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ÚSTAV MATEMATIKY

ASYMPTOTIC PROPERTIES OF SOLUTIONS OF THE SECOND-ORDER DISCRETE EMDEN-FOWLER EQUATION

ASYMPTOTICKÉ VLASTNOSTI ŘEŠENÍ DISKRÉTNÍ EMDEN-FOWLEROVY ROVNICE DRUHÉHO ŘÁDU

DOCTORAL THESIS TOPIC

POJEDNÁNÍ

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ABSTRACT

In the literature a differential second–order nonlinear Emden–Fowler equation

$$y'' \pm x^{\alpha} y^{m} = 0$$

is often investigated, where α and m are constants.

This thesis deals with a discrete equivalent of the second order Emden-Fowler differential equation

$$\Delta^2 u(k) \pm k^{\alpha} u^{m}(k) = 0,$$

where $k \in \mathbb{N}(k_0) := \{k_0, k_0 + 1, ...\}$ is an independent variable, k_0 is an integer and $u\colon \mathbb{N}(k_0) \to \mathbb{R}$ is an unknown solution. In this equation, $\Delta^2 u(k) = \Delta(\Delta u(k))$, $\Delta u(k)$ is the the first-order forward difference of $u(k)$, i.e., $\Delta u(k) = u(k+1) - u(k)$, and $\Delta^2(k)$ is its second–order forward difference, i.e., $\Delta^2 u(k) = u(k + 2) - 2u(k + 1) + u(k)$, α, m are real numbers. The asymptotic behaviour of the solutions to this equation is discussed and the conditions are found such that there exists a power-type asymptotic:

$$u(k) \sim 1/k^s,$$

where s is some constant.

We also discuss a discrete analogy of so-called "blow-up" solutions in the classical theory of differential equations, i.e., the solutions for which there exists a point x^* such that

$$\lim_{x \to x^*} y(x) = \infty,$$

where $y(x)$ is a solution of the Emden-Fowler differential equation

$$y''(x) = y^s(x),$$

with $s \neq 1$ being a real number.

The results obtained are compared to those already known and illustrated with examples.

KEYWORDS

Discrete equation, Emden-Fowler equation, nonlinear equation, system of discrete equations, asymptotic properties, retract principle.

ABSTRAKT

V literatuře je často studována Emden–Fowlerova nelineární diferenciální rovnice druhého řádu

$$y'' \pm x^\alpha y^m = 0,$$

kde α a m jsou konstanty.
V disertační práci je analyzována diskrétní analogie Emden-Fowlerovy diferenciální rovnice

$$\Delta^2 u(k) \pm k^\alpha u^m(k) = 0,$$

kde $k \in \mathbb{N}(k_0) := \{k_0, k_0+1, ...\}$ je nezávislá proměnná, k_0 je celé číslo a $u\colon \mathbb{N}(k_0) \to \mathbb{R}$ je řešení. V této rovnici je $\Delta^2 u(k) = \Delta(\Delta u(k))$, kde $\Delta u(k)$ je diference vpřed prvního řádu funkce $u(k)$, tj. $\Delta u(k) = u(k+1) - u(k)$ a $\Delta^2(k)$ je její diference vpřed druhého řádu, tj. $\Delta^2 u(k) = u(k+2) - 2u(k+1) + u(k)$, a α, m jsou reálná čísla. Je diskutováno asymptotické chování řešení této rovnice a jsou stanoveny podmínky, garantující existence řešení s asymptotikou mocninného typu:

$$u(k) \sim 1/k^s,$$

kde s je vhodná konstanta.
Je také zkoumána diskrétní analogie tzv. "blow-up" řešení (neohraničených řešení) známých v klasické teorii diferenciálních rovnic, tj. řešení pro která v některém bodě x^* platí

$$\lim_{x \to x^*} y(x) = \infty,$$

kde $y(x)$ je řešení Emden-Fowlerovy diferenciální rovnice

$$y''(x) = y^s(x),$$

kde $s \neq 1$ je reálné číslo.
Výsledky jsou ilustrovány příklady a porovnávány s výsledky doposud známými.

KLÍČOVÁ SLOVA

Diskrétní rovnice, Emden-Fowlerova rovnice, nelineární rovnice, systém diskrétních rovnic, asymptoticke chování, princip retraktu.

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DECLARATION

I declare that I have written the Doctoral Thesis titled "Asymptotic properties of the solutions to the second-order discrete Emden-Fowler type equation" independently, under the guidance of the supervisor and using exclusively the technical references and other sources of information cited in the thesis and listed in the comprehensive bibliography at the end of the thesis.

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1 Introduction

Classical differential equations are widely used in different processes. For example, the input continuous signal of the linear system $x(t)$ and the corresponding output signal $y(t)$ can be connected by some differential equation. But if we want to replace a continuous variable t with a discrete one, it leads to the replacement of the differential equation with a difference equation.

To analyse difference equations, we can also use different analytical methods, most of them using approaches similar to those of the classical differential equation. We can also use numerical methods of solving obtaining a result in the form of a numerical sequence, therefore, the difference equation in this case is perceived as an algorithm for the functioning of a discrete system for which a suitable computer programs can be devised.

We also mention the contribution of the mathematicians Bohner M., Georgiev, S.G. and Peterson A.C [8], [9] and [10] to the creation of a theory that combines both classical calculus and the theory of difference equations, expanding the scope of application to continuous scales, as well as allowing us to consider both more complex discrete scales and a combination of discrete-continuous time scales.

In the doctoral thesis we discuss the asymptotic properties of the Emden-Fowler discrete equation. This equation is an extension to the theory of difference equation of a well-known Lane-Emden-Fowler differential equation, which has a great deal of applications in physics, cosmology, meteorology and chemistry. In [22], the form of this equation was

$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} + \beta^2 u^n = 0, \tag{1.1}$$

where r is the radius of a polytropic gas sphere, $n = 1/(k-1)$, with k being the polytropic index and β some physical constant.

The change of variables $u = y/r$ transforms (1.1) into the following equation

$$y'' + \beta^2 r^{1-n} y^n = 0.$$

Now we get the form that is often used in mathematical literature:

$$y'' + x^\sigma |y|^{k-1} y = 0,$$

where k and σ are constants. Later, this equation was generalized for the case of n-th order differential equation

$$y^{(n)} + p(x)|y|^k \mathrm{sgn} y = 0, \tag{1.2}$$

where $n > 2$ is an integer, $p(x)$ is a continuous function, and k is a constant.

Different properties of the solutions of Emden-Fowler differential equations were investigated by many authors. The R.Bellman's monograph [6] had a great influence on the investigation of the Emden-Fowler equations, where he discussed the asymptotic properties of the solutions tending to infinity. F.V.Atkinson in [5] also made a significant contribution to the theory of Emden-Fowler equations. The list of works devoted to the Emden-Fowler type equations is very wide, we will mention some of them: H. J. Lane [38], H. Fowler [24], I. T. Kiguradze, T. A. Chanturia [34], V. A. Kondratev, V. S. Samovol [35], I.V. Astashova [3], H. Goenner, P. Havas [25], S. C. Mancas, H. C. Rost [39], C. M. Khalique [28] and P. Guha [27].

1.1 The current state

In previous chapter, we have already mentioned that there are many papers and books on the Emden-Fowler differential equation. However, turning our attention to the discrete case, we see that there are not so many articles about this type of equation. We can refer a paper by L. Erbe, J. Baoguo and J. Peterson [23] dealing with non-oscillatory solutions of Emden-Fowler type discrete equations providing asymptotic properties of a similar equation on time scales.

V. Kharkov (we refer to [29, 30, 31, 32, 33]) has discussed, except other, the asymptotic properties of the equation of Emden-Fowler type

$$\Delta^2 y_n = \alpha p_n |y_{n+1}|^{\sigma} \mathrm{sgn} y_{n+1},$$

where $\alpha \in \{-1.1\}$, $\sigma \in \mathbb{R}\backslash\{0,1\}$ and the sequence p_n satisfies the following condition

$$\lim_{n\to+\infty} \frac{n\Delta p_n}{p_n} = k, \quad k \in \mathbb{R} \setminus \{-2, -1-\sigma\}.$$

In the thesis we will discuss the asymptotic properties of the solutions to the another discrete equivalent of the Emden-Fowler equation. Let k_0 be a natural number. By $\mathbb{N}(k_0)$ we denote the set of all natural numbers greater than or equal to k_0, that is,

$$\mathbb{N}(k_0) := \{k_0, k_0+1, ...\}.$$

We will study the asymptotic behaviour of the solutions of a second-order non-linear discrete equation of Emden-Fowler type

$$\Delta^2 u(k) \pm k^{\alpha} u^m(k) = 0, \tag{1.3}$$

where $u\colon \mathbb{N}(k_0) \to \mathbb{R}$ is an unknown solution, $\Delta u(k)$ is its first-order forward difference, i.e.,

$$\Delta u(k) = u(k+1) - u(k), \tag{1.4}$$

$\Delta^2(k)$ is its second-order forward difference, i.e.,

$$\Delta^2 u(k) = \Delta(\Delta u(k)) = u(k+2) - 2u(k+1) + u(k),$$

and α, m are real numbers. A function $u = u^*\colon \mathbb{N}(k_0) \to \mathbb{R}$ is called a solution of equation (1.3) if the equality

$$\Delta^2 u^*(k) \pm k^{\alpha}(u^*(k))^m = 0$$

holds for every $k \in \mathbb{N}(k_0)$.

Equation (1.3) is a discretization of the classical Emden-Fowler second-order differential equation (we refer, e.g., to [6])

$$y'' \pm x^{\alpha} y^m = 0, \tag{1.5}$$

where the second-order derivative is replaced by a second-order forward difference and the continuous independent variable is replaced by a discrete one.

One special case of the discrete Emden-Fowler type equation has been discussed in a recent article by Christianen, M.H.M., Janssen, A.J.E.M., Vlasiou, M., and Zwart, B. [12], which describes the charging process of electric vehicles, considering their random arrivals, their stochastic demand for energy at charging stations, and the characteristics of the electricity distribution network. The equation

$$v_{j+1} - 2v_j + v_{j-1} = \frac{k}{v_j}$$

is considered, where $j = 1, 2, ...;$ $v_0 = 1,$ $v_1 = 1 + k$ proving that there exists a solution with "logarithmic" asymptotic behaviour, i.e.

$$v_j \sim j(2k\ln(j))^{1/2},$$

when $j \to \infty$.

1.2 Preliminaries

This section introduces the notation, definitions and theorems used in the thesis.

Definition 1. A function $u_{upp} : \mathbb{B} \to \mathbb{R}$ is said to be an *approximate solution* to equation (1.3) of an order g where $g : \mathbb{N}(k_0) \to \mathbb{R}$ if

$$\lim_{k\to\infty} [\Delta^3 u_{upp}(k) \pm k^\alpha u^n_{upp}(k)]g(k) = 0.$$

If the main term (i.e. the term being asymptotically leading) in $u_{upp}(k)$ is a power-type function, we say that it is a *power-type* approximate solution.

Definition 2. We say that a function $x(k)$ is of order $O(y(k))$ if there exists a constant K, such that

$$|x(k)| \leq |M(y(k))|$$

on $\mathbb{N}(k_0)$. We use the shorter notation $O(y(k))$.

Definition 3. We say that a function $x(k)$ is of order $o(y(k))$ if $y(k) \neq 0$ for all sufficiently large $k \in \mathbb{N}(k_0)$ and

$$\lim_{k\to\infty} \frac{x(k)}{y(k)} = 0.$$

This property is more simply written as $x(k) = o(y(k))$.

In computations below, we will also use the following modification of the Landau order symbol big "O".

Definition 4. Let $f \colon \mathbb{N}(k_0) \to \mathbb{R}$, $g \colon \mathbb{N}(k_0) \to (0, \infty)$. We write $f = O^+(g)$ if there exists an index $k_1 \geq k_0$ such that inequality

$$|f(k)| \leqslant g(k), \quad \forall k \in \mathbb{N}(k_1)$$

holds.

Definition 5. A solution of the equation (1.2) is called a *blow-up* one if there exists some point $x_0 \in \mathbb{R}$, such that

$$\lim_{x\to x_0-0} y(x) = \infty.$$

1.2.1 Binomial series

In the proof of the main results, we use the following formula for the decomposition of a binom into a "binomial series".

Let $r \in \mathbb{R}$, $p \in \mathbb{R}$, $k \in \mathbb{N}(k_0)$ and let

$$\left|\frac{r}{k}\right| < 1.$$

Then,

$$\left(1+\frac{r}{k}\right)^p = 1 + \binom{p}{1}\frac{r}{k} + \binom{p}{2}\frac{r^2}{k^2} + \binom{p}{3}\frac{r^3}{k^3} + ... + \binom{p}{l}\frac{r^l}{k^l} + ... \tag{1.6}$$

where

$$\binom{p}{l} := p(p-1)...(p-l+1)\frac{1}{l!}.$$

1.2.2 Discrete retract principle

In the proofs of the results on the asymptotic behaviour of solutions to equation (1.3), we use an auxiliary apparatus taken from [13, 15] and described below. Consider a system of discrete equations

$$\Delta Y(k) = F(k, Y(k)), \quad k \in \mathbb{N}(k_0) \tag{1.7}$$

where $Y = (Y_0, \dots, Y_{n-1})^T$ and

$$F(k, Y) = (F_1(k, Y), \dots, F_n(k, Y))^T \colon \mathbb{N}(k_0) \times \mathbb{R}^n \to \mathbb{R}^n. \tag{1.8}$$

A solution $Y = Y(k)$ of system (1.7) is defined as a function $Y \colon \mathbb{N}(k_0) \to \mathbb{R}^n$ satisfying (1.7) for each $k \in \mathbb{N}(k_0)$. The initial problem

$$Y(k_0) = Y^0 = (Y_0^0, \dots, Y_{n-1}^0)^T \in \mathbb{R}^n$$

defines a unique solution to (1.7). Obviously, if $F(k, Y)$ is continuous with respect to Y, then the initial problem (1.7), (1.8) defines a unique solution $Y = Y(k_0, Y^0)(k)$, where $Y(k_0, Y^0)$ indicates a dependence of the solution on the initial point (k_0, Y^0), which depends continuously on the value Y^0. Let $b_i, c_i \colon \mathbb{N}(k_0) \to \mathbb{R}$, $i = 1, ..., n$ be given functions satisfying

$$b_i(k) < c_i(k), \quad k \in \mathbb{N}(k_0), \quad i = 1, ..., n. \tag{1.9}$$

Define auxiliary functions $B_i, C_i \colon \mathbb{N}(k_0) \times \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, n$ as

$$B_i(k, Y) := -Y_{i-1} + b_i(k), \quad C_i(k, Y) := Y_{i-1} - c_i(k) \tag{1.10}$$

and auxiliary sets

$$\Omega_B^i := \{(k,Y)\colon k \in \mathbb{N}(k_0), B_i(k,Y) = 0, B_j(k,Y) \le 0, C_p(k,Y) \le 0, \\ \forall\ j,p = 1,\dots,n,\ j \ne i\}, \quad (1.11)$$

$$\Omega_C^i := \{(k,Y)\colon k \in \mathbb{N}(k_0), C_i(k,Y) = 0, B_j(k,Y) \le 0, C_p(k,Y) \le 0, \\ \forall\ j,p = 1,\dots,n,\ p \ne i\} \quad (1.12)$$

where $i = 1, ..., n$.

Playing a crucial role in the proofs and being suitable for applications, the following lemma is a slight modification of [13, Theorem 1] (see [15, Theorem 2] also).

1.2.3 Auxiliary result of a Liapunov type

A result formulated below is proved in [14] by Liapunov-like reasonings.

Definition 6. The set Ω is called the *regular polyfacial set* with respect to the discrete system (1.7) if

$$b_i(k+1) - b_i(k) < F_i(k,Y) < c_i(k+1) - b_i(k), \quad (1.13)$$

for every $i = 1,\dots,n$ and every $(k,Y) \in \Omega_B^i$ and if

$$b_i(k+1) - c_i(k) < F_i(k,Y) < c_i(k+1) - c_i(k), \quad (1.14)$$

for every $i = 1,\dots,n$ and every $(k,Y) \in \Omega_C^i$.

To formulate the following theorem, we need to define sets

$$\Omega(k) = \{(k,Y), Y = (Y_1,\dots,Y_n) \in \mathbb{R}^n, b_i(k) < Y_i < c_i(k), i = 1,\dots,n\},$$

$$\Omega_i(k) = \{(Y)\colon Y \in \mathbb{R}, b_i(k) < Y_i < c_i(k),\ \ i = 1,\dots,n\}.$$

Theorem 1. *[14, Theorem 4] Let* $F\colon \mathbb{N}(k_0) \times \overline{\Omega} \to \mathbb{R}^n$*. Let, moreover,* Ω *be regular with respect to the discrete system* (1.7)*, and let the function*

$$G_i(w) := w + F_i(k, Y_1,\dots,Y_{i-1}, w, Y_{i+1},\dots,Y_n)$$

be monotone on $\overline{\Omega}_i(k)$ *for every fixed* $k \in \mathbb{N}(k_0)$*, each fixed* $i \in \{1,\dots,n\}$*, and every fixed*

$$(Y_1,\dots,Y_{i-1}, Y_{i+1},\dots,Y_n)$$

such that $(k, Y_1,\dots,Y_{i-1}, w, Y_{i+1},\dots,Y_n) \in \Omega$*. Then, every initial problem* $Y(k_0) = Y^*$ *with* $Y^* \in \Omega(k_0)$ *defines a solution* $Y = Y^*(k)$ *of the discrete system* (1.7) *satisfying the relation*

$$Y^*(k) \in \Omega(k)$$

for every $k \in \mathbb{N}(k_0)$*.*

1.2.4 Auxiliary results of an Anti-Liapunov type

Now we formulate a result which is in [13] proved by a retract method sometimes called an Anti-Liapunov method due to the assumptions used being often an opposite to those used when Liapunov method is applied (such an approach goes back to Ważewski, who formulated his topological method formulated for ordinary differential equations). The following theorem is a slight modification of [13, Theorem 1] (see [15, Theorem 2] also).

Theorem 2. *Assume that the function $F(k, Y)$ satisfies* (1.7) *and is continuous with respect to Y. Let the inequality*

$$F_i(k, Y) < b_i(k+1) - b_i(k) \tag{1.15}$$

hold for every $i = 1, \dots, n$ and every $(k, Y) \in \Omega_B^i$. Let, moreover, inequality

$$F_i(k, Y) > c_i(k+1) - c_i(k) \tag{1.16}$$

hold for every $i = 1, \dots, n$ and every $(k, Y) \in \Omega_C^i$. Then, there exists a solution $Y = Y(k)$, $k \in \mathbb{N}(k_0)$ of system (1.7) *satisfying the inequalities*

$$b_i(k) < Y_{i-1}(k) < c_i(k)$$

for every $k \in \mathbb{N}(k_0)$ and $i = 1, \dots, n$.

1.3 The thesis aims

First, Chapter 2 gives us all the technical details of transforming the Emden-Fowler difference equation (1.3) into a system of two first-order difference equations. We will need this transformation to prove the theorems about the power-type asymptotic behaviour.

Then, in Chapter 3 we get sufficient conditions on coefficients α and m of the Emden-Fowler difference equation (1.3), such that there exists a solution with a power asymptotic behaviour. Here we get the results using constants as upper and lower functions. The results of this chapter were published in [4, 36].

Next Chapter 4 shows us that if we change upper and lower functions we can expand the area of appropriate conditions. We divide this chapter into 4 different parts, depending on values $s+1$ and ms. The results of this chapter also include the conditions on α and m, such that there exists a power-asymptotic solution. Some of the results of this chapter were published in [16, 17, 20, 37].

In Chapter 5 we construct the discrete analogy of the blow-up solutions of the Emden-Fowler equation. Part of the results corresponding to this chapter was published in [18, 19].

Finally, in Chapter 6, some conclusions and comparisons are given.

2 Preliminary calculations and theorems

2.1 Constructing an asymptotic power-type solution.

In this chapter we will construct an approximate solution to equation (1.3) in a power form.

Let us define

$$s = \frac{\alpha + 2}{m - 1}, \tag{2.1}$$

$$a = [\mp s(s+1)]^{1/(m-1)} \tag{2.2}$$

and

$$b = \frac{as(s+1)}{s + 2 - ms}. \tag{2.3}$$

Remark 1. We need to assume $m \neq 0$, $m \neq 1$, $s + 2 \neq 0$, and $s + 2 - ms \neq 0$, that is, $m \neq 0$, $m \neq 1$, $\alpha \neq -2$, and $\alpha \neq -2m$.

Remark 2. If, in formula (2.2), either the upper variant of sign is in force (i.e. $-$) and $s(s+1) > 0$ or in (2.2) lower variant of sign in force (i.e. $+$) and $s(s+1) < 0$, then the constant m has the form of a ratio m_1/m_2 of relatively prime integers m_1, m_2, and m_2 is odd, the difference $m_1 - m_2$ is odd as well. If this convention holds, the formula (2.2) defines two or at least one value. As equation (1.3) splits into two equations, when formulating the results, we assume that a concrete variant is fixed (either with the sign $+$ or with the sign $-$).

Remark 3. The equation (1.5) has an exact solution

$$y = \frac{a}{x^s}, \quad x > 0. \tag{2.4}$$

Quite natural is an expectation that the discrete equation of the Emden-Fowler type (1.3), having the formal form coinciding with equation (1.5), that is the equation

$$\Delta^2 u(k) \pm k^\alpha u^m(k) = 0,$$

should have an exact solution of the form (2.4) as well, that is, in our case an exact solution

$$u(k) = \frac{a}{k^s}, \quad k \in \mathbb{N}(k_0). \tag{2.5}$$

But this is no more true because of the different character of both equations. Unfortunately, even looking for an exact solution of the form (2.5) with the values of a and s possibly different from those given by formulas (2.1) and (2.2) does not lead to the desired result. So, we conclude that, unlike the classical Emden-Fowler type differential equation, the discrete analog does not have an exact solution of this form.

Theorem 3. *Let* a*,* b *and* s *be defined by the formulas* (2.1) – (2.3)*. Then, the function*

$$u_{app}(k) \propto \frac{a}{k^s} + \frac{b}{k^{s+1}} \tag{2.6}$$

is an approximate power-type solution of equation (1.3) *of order* $g(k) = k^{s+3}$.

Proof. We are looking for a solution of the form (2.6). Substituting $u(k) = u_{app}(k)$ in equation (1.3) we get

$$\frac{a}{(k+2)^s} - \frac{2a}{(k+1)^s} + \frac{a}{k^s} + \frac{b}{(k+2)^{s+1}} - \frac{2b}{(k+1)^{s+1}} + \frac{b}{k^{s+1}} \pm k^{\alpha}\left(\frac{a}{k^s} + \frac{b}{k^{s+1}}\right)^m = 0$$

or, equivalently,

$$\frac{a}{k^s}\left[\left(1+\frac{2}{k}\right)^{-s} - 2\left(1+\frac{1}{k}\right)^{-s} + 1\right] + \frac{b}{k^{s+1}}\left[\left(1+\frac{2}{k}\right)^{-(s+1)} - 2\left(1+\frac{1}{k}\right)^{-(s+1)} + 1\right] \pm \frac{a^m}{k^{ms-\alpha}}\left[1+\frac{b}{ak}\right]^m = 0.$$

Assuming k_0 sufficiently large and using asymptotic decompositions of the terms in square brackets given by (1.6), we derive

$$\begin{aligned}
&\frac{a}{k^s}\left[1 - \frac{2s}{k} + \frac{2s(s+1)}{k^2} - \frac{4s(s+1)(s+2)}{3k^3} + O\left(\frac{1}{k^4}\right)\right] \\
&-\frac{2a}{k^s}\left[1 - \frac{s}{k} + \frac{s(s+1)}{2k^2} - \frac{s(s+1)(s+2)}{6k^3} + O\left(\frac{1}{k^4}\right)\right] + \frac{a}{k^s} \\
&+\frac{b}{k^{s+1}}\left[1 - \frac{2(s+1)}{k} + \frac{2(s+1)(s+2)}{k^2} + O\left(\frac{1}{k^3}\right)\right] \\
&-\frac{2b}{k^{s+1}}\left[1 - \frac{s+1}{k} + \frac{(s+1)(s+2)}{2k^2} + O\left(\frac{1}{k^4}\right)\right] + \frac{b}{k^{s+1}} \\
&\pm\frac{a^m}{k^{ms-\alpha}}\left[1 + \frac{bm}{ak} + O\left(\frac{1}{k^2}\right)\right] = 0.
\end{aligned} \tag{2.7}$$

It is easy to see that the coefficients of the terms k^{-s} and k^{-s-1} equal zero and the last equality reduces to

$$\begin{aligned}
&\frac{a}{k^s}\left[\frac{2s(s+1)}{k^2} - \frac{4s(s+1)(s+2)}{3k^3} + O\left(\frac{1}{k^4}\right)\right] \\
&-\frac{2a}{k^s}\left[\frac{s(s+1)}{2k^2}\frac{s(s+1)(s+2)}{6k^3} + O\left(\frac{1}{k^4}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{b}{k^{s+1}}\left[-\frac{2(s+1)}{k}+\frac{2(s+1)(s+2)}{k^2}+O\left(\frac{1}{k^3}\right)\right]\\
&-\frac{2b}{k^{s+1}}\left[-\frac{s+1}{k}+\frac{(s+1)(s+2)}{2k^2}+O\left(\frac{1}{k^4}\right)\right]\\
&\pm\frac{a^m}{k^{ms-\alpha}}\left[1+\frac{bm}{ak}+O\left(\frac{1}{k^2}\right)\right]=0. \qquad (2.8)
\end{aligned}
$$

Next, assume that the powers $-(s+2)$ and $-(ms-\alpha)$ are equal. Then, the equation

$$ms-\alpha = s+2$$

implies formula (2.1), that is,

$$s=\frac{\alpha+2}{m-1}.$$

Then,

$$\frac{1}{k^{s+2}}\left[as(s+1)\pm a^m\right]+\frac{1}{k^{s+3}}\left[-as(s+1)(s+2)+b(s+1)(s+2)\pm bma^{m-1}\right] + O\left(\frac{1}{k^{s+4}}\right)=0.$$

If

$$as(s+1)\pm a^m=0,$$

then we get formula (2.2), that is,

$$a=\left[\mp s(s+1)\right]^{1/(m-1)}.$$

Assuming also

$$-as(s+1)(s+2)+b(s+1)(s+2)\pm bma^{m-1}=0, \qquad (2.9)$$

we get

$$b=\frac{as(s+2)}{s+2-ms}$$

and formula (2.3) is proved as well. Therefore, if $u(k)$ in equation (1.3) is replaced by the approximate solution $u_{app}(k)$ as given by formula (2.6), then, in the left-hand side of (2.8), the coefficients of terms k^{-s}, k^{-s-1}, k^{-s-2} and k^{-s-3} will be eliminated. Then, it is possible to set $g(k)=k^{s+3}$. □

2.2 System of difference equations equivalent to a differential equation

Below, rather than of equation (1.3), we will analyse an equivalent system of two difference equations. This system will be constructed using the below auxiliary transformations

$$u(k) = \frac{a}{k^s} + \frac{b}{k^{s+1}}(1 + Y_0(k)), \tag{2.10}$$

$$\Delta u(k) = \Delta\left(\frac{a}{k^s}\right) + \Delta\left(\frac{b}{k^{s+1}}\right)(1 + Y_1(k)), \tag{2.11}$$

$$\Delta^2 u(k) = \Delta^2\left(\frac{a}{k^s}\right) + \Delta^2\left(\frac{b}{k^{s+1}}\right)(1 + Y_2(k)). \tag{2.12}$$

where s, a and b are defined by formulas (2.1) – (2.3), and $Y_i(k)$, $i = 0, 1, 2$ are new dependent functions. Below, we derive relations connecting them. Recall a useful known formula (we refer, e.g., to [21]), used in computations. If x and y are defined on $\mathbb{N}(k_0)$, then

$$\Delta(x(k)y(k)) = x(k+1)\Delta y(k) + (\Delta x(k))y(k), \;\; k \in \mathbb{N}(k_0).$$

Taking the first differences of the left-hand and right-hand sides of (2.10), we derive

$$\Delta u(k) = \Delta\left(\frac{a}{k^s}\right) + \frac{b}{(k+1)^{s+1}}\Delta Y_0(k) + \Delta\left(\frac{b}{k^{s+1}}\right)(1 + Y_0(k)).$$

Comparing the result with (2.11), we get the equation

$$\frac{b}{(k+1)^{s+1}}\Delta Y_0(k) + \Delta\left(\frac{b}{k^{s+1}}\right)(1 + Y_0(k)) = \Delta\left(\frac{b}{k^{s+1}}\right)(1 + Y_1(k)),$$

which is equivalent with

$$\Delta Y_0(k) = (k+1)^{s+1}\Delta\left(\frac{1}{k^{s+1}}\right)(-Y_0(k) + Y_1(k)). \tag{2.13}$$

Taking the first differences of the left-hand and right-hand sides of (2.11), we obtain

$$\Delta^2 u(k) = \Delta^2\left(\frac{a}{k^s}\right) + \Delta\left(\frac{b}{(k+1)^{s+1}}\right)\Delta Y_1(k) + \Delta^2\left(\frac{b}{k^{s+1}}\right)(1 + Y_1(k)).$$

Comparing the result with (2.12), we get

$$\Delta\left(\frac{b}{(k+1)^{s+1}}\right)\Delta Y_1(k) + \Delta^2\left(\frac{b}{k^{s+1}}\right)(1 + Y_1(k)) = \Delta^2\left(\frac{b}{k^{s+1}}\right)(1 + Y_2(k)),$$

and an equivalent equation is

$$\Delta Y_1(k) = \frac{\Delta^2\left(\dfrac{1}{k^{s+1}}\right)}{\Delta\left(\dfrac{1}{(k+1)^{s+1}}\right)}(-Y_1(k)+Y_2(k)). \tag{2.14}$$

The derived system of difference equations (2.13), (2.14) defines the relationships between $Y_i(k)$, $i=0,1,2$ implied by transformations (2.10)–(2.12). Next, we will get a system equivalent with equation (1.3). To do this, we must express $Y_2(k)$ in (2.14) in terms of $Y_0(k)$ using initial equation (1.3). Substitute (2.10) and (2.12) into equation (1.3). Then,

$$\Delta^2\left(\frac{a}{k^s}\right)+\Delta^2\left(\frac{b}{k^{s+1}}\right)(1+Y_2(k)) \pm k^\alpha\left(\frac{a}{k^s}+\frac{b}{k^{s+1}}(1+Y_0(k))\right)^m = 0 \tag{2.15}$$

and, expressing $Y_2(k)$ from (2.15), an equivalent system to equation (1.3) is

$$\Delta Y_0(k) = (k+1)^{s+1}\Delta\left(\frac{1}{k^{s+1}}\right)(-Y_0(k)+Y_1(k)) \tag{2.16}$$

$$\begin{aligned}\Delta Y_1(k) = &\frac{\Delta^2\left(\dfrac{1}{k^{s+1}}\right)}{\Delta\left(\dfrac{1}{(k+1)^{s+1}}\right)}\left(-Y_1(k)-\left(\Delta^2\left(\frac{a}{k^s}\right)+\Delta^2\left(\frac{b}{k^{s+1}}\right)\right.\right.\\ &\left.\left.\mp k^\alpha\left(\frac{a}{k^s}+\frac{b}{k^{s+1}}(1+Y_0(k))\right)^m\right)\frac{1}{\Delta^2\left(\dfrac{b}{k^{s+1}}\right)}\right).\end{aligned} \tag{2.17}$$

System (2.16), (2.17) is too cumbersome and not suitable for a direct investigation. Therefore, we will simplify it by performing some asymptotic transformations. Equation (2.15) takes the form

$$\begin{aligned}\frac{a}{(k+2)^s}-\frac{2a}{(k+1)^s}+\frac{a}{k^s}+\left(\frac{b}{(k+2)^{s+1}}-\frac{2b}{(k+1)^{s+1}}+\frac{b}{k^{s+1}}\right)(1+Y_2(k))&\\ \pm\frac{a^m}{k^{ms-\alpha}}\left(1+\frac{b}{ak}(1+Y_0(k))\right)^m = 0.&\end{aligned} \tag{2.18}$$

Let $Y_0(k)=O(1)$ in (2.18). This property will be assumed when proving the results. This assumption implies, as will be visible from formula (2.19) derived below, the property $Y_2(k)=O(1)$ as well. Expressing asymptotically (using formula (1.6) and auxiliary computations in (2.7)) each of the expressions in the previous equation, we obtain

$$\frac{a}{k^s}\left(1-\frac{2s}{k}+\frac{s(s+1)}{2}\frac{4}{k^2}-\frac{s(s+1)(s+2)}{6}\frac{8}{k^3}+O\left(\frac{1}{k^4}\right)\right)$$

$$
\begin{aligned}
&-\frac{2a}{k^s}\left(1-\frac{s}{k}+\frac{s(s+1)}{2}\frac{1}{k^2}-\frac{s(s+1)(s+2)}{6}\frac{1}{k^3}+O\left(\frac{1}{k^4}\right)\right)+\frac{a}{k^s}\\
&+(1+Y_2(k))\left[\frac{b}{k^{s+1}}\left(1-\frac{2(s+1)}{k}+\frac{(s+1)(s+2)}{2}\frac{4}{k^2}+O\left(\frac{1}{k^3}\right)\right)\right.\\
&\qquad\left.-\frac{2b}{k^{s+1}}\left(1-\frac{(s+1)}{k}+\frac{(s+1)(s+2)}{2}\frac{1}{k^2}+O\left(\frac{1}{k^3}\right)\right)+\frac{b}{k^{s+1}}\right]\\
&\pm\frac{a^m}{k^{s+2}}\left(1+\frac{mb}{ak}(1+Y_0(k))+O\left(\frac{1}{k^2}\right)\right)=0.
\end{aligned}
$$

Carefully grouping the coefficients multiplying the same power functions, we simplify this relation to

$$
\begin{aligned}
&\frac{1}{k^s}(a-2a+a)+\frac{1}{k^{s+1}}(-2as+2as+(1+Y_2(k))(b-2b+b))\\
&+\frac{1}{k^{s+2}}\left[2as(s+1)-as(s+1)+(1+Y_2(k))(-2b(s+1)+2b(s+1))\pm a^m\right]\\
&+\frac{1}{k^{s+3}}\left[-as(s+1)(s+2)\frac{4}{3}+as(s+1)(s+2)\frac{1}{3}+(1+Y_2(k))(2b(s+1)(s+2)\right.\\
&\qquad\left.-b(s+1)(s+2))\pm mba^{m-1}(1+Y_0(k))\right]+O\left(\frac{1}{k^{s+4}}\right)=0.
\end{aligned}
$$

Hence, we have arrived at the equation

$$
\begin{aligned}
&-as(s+1)(s+2)+b(s+1)(s+2))+Y_2(k)b(s+1)(s+2)\\
&\qquad\qquad\pm mba^{m-1}(1+Y_0(k))+O\left(\frac{1}{k}\right)=0.
\end{aligned}
$$

Because (see (2.9))

$$
-as(s+1)(s+2)+b(s+1)(s+2)\pm mba^{m-1}=0,
$$

we have

$$
Y_2(k)b(s+1)(s+2)-mbs(s+1)Y_0(k)+O\left(\frac{1}{k}\right)=0.
$$

Hence,

$$
Y_2(k)=\frac{ms}{s+2}Y_0(k)+O\left(\frac{1}{k}\right). \tag{2.19}
$$

System of equations (2.13), (2.14) if $Y_2(k)$ is replaced by formula (2.19), i.e., the system (2.16), (2.17) takes the form

$$
\Delta Y_0(k)=(k+1)^{s+1}\Delta\left(\frac{1}{k^{s+1}}\right)(-Y_0(k)+Y_1(k)), \tag{2.20}
$$

$$
\Delta Y_1(k)=\frac{\Delta^2\left(\frac{1}{k^{s+1}}\right)}{\Delta\left(\frac{1}{(k+1)^{s+1}}\right)}\left(\frac{ms}{s+2}Y_0(k)-Y_1(k)+O\left(\frac{1}{k}\right)\right). \tag{2.21}
$$

It is easy to verify (using (1.6)) that

$$(k+1)^{s+1}\Delta\left(\frac{1}{k^{s+1}}\right) = -\frac{s+1}{k} + O\left(\frac{1}{k^2}\right)$$

and

$$\frac{\Delta^2\left(\dfrac{1}{k^{s+1}}\right)}{\Delta\left(\dfrac{1}{(k+1)^{s+1}}\right)} = -\frac{s+2}{k} + O\left(\frac{1}{k^2}\right).$$

Applying these formulas to (2.20), (2.21), we have

$$\Delta Y_0(k) = \left(-\frac{s+1}{k} + O\left(\frac{1}{k^2}\right)\right)\left(-Y_0(k) + Y_1(k)\right), \tag{2.22}$$

$$\Delta Y_1(k) = \left(-\frac{s+2}{k} + O\left(\frac{1}{k^2}\right)\right)\left(\frac{ms}{s+2}Y_0(k) - Y_1(k) + O\left(\frac{1}{k}\right)\right). \tag{2.23}$$

The system (2.22), (2.23) will be used in future investigations rather than system (2.16), (2.17).

