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CONSTRUCTION OF THE OPTIMAL CONTROL STRATEGY FOR AN ELECTRIC-POWERED TRAIN

KONSTRUKCE OPTIMÁLNÍ STRATEGIE ŘÍZENÍ ELEKTRICKÉHO VLAKU

DISERTAČNÍ PRÁCE
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Abstrakt
Předložená disertační práce se zabývá popisem charakteru optimální strategie řízení pro elektrický vlak a výpočtem přepínacích okamžiků mezi jednotlivými optimálními jízdními režimy pro standardní typy odporové funkce. S využitím Pontrjaginova principu a souvisejících nástrojů teorie optimálního řízení odvodíme optimální strategii řízení a rovnice pro výpočet přepínacích okamžiků včetně odpovídajících rychlostních profilů. Kromě základního tvaru úlohy o energeticky optimální jízdě vlaku budeme uvažovat i její modifikace zahrnující globální rychlostní omezení, sklon trati i časově-energeticky optimální řízení vlaku. Navíc uvedeme i analýzu řešení s využitím teorie nelineární parametrické optimalizace. Důraz je kladen na exaktní tvar řešení s minimálním využitím numerických metod.

Summary
This thesis deals with the description of the nature of optimal driving strategy for an electric-powered train as well as the calculation of switching times of optimal driving regimes for standard types of resistance function. We apply the Pontryagin principle and related tools of optimal control theory to develop the optimal driving strategy and to derive equations for computation of switching times and the corresponding speed profiles. Besides the basic form of the energy efficient train control problem we consider also its modifications including the global speed constraint, track gradient as well as time-energy efficient train control. Moreover, we analyse also the solution with use of the theory of nonlinear parametric programming. The emphasize is put on exact forms of solutions with a minimal use of numerical methods.

Klíčová slova
Časově-energeticky optimální řízení vlaku, Optimální strategie řízení, Pontrjaginův princip, Kritický parametr, Nelineární parametrická optimalizace, Přepínací okamžiky, Odporová funkce, Rychlostní profil

Keywords
Time-energy efficient train control, Optimal driving strategy, Pontryagin principle, Critical parameter, Non-linear parametric optimization, Switching times, Resistance function, Speed profile

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Content

1. Introduction

The basic problem of the energy efficient train control was formulated and solved in some particular cases by Horn [8] in 1971 with use of the general form of the Pontryagin principle and relating mathematical tools. Since then, it has become a typical problem that can be solved with use of these means.

Many articles discussing this topic appeared especially during the nineties. The type of the optimal strategy consisting of four successive control levels (full power, speed holding, coasting and full braking) was introduced by Howlett et al. [13, 14]. Among articles dealing with various modifications of the basic problem we recall that Pudney et al. [26] considered a vehicle with discrete control settings and speed limits. Howlett et al. [11] discussed a track with non-zero gradient. Both of these assumptions were assumed by Cheng et al. [6] and Khmelnitsky [15]. Howlett and Pudney [12] summarized the above mentioned results. This theoretical background enabled the development of on-board computational systems (such as Metromiser or Freightmiser) for calculating of the efficient driving strategy which were successfully implemented in timetabled suburban and long-haul trains, e.g. in Brisbane or Toronto (see Yee et al. [27]). Let us note that some alternative approaches to these and relating problems were discussed e.g. by Han et al. [7], Howlett et al. [9, 10], Ko et al. [16], Li et al. [18], Liu et al. [19] and Pickhardt [20].

This thesis deals with the energy efficient train control problem and its modifications and introduces a different approach to developing the optimal control strategy along with exact calculation of the switching times and analysis of the solution based on the mathematical tools of nonlinear parametric programming. Under assumption of most common and typically used resistance functions and with use of the properties of the Hamilton function and Lagrange multipliers we derive algebraic equations for computation of the switching times for both feasible control strategies. Thereafter, we introduce the notion of the critical time (or critical parameter in case of the time-energy efficient train control problem) and explain its significance for determination of optimal control strategy.

The modifications of the basic energy optimal control problem discussed in this thesis represent natural enhancement of the basic problem and yield significant results leading to more realistic behaviour of the mathematical model. Assumption of the speed constraints and track gradient broadens the applicability of the model. The time-energy efficient train control enables a suitable combination of the energy optimal drive with respect to time requirements and thus can be used to develop effective timetables.

Let us emphasize that the results presented in previous papers were more or less based on use of numerical methods for solving optimal control problems and thus the analysis of solution in the way introduced in this thesis could not have been performed. Most of the results presented in this thesis were introduced in the papers [21]–[24]. All results are illustrated with use of sample speed profiles.

