$(a,b) \in \Phi(J) \quad \text{iff} \quad a \ominus b, b \ominus a \in J$$

is a congruence of the lattice $\ell(A)$ which satisfies the following compatibility condition, for all $a, b, c \in A$:

$$(a, b) \in \Phi(J) \quad \Rightarrow \quad (c \ominus a, c \ominus b) \in \Phi(J). \tag{2.3}$$

Generally, it is not a congruence of $A$, but if $A$ is an effect basic algebra, then every weak ideal $J$ is an ideal and hence $\Phi(J)$ is a congruence (see [**16**]). The same is obviously true in case when $A$ is a commutative basic algebra because then (2.3) amounts to saying that $\Phi(J)$ is a congruence of $A$.

Since $\Phi(J)$ is a lattice congruence, its 0-class $J$ must be a lattice ideal, and so it is clear that a pre-ideal need not be a weak ideal (cf. Remark 2.1.2). On the other hand, we have:

PROPOSITION 2.1.13. *Let $A$ be a basic algebra. Every weak ideal $J$ is a pre-ideal, and if $A$ satisfies* (C)*, then $\Phi(J) = \Theta_l(J)$.*

PROOF. We first observe that $(a, b) \in \Phi(J)$ implies $(a \oplus c, b \oplus c) \in \Phi(J)$ for every $c \in A$, because $a \oplus c = \neg(\neg c \ominus a) \equiv_{\Phi(J)} \neg(\neg c \ominus b) = b \oplus c$ by (2.3).

Let $a, b \in J$. Then $(a, 0) \in \Phi(J)$ implies $(a \oplus b, b) \in \Phi(J)$. Since $(b, 0) \in \Phi(J)$, it follows $(a \oplus b, 0) \in \Phi(J)$, so $a \oplus b \in [0]_{\Phi(J)} = J$. Further, if $a \leq b \in J$, then $a \ominus b = 0 \in J$ yields $a \in J$. Thus $J \in Pr(A)$.

Now, assume that $A$ satisfies (C). Let $(a, b) \in \Phi(J)$. Then $(\neg a, \neg b) \in \Phi(J)$ by (2.3), and so $a \oslash b = \neg b \ominus \neg a \in J$ and $b \oslash a = \neg a \ominus \neg b \in J$. Hence $(a, b) \in \Theta_l(J)$ since we have $a = (a \oslash b) \oplus (a \wedge b)$ and $b = (b \oslash a) \oplus (a \wedge b)$. Conversely, let $(a, b) \in \Theta_l(J)$, i.e., $a = x_1 \oplus (\cdots \oplus (x_m \oplus b') \dots)$ and $b = y_1 \oplus (\cdots \oplus (y_n \oplus a') \dots)$ for some $x_i, y_j \in J$, $a' \leq a$ and $b' \leq b$. Since $(x_i, 0) \in \Phi(J)$ and $(y_j, 0) \in \Phi(J)$, we get $(a, b') \in \Phi(J)$ and $(b, a') \in \Phi(J)$, whence $a' = a \wedge a' \equiv_{\Phi(J)} b' \wedge a' \equiv_{\Phi(J)} b' \wedge b = b'$ and thus $(a, b) \in \Phi(J)$. $\square$

## 2.2. Prime pre-ideals

In this section, we characterize the meet-prime elements of pre-ideal lattices of basic algebras satisfying (M).

DEFINITION 2.2.1. We say that a pre-ideal $P$ of a basic algebra $A$ is *prime* when $P$ is meet-prime in the lattice $Pr(A)$, i.e., if for all $Q_1, Q_2 \in Pr(A)$, $Q_1 \cap Q_2 \subseteq P$ implies $Q_1 \subseteq P$ or $Q_2 \subseteq P$.

Since $Pr(A)$ is a distributive lattice if $A$ satisfies (M), it easily follows that $P \in Pr(A)$ is prime if and only if it is meet-irreducible in $Pr(A)$, that is, when proving that $P$ is prime, it suffices to show that whenever $P = Q_1 \cap Q_2$ for some $Q_1, Q_2 \in Pr(A)$, then $P = Q_1$ or $P = Q_2$.

LEMMA 2.2.2. *Let $A$ be a basic algebra that satisfies* (M)*. If $F$ is a filter in the lattice $\ell(A)$ and $P$ is a pre-ideal in $A$ with $P \cap F = \emptyset$, then there exists a prime pre-ideal $Q$ of $A$ such that $P \subseteq Q$ and $Q \cap F = \emptyset$. Hence, for every pre-ideal $P$ and every $a \in A \setminus P$ there exists a prime pre-ideal $Q$ such that $P \subseteq Q$ and $a \notin Q$.*

PROOF. Let $\mathfrak{M}$ be the set of all pre-ideals which are disjoint from $F$ and contain $P$. A routine use of Zorn's lemma shows that $\mathfrak{M}$ has a maximal element, $M$ say. Then $M$ is prime. Indeed, if $M = Q_1 \cap Q_2$ where $Q_1, Q_2 \in Pr(A)$ and $Q_1 \neq M \neq Q_2$, then $Q_i \cap F \neq \emptyset$ $(i = 1, 2)$ and for $a_i \in Q_i \cap F$ we have $a_1 \wedge a_2 \in Q_1 \cap Q_2 \cap F = M \cap F = \emptyset$, a contradiction. Thus, by the remark following Definition 2.2.1, $M$ is a prime pre-ideal. □

THEOREM 2.2.3. *Let $A$ be a basic algebra satisfying* (M)*. Then for every $P \in Pr(A)$, the following are equivalent:*

(i) *$P$ is a prime pre-ideal;*
(ii) *for all $a, b \in A$, if $Pg(a) \cap Pg(b) \subseteq P$, then $a \in P$ or $b \in P$;*
(iii) *for all $a, b \in A$, if $a \wedge b \in P$, then $a \in P$ or $b \in P$;*
(iv) *for all $a, b \in A$, if $a \wedge b = 0$, then $a \in P$ or $b \in P$;*
(v) *for all $a, b \in A$, $a \oslash b \in P$ or $b \oslash a \in P$;*
(vi) *$A/\Theta_l(P)$ is linearly ordered;*
(vii) *the set of all pre-ideals exceeding $P$ is linearly ordered by set-inclusion;*
(viii) *$\{a\} \to P = P$ for all $a \in A \setminus P$.*

PROOF. The implications (i) ⇒ (ii) and (iii) ⇒ (iv) are obvious.

(ii) ⇒ (iii) It suffices to note that $Pg(a) \cap Pg(b) = Pg(a \wedge b)$ by Proposition 2.1.8.

(iv) ⇒ (v) Since the underlying lattice $\ell(A)$ is distributive, by Lemma 1.3.10 and (C') we have $0 = (a \wedge b) \oslash (a \wedge b) = (a \oslash (a \wedge b)) \wedge (b \oslash (a \wedge b)) = (a \oslash b) \wedge (b \oslash a)$.

(v) ⇒ (vi) If $a \oslash b \in P$, then $a \leq a \vee b = (a \oslash b) \oplus b$ entails $[a]_{\Theta_l(P)} \leq [b]_{\Theta_l(P)}$.

(vi) ⇒ (vii) Let $Q, R$ be two distinct pre-ideals exceeding $P$ and suppose that $Q \nsubseteq R$ and $R \nsubseteq Q$. Then we can find $a \in Q \setminus R$ and $b \in R \setminus Q$. We have $[a]_{\Theta_l(P)} \leq [b]_{\Theta_l(P)}$ or $[b]_{\Theta_l(P)} \leq [a]_{\Theta_l(P)}$. In the former case, there exist $c_1, \dots, c_n \in P$ and $b' \in A$ with $b' \leq b$ such that $a = c_1 \oplus (\cdots \oplus (c_n \oplus b') \dots)$. Since $b \in R$ and $c_i \in P \subseteq R$, we get $a = c_1 \oplus (\cdots \oplus (c_n \oplus b') \dots) \in R$, a contradiction. In the latter case we would analogously reach the contradiction $b \in Q$.

(vii) ⇒ (i) Since the pre-ideals that contain $P$ form a chain under set-inclusion, $P$ is meet-irreducible and hence prime.

(iii) ⇒ (viii) Let $a \notin P$. If $b \in \{a\} \to P$, then $b \wedge a \in P$, which yields $b \in P$. Hence $\{a\} \to P \subseteq P$. The inclusion $P \subseteq \{a\} \to P$ is evident.

(viii) ⇒ (iv) If $a \wedge b = 0$ where $a \notin P$, then $b \in \{a\} \to P = P$. □

COROLLARY 2.2.4. *For a basic algebra $A$ satisfying* (M)*, the following are equivalent:*

(a) *(the underlying lattice of) $A$ is a chain;*
(b) *$\{0\}$ is a prime pre-ideal;*
(c) *$Pr(A)$ is a chain.*

THEOREM 2.2.5. *Let $A$ be a basic algebra satisfying* (C)*. Then for every pre-ideal $P \in Pr(A)$, the conditions (iii)–(vi) of Theorem 2.2.3 are mutually equivalent and imply (vii).*

PROOF. We only have to show (vi) ⇒ (iii); the proofs of the other implications remain unchanged. Let $A/\Theta_l(P)$ be linearly ordered and assume $a \wedge b \in P$. If $[a]_{\Theta_l(P)} \leq [b]_{\Theta_l(P)}$, then $a = c_1 \oplus (\cdots \oplus (c_n \oplus b') \dots)$ for some $c_i \in P$ and $b' \leq b$. But $c_1 \oplus (\cdots \oplus (c_n \oplus b') \dots) \geq b'$, thus $b' \leq a \wedge b$ and $b' \in P$, which yields $a = c_1 \oplus (\cdots \oplus (c_n \oplus b') \dots) \in P$. Analogously, $[b]_{\Theta_l(P)} \leq [a]_{\Theta_l(P)}$ implies $b \in P$. □

## 2.3. Minimal prime pre-ideals

A lower bounded distributive lattice is *relatively normal* if its set of prime ideals is a root system under set-inclusion. The class of all algebraic distributive lattice whose compact elements form a relatively normal sublattice is denoted by IRN, see [**34**], [**35**].

LEMMA 2.3.1 ([**34**], Corollary 2.2; [**35**], Corollary 3.2). *Let $D$ be an algebraic distributive lattice such that the set of compact elements of $D$ is a sublattice of $D$. The following statements are equivalent.*

(i) *$D$ is a member of the class* IRN.
(ii) *The meet-prime elements of $D$ form a root-system.*
(iii) *The collection of all elements exceeding a given meet-prime element is a chain of meet-prime elements.*
(iv) *For all compact elements $a, b \in D$, there exist compact elements $a', b' \in D$ such that*

$$a' \wedge b' = 0 \quad \text{and} \quad a \vee b' = a' \vee b = a \vee b.$$

Let $A$ be a basic algebra satisfying (M). Then by Proposition 2.1.6 the lattice $Pr(A)$ of all pre-ideals of $A$ is distributive. By Proposition 2.1.8 and Corollary 2.1.9 the compact elements of $Pr(A)$ are just the principal pre-ideals $Pg(a)$ for $a \in A$.

THEOREM 2.3.2. *Let $A$ be a basic algebra satisfying* (M). *Then the lattice of pre-ideals $Pr(A)$ is a member of the class* IRN.

PROOF. By Theorem 2.2.3 (v), a pre-ideal which contains a prime pre-ideal is prime, too, and hence, by Theorem 2.2.3 (vii), the prime pre-ideals of $A$ form a root system. Therefore, $Pr(A) \in$ IRN. □

From the previous theorem and [**35**], Lemma 2.3, we have

COROLLARY 2.3.3. *A prime pre-ideal $P$ of a basic algebra satisfying* (M) *is minimal iff $P = \bigcup\{\{a\}^{\perp} \mid a \notin P\}$.*

LEMMA 2.3.4 ([**26**]; [**36**], Lemma 3.1 (i); [**32**], Corollary 4.14). *Let $D$ be a distributive lattice with* 0. *A prime ideal $P$ of $D$ is a minimal prime ideal iff for every $x \in P$ there exists $y \in D \setminus P$ such that $x \wedge y = 0$.*

PROPOSITION 2.3.5. *Let $A$ be a basic algebra satisfying* (M). *Then $P \subseteq A$ is a minimal prime pre-ideal of $A$ if and only if $P$ is a minimal prime ideal of the lattice $\ell(A)$.*

PROOF. First, we observe that, by the previous lemma, every minimal prime ideal of the lattice $\ell(A)$ is a pre-ideal of the algebra $A$. Indeed, if $x, y \in P$, then there exist $a, b \in A \setminus P$ with $x \wedge a = 0 = y \wedge b$. Then by Lemma 1.3.13 (b), $(x \oplus y) \wedge a \wedge b \leq (x \wedge a \wedge b) \oplus (y \wedge a \wedge b) = 0$. Since $P$ is a prime ideal and $a, b \notin P$, we have $a \wedge b \notin P$, which together with $(x \oplus y) \wedge a \wedge b = 0$ entails $x \oplus y \in P$. Thus $P \in Pr(A)$.

Now, it easily follows that if $P$ is a minimal prime ideal of $\ell(A)$, then it is a minimal prime pre-ideal of $A$. Conversely, let $P$ be a minimal prime pre-ideal of $A$. Then $P$ is a prime ideal of $\ell(A)$ which is minimal because if $P$ is not a minimal prime ideal of $\ell(A)$, then there exists a minimal prime ideal $Q \subset P$ of $\ell(A)$, but $Q$ is a prime pre-ideal of $A$, hence, we have a contradiction with $P$ is a minimal prime pre-ideal of $A$. If $P$ were not minimal, there would exist a minimal prime ideal $Q$ of $\ell(A)$ with $Q \subset P$. But we know that such a $Q$ must be a prime pre-ideal of $A$, which contradicts the assumption that $P$ is a minimal prime pre-ideal of $A$. Hence $P$ is a minimal prime ideal of the lattice $\ell(A)$. $\square$

THEOREM 2.3.6. *Let $D$ be a non-trivial distributive lattice with a least element $0$. The following conditions are equivalent:*

(i) $0$ *is a dually compact element;*
(ii) $D$ *is atomic and every minimal prime ideal is a polar;* [1]
(iii) *for every minimal prime ideal $M$ there exists an atom $a \in D \setminus M$;*
(iv) *every maximal filter is principal.*

PROOF. (i) $\Rightarrow$ (ii) Firstly, if $B$ is a maximal chain in $D \setminus \{0\}$, then $\bigwedge B \neq 0$, since $\bigwedge B' \neq 0$ for every finite subset $B'$ of $B$. Hence, $\bigwedge B$ is an atom for every maximal chain $B$ in $D \setminus \{0\}$.

Secondly, let $M$ be a minimal prime ideal of $D$. Now, suppose that $L(D \setminus M)$ is $\{0\}$, where $L(D \setminus M)$ is the set of all lower bounds of $D \setminus M$. Then $\bigwedge(D \setminus M) = 0$ and there exists a finite subset $N$ of $D \setminus M$ such that $\bigwedge N = 0$. But $M$ is prime, so there exists $x \in N$ such that $x \in M$, a contradiction.

Let $0 \neq a \in L(D \setminus M)$. Then, by Lemma 2.3.4, $a \wedge x = 0$ for every $x \in M$, whence $M \subseteq \{a\}^{\perp}$ and $a \notin M$. Since $M$ is prime, $a \notin M$ entails $\{a\}^{\perp} \subseteq M$. Hence $M = \{a\}^{\perp}$.

(ii) $\Rightarrow$ (iii) Let $M$ be a minimal prime ideal. By (ii), $M = B^{\perp}$ for some $B \subseteq D$. Since $M \neq D$, $B \neq \{0\}$ and so there exists an atom $a \leq b$ for some $b \in B$. Then clearly $a \notin B^{\perp} = M$.

(iii) $\Rightarrow$ (iv) If $P$ is a maximal filter, then $P$ is prime since $D$ is distributive. Thus $D \setminus P$ is a minimal prime ideal. Hence, there exists an atom $a \in P$ and so $P = \{x \in D \mid x \geq a\}$.

(iv) $\Rightarrow$ (i) Let $S$ be a set such that $\bigwedge S' > 0$ for every finite subset $S'$ of $S$. Then $S$ is a subset of some maximal filter $P$. Hence, we have $\bigwedge S > 0$ since $P$ is principal. $\square$

COROLLARY 2.3.7. *Let $A$ be a basic algebra satisfying* (M). *The following conditions are equivalent:*

(i) $0$ *is a dually compact element;*
(ii) $1$ *is a compact element;*
(iii) *$A$ is compactly generated, i.e., if $a = \bigvee X$ exists in $A$, then $a = \bigvee X'$ for some finite subset $X' \subseteq X$;*
(iv) *$A$ is atomic and every minimal prime pre-ideal is a polar;*
(v) *for every minimal prime pre-ideal $M$ there exists an atom $a \in A \setminus M$;*
(vi) *every maximal filter of the lattice $\ell(A)$ is principal.*

PROOF. By Lemma 1.3.8, the underlying lattice $\ell(A)$ of a basic algebra $A$ that satisfies (M) is distributive. Hence, by the previous theorem, (i), (iv), (v), (vi) are equivalent.

[1] We define the polar $B^{\perp}$ of a subset $B$ of a lattice $D$ as $B^{\perp} = \{a \in D \mid a \wedge b = 0 \text{ for all } b \in B\}$. Clearly, if $D$ is a distributive lattice, then $B^{\perp}$ is an ideal of $D$, for every $\emptyset \neq B \subseteq D$.

(i), (ii), (iii) are equivalent since there are antitone involutions on all intervals $[0, a]$, and if $a = \bigvee X$ for some $X \subseteq A$ then $X \subseteq [0, a]$. $\square$

CHAPTER 3

# Basic algebras and lattice-ordered commutative loops

It is well-known that MV-algebras are closely related to lattice-ordered Abelian groups. Namely, if $G$ is a lattice-ordered Abelian group (additively written) and $u$ any positive element of $G$, then the interval $[0, u]$ of $G$ equipped with the operations $\neg x = u - x$ and $x \oplus y = (x+y) \wedge u$ forms an MV-algebra which is commonly denoted by $\Gamma(G, u)$, and more importantly, due to Mundici **[30]**, every MV-algebra is isomorphic to $\Gamma(G, u)$ for some lattice-ordered Abelian group $G$ with a distinguished strong order-unit $u$ in $G$. Essentially the same remains true for pseudo-MV-algebras (also called GMV-algebras), which are a non-commutative generalization of MV-algebras, and lattice-ordered (non-commutative) groups.

The goal for this chapter is to establish an analogous connection between (commutative) basic algebras and lattice-ordered commutative loops. We show that every interval $[0, u]$ of any commutative lattice-ordered loop $L$ is a monotone basic algebra which we denote by $\Gamma(L, u)$ despite the fact that $x \oplus y$ is not defined as in MV-algebras. On the other hand, for any semilinear commutative basic algebra $A$ we construct a lattice-ordered commutative loop $L$ with a strong order-unit $u$ such that $A$ is isomorphic to $\Gamma(L, u)$. The main tool used in the construction are the so-called good functions (see Definition 3.3.2). In the final section, we present a new example of a commutative basic algebra.

## 3.1. From lattice-ordered commutative loops to basic algebras

In this section, we generalize the passage from Abelian $\ell$-groups to MV-algebras (Mundici's functor $\Gamma$), that is, given a lattice-ordered commutative loop $(L, \vee, \wedge, +, -, 0)$ and its positive element $u \in L$, we equip the interval $[0, u]$ with operations $\oplus$ and $\neg$ so that it becomes a basic algebra. While $\neg x$ is defined as $u - x$, $x \oplus y$ cannot be defined as $(x + y) \wedge u$ in general, but it must be derived from the natural antitone involutions $\delta_a$ on the intervals $[0, a]$; thus the basic algebra $\Gamma(L, u) = ([0, u], \oplus, \neg, 0, u)$ is constructed according to Proposition 1.1.2 (iii). We restrict ourselves to commutative $\ell$-loops because in the non-commutative case the intervals $[0, a]$ do not bear natural antitone involutions, which are necessary in order to define a basic algebra.

First, we recall the basic concepts (cf. **[3]**, **[20]**). A *commutative loop* is an algebra $(L, +, -, 0)$ of type $(2, 2, 0)$ satisfying the identities $x+y = y+x$, $x+0 = x$, $(x+y)-y = x$ and $(x-y)+y = x$. If every element $x \in L$ has the *inverse* $x' \in L$ such that $x'+(x+y) = y$ for all $y \in L$, then $(L, +, -, 0)$ is a commutative *inverse loop*. It is clear that $x'' = x$, $(x + y)' = x' + y'$ and $x - y = x + y'$ for all $x, y \in L$.

A *partially ordered commutative loop* is a structure $(L, \leq, +, -, 0)$ such that $(L, \leq)$ is a poset, $(L, +, -, 0)$ is a commutative loop, and

$$x \leq y \quad \text{iff} \quad x + z \leq y + z \tag{3.1}$$

for all $x, y, z \in L$. If $(L, \leq)$ is a lattice with associated lattice operations $\vee$ and $\wedge$, then $(L, \vee, \wedge, +, -, 0)$ is a *lattice-ordered commutative loop*, or a *commutative $\ell$-loop* for short. In this case, the condition (3.1) is equivalent to saying that $+$ distributes over the lattice operations.

The *positive cone* of $L$ is the set $L^+ = \{x \in L \mid 0 \leq x\}$ of all *positive elements* of $L$. In contrast to Abelian $\ell$-groups, the positive cone need not determine the whole commutative $\ell$-loop—it is possible that two non-isomorphic commutative $\ell$-loops have the same positive cone. For example, if we define $x \circ y = x + y - xy$ if $x, y \leq 0$, and $x \circ y = x + y$ otherwise, then we get a linearly ordered commutative loop which has the same positive cone as the linearly ordered Abelian group $(\mathbb{R}, \leq, +, -, 0)$, see [**3**].

It easily follows from (3.1) that in partially ordered commutative loops we have

$$x \leq y \quad \text{iff} \quad z - y \leq z - x, \tag{3.2}$$

$$x \leq y \quad \text{iff} \quad x - z \leq y - z; \tag{3.3}$$

in lattice-ordered commutative loops we have the 'distributive laws'

$$(x \vee y) - z = (x - z) \vee (y - z) \quad \text{and} \quad z - (x \vee y) = (z - x) \wedge (z - y), \tag{3.4}$$

and dually.