3 Power-type asymptotic behaviour in case of constant upper and lower functions

The aim of this chapter is to find conditions for the existence of solutions to equation (1.3) with the power-type asymptotic behaviour when Theorem 2 is applied with constant upper and lower functions $b_1(k)$, $b_2(k)$, $c_1(k)$ and $c_2(k)$. We use the approximate power-type solution described by formula (2.6), where s, a and b are defined by formulas(2.1), (2.2) and (2.3). The results of this chapter were published in [4].

We will prove the theorem, formulated below. Here we deal only with the case $s+1>0$.

Theorem 4. *Let $s>-1$, $m\neq 0$ and $m\neq 1$. Assume that there exist positive numbers ε_i, $i=1,\dots,4$, such that either*

$$ms>0,\;\; \varepsilon_3<\varepsilon_1,\;\; \varepsilon_2>\varepsilon_4,\;\; \varepsilon_3>\frac{ms}{s+2}\varepsilon_1,\;\; \varepsilon_4>\frac{ms}{s+2}\varepsilon_2, \tag{3.1}$$

or

$$ms<0,\;\; \varepsilon_3<\varepsilon_1,\;\; \varepsilon_2>\varepsilon_4,\;\; \varepsilon_3>-\frac{ms}{s+2}\varepsilon_2,\;\; \varepsilon_4>-\frac{ms}{s+2}\varepsilon_1. \tag{3.2}$$

Then, for a sufficiently large fixed k_0, there exists a solution $u\colon \mathbb{N}(k_0)\to\mathbb{R}$ of equation (1.3) *such that, for every $k\in\mathbb{N}(k_0)$,*

$$-\varepsilon_1<\left[u(k)-\frac{a}{k^s}-\frac{b}{k^{s+1}}\right]\left[\frac{b}{k^{s+1}}\right]^{-1}<\varepsilon_2\,, \tag{3.3}$$

$$-\varepsilon_3<\left[\Delta u(k)-\Delta\left(\frac{a}{k^s}\right)-\Delta\left(\frac{b}{k^{s+1}}\right)\right]\left[\Delta\left(\frac{b}{k^{s+1}}\right)\right]^{-1}<\varepsilon_4\,, \tag{3.4}$$

and

$$-\varepsilon_1+O\left(\frac{1}{k}\right)<\left[\Delta^2u(k)-\Delta^2\left(\frac{a}{k^s}\right)-\Delta^2\left(\frac{b}{k^{s+1}}\right)\right]\left[\Delta^2\left(\frac{b}{k^{s+1}}\right)\frac{ms}{s+2}\right]^{-1}<\varepsilon_2+O\left(\frac{1}{k}\right). \tag{3.5}$$

Remark 4. In the proof of Theorem 4, we will apply Theorem 2 from Chapter 1, where system (2.22), (2.23) is considered instead of a system of discrete equations (1.7). That is, in system (1.7) we set $n=2$ and

$$F_1(k,Y_0(k),Y_1(k)):=\left(-\frac{s+1}{k}+O\left(\frac{1}{k^2}\right)\right)\left(-Y_0(k)+Y_1(k)\right),$$
$$F_2(k,Y_0(k),Y_1(k)):=\left(-\frac{s+2}{k}+O\left(\frac{1}{k^2}\right)\right)\left(\frac{ms}{s+2}Y_0(k)-Y_1(k)+O\left(\frac{1}{k}\right)\right).$$

The core of the proof consists of verifying inequalities (1.15), (1.16) estimating functions F_1 and F_2 for properly defined functions $b_i, c_i\colon \mathbb{N}(k_0) \to \mathbb{R}$, $i = 1, 2$ (see (1.9)) satisfying $b_i(k) < c_i(k)$, $k \in \mathbb{N}(k_0)$, $i = 1, 2$. By b_i and c_i, $i = 1, 2$ functions $B_i(k, Y)$ and $C_i(k, Y)$, $i = 1, 2$ in (1.10) and sets Ω^i_B, Ω^i_C, $i = 1, 2$ in (1.11), (1.12) are defined.

3.1 Proof of the theorem

Let $\varepsilon_i > 0$, $i = 1, \dots, 4$ be fixed. Assume that k_0 is positive and sufficiently large such that the asymptotic computations in the proof are correct for every $k \in \mathbb{N}(k_0)$. Now define functions b_i, c_i, $i = 1, 2$, satisfying (1.9), by formulas

$$b_1(k) := -\varepsilon_1\,, \;\; c_1(k) := \varepsilon_2, \tag{3.6}$$

$$b_2(k) := -\varepsilon_3\,, \;\; c_2(k) := \varepsilon_4. \tag{3.7}$$

Then,

$$\begin{aligned}
B_1(k, Y) &:= -Y_0 + b_1(k) = -Y_0 - \varepsilon_1,\\
B_2(k, Y) &:= -Y_1 + b_2(k) = -Y_1 - \varepsilon_3,\\
C_1(k, Y) &:= \quad Y_0 - c_1(k) = \quad Y_0 - \varepsilon_2,\\
C_2(k, Y) &:= \quad Y_1 - c_2(k) = \quad Y_1 - \varepsilon_4
\end{aligned}$$

and

$$\Omega^1_B = \{(k, Y)\colon k \in \mathbb{N}(k_0),\;\; Y_0 = -\varepsilon_1,\;\; -\varepsilon_3 \le Y_1 \le \varepsilon_4\}\,, \tag{3.8}$$

$$\Omega^2_B = \{(k, Y)\colon k \in \mathbb{N}(k_0),\;\; Y_1 = -\varepsilon_3,\;\; -\varepsilon_1 \le Y_0 \le \varepsilon_2\}\,, \tag{3.9}$$

$$\Omega^1_C = \{(k, Y)\colon k \in \mathbb{N}(k_0),\;\; Y_0 = \quad \varepsilon_2,\;\; -\varepsilon_3 \le Y_1 \le \varepsilon_4\}\,, \tag{3.10}$$

$$\Omega^2_C = \{(k, Y)\colon k \in \mathbb{N}(k_0),\;\; Y_1 = \quad \varepsilon_4,\;\; -\varepsilon_1 \le Y_0 \le \varepsilon_2\}\,. \tag{3.11}$$

To apply Theorem 2, inequalites (1.15) and (1.16) must hold. Since inequality (1.15) assumes $(k, Y) \in \Omega^i_B$, $i = 1, \dots, n$ and inequality (1.16) assumes $(k, Y) \in \Omega^i_C$, $i = 1, \dots, n$, we need to verify (taking into account specifications (3.8)–(3.11)) the following:

$$F_1(k, b_1(k), Y_1) = F_1\,(k, -\varepsilon_1, \quad Y_1) < b_1(k+1) - b_1(k) = -\varepsilon_1 + \varepsilon_1 = 0, \tag{3.12}$$

$$F_1(k, c_1(k), Y_1) = F_1\,(k, \quad \varepsilon_2, \quad Y_1) > c_1(k+1) - c_1(k) = \quad \varepsilon_2 - \varepsilon_2 = 0, \tag{3.13}$$

$$F_2(k, Y_0, b_2(k)) = F_2\,(k, \quad Y_0, -\varepsilon_3) < b_2(k+1) - b_2(k) = -\varepsilon_3 + \varepsilon_3 = 0, \tag{3.14}$$

$$F_2(k, Y_0, c_2(k)) = F_2\,(k, \quad Y_0, \quad \varepsilon_4) > c_2(k+1) - c_2(k) = \quad \varepsilon_4 - \varepsilon_4 = 0 \tag{3.15}$$

whenever

$$-\varepsilon_3 \le Y_1 \le \varepsilon_4 \tag{3.16}$$

in (3.12), (3.13) and

$$-\varepsilon_1 \le Y_0 \le \varepsilon_2 \tag{3.17}$$

in (3.14), (3.15).

As $s+1>0$, we can estimate the function F_1 in the following way:

$$F_1(k, b_1, Y_1) = F_1(k, -\varepsilon_1, Y_1) = \frac{s+1}{k}(-\varepsilon_1) - \frac{s+1}{k}Y_1 + O\left(\frac{1}{k^2}\right).$$

Then, (3.12) will hold if

$$F_1(k, b_1, Y_1) \cdot k < \max F_1(k, b_1, Y_1) \cdot k = (s+1)(-\varepsilon_1) + (s+1)\varepsilon_3 + O\left(\frac{1}{k^2}\right)$$
$$< b_1(k+1) - b_1(k) = -\varepsilon_1 + \varepsilon_1 = 0. \tag{3.18}$$

Hence, (3.18) will hold if

$$\varepsilon_3 < \varepsilon_1. \tag{3.19}$$

Similarly,

$$F_1(k, c_1, Y_1) = F_1(k, \varepsilon_2, Y_1) = \frac{s+1}{k}\varepsilon_2 - \frac{s+1}{k}Y_1 + O\left(\frac{1}{k^2}\right)$$

and (3.13) will hold if

$$F_1(k, c_1, Y_1) \cdot k > \min F_1(k, c_1, Y_1) \cdot k = (s+1)\varepsilon_2 + (s+1)(-\varepsilon_4) + +O\left(\frac{1}{k^2}\right)$$
$$> c_1(k+1) - c_1(k) = \varepsilon_2 - \varepsilon_2 = 0. \tag{3.20}$$

Then, for (3.20) to hold,

$$\varepsilon_2 > \varepsilon_4 \tag{3.21}$$

is sufficient.

By Theorem 2, we also need to estimate function F_2, i.e., we must prove that inequalities (3.14) and (3.15) hold. The cases $ms > 0$ and $ms < 0$ will be considered separately.

3.1.1 The case $ms > 0$.

In this case,

$$F_2(k, Y_0, b_2) = F_2(k, Y_0, -\varepsilon_3) = -\frac{ms}{k}Y_0 + \frac{s+2}{k}(-\varepsilon_3) + O\left(\frac{1}{k^2}\right).$$

Inequality (3.14) will hold if

$$F_2(k,Y_0,b_2)\cdot k < \max F_2(k,Y_0,b_2)\cdot k = ms\varepsilon_1 - (s+2)\varepsilon_3 + O\left(\frac{1}{k^2}\right)$$
$$< b_2(k+1) - b_2(k) = -\varepsilon_3 + \varepsilon_3 = 0. \quad (3.22)$$

Inequality (3.22) implies

$$\varepsilon_3 > \frac{ms}{s+2}\varepsilon_1. \quad (3.23)$$

Continuing the analysis, consider

$$F_2(k,Y_0,c_2) = F_2(k,Y_0,\varepsilon_4) = -\frac{ms}{k}Y_0 + \frac{s+2}{k}\varepsilon_4 + O\left(\frac{1}{k^2}\right).$$

Inequality (3.15) will hold if

$$F_2(k,Y_0,c_2)\cdot k > \min F_2(k,Y_0,\varepsilon_4)\cdot k = -ms\varepsilon_2 + (s+2)\varepsilon_4 + O\left(\frac{1}{k^2}\right)$$
$$> c_2(k+1) - c_2(k) = \varepsilon_4 - \varepsilon_4 = 0. \quad (3.24)$$

Then, inequality (3.24) will hold if

$$\varepsilon_4 > \frac{ms}{s+2}\varepsilon_2. \quad (3.25)$$

3.1.2 The case $ms < 0$.

In this case,

$$F_2(k,Y_0,b_2) = F_2(k,Y_0,-\varepsilon_3) = -\frac{ms}{k}Y_0 + \frac{s+2}{k}(-\varepsilon_3) + O\left(\frac{1}{k^2}\right).$$

Inequality (3.14) will hold if

$$F_2(k,Y_0,b_2)\cdot k < \max F_2(k,Y_0,b_2)\cdot k = -ms\varepsilon_2 - (s+2)\varepsilon_3 + O\left(\frac{1}{k^2}\right)$$
$$< b_2(k+1) - b_2(k) = -\varepsilon_3 + \varepsilon_3 = 0. \quad (3.26)$$

Then, inequality (3.26) will hold if

$$\varepsilon_3 > -\frac{ms}{s+2}\varepsilon_2. \quad (3.27)$$

Further, consider function

$$F_2(k,Y_0,c_2) = F_2(k,Y_0,\varepsilon_4) = -\frac{ms}{k}Y_0 + \frac{s+2}{k}\varepsilon_4 + O\left(\frac{1}{k^2}\right).$$

Inequality (3.15) will hold if

$$F_2(k,Y_0,c_2)\cdot k > \min F_2(k,Y_0,\varepsilon_4)\cdot k = ms\varepsilon_1 + (s+2)\varepsilon_4 + O\left(\frac{1}{k^2}\right)$$
$$> c_2(k+1) - c_2(k) = \varepsilon_4 - \varepsilon_4 = 0. \quad (3.28)$$

Then, inequality (3.28) will hold if

$$\varepsilon_4 > -\frac{ms}{s+2}\varepsilon_1. \quad (3.29)$$

3.1.3 Summary of the restrictions derived and application of Theorem 2.

If $ms > 0$, then the inequalities for F_1, F_2 hold if inequalities (3.19), (3.21), (3.23) and (3.25) do. That is, we have derived the system of inequalities (3.1). If $ms < 0$, then the inequalities for F_1, F_2 hold if (3.19), (3.21), (3.27), and (3.29) do. That is, we have derived the system of inequalities (3.2).

Finally, all the hypotheses of Theorem 2 are true and, therefore, there exists a solution

$$Y = Y(k) = (Y_1(k), Y_2(k))^T$$

of system (2.22), (2.23) satisfying the inequalities

$$b_i(k) < Y_{i-1}(k) < c_i(k), \quad i = 1, 2$$

for every $k \in \mathbb{N}(k_0)$, that is, by (3.16), (3.17),

$$-\varepsilon_3 \le Y_1(k) \le \varepsilon_4\,, \tag{3.30}$$

$$-\varepsilon_1 \le Y_0(k) \le \varepsilon_2 \tag{3.31}$$

for every $k \in \mathbb{N}(k_0)$.

We conclude, by the transformation formulas (2.10), (2.11), (2.12) and by the relation (2.19), that there exists a solution $u\colon \mathbb{N}(k_0) \to \mathbb{R}$ of equation (1.3) such that, for every $k \in \mathbb{N}(k_0)$,

$$\left[u(k) - \frac{a}{k^s} - \frac{b}{k^{s+1}}\right]\left[\frac{b}{k^{s+1}}\right]^{-1} = Y_0(k) \tag{3.32}$$

$$\left[\Delta u(k) - \Delta\left(\frac{a}{k^s}\right) - \Delta\left(\frac{b}{k^{s+1}}\right)\right]\left[\Delta\left(\frac{b}{k^{s+1}}\right)\right]^{-1} = Y_1(k), \tag{3.33}$$

$$\left[\Delta^2 u(k) - \Delta^2\left(\frac{a}{k^s}\right) + \Delta^2\left(\frac{b}{k^{s+1}}\right)\right]\left[\Delta^2\left(\frac{b}{k^{s+1}}\right)\right]^{-1} = Y_2(k) \tag{3.34}$$

where

$$Y_2(k) = \frac{ms}{s+2} Y_0(k) + O\left(\frac{1}{k}\right). \tag{3.35}$$

From (3.30)–(3.35), inequalities (3.3)–(3.5) follow.

3.2 Clarification on the conditions in Theorem 4 using only values α and m.

Theorem 4 uses assumptions on m and s. Nevertheless, because the parameters in Emden-Fowler equation (1.3) are m and α, it seems reasonable to analyse their admissible values deduced from this theorem and visualize the derived results in an (m, α)-plane. In other words, we need to find α and m for which the conclusion of Theorem 4 holds.

In Theorem 4 two sets of hypotheses 3.1 and 3.2 are used. These together with the assumption $s > -1$ guarantee the existence of a solution $u = u(k)$ of Emden-Fowler equation with asymptotic behaviour described by formulas (3.3)–(3.5). Below we analyze each set separately.

3.2.1 The case of inequalities (3.1).

Consider the system of inequalities (3.1). Then

$$(i)\ \ s+1>0,\quad (ii)\ \ ms>0,\quad (iii)\ \ \frac{ms}{s+2}\varepsilon_1<\varepsilon_3<\varepsilon_1,\quad (iv)\ \ \frac{ms}{s+2}\varepsilon_2<\varepsilon_4<\varepsilon_2.$$

Since $\varepsilon_i > 0$, $i = 1, \dots, 4$, inequalities (iii) and (iv) are equivalent to

$$\frac{ms}{s+2} < 1$$

and an equivalent system of inequalities

$$s+1>0,\quad ms>0,\quad s(m-1)-2<0 \tag{3.36}$$

can be considered instead of system (i)–(iv).

Moreover, using formula (2.1), system (3.36) yields

$$\frac{\alpha+m+1}{m-1}>0\,,\quad \frac{m(\alpha+2)}{m-1}>0\,,\quad \alpha<0. \tag{3.37}$$

To analyze inequalities (3.37), we consider subcases: $m > 1$ and $m < 1$.

The subcase $m > 1$. The system (3.37) is equivalent to the following one

$$m>1,\quad -2<\alpha<0. \tag{3.38}$$

The result is shown in Figure 3.1 visualized in (m, α)-plane by a yellow domain.

The subcase $m < 1$.

The system (3.37) takes the form

$$\alpha+m+1<0,\ \ m(\alpha+2)<0,\ \ \alpha<0$$

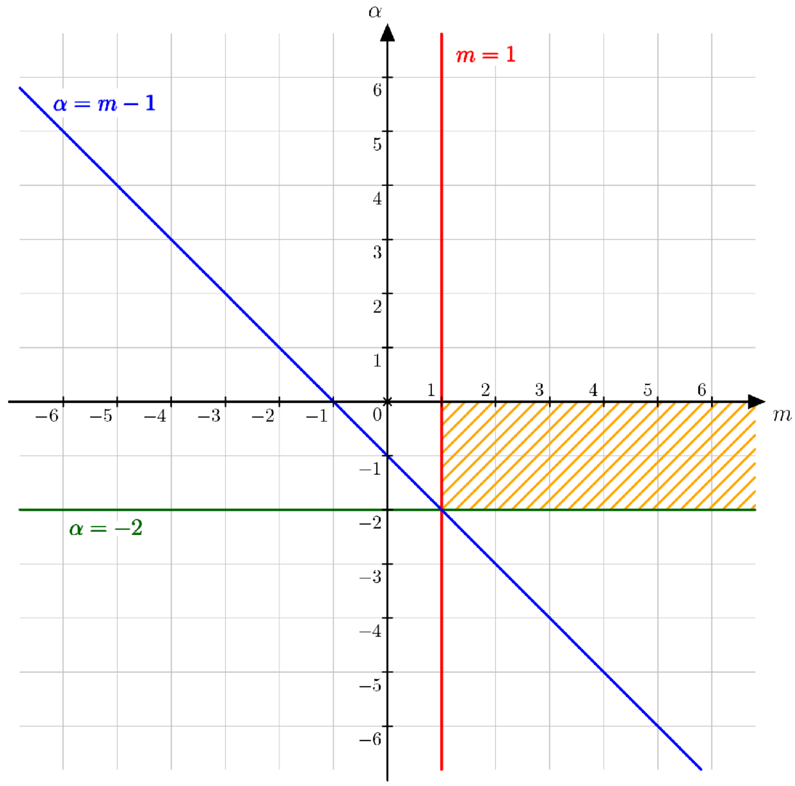

Fig. 3.1: Solution of the system (3.38)

being equivalent to the following two possibilities: either

$$m < 0, \quad -2 < \alpha < \min\{0, -m-1\} \tag{3.39}$$

or

$$0 < m < 1, \quad \alpha < -2. \tag{3.40}$$

Figure 3.2 highlights the resulting domains in (m, α)-plane in pink.

3.2.2 The case of inequalities (3.2)

Consider the system of inequalities (3.2). This system implies

$$(i) \;\; s + 1 > 0,$$
$$(ii) \;\; ms < 0,$$
$$(iii) \;\; \varepsilon_3 > -\frac{ms}{s+2}\varepsilon_2 > -\frac{ms}{s+2}\varepsilon_4 > \frac{m^2 s^2}{(s+2)^2}\varepsilon_1 > \frac{m^2 s^2}{(s+2)^2}\varepsilon_3.$$

As $\varepsilon_i > 0$, $i = 1, \ldots, 4$, we get from (iii)

$$1 - \frac{m^2 s^2}{(s+2)^2} > 0.$$

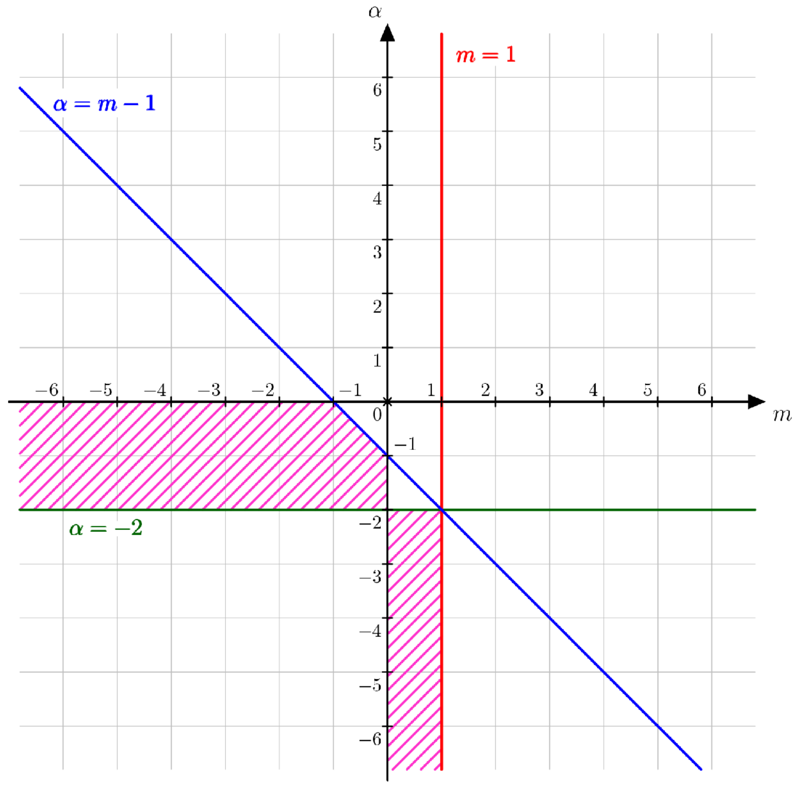

Fig. 3.2: Solution of the systems (3.39), (3.40)

Since $(s+2)^2 > 0$, system (i)–(iii) reduces to

$$s+1>0, \quad ms<0, \quad (s+2+ms)>0$$

and, applying the formula (2.1), to

$$\frac{\alpha+m+1}{m-1}>0, \quad \frac{m(\alpha+2)}{m-1}<0, \quad \frac{\alpha+m\alpha+4m}{m-1}>0. \tag{3.41}$$

To analyze inequalities (3.41), we consider subcases: $m>1$ and $m<1$.

The subcase $m > 1$. The system (3.41) takes the form

$$\alpha > -m-1, \quad \alpha < -2, \quad \alpha > -\frac{4m}{m+1}\,. \tag{3.42}$$

Figure 3.3 highlights the resulting domain, described by these inequalities, in (m,α)-plane in violet.

The subcase $m < 1$. The system (3.41) takes the form

$$\alpha+m+1<0, \quad m(\alpha+2)>0, \quad \alpha+m\alpha+4m<0 \tag{3.43}$$

implying the following 3 possibilities:

$$-1<m<0, \quad \alpha<-2, \quad \alpha<-\frac{4m}{1+m}\,, \tag{3.44}$$

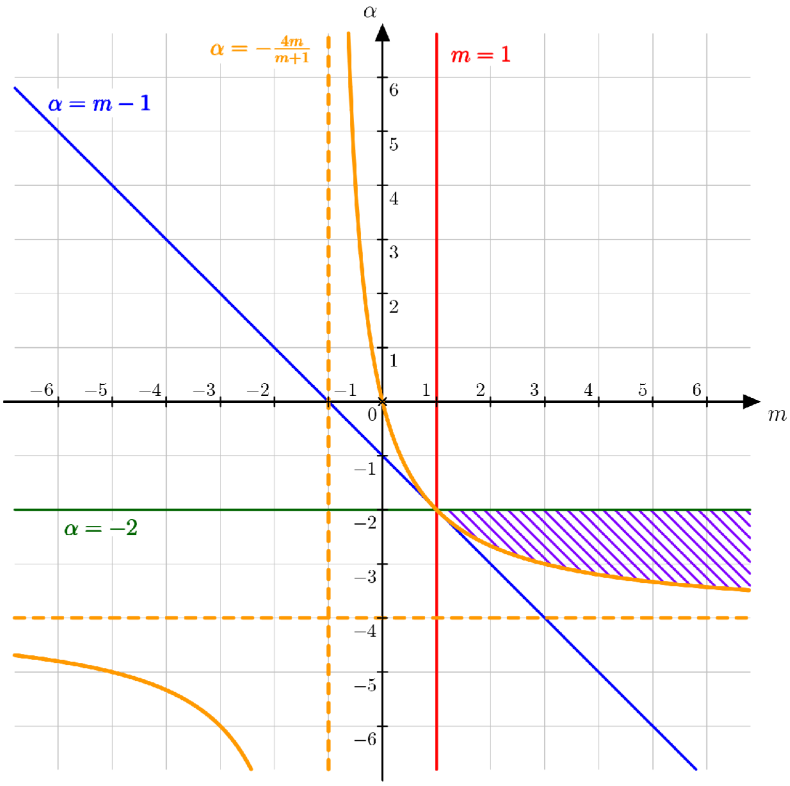

Fig. 3.3: Solution of the system (3.42)

$$m < -1, \quad \alpha < -2, \quad \alpha > -\frac{4m}{1+m}, \tag{3.45}$$

$$m > 0, \quad \alpha > -2, \quad \alpha < -\frac{4m}{1+m}. \tag{3.46}$$

Figure 3.4 highlights the three resulting domains in (m,α)-plane in green. The area corresponding to the solution of system (3.43) can be visualized in (m,α)-plane as follows.

All particular cases are highlighted in Figure 3.5 in (m,α)-plane in corresponding colours. If a fixed (m,α) belongs to the domain of admissible values, all hypotheses of Theorem 4 are true and, for a sufficiently large fixed k_0, there exists a solution $u\colon \mathbb{N}(k_0) \to \mathbb{R}$ of equation (1.3) satisfying, for every $k \in \mathbb{N}(k_0)$, inequalities (3.3)–(3.5).

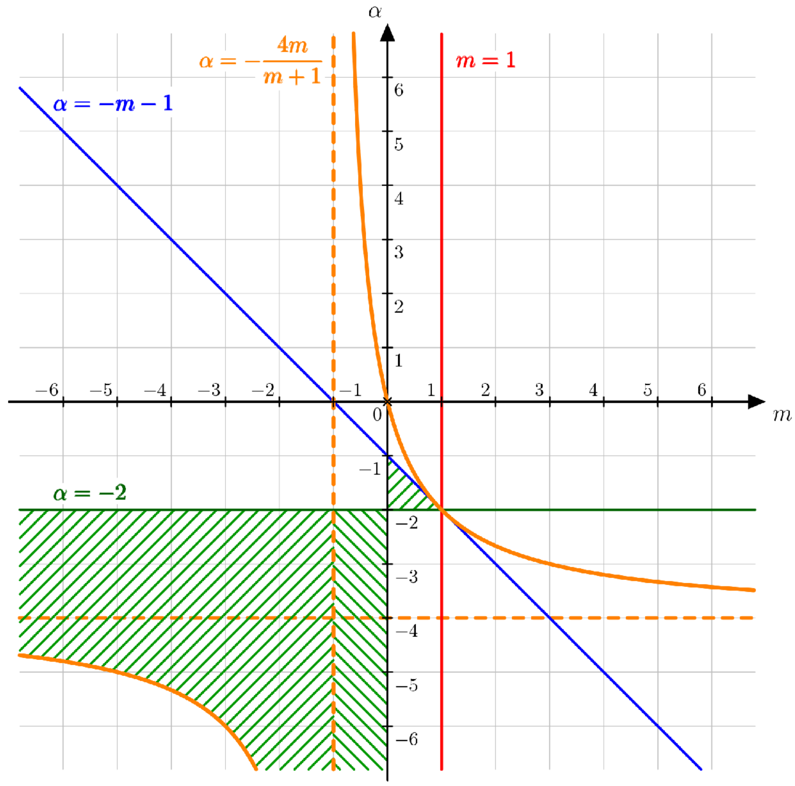

Fig. 3.4: Solution of the system (3.43)

3.3 Examples

In this section we consider seven examples of Emden-Fowler type equations. These are constructed in such a way that, step by step, the values (m, α) belong to each of the seven domains shown in Figure 3.5.

Example 1. In the following example, values (m, α) belong to the domain shown in Figure 3.1 (red domain).

Consider equation (1.3) where $\alpha = -1$, $m = 2$, that is, the equation

$$\Delta^2 u(k) \pm \frac{1}{k} u^2(k) = 0. \tag{3.47}$$

In this example, α and m satisfy (3.38). If we put

$$\varepsilon_1 = \varepsilon_2 = 1, \quad \varepsilon_3 = \varepsilon_4 = \frac{5}{6},$$

then

$$s = \frac{\alpha + 2}{m - 1} = \frac{1}{1} = 1, \quad ms = 2, \quad s + 1 = 2, \quad s + 2 = 3.$$

All inequalities (3.1) hold because

$$s + 1 = 2 > 0, \quad ms = 2 > 0,$$

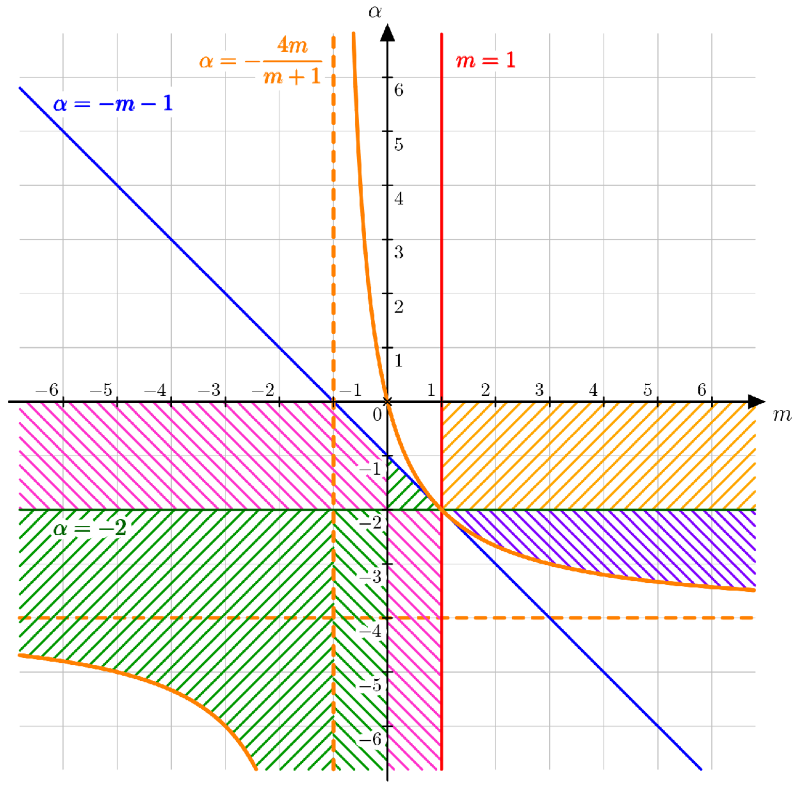

Fig. 3.5: Summary of admissible values

$$\varepsilon_3 = \frac{5}{6} < \varepsilon_1 = 1, \quad \varepsilon_2 = 1 > \varepsilon_4 = \frac{5}{6}\,,$$

$$\varepsilon_3 = \frac{5}{6} > \frac{ms}{s+2}\varepsilon_1 = \frac{2}{3}\cdot 1 = \frac{2}{3}, \quad \varepsilon_4 = \frac{5}{6} > \frac{ms}{s+2}\varepsilon_2 = \frac{2}{3}\cdot 1 = \frac{2}{3}$$

and Theorem 4 is applicable. By formula (2.2)

$$a = [\mp s(s+1)]^{1/(m-1)} = [\mp(1)(1+1)]^{1/(2-1)} = \mp 2$$

and, by formula (2.3),

$$b = \frac{as(s+2)}{s+2-ms} = \frac{(\mp 2)\cdot 1\cdot 3}{1+2-2} = \mp 6\,.$$

Then, the equation (3.47) has a solution $u = u(k)$, $k \in \mathbb{N}(k_0)$ satisfying inequalitites (3.3)–(3.5), that is,

$$-1 < \left[u(k) \pm \frac{2}{k} \pm \frac{6}{k^2}\right]\left[\mp\frac{6}{k^2}\right]^{-1} < 1\,,$$

$$-\frac{5}{6} < \left[\Delta u(k) \pm \Delta\left(\frac{2}{k}\right) \pm \Delta\left(\frac{6}{k^2}\right)\right]\left[\mp\Delta\left(\frac{6}{k^2}\right)\right]^{-1} < \frac{5}{6},$$

$$-1 + O\left(\frac{1}{k}\right) < \left[\Delta^2 u(k) \pm \Delta^2\left(\frac{2}{k}\right) \pm \Delta^2\left(\frac{6}{k^2}\right)\right]\left[\mp\Delta^2\left(\frac{6}{k^2}\right)\frac{2}{3}\right]^{-1} < 1 + O\left(\frac{1}{k}\right)\,.$$

These formulas can be simplified to

$$
\begin{aligned}
u(k) &= \mp\frac{2}{k} \mp \frac{6}{k^2} \cdot O\left(1\right), \\
\Delta u(k) &= \mp\Delta\left(\frac{2}{k}\right) + \Delta\left(\frac{6}{k^2}\right) \cdot O\left(1\right), \\
\Delta^2 u(k) &= \mp\Delta^2\left(\frac{2}{k}\right) + \Delta^2\left(\frac{6}{k^2}\right) \cdot O\left(1\right).
\end{aligned}
$$

Example 2. In the following example, the values (m, α) belong to the domain shown in Figure 3.2 (left yellow domain).