Let us shortly mention the structure of the thesis. This introduction chapter is followed by the overview of the basic theoretical results that we shall use to derive solution of the later introduced optimal control problems and their analysis. There are basic concepts and theorems of the optimal control theory including constraints on state variables. We recall an interesting area of nonlinear parametric programming as well as general notions relating to controllability and reachability of controlled systems necessary for analysis of the feasible solutions of optimal control problems. The following Chapter 3 describes the optimal control strategy for the energy efficient train control problem. This chapter as well as the next three chapters dealing with various modifications of the basic problem are organized as follows. First, we state the formulation of the problem including its interpretation. Thereafter, we introduce the general optimal control strategy. The next section describes the calculation of the switching times. Then we analyse the solution with use of the nonlinear parametric programming. In the Chapter 4 we discuss the energy-efficient train control problem with speed constraints. The following Chapter 5 deals with the time-energy efficient train control problem. The Chapter 6 is devoted to the problem of energy optimal train control under additional assumption of track gradient (downhill or uphill drive). The last chapter summarizes the obtained results and introduces directions for future investigations.

2. Some preliminaries

2.1. Optimal control theory and Pontryagin principle

This section introduces some essential concepts and assertions of optimal control theory, especially the Pontryagin maximum principle, that we shall use in the following chapters of this thesis to derive the optimal control strategy for the later specified train control problems. Most of the theorems and notions from this section as well as the corresponding proofs can be found in Cermak [5] and Pontrjagin et al. [25].

We are going to investigate the behaviour of an object whose specific state can be described for a fixed point in time with use of n real numbers $x_1, x_2, \dots, x_n$. Let us introduce a vector space X of the variable $\mathbf{x} = (x_1, \dots, x_n)$ as the state space of the investigated object. The *motion* of the object denotes the change of the variables $x_1, \dots, x_n$ in time t. In order to introduce the optimal control problem let us assume that the motion of the object can be *controlled* by setting of certain parameters, that can vary in specific boundaries throughout the course of time, i.e. by use of control. Let us further assume, that this control can be specified by r real numbers $u_1, \dots, u_r$ depending on time. We shall assume that their values belong to the set $U \subset E_r$, where E_r denotes an r-dimensional Euclidian space. The set U will be referred to as the *control space*.

Definition 1. (Control) *Vector variable $\mathbf{u}(t) = (u_1(t), \dots, u_r(t))$, defined on a certain interval $\langle 0, T\rangle$ and with values in the control space $U \subset E_r$, will be denoted as the control.*

Further, we will asume that the behaviour of the given object can be specified by the following system of differential equations which can be rewritten in the vector form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}). \tag{2.1}$$

The dot in the previous equation represents a time derivation, $\mathbf{f}(\mathbf{x}, \mathbf{u})$ is a vector consisting of elements $f_1(\mathbf{x}, \mathbf{u}), f_2(\mathbf{x}, \mathbf{u}), \dots, f_n(\mathbf{x}, \mathbf{u})$, where $\mathbf{x} = (x_1, x_2, \dots, x_n) \in X$ and $\mathbf{u} = (u_1, \dots, u_r) \in U$. The system (2.1) will be denoted as the *controlled system*. Let us further assume that the functions $f_1, \dots, f_n$ are defined and continuous in all variables and continuously differentiable in $x_1, x_2, \dots, x_n$. We shall restrict our further considerations only on the following set of controls:

Definition 2. (Feasible controls) *The set of controls $\mathbf{u}(t)$ will be called the set of feasible controls, if all its elements are piecewise continuous functions defined on the interval $\langle 0, T\rangle$.*

Definition 3. *Let $\mathbf{a}, \mathbf{b} \in X$. We say that a feasible control $\mathbf{u}(t), t \in \langle 0, T\rangle$ transfers the point from a location $\mathbf{a}$ to a location $\mathbf{b}$, if the corresponding solution $\mathbf{x}(t)$ of the equation*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}(t)) \tag{2.2}$$

satisfying the initial condition

$$\mathbf{x}(0) = \mathbf{a}, \qquad \mathbf{a} \in X \tag{2.3}$$

is defined on the interval $\langle 0, T\rangle$ and in the time T it crosses the point $\mathbf{b}$, i.e. it satisfies the condition

$$\mathbf{x}(T) = \mathbf{b}, \qquad \mathbf{b} \in X.$$

The pair $(\mathbf{x}(t); \mathbf{u}(t))$, $t \in \langle 0, T\rangle$ *will be denoted as the* controlled process *transferring the point from location* $\mathbf{a}$ *to* $\mathbf{b}$.