Now, the next observation is an easy consequence of (3.2):

LEMMA 3.1.1. *Let $(L, \leq, +, -, 0)$ be a partially ordered commutative loop. Then for every positive element $a \in L^+$, the map $\delta_a \colon x \mapsto a - x$ is an antitone involution on the interval $[0, a]$.*

PROOF. By (3.2), $\delta_a$ is an antitone map on $[0, a]$ because $0 \leq x \leq y \leq a$ implies $a = a - 0 \geq a - x \geq a - y \geq a - a = 0$. Also, $\delta_a$ is an involution because $a - (a - x) = x$. $\square$

In the light of the above lemma and Proposition 1.1.2 (iii), it is clear that every interval $[0, u]$ of a commutative $\ell$-loop is a basic algebra. More precisely:

PROPOSITION 3.1.2. *Let $(L, \vee, \wedge, +, -, 0)$ be a commutative $\ell$-loop and $u \in L^+$ an arbitrary positive element of $L$. If we define*

$$\neg x = u - x \quad \textit{and} \quad x \oplus y = u - (((u - y) - x) \vee 0) = (u - ((u - y) - x)) \wedge u,$$

*for $x, y \in [0, u]$, then $\Gamma(L, u) = ([0, u], \oplus, \neg, 0, u)$ is a monotone basic algebra in which*

$$x \ominus y = (x - y) \vee 0 = x - (x \wedge y) = (x \vee y) - y.$$

PROOF. The internal $[0, u]$ is a bounded lattice with antitone involutions $\delta_a \colon x \mapsto a - x$ on the principal ideals $[0, a] \subseteq [0, u]$. Hence, upon defining $\neg x = \delta_u(x) = u - x$ and

$$\begin{aligned} x \oplus y = \neg \delta_{\neg y}(\neg y \wedge x) &= u - [(u - y) - ((u - y) \wedge x)] = \\ &= u - (0 \vee ((u - y) - x)) = u \wedge (u - ((u - y) - x)), \end{aligned}$$

the algebra $\Gamma(L,u)$ is a basic algebra by Proposition 1.1.2 (iii).

If $x \leq y$, then $(u-x)-z \geq (u-y)-z$, whence $z \oplus x = u - (((u-x)-z) \vee 0) \leq u - (((u-y)-z) \vee 0) = z \oplus y$. Thus $\Gamma(L,u)$ is a monotone basic algebra. Finally, we have $x \ominus y = \neg(y \oplus \neg x) = ((u-(u-x))-y) \vee 0 = (x-y) \vee 0$. $\square$

Note that if $L$ is an Abelian $\ell$-group, then $x \oplus y = (u - ((u-y)-x)) \wedge u = (x+y) \wedge u$, but as the following example shows, if $L$ is only a commutative $\ell$-loop, then $\Gamma(L,u)$ with $\oplus$ defined by $x \oplus y = (x+y) \wedge u$ need not be a basic algebra at all. The example also shows that the basic algebra $\Gamma(L,u)$ need not be commutative in general.

EXAMPLE 3.1.3. Let $\mathbb{R}$ be equipped with the usual linear order and with the operations $\circ$ and $\sim$ that are defined as follows:

$$x \circ y = \begin{cases} x+y+\min(x,y) & \text{if } x,y \geq 0, \\ x+y & \text{otherwise,} \end{cases}$$

$$x \sim y = \begin{cases} x-2y & \text{if } 0 \leq y \leq \frac{x}{3}, \\ \frac{1}{2}(x-y) & \text{if } 0 \leq \frac{x}{3} \leq y \leq x, \\ x-y & \text{otherwise.} \end{cases}$$

Then $(\mathbb{R}, \leq, \circ, \sim, 0)$ is a linearly ordered commutative loop.

Obviously, $(\mathbb{R}, \circ, 0)$ is a commutative groupoid with identity 0, and it is easy to see that $x \leq y$ iff $x \circ z \leq y \circ z$, for all $x,y,z \in \mathbb{R}$.

In order to verify the identity $(x \sim y) \circ y = x$, we basically distinguish three cases:

(i) If $0 \leq y \leq \frac{x}{3}$, then $(x \sim y) \circ y = x - 2y + \min(x-2y, y) = x$ because $y \leq x - 2y$.

(ii) If $0 \leq \frac{x}{3} \leq y \leq x$, then $(x \sim y) \circ y = \frac{x-y}{2} + y + \min(\frac{x-y}{2}, y) = x$ because $\frac{x-y}{2} \leq y$.

(iii) In all other cases we have $x \sim y = x - y$ and $(x \sim y) \circ y = x - y + y = x$.

Analogously, the identity $(x \circ y) \sim y = x$ is verified by considering the following three cases:

(i) If $0 \leq x \leq y$, then $x \circ y = 2x + y$ and $0 \leq \frac{2x+y}{3} \leq y \leq 2x + y$, and hence $(x \circ y) \sim y = \frac{1}{2}(2x + y - y) = x$.

(ii) If $0 \leq y \leq x$, then $x \circ y = x + 2y$ and $0 \leq y \leq \frac{x+2y}{3}$, so $(x \circ y) \sim y = x + 2y - 2y = x$.

(iii) In all other cases we have $x \circ y = x + y$ and $(x \circ y) \sim y = x + y - y = x$.

Thus $(\mathbb{R}, \leq, \circ, \sim, 0)$ is indeed a linearly ordered commutative loop, and so for any $u \in \mathbb{R}^+$, $\Gamma(\mathbb{R}, u) = ([0,u], \oplus, \neg, 0, u)$ is a monotone basic algebra. For instance, for $u = 1$ we have $\frac{1}{2} \oplus \frac{1}{6} = \frac{5}{6}$ while $\frac{1}{6} \oplus \frac{1}{2} = \frac{11}{12}$. Hence $\Gamma(\mathbb{R}, u)$ need not be commutative.

On the other hand, if we equip $[0,u]$ with $x \boxplus y = \min(x+y, u)$, then $([0,u], \boxplus, \neg, 0, u)$ need not be a basic algebra. For instance, again for $u = 1$ we have $\neg(\neg\frac{1}{4} \boxplus \frac{1}{6}) \boxplus \frac{1}{6} = \frac{1}{3}$ but $\neg(\neg\frac{1}{6} \boxplus \frac{1}{4}) \boxplus \frac{1}{4} = \frac{1}{4}$. Thus the identity (1.3) does not hold.

LEMMA 3.1.4. *Let $L$ be an inverse commutative $\ell$-loop. For any $u \in L^+$, the following conditions are equivalent:*

(a) $u + (x+y) = (u+x) + y$ *for all* $x, y \in L$*;*

(b) $u' + (x+y) = (u'+x) + y$ *for all* $x, y \in L$.

PROOF. To prove that (a) $\Rightarrow$ (b), it suffices to note that $u' + (x+y) = (u+(x+y)')' = (u + (x' + y'))' = ((u + x') + y')' = (u + x')' + y = (u' + x) + y$. The argument for (b) $\Rightarrow$ (a) is parallel. $\square$

Note that when $L$ and $u \in L^+$ are as above, then owing to the condition (a) of Lemma 3.1.4, for any $k \in \mathbb{N}$ we can unambiguously write $k \cdot u = u + \cdots + u$, though $L$ is not a group.

LEMMA 3.1.5. *Let $L$ be an inverse commutative $\ell$-loop. If $u \in L^+$ satisfies the equivalent conditions* (a) *and* (b) *of Lemma 3.1.4, then the basic algebra $\Gamma(L, u)$ is commutative and we have*

$$x \oplus y = (x + y) \wedge u$$

*for all $x, y \in [0, u]$. More generally, for all $x_1, \ldots, x_n \in [0, u]$,*

$$x_1 \oplus (\cdots \oplus (x_{n-1} \oplus x_n) \ldots) = (x_1 + (\cdots + (x_{n-1} + x_n) \ldots)) \wedge u.$$

PROOF. By Proposition 3.1.2 we have $x \oplus y = u \wedge (u - ((u - y) - x))$. But $u - ((u - y) - x) = u + ((u + y') + x')' = u + ((u + y')' + x) = (u + (u + y')') + x = (u + (u' + y)) + x = y + x$ by the condition (a). Hence $x \oplus y = (x + y) \wedge u = y \oplus x$. The rest an easy induction on $m$. We know that $x_1 \oplus x_2 = (x_1 + x_2) \wedge u$. Let $n \geq 3$ and suppose that the equality holds for all $k < n$. Then

$$\begin{aligned}
x_1 \oplus (x_2 \oplus (\cdots \oplus (x_{n-1} \oplus x_n) \ldots)) &= \\
&= x_1 \oplus ((x_2 + (\cdots + (x_{n-1} + x_n) \ldots)) \wedge u) \\
&= [x_1 + ((x_2 + (\cdots + (x_{n-1} + x_n) \ldots)) \wedge u)] \wedge u \\
&= (x_1 + (x_2 + (\cdots + (x_{n-1} + x_n) \ldots))) \wedge (x_1 + u) \wedge u \\
&= (x_1 + (x_2 + (\cdots + (x_{n-1} + x_n) \ldots))) \wedge u,
\end{aligned}$$

as required. $\square$

As in MV-algebras (see [**18, 23, 30**]), by a *good sequence* of elements of a commutative basic algebra $A$ be mean a sequence $(a_1, a_2, \ldots)$ of elements of $A$ such that

(i) $a_{i+1} \oplus a_i = a_i$ for all $i \in \mathbb{N}$, and
(ii) there exists $n \in \mathbb{N}$ such that $a_i = 0$ for all $i > n$.

The next result says that if $u \in L^+$ is a *strong order-unit* for $L$, in the sense that for every $a \in L^+$ there is $k \in \mathbb{N}$ such that $a \leq k \cdot u$, then every positive element of $L$ can be uniquely written as the sum of a good sequence of elements of $\Gamma(L, u)$.

LEMMA 3.1.6. *Let $L$ be an inverse commutative $\ell$-loop. Let $u \in L^+$ be a positive element satisfying the equivalent conditions of Lemma 3.1.4. If $a \in L^+$ and $a \leq k \cdot u$ for some $k \in \mathbb{N}$, then there exists a unique good sequence $(a_1, \ldots, a_k, 0, \ldots)$ of elements of the basic algebra $\Gamma(L, u)$ such that $a = a_1 + (\cdots + (a_{k-1} + a_k) \ldots)$.*

PROOF. By induction on $k \in \mathbb{N}$. There is nothing to prove where $k = 1$. Let $k \geq 2$ and suppose that the statement holds true for all $l \leq k$. Since $a \leq k \cdot u$, we have $b = a - (a \wedge u) = 0 \vee (a - u) \leq (k - 1) \cdot u$. By the induction hypothesis there exists a unique good sequence $(b_1, \dots, b_{k-1}, 0, \dots)$ such that $b = b_1 + (\dots + (b_{k-2} + b_{k-1}) \dots)$. Let $a_1 = a \wedge u$ and $a_i = b_{i-1}$ for $i \geq 2$. Then $a_1 + (a_2 + (\dots + (a_{k-1} + a_k) \dots)) = a_1 + (b_1 + (\dots + (b_{k-2} + b_{k-1}) \dots)) = a_1 + b = (a \wedge u) + (a - (a \wedge u)) = a$. Moreover, $a_1 \leq a_1 \oplus a_2 = a_1 \oplus b_1 = (a_1 + b_1) \wedge u \leq (a_1 + b) \wedge u = a \wedge u = a_1$, hence $a_1 \oplus a_2 = a_1$, showing that the sequence is $(a_1, a_2, \dots, a_k, 0, \dots)$ is good.

Uniqueness: Suppose $(x_1, \dots, x_k, 0, \dots)$, $(y_1, \dots, y_k, 0, \dots)$ are two good sequences such that $a = x_1 + (\dots + (x_{k-1} + x_k) \dots) = y_1 + (\dots + (y_{k-1} + y_k) \dots)$. Then $x_1 = x_1 \oplus x_2 = x_1 \oplus (x_2 \oplus x_3) = \dots = x_1 \oplus (\dots \oplus (x_{k-1} \oplus x_k) \dots) = (x_1 + (\dots + (x_{k-1} + x_k) \dots)) \wedge u = a \wedge u$ and, analogously, $y_1 = a \wedge u$.

Note that $a - x_1 = x_2 + (\dots + (x_{k-1} + x_k) \dots)$, $(a - x_1) - x_2 = x_3 + (\dots + (x_{k-1} + x_k) \dots)$, ..., $(\dots (a - x_1) - \dots) - x_{i-1} = x_i + (\dots + (x_{k-1} + x_k) \dots)$ for any $i < k$. Also, $(\dots (a - y_1) - \dots) - y_{i-1} = y_i + (\dots + (y_{k-1} + y_k) \dots)$.

Now, let $i \geq 2$ and suppose that $y_1 = x_1$, ..., $y_{i-1} = x_{i-1}$. Then

$$\begin{aligned}
x_i = x_i \oplus x_{i+1} = \dots &= x_i \oplus (\dots \oplus (x_{k-1} \oplus x_k) \dots) \\
&= (x_i + (\dots + (x_{k-1} + x_k) \dots)) \wedge u \\
&= ((\dots (a - x_1) - \dots) - x_{i-1}) \wedge u \\
&= ((\dots (a - y_1) - \dots) - y_{i-1}) \wedge u \\
&= (y_i + (\dots + (y_{k-1} + y_k) \dots)) \wedge u \\
&= y_i \oplus (\dots \oplus (y_{k-1} \oplus y_k)) \\
&= y_i
\end{aligned}$$

Thus we have $x_i = y_i$ for all $i \leq k$. The proof is complete. $\square$

## 3.2. From linearly ordered basic algebras to linearly ordered loops

We begin with linearly ordered (commutative) basic algebras. First, we mimic Chang's original construction of the enveloping linearly ordered Abelian group of a given linearly ordered MV-algebra (see [**17**]).

Let $A$ be a linearly ordered basic algebra. Now, we do not require that $A$ is commutative. Recall that the subtractions in $A$ are defined by $x \ominus y = \neg(y \oplus \neg x)$ and $x \oslash y = \neg(\neg x \oplus y)$. Let $\mathfrak{L}_A$ be the Cartesian product $\mathbb{Z} \times A$ where, however, for each $m \in \mathbb{Z}$, the pairs $(m, 1)$ and $(m + 1, 0)$ are identified. To be more precise, $\mathfrak{L}_A$ is the quotient set $(\mathbb{Z} \times A)/{\equiv}$ where $\equiv$ is the smallest equivalence on $\mathbb{Z} \times A$ identifying $(m, 1)$ and $(m + 1, 0)$, for all $m \in \mathbb{Z}$. Then $\mathfrak{L}_A$ bears the following natural lexicographic order:

$$(m, a) \leq (n, b) \quad \text{iff} \quad m < n \text{ or } (m = n \text{ and } a \leq b).$$

Further, we equip $\mathfrak{L}_A$ with two binary operations, $+$ and $-$, as follows:

$$(m,a)+(n,b)=\begin{cases}(m+n, a\oplus b) & \text{if } a\oplus b<1 \text{ (i.e. } \neg a>b),\\ (m+n+1, a\odot b) & \text{if } a\oplus b=1 \text{ (i.e. } \neg a\leq b),\end{cases}$$

$$(m,a)-(n,b)=\begin{cases}(m-n, a\oslash b) & \text{if } a\geq b,\\ (m-n-1, a\oplus\neg b) & \text{if } a\leq b.\end{cases}$$

Note that for $a=b$ we have $(m-n, a\oslash b)=(m-n,0)=(m-n-1,1)=(m-n-1,a\oplus\neg b)$.

First of all, we must verify that $+$ and $-$ are correctly defined (i.e., that $\equiv$ is compatible with $+$ and $-$). We have $(m,1)+(n,b)=(m+n+1,b)$ and $(m+1,0)+(n,b)=(m+n+1,b)$ if $b<1$, and $(m+1,0)+(n,1)=(m+n+2,0)=(m+n+1,1)$. Likewise, $(m,a)+(n,1)=(m+n+1,a)$ and $(m,a)+(n+1,0)=(m+n+1,a)$ if $a<1$, and $(m,1)+(n+1,0)=(m+n+2,0)=(m+n+1,1)$. Thus the definition of $+$ correct. The definition of $-$ is correct, too, because $(m,1)-(n,b)=(m-n,\neg b)$ and $(m+1,0)-(n,b)=(m-n,\neg b)$, and analogously, $(m,a)-(n,1)=(m-n-1,a)$ and $(m,a)-(n+1,0)=(m-n-1,a)$.

Now, we describe the properties of the structure $(\mathfrak{L}_A,\leq,+,-,(0,0))$ we have just defined. There is an evident similarity with linearly ordered commutative loops.

LEMMA 3.2.1. *Let $A$ be a linearly ordered basic algebra. Let $\mathfrak{L}_A$, $\leq$, $+$ and $-$ be as above. Then:*

(i) $(0,0)$ *is an identity element for the addition* $+$*;*
(ii) $((m,a)-(n,b))+(n,b)=(m,a)$ *for all* $(m,a),(n,b)\in\mathfrak{L}_A$*;*
(iii) $((m,a)+(n,b))-(n,b)=(m,a)$ *for all* $(m,a),(n,b)\in\mathfrak{L}_A$*;*
(iv) *the lexicographic order $\leq$ is right-compatible with $+$, in the sense that $(m,a)\leq(n,b)$ iff $(m,a)+(p,c)\leq(n,b)+(p,c)$, for all $(m,a),(n,b),(p,c)\in\mathfrak{L}_A$;*
(v) *for every $(m,a)\in\mathfrak{L}_A$, $(-m-1,\neg a)$ is the inverse of $(m,a)$.*

PROOF. (i) We have $(m,a)+(0,0)=(m,a)$ if $a<1$, and $(m,1)+(0,0)=(m+1,0)=(m,1)$, and also $(0,0)+(n,b)=(n,b)$ if $b<1$, and $(0,0)+(n,1)=(n+1,0)=(n,1)$.

(ii) We distinguish the following cases:

- if $1>a\geq b$, then $((m,a)-(n,b))+(n,b)=(m-n,a\oslash b)+(n,b)=(m,(a\oslash b)\oplus b)=(m,a)$ because $(a\oslash b)\oplus b=a\vee b=a<1$;
- if $1=a\geq b$, then $((m,a)-(n,b))+(n,b)=(m+n,\neg b)+(n,b)=(m-n+n+1,\neg b\odot b)=(m+1,0)=(m,1)=(m,a)$; and
- if $a\leq b$, then $((m,a)-(n,b))+(n,b)=(m-n-1,a\oplus\neg b)+(n,b)=(m,(a\oplus\neg b)\odot b)=(m,a)$ because $(a\oplus\neg b)\odot b=\neg(\neg(a\oplus\neg b)\oplus\neg b)=\neg(\neg a\vee\neg b)=a\wedge b=a$.

(iii) Here, we distinguish two cases:

- if $a\oplus b\leq 1$, then $((m,a)+(n,b))-(n,b)=(m+n,a\oplus b)-(n,b)=(m,(a\oplus b)\oslash b)=(m,a)$ because $(a\oplus b)\oslash b=\neg(\neg(a\oplus b)\oplus b)=\neg(\neg a\vee b)=a\wedge\neg b=a$, since $a\oplus b<1$ iff $\neg a\not\leq b$ iff $\neg a>b$ iff $a<\neg b$;

- if $a \oplus b = 1$, then $((m,a)+(n,b))-(n,b) = (m+n+1, a \odot b)-(n,b) = (m,(a \odot b) \oplus \neg b) = (m,a)$ because $(a \odot b) \oplus \neg b = \neg(\neg a \oplus \neg b) \oplus \neg b = a \vee \neg b = a$, since $a \oplus b = 1$ iff $\neg a \leq b$ iff $a \geq \neg b$.

(iv) It will suffice to show that $(m,a) \leq (n,b)$ implies $(m,a)+(p,c) \leq (n,b)+(p,c)$; the converse implication then directly follows from the fact that $\mathfrak{L}_A$ is linearly ordered. To this end, suppose that $(m,a) \leq (n,b)$. There are two possible cases:

- If $m = n$ and $a \leq b$, then $(m,a)+(p,c) \leq (n,b)+(p,c)$. Indeed, if $a \oplus c < 1$, then $(m,a)+(p,c) = (m+p, a \oplus c) \leq (m,b)+(p,c)$ where $(m,b)+(p,c) = (m+p, b \oplus c)$ if $b \oplus c < 1$, and $(m,b)+(p,c) = (m+p+1, b \odot c)$ if $b \oplus c = 1$, since $a \oplus c \leq b \oplus c$. If $a \oplus c = 1$, then also $b \oplus c = 1$ and we have $(m,a)+(p,c) = (m+p+1, a \odot c) \leq (m+p+1, b \odot c) = (m,b)+(p,c)$ since $a \odot c \leq b \odot c$.
- Let $m < n$. If $a \oplus c = 1$ but $b \oplus c < 1$, then $(m,a)+(p,c) = (m+p+1, a \odot c)$ while $(n,b)+(p,c) = (n+p, b \oplus c)$. In this case, $m+p+1 \leq n+p$ and $a \odot c \leq c \leq b \oplus c$, thus $(m,a)+(p,c) \leq (n,b)+(m,c)$. If $a \oplus c < 1$, then one readily sees that $(m,a)+(p,c) = (m+p, a \oplus c) \leq (n,b)+(p,c)$ where $(n,b)+(p,c) = (n+p, b \oplus c)$ if $b \oplus c < 1$, and $(n,b)+(p,c) = (n+p+1, b \odot c)$ if $b \oplus c = 1$.

(v) This is an easy consequence of the definition of $+$. Indeed, $(m,a)+(-m-1, \neg a) = (0, a \odot \neg a) = (0,0)$ and $(-m-1, \neg a)+(m,a) = (0, \neg a \odot a) = (0,0)$. □

THEOREM 3.2.2. *Let $A$ be a linearly ordered commutative basic algebra. Then $\mathfrak{L}_A$ is a linearly ordered commutative inverse loop and $A$ is isomorphic to the basic algebra $\Gamma(\mathfrak{L}_A, (1,0))$. Moreover, $(1,0) = (0,1)$ is a strong order-unit for $\mathfrak{L}_A$ and we have*

$$((1,0)+(m,a))+(n,b) = (1,0)+((m,a)+(n,b))$$

*for all $(m,a),(n,b) \in \mathfrak{L}_A$ (i.e., $(1,0)$ satisfies the equivalent conditions of Lemma 3.1.4).*

PROOF. Since $A$ is commutative, it is clear by the previous lemma that $\mathfrak{L}_A$ is a linearly ordered commutative inverse loop. The map $\varphi\colon a \mapsto (0,a)$ is a bijection from $A$ onto the interval $[(0,0),(1,0)]$ of $\mathfrak{L}_A$. In order to prove that $\varphi$ is an isomorphism between the basic algebras $A$ and $\Gamma(\mathfrak{L}_A,(1,0))$, since $\varphi(0) = (0,0)$ and $\varphi(1) = (0,1) = (1,0)$, it suffices to show that $\varphi$ preserves the operators $\ominus = \oslash$. To this end, let $a,b \in A$. If $a \leq b$, then $\varphi(a \ominus b) = \varphi(0) = (0,0)$ since $(0,a) \leq (0,b)$. Also, if $a \geq b$, then $\varphi(a \ominus b) = (0, a \ominus b)$ and $\varphi(a) \ominus \varphi(b) = (0,a) \ominus (0,b) = (0,a)-(0,b) = (0, a \ominus b)$. Therefore, $A \cong \Gamma(\mathfrak{L}_A,(1,0))$.