Consider equation (1.3) where $\alpha = -1$, $m = -2$, that is, the equation

$$
\Delta^2 u(k) \pm \frac{1}{k} u^{-2}(k) = 0. \tag{3.48}
$$

In this example, α and m satisfy (3.39). If we put

$$
\varepsilon_1 = \varepsilon_2 = 1, \quad \varepsilon_3 = \varepsilon_4 = \frac{2}{3},
$$

then

$$
s = \frac{\alpha + 2}{m - 1} = -\frac{1}{3}, \quad ms = \frac{2}{3}, \quad s + 1 = \frac{2}{3}, \quad s + 2 = \frac{5}{3}.
$$

All inequalities (3.1) hold because

$$
s + 1 = \frac{2}{3} > 0, \quad ms = \frac{2}{3} > 0,
$$

$$
\varepsilon_3 = \frac{2}{3} < \varepsilon_1 = 1, \quad \varepsilon_2 = 1 > \varepsilon_4 = \frac{2}{3},
$$

$$
\varepsilon_3 = \frac{2}{3} > \frac{ms}{s+2}\varepsilon_1 = \frac{2}{5} \cdot 1 = \frac{2}{5}, \quad \varepsilon_4 = \frac{2}{3} > \frac{ms}{s+2}\varepsilon_2 = \frac{2}{5} \cdot 1 = \frac{2}{5}
$$

and Theorem 4 is applicable. By formula (2.2),

$$
a = [\mp s(s+1)]^{1/(m-1)} = \left[\mp\left(-\frac{1}{3}\right)\left(\frac{2}{3}\right)\right]^{1/(-3)} = \pm\left(\frac{2}{9}\right)^{-1/3} = \pm\left(\frac{9}{2}\right)^{1/3}
$$

and, by formula (2.3),

$$
b = \frac{as(s+2)}{s + 2 - ms} = \frac{\pm\,(9/2)^{1/3} \cdot (-1/3) \cdot (5/3)}{(5/3) - (1/3)} = \mp\frac{5}{9}\left(\frac{9}{2}\right)^{1/3}.
$$

Then, the equation (3.48) has a solution $u = u(k)$, $k \in \mathbb{N}(k_0)$ satisfying inequalitites (3.3)–(3.5), that is,

$$
-1 < \left[u(k) \mp \left(\frac{9}{2}\right)^{1/3} k^{1/3} \pm \frac{5}{9}\left(\frac{9}{2}\right)^{1/3} \frac{1}{k^{2/3}}\right] \left[\mp\frac{5}{9}\left(\frac{9}{2}\right)^{1/3} \frac{1}{k^{2/3}}\right]^{-1} < 1,
$$

$$-\frac{2}{3} < \left[\Delta u(k) \mp \Delta\left(\left(\frac{9}{2}\right)^{1/3} k^{1/3}\right) \pm \Delta\left(\frac{5}{9}\left(\frac{9}{2}\right)^{1/3}\frac{1}{k^{2/3}}\right)\right] \cdot \left[\mp\Delta\left(\frac{5}{9}\left(\frac{9}{2}\right)^{1/3}\frac{1}{k^{2/3}}\right)\right]^{-1} < \frac{2}{3},$$

$$-1+O\left(\frac{1}{k}\right) < \left[\Delta^2 u(k) \mp \Delta^2\left(\left(\frac{9}{2}\right)^{1/3} k^{1/3}\right) \pm \Delta^2\left(\frac{5}{9}\left(\frac{9}{2}\right)^{1/3}\frac{1}{k^{2/3}}\right)\right] \cdot \left[\mp\Delta^2\left(\frac{5}{9}\left(\frac{9}{2}\right)^{1/3}\frac{1}{k^{2/3}}\right)\frac{2}{5}\right]^{-1} < 1+O\left(\frac{1}{k}\right).$$

These formulas can be simplified to

$$u(k) = \pm\left(\frac{9}{2}\right)^{1/3} \cdot k^{1/3} \mp \frac{5}{9}\left(\frac{9}{2}\right)^{1/3}\frac{1}{k^{2/3}} \cdot O(1),$$

$$\Delta u(k) = \pm\,\Delta\left(\left(\frac{9}{2}\right)^{1/3} \cdot k^{1/3}\right) - \Delta\left(\frac{5}{9}\left(\frac{9}{2}\right)^{1/3}\frac{1}{k^{2/3}}\right) \cdot O(1),$$

$$\Delta^2 u(k) = \pm\,\Delta^2\left(\left(\frac{9}{2}\right)^{1/3} \cdot k^{1/3}\right) - \Delta^2\left(\frac{5}{9}\left(\frac{9}{2}\right)^{1/3}\frac{1}{k^{2/3}}\right) \cdot O(1).$$

Example 3. In the following example, values (m, α) belong to the domain shown in Figure 3.2 (right yellow domain).

Consider equation (1.3) where $\alpha = -3$, $m = 1/2$, that is, the equation

$$\Delta^2 u(k) \pm \frac{1}{k^3} u^{1/2}(k) = 0. \tag{3.49}$$

In this example α and m satisfy (3.40). If we put

$$\varepsilon_1 = \varepsilon_2 = 1, \quad \varepsilon_3 = \varepsilon_4 = \frac{1}{2},$$

then

$$s = \frac{\alpha+2}{m-1} = \frac{-3+2}{1/2-1} = 2, \;\; ms = 1, \quad s+1 = 3, \quad s+2 = 4$$

and all inequalities (3.1) hold because

$$s+1 = 3 > 0, \quad ms = 1 > 0,$$

$$\varepsilon_3 = \frac{1}{2} < \varepsilon_1 = 1, \quad \varepsilon_2 = 1 > \varepsilon_4 = \frac{1}{2},$$

$$\varepsilon_3 = \frac{1}{2} > \frac{ms}{s+2}\varepsilon_1 = \frac{1}{4}, \quad \varepsilon_4 = \frac{1}{2} > \frac{ms}{s+2}\varepsilon_2 = \frac{1}{4}.$$

By formula (2.2),

$$a = [\mp s(s+1)]^{1/(m-1)} = [\mp 2\cdot 3]^{-2} = \frac{1}{36}$$

and, by formula (2.3),

$$b = \frac{as(s+2)}{s+2-ms} = \frac{1/36\cdot 6}{4-1} = \frac{1}{18}\,.$$

Theorem 4 is applicable and equation (3.49) has a solution $u = u(k)$, $k \in \mathbb{N}(k_0)$ satisfying inequalities (3.3)–(3.5), that is,

$$-1 < \left[u(k) - \frac{1}{36}\cdot\frac{1}{k^2} - \frac{1}{18}\cdot\frac{1}{k^3}\right]\left[\frac{1}{18}\cdot\frac{1}{k^3}\right]^{-1} < 1\,,$$

$$-\frac{1}{2} < \left[\Delta u(k) - \Delta\left(\frac{1}{36k^2}\right) - \Delta\left(\frac{1}{18k^3}\right)\right]\cdot\left[\Delta\left(\frac{1}{18k^3}\right)\right]^{-1} < \frac{1}{2}\,,$$

$$-1 + O\left(\frac{1}{k}\right) < \left[\Delta^2 u(k) - \Delta^2\left(\frac{1}{36k^2}\right) - \Delta^2\left(\frac{1}{18k^3}\right)\right]\cdot\left[\Delta^2\left(\frac{1}{18k^3}\cdot\frac{1}{4}\right)\right]^{-1} <$$
$$< 1 + O\left(\frac{1}{k}\right)\,.$$

These formulas can be simplified to

$$u(k) = \frac{1}{36k^2} + \frac{1}{18k^3}\cdot O(1)\,,$$
$$\Delta u(k) = \Delta\left(\frac{1}{36k^2}\right) + \Delta\left(\frac{1}{18k^3}\right)\cdot O(1)\,,$$
$$\Delta^2 u(k) = \Delta^2\left(\frac{1}{36k^2}\right) + \Delta^2\left(\frac{1}{18k^3}\right)\cdot O(1)\,.$$

Example 4. In the following example, values (m,α) belong to the domain shown in Figure 3.3 (green domain). Consider equation (1.3) where $\alpha = -3$, $m = 4$, that is, the equation

$$\Delta^2 u(k) \pm \frac{1}{k^3}u^4(k) = 0. \tag{3.50}$$

In this example, α and m satisfy (3.42). If we put

$$\varepsilon_1 = \varepsilon_2 = 1, \quad \varepsilon_3 = \varepsilon_4 = \frac{9}{10}\,,$$

then

$$s = \frac{\alpha+2}{m-1} = \frac{-3+2}{4-1} = -\frac{1}{3}\,, \quad ms = -\frac{4}{3}\,, \quad s+1 = \frac{2}{3}\,, \quad s+2 = \frac{5}{3}$$

and all inequalities (3.2) hold because

$$s+1=\frac{2}{3}>0,\quad ms=-\frac{4}{3}<0,$$
$$\varepsilon_3=\frac{9}{10}<\varepsilon_1=1,\quad \varepsilon_2=1>\varepsilon_4=\frac{9}{10}\,,$$
$$\varepsilon_3=\frac{9}{10}>-\frac{ms}{s+2}\varepsilon_1=\frac{4}{5}\,,\quad \varepsilon_4=\frac{9}{10}>-\frac{ms}{s+2}\varepsilon_2=\frac{4}{5}\,.$$

By formula (2.2)

$$a=[\mp s(s+1)]^{1/(m-1)}=\left[\mp\left(-\frac{1}{3}\right)\frac{2}{3}\right]^{1/(4-1)}=\left(\pm\frac{2}{9}\right)^{1/3}$$

and, by formula (2.3),

$$b=\frac{as(s+2)}{s+2-ms}=\frac{(\pm 2/9)^{1/3}\cdot(-1/3)\cdot(5/3)}{(5/3)+(4/3)}=\frac{5}{27}\cdot\left(\mp\frac{2}{9}\right)^{1/3}\,.$$

Theorem 4 is applicable and equation (3.50) has a solution $u=u(k)$, $k\in\mathbb{N}(k_0)$ satisfying inequalitites (3.3)–(3.5), that is,

$$-1<\left[u(k)-\left(\pm\frac{2}{9}\right)^{1/3}\cdot k^{1/3}-\frac{5}{27}\left(\mp\frac{2}{9}\right)^{1/3}\frac{1}{k^{2/3}}\right]\left[\frac{5}{27}\left(\mp\frac{2}{9}\right)^{1/3}\frac{1}{k^{2/3}}\right]^{-1}<1\,,$$

$$-\frac{9}{10}<\left[\Delta u(k)-\Delta\left(\left(\pm\frac{2}{9}\right)^{1/3}\cdot k^{1/3}\right)-\Delta\left(\frac{5}{27}\cdot\left(\mp\frac{2}{9}\right)^{1/3}\cdot\frac{1}{k^{2/3}}\right)\right]$$
$$\cdot\left[\Delta\left(\frac{5}{27}\cdot\left(\mp\frac{2}{9}\right)^{1/3}\cdot\frac{1}{k^{2/3}}\right)\right]^{-1}<\frac{9}{10}\,,$$

$$-1+O\left(\frac{1}{k}\right)<\left[\Delta^2u(k)-\Delta^2\left(\left(\pm\frac{2}{9}\right)^{1/3}\cdot k^{1/3}\right)-\Delta^2\left(\frac{5}{27}\cdot\left(\mp\frac{2}{9}\right)^{1/3}\cdot\frac{1}{k^{2/3}}\right)\right]\cdot$$
$$\cdot\left[\Delta^2\left(\frac{4}{27}\left(\pm\frac{2}{9}\right)^{1/3}\frac{1}{k^{2/3}}\right)\right]^{-1}<1+O\left(\frac{1}{k}\right)\,.$$

These formulas can be simplified to

$$u(k)=\left(\pm\frac{2}{9}\right)^{1/3}\cdot k^{1/3}+\frac{5}{9}\cdot\left(\mp\frac{2}{9}\right)^{1/3}\frac{1}{k^{2/3}}\cdot O\left(1\right),$$
$$\Delta u(k)=\Delta\left(\left(\pm\frac{2}{9}\right)^{1/3}\cdot k^{1/3}\right)+\Delta\left(\frac{5}{9}\cdot\left(\mp\frac{2}{9}\right)^{1/3}\frac{1}{k^{2/3}}\right)\cdot O\left(1\right),$$
$$\Delta^2u(k)=\Delta^2\left(\left(\pm\frac{2}{9}\right)^{1/3}\cdot k^{1/3}\right)+\Delta^2\left(\frac{5}{9}\cdot\left(\mp\frac{2}{9}\right)^{1/3}\frac{1}{k^{2/3}}\right)\cdot O\left(1\right).$$

Example 5. In the following example, values (m, α) belong to the domain shown in Figure 3.4 (middle rectangular blue domain). Consider equation (1.3), where $\alpha = -3$, $m = -1/2$, that is, the equation

$$\Delta^2 u(k) \pm \frac{1}{k^3} u^{-2}(k) = 0. \tag{3.51}$$

In this example, α and m satisfy (3.44). If we put

$$\varepsilon_1 = \varepsilon_2 = 1, \quad \varepsilon_3 = \varepsilon_4 = \frac{1}{2},$$

then

$$s = \frac{\alpha+2}{m-1} = \frac{2}{3}, \quad ms = -\frac{1}{3}, \quad s+1 = \frac{5}{3}, \quad s+2 = \frac{8}{3}$$

and inequalities (3.2) hold because

$$s+1 = \frac{5}{3} > 0, \quad ms = -\frac{1}{3} < 0,$$

$$\varepsilon_3 = \frac{1}{2} < \varepsilon_1 = 1, \quad \varepsilon_2 = 1 > \varepsilon_4 = \frac{1}{2},$$

$$\varepsilon_3 = \frac{1}{2} > -\frac{ms}{s+2}\varepsilon_1 = \frac{1}{8}, \quad \varepsilon_4 = \frac{1}{2} > -\frac{ms}{s+2}\varepsilon_2 = \frac{1}{8}.$$

By formula (2.2)

$$a = [\mp s(s+1)]^{1/(m-1)} = \left[\mp\frac{10}{9}\right]^{-2/3} = \left[\mp\frac{9}{10}\right]^{2/3} = \left(\frac{9}{10}\right)^{2/3}$$

and, by formula (2.3),

$$b = \frac{as(s+2)}{s+2-ms} = \frac{(9/10)^{2/3}\cdot(2/3)\cdot(8/3)}{(8/3)+(1/3)} = \frac{16}{27}\cdot\left(\frac{9}{10}\right)^{2/3}.$$

Theorem 4 is applicable and equation (3.51) has a solution $u = u(k)$, $k \in \mathbb{N}(k_0)$ satisfying inequalitites (3.3)–(3.5), that is,

$$-1 < \left[u(k) - \left(\frac{9}{10k}\right)^{2/3}\frac{1}{k^{2/3}} - \frac{16}{27}\left(\frac{9}{10}\right)^{2/3}\frac{1}{k^{5/3}}\right]\left[\frac{16}{27}\left(\frac{9}{10}\right)^{2/3}\frac{1}{k^{5/3}}\right]^{-1} < 1,$$

$$-\frac{1}{2} < \left[\Delta u(k) - \Delta\left(\left(\frac{9}{10}\right)^{2/3}\frac{1}{k^{2/3}}\right) - \Delta\left(\frac{16}{27}\cdot\left(\frac{9}{10}\right)^{2/3}\frac{1}{k^{5/3}}\right)\right]\cdot$$
$$\cdot\left[\Delta\left(\frac{16}{27}\left(\frac{9}{10}\right)^{2/3}\frac{1}{k^{5/3}}\right)\right]^{-1} < \frac{1}{2},$$

$$-1+O\left(\frac{1}{k}\right) < \left[\Delta^2 u(k) - \Delta^2\left(\left(\frac{9}{10}\right)^{2/3}\frac{1}{k^{2/3}}\right) - \Delta^2\left(\frac{16}{27}\cdot\left(\frac{9}{10}\right)^{2/3}\frac{1}{k^{5/3}}\right)\right]\cdot$$
$$\cdot\left[\Delta^2\left(-\frac{2}{27}\left(\frac{9}{10}\right)^{2/3}\frac{1}{k^{5/3}}\right)\right]^{-1} < 1+O\left(\frac{1}{k}\right).$$

These formulas can be simplified to

$$u(k) = \left(\frac{9}{10}\right)^{2/3}\frac{1}{k^{2/3}} + \frac{16}{27}\left(\frac{9}{10}\right)^{2/3}\frac{1}{k^{5/3}}\cdot O(1),$$
$$\Delta u(k) = \Delta\left(\left(\frac{9}{10}\right)^{2/3}\frac{1}{k^{2/3}}\right) + \Delta\left(\frac{16}{27}\left(\frac{9}{10}\right)^{2/3}\frac{1}{k^{5/3}}\right)\cdot O(1),$$
$$\Delta^2 u(k) = \Delta^2\left(\left(\frac{9}{10}\right)^{2/3}\frac{1}{k^{2/3}}\right) + \Delta^2\left(\frac{16}{27}\left(\frac{9}{10}\right)^{2/3}\frac{1}{k^{5/3}}\right)\cdot O(1).$$

Example 6. In the following example, values (m, α) belong to the domain shown in Figure 3.4 (left blue domain). Consider equation (1.3), where $\alpha = -5$, $m = -4$, that is, the equation

$$\Delta^2 u(k) \pm \frac{1}{k^5} u^{-4}(k) = 0. \tag{3.52}$$

In this example, α and m satisfy (3.45). If we put

$$\varepsilon_1 = \varepsilon_2 = 1, \quad \varepsilon_3 = \varepsilon_4 = \frac{13}{14},$$

then

$$s = \frac{\alpha+2}{m-1} = \frac{3}{5}, \quad ms = -\frac{12}{5}, \quad s+1 = \frac{8}{5}, \quad s+2 = \frac{13}{5}$$

and inequalities (3.2) hold because

$$s+1 = \frac{8}{5} > 0, \quad ms = -\frac{12}{5} < 0,$$
$$\varepsilon_3 = \frac{13}{14} < \varepsilon_1 = 1, \quad \varepsilon_2 = 1 > \varepsilon_4 = \frac{13}{14},$$
$$\varepsilon_3 = \frac{13}{14} > -\frac{ms}{s+2}\varepsilon_1 = \frac{12}{13}, \quad \varepsilon_4 = \frac{13}{14} > -\frac{ms}{s+2}\varepsilon_2 = \frac{12}{13}.$$

By formula (2.2),

$$a = [\mp s(s+1)]^{1/(m-1)} = \left(\mp\frac{24}{25}\right)^{-1/5} = \mp\left(\frac{24}{25}\right)^{-1/5}$$

and, by formula (2.3),

$$b = \frac{as(s+2)}{s+2-ms} = \frac{\mp(24/25)^{-1/5}\cdot(3/5)\cdot(8/5)}{(13/5)+(12/5)} = \mp\frac{24}{125}\cdot\left(\frac{24}{25}\right)^{-1/5}.$$

Theorem 4 is applicable and equation (3.52) has a solution $u = u(k)$, $k \in \mathbb{N}(k_0)$ satisfying inequalitites (3.3)–(3.5), that is,

$$-1 < \left[u(k) \pm \left(\frac{24}{25}\right)^{-1/5} \frac{1}{k^{3/5}} \pm \frac{24}{125}\left(\frac{24}{25}\right)^{-1/5} \frac{1}{k^{8/5}}\right] \left[\mp \frac{24}{125}\left(\frac{24}{25}\right)^{-1/5} \frac{1}{k^{8/5}}\right]^{-1} < 1\,,$$

$$-\frac{13}{14} < \left[\Delta u(k) \pm \Delta\left(\left(\frac{24}{25}\right)^{-1/5} \cdot \frac{1}{k^{3/5}}\right) \pm \Delta\left(\frac{24}{125}\cdot\left(\frac{24}{25}\right)^{-1/5} \frac{1}{k^{8/5}}\right)\right] \cdot \\ \cdot \left[\Delta\left(\mp\frac{24}{125}\left(\frac{24}{25}\right)^{-1/5} \frac{1}{k^{8/5}}\right)\right]^{-1} < \frac{13}{14},$$

$$-1 + O\left(\frac{1}{k}\right) < \left[\Delta^2 u(k) \pm \Delta^2\left(\left(\frac{24}{25}\right)^{-1/5} \frac{1}{k^{3/5}}\right) \pm \Delta^2\left(\frac{24}{125}\left(\frac{24}{25}\right)^{-1/5} \frac{1}{k^{8/5}}\right)\right] \cdot \\ \cdot \left[\Delta^2\left(\pm\frac{288}{1625}\cdot\left(\frac{24}{25}\right)^{-1/5} \frac{1}{k^{8/5}}\right)\right]^{-1} < 1 + O\left(\frac{1}{k}\right)\,.$$

These formulas can be simplified to

$$\begin{aligned} u(k) &= \mp\left(\frac{24}{25}\right)^{-1/5} \frac{1}{k^{3/5}} \mp \frac{24}{125}\cdot\left(\frac{24}{25}\right)^{-1/5} \frac{1}{k^{8/5}} \cdot O\left(1\right), \\ \Delta u(k) &= \mp\Delta\left(\left(\frac{24}{25}\right)^{-1/5} \frac{1}{k^{3/5}}\right) \mp \Delta\left(\frac{24}{125}\cdot\left(\frac{24}{25}\right)^{-1/5} \frac{1}{k^{8/5}}\right) \cdot O\left(1\right), \\ \Delta^2 u(k) &= \mp\Delta^2\left(\left(\frac{24}{25}\right)^{-1/5} \frac{1}{k^{3/5}}\right) \mp \Delta^2\left(\frac{24}{125}\cdot\left(\frac{24}{25}\right)^{-1/5} \frac{1}{k^{8/5}}\right) \cdot O\left(1\right). \end{aligned}$$

Example 7. In the following example, values (m, α) belong to the domain shown in Figure 3.4 (blue triangle-domain). Consider equation (1.3), where $\alpha = -7/4$, $m = 1/2$, that is, the equation

$$\Delta^2 u(k) \pm \frac{1}{k^{7/4}} u^{1/2}(k) = 0. \tag{3.53}$$

In this example, α and m satisfy (3.46). If we put

$$\varepsilon_1 = \varepsilon_2 = 1, \quad \varepsilon_3 = \varepsilon_4 = \frac{1}{3}\,,$$

then

$$s = \frac{\alpha+2}{m-1} = -\frac{1}{2}\,, \quad ms = -\frac{1}{4}\,, \quad s+1 = \frac{1}{2}\,, \quad s+2 = \frac{3}{2}$$

and inequalities (3.2) hold because

$$s+1=\frac{1}{2}>0, \quad ms=-\frac{1}{4}<0,$$

$$\varepsilon_3=\frac{1}{3}<\varepsilon_1=1, \quad \varepsilon_2=1>\varepsilon_4=\frac{1}{3},$$

$$\varepsilon_3=\frac{1}{3}>-\frac{ms}{s+2}\varepsilon_1=\frac{1}{6}, \quad \varepsilon_4=\frac{1}{3}>-\frac{ms}{s+2}\varepsilon_2=\frac{1}{6}\,.$$

By formula (2.2),

$$a=[\mp s(s+1)]^{1/(m-1)}=[\pm 1/4]^{-2}=16$$

and, by formula (2.3),

$$b=\frac{as(s+2)}{s+2-ms}=\frac{-16\cdot(3/4)}{(3/2)+(1/4)}=-\frac{48}{7}\,.$$

Theorem 4 is applicable and equation (3.53) has a solution $u=u(k)$, $k\in\mathbb{N}(k_0)$ satisfying inequalitites (3.3)–(3.5), that is,

$$-1<\left[u(k)-16\cdot k^{1/2}+\frac{48}{7k^{1/2}}\right]\left[-\frac{48}{7k^{1/2}}\right]^{-1}<1\,,$$

$$-\frac{1}{3}<\left[\Delta u(k)-\Delta\left(16\cdot k^{1/2}\right)+\Delta\left(\frac{7}{27}\cdot\left(\mp\frac{9}{4}\right)^{1/3}\frac{1}{k^{4/3}}\right)\right]\cdot$$

$$\cdot\left[\Delta\left(-\frac{48}{7k^{1/2}}\right)\right]^{-1}<1,$$

$$-\,1\,+\,O\left(\frac{1}{k}\right)\;<\;\left[\Delta^2u(k)+\Delta^2\left(\frac{48}{7k^{1/2}}\right)\right]\cdot\left[\Delta^2\left(\frac{8}{7k^{1/2}}\right)\right]^{-1}\;<\;1\,+\,O\left(\frac{1}{k}\right)\,.$$

These formulas can be simplified to

$$u(k)=16\cdot k^{1/2}-\frac{48}{7k^{1/2}}\cdot O\left(1\right),$$

$$\Delta u(k)=\Delta\left(16\cdot k^{1/2}\right)-\Delta\left(\frac{48}{7k^{1/2}}\right)\cdot O\left(1\right),$$

$$\Delta^2u(k)=\Delta^2\left(16\cdot k^{1/2}\right)-\Delta^2\left(\frac{48}{7k^{1/2}}\right)\cdot O\left(1\right).$$

3.4 A remark on the case $s + 1 < 0$.

In this section, we will show why this case is an exception. We will try to estimate functions $F_1(k, b_1, Y_1)$ and $F_1(k, c_1, Y_1)$ similar to formulas (3.18) and (3.20). Assuming $s < -1$, we get

$$F_1(k, b_1, Y_1) \cdot k < \max F_2(k, b_1, Y_1) \cdot k = (s+1)(-\varepsilon_1) + (s+1)(-\varepsilon_4) + O\left(\frac{1}{k^2}\right)$$
$$< b_1(k+1) - b_1(k) = -\varepsilon_1 + \varepsilon_1 = 0. \quad (3.54)$$

Hence, (3.54) holds if

$$\varepsilon_1 + \varepsilon_4 < 0.$$

This is a contradiction since ε_1 and ε_4 are positive numbers.
A similar contradiction we get if we try to estimate $F_1(k, c_1, Y_1)$:

$$F_1(k, c_1, Y_1) \cdot k > \min F_2(k, c_1, Y_1) \cdot k = (s+1)\varepsilon_2 + (s+1)\varepsilon_3 + O\left(\frac{1}{k^2}\right)$$
$$> c_1(k+1) - c_1(k) = \varepsilon_2 - \varepsilon_2 = 0. \quad (3.55)$$

Hence, (3.55) holds if

$$\varepsilon_2 + \varepsilon_3 < 0.$$

This inequality contradicts the positivity of the constants ε_2 and ε_3.

4 Power-type asymptotic behaviour for zero upper and lower function tending to zero

In this chapter, we will show that the areas of coefficient values for which equation (1.3) has solutions asymptotically expressed by a power-type function may change depending on the type of the upper and lower functions. We will search for the conditions such that there exists a solution to equation (1.3) with the following asymptotic behaviour:

$$u(k) = \frac{a}{k^s} + \frac{b}{k^{s+1}} + O\left(\frac{1}{k^{\gamma+s+1}}\right), \tag{4.1}$$

where a, b and s are defined in (2.2), (2.3) and (2.1) and γ is a positive constant.

In this chapter, we have chosen power-type upper and lower functions $b_1(k)$, $b_2(k)$, $c_1(k)$ and $c_2(k)$ tending to zero.

The idea of the proof is similar to the one in the previous chapter while requiring more complex calculations. The scheme of all investigations is the following. The transformations (2.10)–(2.12), where $a_\pm$, $b_\pm$ are computed by formulas (2.2), (2.3), are used to transform the equation (1.3) into an auxiliary system of two equations (2.22), (2.23).

Then, some particular results of those published in [13, 15]) are applied to investigate system (2.22), (2.23). A correct use of Theorem 2 necessitates the proper choice of the functions $b_i(k)$, $c_i(k)$, $i = 1, 2$. In this chapter, we will assume

$$b_1(k) := -\frac{\varepsilon_1}{k^\gamma}, \quad c_1(k) := \frac{\varepsilon_2}{k^\gamma}, \quad b_2(k) := -\frac{\varepsilon_3}{k^\beta}, \quad c_2(k) := \frac{\varepsilon_4}{k^\beta} \tag{4.2}$$

where ε_j, $j = 1, \dots, 4$ are positive constants.

This chapter is divided into 4 parts depending on the values $s+1$ and ms, where s is defined in (2.1). Now we can consider the following Table 4.1.

To prove all the below theorems we need to define some auxiliary sets and functions identical for all four cases.

Let $\varepsilon_i > 0$, $i = 1, \dots, 4$ and let β and γ be fixed. Assuming k_0 positive and sufficiently large such that the asymptotic computations in the proof are correct for every $k \in \mathbb{N}(k_0)$, define functions b_i, c_i, $i = 1, 2$, satisfying (1.9), by formulas

$$b_1(k) := -\frac{\varepsilon_1}{k^\gamma}, \quad c_1(k) := \frac{\varepsilon_2}{k^\gamma},$$
$$b_2(k) := -\frac{\varepsilon_3}{k^\beta}, \quad c_2(k) := \frac{\varepsilon_4}{k^\beta}.$$

Then,

$$B_1(k, Y) := -Y_0 + b_1(k) = -Y_0 - \varepsilon_1,$$

the case	$ms < 0$	$ms > 0$
$s+1>0$	Theorem 7 Theorem 8	Theorem 5 Theorem 6
$s+1<0$	Theorem 9, Theorem 10	Theorem 11, Theorem 12

Tab. 4.1: The structure of the cases.

$$\begin{aligned}
B_2(k,Y) &:= -Y_1 + b_2(k) = -Y_1 - \varepsilon_3,\\
C_1(k,Y) &:= \quad Y_0 - c_1(k) = \quad Y_0 - \varepsilon_2,\\
C_2(k,Y) &:= \quad Y_1 - c_2(k) = \quad Y_1 - \varepsilon_4
\end{aligned}$$

and

$$\Omega_B^1 = \left\{(k,Y)\colon k \in \mathbb{N}(k_0),\ Y_0 = -\frac{\varepsilon_1}{k^\gamma},\ -\frac{\varepsilon_3}{k^\beta} \le Y_1 \le \frac{\varepsilon_4}{k^\beta}\right\}, \tag{4.3}$$

$$\Omega_B^2 = \left\{(k,Y)\colon k \in \mathbb{N}(k_0),\ Y_1 = -\frac{\varepsilon_3}{k^\beta},\ -\frac{\varepsilon_1}{k^\gamma} \le Y_0 \le \frac{\varepsilon_2}{k^\gamma}\right\}, \tag{4.4}$$

$$\Omega_C^1 = \left\{(k,Y)\colon k \in \mathbb{N}(k_0),\ Y_0 = \frac{\varepsilon_2}{k^\gamma},\ -\frac{\varepsilon_3}{k^\beta} \le Y_1 \le \frac{\varepsilon_4}{k^\beta}\right\}, \tag{4.5}$$

$$\Omega_C^2 = \left\{(k,Y)\colon k \in \mathbb{N}(k_0),\ Y_1 = \frac{\varepsilon_4}{k^\beta},\ -\frac{\varepsilon_1}{k^\gamma} \le Y_0 \le \frac{\varepsilon_2}{k^\gamma}\right\}. \tag{4.6}$$

For later formulation, we will need to verify four differences: $b_1(k+1) - b_1(k)$, $b_2(k+1) - b_2(k)$, $c_1(k+1) - c_1(k)$ and $c_2(k+1) - c_2(k)$. As functions $b_1(k)$, $b_2(k)$, $c_1(k)$ and $c_2(k)$ are similar, we will show the calculation for only one case using the binomial formula (1.6):

$$\begin{aligned}
b_1(k+1) - b_1(k) = -\frac{\varepsilon_1}{(k+1)^\gamma} + \frac{\varepsilon_1}{k^\gamma} = -\varepsilon_1 k^{-\gamma}\left(\left(1+\frac{1}{k}\right)^{-\gamma} - 1\right) = \\
-\frac{\varepsilon_1}{k^\gamma}\left(1 - \frac{\gamma}{k} + O\left(\frac{1}{k^2}\right) - 1\right) = \frac{\varepsilon_1\gamma}{k^{\gamma+1}}\left(1 + O\left(\frac{1}{k}\right)\right).
\end{aligned} \tag{4.7}$$

To apply Theorem 1.15, inequalites (1.15) and (1.16) must hold.

Since inequality (1.15) assumes $(k,Y) \in \Omega_B^i$, $i = 1, \dots, n$ and inequality (1.16) assumes $(k,Y) \in \Omega_C^i$, $i = 1, \dots, n$, we need to verify (taking into account specifications (4.3)–(4.6)) and (4.7) the following:

$$F_1(k, b_1(k), Y_1)|_{(k,Y_0,Y_1)\in\Omega_B^1} = F_1\left(k, -\frac{\varepsilon_1}{k^\gamma}, \quad Y_1\right)\Big|_{b_2(k)\le Y_1\le c_2(k)} <$$
$$< b_1(k+1) - b_1(k) = \quad \frac{\varepsilon_1\gamma}{k^{\gamma+1}}\left(1 + O\left(\frac{1}{k}\right)\right), \quad (4.8)$$

$$F_1(k, c_1(k), Y_1)|_{(k,Y_0,Y_1)\in\Omega_C^1} = F_1\left(k, \quad \frac{\varepsilon_2}{k^\gamma}, \quad Y_1\right)\Big|_{b_2(k)\le Y_1\le c_2(k)} >$$
$$> c_1(k+1) - c_1(k) = -\frac{\varepsilon_2\gamma}{k^{\gamma+1}}\left(1 + O\left(\frac{1}{k}\right)\right), \quad (4.9)$$

$$F_2(k, Y_0, b_2(k))|_{(k,Y_0,Y_1)\in\Omega_B^2} = F_2\left(k, \quad Y_0, -\frac{\varepsilon_3}{k^\gamma}\right)\Big|_{b_1(k)\le Y_0\le c_1(k)} <$$
$$< b_2(k+1) - b_2(k) = \quad \frac{\varepsilon_3\beta}{k^{\beta+1}}\left(1 + O\left(\frac{1}{k}\right)\right), \quad (4.10)$$

$$F_2(k, Y_0, c_2(k))|_{(k,Y_0,Y_1)\in\Omega_C^2} = F_2\left(k, \quad Y_0, \quad \frac{\varepsilon_4}{k^\gamma}\right)\Big|_{b_1(k)\le Y_0\le c_1(k)} >$$
$$> c_2(k+1) - c_2(k) = -\frac{\varepsilon_4\beta}{k^{\beta+1}}\left(1 + O\left(\frac{1}{k}\right)\right) \quad (4.11)$$

whenever

$$-\frac{\varepsilon_3}{k^\beta} \le Y_1 \le \frac{\varepsilon_4}{k^\beta} \quad (4.12)$$

in (4.8), (4.9) and

$$-\frac{\varepsilon_1}{k^\gamma} \le Y_0 \le \frac{\varepsilon_2}{k^\gamma} \quad (4.13)$$

in (4.10), (4.11).

The scheme of each of the following fourth sections (sections 4.1–4.4) is similar. In each part, we give two theorems on the existence of a power-type solution. The first theorem considers the conditions, including the values and variables not defined in the formulation of the equation (1.3). The second theorem will define the strict values of m and α and will be represented in the plane.

Examples illustrating all theorems can be found in section 4.6.