Basic optimal control problem. Let us choose points $\mathbf{a}$, $\mathbf{b}$ in the state space X. Among all feasible controls $\mathbf{u}(t)$, transferring the point from location $\mathbf{a}$ to location $\mathbf{b}$, we ought to find such a control $\hat{\mathbf{u}}(t)$ that the value of the functional

$$J = J(T, \mathbf{u}) = \int_0^T f_0(\mathbf{x}(t), \mathbf{u}(t))\,\mathrm{d}t$$

is minimized. Here $\mathbf{x}(t)$ denotes the solution of the Equation (2.2) with the initial condition (2.3), which corresponds to the control $\mathbf{u}(t)$ and in time T this solution passes through the point $\mathbf{b}$. Simply, it could be expressed in the following form

$$J = \int_0^T f_0(\mathbf{x}(t), \mathbf{u}(t))\,\mathrm{d}t \;\rightarrow\; \min \tag{2.4}$$

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \tag{2.5}$$

$$\mathbf{x}(0) = \mathbf{a}, \qquad \mathbf{x}(T) = \mathbf{b}, \qquad \mathbf{a}, \mathbf{b} \in X, \qquad \mathbf{u} \in U. \tag{2.6}$$

Definition 4. *The control* $\hat{\mathbf{u}}(t)$, $t \in \left\langle 0, \hat{T}\right\rangle$, *which is a solution of the problem (2.4)–(2.6), is called optimal control and the corresponding trajectory* $\hat{\mathbf{x}}(t)$ *is denoted as optimal trajectory. The pair* $(\hat{\mathbf{x}}(t); \hat{\mathbf{u}}(t))$, $t \in \left\langle 0, \hat{T}\right\rangle$ *will be called the optimal control process.*

The maximum principle. For an easier formulation of this theorem we shall enhance the system (2.1) with the equation

$$\dot{x}_0 = f_0(\mathbf{x}, \mathbf{u}),$$

where the function f_0 was introduced in the definition of the functional J. Thus, we obtain the following enhanced control system of equations

$$\dot{x}_i = f_i(x_1, x_2, \ldots, x_n, \mathbf{u}), \qquad i = 0, 1, \ldots, n. \tag{2.7}$$

Further, we will investigate the system of equations for adjoint variables in the form

$$\dot{\lambda}_i = -\sum_{k=0}^{n} \frac{\partial f_k}{\partial x_i}(\mathbf{x}, \mathbf{u})\lambda_k, \qquad i = 0, 1, \ldots, n. \tag{2.8}$$

Let us denote $\lambda^* = (\lambda_0, \lambda_1, \ldots, \lambda_n)$, $\mathbf{f}^*(\mathbf{x}, \mathbf{u}) = (f_0(\mathbf{x}, \mathbf{u}), f_1(\mathbf{x}, \mathbf{u}), \ldots, f_n(\mathbf{x}, \mathbf{u}))$ and further we introduce the *Hamilton function* H in variables $x_1, x_2, \ldots, x_n, u_1, \ldots, u_r, \lambda_0, \lambda_1, \ldots, \lambda_n$ in the form (by symbol $(.,.)$ we shall denote the scalar product)

$$H = H(\lambda^*, \mathbf{x}, \mathbf{u}) = (\lambda^*, \mathbf{f}^*(\mathbf{x}, \mathbf{u})) = \sum_{k=0}^{n} \lambda_k f_k(\mathbf{x}, \mathbf{u}). \tag{2.9}$$

With use of this function we can easily rewrite the systems (2.7) and (2.8) into the form of the *Hamilton system*

$$\dot{x}_i = \frac{\partial H}{\partial \lambda_i}, \qquad i = 0, 1, \ldots, n,$$

$$\dot{\lambda}_i = -\frac{\partial H}{\partial x_i}, \qquad i = 0, 1, \ldots, n.$$

Theorem 1. (The maximum principle) *Let us choose points* $\mathbf{a}, \mathbf{b}$ *in the state space* X*. Let* $(\hat{\mathbf{x}}(t); \hat{\mathbf{u}}(t))$*,* $t \in \left\langle 0, \hat{T} \right\rangle$ *be the optimal control process transferring the point from location* $\mathbf{a}$ *to location* $\mathbf{b}$*. Then, there exists a continuous non-zero solution* $\lambda^*(t) = (\lambda_0(t), \lambda_1(t), \ldots, \lambda_n(t))$ *on the interval* $\left\langle 0, \hat{T} \right\rangle$ *of the system*

$$\dot{\lambda}_i = -\frac{\partial H}{\partial x_i}(\lambda^*, \hat{\mathbf{x}}, \hat{\mathbf{u}}), \qquad i = 0, 1, \ldots, n \tag{2.10}$$

such that the Hamilton function H *satisfies for all* $t \in \left\langle 0, \hat{T} \right\rangle$ *the maximum condition*

$$H(\lambda^*(t), \hat{\mathbf{x}}(t), \hat{\mathbf{u}}(t)) = \max_{\mathbf{u} \in U} H(\lambda^*(t), \hat{\mathbf{x}}(t), \mathbf{u}). \tag{2.11}$$