The pair $(0,1) = (1,0)$ is a strong order-unit for $\mathfrak{L}_A$ because for every $(m,a) \in \mathfrak{L}_A$, if $(0,0) \leq (m,a)$, then $m \geq 0$ and $(m,a) \leq (m,1) = (m+1,0) = (\dots((1,0)+(1,0))+\dots)+(1,0)$, with $m+1$ occurrences of $(1,0)$.

Lastly, if $a \oplus b < 1$, thus $((0,1)+(m,a))+(n,b) = (m+1,a)+(n,b) = (m+n+1, a \oplus b)$ and $(1,0)+((m,a)+(n,b)) = (1,0)+(m+n, a \oplus b) = (m+n+1, a \oplus b)$, and analogously, if $a \oplus b = 1$, then $((0,1)+(m,a))+(n,b) = (m+n+2, a \odot b)$ and $(0,1)+((m,a)+(n,b)) = (0,1)+(m+n+1, a \odot b) = (m+n+2, a \odot b)$. Thus $((1,0)+(m,a))+(n,b) = (1,0)+((m,a)+(n,b))$ for all $(m,a),(n,b) \in \mathfrak{L}_A$. □

Note that (owing to the satisfaction of the conditions of Lemma 3.1.4) the addition $\oplus$ in $\Gamma(\mathfrak{L}_A, (1,0))$ is given by the rule

$$(0,a) \oplus (0,b) = \begin{cases} (0,a) + (0,b) = (0, a \oplus b) & \text{if } a \oplus b < 1, \\ (1,0) = (0,1) & \text{if } a \oplus b = 1. \end{cases}$$

By a *semilinear* (or *representable*) commutative $\ell$-loop we mean a commutative $\ell$-loop which is isomorphic to a subdirect product of linearly ordered commutative loops.

COROLLARY 3.2.3. *Let $A$ be semilinear commutative basic algebra. There exists semilinear commutative inverse $\ell$-loop $L$ such that $A$ can be embedded into the algebra $\Gamma(L,u)$ for some positive element $u \in L^+$.*

PROOF. Let $A$ be a subdirect product of $\{A_i \mid i \in I\}$ where each $A_i$ is a linearly ordered commutative basic algebra. Let $L_i = \mathfrak{L}_{A_i}$ where $\mathfrak{L}_{A_i}$ is constructed as above. Then $A_i$ is isomorphic to $\Gamma(L_i, u_i)$ where $u_i$ is a strong order-unit for $L_i$. If we put $L = \prod_{i\in I} L_i$, $0 = (0_i \mid i \in I)$ and $u = (u_i \mid i \in I)$, then $A$ can be embedded into $\Gamma(L,u) \cong \prod_{i \in I} A_i$. $\square$

## 3.3. From basic algebras to $\ell$-loops—good functions

We have seen that every semilinear commutative basic algebra $A$ is isomorphic to a subalgebra of $\Gamma(L,u)$ for a suitable inverse commutative $\ell$-loop $L$ and a positive element $u \in L^+$. The goal for the present section is to construct $L$ such that $A \cong \Gamma(L,u)$ where $u$ is a strong order-unit from $L$. It would be possible to use good sequences just as in MV-algebras (cf. [**18, 30**]), but the disadvantage of this approach is that it only leads to the positive cone of $L$ and $L$ need not be fully determined by $L^+$. Instead, we present another construction which directly leads to the $\ell$-loop $L$.

First, a technical lemma:

LEMMA 3.3.1. *In any basic algebra $A$, we have:*

(i) *$x \oplus y = y$ iff $\neg x \vee y = 1$ iff $\neg y \oplus \neg x = \neg x$;*

(ii) *if $x \oplus y = y$, then $a \oplus y = y$ for all $a \leq x$ and $x \oplus b = b$ for all $b \geq y$.*

*If $A$ is distributive, then*

(iii) *$x \oplus z = y \oplus z$ and $x \odot z = y \odot z$ imply $x = y$.*

*Moreover, if $A$ is linearly ordered, then*

(iv) *$x \oplus y = y$ iff $x = 0$ or $y = 1$;*

(v) *for all $x, y \in A$, $x \oplus y = 1$ or $x \odot y = 0$;*

(vi) *if $x \oplus z = y \oplus z < 1$, then $x = y$.*

PROOF. (i) If $x \oplus y = y$, then $\neg x \vee y = \neg(x \oplus y) \oplus y = \neg y \oplus y = 1$. Conversely, if $\neg x \vee y = \neg(x \oplus y) \oplus y = 1$, then $x \oplus y \leq y$, so $x \oplus y = y$. Clearly, $\neg y \oplus \neg x = \neg x$ is equivalent to $\neg x \vee y = 1$.

(ii) Let $x \oplus y = y$. If $a \leq x$, then $y \leq a \oplus y \leq x \oplus y = y$, thus $a \oplus y = y$. If $b \geq y$, then $1 = \neg x \vee y \leq \neg x \vee b$, so $\neg x \vee b = 1$, which is equivalent to $x \oplus b = b$ by (i).

Now, let $A$ be a distributive lattice. If $x \oplus z = y \oplus z$ and $x \odot z = y \odot z$, then $x \vee \neg z = \neg(\neg x \oplus \neg z) \oplus \neg z = (x \odot z) \oplus \neg z = (y \odot z) \oplus \neg z = y \vee \neg z$ and also $x \wedge \neg z =$

$\neg(\neg x \vee z) = \neg(\neg(x \oplus z) \oplus z) = \neg(\neg(y \oplus z) \oplus z) = y \wedge \neg z$. Distributivity entails $x = y$. We have established (iii).

Lastly, suppose that $A$ is linearly ordered.

(iv) By (i), $x \oplus y = y$ iff $\neg x \vee y = 1$, which is possible any if $\neg x = 1$ or $y = 1$, so $x = 0$ or $y = 1$.

(v) Suppose that $x \oplus y \neq 1$. Then $\neg x \not\leq y$, so $\neg x > y$, or equivalently, $x \leq \neg y$. But then $\neg x \oplus \neg y = 1$ and so $x \odot y = \neg(\neg x \oplus \neg y) = 0$ as required.

(vi) Let $x \oplus z = y \oplus z < 1$. Then by (v) we have $x \odot z = y \odot z$, and hence by (iii) we conclude that $x = y$. □

Now, we define the central concept of this section:

Definition 3.3.2. Let $A$ be a basic algebra. A *good function* in $A$ is a function $\mathfrak{g}\colon \mathbb{Z} \to A$ such that

(i) $\mathfrak{g}(i+1) \oplus \mathfrak{g}(i) = \mathfrak{g}(i)$ for all $i \in \mathbb{Z}$, and
(ii) there exist $k, l \in \mathbb{Z}$ with $k \leq l$ such that $\mathfrak{g}(i) = 1$ for all $i < k$ and $\mathfrak{g}(j) = 0$ for all $j > l$.

Such a good function can be visualized as $(\dots, 1, \mathfrak{g}(k), \dots, \mathfrak{g}(l), 0, \dots)$. We will often write $\mathfrak{g} = (\mathfrak{g}(k), \dots, \mathfrak{g}(l))_k$, i.e. $\mathfrak{g} = (a_1, \dots, a_m)_k$ means that $\mathfrak{g}(i) = 1$ for $i < k$, $\mathfrak{g}(k) = a_1, \dots, \mathfrak{g}(k+m-1) = a_m$ and $\mathfrak{g}(j) = 0$ for $j > k+m-1$.

Note that $\mathfrak{g}(i) = 0$ yields $\mathfrak{g}(j) = 0$ for all $j > i$, and $\mathfrak{g}(j) = 1$ yields $\mathfrak{g}(i) = 1$ for all $i < j$.

Remark 3.3.3. In **[2]**, the concept of a good sequence on an MV-algebra is defined essentially in the same way as our good functions, except that the condition (ii) requires the existence of $n \in \mathbb{N}$ such that $\mathfrak{g}(i) = 0$ for all $i \geq n$ and $\mathfrak{g}(i) = 1$ for all $i < -n$. Also the operations with good functions that we define below are basically the same as those in **[2]**.

Lemma 3.3.4. *If $A$ is linearly ordered, then all good functions in $A$ are of the form $(a)_k$, for some $a \in A$ and $k \in \mathbb{Z}$.*

Proof. Let $\mathfrak{g}\colon \mathbb{Z} \to A$ be a good function in $A$. For each $i \in \mathbb{Z}$ we have $\mathfrak{g}(i+1) \oplus \mathfrak{g}(i) = \mathfrak{g}(i)$, which is equivalent to $\mathfrak{g}(i+1) = 0$ or $\mathfrak{g}(i) = 1$, by Lemma 3.3.1 (iv). Hence $\mathfrak{g}$ must be of the form $\mathfrak{g} = (a)_k$ for some $a \in A$ and $k \in \mathbb{Z}$. □

Proposition 3.3.5. *Let $A$ be a subdirect product of basic algebras $A_t$ $(t \in T)$. Then a function $\mathfrak{g}\colon \mathbb{Z} \to A$ is a good function iff*

(i) *for every $t \in T$, the projection of $\mathfrak{g}$, $\pi_t(\mathfrak{g})\colon i \mapsto \pi_t(\mathfrak{g}(i))$, is a good function in $A_t$,*
(ii) *there exist $k, l \in \mathbb{Z}$ with $k \leq l$ such that $\pi_t(\mathfrak{g}(i)) = 1$ and $\pi_t(\mathfrak{g}(j)) = 0$ for all $t \in T$ and $i, j \in \mathbb{Z}$ with $i < k$ and $j > l$.*

Proof. This is obvious since $\mathfrak{g}(i+1) \oplus \mathfrak{g}(i) = \mathfrak{g}(i)$ iff $\pi_t(\mathfrak{g}(i+1)) \oplus \pi_t(\mathfrak{g}(i)) = \pi_t(\mathfrak{g}(i))$ for all $t \in T$. □

Notation. In what follows, in order to simplify the notation, we let $\bigoplus_{i=1}^{n} x_i$ denote the 'left sum' $x_1 \oplus (\cdots \oplus (x_{n-1} \oplus x_n) \dots)$.

Now, let $A$ be a basic algebra. Given $\mathfrak{a} = (a_1, \dots, a_m)_k$ and $\mathfrak{b} = (b_1, \dots, b_n)_l$ two good functions in $A$, we define the sum of $\mathfrak{a}$ and $\mathfrak{b}$ as the function $\mathfrak{a} + \mathfrak{b} = (c_1, c_2, \dots)_{k+l}$ where

$$c_i = \bigoplus_{j=0}^{i} (a_{i-j} \odot b_j)$$

with $a_0 = b_0 = 1$. Thus the $c_i$'s are given by $c_1 = a_1 \oplus b_1$, $c_2 = a_2 \oplus ((a_1 \odot b_1) \oplus b_2)$, $c_3 = a_3 \oplus [(a_2 \odot b_1) \oplus ((a_1 \odot b_2) \oplus b_3)]$, etc.

Note that $c_i = 0$ for $i > m + n$ because if $i = (i - j) + j > m + n$, then $i - j > m$ or $j > n$, so $a_{i-j} = 0$ or $b_j = 0$, thus $a_{i-j} \odot b_j = 0$ and hence $c_i = 0$. Hence we can write $\mathfrak{a} + \mathfrak{b} = (c_1, \dots, c_{m+n})_{k+l}$, and the function $\mathfrak{a} + \mathfrak{b}$ fulfills the condition (ii) from the definition of good functions.

Also note that if $\mathfrak{a} = (1, \dots, 1, a_{r+1}, \dots, a_m)_k$ and $\mathfrak{b} = (1, \dots, 1, b_{s+1}, \dots, b_n)_l$, then $c_i = 1$ for $i \le r + s$. Indeed, if $i \le r + s$, then $i = p + q$ where $0 \le p \le r$ and $0 \le q \le s$; thus $a_p = 1 = b_q$, so $a_p \odot b_q = 1$ and hence $c_i = \bigoplus_{j=0}^{i} (a_{i-j} \odot b_j) = 1$.

By (i) of Lemma 3.3.1, if $\mathfrak{a} = (a_1, \dots, a_m)_k$ is a good function in $A$, then so is the function $(\neg a_m, \dots, \neg a_1)_l$, for any $l \in \mathbb{Z}$. Hence, for any good function $\mathfrak{a} = (a_1, \dots, a_m)_k$, we can define the good function $\mathfrak{a}^\sharp = (\neg a_m, \dots, \neg a_1)_{-k-m}$.

LEMMA 3.3.6. *Let $A$ be a subdirect product of linearly ordered basic algebras $A_t$ ($t \in T$). Let $\mathfrak{a} = (a_1, \dots, a_m)_k$ and $\mathfrak{b} = (b_1, \dots, b_n)_l$ be good functions in $A$. If the $t$-th projections of $\mathfrak{a}$ and $\mathfrak{b}$ are, respectively, $\pi_t(\mathfrak{a}) = (a_t)_{k_t}$ and $\pi_t(\mathfrak{b}) = (b_t)_{l_t}$, then the $t$-th projection of $\mathfrak{a} + \mathfrak{b}$ is $\pi_t(\mathfrak{a} + \mathfrak{b}) = (a_t \oplus b_t, a_t \odot b_t)_{k_t + l_t}$, and the $t$-th projection of $\mathfrak{a}^\sharp$ is $\pi_t(\mathfrak{a}^\sharp) = (\neg a_t)_{-k_t - 1}$.*

PROOF. We have

$$\pi_t(\mathfrak{a}) = (\pi_t(a_1), \dots, \pi_t(a_p), \dots, \pi_t(a_m))_k = (1, \dots, 1, a_t, 0, \dots, 0)_k = (a_t)_{k_t}$$

where $k_t = k + p - 1$, and

$$\pi_t(\mathfrak{b}) = (\pi_t(b_1), \dots, \pi_t(b_q), \dots, \pi_t(b_n))_k = (1, \dots, 1, b_t, 0, \dots, 0)_l = (b_t)_{l_t}$$

where $l_t = l + q - 1$.

By definition, $\mathfrak{a} + \mathfrak{b} = (c_1, \dots, c_{m+n})_{k+l}$ where $c_i = \bigoplus_{j=0}^{i} (a_{i-j} \odot b_j)$. Thus $\pi_t(\mathfrak{a} + \mathfrak{b}) = (\pi_t(c_1), \dots, \pi_t(c_{m+n}))_{k+l}$ where $\pi_t(c_i) = \bigoplus_{j=0}^{i} (\pi_t(a_{i-j}) \odot \pi_t(b_j))$ for all $i = 1, 2, \dots$. But $\pi_t(a_p) = a_t$, $\pi_t(b_q) = b_t$, $\pi_t(a_i) = \pi_t(b_j) = 1$ for $i < p$ and $j < q$, and likewise $\pi_t(a_i) = \pi_t(b_j) = 0$ for $i > p$ and $j > q$, and hence $\pi_t(\mathfrak{a} + \mathfrak{b}) = (1, \dots, 1, a_t \oplus b_t, a_t \odot b_t)_{k+l} = (a_t \oplus b_t, a_t \odot b_t)_{k_t + l_t} = (a_t)_{k_t} + (b_t)_{l_t} = \pi_t(\mathfrak{a}) + \pi_t(\mathfrak{b})$.

Furthermore, recalling that $\pi_t(\mathfrak{a}) = (1, \dots, 1, \pi_t(a_p), 0, \dots, 0)_k = (a_t)_{k_t}$ with $k_t = k + p - 1$, we get $\pi_t(\mathfrak{a}^\sharp) = (\neg \pi_t(a_m), \dots, \neg \pi_t(a_p), \dots, \neg \pi_t(a_1))_{-k-m} = (1, \dots, 1, \neg \pi_t(a_p), 0, \dots \dots, 0)_{-k-m} = (\neg \pi_t(a_p))_{-k_t - 1} = (\neg a_t)_{-k_t - 1}$ because $k_t = k + p - 1$ yields $-k_t - 1 = -k - p$. □

PROPOSITION 3.3.7. *Let $A$ be a semilinear basic algebra. The sum $\mathfrak{a} + \mathfrak{b}$ of two good functions $\mathfrak{a}, \mathfrak{b}$ in $A$ is a good function in $A$, too.*

PROOF. First, suppose that the algebra $A$ is linearly ordered. Then $\mathfrak{a} = (a)_k$ and $\mathfrak{b} = (b)_l$ for some $a, b \in A$ and $k, l \in \mathbb{Z}$. It is evident that $\mathfrak{a} + \mathfrak{b} = (a \oplus b, a \odot b)_{k+l}$. Since $A$ is linearly ordered, we have $a \oplus b = 1$ or $a \odot b = 0$, thus $\mathfrak{a} + \mathfrak{b}$ is a good function.

Second, let $A$ be a subdirect product of linearly ordered basic algebras $A_t$ $(t \in T)$. The sum $\mathfrak{a} + \mathfrak{b} = (c_1, \dots, c_{m+n})_{k+l}$ of $\mathfrak{a} = (a_1, \dots, a_m)_k$ and $\mathfrak{b} = (b_1, \dots, b_n)_l$ satisfies the condition (ii) of Proposition 3.3.5. For every $t \in T$, the $t$-th projection of $\mathfrak{a} + \mathfrak{b}$ is $(a_t \oplus b_t, a_t \odot b_t)_{k_t+l_t}$, which is good function in $A_t$. Thus, by Proposition 3.3.5 we conclude that $\mathfrak{a} + \mathfrak{b}$ is a good function in $A$. □

LEMMA 3.3.8. *Let $A$ be a semilinear basic algebra. If $\mathfrak{a}, \mathfrak{b}$ are good functions in $A$, then*

(i) $\mathfrak{a} + (0)_0 = \mathfrak{a} = (0)_0 + \mathfrak{a}$,
(ii) $\mathfrak{a} + \mathfrak{a}^\sharp = (0)_0 = \mathfrak{a}^\sharp + \mathfrak{a}$,
(iii) $(\mathfrak{a} + \mathfrak{b}) + \mathfrak{b}^\sharp = \mathfrak{a}$,
(iv) $(\mathfrak{a} + \mathfrak{b}^\sharp) + \mathfrak{b} = \mathfrak{a}$, *and*
(v) $((1)_0 + \mathfrak{a}) + \mathfrak{b} = (1)_0 + (\mathfrak{a} + \mathfrak{b})$.

PROOF. If $A$ is a subdirect product of linearly ordered basic algebras $A_t$ $(t \in T)$, then in view of Proposition 3.3.5 it suffices to verify the equalities (i)–(v) for each projection $\pi_t$ of $A$ onto $A_t$. Therefore, we may assume that $A$ is linearly ordered and $\mathfrak{a}, \mathfrak{b}$ are of the form $\mathfrak{a} = (a)_k$ and $\mathfrak{b} = (b)_l$. Then $\mathfrak{a} + \mathfrak{b} = (a \oplus b, a \odot b)_{k+l}$, $\mathfrak{a}^\sharp = (\neg a)_{-k-1}$ and $\mathfrak{b}^\sharp = (\neg b)_{-l-1}$.

(i) Obviously, $(a)_k + (0)_0 = (a)_k = (0)_0 + (a)_k$.
(ii) We have $(a)_k + (\neg a)_{-k-1} = (a \oplus \neg a, a \odot \neg a)_{-1} = (1, 0)_{-1} = (0)_0$ and $(\neg a)_{-k-1} + (a)_k = (\neg a \oplus a, \neg a \odot a)_{-1} = (1, 0)_{-1} = (0)_0$.
(iii) $((a)_k + (b)_l) + (\neg b)_{-l-1} = (a \oplus b, a \odot b)_{k+l} + (\neg b)_{-l-1} = (c_1, c_2, c_3)_{k-1}$ where $c_1 = (a \oplus b) \oplus \neg b = 1$, $c_2 = (a \odot b) \oplus ((a \oplus b) \odot \neg b) = a$ because in linearly ordered basic algebras either $a \odot b = 0$ if $a \leq \neg b$ or $a \oplus b = 1$ if $a \geq \neg b$, and $c_3 = (a \odot b) \odot \neg b = 0$. Thus $(c_1, c_2, c_3)_{k-1} = (1, a, 0)_{k-1} = (a)_k$.
(iv) This actually follows from (ii) and (iii) because $\mathfrak{b}^{\sharp\sharp} = \mathfrak{b}$.
(v) We have $((1)_0 + (a)_k) + (b)_l = (1, a)_k + (b)_l = (a)_{k+1} + (b)_l = (a \oplus b, a \odot b)_{k+l+1}$, and on the other hand, $(1)_0 + ((a)_k + (b)_l) = (1)_0 + (a \oplus b, a \odot b)_{k+l} = (1, (a \oplus b) \oplus (a \odot b), a \odot b)_{k+l} = (a \oplus b, a \odot b)_{k+l+1}$ because $(a \oplus b) \oplus (a \odot b) = a \oplus b$, which follows from (v) of Lemma 3.3.1 (we have either $a \oplus b = 1$ or $a \odot b = 0$).

□

In what follows, given a commutative basic algebra $A$, we use $\mathfrak{G}_A$ to denote the set of good functions in $A$.