4.1 The case of $ms > 0$ and $s + 1 > 0$

Theorem 5. *Let either*

$$s > 0, \quad m > 0 \tag{4.14}$$

or

$$-1 < s < 0, \quad m < 0. \tag{4.15}$$

Assume that there exists a constant γ satisfying $0 < \gamma < 1$ and positive numbers ε_i, $i = 1, 2, 3, 4$, such that

$$\varepsilon_3 < \varepsilon_1 \frac{\gamma + s + 1}{s + 1}\,, \tag{4.16}$$

$$\varepsilon_4 < \varepsilon_2 \frac{\gamma + s + 1}{s + 1}\,, \tag{4.17}$$

$$\varepsilon_1 < \varepsilon_3 \frac{\gamma + s + 2}{ms}\,, \tag{4.18}$$

$$\varepsilon_2 < \varepsilon_4 \frac{\gamma + s + 2}{ms}\,. \tag{4.19}$$

Then, for a sufficiently large fixed $k_0 > 0$, there exists a solution $u\colon \mathbb{N}(k_0) \to \mathbb{R}$ of equation (1.3) *such that, for every $k \in \mathbb{N}(k_0)$, asymptotic representation* (4.1) *holds or, more presisely, this solution satisfies*

$$-\frac{\varepsilon_1}{k^\gamma} < \left[u(k) - \frac{a}{k^s} - \frac{b}{k^{s+1}}\right]\left[\frac{b}{k^{s+1}}\right]^{-1} < \frac{\varepsilon_2}{k^\gamma}\,, \tag{4.20}$$

$$-\frac{\varepsilon_3}{k^\gamma} < \left[\Delta u(k) - \Delta\left(\frac{a}{k^s}\right) - \Delta\left(\frac{b}{k^{s+1}}\right)\right]\left[\Delta\left(\frac{b}{k^{s+1}}\right)\right]^{-1} < \frac{\varepsilon_4}{k^\gamma}\,, \tag{4.21}$$

$$-\frac{\varepsilon_1}{k^\gamma} + O\left(\frac{1}{k}\right) < \left[\Delta^2 u(k) - \Delta^2\left(\frac{a}{k^s}\right) - \Delta^2\left(\frac{b}{k^{s+1}}\right)\right]\left[\Delta^2\left(\frac{b}{k^{s+1}}\right)\frac{ms}{s+2}\right]^{-1} < \frac{\varepsilon_2}{k^\gamma} + O\left(\frac{1}{k}\right). \tag{4.22}$$

Theorem 6. *Let at least one of following assumptions hold:*

$$m \in \left(-7 - 4\sqrt{3}, -7 + 4\sqrt{3}\right), \quad -2 < \alpha < -m - 1, \tag{4.23}$$

$$0 < m < 1, \quad \alpha < -2, \tag{4.24}$$

$$m > 1, \quad -2 < \alpha < \frac{1}{2}\left(-(m-1) + \sqrt{(m-1)^2 + 16m}\right), \tag{4.25}$$

$$-2 < \alpha < -m - 1, \quad m < 0, \quad (m-1)^2 + 16m > 0 \tag{4.26}$$

and either

$$\alpha < \frac{1}{2}\left(-(m-1) - \sqrt{(m-1)^2 + 16m}\right)$$

or

$$\alpha > \frac{1}{2}\left(-(m-1) + \sqrt{(m-1)^2 + 16m}\right).$$

Then, the conclusion of Theorem 5 holds.

4.1.1 Proof of Theorem 5

From assumptions (4.14) and (4.15), we have $ms > 0$ and $s+1 > 0$. These inequalities are used tacitly below. Now, we will verify inequalities (4.8)–(4.11).

Let us verify inequality (4.8). It will hold if

$$\begin{aligned} F_1(k, b_1(k), Y_1)|_{(k,Y_0,Y_1)\in\Omega_B^1} &\leq \max_{(k,Y_0,Y_1)\in\Omega_B^1} F_1(k, b_1(k), Y_1) \\ &= \left(-\frac{s+1}{k} + O\left(\frac{1}{k^2}\right)\right) \cdot \left(\frac{\varepsilon_1}{k^\gamma} - \frac{\varepsilon_3}{k^\beta}\right) \\ &\qquad < b_1(k+1) - b_1(k) = \frac{\varepsilon_1 \gamma}{k^{\gamma+1}}\left(1 + O\left(\frac{1}{k}\right)\right). \end{aligned}$$

This inequality will hold if either

$$\gamma < \beta \tag{4.27}$$

or

$$\gamma = \beta, \quad \varepsilon_3 < \varepsilon_1 \frac{\gamma + s + 1}{s+1}. \tag{4.28}$$

Now, verify inequality (4.9). It will hold if

$$\begin{aligned} F_1(k, c_1(k), Y_1)|_{(k,Y_0,Y_1)\in\Omega_C^1} &\geq \min_{(k,Y_0,Y_1)\in\Omega_C^1} F_1(k, c_1(k), Y_1) \\ &= \left(-\frac{s+1}{k} + O\left(\frac{1}{k^2}\right)\right) \cdot \left(\frac{-\varepsilon_2}{k^\gamma} + \frac{\varepsilon_4}{k^\beta}\right) \\ &\qquad < c_1(k+1) - c_1(k) = \frac{\varepsilon_2 \gamma}{k^{\gamma+1}}\left(1 + O\left(\frac{1}{k}\right)\right). \end{aligned}$$

This inequality will hold if either

$$\gamma < \beta \tag{4.29}$$

or

$$\gamma = \beta, \quad \varepsilon_4 < \varepsilon_2 \frac{\gamma + s + 1}{s+1}. \tag{4.30}$$

Let us verify inequality (4.10). It will hold if

$$\begin{aligned} F_2(k, Y_0, b_2(k))|_{(k,Y_0,Y_1)\in\Omega_B^2} &\leq \max_{(k,Y_0,Y_1)\in\Omega_B^2} F_2(k, Y_0, b_2(k)) \\ &= \left(-\frac{s+2}{k} + O\left(\frac{1}{k^2}\right)\right)\left(\frac{ms}{s+2}\frac{-\varepsilon_1}{k^\gamma} + \frac{\varepsilon_3}{k^\beta} + O\left(\frac{1}{k}\right)\right) \end{aligned}$$

$$< b_2(k+1) - b_2(k) = \frac{\varepsilon_3\beta}{k^{\beta+1}}\left(1 + O\left(\frac{1}{k}\right)\right). \quad (4.31)$$

This inequality will hold if either

$$\gamma > \beta \quad (4.32)$$

or

$$\gamma = \beta, \quad \gamma < 1, \quad \varepsilon_1 < \varepsilon_3 \frac{\gamma + s + 2}{ms}. \quad (4.33)$$

Let us note that (4.32) contradicts to (4.27). Now, verify inequality (4.11). It will hold if

$$F_2(k, Y_0, c_2(k))|_{(k,Y_0,Y_1)\in\Omega_C^2} \geq \min_{(k,Y_0,Y_1)\in\Omega_C^2} F_2(k, Y_0, c_2)$$
$$= \left(-\frac{s+2}{k} + O\left(\frac{1}{k^2}\right)\right)\left(\frac{ms}{s+2}\frac{\varepsilon_2}{k^\gamma} - \frac{\varepsilon_4}{k^\beta} + O\left(\frac{1}{k}\right)\right)$$
$$> c_2(k+1) - c_2(k) = \frac{\varepsilon_4\beta}{k^{\beta+1}}\left(1 + O\left(\frac{1}{k}\right)\right). \quad (4.34)$$

This inequality will hold if either

$$\gamma > \beta \quad (4.35)$$

or

$$\gamma = \beta, \quad \gamma < 1, \quad \varepsilon_2 < \varepsilon_4 \frac{\gamma + s + 2}{ms}. \quad (4.36)$$

Note again that (4.35) contradicts to (4.27).

Summing up all restrictions (4.27)–(4.36), we get the conditions (4.16)–(4.19). Inequalities (4.20)–(4.22) follow from inequalities (4.12) – (4.13) and formulas (3.32) – (3.34).

This concludes the proof of Theorem 5.

4.1.2 Proof of Theorem 6

Lemma 1. *Let either* (4.14) *or* (4.15) *hold. If, moreover,*

$$ms < \frac{(s+2)(s+3)}{s+1}, \quad (4.37)$$

then, the conclusion of Theorem 5 holds.

Proof. The system (4.16)–(4.19) is equivalent to the following (we need to remember the conditions of this case: $ms > 0$ and $s+1 > 0$). From (4.16) and (4.18), it follows

$$\varepsilon_3 < \varepsilon_1 \frac{\gamma + s + 1}{s+1} < \varepsilon_3 \frac{\gamma + s + 2}{s+1}\frac{\gamma + s + 1}{s+1}.$$

And, from (4.17) and (4.19), it follows

$$\varepsilon_4 < \varepsilon_2 \frac{\gamma + s + 1}{s+1} < \varepsilon_4 \frac{\gamma + s + 2}{s+1}\frac{\gamma + s + 1}{s+1}.$$

Hence,

$$1 < \frac{(\gamma+s+1)(\gamma+s+2)}{ms(s+1)},$$

and, as $ms > 0$ and $s+1>0$, we get

$$\begin{aligned}(s+1)ms - (\gamma+s+1)(\gamma+s+2) < 0 \\ \gamma^2+\gamma(s+1)+\gamma(s+2)+(s+1)(s+2)-ms(s+1) > 0 \\ \gamma^2+\gamma(2s+3)+(s+1)(s+2-ms) > 0.\end{aligned}$$

The discriminant

$$D = (2s+3)^2 - 4(s+1)(s+2-ms)$$

of the quadratic equation

$$\gamma^2+\gamma(2s+3)+(s+1)(s+2-ms) = 0$$

will be positive for $ms > 0$ and $s+1>0$. We have

$$\begin{aligned}D = (2s+3)^2-4(s+1)(s+2-ms) &= 4s^2+12s+9-4s^2-12s+4ms^2-8+4ms \\ &= 4ms^2+4ms+1 = 4ms(s+1)+1 > 0.\end{aligned}$$

Then, as we need $\gamma \in (0,1)$, at least one of the following inequalities should hold:

$$\frac{-(2s+3)-\sqrt{4ms(s+1)+1}}{2} > 0 \tag{4.38}$$

and

$$\frac{-(2s+3)+\sqrt{4ms(s+1)+1}}{2} < 1. \tag{4.39}$$

The first inequality (4.38) does not hold due to $-(2s+3) < 0$ and $\sqrt{D} > 0$, that is

$$-(2s+3)-\sqrt{4ms(s+1)+1} < 0.$$

The second inequality (4.39) is equivalent to the following one

$$\sqrt{4ms(s+1)+1} < 2s+5$$

and, after some simplification, we get

$$4ms^2+4ms+1 < 4s^2+20s+25.$$

Finally,

$$ms(s+1) < s^2+5s+6.$$

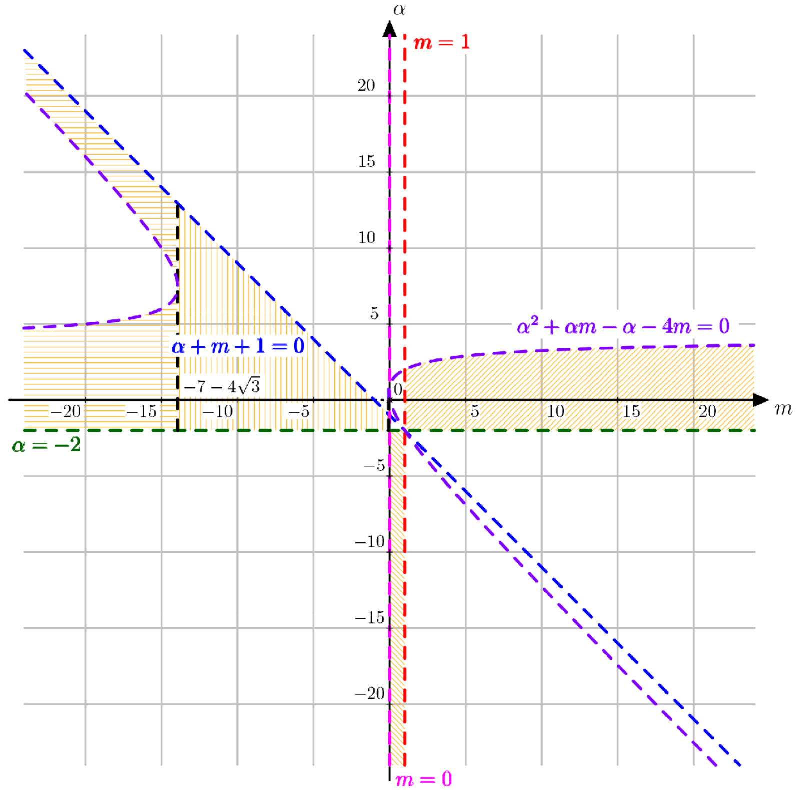

Fig. 4.1: Summary of admissible values (Theorem 6)

Next, if

$$ms < \frac{(s+2)(s+3)}{(s+1)}$$

the system of inequalities (4.16)–(4.19) holds (we can find some ε_i, $i = 1, ..., 4$ and $\gamma \in (0, 1)$) and we have the formulation of the main result. □

Now we are ready to prove Theorem 6. Condition (4.37) holds if

$$ms(s+1) < s^2 + 5s + 6.$$

According to the form of s from (2.1), we get

$$m(\alpha+2)(\alpha+m+1) < (\alpha+2)^2 + 5(\alpha+2)(m-1) + 6(m-1)^2,$$
$$\alpha^2(m1) + \alpha(m-1)^2 - 4m(m-1) < 0,$$
$$(m-1)\left[\alpha^2 + \alpha(m-1) - 4m\right] < 0.$$

As $m > 0$ and $s > 0$, we get $\{0 < m < 1$ and $\alpha < -2\}$ or $\{m > 1$ and $\alpha > -2\}$ and $D = (m-1)^2 + 16m > 0$.

Now we must analyse the following two cases.

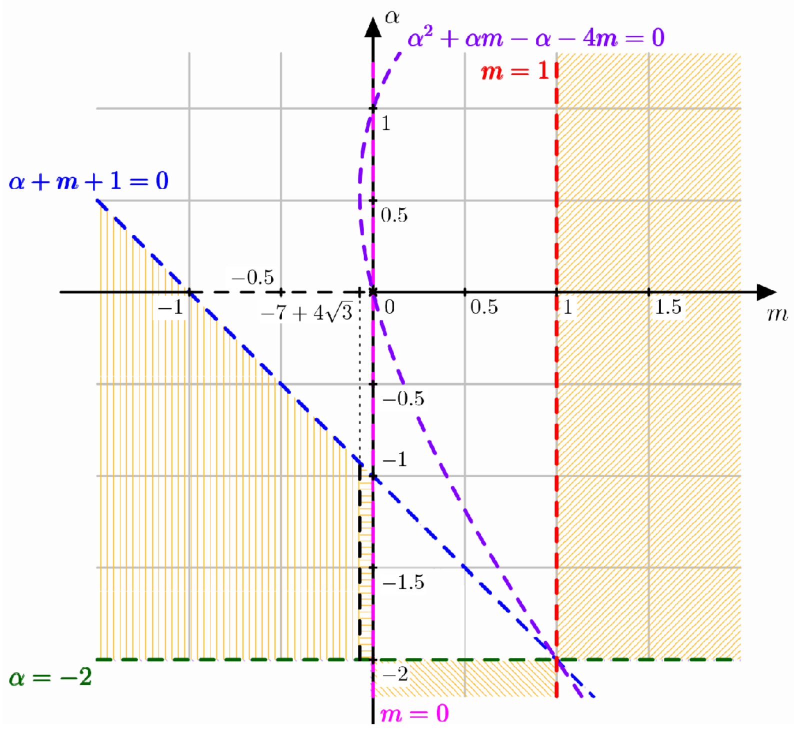

Fig. 4.2: Summary of admissible values - zoom (Theorem 6)

Case A. $m > 1$ and $\alpha > -2$.
The formal solution of the inequality is

$$\frac{-(m-1) - \sqrt{(m-1)^2 + 16m}}{2} < \alpha < \frac{-(m-1) + \sqrt{(m-1)^2 + 16m}}{2},$$

but

$$\frac{-(m-1) - \sqrt{(m-1)^2 + 16m}}{2} < -2$$

and we get the second condition of the Theorem.

Case B. $0 < m < 1$ and $\alpha < -2$.
We need to prove the following inequality:

$$\alpha^2 + m(\alpha - 2) - \alpha - 2m > 0.$$

The proof will be divided into two parts. First, let us show that $\alpha - 2m < 0$. This is obvious because $\alpha < -2$ and $0 < 2m < 2$. Next, let us show that $\alpha^2 + m(\alpha - 2) - \alpha - 2m > 0$. This is equal to the following $(\alpha - m)(\alpha + 2m) > 0$, which holds if $0 < m < 0$ and $\alpha < -2$.

The theorem is proved.

All suitable areas on the (α, m)-plane indicated in Theorem 6 are visualized on the figures 4.1, 4.2.

4.2 The case of $ms < 0$ and $s + 1 > 0$

Theorem 7. *Let either*

$$s > 0, \quad m < 0$$

or

$$-1 < s < 0, \quad m > 0.$$

Assume that there exists a constant γ *satisfying* $0 < \gamma < 1$ *and positive numbers* ε_i, $i = 1, 2, 3, 4$, *such that*

$$\varepsilon_3 < \varepsilon_1 \frac{\gamma + s + 1}{s + 1}, \tag{4.40}$$

$$\varepsilon_4 < \varepsilon_2 \frac{\gamma + s + 1}{s + 1}, \tag{4.41}$$

$$\varepsilon_1 < -\varepsilon_3 \frac{\gamma + s + 2}{ms}, \tag{4.42}$$

$$\varepsilon_2 < -\varepsilon_4 \frac{\gamma + s + 2}{ms}. \tag{4.43}$$

Then, for a sufficiently large fixed $k_0 > 0$, *there exists a solution* $u\colon \mathbb{N}(k_0) \to \mathbb{R}$ *of equation* (1.3) *such that, for every* $k \in \mathbb{N}(k_0)$, *asymptotic representation* (4.1) *holds or, more presisely, this solution satisfies*

$$-\frac{\varepsilon_1}{k^\gamma} < \left[u(k) - \frac{a}{k^s} - \frac{b}{k^{s+1}}\right]\left[\frac{b}{k^{s+1}}\right]^{-1} < \frac{\varepsilon_2}{k^\gamma}, \tag{4.44}$$

$$-\frac{\varepsilon_3}{k^\gamma} < \left[\Delta u(k) - \Delta\left(\frac{a}{k^s}\right) - \Delta\left(\frac{b}{k^{s+1}}\right)\right]\left[\Delta\left(\frac{b}{k^{s+1}}\right)\right]^{-1} < \frac{\varepsilon_4}{k^\gamma}, \tag{4.45}$$

$$-\frac{\varepsilon_1}{k^\gamma} + O\left(\frac{1}{k}\right) < \left[\Delta^2 u(k) - \Delta^2\left(\frac{a}{k^s}\right) - \Delta^2\left(\frac{b}{k^{s+1}}\right)\right]\left[\Delta^2\left(\frac{b}{k^{s+1}}\right)\frac{ms}{s+2}\right]^{-1} < \frac{\varepsilon_2}{k^\gamma} + O\left(\frac{1}{k}\right). \tag{4.46}$$

Theorem 8. *Let* m *and* α *satisfy one of the following conditions* (4.47)–(4.49)*:*

$$m < 0 \quad \wedge \quad \alpha < -2, \tag{4.47}$$

$$0 < m < 1 \quad \wedge \quad -2 < \alpha < -m - 1, \tag{4.48}$$

$$m > 1 \quad \wedge \quad -m - 1 < \alpha < -2, \tag{4.49}$$

and let, moreover,

$$\alpha^2(1 + m) + \alpha(m^2 + 8m - 1) + 8m^2 > 0. \tag{4.50}$$

Then, for a sufficiently large fixed $k_0 > 0$, *there exists a solution* $u\colon \mathbb{N}(k_0) \to \mathbb{R}$ *of equation* (1.3) *such that, for every* $k \in \mathbb{N}(k_0)$, *asymptotic representation* (4.44)–(4.46) *holds.*

4.2.1 Proof of Theorem 7

Let us verify inequalities (4.8)–(4.11). Using formula (4.7) and assumptions of this section that could be transformed to the following inequalities

$$m < 0 \quad \wedge \quad s > 0$$

or

$$m > 0 \quad \wedge \quad -1 < s < 0.$$

we get

$$\begin{aligned} F_1(k, b_1, Y_1)|_{(k,Y_0,Y_1)\in\Omega_B^1} &\le \max_{(k,Y_0,Y_1)\in\Omega_B^1} F_1(k, b_1, Y_1) \\ &= \left(-\frac{s+1}{k} + O\left(\frac{1}{k^2}\right)\right)\left(\frac{\varepsilon_1}{k^\gamma} - \frac{\varepsilon_3}{k^\beta}\right) \\ &< b_1(k+1) - b_1(k) = \frac{\varepsilon_1\gamma}{k^{\gamma+1}}\left(1 + O\left(\frac{1}{k}\right)\right), \end{aligned} \tag{4.51}$$

$$\begin{aligned} F_1(k, c_1, Y_1)|_{(k,Y_0,Y_1)\in\Omega_C^1} &\ge \min_{(k,Y_0,Y_1)\in\Omega_C^1} F_1(k, c_1, Y_1) \\ &= \left(-\frac{s+1}{k} + O\left(\frac{1}{k^2}\right)\right)\left(-\frac{\varepsilon_2}{k^\gamma} + \frac{\varepsilon_4}{k^\beta}\right) \\ &> c_1(k+1) - c_1(k) = -\frac{\varepsilon_2\gamma}{k^{\gamma+1}}\left(1 + O\left(\frac{1}{k^\gamma}\right)\right), \end{aligned} \tag{4.52}$$

$$\begin{aligned} F_2(k, Y_0, b_2)|_{(k,Y_0,Y_1)\in\Omega_B^2} &\le \max_{(k,Y_0,Y_1)\in\Omega_B^2} F_2(k, Y_0, b_2) \\ &= \left(-\frac{s+2}{k} + O\left(\frac{1}{k^2}\right)\right)\left(\frac{ms}{s+2}\frac{\varepsilon_2}{k^\gamma} + \frac{\varepsilon_3}{k^\beta} + O\left(\frac{1}{k}\right)\right) \\ &< b_2(k+1) - b_2(k) = \frac{\varepsilon_3\beta}{k^{\beta+1}}\left(1 + O\left(\frac{1}{k}\right)\right), \end{aligned} \tag{4.53}$$

$$\begin{aligned} F_2(k, Y_0, c_2)|_{(k,Y_0,Y_1)\in\Omega_C^2} &\ge \min_{(k,Y_0,Y_1)\in\Omega_C^2} F_2(k, Y_0, c_2) \\ &= \left(-\frac{s+2}{k} + O\left(\frac{1}{k^2}\right)\right)\left(-\frac{ms}{s+2}\frac{\varepsilon_1}{k^\gamma} - \frac{\varepsilon_4}{k^\beta} + O\left(\frac{1}{k}\right)\right) > \\ &c_2(k+1) - c_2(k) = -\frac{\varepsilon_4\beta}{k^{\beta+1}}\left(1 + O\left(\frac{1}{k}\right)\right). \end{aligned} \tag{4.54}$$

Now, we will study each of the inequalities separately. The first of them (inequalities (4.51), (4.52)) were studied in the previous section in Theorem 5 where the following restrictions were derived:

i) $\beta > \gamma$ ((4.27) and (4.29)),
ii) $\beta = \gamma$ and $\varepsilon_3 < \varepsilon_1 \dfrac{\gamma + s + 1}{s+1}$ (4.28),
iii) $\beta = \gamma$ and $\varepsilon_4 < \varepsilon_2 \dfrac{\gamma + s + 1}{s+1}$ (4.30) .

The third inequality (4.53) is equivalent with

$$-\frac{ms\varepsilon_2}{k^{\gamma+1}} - \frac{\varepsilon_3(s+2)}{k^{\beta+1}} + O\left(\frac{1}{k^{\gamma+2}}\right) + O\left(\frac{1}{k^{\beta+2}}\right) + O\left(\frac{1}{k^2}\right) < \frac{\varepsilon_3 \beta}{k^{\beta+1}} + O\left(\frac{1}{k^{\beta+2}}\right)$$

or with

$$-\frac{ms}{k^{\gamma+1}}\varepsilon_2 + O\left(\frac{1}{k^{\gamma+2}}\right) + O\left(\frac{1}{k^2}\right) < \frac{\varepsilon_3(\beta+s+2)}{k^{\beta+1}} + O\left(\frac{1}{k^{\beta+2}}\right).$$

The last inequality obviously implies $\beta < 1$, $\gamma < 1$, and either

iv) $\beta < \gamma$ (this restriction contradicts to *i*))
or
v) $\beta = \gamma$ and

$$\varepsilon_2 < -\varepsilon_3 \frac{\beta + s + 2}{ms} .$$

Finally, the last inequality (4.54) is equivalent to

$$\frac{ms\varepsilon_1}{k^{\gamma+1}} + \frac{\varepsilon_4(s+2)}{k^{\beta+1}} + O\left(\frac{1}{k^{\gamma+2}}\right) + O\left(\frac{1}{k^{\beta+2}}\right) + O\left(\frac{1}{k^2}\right) > -\frac{\varepsilon_4 \beta}{k^{\beta+1}} + O\left(\frac{1}{k^{\beta+2}}\right),$$

or

$$-\frac{ms}{k^{\gamma+1}}\varepsilon_1 + O\left(\frac{1}{k^{\gamma+2}}\right) + O\left(\frac{1}{k^2}\right) < \frac{\varepsilon_4(\beta+s+2)}{k^{\beta+1}} + O\left(\frac{1}{k^{\beta+2}}\right). \qquad (4.55)$$

Analyzing (4.55), we conclude that inequalities $\beta < 1$, $\gamma < 1$ must hold. Moreover, one of the following restriction must be fulfilled: either

vi) $\beta < \gamma$ (this restriction contradicts (*i*))
or
vii) $\beta = \gamma$ and $\varepsilon_1 < -\varepsilon_4 \dfrac{\beta + s + 2}{ms}$.

Combining the conditions $i)$ – $vii)$ we get the system of inequalities (4.40)–(4.43):

$$\begin{aligned}
0 &< \gamma = \beta < 1,\\
\varepsilon_3 &< \varepsilon_1 \frac{\gamma+s+1}{s+1},\\
\varepsilon_4 &< \varepsilon_2 \frac{\gamma+s+1}{s+1},\\
\varepsilon_2 &< -\varepsilon_3 \frac{\beta+s+2}{ms},\\
\varepsilon_1 &< -\varepsilon_4 \frac{\beta+s+2}{ms}.
\end{aligned}$$

Inequalities (4.44)–(4.46) follow from inequalities (4.12) – (4.13) and formulas (3.32) – (3.34).

This concludes the proof of Theorem 7.

4.2.2 Proof of Theorem 8

Below, we analyse this system (4.40)–(4.43). We derive

$$\varepsilon_3 < \varepsilon_1 \frac{\gamma+s+1}{s+1} < -\varepsilon_4 \frac{\gamma+s+2}{ms}\frac{\gamma+s+1}{s+1} < \\ -\varepsilon_2 \frac{\gamma+s+2}{ms}\frac{(\gamma+s+1)^2}{(s+1)^2} < \varepsilon_3 \frac{(\gamma+s+2)^2}{(ms)^2}\frac{(\gamma+s+1)^2}{(s+1)^2}.$$

Because $\varepsilon_3 > 0$,

$$1 < \left|\frac{\gamma+s+2}{ms}\right|\left|\frac{\gamma+s+1}{s+1}\right|,$$

or

$$|ms||s+1| < |\gamma+s+1||\gamma+s+2|.$$

The additional conditions that we assumed earlier ($\gamma > 0$, $s+1 > 0$ and $ms < 0$) help us get rid of absolute values resulting in

$$-ms(s+1) < (\gamma+s+1)(\gamma+s+2).$$

Simplifying this inequality, we obtain

$$\gamma^2 + (2s+3)\gamma + (s+1)(s+2) + ms(s+1) > 0$$

or

$$\gamma^2 + (2s+3)\gamma + (s+1)(ms+s+2) > 0. \tag{4.56}$$

Consider equation corresponding to (4.56)

$$\gamma^2 + (2s+3)\gamma + (s+1)(ms+s+2) = 0. \tag{4.57}$$

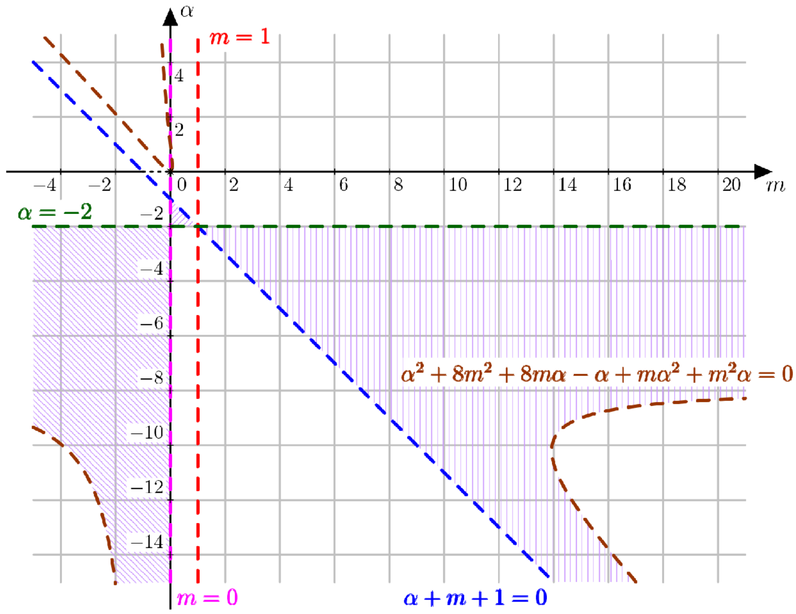

Fig. 4.3: Summary of admissible values (Theorem 8)

The discriminant of equation (4.57)

$$D := (2s+3)^2 - 4(s+1)(ms+s+2) = -4ms(s+1) + 1 > 0$$

is positive because $ms < 0$ and $s+1 > 0$. For the existence of a $\gamma \in (0,1)$, satisfying (4.56), the validity of at least one of the following two conditions is necessary

$$\gamma_1 = \frac{-(2s+3) - \sqrt{D}}{2} > 0 \tag{4.58}$$

or

$$\gamma_2 = \frac{-(2s+3) + \sqrt{D}}{2} < 1. \tag{4.59}$$

The assumption $s+1 > 0$ provides the following chain of inequalities

$$0 < 2s+2 < 2s+3.$$

Hence, it is easy to see that inequality (4.58) does not hold for any m and s because

$$0 > -(2s+3) - \sqrt{D} > 0.$$

Consider the inequality (4.59). We derive an inequality

$$-(2s+3) + \sqrt{D} < 2$$

which can be simplified to

$$-ms(s+1) < s^2 + 5s + 6.$$

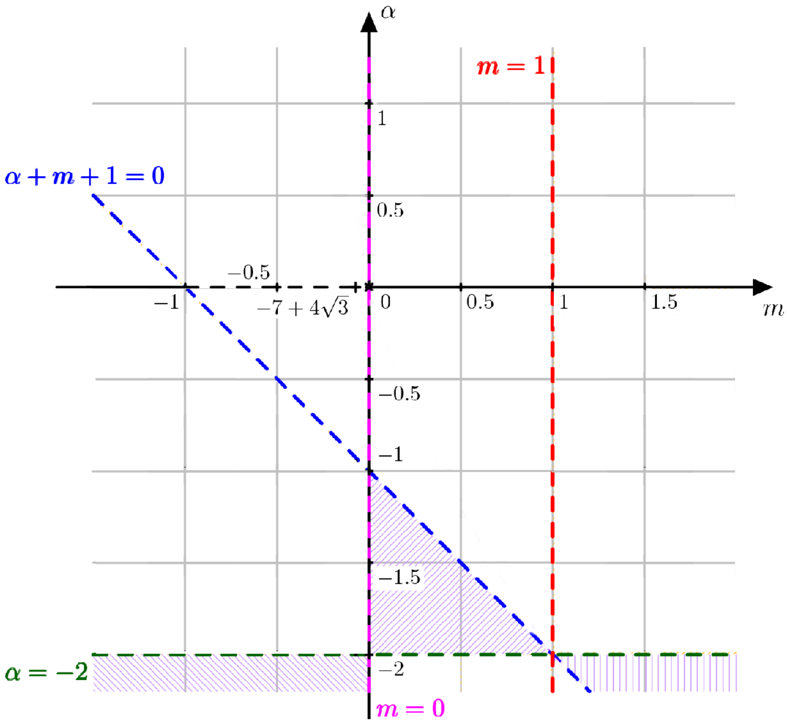

Fig. 4.4: Summary of admissible values - zoom (Theorem 8)

Replacing s by formula (2.1), we get

$$-m\frac{\alpha+2}{m-1}\cdot\frac{\alpha+m+1}{m-1}<\frac{(\alpha+2)^2}{(m-1)^2}+5\frac{\alpha+2}{m-1}+6.$$

This can be simplified to

$$-m(\alpha+2)(\alpha+m+1)<(\alpha+2)^2+5(\alpha+2)(m-1)+6(m-1)^2.$$

Further simplification gives

$$\alpha^2(1+m)+\alpha(m^2+8m-1)+8m^2>0.$$

Finally, we conclude that Theorem 8 is proved.

In figures 4.3, 4.4 the resulting domains in (m,α)-plane is highlighted in violet.

4.3 The case of $ms < 0$ and $s+1 < 0$

Theorem 9. *Let $\alpha \neq 0$ and*

$$s < -1, \quad m > 0, \quad s \neq -2$$

Assume that there exists a constant γ, satisfying $0 < \gamma < 1$ and positive numbers ε_i, $i = 1, 2, 3, 4$, such that

$$\varepsilon_4 < -\varepsilon_1 \frac{\gamma + s + 1}{s + 1}, \tag{4.60}$$

$$\varepsilon_3 < -\varepsilon_2 \frac{\gamma + s + 1}{s + 1}, \tag{4.61}$$

$$\varepsilon_2 < -\varepsilon_3 \frac{\gamma + s + 2}{ms}, \tag{4.62}$$

$$\varepsilon_1 < -\varepsilon_4 \frac{\gamma + s + 2}{ms}. \tag{4.63}$$

Then, for a sufficiently large fixed $k_0 > 0$, there exists a solution $u\colon \mathbb{N}(k_0) \to \mathbb{R}$ of equation (1.3) *such that, for every $k \in \mathbb{N}(k_0)$, asymptotic representation* (4.1) *holds or, more presisely, this solution satisfies*

$$-\frac{\varepsilon_1}{k^\gamma} < \left[u(k) - \frac{a}{k^s} - \frac{b}{k^{s+1}}\right]\left[\frac{b}{k^{s+1}}\right]^{-1} < \frac{\varepsilon_2}{k^\gamma},$$

$$-\frac{\varepsilon_3}{k^\gamma} < \left[\Delta u(k) - \Delta\left(\frac{a}{k^s}\right) - \Delta\left(\frac{b}{k^{s+1}}\right)\right]\left[\Delta\left(\frac{b}{k^{s+1}}\right)\right]^{-1} < \frac{\varepsilon_4}{k^\gamma},$$

$$-\frac{\varepsilon_1}{k^\gamma} + O\left(\frac{1}{k}\right) < \left[\Delta^2 u(k) - \Delta^2\left(\frac{a}{k^s}\right) - \Delta^2\left(\frac{b}{k^{s+1}}\right)\right]\left[\Delta^2\left(\frac{b}{k^{s+1}}\right)\frac{ms}{s+2}\right]^{-1} < \frac{\varepsilon_2}{k^\gamma} + O\left(\frac{1}{k}\right).$$

Theorem 10. *Let the numbers α and m satisfy*

$$\alpha \neq \{0, -2m\},$$

$$\frac{\alpha + m + 1}{m - 1} < 0, \tag{4.64}$$

$$m\frac{\alpha + 2}{m - 1} < 0, \tag{4.65}$$

$$\frac{2\alpha + 5m - 1}{m - 1} > 0, \tag{4.66}$$

$$(m-1)(\alpha^2 + \alpha m - \alpha - 4m) < 0 \tag{4.67}$$

and

$$\gamma + \frac{\alpha+m+1}{m-1} > 0 \tag{4.68}$$

where γ is a fixed number such that $\gamma \in (\gamma^*, 1)$ and

$$\gamma^* = \frac{1}{2}\left(-\frac{2\alpha+3m+1}{m-1} + \sqrt{4m\frac{\alpha+2}{m-1}\cdot\frac{\alpha+m+1}{m-1}+1}\right).$$

Then, the conclusion of Theorem 9 holds.