Moreover, $H(\lambda^*(t), \hat{\mathbf{x}}(t), \hat{\mathbf{u}}(t)) \equiv 0$ *and* $\lambda_0(t)$ *is nonpositive and constant on* $\left\langle 0, \hat{T} \right\rangle$*.*

Remark. *The maximum principle represents a necessary condition for existence of the optimal control process, not a sufficient one. It is a sufficient condition e.g. in case of time optimization of linear control systems.*

In case of a fixed value of the time T for the transfer of the point from location $\mathbf{a}$ to location $\mathbf{b}$, we obtain the following formulation of the maximum principle:

Theorem 2. (The maximum principle for the problem with a fixed time) *Let us choose points* $\mathbf{a}, \mathbf{b}$ *in the state space* X*. Let* $(\hat{\mathbf{x}}(t); \hat{\mathbf{u}}(t))$*,* $t \in \langle 0, T \rangle$ *be the optimal control process transferring the point from location* $\mathbf{a}$ *to location* $\mathbf{b}$ *(for a fixed time* T*). Then, there exists a continuous non-zero solution* $\lambda^*(t) = (\lambda_0(t), \lambda_1(t), \ldots, \lambda_n(t))$ *on the interval* $\langle 0, T \rangle$ *of the system (2.10) such that the Hamilton function* H *(see (2.9)) satisfies for all* $t \in \langle 0, T \rangle$ *the maximum condition (2.11). Moreover,* $H(\lambda^*(t), \hat{\mathbf{x}}(t), \hat{\mathbf{u}}(t)) \equiv$ *const. and* $\lambda_0(t)$ *is nonpositive and constant on* $\langle 0, T \rangle$*.*

Definition 5. (Singular control) *Let* $\bar{\mathbf{u}}(t)$*,* $t \in \left\langle 0, \bar{T} \right\rangle$ *be an extremal control, i.e. a control, which satisfies for all* $t \in \left\langle 0, \bar{T} \right\rangle$ *the maximum condition*

$$H(\lambda^*(t), \bar{\mathbf{x}}(t), \bar{\mathbf{u}}(t)) = \max_{\mathbf{u} \in U} H(\lambda^*(t), \bar{\mathbf{x}}(t), \mathbf{u}).$$

Let there exist a nontrivial interval I *and a set* $\omega(t) \subseteq U$ *which for every* $t \in I$ *consists of at least two elements and the following condition is satisfied*

$$H(\lambda^*(t), \bar{\mathbf{x}}(t), \bar{\mathbf{u}}(t)) = H(\lambda^*(t), \bar{\mathbf{x}}(t), \mathbf{u})$$

for every $\mathbf{u} \in \omega(t)$*,* $t \in I$*. Then the control* $\bar{\mathbf{u}}(t)$ *is called the singular control on* I *and the interval* I *is denoted as singular interval. We can determine the values of the singular control on the interval* I *with use of derivations of the Hamilton function, if they exist.*

2.2. A problem with constrained state variables

In this section we shall introduce the main concepts and theorems relating to the solution of the optimal control problems with constrained state variables. The following notions can be found mainly in Bryson et al. [4]. A more general attitude to this area is mentioned in Pontrjagin et al. [25].

Let us enhance the problem (2.4)–(2.6) with the constraint on the state variable of the inequality type

$$S(\mathbf{x}, t) \leq 0\,, \tag{2.12}$$

where S is a scalar function. The Hamilton function is defined by the relation:

$$H = (\lambda^*, \mathbf{f}^*) + \mu S^{(q)},$$

where $S^{(q)}$ is obtained by derivation of the function S given by (2.12) according to time t until the resulting function is explicitly dependent on variable $\mathbf{u}$ ($\dot{\mathbf{x}}$ is substituted by $\mathbf{f}(\mathbf{x}, \mathbf{u}, t)$ where necessary). If the control variable $\mathbf{u}$ appears explicitly in the q-th derivation of S according to time, we shall denote the constraint as of the *q-th order*. Then $S^{(q)}$ denotes the corresponding q-th derivation. The function $\mu(t)$ is an additional Lagrange multiplier. Further, **on the constraint boundary** it holds:

$$S^{(q)} = 0, \qquad S = 0, \qquad \mu(t) \leq 0.$$

Off the constraint boundary the following relations hold:

$$S < 0, \qquad \mu = 0.$$

The adjoint system of equations is in the form

$$\dot{\lambda}^{*T} = -H_x \equiv \begin{cases} -(\lambda^*, \mathbf{f}_x^*) - \mu S_x^{(q)}, & S^{(q)} = 0 \\ -(\lambda^*, \mathbf{f}_x^*), & S^{(q)} < 0. \end{cases}$$