COROLLARY 3.3.9. *If $A$ is a semilinear commutative basic algebra, then the algebra $(\mathfrak{G}_A, +, -, (0)_0)$, where $\mathfrak{a} - \mathfrak{b} = \mathfrak{a} + \mathfrak{b}^\sharp$, is an inverse commutative loop.*

LEMMA 3.3.10. *If $A$ is semilinear, then $\mathfrak{G}_A$ is a lattice with the respect to the point-wise ordering; the joins and meets in $\mathfrak{G}_A$ are point-wise, too.*

PROOF. Let $A$ be a subdirect product of linearly ordered basic algebras $A_t$ $(t \in T)$. Let $\mathfrak{a}, \mathfrak{b} \in \mathfrak{G}_A$. It suffices to show that the functions $\mathfrak{a} \vee \mathfrak{b} = (\dots, \mathfrak{a}(i) \vee \mathfrak{b}(i), \dots)$ and

$\mathfrak{a} \wedge \mathfrak{b} = (\ldots, \mathfrak{a}(i) \wedge \mathfrak{b}(i), \ldots)$ are good. To this end, it suffices to show that each projection is a good function in $A_t$. Let $\pi_t(\mathfrak{a}) = (a_t)_{k_t}$ and $\pi_t(\mathfrak{b}) = (b_t)_{l_t}$. We have $\pi_t(\mathfrak{a} \vee \mathfrak{b}) = (\ldots, \pi_t(\mathfrak{a}(i)) \vee \pi_t(\mathfrak{b}(i)), \ldots)$, $\pi_t(\mathfrak{a} \wedge \mathfrak{b}) = (\ldots, \pi_t(\mathfrak{a}(i)) \wedge \pi_t(\mathfrak{b}(i)), \ldots)$, $\pi_t(\mathfrak{a}) = (a_t)_{k_t} = (\ldots, 1, \pi_t(\mathfrak{a}(k_t)), 0, \ldots)$ and $\pi_t(\mathfrak{b}) = (b_t)_{l_t} = (\ldots, 1, \pi_t(\mathfrak{b}(l_t)), 0, \ldots)$. If $k_t < l_t$, then $(a_t)_{k_t} \vee (b_t)_{l_t} = (b_t)_{l_t}$ and $(a_t)_{k_t} \wedge (b_t)_{l_t} = (a_t)_{k_t}$. If $k_t = l_t$ and $a_t \leq b_t$ (or $a_t \geq b_t$), then $(a_t)_{k_t} \vee (b_t)_{l_t} = (b_t)_{k_t}$ and $(a_t)_{k_t} \wedge (b_t)_{l_t} = (a_t)_{k_t}$ (or $(a_t)_{k_t} \vee (b_t)_{l_t} = (a_t)_{k_t}$ and $(a_t)_{k_t} \wedge (b_t)_{l_t} = (b_t)_{k_t}$, respectively). In any case, $\pi_t(\mathfrak{a} \vee \mathfrak{b}) = (a_t)_{k_t} \vee (b_t)_{l_t}$ and $\pi_t(\mathfrak{a} \wedge \mathfrak{b}) = (a_t)_{k_t} \wedge (b_t)_{l_t}$ are good functions in $A_t$. □

Note that $\mathfrak{a} \leq \mathfrak{b}$ iff $\mathfrak{a}(i) \leq \mathfrak{b}(i)$ for all $i \in \mathbb{Z}$ iff $\pi_t(\mathfrak{a}) \leq \pi_t(\mathfrak{b})$ for all $t \in T$. Also, if $A$ is linearly ordered, then $(a)_k \leq (b)_l$ iff $k < l$ or ($k = l$ and $a \leq b$); thus $\mathfrak{G}_A$ is linearly ordered provided that $A$ is linearly ordered.

Now, we can state our main theorem.

THEOREM 3.3.11. *Let $A$ be a semilinear commutative basic algebra. Then the structure $(\mathfrak{G}_A, \vee, \wedge, +, -, (0)_0)$ is a lattice-ordered commutative inverse loop, and the basic algebra $A$ is isomorphic to $\Gamma(\mathfrak{G}_A, (1)_0)$, where $(1)_0$ is a strong order-unit for $\mathfrak{G}_A$.*

PROOF. We have already seen that $\mathfrak{G}_A$ is both a lattice and a commutative inverse loop, so there remain to show that the point-wise ordering is compatible with $+$. Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathfrak{G}_A$. If $\mathfrak{a} \leq \mathfrak{b}$, then $\pi_t(\mathfrak{a}) = (a_t)_{k_t} \leq (b_t)_{l_t} = \pi_t(\mathfrak{b})$ for each $t \in T$; that is, $k_t < l_t$ or ($k_t = l_t$ and $a_t \leq b_t$). If $\pi_t(\mathfrak{c}) = (c_t)_{p_t}$, then $(a_t)_{k_t} + (c_t)_{p_t} = (a_t \oplus c_t, a_t \odot c_t)_{k_t+p_t} \leq (b_t \oplus c_t, b_t \odot c_t)_{l_t+p_t} = (b_t)_{l_t} + (c_t)_{p_t}$. Indeed, first, if $k_t < l_t$, then the only possibility which is not obvious at first glance is that the left hand side is $(1, a_t \odot c_t)_{k_t+p_t} = (a_t \odot c_t)_{k_t+p_t+1}$ while the right-hand side is $(b_t \oplus c_t)_{l_t+p_t}$ where $k_t + p_t + 1 = l_t + p_t$. But in this case the inequality holds, too, because $a_t \odot c_t \leq c_t \leq b_t \oplus c_t$. Second, if $k_t = l_t$ and $a_t \leq b_t$, then this case is impossible since $1 = a_t \oplus c_t \leq b_t \oplus c_t$ implies $b_t \oplus c_t = 1$, so the right-hand side is $(b_t \odot c_t)_{k_t+p_t+1}$.

We have just shown that $\mathfrak{a} \leq \mathfrak{b}$ implies $\mathfrak{a} + \mathfrak{c} \leq \mathfrak{b} + \mathfrak{c}$, because this is true in all projections. Also conversely, if $\mathfrak{a} + \mathfrak{c} \leq \mathfrak{b} + \mathfrak{c}$, then $\mathfrak{a} \leq \mathfrak{b}$ since for every projection $\pi_t$, $\pi_t(\mathfrak{a}) + \pi_t(\mathfrak{c}) \leq \pi_t(\mathfrak{b}) + \pi_t(\mathfrak{c})$ implies $\pi_t(\mathfrak{a}) \leq \pi_t(\mathfrak{b})$ because $\mathfrak{G}_{A_t}$ is a linearly ordered commutative loop.

As for the latter statement, it is clear that the map $\varphi\colon a \mapsto (a)_0$ is an order-preserving bijection from $A$ onto the interval $[(0)_0, (1)_0]$ of $\mathfrak{G}_A$. Moreover, $\varphi(\neg a) = (\neg a)_0$ and $\neg\varphi(a) = \neg(a)_0 = (1)_0 - (a)_0 = (1)_0 + (\neg a)_{-1} = (1, \neg a)_{-1} = (\neg a)_0$, so $\varphi(\neg a) = \neg\varphi(a)$, and finally, $\varphi(a) \oplus \varphi(b) = ((a)_0 + (b)_0) \wedge (1)_0 = (a \oplus b, a \odot b)_0 \wedge (1)_0 = (a \oplus b, 0)_0 = (a \oplus b)_0 = \varphi(a \oplus b)$. Therefore, $\varphi\colon a \mapsto (a)_0$ is isomorphism between the basic algebras $A$ and $\Gamma(\mathfrak{G}_A, (1)_0)$ as claimed.

That $(1)_0$ is strong order-unit for $\mathfrak{G}_A$ follows from the fact that $m \cdot (1)_0 = (1)_0 + [\cdots + ((1)_0 + (1)_0) \ldots] = (1)_{m-1}$, thus if $\mathfrak{a} = (a_1, \ldots, a_m)_k$ is a positive element of $\mathfrak{G}_A$ (i.e. $k \geq 0$), then $\mathfrak{a} \leq (1, \ldots, 1)_k = (1)_{k+m-1} = (k + m) \cdot (1)_0$. □

## 3.4. Lexicographic products

In a sense, commutative (and also monotone) basic algebras are very similar to MV-algebras, but they are not the same, though it is no so easy to find a *proper* commutative basic algebra which is not an MV-algebra. The first example was given in [**4**]; roughly speaking, Botur's algebra was constructed by 'deforming' the addition $x \oplus y = \min(x+y, 1)$ in the standard MV-algebra $[0,1]_{\mathrm{MV}}$. In [**5**], this method was generalized and it was proved that there exist subdirectly irreducible linearly ordered proper commutative basic algebras of any infinite cardinality.

In this section, we present a new class of examples of proper commutative basic algebras; our construction is based on lexicographic products $\mathbb{Z} \vec{\times} L$ where $L$ is a commutative $\ell$-loop.

Let $L$ be a commutative $\ell$-loop and let $\mathbb{Z}$ be the additive group of integers with the usual linear order. Certainly, the direct product $\mathbb{Z} \times L$ of $\mathbb{Z}$ and $L$ as loops is a commutative loop; it is an inverse one if $L$ is an inverse loop. The *lexicographic product* $\mathbb{Z} \vec{\times} L$ is the direct product $\mathbb{Z} \times L$ equipped with the lexicographic order

$$(m, a) \leq (n, b) \quad \text{iff} \quad m < n \text{ or } (m = n \text{ and } a \leq b).$$

It is easy to see that $\mathbb{Z} \vec{\times} L$ is an $\ell$-loop in which $(m, a) \vee (n, b) = (m, a)$ if $m > n$, $(m, a) \vee (n, b) = (n, b)$ if $m < n$, and $(m, a) \vee (m, b) = (m, a \vee b)$, and $(m, a) \wedge (n, b)$ is given dually.

Moreover, for any $k \in \mathbb{N}$, $(k, 0)$ is a strong order-unit satisfying the conditions (a) and (b) of Lemma 3.1.4. Indeed, it is plain that $(k, 0) + ((m, a) + (n, b)) = (k + m + n, a + b) = ((k, 0) + (m, a)) + (n, b)$ for all $(m, a), (n, b) \in \mathbb{Z} \vec{\times} L$, and if $(m, a)$ is a positive element in $\mathbb{Z} \vec{\times} L$, then $(m, a) \leq (m + 1) \cdot (k, 0)$. Hence, if $L$ is an inverse loop, then the basic algebra $\Gamma(\mathbb{Z} \vec{\times} L, (k, 0))$ is commutative.

Example 3.4.1 (cf. Example 3.1.3). Let $\mathbb{R}^+$, the set of all reals $\geq 0$, be equipped with the following operations:

$$x \circ y = x + y + \min(x, y) \quad \text{and} \quad x \div y = \begin{cases} x - 2y & \text{if } y \leq \frac{x}{3}, \\ \frac{1}{2}(x - y) & \text{if } \frac{x}{3} \leq y \leq x, \\ 0 & \text{if } x \leq y. \end{cases}$$

That is, $x \div y = \max(x \sim y, 0)$ where $\sim$ is defined in Example 3.1.3. Then the structure $(\mathbb{R}^+, \leq, \circ, \div, 0)$ is a commutative *positive divisibility semiloop* in the sense of Bosbach [**3**] (also see Appendix), and hence, by [**3**, Prop. 2.11], it is the positive cone a linearly ordered commutative loop, for instance, of the linearly ordered commutative loop $(\mathbb{R}, \leq, \circ, \sim, 0)$ from Example 3.1.3. In addition, by [**3**, Prop. 2.14], it is the positive cone of a *unique* linearly ordered commutative *inverse* loop, say $(L, \leq, \circ, \sim, 0)$. We don't need to recall the entire construction of $L$, but it is worth observing that in this linearly ordered commutative loop $L$ we have $x \circ y = x + y + \max(x, y)$ for $x, y \in \mathbb{R}^-$ (here, $\mathbb{R}^-$ is the set of all reals $\leq 0$).

Now, the lexicographic product $\mathbb{Z} \vec{\times} L$ is a linearly ordered commutative inverse loop and the basic algebra $A = \Gamma(\mathbb{Z} \vec{\times} L, (1, 0))$ is commutative. Note that $(0, a) \oplus (0, b) = (0, a \circ b)$, while $(1, a) \oplus (1, b) = (1, 0)$.

The structure of the basic algebra $A$ can of course be described in terms of antitone involutions. We have $A = \{(0, a) \mid a \in \mathbb{R}^+\} \cup \{(1, a) \mid a \in \mathbb{R}^-\}$. The antitone involutions

$\delta_{(k,a)}\colon (n,x) \mapsto (1,0) \ominus (n,x)$, for $(k,a) \in A$, are given as follows:

$$\delta_{(k,a)}(n,x) = \begin{cases} (0, a \div x) & \text{for } k = n = 0 \text{ and } 0 \leq x \leq a, \\ (0, (-x) \div (-a)) & \text{for } k = n = 1 \text{ and } x \leq a \leq 0, \\ (1, a \circ (-x)) & \text{for } k = 1, n = 0 \text{ and } a \leq 0 \leq x. \end{cases}$$

CHAPTER 4

# Derivations on basic algebras

In the last decade, there have been several papers about derivations on MV-algebras and other related algebras; see [**1**,**22**,**38**]. The concept has been inspired by derivations on rings, hence a derivation on an MV-algebra $A$ is a map $d\colon A \to A$ satisfying $d(x \oplus y) = d(x) \oplus d(y)$ and $d(x \odot y) = (d(x) \odot y) \oplus (x \odot d(y))$ for all $x, y \in A$. Sometimes, by a derivation is meant a map satisfying the latter condition only, while a map that satisfies both conditions is called an additive derivation. In this short note, we give a complete characterization of (additive) derivations on MV-algebras. Actually, we prove the result for derivations on basic algebras.

Adopting the definition from MV-algebras, we prove that every derivation $d$ on a basic algebra $A$ is of the form $d\colon x \mapsto x \wedge e$ where $e = d(1)$. In fact, $d$ is a homomorphism onto the interval basic algebra $[0, e]$ which has the property that all its elements are sharp (idempotent) in $A$. In some particular cases, the element $e$ is even central, so the algebra $A$ is isomorphic to the direct product $[0, e] \times [0, \neg e]$ and, roughly speaking, $d$ is the 'projection' onto $(x, y) \mapsto (x, 0)$. On the other hand, if $a \in A$ is such that the map $f\colon x \mapsto x \wedge a$ is a homomorphism from $A$ onto $[0, a]$ and all elements of the interval algebra are sharp in $A$, then $f$ is a derivation on the basic algebra $A$.

## 4.1. Sharp and central elements

An element $a$ of a basic algebra $A$ is said to be *sharp* if $a \wedge \neg a = 0$ or, equivalently, if $a \vee \neg a = 1$. Thus $a \in A$ is a sharp element iff $\neg a$ is a complement of $a$ in $A$, in which case, however, $a$ can have other complements besides $\neg a$. On the other hand, it is possible that $\neg a$ is not a complement of $a$, though $a$ has some complement(s).

An element $a \in A$ is *central* if $a = \varphi^{-1}(1, 0)$ or $a = \varphi^{-1}(0, 1)$ for some isomorphism $\varphi\colon A \to A_1 \times A_2$. Clearly, if $a = \varphi^{-1}(1, 0)$, then $\neg a = \varphi^{-1}(0, 1)$, and vice versa. Suppose that $a = \varphi^{-1}(1, 0)$. Then $[0, a] = \varphi^{-1}(A_1 \times \{0\})$ and $[0, \neg a] = \varphi^{-1}(\{0\} \times A_2)$, whence $[0, a] \cong A_1$ and $[0, \neg a] \cong A_2$, and so $[0, a] \times [0, \neg a] \cong A$. It can easily be seen that the isomorphism in question is $\eta\colon (x, y) \mapsto x \vee y$; its inverse is $\eta^{-1}\colon x \mapsto (x \wedge a, x \wedge \neg a)$.

The central elements of $A$ correspond to the decomposition of $A$ as a direct product of two basic algebras, hence to factor congruences of $A$. Recall that $\Theta \in Con(A)$ is a factor congruence if there exists $\Theta^* \in Con(A)$ such that $\Theta \cap \Theta^* = \Delta_A$ and $\Theta \circ \Theta^* = \nabla_A$ where $\Delta_A$ and $\nabla_A$ denote the least and the greatest congruence on $A$, respectively.

It is well-known that sharp and central elements coincide in MV-algebras. A simple characterization of central elements is known also for lattice effect algebras (= basic algebras satisfying (1.7)); namely, an element $a \in A$ is central iff $x = (x \wedge a) \vee (x \wedge \neg a)$ for all $x \in A$ (see [**19**, **33**]). A description of central elements of general basic algebras can be found in [**12**] where the term 'decomposing' was used instead of 'central'.

We use $S(A)$ and $C(A)$ to denote respectively the set of the sharp elements and the set of the central elements of the algebra $A$. Obviously, we have $C(A) \subseteq S(A)$ with equality in some particular cases, e.g. in MV-algebras. Whereas $C(A)$ is always a subalgebra of $A$, $S(A)$ is neither a sublattice nor a subalgebra in general. But $S(A)$ is a sublattice when $A$ is distributive.

It is easy to prove that $a \in S(A)$ iff $a \oplus a = a$ (see [**11**]). Since the variety of basic algebras is arithmetical, we have $\Theta \vee \Phi = \Theta \circ \Phi$ for all $\Theta, \Phi \in Con(A)$, whence the factor congruences of $A$ form a Boolean sublattice of $Con(A)$.

LEMMA 4.1.1. *Let $A$ be a basic algebra. Then $a \in A$ is a central element iff*

$$\Theta_a = \{(x, y) \mid x \wedge a = y \wedge a\}$$

*is a factor congruence of $A$.*

PROOF. Let $\varphi\colon A \to A_1 \times A_2$ be an isomorphism, $a_1 = \varphi^{-1}(1, 0)$ and $a_2 = \varphi^{-1}(0, 1)$. If $\Theta_i$ is the kernel congruence of the homomorphism $\varphi \circ \pi_i\colon A \to A_i$ where $\pi_i$ is the projection of $A_1 \times A_2$ onto $A_i$, then clearly $\Theta_1 \cap \Theta_2 = \Delta_A$ and $\Theta_1 \circ \Theta_2 = \nabla_A$. Moreover, the map $\pi_{a_i}\colon x \mapsto x \wedge a_i$ is a homomorphism of $A$ onto the interval algebra $[0, a_i]$; its kernel congruence $\Theta_{a_i}$ is just $\Theta_i$ (because $\pi_i(\varphi(x)) = \pi_i(\varphi(y))$ iff $\varphi(x \wedge a_i) = \varphi(y \wedge a_i)$ iff $x \wedge a_i = y \wedge a_i$). This shows that if $a \in A$ is a central element, then $\Theta_a$ is a factor congruence of $A$ with $\Theta_a^* = \Theta_{\neg a}$.

Conversely, let $\Theta \in Con(A)$ be a factor congruence; then there exists a unique $a \in A$ such that $(1, a) \in \Theta$ and $(a, 0) \in \Theta^*$. The map $\varphi\colon x \mapsto ([x]_\Theta, [x]_{\Theta^*})$ is an isomorphism of $A$ onto $A/\Theta \times A/\Theta^*$ such that $\varphi(a) = ([1]_\Theta, [0]_{\Theta^*})$. Thus $a$ is a central element of $A$. Moreover, it is easy to see that $\Theta = \Theta_a$ and $\Theta^* = \Theta_{\neg a}$. Now, if $b \in A$ such that $\Theta_b$ is a factor congruence, then $\Theta_b = \Theta_a$ where $a \in C(A)$ and, obviously, $b = a$. $\square$

Consequently, the map $a \mapsto \Theta_a$ is a bijection between the central elements of $A$ and the factor congruences of $A$.

PROPOSITION 4.1.2. *For every basic algebra $A$, $C(A)$ is a subalgebra of $A$ isomorphic to the Boolean algebra of factor congruences of $A$.*

PROOF. Let $a, b \in C(A)$. We known that both $\Theta_a \cap \Theta_b$ and $\Theta_a \vee \Theta_b = \Theta_a \circ \Theta_b$ are factor congruences, thus, $\Theta_a \cap \Theta_b = \Theta_c$ and $\Theta_a \circ \Theta_b = \Theta_d$ for some $c, d \in C(A)$. In fact, $c$ is the least element of $[1]_{\Theta_a \cap \Theta_b}$ and $d$ is the least element of $[1]_{\Theta_a \circ \Theta_b}$.

Clearly, $(a \vee b, 1) \in \Theta_a \cap \Theta_b$, so $c \leq a \vee b$, and $(c, 1) \in \Theta_a \cap \Theta_b$ means $c \geq a \vee b$. Hence $\Theta_a \cap \Theta_b = \Theta_{a \vee b}$.

Furthermore, $(a \wedge b, 1) \in \Theta_a \circ \Theta_b$ because $(a \wedge b, b) \in \Theta_a$ and $(b, 1) \in \Theta_b$, thus $d \leq a \wedge b$. But there exists $x \in A$ such that $(d, x) \in \Theta_a$ and $(x, 1) \in \Theta_b$, so $d \wedge a = x \wedge a$ and $x \geq b$, whence $d \wedge a \wedge b = x \wedge a \wedge b = a \wedge b$, i.e. $d \geq a \wedge b$. Hence $\Theta_a \circ \Theta_b = \Theta_{a \wedge b}$.

This shows that $C(a)$ is a sublattice of $A$. However, it is clear that $a \vee b = a \oplus b$ (in fact, $a \vee x = a \oplus x$ for every $x \in A$), and also $\neg a \in C(A)$ for any $a \in C(A)$. Thus $C(A)$ is a subalgebra.

Finally, the map $\varphi\colon a \mapsto \Theta_{\neg a}$ is an isomorphism between $C(A)$ and the Boolean algebra of factor congruences. Indeed, $\varphi(0) = \Theta_1 = \Delta_A$, $\varphi(1) = \Theta_0 = \nabla_A$, $\varphi(a \oplus b) = \varphi(a \vee b) = \Theta_{\neg a \wedge \neg b} = \Theta_{\neg a} \vee \Theta_{\neg b} = \varphi(a) \vee \varphi(b) = \varphi(a) \circ \varphi(b)$, and $\varphi(\neg a) = \Theta_a = \Theta^*_{\neg a} = \varphi(a)^*$. $\square$

LEMMA 4.1.3. *If a basic algebra $A$ satisfies* (M), *then $S(A) = C(A)$.*

PROOF. Let $a \in S(A)$. Since $A$ satisfies (M), it is a distributive lattice and it follows that $\eta\colon (x, y) \mapsto x \vee y$ is a lattice isomorphism between $[0, a] \times [0, \neg a]$ and $A$. If $x \leq a$ and $y \leq \neg a$, then $x \wedge y = 0$, whence $x \ominus y = x \ominus (x \wedge y) = x \ominus 0 = x$ and also $y \ominus x = y$. Distributivity is equivalent to (D), and hence for any $(x_i, y_i) \in [0, a] \times [0, \neg a]$ we have:

$$\begin{aligned}
\eta(x_1, y_1) \ominus \eta(x_2, y_2) &= (x_1 \vee y_1) \ominus (x_2 \vee y_2)\\
&= (x_1 \ominus (x_2 \vee y_2)) \vee (y_1 \ominus (x_2 \vee y_2))\\
&= ((x_1 \ominus x_2) \wedge (x_1 \ominus y_2)) \vee ((y_1 \ominus x_2) \wedge (y_1 \ominus y_2))\\
&= ((x_1 \ominus x_2) \wedge x_1) \vee (y_1 \wedge (y_1 \ominus y_2))\\
&= (x_1 \ominus x_2) \vee (y_1 \ominus y_2)\\
&= \eta(x_1 \ominus x_2, y_1 \ominus y_2)\\
&= \eta((x_1, y_1) \ominus (x_2, y_2)).
\end{aligned}$$

Therefore, the map $\eta$ is also an isomorphism of basic algebras, proving that $a \in C(A)$. $\square$

Given a basic algebra $A$, a map $h\colon A \to A$ is *additive* if $h(x \oplus y) = h(x) \oplus h(y)$ and *isotone* $x \leq y$ implies $h(x) \leq h(y)$, for all $x, y \in A$. Every additive map is isotone because $x \leq y$ iff $y = z \oplus x$ for some $z$, whence $h(x) \leq h(z) \oplus h(x) = h(y)$.