4.3.1 Proof of Theorem 9

Let us again verify inequalities (4.8)–(4.11). Using formula (4.7) and the assumptions of this section that could be transformed into the following inequalities

$$\begin{aligned} F_1(k,Y_0,Y_1)|_{(k,Y_0,Y_1)\in\Omega_B^1} &\le \max_{(k,Y_0,Y_1)\in\Omega_B^1} F_1(k,b_1(k),Y_1) \\ &= \left(-\frac{s+1}{k}+O\left(\frac{1}{k^2}\right)\right)\left(\frac{\varepsilon_1}{k^\gamma}+\frac{\varepsilon_4}{k^\beta}\right) \\ &< b_1(k+1)-b_1(k) = \frac{\varepsilon_1\gamma}{k^{\gamma+1}}\left(1+O\left(\frac{1}{k}\right)\right), \end{aligned} \tag{4.69}$$

$$\begin{aligned} F_1(k,Y_0,Y_1)|_{(k,Y_0,Y_1)\in\Omega_C^1} &\ge \min_{(k,Y_0,Y_1)\in\Omega_C^1} F_1(k,c_1(k),Y_1) = \\ &= \left(-\frac{s+1}{k}+O\left(\frac{1}{k^2}\right)\right)\left(-\frac{\varepsilon_2}{k^\gamma}-\frac{\varepsilon_3}{k^\beta}\right) \\ &> c_1(k+1)-c_1(k) = -\frac{\varepsilon_2\gamma}{k^{\gamma+1}}\left(1+O\left(\frac{1}{k}\right)\right), \end{aligned} \tag{4.70}$$

$$\begin{aligned} F_2(k,Y_0,Y_1)|_{(k,Y_0,Y_1)\in\Omega_B^2} &\le \max_{(k,Y_0,Y_1)\in\Omega_B^2} F_2(k,Y_0,b_2(k)) \\ &= \left(-\frac{s+2}{k}+O\left(\frac{1}{k^2}\right)\right)\left(\frac{ms}{s+2}\frac{\varepsilon_2}{k^\gamma}+\frac{\varepsilon_3}{k^\beta}+O\left(\frac{1}{k}\right)\right) \\ &< c_2(k+1)-c_2(k) = \frac{\varepsilon_3\beta}{k^{\beta+1}}\left(1+O\left(\frac{1}{k}\right)\right), \end{aligned}$$

and

$$\begin{aligned} F_2(k,Y_0,Y_1)|_{(k,Y_0,Y_1)\in\Omega_C^2} &\ge \min_{(k,Y_0,Y_1)\in\Omega_C^2} F_2(k,Y_0,c_2) = \\ &= \left(-\frac{s+2}{k}+O\left(\frac{1}{k^2}\right)\right)\left(-\frac{ms}{s+2}\frac{\varepsilon_1}{k^\gamma}-\frac{\varepsilon_4}{k^\beta}+O\left(\frac{1}{k}\right)\right) \\ &> c_2(k+1)-c_2(k) = -\frac{\varepsilon_4\beta}{k^{\beta+1}}\left(1+O\left(\frac{1}{k}\right)\right). \end{aligned}$$

Now, we will study each of the inequalities separately.

Inequality (4.69) is equivalent with

$$-\frac{s+1}{k^{\gamma+1}}\varepsilon_1 - \frac{s+1}{k^{\beta+1}}\varepsilon_4 + O\left(\frac{1}{k^{\gamma+2}}\right) + O\left(\frac{1}{k^{\beta+2}}\right) < \frac{\varepsilon_1\gamma}{k^{\gamma+1}} + O\left(\frac{1}{k^{\gamma+2}}\right)$$

or with

$$-\frac{s+1}{k^{\beta+1}}\varepsilon_4 + O\left(\frac{1}{k^{\beta+2}}\right) < \varepsilon_1\frac{\gamma+s+1}{k^{\gamma+1}} + O\left(\frac{1}{k^{\gamma+2}}\right).$$

The last inequality implies

i) $\beta > \gamma$

or

ii) $\beta = \gamma$ and

$$\varepsilon_4 < -\varepsilon_1\frac{\gamma+s+1}{s+1}.$$

Inequality (4.70) is equivalent with

$$\frac{s+1}{k^{\gamma+1}}\varepsilon_2 + \frac{s+1}{k^{\beta+1}}\varepsilon_3 + O\left(\frac{1}{k^{\gamma+2}}\right) + O\left(\frac{1}{k^{\beta+2}}\right) > -\frac{\varepsilon_2\gamma}{k^{\gamma+1}} + O\left(\frac{1}{k^{\gamma+2}}\right)$$

or with

$$\frac{s+1}{k^{\beta+1}}\varepsilon_3 + O\left(\frac{1}{k^{\beta+2}}\right) > -\varepsilon_2\frac{\gamma+s+1}{k^{\gamma+1}} + O\left(\frac{1}{k^{\gamma+2}}\right).$$

The last inequality implies

iii) $\beta > \gamma$

or

iv) $\beta = \gamma$ and

$$\varepsilon_3 < -\varepsilon_2\frac{\gamma+s+1}{s+1}.$$

The last two inequalities are the same as in Theorem 7 ((4.53) and (4.54)). Therefore, we must consider the following conditions:

v) $\beta < \gamma$ (contradicts i) and iii))

or

vi) $\beta = \gamma$ and

$$\varepsilon_2 < -\varepsilon_3\frac{\beta+s+2}{ms}$$

vii) $\beta = \gamma$ and

$$\varepsilon_1 < -\varepsilon_4\frac{\beta+s+2}{ms}$$

Hence, we have the system of conditions (4.60)–(4.63).

Remark 5. Note the following. For the solvability of the system of inequalities (4.60)–(4.63), the inequality

$$\gamma + s + 1 > 0 \tag{4.71}$$

is necessary as, in the opposite case, inequalities (4.60), (4.61) cannot be satisfied due to the positivity of ε_i, $i = 1, 2, 3, 4$ and the property $s + 1 < 0$.

4.3.2 Proof of Theorem 10

First, lest us mention that, due to a symmetry between the sub-system of inequalities (4.60), (4.63) and the sub-system of inequalities (4.61), (4.62) as well as the first one being independent of the second and vice versa, it is sufficient to analyse the solvability of only one of these two sub-systems. Below, sub-system of inequalities (4.60), (4.63) is considered. We get

$$\varepsilon_4 < -\varepsilon_1 \frac{\gamma+s+1}{s+1} < \varepsilon_4 \frac{\gamma+s+2}{ms} \frac{\gamma+s+1}{s+1}$$

or

$$(\gamma+s+1)(\gamma+s+2) - ms(s+1) > 0. \tag{4.72}$$

We rewrite (4.72) as a quadratic inequality with respect to γ,

$$\Gamma(\gamma) := \gamma^2 + \gamma(2s+3) + (s+1)(-ms+s+2) > 0 \tag{4.73}$$

with discriminant D of quadratic equation $\Gamma(\gamma) = 0$,

$$D = (2s+3)^2 - 4(s+1)(-ms+s+2) = 4ms(s+1) + 1 > 0.$$

Two real roots γ_1, γ_2, $\gamma_1 < \gamma_2$ of equation $\Gamma(\gamma) = 0$ are

$$\gamma_{1,2} = \frac{1}{2}\left(-(2s+3) \mp \sqrt{4ms(s+1)+1}\right). \tag{4.74}$$

Inequalities (4.60)–(4.63) will be solvable (i.e. suitable ε_i, $i = 1, 2, 3, 4$ will exist) if $\gamma_1 > 0$ or $\gamma_2 < 1$. Below, both cases are discussed.

The case of $\gamma_1 > 0$

If $\gamma_1 > 0$ then, as it follows from (4.74),

$$\sqrt{4ms(s+1)+1} < -(2s+3) \tag{4.75}$$

and, consequently, inequality

$$2s+3 < 0 \tag{4.76}$$

must be fulfilled. Replacing in (4.75) the value s by (2.1),

we get inequality

$$4m\frac{\alpha+2}{m-1}\frac{\alpha+m+1}{m-1} + 1 < \frac{(2(\alpha+2)+3(m-1))^2}{(m-1)^2}$$

which can be reduced to

$$\alpha(\alpha+m+1)(m-1) < 0$$

or to

$$\alpha(s+1) < 0. \tag{4.77}$$

Inequality (4.77) can be valid only if $\alpha > 0$ and, from the (2.1), we have $m \in (0, 1)$. Therefore, considering all the assumptions we get

$$\alpha > 0, \ \ 0 < m < 1. \tag{4.78}$$

It is easy to verify that (4.78) implies the validity of inequality (4.76). However, inequality (4.71) is not satisfied for a $\gamma \in (0, \gamma_1)$ because we have

$$\gamma_1 + s + 1 = \frac{1}{2}\left(-(2s+3) - \sqrt{4ms(s+1)+1}\right) + s + 1 = \\ -\frac{1}{2}\left(1 + \sqrt{4ms(s+1)+1}\right) < 0.$$

The case of $\boldsymbol{\gamma_2 < 1}$ Let $\gamma_2 < 1$. Then, by formula (4.74), we will analyse the inequality

$$\sqrt{4ms(s+1)+1} < 2s + 5 \tag{4.79}$$

to see that a necessary condition for its solvability is

$$2s + 5 > 0. \tag{4.80}$$

Inequality (4.79) is equivalent with

$$4ms(s+1) + 1 < (2s+5)^2. \tag{4.81}$$

Replacing s in (4.81) by formula (2.1), we have

$$4m\frac{\alpha+2}{m-1} \cdot \frac{\alpha+m+1}{m-1} + 1 < \left(2\frac{\alpha+2}{m-1} + 5\right)^2. \tag{4.82}$$

Calculating inequality (4.82), we derive its equivalent form

$$(m-1)(\alpha^2 + \alpha m - \alpha - 4m) < 0. \tag{4.83}$$

Considering all assumptions, we see that the theorem holds if inequalities $s + 1 < 0$, $ms < 0$, (4.71), (4.80) and (4.83) hold. These are the conditions of our Theorem (9).

4.3.3 Some remarks to this section

Remark 6. The systems of inequalities (4.64)–(4.68) and (4.60)–(4.63) are solvable. We show that the system of inequalities (4.64)–(4.68) is satisfied, e.g., for the choice $m = 1/2$, $\alpha = -27/20$. In such a case, inequality (4.64) holds since

$$s = \frac{\alpha+2}{m-1} = -\frac{13}{10}, \quad s + 1 = \frac{\alpha+m+1}{m-1} = -\frac{3}{10} < 0,$$

inequality (4.65) holds since

$$ms = m\frac{\alpha+2}{m-1} = -\frac{13}{20} < 0,$$

inequality (4.66) holds since

$$2s+5=\frac{2\alpha+5m-1}{m-1}=\frac{12}{5}>0,$$

inequality (4.67) holds since

$$(m-1)(\alpha^2+\alpha m-\alpha-4m)=-\frac{199}{800}<0.$$

Moreover

$$\gamma_2=\frac{1}{2}\left(-\frac{2\alpha+3m+1}{m-1}+\sqrt{4m\frac{\alpha+2}{m-1}\cdot\frac{\alpha+m+1}{m-1}+1}\right)=-\frac{1}{5}+\frac{1}{2}\sqrt{1.78}\doteq 0.467$$

and inequality (4.68) holds since

$$\gamma+s+1=\gamma-\frac{3}{10}>0,$$

where γ is a fixed number such that $\gamma\in(\gamma_2,1)$. Let, e.g., $\gamma=0.8$. Then the system of inequalities (4.60) – (4.63) equals

$$\varepsilon_4<-\,\varepsilon_1\frac{\gamma+s+1}{s+1}=-\varepsilon_1\frac{0.8-0.3}{-0.3}=\frac{5}{3}\,\varepsilon_1, \tag{4.84}$$

$$\varepsilon_1<-\,\varepsilon_4\frac{\gamma+s+2}{ms}=-\varepsilon_4\frac{0.8-0.3+1}{-13/20}=\frac{30}{13}\,\varepsilon_4. \tag{4.85}$$

The choice, e.g., $\varepsilon_1=\varepsilon_4=1$, while solving the sub-system (4.84), (4.85), solves the sub-system (4.60), (4.63) as well.

Lemma 2. *Let inequalities* (4.64)–(4.67) *hold. Then, the root* γ_2*, defined by formula* (4.74)*, is positive.*

Proof. First, assume $\gamma_2=0$. From (4.73), we have

$$\Gamma(0)=(s+1)(-ms+s+2)=0. \tag{4.86}$$

Because $s+1<0$, (4.86) implies

$$-ms+s+2=0$$

and, by (2.1),

$$-ms+s+2=-m\frac{\alpha+2}{m-1}+\frac{\alpha+2}{m-1}+2=-\alpha=0.$$

But $\alpha\neq 0$ as, in the opposite case, inequalities (4.65) and (4.67), i.e.,

$$\frac{2m}{m-1}<0\,,\quad (m-1)(-4m)<0$$

give a contradiction. Therefore, $\gamma_2 \neq 0$. Next, let $\gamma_2 < 0$. Then, (4.74) implies $2s+3>0$ and the inequality

$$\sqrt{4ms(s+1)+1} < 2s+3$$

yields

$$ms(s+1) < (s+1)(s+2).$$

As $s+1<0$, the last inequality is equivalent to

$$s(m-1) > 2.$$

Replacing s by the formula (2.1), we get $\alpha > 0$. Now, let us show that the positivity of α leads to a contradiction.

If $m>1$, then the condition $s+1<0$ implies $\alpha < -2$ and we get a contradiction. Let $m<1$. The conditions $s+1<0$ and $ms<0$ imply $0<m<1$. The assumption $2s+3>0$ can be transformed into

$$2\alpha + 3m + 1 < 0,$$

which is not possible as $\alpha > 0$ and $m > 0$. □

Remark 7. The domain defined by inequalities (4.64)–(4.68) in Theorem 10 is visualized in (m,α)-plane by Figure 4.5. This domain splits into two open subdomains, one of them being blue color and other green.

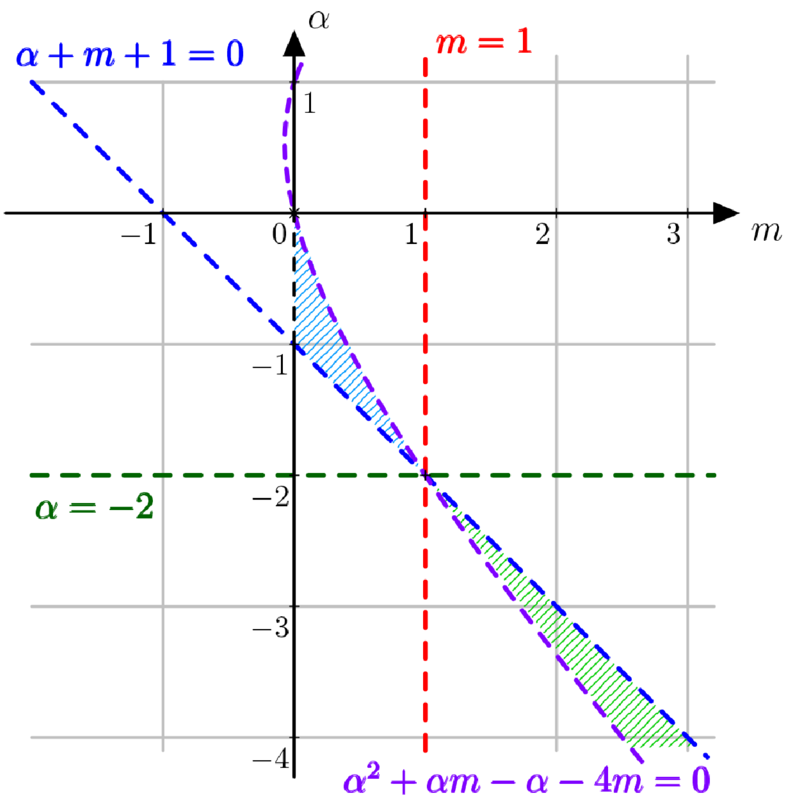

Fig. 4.5: Summary of admissible values (Theorem 10)

4.4 The case of $ms > 0$ and $s + 1 < 0$

Theorem 11. *Let* $\alpha \neq 0$ *and*

$$s < -1, \quad m > 0, \quad s \neq -2.$$

Assume that there exists a constant γ*, satisfying* $0 < \gamma < 1$ *and positive numbers* ε_i*,* $i = 1, 2, 3, 4$*, such that*

$$\varepsilon_4 < -\varepsilon_1 \frac{\gamma + s + 1}{s + 1}, \tag{4.87}$$

$$\varepsilon_3 < -\varepsilon_2 \frac{\gamma + s + 1}{s + 1}, \tag{4.88}$$

$$\varepsilon_1 < \varepsilon_3 \frac{\gamma + s + 2}{ms}, \tag{4.89}$$

$$\varepsilon_2 < \varepsilon_4 \frac{\gamma + s + 2}{ms}. \tag{4.90}$$

Then, for a sufficiently large fixed $k_0 > 0$*, there exists a solution* $u \colon \mathbb{N}(k_0) \to \mathbb{R}$ *of equation* (1.3) *such that, for every* $k \in \mathbb{N}(k_0)$*, asymptotic representation* (4.1) *holds* *or, more precisely, such a solution satisfies*

$$-\frac{\varepsilon_1}{k^\gamma} < \left[u(k) - \frac{a}{k^s} - \frac{b}{k^{s+1}}\right]\left[\frac{b}{k^{s+1}}\right]^{-1} < \frac{\varepsilon_2}{k^\gamma},$$

$$-\frac{\varepsilon_3}{k^\gamma} < \left[\Delta u(k) - \Delta\left(\frac{a}{k^s}\right) - \Delta\left(\frac{b}{k^{s+1}}\right)\right]\left[\Delta\left(\frac{b}{k^{s+1}}\right)\right]^{-1} < \frac{\varepsilon_4}{k^\gamma},$$

$$-\frac{\varepsilon_1}{k^\gamma} + O\left(\frac{1}{k}\right) < \left[\Delta^2 u(k) - \Delta^2\left(\frac{a}{k^s}\right) - \Delta^2\left(\frac{b}{k^{s+1}}\right)\right]\left[\Delta^2\left(\frac{b}{k^{s+1}}\right)\frac{ms}{s+2}\right]^{-1} < \frac{\varepsilon_2}{k^\gamma} + O\left(\frac{1}{k}\right).$$

Theorem 12. *Let numbers α and m satisfy*

$$\alpha \neq \{0, -2m\}$$

$$\frac{\alpha+m+1}{m-1} < 0, \tag{4.91}$$

$$m\frac{\alpha+2}{m-1} > 0, \tag{4.92}$$

$$\frac{2\alpha+5m-1}{m-1} > 0, \tag{4.93}$$

$$\alpha^2 + 8m^2 + 8m\alpha - \alpha + m\alpha^2 + m^2\alpha > 0 \tag{4.94}$$

and

$$\gamma + \frac{\alpha+m+1}{m-1} > 0 \tag{4.95}$$

where γ is a fixed number such that $\gamma \in (\gamma^*, 1)$ and

$$\gamma^* = \frac{1}{2}\left(-\frac{2\alpha+3m+1}{m-1} + \sqrt{1 - 4m\frac{\alpha+2}{m-1}\cdot\frac{\alpha+m+1}{m-1}}\right).$$

Then, the conclusion of Theorem 11 holds.

4.4.1 Proof of the Theorem 11

Let us again verify inequalities (4.8)–(4.11). Using formula (4.7) and assumptions of this section, which could be transformed into the following inequalities

$$F_1(k, Y_0, Y_1)|_{(k,Y_0,Y_1)\in\Omega^1_B} \le \max_{(k,Y_0,Y_1\in\Omega^1_B} F_1(k, b_1(k), Y_1)$$
$$= \left(-\frac{s+1}{k} + O\left(\frac{1}{k^2}\right)\right)\left(\frac{\varepsilon_1}{k^\gamma} + \frac{\varepsilon_4}{k^\beta}\right)$$

$$< b_1(k+1) - b_1(k) = \frac{\varepsilon_1 \gamma}{k^{\gamma+1}} \left(1 + O\left(\frac{1}{k}\right)\right), \quad (4.96)$$

$$\begin{aligned} F_1(k, Y_0, Y_1)|_{(k,Y_0,Y_1)\in\Omega_C^1} \geq \min_{(k,Y_0,Y_1)\in\Omega_C^1} F_1(k, c_1(k), Y_1) = \\ = \left(-\frac{s+1}{k} + O\left(\frac{1}{k^2}\right)\right)\left(-\frac{\varepsilon_2}{k^\gamma} - \frac{\varepsilon_3}{k^\beta}\right) \\ > c_1(k+1) - c_1(k) = -\frac{\varepsilon_2 \gamma}{k^{\gamma+1}}\left(1 + O\left(\frac{1}{k}\right)\right), \quad (4.97) \end{aligned}$$

$$\begin{aligned} F_2(k, Y_0, Y_1)|_{(k,Y_0,Y_1)\in\Omega_B^2} \leq \max_{(k,Y_0,Y_1)\in\Omega_B^2} F_2(k, Y_0, b_2(k)) \\ = \left(-\frac{s+2}{k} + O\left(\frac{1}{k^2}\right)\right)\left(\frac{ms}{s+2}\frac{\varepsilon_1}{k^\gamma} + \frac{\varepsilon_3}{k^\beta} + O\left(\frac{1}{k}\right)\right) \\ < b_2(k+1) - b_2(k) = \frac{\varepsilon_3 \beta}{k^{\beta+1}}\left(1 + O\left(\frac{1}{k}\right)\right) \quad (4.98) \end{aligned}$$

and

$$\begin{aligned} F_2(k, Y_0, Y_1)|_{(k,Y_0,Y_1)\in\Omega_C^2} \geq \min_{(k,Y_0,Y_1)\in\Omega_C^2} F_2(k, Y_0, c_2) \\ = \left(-\frac{s+2}{k} + O\left(\frac{1}{k^2}\right)\right)\left(\frac{ms}{s+2}\frac{\varepsilon_2}{k^\gamma} - \frac{\varepsilon_4}{k^\beta} + O\left(\frac{1}{k}\right)\right) \\ > c_2(k+1) - c_2(k) = -\frac{\varepsilon_4 \beta}{k^{\beta+1}}\left(1 + O\left(\frac{1}{k}\right)\right). \quad (4.99) \end{aligned}$$

Since inequalities 4.96 and (4.97) duplicate inequalities (4.69) and (4.70), we get the following conditions
i) $\beta > \gamma$
or
ii) $\beta = \gamma$ and

$$\varepsilon_4 < -\varepsilon_1 \frac{\gamma + s + 1}{s+1}.$$

iii) $\beta = \gamma$ and

$$\varepsilon_3 < -\varepsilon_2 \frac{\gamma + s + 1}{s+1}.$$

Since inequalities 4.98 and (4.99) duplicate inequalities (4.31) and (4.34), we get the following conditions
iv) $\gamma > \beta$ - contradiction to (i)
or
v) $\beta = \gamma$ and

$$\varepsilon_1 < \varepsilon_3 \frac{\gamma + s + 2}{ms}.$$

vi) $\beta = \gamma$ and

$$\varepsilon_2 < \varepsilon_4 \frac{\gamma + s + 2}{ms}.$$

Hence, we get the hypotheses of Theorem (11).

Note that, in this case, we can make the same remark as in Theorem (9).

Remark 8. For the solvability of the system of inequalities (4.87)–(4.90), the inequality

$$\gamma + s + 1 > 0 \tag{4.100}$$

is necessary as, in the opposite case, inequalities (4.87), (4.88) cannot be satisfied due to the positivity of ε_i, $i = 1, 2, 3, 4$ and the property $s + 1 < 0$.

4.4.2 Proof of Theorem 12

To solve the system (4.87)–(4.90) we can write out the chain of inequalities

$$\varepsilon_4 < -\varepsilon_1 \frac{\gamma + s + 1}{s + 1} < -\varepsilon_3 \frac{\gamma + s + 1}{s + 1} \frac{\gamma + s + 2}{ms} < \varepsilon_2 \frac{(\gamma + s + 1)^2(\gamma + s + 2)}{(s + 1)^2 ms} < \varepsilon_4 \frac{(\gamma + s + 1)^2(\gamma + s + 2)^2}{(s + 1)^2(ms)^2}. \tag{4.101}$$

As $\varepsilon_4 > 0$, (4.101) implies

$$1 < \frac{(\gamma + s + 1)^2(\gamma + s + 2)^2}{(s + 1)^2(ms)^2}$$

and

$$[(\gamma + s + 1)(\gamma + s + 2) - (s + 1)ms)]\,[(\gamma + s + 1)(\gamma + s + 2) + (s + 1)ms] > 0. \tag{4.102}$$

Put

$$G_1(\gamma) := (\gamma + s + 1)(\gamma + s + 2) - (s + 1)ms,$$
$$G_2(\gamma) := (\gamma + s + 1)(\gamma + s + 2) + (s + 1)ms.$$

Inequality (4.102) will hold if either

$$G_1(\gamma) > 0 \text{ and } G_2(\gamma) > 0 \tag{4.103}$$

or

$$G_1(\gamma) < 0 \text{ and } G_2(\gamma) < 0.$$

Let us consider each of the above possibilities separately.

The case $G_1(\gamma) > 0$, $G_2(\gamma) > 0$. Consider system of inequalities (4.103). Because $s + 1 < 0$ and $ms > 0$, inequality $G_2(\gamma) > 0$ implies $G_1(\gamma) > 0$. Consequently, it is sufficient to consider the inequality $G_2(\gamma) > 0$ only. Rewrite the last inequality as a quadratic one with respect to γ,

$$G_2(\gamma) := \gamma^2 + \gamma(2s + 3) + (s + 1)(s + ms + 2) > 0$$

with discriminant D of a quadratic equation $G_2(\gamma) = 0$

$$D = (2s+3)^2 - 4(s+1)(s+ms+2) = 1 - 4ms(s+1) > 0.$$

The two real roots γ_1, γ_2, $\gamma_1 < \gamma_2$ of the equation $G_2(\gamma) = 0$ are

$$\gamma_{1,2} := \frac{-(2s+3) \mp \sqrt{1-4ms(s+1)}}{2}. \tag{4.104}$$

System (4.87)–(4.90) will be solvable (i.e. suitable ε_i, $i = 1,2,3,4$ will exist) if $\gamma_1 > 0$ or if $\gamma_2 < 1$.

The case $\boldsymbol{\gamma_1 > 0}$. In this case, the necessary condition (4.100) does not hold because, for $\gamma \leq \gamma_1$, we have

$$\begin{aligned}\gamma + s + 1 \leq \gamma_1 + s + 1 = \frac{-(2s+3) - \sqrt{1-4ms(s+1)}}{2} + s + 1 \\ = \frac{-1 - \sqrt{1-4ms(s+1)}}{2} < 0.\end{aligned}$$

The case $\boldsymbol{\gamma_2 < 1}$. This inequality is equivalent with inequality

$$\sqrt{1-4ms(s+1)} < 2s+5. \tag{4.105}$$

The necessary condition for its solvability is the inequality

$$2s + 5 > 0.$$

If it is fulfilled, then inequality (4.105) is equivalent to

$$1 - 4ms(s+1) < (2s+5)^2$$

and, replacing s by formula (2.1), we get the following condition

$$\alpha^2 + 8m^2 + 8m\alpha - \alpha + m\alpha^2 + m^2\alpha > 0. \tag{4.106}$$

Considering all the assumptions, we state that Lemma 1 is applicable if inequalities $s+1 < 0$, $ms > 0$, (4.105), (4.106) and (4.100) hold, that is, if

$$s + 1 = \frac{\alpha + m + 1}{m - 1} < 0, \tag{4.107}$$

$$ms = m\frac{\alpha + 2}{m - 1} > 0, \tag{4.108}$$

$$2s + 5 = \frac{2\alpha + 5m - 1}{m - 1} > 0, \tag{4.109}$$

$$\alpha^2 + 8m^2 + 8m\alpha - \alpha + m\alpha^2 + m^2\alpha > 0 \tag{4.110}$$

and

$$\gamma + s + 1 = \gamma + \frac{\alpha + m + 1}{m - 1} > 0, \tag{4.111}$$

where γ is a fixed number such that $\gamma \in (\gamma_2, 1)$ and

$$\gamma_2 = \frac{1}{2}\left(-(2s+3) + \sqrt{1 - 4ms(s+1)}\right) = \frac{1}{2}\left(-\frac{2\alpha + 3m + 1}{m - 1} + \sqrt{1 - 4m\frac{\alpha + 2}{m - 1}\frac{\alpha + m + 1}{m - 1}}\right).$$

4.4.3 Some remarks to this section

Remark 9. The system of inequalities (4.107)–(4.111) is solvable and so is the system of inequalities (4.87)–(4.90). We show that system of inequalities (4.107)–(4.111) is satisfied, e.g., for the choice $m = -2$, $\alpha = 3/2$. In such a case, inequality (4.107) will hold since

$$s = \frac{\alpha + 2}{m - 1} = -\frac{7}{6}, \quad s + 1 = \frac{\alpha + m + 1}{m - 1} = -\frac{1}{6} < 0,$$

inequality (4.108) will hold since

$$ms = m\frac{\alpha + 2}{m - 1} = \frac{7}{3} > 0,$$

inequality (4.109) will hold since

$$2s + 5 = \frac{2\alpha + 5m - 1}{m - 1} = \frac{8}{3} > 0,$$

inequality (4.110) will hold since

$$\alpha^2 + 8m^2 + 8m\alpha - \alpha + m\alpha^2 + m^2\alpha = \frac{41}{4} > 0.$$

Moreover,

$$\gamma_2 = \frac{1}{2}\left(-\frac{2\alpha + 3m + 1}{m - 1} + \sqrt{1 - 4m\frac{\alpha + 2}{m - 1}\frac{\alpha + m + 1}{m - 1}}\right) \doteq 0.46597$$

and inequality (4.111) holds since

$$\gamma + s + 1 = \gamma - \frac{1}{12} > 0,$$

where γ is a fixed number such that $\gamma \in (\gamma_2, 1)$. Let, e.g., $\gamma = 5/6$. Then, system (4.87)–(4.90) has the form

$$\varepsilon_4 < -\varepsilon_1\frac{\gamma + s + 1}{s + 1} = 4\varepsilon_1,$$

$$\varepsilon_3 < -\varepsilon_2 \frac{\gamma + s + 1}{s + 1} = 4\varepsilon_2,$$
$$\varepsilon_1 < \varepsilon_3 \frac{\gamma + s + 2}{ms} = \frac{5}{7}\varepsilon_3,$$
$$\varepsilon_2 < \varepsilon_4 \frac{\gamma + s + 2}{ms} = \frac{5}{7}\varepsilon_4.$$

The choice, e.g., $\varepsilon_1 = \varepsilon_2 = 1$, $\varepsilon_3 = \varepsilon_4 = 2$ solves this system.

Lemma 3. *Let inequalities* (4.107)–(4.111) *hold. Then, the root* γ_2 *defined by formula* (4.104)*is positive.*

Proof. Assume that $\gamma_2 \leq 0$. Then

$$\sqrt{1 - 4ms(s + 1)} \leq 2s + 3. \tag{4.112}$$

Inequality (4.112) can hold only if

$$2s + 3 \geq 0 \tag{4.113}$$

holds. Moreover, (4.112) implies

$$-\, ms \geq s + 2. \tag{4.114}$$

From (4.113) and (4.114), we can derive a chain of inequalities

$$-ms \geq s + 2 \geq -\frac{3}{2} + 2 = \frac{1}{2} > 0.$$

This is in contradiction with the assumption $ms > 0$. Therefore, $\gamma_2 > 0$.

□

Remark 10. The domain defined by inequalities (4.91)–(4.95) in Theorem 12 is visualized in (m, α)-plane by Figure 4.6. This domain splits into two open subdomains, one of them shown in red while the other in blue.

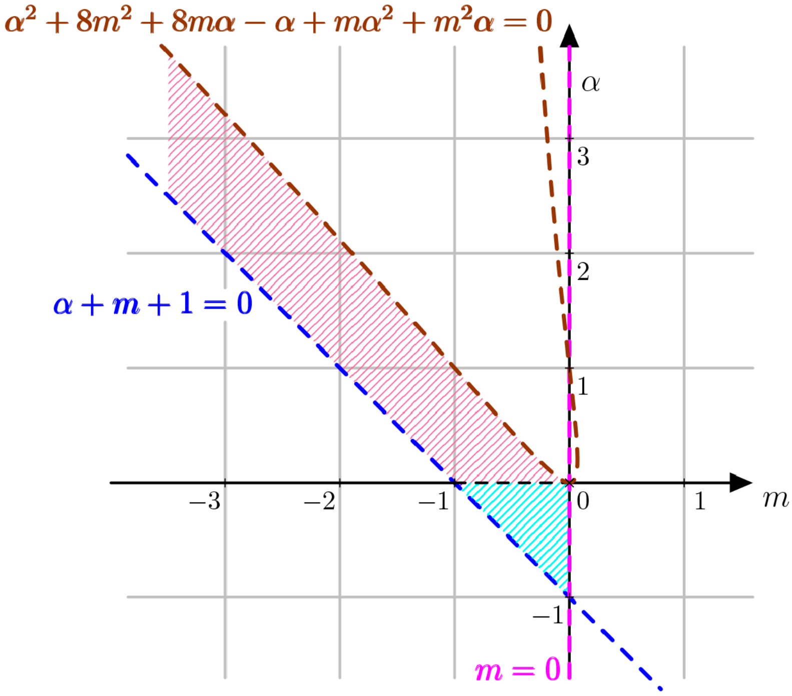

Fig. 4.6: Summary of admissible values (Theorem 12)

4.5 All the above cases unified and compared with the case of constant upper an lower functions

In this section, we will compare the above results. The results of Theorems 5 – 12 can all represented by the below Figures 4.7 and 4.8.