The necessary optimality condition is given by (2.11). Within the entry on the boundary constraint there have to be fulfilled the so-called *tangency conditions*

$$N(\mathbf{x}, t) \stackrel{\text{def}}{=} \begin{bmatrix} S(\mathbf{x}, t) \\ S^{(1)}(\mathbf{x}, t) \\ \vdots \\ S^{(q-1)}(\mathbf{x}, t) \end{bmatrix} = 0.$$

Let us denote the entry point in time onto the constraint boundary as t_1 and the exit point in time as t_2. Let us choose the time t_1 as the point where the Lagrange multipliers and the Hamilton function do not have to be continuous, whereas in time t_2 the continuity must be fulfilled. Let us further denote with symbol t_1^- the corresponding left-sided limit of time t_1 and with t_1^+ the right-sided limit of t_1. Then, the discontinuity of the Lagrange multipliers λ^* and the Hamilton function H can be expressed with use of the following relations

$$\lambda^{*T}(t_1^-) = \lambda^{*T}(t_1^+) + \pi^T \frac{\partial N}{\partial \mathbf{x}(t_1)},$$

$$H(t_1^-) = H(t_1^+) - \pi^T \frac{\partial N}{\partial t_1},$$

where π is a q-dimensional vector of constant multipliers. These relations are called the *jump conditions*. Let us note that the state variables $\mathbf{x}$ are continuous in time t_1, i.e. it holds $\mathbf{x}(t_1^-) = \mathbf{x}(t_1^+)$.

2.3. Nonlinear parametric optimization

The last section of this chapter deals with the concepts and theorems from the area of nonlinear parametric optimization, which enable us later to calculate a certain critical parameter. They are mainly related to continuity of the elements of solution to the nonlinear optimization problem with a parameter. Most of this theoretical results with corresponding proofs can be found in Bank [1].

In this section we assume the following nonlinear parametric optimization problem:

$$\min \{f(\mathbf{x},\lambda) \mid \mathbf{x} \in M(\lambda)\}\,, \quad \lambda \in \Lambda\,, \tag{2.13}$$

where $M(\lambda) \subset X$, X and Λ are metric spaces and f is a function mapping $X \times \Lambda$ into $\mathbf{R} \cup \{+\infty, -\infty\}$. Let us further denote

$$\varphi : \lambda \to \varphi(\lambda) := \inf_{\mathbf{x}\in M(\lambda)} f(\mathbf{x},\lambda)$$

the function corresponding to the optimal value of the cost functional f relating to problem (2.13) depending on the vector of parameters λ. Let us further denote

$$\psi : \lambda \to \psi(\lambda) := \{\mathbf{x} \in M(\lambda) \mid f(\mathbf{x},\lambda) = \varphi(\lambda)\}$$

the mapping which assigns to every vector of parameters λ a set of all optimal solutions $\mathbf{x} \in X$ of the problem (2.13).

Definition 6. *Let (X, d_X) and (Λ, d_Λ) be metric spaces. The point-to-set mapping $\Gamma : \Lambda \to 2^X$ is a function mapping each $\lambda \in \Lambda$ into a (possibly empty) subset $\Gamma(\lambda)$ of X.*

Remark. *As it is customary, for a subset A of the metric space X and for arbitrary $\epsilon > 0$ the ϵ-neighbourhood of the set A is the set*

$$U_\epsilon A := \{\mathbf{x} \in X \mid d_X(\mathbf{x}, A) < \epsilon\}\,, \quad \textit{where} \quad d_X(\mathbf{x}, A) = \inf_{\mathbf{y}\in A} d_X(\mathbf{x},\mathbf{y})$$

and d_X denotes the corresponding metric. If A is an empty set, then $d_X(\mathbf{x}, A)$ is by definition equal to $+\infty$. To avoid misunderstanding, we denote by the symbol $V_\epsilon B$ the ϵ-neighbourhood of the set $B \subset \Lambda$. We shall further assume the euclidean metric.

Definition 7. *A point-to-set mapping $\Gamma : \Lambda \to 2^X$ is said to be*

1. *closed at a point λ^0 if for each pair of sequences $\{\lambda^t\} \subset \Lambda$ and $\{\mathbf{x}^t\} \subset X$, $t = 1, 2, \ldots$ with the properties*
$$\lambda^t \to \lambda^0, \quad \mathbf{x}^t \in \Gamma\lambda^t, \quad \mathbf{x}^t \to \mathbf{x}^0$$
it follows that $\mathbf{x}^0 \in \Gamma\lambda^0$;

2. *upper semicontinuous (according to Berge or, simply, B) at a point λ^0 if for each open set Ω containing $\Gamma\lambda^0$ there exists a $\delta = \delta(\Omega) > 0$ such that $\Gamma\lambda \subset \Omega$ for every $\lambda \in V_\delta\{\lambda^0\}$;*