LEMMA 4.1.4. *Let $A$ be a basic algebra and let $a \in A$. If the map $f\colon x \mapsto x \wedge a$ or the map $g\colon x \mapsto x \odot a$ is additive, then $a \in S(A)$ and $x \wedge a = x \odot a$, i.e. $f(x) = g(x)$, for all $x \in A$.*

PROOF. Suppose that $f$ is additive. Then $a \oplus a = f(a) \oplus f(a) = f(a \oplus a) = (a \oplus a) \wedge a = a$, thus $a \in S(A)$. Since $f(a) = a = f(1)$, we have $(a \oplus x) \wedge a = f(a \oplus x) = f(a) \oplus f(x) = f(1) \oplus f(x) = f(1 \oplus x) = f(1) = a$. Thus $a \leq a \oplus x$, or equivalently, $\neg a \geq \neg(a \oplus x) = \neg a \oslash x$, whence $f(\neg a \oslash x) = (\neg a \oslash x) \wedge a = 0$. Since $(x \odot a) \oplus \neg a = \neg a \vee x = (\neg a \oslash x) \oplus x$, it follows that $x \odot a = ((x \odot a) \wedge a) \oplus 0 = f(x \odot a) \oplus f(\neg a) = f((x \odot a) \oplus \neg a) = f((\neg a \oslash x) \oplus x) = f(\neg a \oslash x) \oplus f(x) = 0 \oplus f(x) = f(x)$.

Conversely, if $g$ is additive, then $x \wedge a = (x \oplus \neg a) \odot a = g(x \oplus \neg a) = g(x) \oplus g(\neg a) = g(x)$ because $g(\neg a) = \neg a \odot a = 0$. $\square$

LEMMA 4.1.5. *Suppose that the map $f\colon x \mapsto x \wedge a$ is additive, for a fixed $a \in A$. Then for all $x, y \in A$:*

(i) *$f(\neg x) = f(\neg f(x))$;*
(ii) *$f(x \odot y) = f(x) \odot f(y)$;*
(iii) *$f(x \odot y) = f(x) \odot y = x \odot f(y)$;*
(iv) *$f(x \ominus y) = f(x) \ominus f(y)$.*

PROOF. (i) Since $f(x) = x \wedge a = x \odot a$ by Lemma 4.1.4, we obviously have $f(\neg f(x)) = \neg(x \odot a) \odot a = \neg x \wedge a = f(\neg x)$.

(ii) By (i) we have

$$f(x \odot y) = f(\neg(\neg x \oplus \neg y)) = f(\neg f(\neg x \oplus \neg y)),$$

and since $f(x) \odot f(y) \leq f(y) \leq a$, also

$$f(x) \odot f(y) = f(f(x) \odot f(y)) = f(\neg(\neg f(x) \oplus \neg f(y))) = f(\neg f(\neg f(x) \oplus \neg f(y))).$$

But, again by (i),

$$f(\neg x \oplus \neg y) = f(\neg x) \oplus f(\neg y) = f(\neg f(x)) \oplus f(\neg f(y)) = f(\neg f(x) \oplus \neg f(y)),$$

and hence $f(x \odot y) = f(x) \odot f(y)$.

(iii) Since $x \odot f(y) \leq f(y) \leq a$, we have $x \odot f(y) = f(x \odot f(y)) = f(x) \odot f(f(y)) = f(x) \odot f(y)$. On the other hand, $f(x) \leq a \leq a \oplus \neg y$ implies $f(x) \odot y \leq (a \oplus \neg y) \odot y = a \wedge y$, whence $f(x) \odot y = f(f(x) \odot y) = f(f(x)) \odot f(y) = f(x) \odot f(y)$.

(iv) Using (i) and $x \ominus y = \neg y \odot x$, this follows from (ii): $f(x \ominus y) = f(\neg y) \odot f(x) = f(\neg f(y)) \odot f(f(x)) = f(\neg f(y) \odot f(x)) = f(f(x) \ominus f(y)) = f(x) \ominus f(y)$. $\square$

COROLLARY 4.1.6. *Let $A$ be a basic algebra, $a \in A$ and suppose that $f\colon x \mapsto x \wedge a$ is additive. Then $f$ is a homomorphism from $A$ onto the interval basic algebra $[0, a]$ in which the operations are given by $\neg_a x = \neg x \wedge a$, $x \oplus_a y = x \oplus y$, $x \odot_a y = x \odot y$ and $x \oslash_a y = x \oslash y$.*

PROOF. In view of Lemma 4.1.5 (iv), the map $f$ is a homomorphism from $A$ onto $[0, a]$. Since $f$ fixes the elements of $[0, a]$, for any $x, y \in [0, a]$ we have $\neg_a x = \neg_a f(x) = f(\neg x) = \neg x \wedge a$, $x \oplus_a y = f(x) \oplus_a f(y) = f(x \oplus y) = f(x) \oplus f(y) = x \oplus y$ as $f$ is additive, $x \odot_a y = f(x) \odot_a f(y) = f(x \odot y) = f(x) \odot f(y) = x \odot y$ by Lemma 4.1.5 (ii), and $x \oslash_a y = f(x) \oslash_a f(y) = f(x \oslash y) = (x \oslash y) \wedge a$. But $x \leq a \leq a \oplus y$ implies $x \oslash y \leq (a \oplus y) \oslash y = a \wedge \neg y$, and so $x \oslash_a y = x \oslash y$. $\square$

## 4.2. Derivations on basic algebras

We have already recalled that a derivation on an MV-algebra is an additive map that satisfies the condition

$$d(x \odot y) = (d(x) \odot y) \oplus (x \odot d(y)). \tag{4.1}$$

However, in the case of non-commutative basic algebras, (4.1) is not the same as

$$d(x \odot y) = (x \odot d(y)) \oplus (d(x) \odot y), \tag{4.2}$$

and hence we can seemingly define two types of derivations.[1] We let $\mathcal{D}_1(A)$ and $\mathcal{D}_2(A)$ denote the set of all additive maps $d\colon A \to A$ satisfying (4.1) and (4.2), respectively, and by a *derivation* on a basic algebra $A$ we mean every map $d$ which belongs to $\mathcal{D}(A) := \mathcal{D}_1(A) \cup \mathcal{D}_2(A)$.

EXAMPLE 4.2.1. The simplest examples of derivations are:

[1]In a non-commutative basic algebra $A$, the conditions (4.1) and (4.2) are not equivalent without additivity of $d$, but we are going to show that if $d\colon A \to A$ is additive, then it satisfies (4.1) iff it satisfies (4.2). Thus there is only one type of derivations on $A$, in symbols, $\mathcal{D}(A) = \mathcal{D}_1(A) = \mathcal{D}_2(A)$.

(a) For any basic algebra $A$, the zero map $0_A\colon x \mapsto 0$ is a derivation (of both types). On the other hand, the identity map $id_A\colon x \mapsto x$ is a derivation iff $S(A) = A$.
(b) Let $A, B$ be two basic algebras such that $S(A) = A$. Then the 'projection' $p_A\colon (x, y) \mapsto (x, 0)$ is a derivation (of both types) on the direct product $A \times B$. We will see that in some particular cases all derivations are of this form.

LEMMA 4.2.2. *Let $A$ be a basic algebra. For any derivation $d \in \mathcal{D}(A)$, we have:*
(i) *$d(0) = 0$;*
(ii) *$d$ is isotone; in particular, $d(x) \leq d(1)$ for every $x \in A$;*
(iii) *$d(x) \leq x$ for every $x \in A$;*
(iv) *$d(1) \in S(A)$;*
(v) *$d(\neg d(1) \odot x) = 0$ for every $x \in A$.*

PROOF. Suppose that $d \in \mathcal{D}_1(A)$; the proof for $d \in \mathcal{D}_2(A)$ is parallel.
(i) We have $d(0) = d(0 \odot 0) = (d(0) \odot 0) \oplus (0 \odot d(0)) = 0 \oplus 0 = 0$.
(ii) Obvious since $d$ is additive.
(iii) Recall that $x \oplus y = 0$ iff $x = y = 0$. Since $0 = d(0) = d(\neg x \odot x) = (d(\neg x) \odot x) \oplus (\neg x \odot d(x))$, it follows $d(x) \ominus x = \neg x \odot d(x) = 0$, so $d(x) \leq x$.
(iv) We have $d(1) = d(1) \oplus d(1)$.
(v) By (b), (c) and (d) we have $d(\neg d(1)) \leq d(1) \wedge \neg d(1) = 0$, so $d(\neg d(1)) = 0$. Consequently, $d(\neg d(1) \odot x) = (d(\neg d(1)) \odot x) \oplus (\neg d(1) \odot d(x)) = \neg d(1) \odot d(x) = 0$ because $d(x) \leq d(1)$.

□

Every derivation $d$ is completely determined by the element $d(1)$, namely:

LEMMA 4.2.3. *Let $A$ be a basic algebra and $d \in \mathcal{D}(A)$. Then for every $x \in A$:*
(i) *$d(x) = x \wedge d(1)$;* [2]
(ii) *$d(x) = x \odot d(1)$;*
(iii) *$d(x) \in S(A)$.*

PROOF. (i) Let $d \in \mathcal{D}_1(A)$. First, $d(x) = d(x \odot 1) = d(x) \oplus (x \odot d(1)) \geq x \odot d(1)$ for any $x \in A$, and hence, by replacing $x$ with $x \oplus \neg d(1)$, we get $d(x \oplus \neg d(1)) \geq (x \oplus \neg d(1)) \odot d(1) = x \wedge d(1)$. Then, by Lemma 4.2.2 (v), $d(x) = d(x) \oplus 0 = d(x) \oplus d(\neg d(1)) = d(x \oplus \neg d(1)) \geq x \wedge d(1)$. But by Lemma 4.2.2 (ii) and (iii) we also have $d(x) \leq x \wedge d(1)$, thus $d(x) = x \wedge d(1)$.

Let $d \in \mathcal{D}_2(A)$. If $x \leq d(1)$, then $\neg x \vee d(1) \geq \neg d(1) \vee d(1) = 1$, so $\neg x \vee d(1) = 1$ and $x = (\neg x \vee d(1)) \odot x = d(1) \odot x$. Then $d(x) = d(1 \odot x) = d(x) \oplus (d(1) \odot x) = d(x) \oplus x \geq x$. Since $d(x) \leq x$, we get $d(x) = d(x) \oplus x = x$. Hence $x \oplus x = x$ for any $x \leq d(1)$. Now, by replacing $x$ with $x \wedge d(1)$, and since $d$ is isotone, we have $d(x) \geq d(x \wedge d(1)) = x \wedge d(1)$. As above, by Lemma 4.2.2 (ii) and (iii) we conclude that $d(x) = x \wedge d(1)$.

(ii) By Lemma 4.2.2 (v) we have $d(\neg d(1) \oslash x) = 0$ for any $x \in A$, because $\neg d(1) \oslash x = \neg d(1) \odot \neg x$. Since $(x \odot d(1)) \oplus \neg d(1) = \neg d(1) \vee x = (\neg d(1) \oslash x) \oplus x$, in view of (i) we

[2] Though it is clear, we emphasize that (i) entails $d(x) = x$ for $x \leq d(1)$, and in particular, $d(d(x)) = d(x)$ for every $x \in A$. Thus $d$ is an interior operator on $A$.

have $x \odot d(1) = [(x \odot d(1)) \wedge d(1)] \oplus 0 = d(x \odot d(1)) \oplus d(\neg d(1)) = d((x \odot d(1)) \oplus \neg d(1)) = d((\neg d(1) \oslash x) \oplus x) = d(\neg d(1) \oslash x) \oplus d(x) = 0 \oplus d(x) = d(x)$.

(iii) If $d \in \mathcal{D}_1(A)$, then $d(x) = d(x \odot 1) = d(x) \oplus (x \odot d(1)) = d(x) \oplus d(x)$ by (ii). In proving (i) for $d \in \mathcal{D}_2(A)$ we have shown that $d(x) = d(x) \oplus x$ for any $x \leq d(1)$. Since $d(d(x)) = d(x) \wedge d(1) = d(x)$ for every $x \in A$, it follows that $d(x) = d(x) \oplus d(x)$. In either case, the element $d(x)$ is idempotent, thus $d(x) \in S(A)$. □

REMARK 4.2.4. The map $f\colon x \mapsto x \wedge a$ (or $g\colon x \mapsto x \odot a$) need not be a derivation, even when it is additive. In fact, recalling Lemma 4.2.3 (iii), it is obvious that if the map is additive, then $f = g$ is a derivation on $A$ iff $[0, a] \subseteq S(A)$. Indeed, if $[0, a] \subseteq S(A)$, then $f(x \odot y) \in S(A)$, so that $f(x \odot y) = (f(x) \odot y) \oplus (x \odot f(y)) = (x \odot f(y)) \oplus (f(x) \odot y)$ by Lemma 4.1.5 (iii).

By Lemma 4.2.3 we know that a derivarion $d$ is an additive map given by $d(x) = x \wedge d(1) = x \odot d(1)$, and that $d(x)$ is always a sharp element. Hence, by the above corollary we obtain:

COROLLARY 4.2.5. *Let $A$ be a basic algebra. Every derivation $d \in \mathcal{D}(A)$ is a homomorphism from the basic algebra $A$ onto the interval algebra $[0, e]$ with $e = d(1)$. Moreover, $[0, e] \subseteq S(A)$ and the operations in $[0, e]$ are given by $\neg_e x = \neg x \wedge e$, $x \oplus_e y = x \oplus y$, $x \odot_e y = x \odot y$ and $x \oslash_e y = x \oslash y$.*

REMARK 4.2.6. It is easily seen that every $d \in \mathcal{D}(A)$ is a lattice derivation in the sense of **[37]** (also **[21]**), i.e., $d(x \vee y) = d(x) \vee d(y)$ and $d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y))$ for all $x, y \in A$. Indeed, the former equality holds since $d\colon x \mapsto x \wedge e$ is a homomorphism onto $[0, e]$, and the latter since $d(x \wedge y) = x \wedge y \wedge e = d(x) \wedge y = x \wedge d(y)$. The converse is not true – a lattice derivation need not be a derivation on the algebra, but it is proved in **[22]** that a lattice derivation $d$ on an MV-algebra is a derivation on the algebra iff all elements $d(x)$ are sharp.

The question arises which homomorphisms onto interval subalgebras are derivations. The next lemma is a partial converse to Corollary 4.1.6.

LEMMA 4.2.7. *Let $A$ be a basic algebra, $a \in A$ and let $f\colon x \mapsto x \wedge a$ be a homomorphism onto the interval algebra $[0, a]$. Then $f$ is additive, the operations in $[0, a]$ are given by $\neg_a x = \neg x \wedge a$ and $x \oplus_a y = x \oplus y$, and moreover, $f \in \mathcal{D}(A)$—in fact, $f \in \mathcal{D}_1(A) \cap \mathcal{D}_2(A)$—if and only if $[0, a] \subseteq S(A)$.*

PROOF. The first part is parallel to the proof of Corollary 4.1.6. For any $x, y \in [0, a]$, $\neg_a x = \neg_a f(x) = f(\neg x) = \neg x \wedge a$ and $x \oplus_a y = f(x) \oplus_a f(y) = f(x \oplus y) = (x \oplus y) \wedge a$. Since $\neg y \wedge a = f(1 \oslash y) = f(1) \oslash_a f(y) = f(a) \oslash_a f(y) = f(a \oslash y) = (a \oslash y) \wedge a$, we have $a \oplus y = (\neg y \wedge a) \oplus y = ((a \oslash y) \wedge a) \oplus y \leq (a \oslash y) \oplus y = a$, whence $x \oplus y \leq a \oplus y \leq a$. Hence $x \oplus_a y = x \oplus y$. Now, it follows that $f$ is additive because, for any $x, y \in A$, $f(x \oplus y) = f(x) \oplus_a f(y) = f(x) \oplus f(y)$.

Since $f(1) = a$, we know that $[0, a] \subseteq S(A)$ if $f \in \mathcal{D}(A)$; see Lemma 4.2.3 (iii) and Corollary 4.2.5. On the other hand, if $[0, a] \subseteq S(A)$, it suffices to apply Remark 4.2.4. □

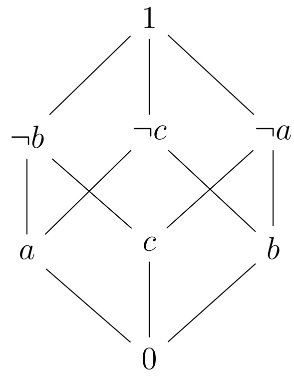


FIGURE 1. The basic algebra from Example 4.2.9

By Corollary 4.2.5 and Lemma 4.2.7 we conclude:

COROLLARY 4.2.8. *For any basic algebra $A$, $\mathcal{D}(A) = \mathcal{D}_1(A) = \mathcal{D}_2(A)$.*

For any derivation $d$ on any basic algebra, every element less than or equal to $d(1)$ is sharp. The following example shows that, in general, $d(1)$ need not be central.

EXAMPLE 4.2.9 (cf. [**15**], Example 3.1). Let $(A, \oplus, \neg, 0)$ be the basic algebra whose underlying lattice is shown in Figure 1, where $a \oplus c = \neg b \oplus c = \neg a$, $b \oplus c = \neg a \oplus c = \neg b$ and $x \oplus y = x \vee y$ in all other cases. The antitone involution $\gamma_c$ on $[c, 1]$ is given by $\gamma_c(\neg a) = \neg a$ and $\gamma_c(\neg b) = \neg b$, while in every other principal filter $[y, 1]$, $\gamma_y$ is just complementation in $[y, 1]$. Note that $S(A) = A$.

It is straightforward to verify that the map $f\colon x \mapsto x \wedge c$ is a homomorphism onto the interval basic algebra $[0, c]$. Since $[0, c] \subseteq S(A)$, $f \in \mathcal{D}(A)$. But the element $c$ is not central in $A$. Indeed, were $\eta\colon (x, y) \mapsto x \vee y$ an isomorphism from $[0, c] \times [0, \neg c]$ onto $A$, we would have $\neg\eta(c, a) = \neg\neg b = b$, while $\eta(\neg(c, a)) = \eta(0, a) = a$, because the negation of $a$ in $[0, \neg c]$ is $\neg_{\neg c} a = \neg c \ominus a = \neg(a \oplus c) = \neg\neg a = a$.

For completeness, $C(A) = \{0, 1\}$.

However, there is a large class of basic algebras where $d(1)$ is central, namely, the class of lattice effect algebras (= basic algebras satisfying (1.7)).

### 4.3. Derivations on lattice effect algebras and $\ell$-groups

Two elements $x, y$ of a lattice effect algebra are said to be *compatible*[3] (see [**19**], Section 10.1) iff $(x \vee y) - y = x - (x \wedge y)$, i.e., if $x \oslash y = x \ominus y$. By [**11**], this is also equivalent to $x \oplus y = y \oplus x$. Obviously, any two comparable elements are compatible.

We need one more property of lattice effect algebras. It is known (e.g. [**19**], Prop. 1.8.9) that if $x \wedge y = 0$ and $x + y$ is defined, then $x \vee y = x + y$. In the language of basic algebras: if $x \wedge y = 0$ and $x \leq \neg y$, then $x \vee y = x \oplus y$.

[3]Compatibility is also briefly discussed in Section 1.3 (p. 11).

LEMMA 4.3.1. *Let $A$ be a lattice effect algebra. For every derivation $d \in \mathcal{D}(A)$, the element $e = d(1)$ is central and $[0, e]$ is an orthomodular lattice, or a Boolean algebra when $A$ is distributive.*

PROOF. Recall from Section 4.1 that $e \in C(A)$ iff $x = (x \wedge e) \vee (x \wedge \neg e)$ for all $x \in A$.

Obviously, $d(x) \vee (x \wedge \neg e) \leq x$, thus we want to show that $d(x) \vee (x \wedge \neg e) \geq x$. Since $d(x) \leq x$, we have $x \oplus d(x) = d(x) \oplus x \geq x$. Since $d(x) \leq e$, it follows directly from (1.7) that $d(x) \oplus \neg e = \neg e \oplus d(x)$. We have $d(x) \oplus \neg e = (x \wedge e) \oplus \neg e = x \oplus \neg e = \neg(\neg x \odot e) = \neg d(\neg x) = \neg(\neg x \wedge e) = x \vee \neg e$, and hence $\neg e \oplus d(x) \geq x$. Now, since $e \in S(A)$ and $d(x) \leq e$, we have $d(x) \wedge (x \wedge \neg e) = 0$ and $d(x) \leq \neg(x \wedge \neg e)$, whence $d(x) \vee (x \wedge \neg e) = (x \wedge \neg e) \oplus d(x) = (x \oplus d(x)) \wedge (\neg e \oplus d(x)) \geq x$, as required.

For the latter statement, since $[0, e] \subseteq S(A)$ and $\oplus_e$ is the restriction of $\oplus$ to $[0, e]$, the interval algebra $[0, e]$ satisfies the identity $x \oplus x = x$ and hence is (equivalent to) an orthomodular lattice, or a Boolean algebra when $A$ is distributive. $\square$

Let $d$ be a derivation on a basic algebra $A$ such that $e = d(1)$ is a central element. Then, by definition, the direct product $[0, e] \times [0, \neg e]$ is isomorphic to $A$ under $\eta\colon (x, y) \mapsto x \vee y$; the inverse isomorphism is $\theta\colon x \mapsto (x \wedge e, x \wedge \neg e)$. Therefore, the derivation $\theta(d)$ on the direct product that corresponds to $d$ under $\theta$ is given by $(x, y) \mapsto (x, y) \wedge (e, 0) = (x, 0)$; see Example 4.2.1 (b). Therefore, by Lemma 4.3.1:

COROLLARY 4.3.2. *In any lattice effect algebra (or any MV-algebra) $A$, there is a one-one correspondence between the derivations on $A$ and the direct product decompositions $A \cong A_1 \times A_2$ where $A_1$ is an orthomodular lattice (or a Boolean algebra, respectively).*

In conclusion, we briefly focus on MV-algebras. We refer the reader to **[18]** or **[23]**.