Now, in addition, we need to compare these results with those of the Theorem 4 of Chapter 3. As the proof of this theorem is structured similarly, it should be mentioned that the crucial role in applying Theorem 2 is played by a proper choice of upper and lower functions $b_i(k)$ and $c_i(k)$, where $i = 1, 2$. Both sets of the upper and lower functions chosen ((3.6) and (3.7)) and (4.2) lead to the identical asymptotic relation

$$-\frac{\varepsilon_1}{k^\gamma} < \left[u(k) - \frac{a}{k^s} - \frac{b}{k^{s+1}}\right]\left[\frac{b}{k^{s+1}}\right]^{-1} < \frac{\varepsilon_2}{k^\gamma},$$

$$-\frac{\varepsilon_3}{k^\gamma} < \left[\Delta u(k) - \Delta\left(\frac{a}{k^s}\right) - \Delta\left(\frac{b}{k^{s+1}}\right)\right]\left[\Delta\left(\frac{b}{k^{s+1}}\right)\right]^{-1} < \frac{\varepsilon_4}{k^\gamma},$$

$$-\frac{\varepsilon_1}{k^\gamma} + O\left(\frac{1}{k}\right) < \left[\Delta^2 u(k) - \Delta^2\left(\frac{a}{k^s}\right) - \Delta^2\left(\frac{b}{k^{s+1}}\right)\right]\left[\Delta^2\left(\frac{b}{k^{s+1}}\right)\frac{ms}{s+2}\right]^{-1}$$
$$< \frac{\varepsilon_2}{k^\gamma} + O\left(\frac{1}{k}\right),$$

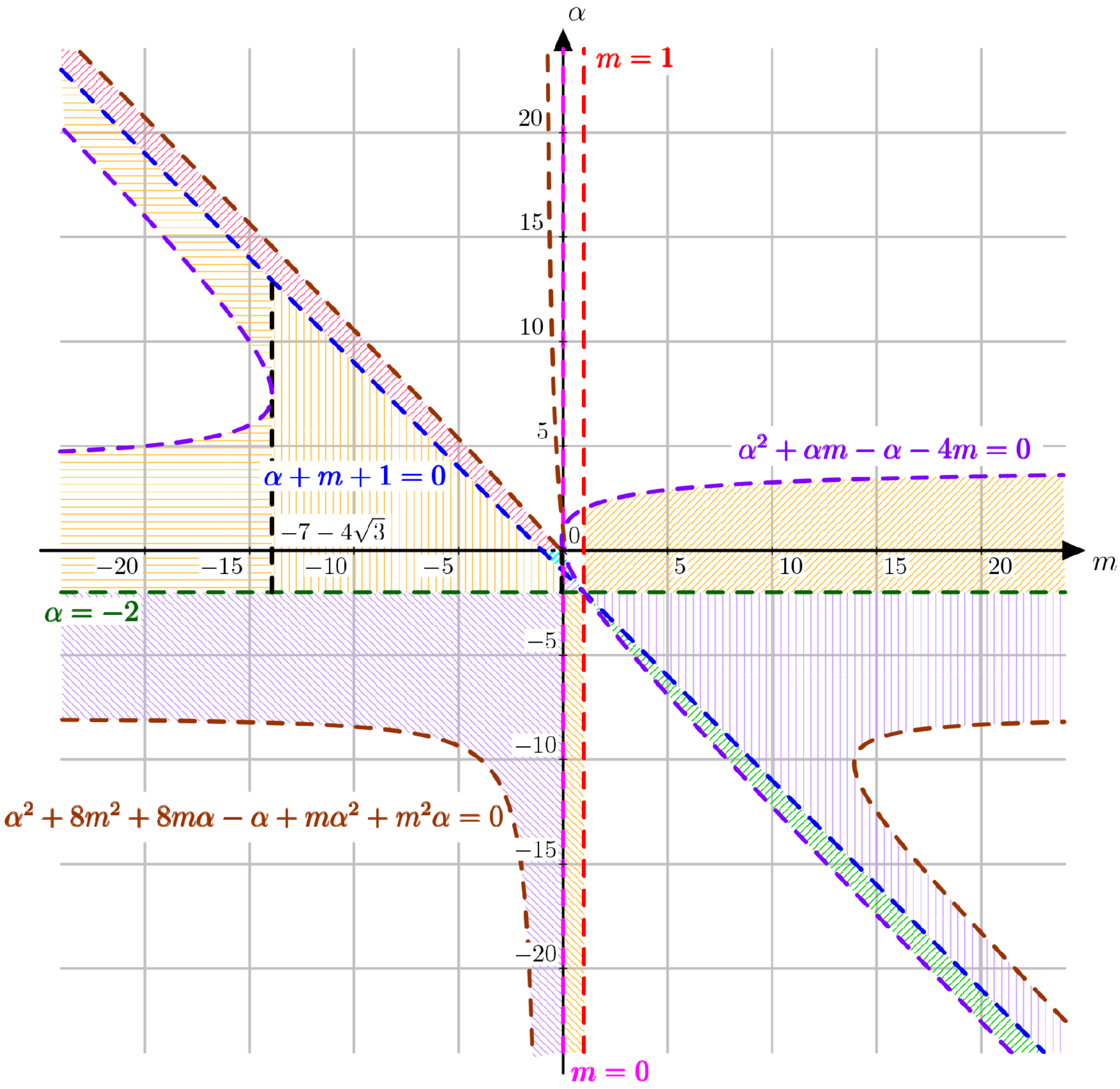

Fig. 4.7: Summary of admissible values (Theorems 5–12)

or more precisely

$$\left|u_\pm(k) - a_\pm k^{-s} - b_\pm k^{-s-1}\right| < \frac{\max\{\varepsilon_1, \varepsilon_2\}|b_\pm|}{k^{s+\gamma+1}},$$

$$\left|\Delta u_\pm(k) - a_\pm \Delta k^{-s} - b_\pm \Delta k^{-s-1}\right| < \left|\Delta\left(\frac{b_\pm}{k^{s+1}}\right)\right| \frac{\max\{\varepsilon_1, \varepsilon_2\}}{k^\gamma},$$

$$\left|\Delta^2 u_\pm(k) - a_\pm \Delta^2 k^{-s} - b_\pm \Delta^2 k^{-s-1}\right| \\ < \left|\Delta^2\left(\frac{b_\pm}{k^{s+1}}\right)\right| \left(\max\{\varepsilon_1, \varepsilon_2\}\frac{ms}{k^\gamma|s+2|} + \left|O\left(\frac{1}{k}\right)\right|\right).$$

However, the change of the form of upper and lower functions from constants to power functions extends the set of appropriate conditions reopening the question

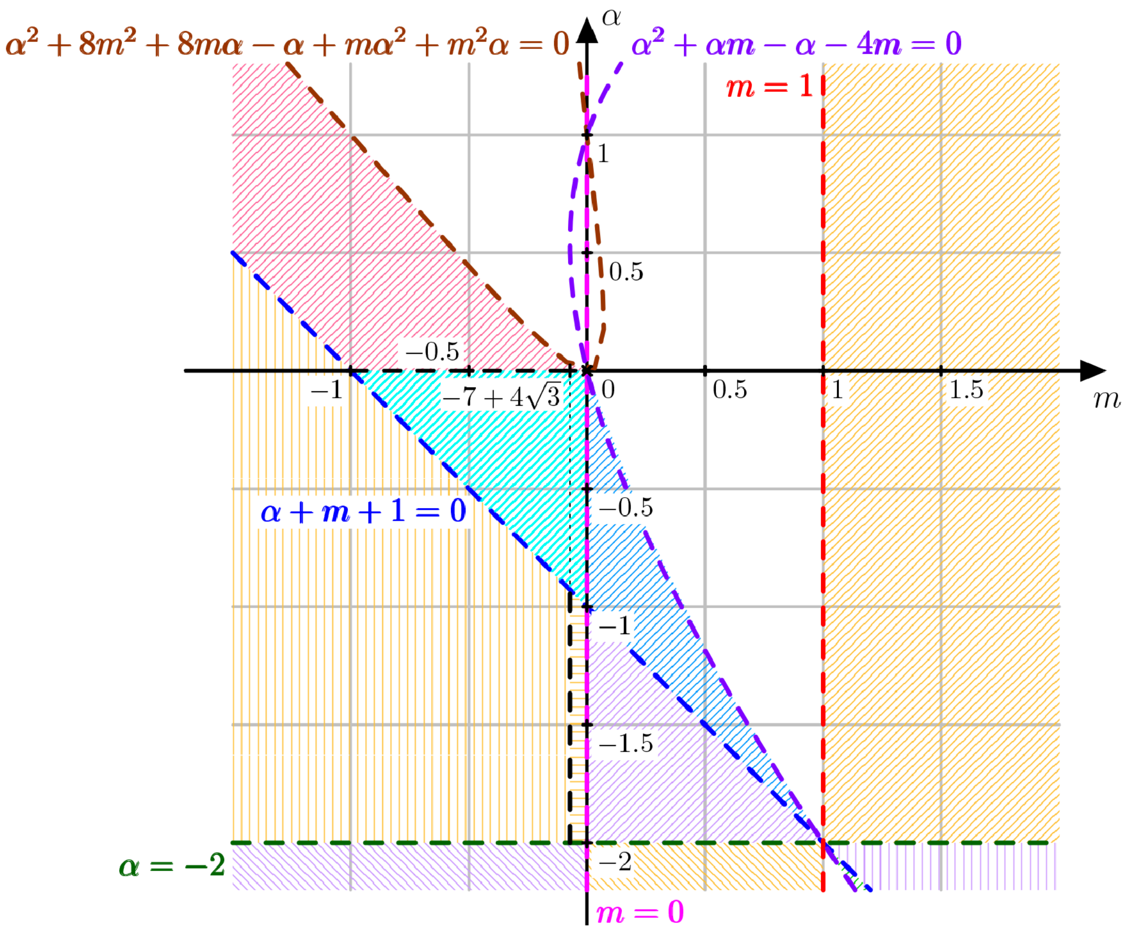

Fig. 4.8: Summary of admissible values - zoom (Theorems 5–12)

of the asymptotic behaviour of the Emden-Fowler equation solutions in the case of $s + 1 < 0$.

To illustrate that the set of appropriate conditions has expanded even in the case of $s + 1 > 0$, all sets are put in a single Figure 4.9. Here the yellow domain is the summary of the results of this chapter (non-constant case) while the green domain summarises the results of Chapter 3 (constant case).

Let us show that the union of all green domains is a subset of the union of all yellow domains. It is sufficient to prove that there exists no solution of the system of equations:

$$\begin{cases} m\alpha + \alpha + 4m = 0, \\ \alpha^2 + 8m^2 + 8m\alpha - \alpha + m\alpha^2 + m^2\alpha = 0, \\ m \neq \{0, 1\}, \\ \alpha \neq -2. \end{cases} \tag{4.115}$$

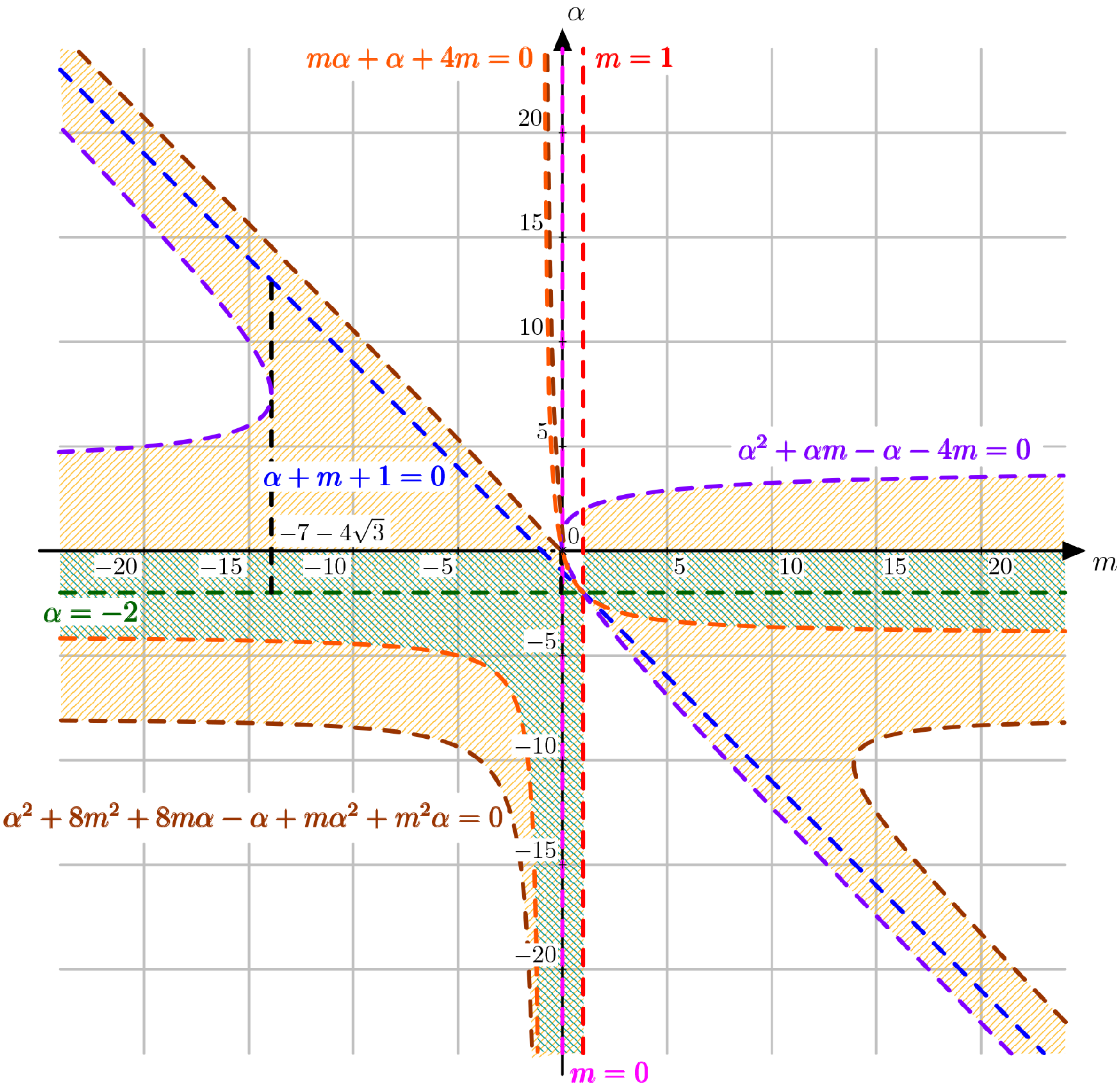

Fig. 4.9: Summary of admissible values (Theorems 5–12) and Chapter 3)

From the first equation of the system (4.115) we get:

$$\alpha = -\frac{4m}{m+1}.$$

Hence, substitution provides the following

$$\frac{16m^2}{(m+1)^2} + 8m^2 - 8m\frac{4m}{m+1} + \frac{4m}{m+1} + m\frac{16m^2}{(m+1)^2} - m^2\frac{4m}{m+1} = 0$$

and, finally, we get

$$m(m-1)^2(m+1) = 0.$$

So, there are no solutions of the system (4.115) and, hence, the border curves do not intersect. It is easy to see that points $(10, -5)$ and $(-10, -5)$ belong to the yellow domain, but not to the blue one.

For the point $(10, -5)$:

$$\alpha + \frac{4m}{1+m} = -\frac{15}{11} < 0$$

contradicts to (3.42) and

$$\alpha^2 + 8m^2 + 8m\alpha - \alpha + m\alpha^2 + m^2\alpha = 180 > 0$$

satisfies condition (4.50).

For the point $(-10, -5)$:

$$\alpha + \frac{4m}{1+m} = -\frac{5}{9} < 0$$

contradicts to (3.42) and

$$\alpha^2 + 8m^2 + 8m\alpha - \alpha + m\alpha^2 + m^2\alpha = 480 > 0$$

satisfies condition (4.50).

4.6 Examples

Example 8. In the following example, values (m, α) belong to the domain shown in Figure 4.2 (blue domain). Consider equation (1.3) where $\alpha = 1$, $m = -4$, that is, the equation

$$\Delta^2 u(k) \pm k u^{-4}(k) = 0. \tag{4.116}$$

In this example, α and m satisfy (4.23). If we put

$$\varepsilon_1 = \varepsilon_2 = \frac{1}{2}, \quad \varepsilon_3 = \varepsilon_4 = 1, \quad \gamma = \frac{3}{4},$$

then, by formula (2.1),

$$s = \frac{\alpha + 2}{m - 1} = -\frac{3}{5},$$

by formula (2.2),

$$a = [\mp s(s+1)]^{1/(m-1)} = \pm \sqrt[5]{\frac{25}{6}}$$

and, finally, by formula (2.3),

$$b = \frac{as(s+2)}{s+2-ms} = \pm \frac{21}{25} \sqrt[5]{\frac{25}{6}}.$$

Theorem 6 is applicable and equation (4.116) has a solution $u = u(k)$, $k \in \mathbb{N}(k_0)$ satisfying inequalitites (4.20)–(4.22), that is,

$$-\frac{1}{2k^{3/4}} < \left[u(k) \mp \sqrt[5]{\frac{25}{6}} k^{3/5} \mp \frac{21}{25} \sqrt[5]{\frac{25}{6}} \frac{1}{k^{2/5}} \right] \left[\pm \frac{25}{21} \sqrt[5]{\frac{6}{25}} k^{2/5} \right]^{-1} < \frac{1}{2k^{3/4}},$$

$$-\frac{1}{k^{3/4}} < \left[\Delta u(k) - \Delta \left(\pm \sqrt[5]{\frac{25}{6}} k^{3/5} \right) - \Delta \left(\frac{\pm \frac{21}{25} \sqrt[5]{\frac{25}{6}}}{k^{2/5}} \right) \right] \cdot \\ \cdot \left[\Delta \left(\frac{\pm \frac{21}{25} \sqrt[5]{\frac{25}{6}}}{k^{2/5}} \right) \right]^{-1} < \frac{1}{k^{3/4}},$$

$$-\frac{1}{2k^{3/4}} + O\left(\frac{1}{k}\right) < \left[\Delta^2 u(k) - \Delta^2 \left(\pm \sqrt[5]{\frac{25}{6}} k^{3/5} \right) - \Delta^2 \left(\frac{\pm \frac{21}{25} \sqrt[5]{\frac{25}{6}}}{k^{2/5}} \right) \right] \cdot \\ \cdot \left[\Delta^2 \left(\frac{\pm \frac{21}{25} \sqrt[5]{\frac{25}{6}}}{k^{2/5}} \right) \right]^{-1} < \frac{1}{2k^{3/4}} + O\left(\frac{1}{k}\right).$$

These formulas can be simplified to

$$u(k) = \pm\sqrt[5]{\frac{25}{6}} \cdot k^{3/5} \pm \frac{21}{25}\sqrt[5]{\frac{25}{6}}\frac{1}{k^{2/5}} + O\left(\frac{1}{k^{23/20}}\right),$$

$$\Delta u(k) = \pm\Delta\left(\sqrt[5]{\frac{25}{6}} \cdot k^{3/5}\right) \pm \Delta\left(\frac{21}{25}\sqrt[5]{\frac{25}{6}}\frac{1}{k^{2/5}}\right) + O\left(\frac{1}{k^{43/20}}\right),$$

$$\Delta^2 u(k) = \pm\Delta^2\left(\sqrt[5]{\frac{25}{6}} \cdot k^{3/5}\right) \pm \Delta^2\left(\frac{21}{25}\sqrt[5]{\frac{25}{6}}\frac{1}{k^{2/5}}\right) + O\left(\frac{1}{k^{63/20}}\right).$$

Example 9. In the following example, values (m, α) belong to the domain shown in Figure 4.2 (red domain). Consider equation (1.3), where $\alpha = -3$, $m = \frac{1}{2}$, that is, the equation

$$\Delta^2 u(k) \pm k^{-3} u^{1/2}(k) = 0. \tag{4.117}$$

In this example, α and m satisfy (4.24). If we put

$$\varepsilon_1 = \varepsilon_2 = 3, \quad \varepsilon_3 = \varepsilon_4 = 1, \quad \gamma = \frac{1}{2},$$

then, by formula (2.1),

$$s = \frac{\alpha + 2}{m - 1} = 2,$$

by formula (2.2),

$$a = [\mp s(s+1)]^{1/(m-1)} = \frac{1}{36}$$

and, finally, by formula (2.3),

$$b = \frac{as(s+2)}{s + 2 - ms} = \frac{2}{27}.$$

Theorem 6 is applicable and equation (4.117) has a solution $u = u(k)$, $k \in \mathbb{N}(k_0)$ satisfying inequalitites (4.20)–(4.22), that is,

$$-\frac{3}{k^{1/2}} < \left[u(k) - \frac{1}{36k^2} - \frac{2}{27k^3}\right]\left[\frac{2}{27k^3}\right]^{-1} < \frac{3}{k^{1/2}},$$

$$-\frac{1}{k^{1/2}} < \left[\Delta u(k) - \Delta\left(\frac{1}{36k^2}\right) - \Delta\left(\frac{2}{27k^3}\right)\right] \cdot \left[\Delta\left(\frac{2}{27k^3}\right)\right]^{-1} < \frac{1}{k^{1/2}},$$

$$-\frac{3}{k^{1/2}} + O\left(\frac{1}{k}\right) < \left[\Delta^2 u(k) - \Delta^2\left(\frac{1}{36k^2}\right) - \Delta^2\left(\frac{2}{27k^3}\right)\right] \cdot \left[\Delta^2\left(\frac{2}{27k^3}\right)\right]^{-1} < \frac{1}{k^{1/2}} + O\left(\frac{1}{k}\right).$$

These formulas can be simplified to

$$u(k) = \frac{1}{36k^2} + \frac{2}{27k^3} + O\left(\frac{1}{k^{7/2}}\right),$$
$$\Delta u(k) = \Delta\left(\frac{1}{36k^2}\right) + \Delta\left(\frac{2}{27k^3}\right) + O\left(\frac{1}{k^{9/2}}\right),$$
$$\Delta^2 u(k) = \Delta^2\left(\frac{1}{36k^2}\right) + \Delta^2\left(\frac{2}{27k^3}\right) + O\left(\frac{1}{k^{11/2}}\right).$$

Example 10. In the following example, values (m,α) belong to the domain shown in Figure 4.2 (red domain). Consider equation (1.3), where $\alpha = 1$, $m = 6$, that is, the equation

$$\Delta^2 u(k) \pm k u^6(k) = 0. \tag{4.118}$$

In this example, α and m satisfy (4.25). If we put

$$\varepsilon_1 = \varepsilon_2 = 1, \quad \varepsilon_3 = \varepsilon_4 = 1.21, \quad \gamma = 0.4\,,$$

then, by formula (2.1),

$$s = \frac{\alpha+2}{m-1} = 0.6,$$

by formula (2.2),

$$a = [\mp s(s+1)]^{1/(m-1)} = \mp\sqrt[5]{1.56}$$

and, finally, by formula (2.3),

$$b = \frac{as(s+2)}{s+2-ms} = \mp\frac{39}{155}\sqrt[5]{1.56}\,.$$

Theorem 6 is applicable and equation (4.118) has a solution $u = u(k)$, $k \in \mathbb{N}(k_0)$ satisfying inequalitites (4.20)–(4.22), that is,

$$-\frac{1}{k^{0.4}} < \left[u(k) \pm \sqrt[5]{1.56}\frac{1}{k^{0.6}} \pm \sqrt[5]{1.56}\frac{39}{155k^{1.6}}\right]\left[\sqrt[5]{1.56}\frac{39}{155k^{1.6}}\right]^{-1} < \frac{1}{k^{0.4}}\,,$$

$$-\frac{1.21}{k^{0.4}} < \left[\Delta u(k) \pm \Delta\left(\sqrt[5]{1.56}\frac{1}{k^{0.6}}\right) \pm \Delta\left(\sqrt[5]{1.56}\frac{39}{155k^{1.6}}\right)\right]\cdot$$
$$\cdot\left[\Delta\left(\mp\sqrt[5]{1.56}\frac{39}{155k^{1.6}}\right)\right]^{-1} < \frac{1.21}{k^{0.4}},$$

$$-\frac{1}{k^{0.4}} + O\left(\frac{1}{k}\right) < \left[\Delta^2 u(k) \pm \Delta^2\left(\sqrt[5]{1.56}\frac{1}{k^{0.6}}\right) \pm \Delta^2\left(\sqrt[5]{1.56}\frac{39}{155k^{1.6}}\right)\right]\cdot$$
$$\cdot\left[\Delta^2\left(\mp\sqrt[5]{1.56}\frac{39}{155k^{1.6}}\right)\right]^{-1} < \frac{1}{k^{0.4}} + O\left(\frac{1}{k}\right).$$

These formulas can be simplified to

$$u(k) = \mp\sqrt[5]{1.56}\frac{1}{k^{0.6}} \mp \sqrt[5]{1.56}\frac{39}{155k^{1.6}} + O\left(\frac{1}{k^2}\right),$$

$$\Delta u(k) = \mp\Delta\left(\sqrt[5]{1.56}\frac{1}{k^{0.6}}\right) \mp \Delta\left(\sqrt[5]{1.56}\frac{39}{155k^{1.6}}\right) + O\left(\frac{1}{k^3}\right),$$

$$\Delta^2 u(k) = \mp\Delta^2\left(\sqrt[5]{1.56}\frac{1}{k^{0.6}}\right) \mp \Delta^2\left(\sqrt[5]{1.56}\frac{39}{155k^{1.6}}\right) + O\left(\frac{1}{k^4}\right).$$

Example 11. In the following example, values (m, α) belong to the domain shown in Figure 4.2 (red domain). Consider equation (1.3), where $\alpha = 2$, $m = -16$, that is, the equation

$$\Delta^2 u(k) \pm k^2 u^{-16}(k) = 0. \tag{4.119}$$

In this example, α and m satisfy (4.26). If we put

$$\varepsilon_1 = \varepsilon_2 = 1, \quad \varepsilon_3 = \varepsilon_4 = 1.8, \quad \gamma = \frac{13}{17},$$

then, by formula (2.1),

$$s = \frac{\alpha + 2}{m - 1} = -\frac{4}{17},$$

by formula (2.2),

$$a = [\mp s(s+1)]^{1/(m-1)} = \pm\sqrt[17]{\frac{289}{52}}$$

and, finally, by formula (2.3),

$$b = \frac{as(s+2)}{s + 2 - ms} = \mp\frac{60}{289}\sqrt[17]{\frac{289}{57}}.$$

Theorem 6 is applicable and equation (4.119) has a solution $u = u(k)$, $k \in \mathbb{N}(k_0)$ satisfying inequalitites (4.20)–(4.22), that is,

$$-\frac{1}{k^{13/17}} < \left[u(k) \mp \sqrt[17]{\frac{289}{52}} \cdot k^{4/17} \pm \sqrt[17]{\frac{289}{52}}\frac{60}{289k^{13/17}}\right]\left[\mp\sqrt[17]{\frac{289}{52}}\frac{60}{289k^{13/17}}\right]^{-1} < \frac{1}{k^{13/17}},$$

$$-\frac{1.8}{k^{13/17}} < \left[\Delta u(k) \mp \Delta\left(\sqrt[17]{\frac{289}{52}} \cdot k^{4/17}\right) \pm \Delta\left(\sqrt[17]{\frac{289}{52}}\frac{60}{289k^{13/17}}\right)\right] \cdot$$
$$\cdot\left[\Delta\left(\mp\sqrt[17]{\frac{289}{52}}\frac{60}{289k^{13/17}}\right)\right]^{-1} < \frac{1.8}{k^{13/17}},$$

$$-\frac{1}{k^{13/17}}+O\left(\frac{1}{k}\right)<\left[\Delta^2u(k)\mp\Delta^2\left(\sqrt[17]{\frac{289}{52}}\cdot k^{4/17}\right)\pm\Delta^2\left(\sqrt[17]{\frac{289}{52}}\frac{60}{289k^{13/17}}\right)\right]\cdot$$

$$\cdot\left[\Delta^2\left(\mp\sqrt[17]{\frac{289}{52}}\frac{60}{289k^{13/176}}\right)\right]^{-1}<\frac{1}{k^{13/17}}+O\left(\frac{1}{k}\right).$$

These formulas can be simplified to

$$u(k)=\pm\sqrt[17]{\frac{289}{52}}\cdot k^{4/17}\mp\sqrt[17]{\frac{289}{52}}\frac{60}{289k^{13/17}}+O\left(\frac{1}{k^{26/17}}\right),$$

$$\Delta u(k)=\pm\Delta\left(\sqrt[17]{\frac{289}{52}}\cdot k^{4/17}\right)\mp\Delta\left(\sqrt[17]{\frac{289}{52}}\frac{60}{289k^{13/17}}\right)+O\left(\frac{1}{k^{43/17}}\right),$$

$$\Delta^2u(k)=\pm\Delta^2\left(\sqrt[17]{\frac{289}{52}}\cdot k^{4/17}\right)\mp\Delta^2\left(\sqrt[17]{\frac{289}{52}}\frac{60}{289k^{13/17}}\right)+O\left(\frac{1}{k^{60/17}}\right).$$

Example 12. In the following example, values (m,α) belong to the domain shown in Figure 4.4 (red domain). Consider equation (1.3), where $\alpha=-4$, $m=-3$, that is, the equation

$$\Delta^2u(k)\pm k^{-4}u^{-3}(k)=0. \tag{4.120}$$

In this example, α and m satisfy (4.47) and (4.50):

$$\alpha^2(1+m)+\alpha(m^2+8m-1)+8m^2=152>0.$$

If we put

$$\varepsilon_1=\varepsilon_2=1,\quad \varepsilon_3=\varepsilon_4=1,\quad \gamma=\frac{1}{5},$$

then, by formula (2.1),

$$s=\frac{\alpha+2}{m-1}=-\frac{1}{5}=0.2,$$

by formula (2.2),

$$a=[\mp s(s+1)]^{1/(m-1)}=\mp\sqrt[5]{\frac{25}{6}}$$

and, finally, by formula (2.3),

$$b=\frac{as(s+2)}{s+2-ms}=\pm\frac{3}{10}\sqrt[5]{\frac{25}{6}}.$$

Theorem 8 is applicable and equation (4.120) has a solution $u=u(k)$, $k\in\mathbb{N}(k_0)$ satisfying inequalitites (4.44)–(4.46), that is,

$$-\frac{1}{k^{0.2}}<\left[u(k)\pm\sqrt[5]{\frac{25}{6}}\frac{1}{k^{0.2}}\mp\sqrt[5]{\frac{25}{6}}\frac{3}{10k^{1.2}}\right]\left[\pm\sqrt[5]{\frac{25}{6}}\frac{3}{10k^{1.2}}\right]^{-1}<\frac{1}{k^{0.2}},$$

$$-\frac{1}{k^{0.2}} < \left[\Delta u(k) \pm \Delta\left(\sqrt[5]{\frac{25}{6}}\frac{1}{k^{0.2}}\right) \mp \Delta\left(\sqrt[5]{\frac{25}{6}}\frac{3}{10k^{1.2}}\right)\right]\cdot$$
$$\cdot\left[\Delta\left(\pm\sqrt[5]{\frac{25}{6}}\frac{3}{10k^{1.2}}\right)\right]^{-1} < \frac{1}{k^{0.2}},$$

$$-\frac{1}{k^{0.2}} + O\left(\frac{1}{k}\right) < \left[\Delta^2 u(k) \pm \Delta^2\left(\sqrt[5]{\frac{25}{6}}\frac{1}{k^{0.2}}\right) \mp \Delta^2\left(\sqrt[5]{\frac{25}{6}}\frac{3}{10k^{1.2}}\right)\right]\cdot$$
$$\cdot\left[\Delta^2\left(\pm\sqrt[5]{\frac{25}{6}}\frac{3}{10k^{1.2}}\right)\right]^{-1} < \frac{1}{k^{0.2}} + O\left(\frac{1}{k}\right).$$

These formulas can be simplified to

$$u(k) = \mp\sqrt[5]{\frac{25}{6}}\cdot\frac{1}{k^{0.2}} \pm \sqrt[5]{\frac{25}{6}}\frac{3}{10k^{1.2}} + O\left(\frac{1}{k^{1.4}}\right),$$

$$\Delta u(k) = \mp\Delta\left(\sqrt[5]{\frac{25}{6}}\frac{1}{k^{0.2}}\right) \pm \Delta\left(\sqrt[5]{\frac{25}{6}}\frac{3}{10k^{1.2}}\right) + O\left(\frac{1}{k^{2.4}}\right),$$

$$\Delta^2 u(k) = \mp\Delta^2\left(\sqrt[5]{\frac{25}{6}}\frac{1}{k^{0.2}}\right) \pm \Delta^2\left(\sqrt[5]{\frac{25}{6}}\frac{3}{10k^{1.2}}\right) + O\left(\frac{1}{k^{3.4}}\right).$$

Example 13. In the following example, values (m, α) belong to the domain shown in Figure 4.4 (red domain). Consider equation (1.3), where $\alpha = -7/4$, $m = 1/2$, that is, the equation

$$\Delta^2 u(k) \pm k^{-7/4} u^{1/2}(k) = 0. \tag{4.121}$$

In this example, α and m satisfy (4.48) and (4.50):

$$\alpha^2(1+m) + \alpha(m^2 + 8m - 1) + 8m^2 = \frac{29}{32} > 0.$$

If we put

$$\varepsilon_1 = \varepsilon_2 = 1, \quad \varepsilon_3 = \varepsilon_4 = 1, \quad \gamma = \frac{1}{2},$$

then, by formula (2.1),

$$s = \frac{\alpha+2}{m-1} = -\frac{1}{2} = -0.5,$$

by formula (2.2),

$$a = [\mp s(s+1)]^{1/(m-1)} = 16$$

and, finally, by formula (2.3),

$$b = \frac{as(s+2)}{s+2-ms} = -\frac{48}{7}.$$

Theorem 8 is applicable and equation (4.121) has a solution $u = u(k)$, $k \in \mathbb{N}(k_0)$ satisfying inequalitites (4.44)–(4.46), that is,

$$-\frac{1}{k^{0.5}} < \left[u(k) - 16\sqrt{k} + \frac{48}{7}\frac{1}{\sqrt{k}}\right]\left[-\frac{48}{7}\frac{1}{\sqrt{k}}\right]^{-1} < \frac{1}{k^{0.5}},$$

$$-\frac{1}{k^{0.5}} < \left[\Delta u(k) - \Delta\left(16\sqrt{k}\right) + \Delta\left(\frac{48}{7}\frac{1}{\sqrt{k}}\right)\right] \cdot \left[\Delta\left(-\frac{48}{7}\frac{1}{\sqrt{k}}\right)\right]^{-1} < \frac{1}{k^{0.5}},$$

$$-\frac{1}{k^{0.5}} + O\left(\frac{1}{k}\right) < \left[\Delta^2 u(k) - \Delta^2\left(16\sqrt{k}\right) + \Delta^2\left(\frac{48}{7}\frac{1}{\sqrt{k}}\right)\right] \cdot$$
$$\cdot \left[\Delta^2\left(-\frac{48}{7}\frac{1}{\sqrt{k}}\right)\right]^{-1} < \frac{1}{k^{0.5}} + O\left(\frac{1}{k}\right).$$

These formulas can be simplified to

$$u(k) = 16\sqrt{k} - \frac{48}{7}\frac{1}{\sqrt{k}} + O\left(\frac{1}{k}\right),$$
$$\Delta u(k) = \Delta\left(16\sqrt{k}\right) - \Delta\left(\frac{48}{7}\frac{1}{\sqrt{k}}\right) + O\left(\frac{1}{k^2}\right),$$
$$\Delta^2 u(k) = \Delta^2\left(16\sqrt{k}\right) - \Delta^2\left(\frac{48}{7}\frac{1}{\sqrt{k}}\right) + O\left(\frac{1}{k^3}\right).$$

Example 14. In the following example, values (m, α) belong to the domain shown in Figure 4.4 (red domain). Consider equation (1.3), where $\alpha = -3$, $m = 4$, that is, the equation

$$\Delta^2 u(k) \pm k^{-3} u^4(k) = 0. \tag{4.122}$$

In this example, α and m satisfy (4.49) and (4.50):

$$\alpha^2(1+m) + \alpha(m^2 + 8m - 1) + 8m^2 = 32 > 0.$$

If we put

$$\varepsilon_1 = \varepsilon_2 = 1, \quad \varepsilon_3 = \varepsilon_4 = 1, \quad \gamma = \frac{1}{3},$$

then, by formula (2.1),

$$s = \frac{\alpha + 2}{m - 1} = -\frac{1}{3},$$

by formula (2.2),

$$a = [\mp s(s+1)]^{1/(m-1)} = \pm\sqrt[3]{\frac{2}{9}}$$

and, finally, by formula (2.3),

$$b = \frac{as(s+2)}{s+2-ms} = \mp\frac{5}{27}\sqrt[3]{\frac{2}{9}}.$$

Theorem 8 is applicable and equation (4.122) has a solution $u = u(k)$, $k \in \mathbb{N}(k_0)$ satisfying inequalitites (4.44)–(4.46), that is,

$$-\frac{1}{k^{1/3}} < \left[u(k) \mp \sqrt[3]{\frac{2}{9}} \cdot k^{1/3} \pm \frac{5}{27}\sqrt[3]{\frac{2}{9}}\frac{1}{k^{2/3}}\right]\left[\mp\frac{5}{27}\sqrt[3]{\frac{2}{9}}\frac{1}{k^{2/3}}\right]^{-1} < \frac{1}{k^{1/3}},$$

$$-\frac{1}{k^{1/3}} < \left[\Delta u(k) \mp \Delta\left(\sqrt[3]{\frac{2}{9}} \cdot k^{1/3}\right) \pm \Delta\left(\frac{5}{27}\sqrt[3]{\frac{2}{9}}\frac{1}{k^{2/3}}\right)\right] \cdot$$
$$\cdot\left[\Delta\left(\mp\frac{5}{27}\sqrt[3]{\frac{2}{9}}\frac{1}{k^{2/3}}\right)\right]^{-1} < \frac{1}{k^{1/3}},$$

$$-\frac{1}{k^{1/3}} + O\left(\frac{1}{k}\right) < \left[\Delta^2 u(k) \mp \Delta^2\left(\sqrt[3]{\frac{2}{9}} \cdot k^{1/3}\right) \pm \Delta^2\left(\frac{5}{27}\sqrt[3]{\frac{2}{9}}\frac{1}{k^{2/3}}\right)\right] \cdot$$
$$\cdot\left[\Delta^2\left(\mp\frac{5}{27}\sqrt[3]{\frac{2}{9}}\frac{1}{k^{2/3}}\right)\right]^{-1} < \frac{1}{k^{1/3}} + O\left(\frac{1}{k}\right).$$