3. *lower semicontinuous (according to Berge or, simply, B) at a point λ^0 if for each open set Ω satisfying $\Omega \cap \Gamma\lambda^0 \neq \emptyset$ there exists a $\delta = \delta(\Omega) > 0$ such that $\Omega \cap \Gamma\lambda \neq \emptyset$ for every $\lambda \in V_\delta\{\lambda^0\}$;*

4. *upper semicontinuous (according to Hausdorff or, simply, H) at a point* λ^0 *if for each* $\epsilon > 0$ *there exits a* $\delta > 0$ *such that* $\Gamma\lambda \subset U_\epsilon \Gamma\lambda^0$ *for every* $\lambda \in V_\delta \{\lambda^0\}$*;*

5. *lower semicontinuous (according to Hausdorff or, simply, H) at a point* λ^0 *if for each* $\epsilon > 0$ *there exits a* $\delta > 0$ *such that* $\Gamma\lambda^{\mathbf{0}} \subset U_\epsilon \Gamma\lambda$ *for every* $\lambda \in V_\delta \{\lambda^0\}$*.*

Remark. *We use, according to Bank [1], the following abbreviations: u.s.c.-B for upper semicontinuous (B) mapping, l.s.c.-B for lower semicontinuous (B) mapping and by analogy u.s.c.-H, l.s.c.-H.*

Remark. *The following relations hold (see Bank [1]):*

$$\textit{u.s.c.-B} \Rightarrow \textit{u.s.c.-H}, \quad \textit{l.s.c.-H} \Rightarrow \textit{l.s.c.-B}.$$

Definition 8. *A point-to-set mapping* $\Gamma : \Lambda \to 2^X$ *is continuous at* λ^0 *if it is u.s.c.-H and l.s.c.-B at* λ^0*.*

Lemma 1. *If the mapping* Γ *is u.s.c.-H at* λ^0 *and if the set* $\Gamma\lambda^0$ *is closed, then the mapping* Γ *is closed at* λ^0*.*

Let us further assume the problem (2.13) again. The following theorems describe the continuity properties of the mappings which determine the optimal solution of the problem.

Theorem 3. *Let* M *be a closed mapping at* λ^0*,* $M(\lambda^0)$ *be a non-empty set,* f *be a continuous function and the metric space* X *be compact. Then* φ *is lower semicontinuous at* λ^0*;* φ *is also upper semicontinuous at* λ^0 *if and only if the mapping* ψ *is u.s.c.-B at* λ^0*.*

Theorem 4. φ *is upper semicontinuous at* λ^0 *if* M *is l.s.c.-B at* λ^0 *and* f *is upper semicontinuous on* $M(\lambda^0) \times \{\lambda^0\}$*.*

2.4. A remark on controlled systems

For the sake of completeness we mention also some basic notions and theorems relating to controllability and reachability of controlled systems. The Pontryagin principle itself does not deal with the existence of the optimal control. Therefore, it is convenient to deal with the question whether there exists a feasible control which transfers the system from an initial state to a target point because a solution which satisfies the maximum principle does not have to be necessarily optimal. We recall that the Pontryagin principle is only a necessary condition. For more detailed explanation of the following concepts and exact proofs of the theorems see e.g. Brunovsky [3].

Let us assume the controlled system (2.1) again.

Definition 9. *A state* $\mathbf{x}_1$ *is called* reachable *if there exists a control* $\mathbf{u} \in U$ *that transfers the state of the system from the initial state* $\mathbf{x}_0$ *to* $\mathbf{x}_1$ *in some finite time* T*.*

Definition 10. *The set* $\Omega(T, \mathbf{x}_0) = \{\mathbf{x}(T, \mathbf{x}_0, \mathbf{u}) \,|\, \mathbf{u} \in U\}$*, i.e. the set of all points* $\mathbf{x}$ *that the initial point* $\mathbf{x}_0$ *can be transferred to in time* T*, is called the* reachability region *(from point* $\mathbf{x}_0$ *in time* T*).*

Definition 11. *The system (2.1) is* reachable *at time T if every state $\mathbf{x}_1$ in the state space X is reachable at time T from the initial point $\mathbf{x}_0$.*

Let us denote

$$\Omega^+(\mathbf{x}_0) = \bigcup_{T \geq 0} \Omega(T, \mathbf{x}_0)$$

the set of all $\mathbf{x} \in X$ that the initial state $\mathbf{x}_0$ can be transferred to with use of a feasible control $\mathbf{u}$.

Definition 12. *The system (2.1) is called*

- locally controllable *from $\mathbf{x}_0$ if $\Omega^+(\mathbf{x}_0)$ contains the surrounding of the point $\mathbf{x}_0$,*
- controllable *from $\mathbf{x}_0$ if $\Omega^+(\mathbf{x}_0) = X$,*
- completely controllable *if $\Omega^+(\mathbf{x}_0) = X$ for every $\mathbf{x}_0 \in X$.*

Let us further assume the linear controlled system in the form

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}\,, \tag{2.14}$$

where A and B are constant matrices and $U = \mathbf{R}^r$.