Let $(G, +, \leq)$ be an Abelian $\ell$-group, i.e., an Abelian group equipped with a compatible lattice order. For any $0 \leq u \in G$, the algebra $\Gamma(G, u) = ([0, u], \oplus, \neg, 0)$, where $x \oplus y = (x + y) \wedge u$ and $\neg x = u - x$, is an MV-algebra. Up to isomorphism, all MV-algebras are of the form $\Gamma(G, u)$ where, moreover, $u$ is a strong unit for $G$, i.e., the convex $\ell$-subgroup of $G$ generated by $u$ is $G$. Even strongly, $\Gamma$ as a functor is an equivalence between the category of unital Abelian $\ell$-groups $(G, u)$ where arrows are $\ell$-group homomorphisms preserving strong units and the category of MV-algebras where arrows are homomorphisms of MV-algebras.

Now, let $A = \Gamma(G, u)$ be an MV-algebra and $d \in \mathcal{D}(A)$ a derivation on it and, as before, $e = d(1)$. Further, let $A_1 = [0, e]$ and $A_2 = [0, \neg e]$. Since $A \cong A_1 \times A_2$, it follows that $G \cong G_1 \times G_2$, where $G_1$ and $G_2$ are the convex $\ell$-subgroups of $G$ generated by $e$ and $\neg e$, respectively. In fact, $G$ is the direct sum of $G_1$ and $G_2$, because $G_1 \cap G_2 = \{0\}$ and $G = G_1 + G_2$. The MV-algebra $\Gamma(G_1, e)$ is just the interval algebra $A_1 = [0, e]$, and since the derivation $d$ is a homomorphism from $A$ onto $A_1$, $d$ can be extended to a morphism, say $\tilde{d}$, from $(G, u)$ onto $(G_1, e)$. It is not hard to show that $\tilde{d}$ agrees with the projection of $G$ onto $G_1$ as a direct summand.

# Appendix

The appendix contains two papers about structures related to basic algebras. In the first article, we introduce the so-called *skew residuated lattices* which are similar to (integral commutative) residuated lattices, except that $\cdot$ may not be associative and the adjointness property is replaced by the condition $(x \vee y) \cdot z = y$ iff $z = x \to y$. We characterize skew residuated lattices as lattices $(L, \vee, \wedge)$ where for every $a \in L$ there exists a map $\psi_a \colon L \to L$ such that

- $\psi_a$ is an involution on $[a) = \{x \in L \mid a \leq x\}$ for every $a \in L$,
- $\psi_x(x) = \psi_y(y)$ for every $x, y \in L$,
- for every $x, y \in L$ there exists a unique $z \in L$ such that $x = \psi_z(y \vee z)$ and $y = \psi_z(x \vee z)$.

We show that the negative cones of commutative $\ell$-loops (CND-semiloops) can be regarded as skew residuated lattices. In Chapter 3, we needed commutative inverse $\ell$-loops, but by **[3]** there is a one-one correspondence between commutative inverse $\ell$-loops and skew residuated lattices satisfying the identity $x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z)$. This is very useful in finding examples of commutative inverse $\ell$-loops and commutative basic algebras, see Section 3.4.

In the second article, we study congruences on directoids. A *directoid* is a partially ordered set with a binary operation $\sqcup$, where $x \sqcup y = \max\{x, y\}$ if $x$ and $y$ are comparable, and $x \sqcup y = z$ for some $z \geq x, y$ otherwise. We present several simplified characterizations of congruences on directoids, directoids with an antitone involution, directoids with sectionally antitone involutions and double directoids.

Since every basic algebra is a lattice (directoid) with sectionally antitone involutions, we can use the results of Section 4 of the second article for congruences on basic algebras.

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# Skew residuated lattices ☆

I. Chajda*, J. Krňávek

*Department of Algebra and Geometry, Faculty of Science, Palacký University Olomouc, 17. listopadu 12, 771 46 Olomouc, Czech Republic*

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**Abstract**

We replace the so-called adjointness in the definition of residuated lattice by its strict version where inequalities are replaced by equalities. We prove that such structures, called skew residuated lattices, can be characterized as lattices with certain involutions in principal filters. Since skew residuated lattices have the cancellation property, they are close to divisibility loops introduced by B. Bosbach in 1988. We show the condition under which the skew residuated lattices can be represented by such loops.


*Keywords:* Skew residuated lattice; Adjointness property; Reversion; Difference property; Sectional involution; Divisibility semiloop

## 1. Introduction

Residuated lattices are very useful algebraic structures because they describe the structure of truth values of fuzzy logic. As a source of basic concepts, the reader is referred to the compendium by Bělohlávek [1].

Recall that a *residuated lattice* is an algebra $\mathcal{R} = (R; \vee, \wedge, \cdot, \rightarrow, 0, 1)$ of type (2, 2, 2, 2, 0, 0) such that

(a) $(R; \vee, \wedge, 0, 1)$ is a bounded lattice;
(b) $(R; \cdot, 1)$ is commutative monoid satisfying $x \cdot 1 = x$;
(c) it satisfies the *adjointness property*, i.e. $x \cdot y \leq z$ if and only if $x \leq y \rightarrow z$.

In the corresponding fuzzy logic, the operation $\cdot$ is recognized as logic connective "conjunction" and $\rightarrow$ is considered as "implication".

However, residuated structures have their own lives and hence non-commutative versions of residuated lattices (called *residuated lattice ordered monoids*) also exist where the neutral element $e$ of the monoid $(R; \cdot, e)$ need not coincide with the greatest element 1 of the lattice $(R; \vee, \wedge)$.

In what follows, we will assume that groupoid $(R; \cdot)$ is commutative but its associativity is not supposed, i.e. it need not be a semigroup. This concept was used by the first author [3] to get an algebraic axiomatization of a certain fuzzy-like logic. Hence, we define:

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* Corresponding author. Tel.: +420 585634653.
*E-mail addresses:* ivan.chajda@upol.cz (I. Chajda), jan.krnavek@upol.cz (J. Krňávek).

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**Definition 1.** An algebra $\mathcal{L} = (L; \vee, \wedge, \cdot, \rightarrow, e)$ is called a *skew residuated lattice* if

(i) $(L; \vee, \wedge)$ is a lattice;
(ii) $(L; \cdot, e)$ is a commutative groupoid satisfying $x \cdot e = x$;
(iii) $(x \vee y) \cdot z = y$ if and only if $z = x \rightarrow y$;
(iv) $x \cdot y \leq y$.

Let us note that the condition (iii) replaces the adjointness in residuated lattices. It was mentioned by G. Champenois (personal communication) that these skew residuated lattices are useful for some applications.

Other applications of skew residuated lattices are in the area of logic of quantum mechanics. The domain of quantum events is described by effect algebra introduced by Foulis and Bennet in [7]. It was proved in [5] that they can be represented by certain conditionally residuated structures. The concept of effect algebra has been generalized in [6] to the non-associative case. Now, non-associative effect algebras can be represented as conditional skew residuated lattices, see [4]; hence it is natural to study them in their own right. As another application it will be shown that skew residuated lattices in which multiplication distributes over join are precisely CND-semiloops: a commutative dual version of certain structures introduced in [2].

Moreover, the condition (iii) is called a *reversion* or a *difference property* in [8]. Under this property, the multiplication "$\cdot$" resembles the lattice operation meet. Let us note that the lattice $(L; \vee, \wedge)$ is not assumed to be bounded. However, we can prove that $e$ is the greatest element with respect to the lattice order $\leq$.

**Lemma 1.** *In every skew residuated lattice* $\mathcal{L} = (L; \vee, \wedge, \cdot, \rightarrow, e)$, *the following holds*:

(a) $x \rightarrow y = (x \vee y) \rightarrow y$;
(b) $x \leq y$ *if and only if* $x \rightarrow y = e$;
(c) $x \cdot y \leq x \wedge y$;
(d) $e$ *is the greatest element w.r.t.* $\leq$.

**Proof.**

(a) is followed directly by (iii) due to the fact that $(x \vee y) \cdot z = ((x \vee y) \vee y) \cdot z$.
(b) Assume $x \leq y$. Then $(x \vee y) \cdot e = x \vee y = y$ and hence $e = x \rightarrow y$. Conversely, $e = x \rightarrow y$ yields $x \vee y = (x \vee y) \cdot e = y$ proving $x \leq y$.
(c) Due to (iv) we have $x \cdot y \leq y$ and, using commutativity, also $x \cdot y = y \cdot x \leq x$. Thus $x \cdot y \leq x \wedge y$.
(d) By (ii) and (iv) we infer $x = x \cdot e \leq e$ for each $x \in L$. $\square$

We are going to show that skew residuated lattices satisfy the *cancellation property*, i.e. the commutative groupoid $(L; \cdot, e)$ is close to a loop. Recall that a loop means a quasigroup with unit element 1, i.e. it is an algebra $(A; \cdot, 1)$ such that $x \cdot 1 = x = 1 \cdot x$ for each $x \in A$ and for each $a \in A$ there exists $b, c \in A$ with $a \cdot b = 1$ and $c \cdot a = 1$.

**Lemma 2.** *Every skew residuated lattice has the cancellation property*, *i.e.* $x \cdot a = x \cdot b$ *implies* $a = b$.

**Proof.** Denote by $y = x \cdot a$ and assume $x \cdot a = x \cdot b$ for $x, a, b \in L$. Using (iv) and commutativity, we have $y = x \cdot a = a \cdot x \leq x$; thus $x = x \vee y$ and hence $(x \vee y) \cdot a = y$. Similarly $(x \vee y) \cdot b = y$ and thus, by (iii), we conclude $a = x \rightarrow y = b$. $\square$

One technical result can be stated.

**Lemma 3.** *Every skew residuated lattice satisfies the following*:

(a) $y \leq x \rightarrow y$;
(b) $(x \rightarrow y) \rightarrow y = x \vee y$.

**Proof.**

(a) Let $z = x \rightarrow y$. By (iii) and (iv) we have $(x \vee y) \cdot z = y$, thus $x \rightarrow y = z \geq (x \vee y) \cdot z = y$.
(b) By (a) we get $x \rightarrow y = (x \rightarrow y) \vee y$ and thus, using (iii) and (ii), $((x \rightarrow y) \vee y) \cdot (x \vee y) = (x \vee y) \cdot ((x \rightarrow y) \vee y) = y$. Applying (iii), we obtain $(x \rightarrow y) \rightarrow y = x \vee y$. $\square$

The next result shows that the "implication reduct" $(L; \to)$ of a skew residuated lattice is very close to a quasigroup. Namely, we have

**Theorem 1.** *Let $\mathcal{L} = (L; \vee, \wedge, \cdot, \to, e)$ be a skew residuated lattice. Then for each $a, b \in L$ there exists a unique element $y \in L$ such that*

$$a = b \to y \quad \text{and} \quad b = a \to y.$$

*Moreover, $y = a \cdot b$.*

**Proof.** Let $a, b \in L$ and denote $a \cdot b = y$. By (ii) and (iv) we have $y \le a$, $y \le b$ thus

$$(a \vee (a \cdot b)) \cdot b = a \cdot b,$$
$$(b \vee (a \cdot b)) \cdot a = b \cdot a = a \cdot b.$$

By (iii) these yield $b = a \to (a \cdot b)$, $a = b \to (a \cdot b)$ proving the existence. For uniqueness, let $z \in L$ and assume $a = b \to z$, $b = a \to z$. By Lemma 3(a) we obtain $z \le a$, $z \le b$ and thus $a \vee z = a$, $b \vee z = b$. Using (iii) we derive $a \cdot b = (a \vee z) \cdot b = z$. $\square$

In what follows, we are going to show that (iii) of Definition 1 can be replaced by two identities. Since (i), (ii) and (iv) are identities, this yields the fact that the class of skew residuated lattices is a variety of algebras.

**Theorem 2.** *An algebra $\mathcal{L} = (L; \vee, \wedge, \cdot, \to, e)$ is a skew residuated lattice if $(L; \vee, \wedge)$ is a lattice satisfying the identities* (ii), (iv) *and*

(I1) $(x \vee y) \cdot (x \to y) = y$;
(I2) $x \to (x \cdot y) = y$.

**Proof.** If $\mathcal{L}$ is a skew residuated lattice then it satisfies $x \to (x \cdot y) = y$ directly by Theorem 1. Since $x \to y = x \to y$, by (iii) we infer the identity $(x \vee y) \cdot (x \to y) = y$.

Conversely, assume that $\mathcal{L}$ is a lattice satisfying (iii), (iv) and the identities (I1), (I2). Assume $x \cdot y = x \cdot z$. Then $y = x \to (x \cdot y) = x \to (x \cdot z) = z$, i.e. the cancellation property holds.

Now, assume $(x \vee y) \cdot z = y$. Since $(x \vee y) \cdot (x \to y) = y$, the cancellation property gets $z = x \to y$. The converse implication is evident. $\square$

**Remark 1.** If a skew residuated lattice $L$ is non-trivial, i.e. if it has more than one element, then it is infinitive. Namely, let $L$ contain an element $a$ distinct from the greatest element 1. Due to (iv), we have $a \cdot a \le a = a \cdot 1$. Due to cancellability, $a \cdot a$ differs from $a \cdot 1$ and hence $a \cdot a < a$. Analogously we obtain an infinite chain $\cdots < a \cdot a \cdot a \cdot a < a \cdot a \cdot a < a \cdot a < a < 1$.

Recall that an *involution* on a set $A$ means a selfmap $f : A \to A$ such that $f\big(f(x)\big) = x$ for each $x \in A$. Of course, every involution on $A$ is a bijection on $A$.

In what follows, we show that skew residuated lattices can be fully determined as lattices $(L; \vee, \wedge)$ equipped by a set $(\psi_y)_{y \in L}$ of involutions on principal filters $[y) = \{a \in L;\ y \le a\}$.

**Theorem 3.** *Let $\mathcal{L} = (L; \vee, \wedge, \cdot, \to, e)$ be a skew residuated lattice. Define $\psi_y(x) = (x \vee y) \to y$. Then*

(a) *$\psi_y$ is an involution on $[y)$;*
(b) *$\psi_y(y) = \psi_x(x)$;*
(c) *for every $a, b \in L$ there exists unique $y \in L$ such that $a = \psi_y(b \vee y)$ and $b = \psi_y(a \vee y)$.*

**Proof.**

(a) Assume $x \in [y)$. Then $y \le x$; i.e. $x \vee y = x$ and hence $\psi_y(x) = x \to y$. By Lemma 3(a) we have $\psi_y(x) \ge y$, i.e. $\psi_y$ maps $[y)$ into itself. Moreover, for $x \in [y)$ we have by Lemma 3(b) also $\psi_y\big(\psi_y(x)\big) = (x \to y) \to y = x \vee y = x$; thus $\psi_y$ is an involution on $[y)$.

(b) By Lemma 1(b) we have $\psi_y(y) = (y \vee y) \to y = y \to y = e$ whence (b) is evident.

(c) Let $a, b \in L$ and take $y = a \cdot b$. Then, by (iv) and (ii), $a \cdot b \leq a, b$ and hence $(a \vee a \cdot b) \cdot b = a \cdot b$. Due to (iii) we conclude $b = a \to a \cdot b = \psi_{a \cdot b}(a)$. Hence, such $y$ exists. Assume now that $a = \psi_z(b \vee z)$ and $b = \psi_z(a \vee z)$. Then $(a \vee z) \cdot b = z = (b \vee z) \cdot a$. Since $a \vee z, b \vee z \in [z)$ and $\psi_z$ maps $[z)$ onto itself, also $a = \psi_z(b \vee z) \in [z)$ and $b = \psi_z(a \vee z) \in [z)$ whence $z \leq a, b$. Then $a \cdot b = (a \vee z) \cdot b = z$ as above. Hence, the element $y = a \cdot b$ is unique. □

To prove that the aforementioned result is a characterization, we have to show the converse:

**Theorem 4.** *Let* $(L; \vee, \wedge)$ *be a lattice such that for each* $y \in L$ *there exists a mapping* $\psi_y: L \to L$ *such that* (a)–(c) *are satisfied. If we define* $x \to y = \psi_y(x \vee y)$, $e = \psi_x(x)$ *and* $x \cdot y = z$ *if and only if* $y = \psi_z(x \vee z)$ *and* $x = \psi_z(y \vee z)$, *then* $\mathcal{L} = (L; \vee, \wedge, \cdot, \to, e)$ *is a skew residuated lattice.*

**Proof.** By (b) it is evident that $e$ is an algebraic constant which is correctly defined.

Let $a, b \in L$. By (c) there exists unique $y \in L$ with $a = \psi_y(b \vee y)$ and $b = \psi_y(a \vee y)$ and hence $a \cdot b = y$ is also correctly defined. Moreover, it yields $a \cdot b = b \cdot a$. Since $e = \psi_x(x) = \psi_x(x \vee x)$, and also $x \cdot e = x$ thus $(L; \cdot, e)$ is a commutative groupoid with a neutral element $e$.

Assume now $a \cdot b = y$. Then $b = \psi_y(a \vee y) \geq y = a \cdot b$ since $\psi_y$ is a selfmap of $[y)$ by (a). We have shown (iv). Due to commutativity, also $a \cdot b \leq a$. Hence, if $x \cdot z = y$ then $y \leq x$ and thus $x = x \vee y$ and $x \cdot z = (x \vee y) \cdot z$. By definition, it yields immediately that $(x \vee y) \cdot z = y$ if and only if $z = \psi_y(x \vee y) = x \to y$ which is (iii). □

It was already mentioned that, due to the cancellation property, the multiplication groupoid $(L; \cdot, e)$ of a skew residuated lattice is close to a loop. In what follows, we show how close it is and we get a loop-like characterization of skew residuated lattices satisfying the *distributivity law*

$$x \cdot (a \vee b) = (x \cdot a) \vee (x \cdot b).$$

At first we borrow one concept from [8] and we modify it for our reasons.

**Definition 2.** By *commutative negative divisibility semiloop* (*CND-semiloop*, for short) we mean an algebra $\mathcal{G} = (G; \cdot, \vee, 1)$ of type (2, 2, 0) satisfying the following conditions:

(D1) $(G; \vee, 1)$ is a semilattice with the greatest element 1;
(D2) $(G; \cdot, 1)$ is a cancellative commutative groupoid satisfying $x \cdot 1 = x$;
(D3) $\mathcal{G}$ satisfies the distributivity law $x \cdot (a \vee b) = (x \cdot a) \vee (x \cdot b)$;
(D4) if $b \leq a \cdot x$ then there exists $y \in G$ where $b = a \cdot y$.

Let us note that if $\mathcal{G} = (G; \cdot, e, \vee, \wedge)$ is a commutative l-group (lattice ordered group) then its negative cone, i.e. $G^- = \{x \in G; x \leq e\}$ is a CND-semiloop. More generally, if $\mathcal{L} = (L; \cdot, e, \vee, \wedge)$ is a lattice ordered commutative loop then $L^- = \{x \in L; x \leq e\}$ forms a CND-semiloop. This is the motivation for the name "commutative negative divisibility semiloop".

Now, we show that skew residuated lattices satisfying distributivity law (D3) are just CND-semiloops.

**Theorem 5.** *Let* $\mathcal{G} = (G; \cdot, \vee, 1)$ *be a CND-semiloop. Define* $x \to y := z$ *iff* $(x \vee y) \cdot z = y$ *and* $x \wedge y := x \cdot (x \to y)$. *Then* $\mathcal{L}(G) = (G; \vee, \wedge, \cdot, \to, 1)$ *is a skew residuated lattice satisfying the distributivity law* (D3).

**Proof.** At first, we show that the operation $\to$ is correctly defined. Namely, $(x \vee y) \cdot 1 \geq y$ and hence there exists $z$ such that $(x \vee y) \cdot z = y$ by (D4). Moreover, if $(x \vee y) \cdot z = y$ and $(x \vee y) \cdot w = y$ then, using the cancellation property (D2), we infer $z = w$.

Further, $x \leq y$ if and only if $z \cdot x \leq z \cdot y$ for each $z \in G$ because $z \cdot x \vee z \cdot y = z \cdot y$ if and only if $z \cdot (x \vee y) = z \cdot y$ if and only if $x \vee y = y$. Hence, $x \leq 1$ yields $x \cdot y \leq 1 \cdot y = y$ proving (iv).

(ii) is followed directly by (D2) and (iii) follows by the definition of $\to$. (i) Remains to be proved. We must show that $x \wedge y$ as defined is an infimum of $\{x, y\}$ with respect to the induced order of $(G; \vee)$.

At first, we have $x \cdot (x \to y) \leq x$ and $x \cdot (x \to y) \leq (x \vee y) \cdot (x \to y) = y$, i.e. $x \wedge y := x \cdot (x \to y)$ is a lower bound of $\{x, y\}$. Assume $z \leq x, z \leq y$. Then $z = x \cdot a$ for some $a \in G$ due to the fact that $z \leq x \cdot 1$ and (D4). Hence, $y = y \vee (y \cdot a) \vee (x \cdot a) = \big((x \vee y) \cdot (x \to y)\big) \vee \big((x \vee y) \cdot a\big) = (x \vee y) \cdot \big((x \to y) \vee a\big)$.

However, $y = (x \vee y) \cdot (x \to y)$ and hence $\big((x \to y) \vee a\big) = x \to y$ thus $a \leq x \to y$. We conclude $z = x \cdot a \leq x \cdot (x \to y)$, i.e. $x \cdot (x \to y)$ is the greatest lower bound of $x, y$, i.e. it is really $\inf\{x, y\}$. We have shown (i). $\square$

We are able to prove the converse statement.

**Theorem 6.** *Let* $\mathcal{L} = (L; \vee, \wedge, \cdot, \to, 1)$ *be a skew residuated lattice satisfying the distributivity law* (D3). *Then* $\mathcal{G}(L) = (L; \cdot, \vee, 1)$ *is a CND-semiloop.*

**Proof.** Since $(z \cdot x) \vee (z \cdot y) = z \cdot y$ if and only if $z \cdot (x \vee y) = z \cdot y$ if and only if $x \vee y = y$ due to the cancellation property and (D3), we get

$$x \leq y \text{ if and only if } z \cdot x \leq z \cdot y \tag{$*$}$$

for each $z \in L$.