These formulas can be simplified to

$$u(k) = \pm\sqrt[3]{\frac{2}{9}} \cdot k^{1/4} \mp \frac{5}{27}\sqrt[3]{\frac{2}{9}}\frac{1}{k^{2/3}} + O\left(\frac{1}{k}\right),$$

$$\Delta u(k) = \pm\Delta\left(\sqrt[3]{\frac{2}{9}} \cdot k^{1/3}\right) \mp \Delta\left(\frac{5}{27}\frac{1}{k^{2/3}}\right) + O\left(\frac{1}{k^2}\right),$$

$$\Delta^2 u(k) = \pm\Delta^2\left(\sqrt[3]{\frac{2}{9}} \cdot k^{1/3}\right) \mp \Delta^2\left(\frac{5}{27}\sqrt[3]{\frac{2}{9}}\frac{1}{k^{2/3}}\right) + O\left(\frac{1}{k^3}\right).$$

Example 15. In the following example, values (m, α) belong to the domain shown in Figure 4.5 (red domain). Consider equation (1.3), where $\alpha = -27/20$, $m = 1/2$, that is, the equation

$$\Delta^2 u(k) \pm k^{-27/20} u^{1/2}(k) = 0. \tag{4.123}$$

In this example, α and m satisfy all the conditions of Theorem 10: condition (4.64) is

$$\frac{\alpha+m+1}{m-1} = -\frac{3}{10} < 0$$

condition (4.65) is

$$m\frac{\alpha+2}{m-1} = -\frac{13}{20} < 0$$

condition (4.66) is

$$\frac{2\alpha+5m-1}{m-1} = \frac{12}{5} > 0$$

and condition (4.67) is

$$(m-1)(\alpha^2+\alpha m-\alpha-4m) = -\frac{199}{800} < 0.$$

If we put

$$\varepsilon_1 = \varepsilon_2 = 1, \quad \varepsilon_3 = \varepsilon_4 = 1, \quad \gamma = \frac{8}{10} = 0.8\,,$$

then, the condition (4.68) is applicable:

$$\gamma + \frac{\alpha+m+1}{m-1} = \frac{5}{10} > 0$$

and ,by formula (2.1),

$$s = \frac{\alpha+2}{m-1} = -\frac{13}{10},$$

by formula (2.2),

$$a = [\mp s(s+1)]^{1/(m-1)} = \frac{1521}{10000} = 0.1521$$

and, finally, by formula (2.3),

$$b = \frac{as(s+2)}{s+2-ms} = -\frac{30758}{300000} \doteq -0.1025267\,.$$

Theorem 8 is applicable and equation (4.123) has a solution $u = u(k)$, $k \in \mathbb{N}(k_0)$ satisfying inequalitites (4.44)–(4.46), that is,

$$-\frac{1}{k^{0.8}} < \left[u(k) - \frac{1521}{10000}\cdot k^{13/10} + \frac{30758}{300000}\cdot k^{3/10}\right]\left[-\frac{30758}{300000}\cdot k^{3/10}\right]^{-1} < \frac{1}{k^{0.8}}\,,$$

$$-\frac{1}{k^{0.8}} < \left[\Delta u(k) - \Delta\left(\frac{1521}{10000}\cdot k^{13/10}\right) + \Delta\left(\frac{30758}{300000}\cdot k^{3/10}\right)\right]\cdot \\ \cdot\left[\Delta\left(-\frac{30758}{300000}\cdot k^{3/10}\right)\right]^{-1} < \frac{1}{k^{0.8}},$$

$$-\frac{1}{k^{0.8}} + O\left(\frac{1}{k}\right) < \left[\Delta^2 u(k) - \Delta^2\left(\frac{1521}{10000}\cdot k^{13/10}\right) + \Delta^2\left(\frac{30758}{300000}\cdot k^{3/10}\right)\right]\cdot \\ \cdot\left[\Delta^2\left(-\frac{30758}{300000}\cdot k^{3/10}\right)\right]^{-1} < \frac{1}{k^{0.8}} + O\left(\frac{1}{k}\right)\,.$$

These formulas can be simplified to

$$u(k) = \frac{1521}{10000} \cdot k^{13/10} - \frac{30758}{300000} k^{3/10} + O\left(\frac{1}{k^{1/2}}\right),$$

$$\Delta u(k) = \Delta\left(\frac{1521}{10000} \cdot k^{13/10}\right) - \Delta\left(\frac{30758}{300000} k^{3/10}\right) + O\left(\frac{1}{k^{3/2}}\right),$$

$$\Delta^2 u(k) = \Delta^2\left(\frac{1521}{10000} \cdot k^{13/10}\right) - \Delta^2\left(\frac{30758}{300000} k^{3/10}\right) + O\left(\frac{1}{k^{5/2}}\right).$$

Example 16. In the following example, values (m, α) belong to the domain shown in Figure 4.5 (red domain). Consider equation (1.3), where $\alpha = -3.1$, $m = 2$, that is, the equation

$$\Delta^2 u(k) \pm k^{-3.1} u^2(k) = 0. \tag{4.124}$$

In this example, α and m satisfy all the conditions of Theorem 10: condition (4.64) is

$$\frac{\alpha + m + 1}{m - 1} = -0.1 < 0$$

condition (4.65) is

$$m\frac{\alpha + 2}{m - 1} = -2.2 < 0$$

condition (4.66) is

$$\frac{2\alpha + 5m - 1}{m - 1} = 2.8 > 0$$

and condition (4.67) is

$$(m - 1)(\alpha^2 + \alpha m - \alpha - 4m) = -1.49 < 0.$$

If we put

$$\varepsilon_1 = \varepsilon_2 = 1, \quad \varepsilon_3 = \varepsilon_4 = 2, \quad \gamma = \frac{1}{2} = 0.5,$$

then, the condition (4.68) is applicable:

$$\gamma + \frac{\alpha + m + 1}{m - 1} = 0.4 > 0$$

and by formula (2.1),

$$s = \frac{\alpha + 2}{m - 1} = -1.1,$$

by formula (2.2),

$$a = [\mp s(s + 1)]^{1/(m-1)} = \mp\frac{11}{100} = \mp 0.11$$

and, finally, by formula (2.3),

$$b = \frac{as(s + 2)}{s + 2 - ms} = \pm\frac{1089}{31000} \doteq \pm 0.0351\,.$$

Theorem 8 is applicable and equation (4.124) has a solution $u = u(k)$, $k \in \mathbb{N}(k_0)$ satisfying inequalitites (4.44)–(4.46), that is,

$$-\frac{1}{k^{1/2}} < \left[u(k) \pm \frac{11}{100}\cdot k^{11/10} \mp \frac{1089}{31000}\cdot k^{1/10}\right]\left[\pm\frac{1089}{31000}\cdot k^{1/10}\right]^{-1} < \frac{1}{k^{1/2}},$$

$$-\frac{2}{k^{1/2}} < \left[\Delta u(k) \pm \Delta\left(\frac{11}{100}\cdot k^{11/10}\right) \mp \Delta\left(\frac{1089}{31000}\cdot k^{1/10}\right)\right]\cdot \\ \cdot\left[\Delta\left(\pm\frac{1089}{31000}\cdot k^{1/10}\right)\right]^{-1} < \frac{2}{k^{1/2}},$$

$$-\frac{1}{k^{1/2}} + O\left(\frac{1}{k}\right) < \left[\Delta^2 u(k) \pm \Delta^2\left(\frac{11}{100}\cdot k^{11/10}\right) \mp \Delta^2\left(\frac{1089}{31000}\cdot k^{1/10}\right)\right]\cdot \\ \cdot\left[\Delta^2\left(\pm\frac{1089}{31000}\cdot k^{1/10}\right)\right]^{-1} < \frac{1}{k^{1/2}} + O\left(\frac{1}{k}\right).$$

These formulas can be simplified to

$$u(k) = \mp\frac{11}{100}\cdot k^{11/10} \pm \frac{1089}{31000}k^{1/10} + O\left(\frac{1}{k^{3/5}}\right),$$

$$\Delta u(k) = \mp\Delta\left(\frac{11}{100}\cdot k^{11/10}\right) \pm \Delta\left(\frac{1089}{31000}k^{1/10}\right) + O\left(\frac{1}{k^{8/5}}\right),$$

$$\Delta^2 u(k) = \mp\Delta^2\left(\frac{11}{100}\cdot k^{11/10}\right) \pm \Delta^2\left(\frac{1089}{31000}k^{1/10}\right) + O\left(\frac{1}{k^{13/5}}\right).$$

Example 17. In the following example, values (m, α) belong to the domain shown in Figure 4.5 (red domain). Consider equation (1.3), where $\alpha = 3/2$, $m = -2$, that is, the equation

$$\Delta^2 u(k) \pm k^{3/2} u^{-2}(k) = 0. \tag{4.125}$$

In this example, α and m satisfy all the conditions of Theorem 10: condition (4.91) is

$$\frac{\alpha + m + 1}{m - 1} = -\frac{1}{6} < 0$$

condition (4.92) is

$$m\frac{\alpha + 2}{m - 1} = \frac{7}{3} > 0$$

condition (4.93) is

$$\frac{2\alpha + 5m - 1}{m - 1} = 24 > 0$$

and condition (4.67) is

$$\alpha^2 + 8m^2 + 8m\alpha - \alpha + m\alpha^2 + m^\alpha = \frac{41}{4} > 0.$$

If we put

$$\varepsilon_1 = \varepsilon_2 = 1, \quad \varepsilon_3 = \varepsilon_4 = 2, \quad \gamma = \frac{5}{6},$$

then, the condition (4.68) is applicable:

$$\gamma + \frac{\alpha + m + 1}{m - 1} = \frac{2}{3} > 0$$

and, by formula (2.1),

$$s = \frac{\alpha + 2}{m - 1} = -\frac{7}{6},$$

by formula (2.2),

$$a = [\mp s(s+1)]^{1/(m-1)} = \mp\sqrt[3]{\frac{36}{7}} \doteq \mp 1.7261$$

and, finally, by formula (2.3),

$$b = \frac{as(s+2)}{s + 2 - ms} = \mp\sqrt[3]{\frac{36}{7}}\frac{35}{54} \doteq \mp 1.1188\,.$$

Theorem 8 is applicable and equation (4.125) has a solution $u = u(k)$, $k \in \mathbb{N}(k_0)$ satisfying inequalitites (4.44)–(4.46), that is,

$$-\frac{1}{k^{5/6}} < \left[u(k) \pm \sqrt[3]{\frac{36}{7}} \cdot k^{7/6} \mp \sqrt[3]{\frac{36}{7}}\frac{35}{54} \cdot k^{1/6}\right] \left[\pm\sqrt[3]{\frac{36}{7}}\frac{35}{54} \cdot k^{1/6}\right]^{-1} < \frac{1}{k^{5/6}}\,,$$

$$-\frac{2}{k^{5/6}} < \left[\Delta u(k) \pm \Delta\left(\sqrt[3]{\frac{36}{7}} \cdot k^{7/6}\right) \mp \Delta\left(\sqrt[3]{\frac{36}{7}}\frac{35}{54} \cdot k^{1/6}\right)\right] \cdot$$
$$\cdot\left[\Delta\left(\pm\sqrt[3]{\frac{36}{7}}\frac{35}{54} \cdot k^{1/6}\right)\right]^{-1} < \frac{2}{k^{5/6}},$$

$$-\frac{1}{k^{5/6}} + O\left(\frac{1}{k}\right) < \left[\Delta^2 u(k) \pm \Delta^2\left(\sqrt[3]{\frac{36}{7}} \cdot k^{7/6}\right) \mp \Delta^2\sqrt[3]{\frac{36}{7}}\left(\frac{35}{54} \cdot k^{1/6}\right)\right] \cdot$$
$$\cdot\left[\Delta^2\left(\pm\sqrt[3]{\frac{36}{7}}\frac{35}{54} \cdot k^{1/6}\right)\right]^{-1} < \frac{1}{k^{5/6}} + O\left(\frac{1}{k}\right).$$

These formulas can be simplified to

$$u(k) = \mp\sqrt[3]{\frac{36}{7}} \cdot k^{7/6} \pm \sqrt[3]{\frac{36}{7}}\frac{35}{54}k^{1/6} + O\left(\frac{1}{k^{2/3}}\right),$$

$$\Delta u(k) = \mp\Delta\left(\sqrt[3]{\frac{36}{7}} \cdot k^{7/6}\right) \pm \Delta\left(\sqrt[3]{\frac{36}{7}}\frac{35}{54}k^{1/6}\right) + O\left(\frac{1}{k^{5/3}}\right),$$

$$\Delta^2 u(k) = \mp\Delta^2\left(\sqrt[3]{\frac{36}{7}} \cdot k^{7/6}\right) \pm \Delta^2\left(\sqrt[3]{\frac{36}{7}}\frac{35}{54}k^{1/6}\right) + O\left(\frac{1}{k^{8/3}}\right).$$

5 A discrete analogy of the blow-up solution

To illustrate an analogous blow-up phenomenon for a discrete second-order equation, we will use an autonomous second-order Emden-Fowler type differential equation

$$y''(x) = y^s(x), \tag{5.1}$$

where $s \neq 1$ is a real number.

Let us show that (5.1) can have blow-up solutions.

First, equation (5.1) is solvable and its general solution can be written in the form

$$\int_{y_0}^{y(x)} \frac{dz}{\sqrt{2\int z^s dz + C}} = x - x_0 \tag{5.2}$$

where C is an arbitrary (but admissible) constant and (x_0, y_0) is an arbitrary admissible point. If, for example, $s = 3$ and $C = 0$, then it is easy to derive from (5.2) a class of solutions

$$y(x) = \pm\frac{\sqrt{2}}{x + K} \tag{5.3}$$

where K is an arbitrary constant and one can see the blow-up phenomenon explicitly if $x \to \pm K$.

In directly transferring the above phenomena to discrete equations, there are some circumstances to be taken in consideration because the independent variable in discrete equations is discrete running over a set of integers.

Therefore, we prove the existence of this phenomenon implicitly as follows. First, we transform equation (5.1) by a transformation

$$x = u(y) \tag{5.4}$$

where u is a new unknown function. This transformation will be such that x has a finite limit as y tends to infinity. For example, writing solution (5.3) in the form (5.4), we derive

$$x = u(y) = \pm\frac{\sqrt{2}}{y} - K. \tag{5.5}$$

If $y \to \infty$, then, by (5.5), $x \to -K$. Next, we will compile a differential equation for u in (5.4) and the form of this equation will serve as a motivation for constructing a related discrete equation.

Differentiating the transformation (5.4) with respect to x, we derive

$$1 = u'_y \cdot y'_x. \tag{5.6}$$

Differentiating (5.6) with respect to x again, we have

$$0 = u''_{yy} \cdot (y'_x)^2 + u'_y \cdot y''_{xx}. \tag{5.7}$$

Assuming $u'_y \neq 0$, from (5.7), we get

$$y'' = -\frac{u'' \cdot (y')^2}{u'} \tag{5.8}$$

and, using (5.1), (5.6), (5.8)

$$y^s = y'' = -\frac{u'' \cdot (y')^2}{u'} = -\frac{u''}{(u')^3}$$

and, finally, for u we derive

$$u'' = -y^s (u')^3 . \tag{5.9}$$

Then, a discrete analogy to differential equation (5.9) is the following

$$\Delta^2 v(k) = -k^s (\Delta v(k))^3 . \tag{5.10}$$

A problem equivalent to blow-up phenomena for differential equation (5.1) is one of proving the existence of a nontrivial solution to equation (5.9) such that the limit $\lim_{y\to\infty} u(y)$ exists and is finite. Therefore, we consider the problem to prove the existence of a nontrivial solution to equation (5.10) such that the limit $\lim_{k\to\infty} v(k)$ exists and is finite. More exactly, under condition $s > 1$, we prove the existence of a solution to equation (5.10) such that

$$\lim_{k\to\infty} v(k) = 0. \tag{5.11}$$

5.1 An approximate solution of second-order discrete Emden-Fowler equation (5.10)

We will search for an approximate solution of discrete equation (5.10) with asymptotic behaviour

$$v(k) \sim V(k) := c \cdot k^{-\alpha}$$

as $k \to \infty$ where c and α are constants still unknown. We assume $c \neq 0$, $\alpha \neq 0$ trying to find these constants. To do this, we must replace $\Delta v(k)$ and $\Delta^2 v(k)$ with $\Delta V(k)$ and $\Delta^2 V(k)$ in (5.10). Let us perform, for $k \to \infty$, auxiliary asymptotic computation of $\Delta V(k)$ and $\Delta^2 V(k)$. With the necessary order of accuracy for $\Delta V(k)$, we obtain

$$\Delta V(k) = c(k+1)^{-\alpha} - ck^{-\alpha} = ck^{-\alpha}\left(1 + \frac{1}{k}\right)^{-\alpha} - ck^{-\alpha} =$$

$$= ck^{-\alpha}\left(1-\frac{\alpha}{k}+\frac{\alpha(\alpha+1)}{2k^2}-\frac{\alpha(\alpha+1)(\alpha+2)}{6k^3}+O\left(\frac{1}{k^4}\right)-1\right)$$

$$= -\frac{c\alpha}{k^{\alpha+1}}+\frac{c\alpha(\alpha+1)}{2k^{\alpha+2}}-\frac{c\alpha(\alpha+1)(\alpha+2)}{6k^{\alpha+3}}+O\left(\frac{1}{k^{\alpha+4}}\right)$$

and, for $\Delta^2 V(k)$, we have

$$\Delta^2 V(k) = c(k+2)^{-\alpha}-2c(k+1)^{-\alpha}+ck^{-\alpha}=\frac{c}{k^\alpha}\left(\left(1+\frac{2}{k}\right)^{-\alpha}-2\left(1+\frac{1}{k}\right)^{-\alpha}+1\right)$$

$$= \frac{c}{k^\alpha}\left(\left(1-\frac{2\alpha}{k}+\frac{2\alpha(\alpha+1)}{k^2}-\frac{4\alpha(\alpha+1)(\alpha+2)}{3k^3}+O\left(\frac{1}{k^4}\right)\right)\right.$$

$$\left.-2\left(1-\frac{\alpha}{k}+\frac{\alpha(\alpha+1)}{2k^2}-\frac{\alpha(\alpha+1)(\alpha+2)}{6k^3}+O\left(\frac{1}{k^4}\right)\right)+1\right)$$

$$= \frac{c\alpha(\alpha+1)}{k^{\alpha+2}}-\frac{c\alpha(\alpha+1)(\alpha+2)}{3k^{\alpha+3}}+O\left(\frac{1}{k^{\alpha+4}}\right).$$

Then, replacing in (5.10) $\Delta v(k)$ and $\Delta^2 v(k)$ with $\Delta V(k)$ and $\Delta^2 V(k)$, we derive

$$\frac{c\alpha(\alpha+1)}{k^{\alpha+2}}+O\left(\frac{1}{k^{\alpha+3}}\right)=-k^s\left(-\frac{c\alpha}{k^{\alpha+1}}+\frac{c\alpha(\alpha+1)}{2k^{\alpha+2}}+O\left(\frac{1}{k^{\alpha+3}}\right)\right)^3$$

and

$$\frac{c\alpha(\alpha+1)}{k^{\alpha+2}}+O\left(\frac{1}{k^{\alpha+3}}\right)$$

$$= -k^s\left(-\frac{c^3\alpha^3}{k^{3\alpha+3}}+\frac{3c^3\alpha^3(\alpha+1)}{2k^{3\alpha+4}}-\frac{3c^3\alpha^3(\alpha+1)^2}{4k^{3\alpha+5}}+O\left(\frac{1}{k^{3\alpha+5}}\right)\right).$$

The last expression implies

$$\frac{c\alpha(\alpha+1)}{k^{\alpha+2}}=\frac{c^3\alpha^3}{k^{3\alpha+3-s}}+O\left(\frac{1}{k^{3\alpha+4-s}}\right)+O\left(\frac{1}{k^{\alpha+3}}\right). \tag{5.12}$$

Relation (5.12) is satisfied for

$$\begin{cases} \alpha+2=3\alpha+3-s, \\ c\alpha(\alpha+1)=c^3\alpha^3. \end{cases} \tag{5.13}$$

The values

$$\alpha=\frac{s-1}{2},\quad c=\pm\frac{\sqrt{\alpha+1}}{\alpha}=\pm\frac{\sqrt{2s+2}}{s-1} \tag{5.14}$$

solve the system (5.13). Since $V(k)$ can assume two values, we denote

$$V(k)=V_\pm(k)=\pm\frac{\sqrt{2s+2}}{s-1}k^{(1-s)/2}.$$

5.2 System equivalent to discrete Emden-Fowler equation (5.10)

Define the following change of variables:

$$v(k) = ck^{-\alpha}(1+Y_1(k)), \tag{5.15}$$

$$\Delta v(k) = (\Delta(ck^{-\alpha}))(1+Y_2(k)), \tag{5.16}$$

$$\Delta^2 v(k) = (\Delta^2 \left(ck^{-\alpha}\right))(1+Y_3(k)) \tag{5.17}$$

where $Y_i(k)$, $i=1,2,3$ are new dependent functions $Y_i\colon \mathbb{N}(k_0) \to \mathbb{R}$, c and α are defined by (5.14). In (5.10) replace $\Delta v(k)$, $\Delta^2 v(k)$ with (5.16), (5.17). First, compute

$$\Delta k^{-\alpha} = (k+1)^{-\alpha} - k^{-\alpha} = k^{-\alpha}\left[\left(1+\frac{1}{k}\right)^{-\alpha} - 1\right] = k^{-\alpha}\left[1 - \frac{\alpha}{k} + \frac{\alpha(\alpha+1)}{2k^2} - \frac{\alpha(\alpha+1)(\alpha+2)}{6k^3} + O\left(\frac{1}{k^4}\right) - 1\right] = -\frac{\alpha}{k^{\alpha+1}} + \frac{\alpha(\alpha+1)}{2k^{\alpha+2}} - \frac{\alpha(\alpha+1)(\alpha+2)}{6k^{\alpha+3}} + O\left(\frac{1}{k^{\alpha+4}}\right)$$

and

$$\begin{aligned}\Delta^2 k^{-\alpha} &= (k+2)^{-\alpha} - 2(k+1)^{-\alpha} + k^{-\alpha} = k^{-\alpha}\left[\left(1+\frac{2}{k}\right)^{-\alpha} - 2\left(1+\frac{1}{k}\right)^{-\alpha} + 1\right]\\ &= k^{-\alpha}\left[1 - \frac{2\alpha}{k} + \frac{2\alpha(\alpha+1)}{k^2} - \frac{4\alpha(\alpha+1)(\alpha+2)}{3k^3} + \frac{2\alpha(\alpha+1)(\alpha+2)(\alpha+3)}{3k^4}\right.\\ &\left. -2\left(1 - \frac{\alpha}{k} + \frac{\alpha(\alpha+1)}{2k^2} - \frac{\alpha(\alpha+1)(\alpha+2)}{6k^3} + \frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}{24k^4}\right) + 1 + O\left(\frac{1}{k^5}\right)\right]\\ &= \frac{\alpha(\alpha+1)}{k^{\alpha+2}} - \frac{\alpha(\alpha+1)(\alpha+2)}{k^{\alpha+3}} + \frac{7\alpha(\alpha+1)(\alpha+2)(\alpha+3)}{12k^{\alpha+4}} + O\left(\frac{1}{k^{\alpha+5}}\right).\end{aligned} \tag{5.18}$$

Let us take the first difference of equation (5.15):

$$\begin{aligned}\Delta v(k) = \Delta\left(ck^{-\alpha}(1+Y_1(k))\right) &= c\left[(\Delta(k^{-\alpha}))(1+Y_1(k)) + (k+1)^{-\alpha}\Delta(1+Y_1(k))\right]\\ &= c\Big[(\Delta(k^{-\alpha}))(1+Y_1(k)) + (k+1)^{-\alpha}\Delta Y_1(k)\Big].\end{aligned}$$

Substituting it to equation (5.16), we get

$$c(\Delta(k^{-\alpha}))(1+Y_2(k)) = c[(\Delta k^{-\alpha})(1+Y_1(k)) + (k+1)^{-\alpha}\Delta Y_1(k)].$$

Simplifying this expression, we have

$$\Delta Y_1(k) = (\Delta k^{-\alpha})(Y_2(k) - Y_1(k))(k+1)^{\alpha} = \left(-\frac{\alpha}{k^{\alpha+1}} + \frac{\alpha(\alpha+1)}{2k^{\alpha+2}} - \frac{\alpha(\alpha+1)(\alpha+2)}{6k^{\alpha+3}} + \right.$$

$$\left. + O\left(\frac{1}{k^{\alpha+4}}\right)\right) k^{\alpha}\left(1+\frac{1}{k}\right)^{\alpha}(Y_2(k) - Y_1(k)) = \left(-\frac{\alpha}{k} + \frac{\alpha(\alpha+1)}{2k^2} - \frac{\alpha(\alpha+1)(\alpha+2)}{6k^3}\right.$$

$$+ O\left(\frac{1}{k^4}\right)\right) \cdot \left(1 + \frac{\alpha}{k} + \frac{\alpha(\alpha-1)}{2k^2} + \frac{\alpha(\alpha-1)(\alpha-2)}{6k^3} + O\left(\frac{1}{k^4}\right)\right)(Y_2(k) - Y_1(k))$$
$$= \left(-\frac{\alpha}{k} + \frac{\alpha(\alpha+1)}{2k^2} - \frac{\alpha(\alpha+1)(\alpha+2)}{6k^3} - \frac{\alpha^2}{k^2} + \frac{\alpha^2(\alpha+1)}{2k^3} - \frac{\alpha^2(\alpha-1)}{2k^3} + O\left(\frac{1}{k^4}\right)\right)$$
$$\cdot(Y_2(k) - Y_1(k)) = \left(-\frac{\alpha}{k} + \frac{\alpha^2+\alpha-2\alpha^2}{2k^2} + \frac{-\alpha(\alpha^2+3\alpha+2)+6\alpha^2}{6k^3} + O\left(\frac{1}{k^4}\right)\right)$$
$$= (Y_2(k) - Y_1(k)) = \left(-\frac{\alpha}{k} - \frac{\alpha(\alpha-1)}{2k^2} - \frac{\alpha(\alpha-1)(\alpha-2)}{6k^3} + O\left(\frac{1}{k^4}\right)\right)(Y_2(k) - Y_1(k))$$

and, finally,

$$\Delta Y_1(k) = \left[\frac{\alpha}{k} + \frac{\alpha(\alpha-1)}{2k^2} + \frac{\alpha(\alpha-1)(\alpha-2)}{3k^3} + O\left(\frac{1}{k^4}\right)\right](Y_1(k) - Y_2(k)]. \quad (5.19)$$

Let us take the first difference of equation (5.16):

$$\Delta^2 v(k) = \Delta((\Delta(ck^{-\alpha}))(1+Y_2(k))) = (\Delta^2(ck^{-\alpha}))(1+Y_2(k))$$
$$+(\Delta(c(k+1)^{-\alpha}))\Delta(1+Y_2(k)) = c(\Delta^2(k^{-\alpha}))(1+Y_2(k)) + c(\Delta((k+1)^{-\alpha}))\Delta Y_2(k). \quad (5.20)$$

Compute

$$\Delta(k+1)^{-\alpha} = (k+2)^{-\alpha} - (k+1)^{-\alpha} = k^{-\alpha}\left(1+\frac{2}{k}\right)^{-\alpha} - k^{-\alpha}\left(1+\frac{1}{k}\right)^{-\alpha}$$
$$= k^{-\alpha}\left[1 - \frac{2\alpha}{k} + \frac{4\alpha(\alpha+1)}{2k^2} - \frac{8\alpha(\alpha+1)(\alpha+2)}{6k^3} + \frac{16\alpha(\alpha+1)(\alpha+2)(\alpha+3)}{24k^4} + O\left(\frac{1}{k^5}\right)\right]$$
$$- k^{-\alpha}\left[1 - \frac{\alpha}{k} + \frac{\alpha(\alpha+1)}{2k^2} - \frac{\alpha(\alpha+1)(\alpha+2)}{6k^3} + \frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}{24k^4} + O\left(\frac{1}{k^5}\right)\right]$$
$$= -\frac{\alpha}{k^{\alpha+1}} + \frac{3\alpha(\alpha+1)}{2k^{\alpha+2}} - \frac{7\alpha(\alpha+1)(\alpha+2)}{6k^{\alpha+3}} + \frac{15\alpha(\alpha+1)(\alpha+2)(\alpha+3)}{24k^{\alpha+4}} + O\left(\frac{1}{k^{\alpha+5}}\right). \quad (5.21)$$

From (5.17) and (5.20), we derive

$$(\Delta^2(ck^{-\alpha}))(1+Y_3(k)) = c(\Delta^2(k^{-\alpha}))(1+Y_2(k)) + c(\Delta(k+1)^{-\alpha})\Delta Y_2(k)$$

and

$$\Delta Y_2(k) = (\Delta^2 k^{-\alpha})(\Delta(k+1)^{-\alpha})^{-1}(Y_3(k) - Y_2(k)). \quad (5.22)$$

In the following two parts, this equation will be simplified. First, we will compute the coefficient

$$(\Delta^2 k^{-\alpha})(\Delta(k+1)^{-\alpha})^{-1}. \quad (5.23)$$

Next, $Y_3(k)$ will be expressed using equation (5.10) and transformation formulas (5.16) and (5.17).

5.2.1 Equation (5.22) - simplification of the coefficient (5.23)

Now, compute the coefficient (5.23) appearing in (5.22). Using (5.21), we have

$$\begin{aligned}
\left(\Delta(k+1)^{-\alpha}\right)^{-1} &= \Bigg[-\frac{\alpha}{k^{\alpha+1}}+\frac{3\alpha(\alpha+1)}{2k^{\alpha+2}}-\frac{7\alpha(\alpha+1)(\alpha+2)}{6k^{\alpha+3}}+ \\
&\qquad \frac{15\alpha(\alpha+1)(\alpha+2)(\alpha+3)}{24k^{\alpha+4}}+O\left(\frac{1}{k^{\alpha+5}}\right)\Bigg]^{-1} \\
&= -\frac{k^{\alpha+1}}{\alpha}\left[1+\left(-\frac{3(\alpha+1)}{2k}+\frac{7(\alpha+1)(\alpha+2)}{6k^2}-\frac{15(\alpha+1)(\alpha+2)(\alpha+3)}{24k^3}+O\left(\frac{1}{k^4}\right)\right)\right]^{-1} \\
&= -\frac{k^{\alpha+1}}{\alpha}\Bigg[1-\left(-\frac{3(\alpha+1)}{2k}+\frac{7(\alpha+1)(\alpha+2)}{6k^2}-\frac{15(\alpha+1)(\alpha+2)(\alpha+3)}{24k^3}+O\left(\frac{1}{k^4}\right)\right) \\
&\quad \left(-\frac{3(\alpha+1)}{2k}+\frac{7(\alpha+1)(\alpha+2)}{6k^2}+O\left(\frac{1}{k^3}\right)\right)^2-\left(-\frac{3(\alpha+1)}{2k}+O\left(\frac{1}{k^2}\right)\right)^3+O\left(\frac{1}{k^4}\right)\Bigg] \\
&= -\frac{k^{\alpha+1}}{\alpha}\Bigg[1+\frac{3(\alpha+1)}{2k}-\frac{7(\alpha+1)(\alpha+2)}{6k^2}+\frac{5(\alpha+1)(\alpha+2)(\alpha+3)}{8k^3}+\frac{9(\alpha+1)^2}{4k^2} \\
&\qquad -\frac{7(\alpha+1)^2(\alpha+2)}{2k^3}+\frac{27(\alpha+1)^3}{8k^3}+O\left(\frac{1}{k^4}\right)\Bigg]
\end{aligned}$$

$$\begin{aligned}
&= -\frac{k^{\alpha+1}}{\alpha}\Bigg[1+\frac{3(\alpha+1)}{2k}+\frac{(\alpha+1)(-14\alpha-28+27\alpha+27)}{12k^2} \\
&\qquad +\frac{(\alpha+1)(5(\alpha^2+5\alpha+6)-28(\alpha^2+3\alpha+2)+27(\alpha^2+2\alpha+1)}{8k^3}+O\left(\frac{1}{k^4}\right)\Bigg]
\end{aligned}$$

$$\begin{aligned}
&-\frac{k^{\alpha+1}}{\alpha}\left[1+\frac{3(\alpha+1)}{2k}+\frac{(\alpha+1)(13\alpha-1)}{12k^2}+\frac{(\alpha+1)(4\alpha^2-5\alpha+1)}{8k^3}+O\left(\frac{1}{k^4}\right)\right] \\
&= -\frac{k^{\alpha+1}}{\alpha}\left[1+\frac{3(\alpha+1)}{2k}+\frac{(\alpha+1)(13\alpha-1)}{12k^2}+\frac{(\alpha+1)(\alpha-1)(4\alpha-1)}{8k^3}+O\left(\frac{1}{k^4}\right)\right]. \qquad (5.24)
\end{aligned}$$

Now, we use (5.18) and (5.24) to calculate the coefficient

$$\begin{aligned}
(\Delta^2k^{-\alpha})(\Delta(k+1)^{-\alpha})^{-1} &= \Bigg[\frac{\alpha(\alpha+1)}{k^{\alpha+2}}-\frac{\alpha(\alpha+1)(\alpha+2)}{k^{\alpha+3}}+\frac{7\alpha(\alpha+1)(\alpha+2)(\alpha+3)}{12k^{\alpha+4}} \\
&+O\left(\frac{1}{k^{\alpha+5}}\right)\Bigg]\cdot\left(-\frac{k^{\alpha+1}}{\alpha}\right)\Bigg[1+\frac{3(\alpha+1)}{2k}+\frac{(\alpha+1)(13\alpha-1)}{12k^2}+\frac{(\alpha+1)(\alpha-1)(4\alpha-1)}{8k^3} \\
&\qquad +O\left(\frac{1}{k^4}\right)\Bigg]
\end{aligned}$$

$$= \left[-\frac{\alpha+1}{k}+\frac{(\alpha+1)(\alpha+2)}{k^2}-\frac{7(\alpha+1)(\alpha+2)(\alpha+3)}{12k^3}+O\left(\frac{1}{k^4}\right)\right]$$

$$\cdot\left[1+\frac{3(\alpha+1)}{2k}+\frac{(\alpha+1)(13\alpha-1)}{12k^2}+\frac{(\alpha+1)(\alpha-1)(4\alpha-1)}{8k^3}+O\left(\frac{1}{k^4}\right)\right]$$

$$= -\frac{\alpha+1}{k}+\frac{(\alpha+1)(\alpha+2)}{k^2}-\frac{7(\alpha+1)(\alpha+2)(\alpha+3)}{12k^3}-\frac{3(\alpha+1)^2}{2k^2} + \frac{3(\alpha+1)^2(\alpha+2)}{2k^3}-\frac{(\alpha+1)^2(13\alpha-1)}{12k^3}+O\left(\frac{1}{k^4}\right)$$

$$= -\frac{\alpha+1}{k}+\frac{(\alpha+1)(2\alpha+4-3\alpha-3)}{2k^2} + \frac{(\alpha+1)(-7\alpha^2-35\alpha-42+18\alpha^2+54\alpha+36-13\alpha^2-12\alpha+1)}{12k^3}+O\left(\frac{1}{k^4}\right)$$
$$= -\frac{\alpha+1}{k}-\frac{(\alpha+1)(\alpha-1)}{2k^2}-\frac{(\alpha+1)(2\alpha^2-7\alpha+5)}{12k^3}+O\left(\frac{1}{k^4}\right)$$
$$= -\frac{\alpha+1}{k}-\frac{(\alpha+1)(\alpha-1)}{2k^2}-\frac{(\alpha+1)(\alpha-1)(2\alpha-5)}{12k^3}+O\left(\frac{1}{k^4}\right) \quad (5.25)$$