Theorem 5. *The system (2.14) is completely controllable if and only if the rank of the matrix $(B, AB, \ldots, A^{n-1}B)$ is n, i.e. if we can choose n linearly independent columns of the matrix $(B, AB, \ldots, A^{n-1}B)$.*

3. Energy efficient train control

This chapter deals with the basic energy efficient train control problem introduced by Horn [8]. This problem has been solved mainly with use of numerical methods. We describe an analytical approach that leads to development of an energy efficient train control with exact relations for computation of the switching times between individual driving modes and to introduction of the critical time as the key factor for determination of the optimal control strategy. Main results of this chapter were introduced in papers [22] and [24].

3.1. Formulation of the problem

Throughout the Chapter 3 we are going to study the problem of the energy efficient train control in the following form. The aim is to minimize the objective functional

$$J = \int_0^T u^+ (t)\, v (t) \,\mathrm{d}t \tag{3.1}$$

with respect to the system of differential equations

$$\dot{x} (t) = v (t) \,, \tag{3.2}$$

$$\dot{v} (t) = u (t) - r (v) \tag{3.3}$$

and boundary conditions

$$x (0) = 0 \,,\ v (0) = 0 \,, \tag{3.4}$$

$$x (T) = L \,,\ v (T) = 0 \,. \tag{3.5}$$

The function u^+ is defined as follows

$$u^+ (t) := \begin{cases} u (t) \,, & \text{for } u (t) > 0 \,, \\ 0 \,, & \text{for } u (t) \leq 0 \,. \end{cases}$$

We assume that the control variable u is a piecewise continuous function mapping the interval $[0, T]$ into $[-\alpha, \beta]$, where $\alpha, \beta > 0$ are given constants, and $r = r (v)$ is a differentiable function (with respect to v) with the properties $r, r' \geq 0$ and $r' (v)\, v$ is a nondecreasing function for $v \geq 0$. We shall illustrate our following considerations utilizing the linear and quadratic form of the resistance function r (which satisfy the above mentioned properties). A generalization to the most common type of resistance function:

$$r (v) = bv + c (v)^2 \,.$$

is only a technical matter.

Let us emphasize that the problem (3.1)–(3.5) describes the motion of a train along a straight level track of length $L > 0$ with minimal consumption of electric energy J. Without loss of generality let us further assume that the mass of the train $m = 1$. The phase coordinates x and v correspond to position and speed of the train, respectively. The given parameter T represents the time that is available according to the timetable for the train to complete the track. The function r represents the frictional resistance.

3.2. Description of optimal control strategy

In this section we develop the optimal control strategy for the problem (3.1)–(3.5). First, we need to determine the value of the minimum time $T_{\min}$ that it is possible to complete the track within. Solving the corresponding minimum time problem (i.e. $J = T \;\rightarrow\; \min.$) we can easily arrive at the standard well-known "bang-bang" control.

As it is obvious, the value of the time $T_{\min}$ can be exactly determined if we specify the form of the resistance function r. Under assumption $r(v) = bv$ $(b > 0)$ we obtain the following relation

$$T_{\min} = \frac{1}{b} \ln \eta \,,$$

where η has to satisfy the equation

$$(\alpha + \beta)\, \mathrm{e}^{Lb^2/(\alpha+\beta)} \cdot \eta^{\alpha/(\alpha+\beta)} - \alpha\eta - \beta = 0 \,.$$

Similarly for quadratic type of resistance function $r(v) = cv^2$ $(c > 0)$ the value $T_{\min}$ can be determined from the equation

$$T_{\min} = t^* + \frac{1}{\sqrt{\alpha c}} \cdot \arctan\left[\sqrt{\frac{\beta}{\alpha}} \tanh \sqrt{\beta c t^*}\right] \,,$$

where t^* is calculated from the equation

$$\alpha \cosh^2\left(\sqrt{\beta c t^*}\right) + \beta \sinh^2\left(\sqrt{\beta c t^*}\right) = \alpha\, \mathrm{e}^{2cL} \;.$$

A sample speed profile for the time optimal problem is shown in the Figure 3.1. Let us