Hence, $x = x \cdot 1 \leq 1$ and, applying (iv), we get (D1). (D2) follows directly from (ii) and Lemma 2. The axiom (D3) is assumed. We need to only prove (D4).

First we show that

$$(x \cdot y) \wedge (x \cdot z) = x \cdot (y \wedge z). \tag{$**$}$$

Of course, $x \cdot (y \wedge z) \leq x \cdot z$, $x \cdot (y \wedge z) \leq x \cdot y$ by ($*$). Assume $c \leq x \cdot y$, $c \leq x \cdot z$. Then $c = x \cdot (x \to c) \leq x \cdot y, x \cdot z$ because $x \vee c = x$. Hence, $x \to c \leq y, z$ and thus $x \to c \leq y \wedge z$. This gets $c = x \cdot (x \to c) \leq x \cdot (y \wedge z)$. This shows that $x \cdot (y \wedge z)$ is infimum of $\{x \cdot y, x \cdot z\}$ proving ($**$).

Now, assume $b \leq a \cdot x$. By ($**$) we infer $b = b \wedge (a \cdot x) = \big(a \cdot (a \to b)\big) \wedge (a \cdot x) = a \cdot \big((a \to b) \wedge x\big)$, i.e. $b = a \cdot y$ for $y = (a \to b) \wedge x$. $\square$

**Example 1.** Let $L$ be a chain of non-positive real numbers with natural ordering. Of course, it is a lattice. Define $x \odot y := x + y - x \cdot y$; $x \to y := \min\{0, (y - x)/(1 - x)\}$ for all rational numbers $x, y \leq 0$ and $x \odot y := x + y$; $x \to y := \min\{0, y - x\}$ if $x$ or $y$ is not a rational number. It is easy to see that $\mathcal{L} = (L, \vee, \wedge, \odot, \to, 0)$ satisfies (i), (ii), (iv).

Now, we show that $\mathcal{L}$ satisfies (I1) and (I2) too.

1. Assume $x \leq y$. Then $(x \vee y) \odot (x \to y) = y \odot 0 = y + 0 = y$ since $y - x \geq 0$.
2. Assume $y < x \leq 0$ and $x, y$ to be rational numbers. Then $(x \vee y) \odot (x \to y) = x + (y - x)/(1 - x) - x \cdot ((y - x)/(1 - x)) = x + (1 - x)((y - x)/(1 - x)) = x + y - x = y$ since $y - x, (y - x)/(1 - x)$ are rational numbers.
3. Assume $y < x \leq 0$ and $x$ or $y$ not to be rational number. Then $(x \vee y) \odot (x \to y) = x + (y - x) = y$ since $x$ or $y - x$ is not a rational number.

Similarly,

1. Assume $x, y$ to be rational numbers. Then $x \to (x \odot y) = (x + y - x \cdot y - x)/(1 - x) = (y - x \cdot y)/(1 - x) = (y \cdot (1 - x))/(1 - x) = y$ since $x + y - x \cdot y$ is a rational number and $x + y - x \cdot y \leq x$.
2. Assume $x$ or $y$ not to be a rational number. Then $x \to (x \odot y) = (x + y) - x = y$ since $x$ or $x + y$ is not a rational number and $x + y \leq x$.

Now, $\mathcal{L}$ is not a residuated lattice because $-1 \odot -1 = -3 \leq -\sqrt{5}$, but $-1 > -1 \to -\sqrt{5} = -\sqrt{5} + 1$. Moreover, $\mathcal{L}$ do not satisfy (D3) because $-1 \odot (-1 \vee -\sqrt{2}) = -1 \odot -1 = -3 \neq -1 + -\sqrt{2} = -3 \vee (-1 + -\sqrt{2}) = (-1 \odot -1) \vee (-1 \odot -\sqrt{2})$.

Recall that an algebra $\mathcal{A}$ is called *arithmetical* if its congruence lattice $Con\,\mathcal{A}$ is distributive and congruences are *permutable*, i.e. $\Theta \circ \Phi = \Phi \circ \Theta$ for each $\Theta, \Phi \in Con\,\mathcal{A}$, where $\circ$ means the composition of binary relations. It is well-known that if there exists a *Pixley term* $p(x, y, z)$ on $\mathcal{A}$, i.e. $p(x, z, z) = x$, $p(x, y, x) = x$ and $p(x, x, z) = z$

then $\mathcal{A}$ is arithmetical. For varieties of algebras, the existence of a Pixley term is a necessary and sufficient condition arithmetically.

We are going to show that the variety $\mathcal{S}$ of skew residuated lattices satisfies an interesting congruence property.

**Theorem 7.** *The variety $\mathcal{S}$ of all skew residuated lattices has a Pixley term $p(x, y, z) = ((x \to y) \to z) \wedge ((z \to y) \to x) \wedge (x \vee z)$ and hence every $\mathcal{L} \in \mathcal{S}$ is arithmetical.*

**Proof.** Of course, $e \to v = (v \to v) \to v = v \vee v = v$ then

$$p(x, z, z) = ((x \to z) \to z) \wedge (e \to x) \wedge (x \vee z) = (x \vee z) \wedge x = x,$$
$$p(x, x, z) = ((x \to x) \to z) \wedge ((z \to x) \to x) \wedge (x \vee z) = z \wedge (x \vee z) = z,$$
$$p(x, y, x) = ((x \to y) \to x) \wedge ((x \to y) \to x) \wedge (x \vee x) = ((x \to y) \to x) \wedge x = x.$$

since $(x \to y) \to x \geq x$ by Lemma 3(a). □

Fuzzy logics form an important tool of applications in numerous areas outside mathematics. They are axiomatized by means of residuated lattices, see e.g. [1]. The logics of quantum mechanics can be axiomatized by means of effect algebras introduced by Foulis and Bennet [7]. A non-associative version of effect algebras, the so-called skew effect algebras (see [5]) form an axiomatization of the corresponding logic. Since skew effect algebras are represented by means of skew residuated lattices, we can consider our skew residuated lattices as a common generalization of both of these structures and hence as a tool for certain unification of the fuzzy logic and the non-associative logic of quantum mechanics.

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# Congruences on directoids

Ivan Chajda, Jan Krňávek and Helmut Länger

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**Abstract.** Directoids are groupoids defined on every upward directed poset. They fully characterize these posets. Hence, in order to study conguences on directed posets, we can convert the poset into a directoid and study congruences on it. The paper is devoted to several characterizations of congruences on directoids, on directoids with an antitone involution, on directoids with sectionally antitone involutions and on double directoids.

## 1. Introduction

There are various (mutually distinct) definitions of congruences on certain posets, a topic treated by several authors (M. Kolibiar, R. Halaš, G. Dorfer, V. Snášel, to mention a few) in the past see, e.g., [2], [3], [5] and [7]. However, every up-directed poset, i.e. poset $(A, \leq)$ where for any $a, b \in A$ the set $U(a, b) := \{x \in A \mid x \geq a, b\}$ is non-empty, can be organized in a (in general) non-unique way into a directoid which is a certain groupoid, see [4] and [6]. In particular, every upper bounded poset (i.e. a poset with greatest element 1) is up-directed since $1 \in U(a, b)$ for all $a, b \in A$. Now congruences on up-directed posets can be defined as the congruences of a corresponding directoid. This is the leading

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idea of our paper. Posets may have several different representations as directoids which (surely will) give rise to different congruences. Therefore, from the point of view of posets, this concept of congruences is not intrinsic. Directoids as algebras inherently comprise a canonical concept for congruences which can be transferred to the corresponding poset. This motivates us to study which binary relations on directoids, directoids with an antitone involution, directoids with sectionally antitone involutions respectively double directoids are congruences.

We start with the definition of a join-directoid.

**Definition 1.1.** A *(join-)directoid* is a groupoid $(D,\sqcup)$ such that there exists a partial order relation $\leq$ on $D$ with $x,y\leq x\sqcup y$ such that $x\sqcup y=\max(x,y)$ in case $x$ and $y$ are comparable $(x,y\in D)$.

**Remark 1.2.** The relation $\leq$ is uniquely determined by $\sqcup$ since for $x,y\in D$ we have $x\leq y$ if and only if $x\sqcup y=y$.

The great advantage of using directoids as a representation of up-directed posets is that directoids can be characterized by equations (cf. [6]):

**Lemma 1.3.** *A groupoid $(A,\sqcup)$ is a directoid if and only if it satisfies the following identities:*

(i) $x\sqcup x=x$,
(ii) $(x\sqcup y)\sqcup x=x\sqcup y$,
(iii) $y\sqcup(x\sqcup y)=x\sqcup y$,
(iv) $x\sqcup((x\sqcup y)\sqcup z)=(x\sqcup y)\sqcup z$.

In view of investigating congruences on directoids let us mention that G. Dorfer (see [3]) characterized congruences on lattices without using lattice operations. This motivated us to characterize congruences on directoids in a similar way.

## 2. Congruences on directoids

**Definition 2.1.** For every directoid $(D,\sqcup)$ the poset $(D,\leq)$ defined by $x\leq y$ if $x\sqcup y=y$ $(x,y\in D)$ will be called the *poset corresponding* to $(D,\sqcup)$.

Although the operation $\sqcup$ in a directoid $\mathcal{D}=(D,\sqcup)$ need not be monotone (in fact $x\leq y$ implies $z\sqcup x\leq z\sqcup y$ for all $z\in D$ if and only if $\mathcal{D}$ is a join-semilattice, see e. g. [4] and [6]), we are able to prove that classes of congruences on $\mathcal{D}$ are convex.

**Lemma 2.2.** *If $\mathcal{D} = (D, \sqcup)$ is a directoid, $a \in D$ and $\Theta \in \operatorname{Con}\mathcal{D}$ then $[a]\Theta$ is a convex subset of the corresponding poset $(D, \leq)$.*

**Proof.** If $b, c \in [a]\Theta$ and $d \in [b, c]$ then $d \in [d]\Theta = [d \sqcup b]\Theta = [d \sqcup c]\Theta = [c]\Theta = [a]\Theta$. ∎

Next we consider some properties that are automatically fulfilled by congruences on directoids:

**Lemma 2.3.** *Let $\mathcal{D} = (D, \sqcup)$ be a directoid and $\Theta$ a binary relation on $D$ satisfying* (i)–(iii) *for all $x, y, z \in D$:*
(i) *$(x, x \sqcup y), (y, x \sqcup y) \in \Theta$ implies $(x, y) \in \Theta$.*
(ii) *$x \leq y \leq z$ and $(x, z) \in \Theta$ together imply $(x, y) \in \Theta$.*
(iii) *$x \leq y$ and $(x, y) \in \Theta$ together imply $(x \sqcup z, y \sqcup z), (z \sqcup x, z \sqcup y) \in \Theta$.*
*Moreover, assume $(a, d) \in \Theta$ and $b, c \in [a, d]$. Then $(b, c) \in \Theta$.*

**Proof.** Since $a \leq c \leq d$ and $(a, d) \in \Theta$ we have $(a, c) \in \Theta$ according to (ii). Now, using (iii), $a \leq b$ together with $(a, c) \in \Theta$ and $a \leq c$ yields $(b, b \sqcup c) = (b \sqcup a, b \sqcup c) \in \Theta$. Interchanging the roles of $b$ and $c$ we obtain $(a, b) \in \Theta$ and thus, with (iii), $(c, b \sqcup c) = (a \sqcup c, b \sqcup c) \in \Theta$. By (i), $(b, b \sqcup c), (c, b \sqcup c) \in \Theta$ implies $(b, c) \in \Theta$. ∎

Now we are able to characterize congruences on directoids:

**Theorem 2.4.** *If $\mathcal{D} = (D, \sqcup)$ is a directoid and $\Theta$ a reflexive binary relation on $D$ then $\Theta \in \operatorname{Con}\mathcal{D}$ if and only if $\Theta$ satisfies* (i)–(iv) *for all $x, y, z \in D$:*
(i) *$(x, y) \in \Theta$ implies $(x, x \sqcup y), (y, x \sqcup y) \in \Theta$.*
(ii) *$x \leq y$ and $(x, y) \in \Theta$ together imply $(x \sqcup z, y \sqcup z), (z \sqcup x, z \sqcup y) \in \Theta$.*
(iii) *$x \leq y \leq z$ and $(x, y), (y, z) \in \Theta$ together imply $(x, z) \in \Theta$.*
(iv) *$x, y \leq z$ and $(x, z), (y, z) \in \Theta$ together imply $(x, y) \in \Theta$.*

**Proof.** It is straightforward to check that a congruence $\Theta$ satisfies (i) – (iv). To prove the converse, let $a, b, c \in D$.

First assume $(a, b) \in \Theta$. Then $(b, a \sqcup b), (a, a \sqcup b) \in \Theta$ according to (i) and hence $(b, a) \in \Theta$ by (iv) proving symmetry of $\Theta$.

Now assume $(a, b), (b, c) \in \Theta$. Since $(b, c) \in \Theta$ we have $(b, b \sqcup c) \in \Theta$ by (i) which together with $b \leq b \sqcup c$ yields

$$(a \sqcup b, (a \sqcup b) \sqcup (b \sqcup c)) = ((a \sqcup b) \sqcup b, (a \sqcup b) \sqcup (b \sqcup c)) \in \Theta$$

according to (ii). Analogously, since $(a,b)\in\Theta$ we have $(b,a\sqcup b)\in\Theta$ by (i) which together with $b\le a\sqcup b$ yields

$$(b\sqcup c,(a\sqcup b)\sqcup(b\sqcup c))=(b\sqcup(b\sqcup c),(a\sqcup b)\sqcup(b\sqcup c))\in\Theta$$

according to (ii). Now $(a,b)\in\Theta$ implies $(a,a\sqcup b)\in\Theta$ according to (i) which together with $(a\sqcup b,(a\sqcup b)\sqcup(b\sqcup c))\in\Theta$ yields $(a,(a\sqcup b)\sqcup(b\sqcup c))\in\Theta$ according to (iii). Analogously, $(b,c)\in\Theta$ implies $(c,b\sqcup c)\in\Theta$ according to (i) which together with $(b\sqcup c,(a\sqcup b)\sqcup(b\sqcup c))\in\Theta$ yields $(c,(a\sqcup b)\sqcup(b\sqcup c))\in\Theta$ according to (iii). Now $(a,(a\sqcup b)\sqcup(b\sqcup c)),(c,(a\sqcup b)\sqcup(b\sqcup c))\in\Theta$ implies $(a,c)\in\Theta$ according to (iv) proving transitivity of $\Theta$.

Finally, assume $(a,b)\in\Theta$. Then $(a,a\sqcup b),(b,a\sqcup b)\in\Theta$ by (i) and hence

$$(a\sqcup c,(a\sqcup b)\sqcup c),(b\sqcup c,(a\sqcup b)\sqcup c),(c\sqcup a,c\sqcup(a\sqcup b)),(c\sqcup b,c\sqcup(a\sqcup b))\in\Theta$$

according to (ii). Now $(a\sqcup c,b\sqcup c),(c\sqcup a,c\sqcup b)\in\Theta$ follows from symmetry and transitivity of $\Theta$ completing the proof of the theorem.

■

## 3. Congruences on directoids with an antitone involution

In applications, directoids with an antitone involution play an essential role. We define

**Definition 3.1.** An *antitone involution* on a poset $(P,\le)$ is a unary operation $'$ on $P$ satisfying (i) and (ii) for all $x,y\in P$:

(i) $x\le y$ implies $x'\ge y'$.

(ii) $(x')'=x$.

**Definition 3.2.** A *directoid with an antitone involution* is an algebra $(D,\sqcup,')$ of type $(2,1)$ such that $(D,\sqcup)$ is a directoid and $'$ is an antitone involution on the corresponding poset.

For directoids with an antitone involution, Lemma 2.3 can be modified as follows:

**Lemma 3.3.** *Let $\mathcal{D} = (D, \sqcup, ')$ be a directoid with an antitone involution, $x \sqcap y := (x' \sqcup y')'$ for all $x, y \in D$ and $\Theta$ a binary relation on $D$ satisfying* (i)–(iv) *for all $x, y, z \in D$:*

(i) $(x, y) \in \Theta$ *implies* $(y', x') \in \Theta$.

(ii) $(x, y) \in \Theta$ *if and only if* $(x \sqcap y, x \sqcup y) \in \Theta$.

(iii) $x \leq y \leq z$ *and* $(x, y), (y, z) \in \Theta$ *together imply* $(x, z) \in \Theta$.

(iv) $x \leq y$ *and* $(x, y) \in \Theta$ *together imply* $(x \sqcup z, y \sqcup z), (z \sqcap x, z \sqcap y) \in \Theta$.

*Moreover, assume $(a, d) \in \Theta$ and $b, c \in [a, d]$. Then $(b, c) \in \Theta$.*

**Proof.** It is evident that (i) and (iv) together imply that for every $x, y, z \in D$ we have

(v) $x \leq y$ and $(x, y) \in \Theta$ together imply $(x \sqcap z, y \sqcap z), (z \sqcup x, z \sqcup y) \in \Theta$.

Since $a \leq b \leq d$, using (iv) and (v), we infer

$$(b, d) = (a \sqcup b, d \sqcup b) \in \Theta \text{ and } (a, b) = (a \sqcap b, d \sqcap b) \in \Theta.$$

Further, by $a \leq c \leq d$, (v) and (iv), we get

$$(b \sqcap c, c) = (b \sqcap c, d \sqcap c) \in \Theta \text{ and } (c, b \sqcup c) = (a \sqcup c, b \sqcup c) \in \Theta.$$

Since $b \sqcap c \leq c \leq b \sqcup c$ we apply (iii) to get $(b \sqcap c, b \sqcup c) \in \Theta$ and, by (ii), $(b, c) \in \Theta$. ■

Now we are able to characterize congruences on directoids with an antitone involution:

**Theorem 3.4.** *If $\mathcal{D} = (D, \sqcup, ')$ is a directoid with an antitone involution and $\Theta$ a reflexive and symmetric binary relation on $D$ then $\Theta \in \mathrm{Con}\mathcal{D}$ if and only if $\Theta$ satisfies conditions* (i)–(iv) *of Lemma* 3.3 *for all $x, y, z \in D$.*

**Proof.** It is straightforward to check that a congruence $\Theta$ satisfies (i)–(iv) of Lemma 3.3. We want to prove the converse.

We first prove transitivity of $\Theta$. Assume $(x, y), (y, z) \in \Theta$. By (ii) of Lemma 3.3 we have

$$(x \sqcap y, x \sqcup y), (y \sqcap z, y \sqcup z) \in \Theta$$

and, due to (iv) and (v)

$$((y \sqcap z) \sqcap (x \sqcap y), y \sqcap z) = ((y \sqcap z) \sqcap (x \sqcap y), (y \sqcap z) \sqcap (x \sqcup y)) \in \Theta$$

and

$$(y \sqcup z, (y \sqcup z) \sqcup (x \sqcup y)) = ((y \sqcup z) \sqcup (x \sqcap y), (y \sqcup z) \sqcup (x \sqcup y)) \in \Theta.$$

Applying (iii) of Lemma 3.3 twice yields $((y \sqcap z) \sqcap (x \sqcap y), (y \sqcup z) \sqcup (x \sqcup y)) \in \Theta$. Since $x, z \in [(y \sqcap z) \sqcap (x \sqcap y), (y \sqcup z) \sqcup (x \sqcup y)]$, we apply Lemma 3.3 to get $(x, z) \in \Theta$. Hence, $\Theta$ is an equivalence relation on $D$.

Now we prove the substitution property. By (i) of Lemma 3.3, $\Theta$ has the substitution property with respect to $'$. Suppose $(x, y) \in \Theta$ and $z \in D$. Then, by (ii) of Lemma 3.3, $(x \sqcap y, x \sqcup y) \in \Theta$ and $x, y \in [x \sqcap y, x \sqcup y]$. Thus, by Lemma 3.3, $(x, x \sqcup y), (y, x \sqcup y) \in \Theta$. According to (iv) of Lemma 3.3,

$$(x \sqcup z, (x \sqcup y) \sqcup z), (y \sqcup z, (x \sqcup y) \sqcup z) \in \Theta.$$

Since $\Theta$ is symmetric and transitive, we conclude $(x \sqcup z, y \sqcup z) \in \Theta$. Analogously, it can be shown that $(z \sqcup x, z \sqcup y) \in \Theta$. Thus $\Theta \in \mathrm{Con}\mathcal{D}$.

■

## 4. Congruences on directoids with sectionally antitone involutions

Directoids with sectionally antitone involutions were used by several authors (e. g. by R. Halaš and L. Plojhar) for introducing algebras axiomatizing certain propositional logics. Moreover, these structures serve as underlying structures for so-called effect algebras which are used for an axiomatization of the domain of probabilities of observables in the logic of quantum mechanics. Hence it is important to study algebraic constructions of these directoids. In particular, we are going to describe their congruences. We were successful in the case of bounded directoids.

**Definition 4.1.** A *directoid with sectionally antitone involutions* is an algebra $(D, \sqcup, \rightarrow, 1)$ of type $(2, 2, 0)$ such that $(D, \sqcup, 1)$ is a directoid with 1 and for every $a \in D$ there exists an antitone involution $^a$ on $([a, 1], \leq)$ with $(x \sqcup y)^y = x \rightarrow y$ for all $x, y \in D$.

**Remark 4.2.** The unary operation $^a$ on $[a, 1]$ is uniquely determined by $\rightarrow$ since $x^a = x \rightarrow a$ for all $x \in [a, 1]$.

**Definition 4.3.** The *kernel of a congruence* $\Theta$ on a directoid $(D, \sqcup, 1)$ with 1 is the class $[1]\Theta$.

The connection between congruences and their kernels by means of sectional involutions is described in the following two lemmas.