5.2.2 Equation (5.22) - computation of $Y_3(k)$

Substituting the changes of variables (5.16) and (5.17) into equation (5.10) gives

$$\Delta^2(ck^{-\alpha})(1+Y_3(k)) = -k^s(\Delta(ck^{-\alpha}))^3(1+Y_2(k))^3$$

or, expressing c by the formula in (5.14),

$$\Delta^2k^{-\alpha}+(\Delta^2k^{-\alpha})Y_3(k) = -k^s\frac{\alpha+1}{\alpha^2}(\Delta k^{-\alpha})^3(1+3Y_2(k)+3Y_2^2(k)+Y_2^3(k)).$$

From the last relation we derive

$$Y_3(k) = \left[-k^s\frac{\alpha+1}{\alpha^2}(\Delta k^{-\alpha})^3(1+3Y_2(k)+3Y_2^2(k)+Y_2^3(k))-\Delta^2k^{-\alpha}\right]\cdot(\Delta^2k^{-\alpha})^{-1}. \quad (5.26)$$

Let us perform auxiliary computations of the expressions appearing in (5.26). We start with $\Delta^2(k^{-\alpha})^{-1}$ using formula (5.18).

$$(\Delta^2k^{-\alpha})^{-1} = \left[\frac{\alpha(\alpha+1)}{k^{\alpha+2}}-\frac{\alpha(\alpha+1)(\alpha+2)}{k^{\alpha+3}}+\frac{7\alpha(\alpha+1)(\alpha+2)(\alpha+3)}{12k^{\alpha+4}}+O\left(\frac{1}{k^{\alpha+5}}\right)\right]^{-1}$$
$$= \frac{k^{\alpha+2}}{\alpha(\alpha+1)}\left[1+\left(-\frac{\alpha+2}{k}+\frac{7(\alpha+2)(\alpha+3)}{12k^2}+O\left(\frac{1}{k^3}\right)\right)\right]^{-1}$$
$$= \frac{k^{\alpha+2}}{\alpha(\alpha+1)}\left[1+\frac{\alpha+2}{k}-\frac{7(\alpha+2)(\alpha+3)}{12k^2}+\frac{(\alpha+2)^2}{k^2}+O\left(\frac{1}{k^3}\right)\right]$$
$$= \frac{k^{\alpha+2}}{\alpha(\alpha+1)}\left[1+\frac{\alpha+2}{k}+\frac{(\alpha+2)(-7\alpha-21+12\alpha+24)}{12k^2}+O\left(\frac{1}{k^3}\right)\right]$$

$$= \frac{k^{\alpha+2}}{\alpha(\alpha+1)}\left[1+\frac{\alpha+2}{k}+\frac{(\alpha+2)(5\alpha+3)}{12k^2}+O\left(\frac{1}{k^3}\right)\right].$$

Using this relation and formula (5.18), we compute

$$\begin{aligned}
&(\Delta^2 k^{-\alpha})^{-1}(\Delta k^{-\alpha})^3(\alpha+1)\alpha^{-2}\\
&\qquad= \frac{\alpha+1}{\alpha^2}\frac{k^{\alpha+2}}{\alpha(\alpha+1)}\left[1+\frac{\alpha+2}{k}+\frac{(\alpha+2)(5\alpha+3)}{12k^2}+O\left(\frac{1}{k^3}\right)\right]\\
&\qquad\cdot\left[-\frac{\alpha}{k^{\alpha+1}}+\frac{\alpha(\alpha+1)}{2k^{\alpha+2}}-\frac{\alpha(\alpha+1)(\alpha+2)}{6k^{\alpha+3}}+O\left(\frac{1}{k^{\alpha+4}}\right)\right]^3\\
&= \frac{k^{\alpha+2}}{\alpha^3}\left[1+\frac{\alpha+2}{k}+\frac{(\alpha+2)(5\alpha+3)}{12k^2}+O\left(\frac{1}{k^3}\right)\right]\cdot\left[-\frac{\alpha^3}{k^{3\alpha+3}}+\frac{3\alpha^3(\alpha+1)}{2k^{3\alpha+4}}+O\left(\frac{1}{k^{3\alpha+5}}\right)\right]\\
&\qquad= \frac{k^{\alpha+2}}{\alpha^3}\left[-\frac{\alpha^3}{k^{3\alpha+3}}+\frac{3\alpha^3(\alpha+1)}{2k^{3\alpha+4}}-\frac{\alpha^3(\alpha+2)}{k^{3\alpha+4}}+O\left(\frac{1}{k^{3\alpha+5}}\right)\right]\\
&\qquad\qquad= -\frac{1}{k^{2\alpha+1}}+\frac{\alpha-1}{2k^{2\alpha+2}}+O\left(\frac{1}{k^{2\alpha+3}}\right).
\end{aligned}$$

Hence, simplification of (5.26) yields

$$Y_3(k) = \left(\frac{1}{k^{2\alpha+1-s}}-\frac{\alpha-1}{k^{2\alpha+2-s}}+O\left(\frac{1}{k^{2\alpha+3-s}}\right)\right)(1+3Y_2(k)+3Y_2^2(k)+Y_2^3(k))-1. \tag{5.27}$$

5.2.3 Equation (5.22) - a simplified form

Finally, we use (5.25) and (5.27) to get the following simplified version of equation (5.22):

$$\begin{aligned}
&\Delta Y_2(k) = -\left(\frac{\alpha+1}{k}+\frac{(\alpha+1)(\alpha-1)}{2k^2}+O\left(\frac{1}{k^3}\right)\right)\\
&\cdot\left(\left(\frac{1}{k^{2\alpha+1-s}}-\frac{\alpha-1}{k^{2\alpha+2-s}}+O\left(\frac{1}{k^{2\alpha+3-s}}\right)\right)(1+3Y_2(k)+3Y_2^2(k)+Y_2^3(k))-1-Y_2(k)\right).
\end{aligned}$$

By (5.14), we have $2\alpha+1-s=0$ so that the last equation can be rewritten in the form

$$\begin{aligned}
&\Delta Y_2(k) = -\left(\frac{\alpha+1}{k}+O\left(\frac{1}{k^2}\right)\right)\\
&\qquad\cdot\left(\left(1-\frac{\alpha-1}{k}+O\left(\frac{1}{k^2}\right)\right)(1+3Y_2(k)+3Y_2^2(k)+Y_2^3(k))-1-Y_2(k)\right)\\
&= -\left(\frac{\alpha+1}{k}+O\left(\frac{1}{k^2}\right)\right)\left(\left(1+O\left(\frac{1}{k}\right)\right)(1+3Y_2(k)+3Y_2^2(k)+Y_2^3(k))-1-Y_2(k)\right)
\end{aligned}$$

or, keeping only the necessary order of accuracy, and assuming $Y_2(k)=O^+(1)$ (this assumption will also remain in force in the below analysis), 5be preserved in analysis below),

$$\Delta Y_2(k) = -\left(\frac{\alpha+1}{k}+O\left(\frac{1}{k^2}\right)\right)\left(2Y_2(k)+3Y_2^2(k)+Y_2^3(k)+O\left(\frac{1}{k}\right)\right). \tag{5.28}$$

Now, the resulting equations (5.19) and (5.28) form the following system of equations with respect to variables $Y_1(k)$ and $Y_2(k)$ defined by the change of variables (5.15) and (5.16) (with a sufficient order of accuracy preserved)

$$\Delta Y_1(k) = \left(\frac{\alpha}{k} + \frac{\alpha(\alpha-1)}{k^2} + O\left(\frac{1}{k^3}\right)\right)(Y_1(k) - Y_2(k)), \tag{5.29}$$

$$\Delta Y_2(k) = -\left(\frac{\alpha+1}{k} + O\left(\frac{1}{k^2}\right)\right)\left(2Y_2(k) + 3Y_2^2(k) + Y_2^3(k) + O\left(\frac{1}{k}\right)\right). \tag{5.30}$$

5.3 Investigation of system (5.29), (5.30)

Analysing the structure of system (5.29), (5.30), we conclude that the second equation of the system depends on variable $Y_2(k)$ and does not depend on variable $Y_1(k)$. This is clear from (5.22) and (5.26). Then, the system (5.29), (5.30) is of a "triangular" type. Therefore, it is possible to consider the second equation (5.30) separately and then continue with the investigation of equation (5.29).

5.3.1 Investigation of equation (5.30)

In this part, we assume $\alpha > -1$ (this assumption is equivalent to the assumption $s > -1$). Consider the second equation (5.30) in the system (5.29), (5.30) separately. That is, we will analyse the equation

$$\Delta Y_2(k) = -\left(\frac{\alpha+1}{k} + O\left(\frac{1}{k^2}\right)\right)\left(2Y_2(k) + 3Y_2^2(k) + Y_2^3(k) + O\left(\frac{1}{k}\right)\right).$$

where $Y_2(k) = O^+(1)$ is assumed. As, after having investigated equation (5.30), we will continue with the investigation of the first equation (5.29), we use following settings, a part of them will be used later.

Set $b_i(k) := -\varepsilon_i$, $c_i(k) := \gamma_i$ where ε_i, γ_i are fixed positive numbers less than 1. Put

$$B_i(k, Y_1, Y_2) := -Y_i - \varepsilon_i, \quad C_i(k, Y_1, Y_2) := Y_i - \gamma_i, \quad i = 1, 2$$

Auxiliary sets Ω_B^2, Ω_C^2 are reduced as follows

$$\Omega_B^2 = \{(k, Y_2) \colon k \in \mathbb{N}(k_0), Y_2 = -\varepsilon_2\},$$
$$\Omega_C^2 = \{(k, Y_2) \colon k \in \mathbb{N}(k_0), Y_2 = \quad \gamma_2\}.$$

We will apply Theorem 1 to equation (5.30). This means that, we need to show that (1.13) and (1.14)) hold for $i = 2$ where

$$F_2(k,Y_1,Y_2)=F_2(k,Y_2)=-\left(\frac{\alpha+1}{k}+O\left(\frac{1}{k^2}\right)\right)\left(2Y_2(k)+3Y_2^2(k)+Y_2^3(k)+O\left(\frac{1}{k}\right)\right). \tag{5.31}$$

The inequality (1.13) now has the following form

$$0=b_2(k+1)-b_2(k)<F_2(k,Y_2)|_{(k,Y_2)\in\Omega_B^2}<\gamma_2+\varepsilon_2. \tag{5.32}$$

The function

$$F_2(k,Y_2)|_{(k,Y_2)\in\Omega_B^2}=F_2(k,-\varepsilon_2)=-\left(\frac{\alpha+1}{k}+O\left(\frac{1}{k^2}\right)\right)\left(-2\varepsilon_2+3\varepsilon_2^2-\varepsilon_2^3+O\left(\frac{1}{k}\right)\right)$$

takes on positive values for all sufficiently large k if

$$2\varepsilon_2-3\varepsilon_2^2+\varepsilon_2^3=\varepsilon_2(\varepsilon_2-1)(\varepsilon_2-2)>0,$$

that is, if $\varepsilon_2\in(0,1)\cup(2,+\infty)$. Because we assume $Y_2(k)=O^+(1)$, only values $\varepsilon_2\in(0,1)$ can be used. Then, the right inequality in (5.32) holds for all sufficiently large k. The left inequality in (5.32) holds as well.

Now, we show that (1.14)) holds for $i=2$. This inequality reduces to

$$-\varepsilon_2-\gamma_2=b_2(k+1)-c_2(k)<F_2(k,Y_2)|_{(k,Y_2)\in\Omega_C^2}<\gamma_2-\gamma_2=0, \tag{5.33}$$

where

$$F_2(k,Y_2)|_{(k,Y_2)\in\Omega_C^2}=F_2(k,\gamma_2)=-\left(\frac{\alpha+1}{k}+O\left(\frac{1}{k^2}\right)\right)\left(2\gamma_2+3\gamma_2^2+\gamma_2^3+O\left(\frac{1}{k}\right)\right).$$

The function $F_2(k,\gamma_2)$ is, for $\gamma_2\in(0,1)$ and for all sufficiently large k negative, so the right inequality in (5.33) holds. The left inequality in (5.33) holds, too, because the function $F_2(k,\gamma_2)$ is vanishing as $k\to\infty$.

Finally, we need to show that the function

$$G_2(w):=w+F_2(k,w) \tag{5.34}$$

is monotone on

$$\overline{\Omega}_2(k)=\{(w)\colon w\in\mathbb{R},b_2(k)\le w\le c_2(k)\}=\{(w)\colon w\in\mathbb{R},-\varepsilon_2\le w\le\gamma_2\}$$

for every fixed $k\in\mathbb{N}(k_0)$. We will verify this monotonicity by computing $G'(w)$. Since, in formula (5.34), the function $F_2(k,w)$ is defined by (5.31), direct computation of the derivative is not possible. The reason is that the variable w is "hidden" in the Lambda order symbol, for which the operation of taking derivative is not defined.

Therefore, we use the original expression for $F_2(k,w)$ given by the formula on the right-hand side of the equation (5.22) where $Y_3(k)$ is expressed by formula (5.26). Then

$$G_2(w) = w + F_2(k,w) = w + (\Delta^2 k^{-\alpha})(\Delta(k+1)^{-\alpha})^{-1}$$
$$\cdot\left(\left(-k^s\frac{\alpha+1}{\alpha^2}(\Delta k^{-\alpha})^3(1+3w+3w^2+w^3) - \Delta^2 k^{-\alpha}\right)\cdot(\Delta^2 k^{-\alpha})^{-1} - w\right)$$

and, therefore,

$$G_2'(w) = 1 + (\Delta^2 k^{-\alpha})\left(\Delta(k+1)^{-\alpha}\right)^{-1}$$
$$\cdot\left(\left(\Delta^2(k^{-\alpha}\right)^{-1}\cdot\left(-k^s\frac{\alpha+1}{\alpha^2}\left(\Delta k^{-\alpha}\right)^3\cdot(3+6w+3w^2)\right) - 1\right)$$
$$= 1 - \left(\Delta(k+1)^{-\alpha}\right)^{-1} k^s\frac{\alpha+1}{\alpha^2}\left(\Delta k^{-\alpha}\right)^3\cdot(3+6w+3w^2) - \left(\Delta^2 k^{-\alpha}\right)\left(\Delta(k+1)^{-\alpha}\right)^{-1}.$$

We calculated (see formulas (5.25), (5.18), (5.24))

$$\left(\Delta^2 k^{-\alpha}\right)\left(\Delta(k+1)^{-\alpha}\right)^{-1} = -\frac{\alpha+1}{k} - \frac{(\alpha+1)(\alpha-1)}{2k^2} + O\left(\frac{1}{k^3}\right),$$

$$\Delta k^{-\alpha} = -\frac{\alpha}{k^{\alpha+1}} + \frac{\alpha(\alpha+1)}{2k^{\alpha+2}} - \frac{\alpha(\alpha+1)(\alpha+2)}{6k^{\alpha+3}} + O\left(\frac{1}{k^{\alpha+4}}\right)$$

and

$$\left(\Delta(k+1)^{-\alpha}\right)^{-1} = -\frac{k^{\alpha+1}}{\alpha}\left[1 + \frac{3(\alpha+1)}{2k} + \frac{(\alpha+1)(13\alpha-1)}{12k^2} + O\left(\frac{1}{k^3}\right)\right].$$

Then

$$G_2'(w) = 1 + \frac{\alpha+1}{k} + \frac{(\alpha+1)(\alpha-1)}{2k^2} + O\left(\frac{1}{k^3}\right)$$
$$+\frac{k^{\alpha+1}}{\alpha}\left(1 + \frac{3(\alpha+1)}{2k} + \frac{(\alpha+1)(13\alpha-1)}{12k^2} + O\left(\frac{1}{k^3}\right)\right)$$
$$\cdot k^s\frac{\alpha+1}{\alpha^2}\left(-\frac{\alpha}{k^{\alpha+1}} + \frac{\alpha(\alpha+1)}{2k^{\alpha+2}} - \frac{\alpha(\alpha+1)(\alpha+2)}{6k^{\alpha+3}} + O\left(\frac{1}{k^{\alpha+4}}\right)\right)^3(3+6w+3w^2)$$
$$= 1 + \frac{\alpha+1}{k} + \frac{(\alpha+1)(\alpha-1)}{2k^2} + O\left(\frac{1}{k^3}\right)$$
$$-\frac{\alpha+1}{\alpha^3}k^{s+\alpha+1}\left[1 + \frac{3(\alpha+1)}{2k} + \frac{(\alpha+1)(13\alpha-1)}{12k^2} + O\left(\frac{1}{k^3}\right)\right]\cdot\frac{\alpha^3}{k^{3\alpha+3}}$$
$$\cdot\left[1 - \frac{\alpha+1}{2k} + \frac{(\alpha+1)(\alpha+2)}{6k^2} + O\left(\frac{1}{k^3}\right)\right]^3(3+6w+3w^2)$$
$$= 1 + \frac{\alpha+1}{k} + \frac{(\alpha+1)(\alpha-1)}{2k^2} + O\left(\frac{1}{k^3}\right)$$
$$-\frac{\alpha+1}{k^{2\alpha-s+2}}\left[1 + \frac{3(\alpha+1)}{2k} + \frac{(\alpha+1)(13\alpha-1)}{12k^2} + O\left(\frac{1}{k^3}\right)\right]$$

$$\cdot \left[1 - \frac{\alpha+1}{2k} + \frac{(\alpha+1)(\alpha+2)}{6k^2} + \frac{(\alpha+1)^2}{4k^2} + O\left(\frac{1}{k^3}\right)\right](3+6w+3w^2).$$

Since, by (5.14), $2\alpha - s + 2 = 1$, we have

$$\begin{aligned} G_2'(w) = 1 &+ \frac{\alpha+1}{k} + \frac{(\alpha+1)(\alpha-1)}{2k^2} + O\left(\frac{1}{k^3}\right) \\ &- \frac{\alpha+1}{k}\left[1 + \frac{3(\alpha+1)}{2k} + \frac{(\alpha+1)(13\alpha-1)}{12k^2} + O\left(\frac{1}{k^3}\right)\right] \\ &\cdot \left[1 - \frac{\alpha+1}{2k} + \frac{(\alpha+1)(5\alpha+7)}{12k^2} + \frac{(\alpha+1)^2}{4k^2} + O\left(\frac{1}{k^3}\right)\right](3+6w+3w^2). \end{aligned}$$

and, for all sufficiently large k, $G_2'(w) \sim 1$. That is, $G_2'(w) > 0$ and G_2 is monotone. Theorem 1 is applicable and, therefore, there exists a solution $Y_2 = Y_2^*(k)$ to equation (5.30) satisfying inequality

$$-\varepsilon_2 < Y_2^*(k) < \gamma_2, \quad k \in \mathbb{N}(k_0) \tag{5.35}$$

where k_0 is sufficiently large and positive numbers ε_2, γ_2, $\varepsilon_2 < 1$, $\gamma_2 < 1$ are fixed. Note that this solution is not trivial as it follows, e.g., from the analysis of relations (5.22), (5.26).

5.3.2 Investigation of equation (5.29)

In this part, we assume $\alpha > 0$ (this assumption is equivalent to assumption $s > 1$). Now we use Theorem 2 to analyse equation (5.29), that is, the equation

$$\Delta Y_1(k) = \left(\frac{\alpha}{k} + \frac{\alpha(\alpha-1)}{k^2} + O\left(\frac{1}{k^3}\right)\right)(Y_1(k) - Y_2(k)). \tag{5.36}$$

In part 5.3.1, we proved that there exists a solution $Y_2 = Y_2^*(k)$ of equation (5.30) with an asymptotic behaviour described by inequality (5.35). Let us assume such a solution in (5.36). Then, we arrive at the equation

$$\Delta Y_1(k) = \left(\frac{\alpha}{k} + \frac{\alpha(\alpha-1)}{k^2} + O\left(\frac{1}{k^3}\right)\right)(Y_1(k) - Y_2^*(k)). \tag{5.37}$$

In Theorem 2, put

$$F_1(k, Y_1, Y_2) = F_1(k, Y_1) = \left(\frac{\alpha}{k} + \frac{\alpha(\alpha-1)}{k^2} + O\left(\frac{1}{k^3}\right)\right)(Y_1(k) - Y_2^*(k))$$

while using the notation defined in part 5.3.1. Auxiliary sets Ω_B^1, Ω_C^1 are reduced as follows

$$\Omega_B^1 = \{(k, Y_2) \colon k \in \mathbb{N}(k_0), Y_1 = -\varepsilon_1\},$$

$$\Omega_C^1 = \{(k, Y_2)\colon k \in \mathbb{N}(k_0), Y_1 = \quad \gamma_1\}.$$

Then, for $k \in \mathbb{N}(k_0)$, inequality (1.15) has the form

$$F_1(k, Y_1)|_{(k,Y_1)\in\Omega_B^1} = F_1(k, -\varepsilon_1) = \left[\frac{\alpha}{k} + \frac{\alpha(\alpha-1)}{k^2} + O\left(\frac{1}{k^3}\right)\right](-\varepsilon_1 - Y_2^*(k)) < \\ < b_1(k+1) - b_1(k) = 0. \quad (5.38)$$

Due to (5.35), we derive

$$\left[\frac{\alpha}{k} + \frac{\alpha(\alpha-1)}{k^2} + O\left(\frac{1}{k^3}\right)\right](-\varepsilon_1 - Y_2^*(k)) < \left[\frac{\alpha}{k} + \frac{\alpha(\alpha-1)}{k^2} + O\left(\frac{1}{k^3}\right)\right](-\varepsilon_1 + \varepsilon_2)$$

and inequality (5.38) will hold if $\varepsilon_2 < \varepsilon_1$. Then, for $k \in \mathbb{N}(k_0)$, inequality (1.16) has the form

$$F_1(k, Y_1)|_{(k,Y_1)\in\Omega_C^1} = F_1(k, \gamma_1) = \left[\frac{\alpha}{k} + O\left(\frac{\alpha-1}{k^2}\right)\right](\gamma_1 - Y_2^*(k)) > c_1(k+1) - c_1(k) = 0. \quad (5.39)$$

Due to (5.35), we derive

$$\left[\frac{\alpha}{k} + O\left(\frac{\alpha-1}{k^2}\right)\right]\cdot(\gamma_1 - Y_2^*(k)) > \left[\frac{\alpha}{k} + O\left(\frac{\alpha-1}{k^2}\right)\right]\cdot(\gamma_1 - \gamma_2))$$

and inequality (5.39) will hold if $\gamma_1 > \gamma_2$. Theorem 2 is applicable. Therefore, there exists a solution $Y_1 = Y_1^*(k)$ to equation (5.37) satisfying inequality

$$-\varepsilon_1 < Y_1^*(k) < \gamma_1, \quad k \in \mathbb{N}(k_0)$$

where k_0 is sufficiently large and numbers $0 < \varepsilon_2 < \varepsilon_1 < 1$, $0 < \gamma_2 < \gamma_1 < 1$ are fixed. Note that this solution is not trivial because $Y_2^*(k)$ is not trivial.

5.3.3 Existence of a bounded solution to system (5.29)–(5.30)

Summarized, the investigations conducted in parts 5.3.1, 5.3.2 in fact prove the following theorem.

Theorem 13. *Let $s > 1$. Let ε_i, γ_i, $i = 1, 2$ be fixed positive numbers such that $\varepsilon_2 < \varepsilon_1 < 1$, $\gamma_2 < \gamma_1 < 1$. Then, there exists a solution $Y(k) = Y^*(k) = (Y_1^*(k), Y_2^*(k))$ to the system* (5.29), (5.30) *such that*

$$-\varepsilon_i < Y_i^*(k) < \gamma_i, \quad i = 1, 2, \quad \forall k \in \mathbb{N}(k_0) \quad (5.40)$$

provided that k_0 is sufficiently large.

5.4 Existence of a nontrivial solution to equation (5.10) with property (5.11)

In this part, we show that Theorem 13 implies the existence of a nontrivial solution to equation (5.10) with property (5.11).

Theorem 14. *Let $s > 1$. Let ε_i, γ_i, $i = 1, 2$ be fixed positive numbers such that $\varepsilon_2 < \varepsilon_1 < 1$, $\gamma_2 < \gamma_1 < 1$. Then, there exists a solution $v = v(k)$ to equation (5.10) such that*

$$-\varepsilon_1 |c| k^{-\alpha} < v(k) - ck^{-\alpha} < \gamma_1 |c| k^{-\alpha},$$

$$-\varepsilon_2 \gamma_2 \Delta(|c| k^{-\alpha})) < \Delta v(k) - (\Delta(ck^{-\alpha})) < \gamma_2 \Delta(|c| k^{-\alpha}))$$

and

$$\Delta^2 v(k) = O(1) \tag{5.41}$$

for all $k \in \mathbb{N}(k_0)$ provided that k_0 is sufficiently large.

Proof. The conclusion of the theorem is a consequence of the transformation formulas (5.15)–(5.17), inequalities (5.40) in Theorem 13 and (in the case of formula (5.41)) formula (5.27). □

6 Conclusion and Comparisons

This doctoral thesis studies the asymptotic behaviour of solutions of a discrete Emden-Fowler equation. Analysis of the results reveals two different types of asymptotic behaviour.

The first one may be termed a power type. The method used consists in the retract principle and we see that the choice of different upper and lower functions provides us with different areas of existence of a power-type asymptotic behaviour.

The second one is an analogy for the blow-up solutions. The method of searching for solutions of this type can be applied to other different non linear discrete equations.

Moreover, the equation

$$\Delta^2 v(k) \pm p k^{\alpha} v^m(k) = 0, \tag{6.1}$$

where p is a positive constant, which is somewhat more general than equation (1.3) (1.3), can obviously be transformed to the form (1.3) by a transformation $v(k) = qu(k)$ where q is a positive number defined as $q = p^{1/(1-m)}$.

We can also extend the results achieved in Chapters 3 and 4, by adding to the equation (1.3) (or (6.1)) a perturbation - function $\omega \colon \mathbb{N}(k_0) \to \mathbb{R}$ assumed to be sufficiently small. Thus, we can study the equation

$$\Delta u(k) \pm k^{\alpha} u^m(k) = \omega(k).$$

Here, "sufficiently small" is understood as:

$$\omega(k) = O\left(\frac{1}{k^{s+4}}\right),$$

where s was defined in (2.1)

From the proofs, we can see that all the calculations can be applied as this "smallness" is hidden in the Landau symbol "big O".

This thesis includes several theorems on the conditions for the existence of solutions to the Emden-Fowler type difference equations with power-type asymptotic behaviour. Each theorem is supplemented with a figure to be more illustrative. Also, examples are given to show applications of the results achieved.

As we have already mentioned in Current State, there are some already existing results on this topic. Thus, it is necessary to relate them to the results of this doctoral dissertation.

First, we refer to the results by L.Erbe, J. Baoguo and J. Peterson [23], where the authors proved that there exists a solution $x(t)$ to equation

$$x^{\Delta\Delta}(t) + p(t)x^m(t) = 0\,, \tag{6.2}$$

such that there exists a nonzero finite limit

$$\lim_{t\to\infty} \frac{x(t)}{t} = A \tag{6.3}$$

provided that

$$\int_{t_0}^{\infty} t^m |p(t)| \Delta t < \infty \,, \tag{6.4}$$

where the integral is understood on a given time scale.

This is the result of time scales calculus where the concept of derivative on time scale is defined as follows. The function

$$\sigma(t) = \inf\{s \; in \mathbb{T} \colon s > 6\},$$

where $\mathbb{T}$ is a time scale (i.e., a closed nonempty subset of $\mathbb{R}$) is called a forward jump operator and the function

$$\rho(t) = \sup\{s \; in \mathbb{T} : s < t\}$$

is called a backward jump operator. Define

$$\mathbb{T}^k = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}] \;\text{ if }\; \sup \mathbb{T} < \infty, \\ \mathbb{T} \;\text{ if }\; \sup \mathbb{T} = \infty. \end{cases}$$

The function $x\colon \mathbb{T} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{T}^k$ provided that the limit

$$x^{\Delta}(t) := \lim_{s\to t} \frac{x(t) - x(s)}{t - s}$$

exists if $\sigma(t) = t$ and x is continuous at t, $x^{\Delta}(t)$ is called the delta derivative. If $\sigma(t) > t$, we put

$$x^{\Delta}(t) := \frac{x(\sigma(t)) - x(t)}{\sigma(t) - t} \,.$$

This investigation is close to our topic because the definition of the first difference is a special case of a time scale delta derivative. The main distinction is that we investigate the asymptotic properties of the solutions without assuming the integral (6.4) being convergent.

Reformulating the result of [23] in terms of a difference equation, we see that the time-scales integral becomes an infinite sum and we have the following lemma.

Lemma 4. *Let $m > 0$ and*

$$\sum_{k=1}^{\infty} k^m |p(k)| < \infty.$$

Then, equation

$$\Delta^2 x(k) + p(k) x^m(k) = 0$$

has a solution $x = x(k)$ such that

$$\lim_{k\to\infty} x(k)/k = A \neq 0.$$

Now we can show that even a weaker Theorem 4 has examples that do not work with the result in [23].

Example 18. Let us consider equation (see Example 1)

$$\Delta^2 u(k) - k^{-1}u^2(k) = 0$$

where $m = 2$.

This equation has a solution described by the asymptotic formula

$$u(k) = \frac{2}{k} + O\left(\frac{1}{k^2}\right).$$

In this case, the left-hand side of condition (6.4) with $p(k) = k^{-1}$ is

$$\sum_{k=1}^{\infty} k^2 \cdot k^{-1} = \sum_{k=1}^{\infty} k = \infty.$$

Therefore, (6.4) does not hold and the result of [23] is not applicable.

Example 19. Let us consider equation (see Example 2)

$$\Delta^2 u(k) + k^{-7/4}u^{1/2}(k) = 0$$

where $m = 1/2$. This equation has a solution described by the asymptotic formula

$$u(k) = 16 \cdot k^{1/2} + O\left(\frac{1}{k^{1/2}}\right). \tag{6.5}$$

In this case, the condition (6.4) with $p(k) = k^{-7/4}$ holds because

$$\sum_{k=1}^{\infty} k^{-7/4} \cdot k^{1/2} = \sum_{k=1}^{\infty} k^{-5/4} < \infty$$

but formulas (6.3) and (6.5) describing the asymptotic behaviour of a solution are different. It means that solutions described by these formulas are different.

Next, we refer to the results by V.Kharkov, where in[30] asymptotic representations of so-called $P(\lambda)$-solutions of equation

$$\Delta^2 y_n = \alpha p_n |y_n|^\sigma \operatorname{sign} y_n$$

are considered, where $\alpha \in \{\pm 1\}$, $\sigma \in \mathbb{R} \setminus \{0, 1\}$ and $\{p_n\}$ is a positive sequence. The results are applied to equation

$$\Delta^2 y_n = \alpha n^k |y_n|^\sigma \operatorname{sign} y_n$$

and, among others, the condition

$$\alpha(k+2)(k+\sigma+1) > 0 \tag{6.6}$$

must be fulfilled. Adapting notation from in [30], we state that (despite equations considered being not equivalent), e.g., for equation (15), where the upper sign variant $+$ is considered, we have $\alpha = -1$, $k = -27/20$, $\sigma = 1/2$ and inequality (6.6) does not hold. Therefore, the results are independent.

Although close in terms of their topics, different equations or asymptotic problems are studied in the following papers [40, 11, 31].

In [40] the author studies a difference equation of Emden–Fowler type

$$\Delta^m x_n = a_n f(x_{\sigma(n)}) + b_n$$

and, assuming f to be a power type function and knowing that $\Delta^m y_n = b_n$, sufficient conditions are given guaranteeing the existence of a solution x such that $x_n = y_n + o(n^s)$, where $s < 0$.

The paper [11] considers a class of equations of Emden-Fowler type

$$\Delta(a_n|\Delta x_n|^\alpha \operatorname{sign} \Delta x_n) + b_n|\Delta x_{n+1}|^\beta \operatorname{sgn} \Delta x_{n+1} = 0$$

where $\alpha > 0$, $\beta > 0$, $\{a_n\}$ and $\{b_n\}$ are positive sequences. Among others the existence of nonoscillatory solutions is studied.

A full classification of positive solutions of equation

$$\Delta^2 y_n = \alpha p_n |y_{n+1}|^\sigma \operatorname{sign} y_{n+1}, \tag{6.7}$$

where $\alpha \in \{\pm 1\}$, $\sigma \in \mathbb{R} \setminus \{0, 1\}$ and $\lim_{n\to\infty}(n\Delta p_n)/p_n = k \in \mathbb{R} \setminus \{-2, -1-\sigma\}$, can be found in [31]. In this paper, unlike the "direct" discretization

$$x \sim k, \ \ y(x) \sim u(k), \ \ y''(x) \sim \Delta^2 u(k)$$

a different one is used. Therefore, the classes of the considered equations (1.3) and (6.7) (regardless of a different coefficient in (6.7)) are different.

We finish our comparisons with referring to books [1, 2, 7, 21, 26, 42, 41] where a variety of results can be found on asymptotic behaviour of solutions of some classes of difference equations.

Let us formulate some open problems related to the topic of the thesis.

Open problem 1. *A discrete analogy to the blow-up solutions was discussed in Chapter 5, where a discrete analogue of differential equation* (5.1) *of Emden-Fowler type was investigated. We expect that, for nonlinear solutions that are more general than* (5.1), *it will be possible to prove the existence of blow-up solutions as well. It seems that the time-scale calculus would be an apparatus more suitable for investigating solutions rapidly tending to infinity near a fixed finite point by the methods used in the thesis. A time-scale* $\mathbb{T}$ *suitable for this case can be, e.g., the set*

$$\mathbb{T}(x_0) = \left\{ x_0 - \frac{1}{x}, x \in \mathbb{N} \right\},$$

where $x_0 \in \mathbb{R}$. *Then, sufficient conditions would be given for the existence of a solution blowing-up at the point* x_0.

Open problem 2. *The clarification of the asymptotic behaviour of solutions to equation* (1.3) *if the definition of the first-order forward difference* (1.4) *is changed to*

$$\Delta u(k) = \frac{u(k+h) - u(k)}{h},$$

where h *is a positive number and is used to discretize equation* (1.5). *Then, the expected results could coincide, if* $h \to 0$, *with the results known for differential Emden-Fowler equation* (1.5). *We also expect that, in the event of* $h \to 0$, *the domain of the respective points* (m, α) *will expand to the whole plain* $\mathbb{R}^2$.

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List of symbols

$\mathbb{N}$	the set of natural numbers $\{1, 2, \dots\}$
$\mathbb{N}(k_0)$	the set of integers $\{k_0, k_0 + 1, \dots\}$
$\mathbb{Z}$	the set of integers
$\mathbb{R}$	the set of real numbers
$\mathbb{T}$	a time scale set
O	Landau symbol big "O"
o	Landau symbol little "o"
$\Delta u(k)$	the first-order forward difference of the function $u(k)$
$\Delta^2 u(k)$	the second-order forward difference of the function $u(k)$
$x^{\Delta}(t)$	the time scale delta derivative of the function $x(t)$