**Lemma 4.4.** *If $\mathcal{D} = (D, \sqcup, \to, 1)$ is a directoid with sectionally antitone involutions, $\Theta \in \operatorname{Con}\mathcal{D}$, $a, b \in D$ and $b \le a$ then $(a, b) \in \Theta$ if and only if $a^b \in [1]\Theta$.*

**Proof.** "$\Rightarrow$": $a^b = a \to b \in [b \to b]\Theta = [b^b]\Theta = [1]\Theta$.

"$\Leftarrow$": $(a, b) = ((a^b)^b, 1^b) = (a^b \to b, 1 \to b) \in \Theta$. ∎

**Lemma 4.5.** *If $\mathcal{D} = (D, \sqcup, \to, 1)$ is a directoid with sectionally antitone involutions, $\Theta \in \operatorname{Con}\mathcal{D}$ and $a, b \in D$ then $(a, b) \in \Theta$ if and only if there exists an element $c$ of $D$ such that $c \ge a, b$ and $c^a, c^b \in [1]\Theta$.*

**Proof.** "$\Rightarrow$": We have $a \sqcup b \ge a, b$,

$$(a \sqcup b)^a = (a \sqcup b) \to a \in [(a \sqcup a) \to a]\Theta = [a \to a]\Theta = [a^a]\Theta = [1]\Theta$$

and

$$(a \sqcup b)^b = (a \sqcup b) \to b \in [(a \sqcup a) \to a]\Theta = [a \to a]\Theta = [a^a]\Theta = [1]\Theta.$$

"$\Leftarrow$": We have

$$(a, c) = (1^a, (c^a)^a) = (1 \to a, c^a \to a) \in \Theta$$

and

$$(c, b) = ((c^b)^b, 1^b) = (c^b \to b, 1 \to b) \in \Theta$$

and hence $(a, b) \in \Theta$. ∎

**Theorem 4.6.** *If $\mathcal{D} = (D, \sqcup, \to, 1)$ is a directoid with sectionally antitone involutions, $\Theta \in \operatorname{Con}\mathcal{D}$ and $F := [1]\Theta$ then for all $x, y, z \in D$,* (i)–(v) *hold:*

(i) *$x \in F$, $x \ge y$ and $x^y \in F$ together imply $y \in F$.*
(ii) *$z \ge x, y$ and $z^x, z^y \in F$ together imply $(x \sqcup y)^x, (x \sqcup y)^y \in F$.*
(iii) *$x \le y \le z$ and $z^y \in F$ together imply $(y^x)^{(z^x)} \in F$.*
(iv) *$x \ge y$ and $x^y \in F$ together imply that there exist $u, v \in D$ with $u \ge x \sqcup z, y \sqcup z$, $v \ge z \sqcup x, z \sqcup y$ and $u^{x \sqcup z}, u^{y \sqcup z}, v^{z \sqcup x}, v^{z \sqcup y} \in F$.*
(v) *$x \le y \le z$ and $y^x \in F$ together imply that there exists an $u \in D$ with $u \ge z^x, z^y$ and $u^{(z^x)}, u^{(z^y)} \in F$.*

**Proof.** (i) $y = 1^y = 1 \to y \in [x^y \to y]\Theta = [(x^y)^y]\Theta = [x]\Theta = [1]\Theta = F$.

(ii) Since

$$x = 1^x = 1 \to x \;\Theta\; z^x \to x = (z^x)^x = z = (z^y)^y = z^y \to y \;\Theta\; 1 \to y = 1^y = y$$

we have

$$(x \sqcup y)^x = (x \sqcup y) \to x \in [(x \sqcup x) \to x]\Theta = [x \to x]\Theta = [x^x]\Theta = [1]\Theta = F$$

and

$$(x \sqcup y)^y = (x \sqcup y) \to y \in [(x \sqcup x) \to x]\Theta = [x \to x]\Theta = [x^x]\Theta = [1]\Theta = F.$$

(iii)

$$\begin{aligned}(y^x)^{(z^x)} &= y^x \to z^x = (y \to x) \to (z \to x) = (1^y \to x) \to ((z^y)^y \to x)\\ &= ((1 \to y) \to x) \to ((z^y \to y) \to x) \in [((1 \to y) \to x) \to ((1 \to y) \to x)]\Theta\\ &= [((1 \to y) \to x)^{(1 \to y) \to x}]\Theta = [1]\Theta = F.\end{aligned}$$

(iv) $(x, y) = ((x^y)^y, 1^y) = (x^y \to y, 1 \to y) \in \Theta$ and hence $(x \sqcup z, y \sqcup z), (z \sqcup x, z \sqcup y) \in \Theta$. According to Lemma 4.5 there exist $u, v \in D$ with $u^{x \sqcup z}, u^{y \sqcup z}, v^{z \sqcup x}, v^{z \sqcup y} \in F$.

(v) $(x, y) = (1^x, (y^x)^x) = (1 \to x, y^x \to x) \in \Theta$ and hence $(z^x, z^y) = (z \to x, z \to y) \in \Theta$. Using Lemma 4.5 we conclude that there exists some $u \in D$ with $u^{(z^x)}, u^{(z^y)} \in F$.

∎

We are going to introduce the concept of a filter in order to explain the role of the kernel of the corresponding congruence.

**Definition 4.7.** A *filter* of a directoid $(D, \sqcup, \to, 1)$ with sectionally antitone involutions is a subset $F$ of $D$ containing the element 1 and satisfying (i) – (v) of Theorem 4.6.

**Definition 4.8.** For a directoid $(D, \sqcup, \to, 1)$ with sectionally antitone involutions and a non-empty subset $F$ of $D$ we define

$$\Theta_F := \{(x, y) \in D^2 \mid \text{there exists an element } z \text{ of } D \text{ with } z \geq x, y \text{ and } z^x, z^y \in F\}.$$

**Lemma 4.9.** *If $\mathcal{D} = (D, \sqcup, \to, 1)$ is a directoid with sectionally antitone involutions, $a, b \in D$, $a \geq b$ and $F$ is a filter of $\mathcal{D}$ then $(a, b) \in \Theta_F$ if and only if $a^b \in F$.*

**Proof.** "$\Rightarrow$": There exists some $c \in D$ with $c \geq a, b$ and $c^a, c^b \in F$. By (ii) of Theorem 4.6 we conclude $a^b = (a \sqcup b)^b \in F$.

"$\Leftarrow$": According to Definition 4.7, $a^a = 1 \in F$ and hence $a^a, a^b \in F$ showing $(a, b) \in \Theta_F$.

∎

**Lemma 4.10.** *If $\mathcal{D} = (D, \sqcup, \rightarrow, 1)$ is a directoid with sectionally antitone involutions, $F$ is a filter of $\mathcal{D}$ and $a, b \in D$ then $(a, b) \in \Theta_F$ if and only if $(a, a \sqcup b), (b, a \sqcup b) \in \Theta_F$.*

**Proof.** "$\Rightarrow$": There exists an element $c$ of $D$ with $c \geq a, b$ and $c^a, c^b \in F$ and, by (ii) of Theorem 4.6, we have $(a \sqcup b)^a, (a \sqcup b)^b \in F$. Using (iv) of Theorem 4.6 with $x = a \sqcup b$, $y = a$ and $z = a$ yields the existence of some $d \in D$ with $d \geq a \sqcup b$ and $d^{a \sqcup b}, d^a \in F$ and, by the definition of $\Theta_F$, $(a, a \sqcup b) \in \Theta_F$. Analogously, $(b, a \sqcup b) \in \Theta_F$ can be shown.

"$\Leftarrow$": According to the definition of $\Theta_F$ there exist $c, d \in D$ with

$$c, d \geq a \sqcup b \text{ and } c^a, c^{a \sqcup b}, d^b, d^{a \sqcup b} \in F.$$

By (ii) of Theorem 4.6 we obtain $(a \sqcup b)^a, (a \sqcup b)^b \in F$. By the definition of $\Theta_F$ we conclude $(a, b) \in \Theta_F$.

∎

Now we are able to characterize congruences on bounded directoids with sectionally antitone involutions.

**Lemma 4.11.** *Let $\mathcal{D} = (D, \sqcup, \rightarrow, 0, 1)$ be a bounded directoid with sectionally antitone involutions and $\Theta$ a reflexive binary relation on $D$. Then $\Theta \in \mathrm{Con}\mathcal{D}$ if and only if $\mathcal{D}$ satisfies* (i)–(iv) *of Theorem* 2.4 *and for all $x, y, z \in D$,* (v) *and* (vi) *hold:*

(v) *$z \leq x \leq y$ and $(x, y) \in \Theta$ together imply $(x^z, y^z) \in \Theta$.*
(vi) *$x \leq y \leq z$ and $(x, y) \in \Theta$ together imply $(z^x, z^y) \in \Theta$.*

**Proof.** It is straightforward to check that a congruence $\Theta$ satisfies (i)–(iv) of Theorem 2.4 as well as (v) and (vi). We want to prove the converse.

From Theorem 2.4 it follows that $\Theta$ is a congruence on $(D, \sqcup)$. Assume $(a, b) \in \Theta$ and let $c \in D$.

Then $c \leq a \sqcup c \leq (a \sqcup c) \sqcup (b \sqcup c)$ and $(a \sqcup c, (a \sqcup c) \sqcup (b \sqcup c)) \in \Theta$ and hence

$$((a \sqcup c)^c, ((a \sqcup c) \sqcup (b \sqcup c))^c) \in \Theta$$

according to (v). Analogously, it follows $((b \sqcup c)^c, ((a \sqcup c) \sqcup (b \sqcup c))^c) \in \Theta$. Now $(a \to c, b \to c) = ((a \sqcup c)^c, (b \sqcup c)^c) \in \Theta$ follows from symmetry and transitivity of $\Theta$.

Moreover, $0 \le a \le a \sqcup b$ and $(a, a \sqcup b) \in \Theta$ and hence $(a^0, (a \sqcup b)^0) \in \Theta$ according to (v). Analogously, $(b^0, (a \sqcup b)^0) \in \Theta$. Now symmetry and transitivity of $\Theta$ yields $(a^0, b^0) \in \Theta$. Moreover, we have $(a^0 \sqcup b^0)^0 \le a \le c \sqcup a$ and $((a^0 \sqcup b^0)^0, a) \in \Theta$ and hence $((c \sqcup a)^{(a^0 \sqcup b^0)^0}, (c \sqcup a)^a) \in \Theta$ according to (vi). Analogously, $((c \sqcup b)^{(a^0 \sqcup b^0)^0}, (c \sqcup b)^b) \in \Theta$. Finally, we have $(a^0 \sqcup b^0)^0 \le c \sqcup a \le (c \sqcup a) \sqcup (c \sqcup b)$ and $(c \sqcup a, (c \sqcup a) \sqcup (c \sqcup b)) \in \Theta$ and hence $((c \sqcup a)^{(a^0 \sqcup b^0)^0}, ((c \sqcup a) \sqcup (c \sqcup b))^{(a^0 \sqcup b^0)^0}) \in \Theta$ according to (v). Analogously, $((c \sqcup b)^{(a^0 \sqcup b^0)^0}, ((c \sqcup a) \sqcup (c \sqcup b))^{(a^0 \sqcup b^0)^0}) \in \Theta$. Now $(c \to a, c \to b) = ((c \sqcup a)^a, (c \sqcup b)^b) \in \Theta$ follows by symmetry and transitivity of $\Theta$ completing the proof of the lemma.

■

**Theorem 4.12.** *If $\mathcal{D} = (D, \sqcup, \to, 0, 1)$ is a bounded directoid with sectionally antitone involutions and $F$ is a filter of $\mathcal{D}$ then $\Theta_F \in \mathrm{Con}\mathcal{D}$.*

**Proof.** It is evident that $\Theta_F$ is reflexive and symmetric. According to Lemma 4.11, we need only check conditions (i)–(iv) of Theorem 2.4 and conditions (v) and (vi) of Lemma 4.11.

Let $a, b, c \in D$.

(i) holds because of Lemma 4.10.

(ii) If $a \le b$ and $(a, b) \in \Theta_F$ then by Lemma 4.9 we have $b^a \in F$. According to Theorem 4.6 (iv) there exists an element $d$ of $D$ with $d \ge a \sqcup c, b \sqcup c$ and $d^{a \sqcup c}, d^{b \sqcup c} \in \Theta_F$. By the definition of $\Theta_F$ it follows $(a \sqcup c, b \sqcup c) \in \Theta_F$. Analogously, $(c \sqcup a, c \sqcup b) \in \Theta_F$ can be proved.

(iii) If $a \le b \le c$ and $(a, b), (b, c) \in \Theta_F$ then $b^a, c^b \in F$ according to Lemma 4.9 and hence $(b^a)^{(c^a)} \in F$ by Theorem 4.6 (iii) whence $c^a \in F$ using Theorem 4.6 (i). This implies $(a, c) \in \Theta_F$ by Lemma 4.9.

(iv) If $a, b \le c$ and $(a, c), (b, c) \in \Theta_F$ then $c^a, c^b \in F$ because of Lemma 4.9 and hence $(a, b) \in \Theta_F$ according to the definition of $\Theta_F$.

(v) If $c \le a \le b$ and $(a, b) \in \Theta_F$ then $b^a \in F$ because of Lemma 4.9 and hence $(a^c)^{(b^c)} \in F$ according to Theorem 4.6 (iii) whence $(a^c, b^c) \in \Theta_F$ by Lemma 4.9.

(vi) If $a \le b \le c$ and $(a, b) \in \Theta_F$ then $b^a \in F$ because of Lemma 4.9 and hence according to Theorem 4.6 (v) there exists an element $d$ of $D$ with $d \ge c^a, c^b$ and $d^{(c^a)}, d^{(c^b)} \in F$ showing $(c^a, c^b) \in \Theta_F$ completing the proof of the theorem.

■

## 5. Congruences on double directoids

As mentioned in Chapter 3, a directoid having an antitone involution can be considered as an algebra with two binary operations where the second one is defined by the first one via the de Morgan laws. However, there exists another approach to get algebras which are directoids with respect to both basic operations and which satisfy some absorption laws, but which need not have an antitone involution. These algebras will be called double directoids. It is worth noticing that they may differ from so-called $\lambda$-lattices (cf. [1]) since the fundamental operations need not be commutative. In fact $\lambda$-lattices coincide with double directoids both operations of which are commutative.

For introducing this concept we need the following definition:

**Definition 5.1.** A *meet-directoid* is a groupoid $(D,\sqcap)$ such that there exists a partial order relation $\leq$ on $D$ with $x\sqcap y\leq x,y$ such that $x\sqcap y=\min(x,y)$ in case $x$ and $y$ are comparable $(x,y\in D)$.

**Remark 5.2.** The relation $\leq$ is uniquely determined by $\sqcap$ since for $x,y\in D$ we have $x\leq y$ if and only if $x\sqcap y=x$.

Analogously to the case of join-directoids, meet-directoids can be characterized by equations (cf. [6]):

**Lemma 5.3.** *A groupoid $(A,\sqcap)$ is a meet-directoid if and only if it satisfies the following identities:*
(i) $x\sqcap x=x$,
(ii) $y\sqcap(x\sqcap y)=x\sqcap y$,
(iii) $(x\sqcap y)\sqcap x=x\sqcap y$,
(iv) $(x\sqcap(y\sqcap z))\sqcap z=x\sqcap(y\sqcap z)$.

Now we define our concept of a double directoid:

**Definition 5.4.** A *double directoid* is an algebra $\mathcal{D}=(D,\sqcup,\sqcap)$ of type $(2,2)$ such that $(D,\sqcup)$ is a join-directoid, $(D,\sqcap)$ is a meet-directoid and the absorption laws

$$x\sqcap(x\sqcup y)=(y\sqcap x)\sqcup x=x$$

hold.

**Remark 5.5.** It is easy to see that $x \sqcup y = y$ if and only if $x \sqcap y = x$. If this is the case we write $x \leq y$. The ordered pair $(D, \leq)$ is then a poset which we call the *poset corresponding* to $(D, \sqcup, \sqcap)$. It follows that the poset corresponding to $(D, \sqcup)$ coincides with the poset corresponding to $(D, \sqcap)$ and with the poset corresponding to $\mathcal{D}$.

Now we are able to characterize congruences on double directoids in a similar way as it was done for lattices by G. Dorfer (cf. [3]).

**Theorem 5.6.** *If $\mathcal{D} = (D, \sqcup, \sqcap)$ is a double directoid and $\Theta$ an equivalence relation on $D$ then the following are equivalent:*

(i) $\Theta \in \mathrm{Con}\mathcal{D}$

(ii) *$\Theta$ satisfies the following conditions for all $x, y, z \in D$:*

(1) *If $x, y$ are comparable and $(x, y) \in \Theta$ then there exists an element $u$ of $[x \sqcup z]\Theta$ with $u \geq y \sqcup z$ and an element $v$ of $[z \sqcup x]\Theta$ with $v \geq z \sqcup y$.*

(2) *If $x, y$ are comparable and $(x, y) \in \Theta$ then there exists an element $u$ of $[x \sqcap z]\Theta$ with $u \leq y \sqcap z$ and an element $v$ of $[z \sqcap x]\Theta$ with $v \leq z \sqcap y$.*

(3) *$[x]\Theta$ is a convex subalgebra of $\mathcal{D}$.*

(4) *$x \leq y$ and $z \in [x]\Theta$ implies that there exists $u \in [y]\Theta$ with $z \leq u$.*

(5) *$x \leq y$ and $z \in [y]\Theta$ implies that there exists $u \in [x]\Theta$ with $u \leq z$.*

(iii) *$\Theta$ satisfies the following conditions for all $x, y, z \in D$:*

(6) *$(x, y) \in \Theta$ implies $(x, x \sqcup y), (y, x \sqcup y) \in \Theta$.*

(7) *$x \leq y$ and $(x, y) \in \Theta$ together imply $(x \sqcup z, y \sqcup z), (z \sqcup x, z \sqcup y) \in \Theta$.*

(8) *$x \leq y$ and $(x, y) \in \Theta$ together imply $(x \sqcap z, y \sqcap z), (z \sqcap x, z \sqcap y) \in \Theta$.*

**Proof.** (i)$\Rightarrow$(ii):

(1) Put $u := y \sqcup z$ and $v := z \sqcup y$.

(2) Put $u := y \sqcap z$ and $v := z \sqcap y$.

(3) If $y, z \in [x]\Theta$ then $y \sqcup z \in [x \sqcup x]\Theta = [x]\Theta$ and $y \sqcap z \in [x \sqcap x]\Theta = [x]\Theta$. If, moreover, $u \in [y, z]$ then $u = y \sqcup u \in [z \sqcup u]\Theta = [z]\Theta = [x]\Theta$.

(4) Put $u := z \sqcup y$.

(5) Put $u := z \sqcap x$.

(ii)$\Rightarrow$(iii):

(6) follows from (3).

(7) Assume $x \leq y$ and $(x, y) \in \Theta$. According to (1) there exists an element $u$ of $[x \sqcup z]\Theta$ with $u \geq y \sqcup z$ and an element $v$ of $[y \sqcup z]\Theta$ with $v \geq x \sqcup z$. According to (5) there exists an element $w$ of $[x \sqcup z]\Theta$ with $w \leq y \sqcup z$. This yields $w, u \in [x \sqcup z]\Theta$ and $w \leq y \sqcup z \leq u$. From (3) it follows $(x \sqcup z, y \sqcup z) \in \Theta$. The fact $(z \sqcup x, z \sqcup y) \in \Theta$ follows analogously.

(8) can be proved similarly as (7).

(iii)⇒(i): Assume $(x,y)\in\Theta$. According to (6) we have $(x,x\sqcup y)\in\Theta$ and (7) implies $(x\sqcup z,(x\sqcup y)\sqcup z)\in\Theta$. Analogously, $(y,x\sqcup y),(y\sqcup z,(x\sqcup y)\sqcup z)\in\Theta$ because of (6) and (7). Together we obtain $(x\sqcup z,y\sqcup z)\in\Theta$, Analogously, $(z\sqcup x,z\sqcup y)\in\Theta$ can be proved. Similarly, $(x,x\sqcup y),(x\sqcap z,(x\sqcup y)\sqcap z)\in\Theta$ by (6) and (8). Analogously, $(y,x\sqcup y),(y\sqcap z,(x\sqcup y)\sqcap z)\in\Theta$ by (6) and (8). This yields $(x\sqcap z,y\sqcap z)\in\Theta$. The relation $(z\sqcap x,z\sqcap y)\in\Theta$ can be proved in an analogous way.

■

**Lemma 5.7.** *Let $(D,\sqcup,\sqcap)$ be a double directoid, $a,b,c\in D$, $\Theta$ an equivalence relation on $D$ satisfying (3) – (5) and assume $(a,b)\in\Theta$. Then there exists an element $d$ of $[a\sqcap c]\Theta$ and an element $e$ of $[a\sqcup c]\Theta$ with $b,c\in[d,e]$.*

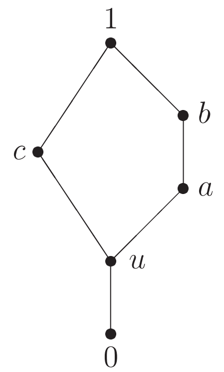
**Proof.** Because of $a\sqcap c\le a$ there exists an element $d$ of $[a\sqcap c]\Theta$ such that $d\le b$ according to (5). Thus $d\sqcap(a\sqcap c)\le b,c$ and $d\sqcap(a\sqcap c)\in[a\sqcap c]\Theta$ according to (3).

Similarly, because of $a\le a\sqcup c$ there exists an element $e$ of $[a\sqcup c]\Theta$ such that $b\le e$ according to (4). Thus $b,c\le e\sqcup(a\sqcup c)$ and $e\sqcup(a\sqcup c)\in[a\sqcup c]\Theta$ according to (3).

■

Contrary to the case of lattices, we need two more conditions (1) and (2). Namely, in [3] it was shown that for a lattice $\mathcal{D}=(D,\sqcup,\sqcap)$, an equivalence relation $\Theta$ on $D$ satisfying (3) – (5) is already a congruence on $\mathcal{D}$. That this does not hold in general for double directoids is shown by the following example:

**Example 5.8.** Let $\mathcal{D}=(D,\sqcup,\sqcap)=(\{0,u,a,b,c,1\},\sqcup,\sqcap)$ be the commutative double directoid (i.e. $\lambda$-lattice) with the following Hasse diagram and $a\sqcap c:=0$ and $b\sqcap c:=u$:




Then the equivalence relation on $D$ having $\{a,b\}$ as its unique non-singleton class satisfies (3) – (5) and even (1), but not (2) and is therefore not a congruence on $\mathcal{D}$.

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I. Chajda, Palacký University Olomouc, Department of Algebra and Geometry, 17.listopadu 12, 77146 Olomouc, Czech Republic; *e-mail*: ivan.chajda@upol.cz

J. Krňávek, Palacký University Olomouc, Department of Algebra and Geometry, 17.listopadu 12, 77146 Olomouc, Czech Republic; *e-mail*: jan.krnavek@upol.cz

H. Länger, Vienna University of Technology, Institute of Discrete Mathematics and Geometry, Wiedner Hauptstraße 8-10, 1040 Vienna, Austria; *e-mail*: h.laenger@tuwien.ac.at

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