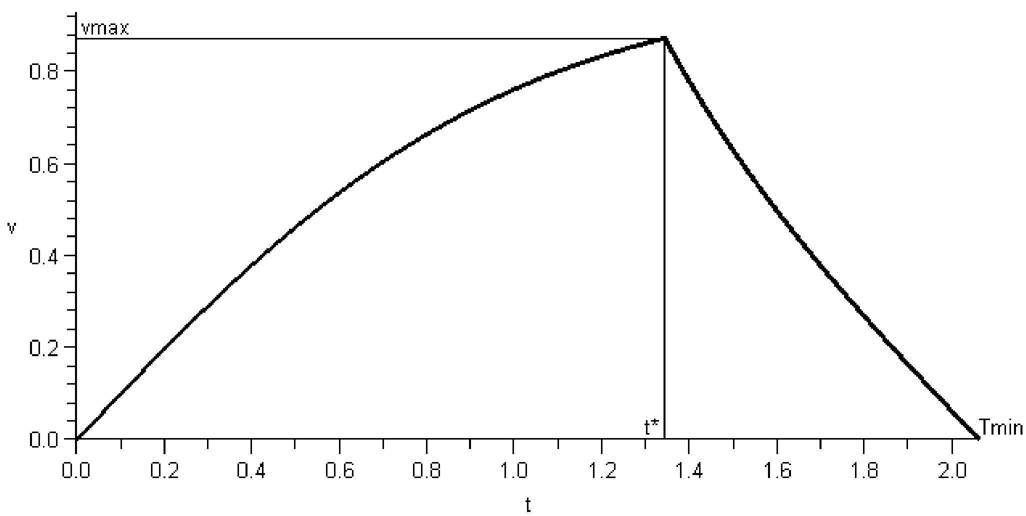

Figure 3.1: *A typical speed profile for time optimization, parameters* $\alpha = 1$*,* $\beta = 1$*,* $c = 1$*,* $L = 1$ *and* $r = cv^2$

further assume that the given time T satisfies the relation $T > T_{\min}$.

Now, let us recall the assertion which yields the energy efficient control strategy for the problem (3.1)–(3.5) (for more details and exact proof see e.g. Howlett [14]).

Theorem 6. *Let* $(\hat{x}(t), \hat{v}(t); \hat{u}(t))$, $t \in \langle 0, T\rangle$ *be the energy optimal solution of the problem (3.1)–(3.5). Then there exist* t_1, t_2, t_3, *where* $0 < t_1 \le t_2 < t_3 < T$, *such that*

$$\hat{u}(t) = \begin{cases} \beta & \textit{for } 0 \le t < t_1 \\ r(\hat{v}(t)) \equiv \textit{const.} & \textit{for } t_1 \le t < t_2 \\ 0 & \textit{for } t_2 \le t < t_3 \\ -\alpha & \textit{for } t_3 \le t \le T. \end{cases}$$

The research of the author was directed mainly on the type of the relation between the switching times t_1 and t_2 (equality or sharp inequality) and other relating topics, especially the calculation of the switching times. *The type of the relation between* t_1 *and* t_2 *cannot be specified directly from Pontryagin principle.*

3.3. The calculation of switching times

Let us assume that $t_1 = t_2$. Then we can easily arrive at the values of the switching times by integration of the relations (3.2) and (3.3) on individual time intervals, comparing values of position and velocity in boundary points of these time intervals (i.e. in $t = t_1 = t_2$ and $t = t_3$) and involving conditions (3.4) and (3.5). Of course, the second phase (speed-holding) is omitted in this consideration.

In particular, let us assume the linear resistance function r. We obtain an equation for the unknown t_3 in the form:

$$Lb^2 + \alpha bT - \alpha bt_3 = \beta \ln\left(\frac{\alpha}{\beta}\,\mathrm{e}^{bT} - \frac{\alpha}{\beta}\,\mathrm{e}^{bt_3} + 1\right). \tag{3.6}$$

Consequently, the value of the switching time $t_1 = t_2$ is determined from the relation

$$t_1 = \frac{1}{b}\ln\left(\frac{\alpha}{\beta}\,\mathrm{e}^{bT} - \frac{\alpha}{\beta}\,\mathrm{e}^{bt_3} + 1\right) \tag{3.7}$$

and the value of the maximum speed $v_{\max}$ within the whole track according to the relation

$$v_{\max} = -\frac{\beta}{b}\left(\frac{\alpha}{\beta}\,\mathrm{e}^{bT} - \frac{\alpha}{\beta}\,\mathrm{e}^{bt_3} + 1\right)^{-1} + \frac{\beta}{b}. \tag{3.8}$$

In case of quadratic resistance function r we obtain similarly the equation for calculation of time t_3 in the form:

$$\sqrt{\frac{c}{\beta}}\operatorname{arcsinh}\left\{\sqrt{\frac{\alpha}{\beta}}\,\mathrm{e}^{cL}\left|\sin\left[\sqrt{\alpha c}\,(T-t_3)\right]\right|\right\} + \sqrt{\frac{c}{\alpha}}\cot\left[\sqrt{\alpha c}\,(T-t_3)\right] =$$
$$\sqrt{\frac{c}{\beta}}\coth\operatorname{arcsinh}\left\{\sqrt{\frac{\alpha}{\beta}}\,\mathrm{e}^{cL}\left|\sin\left[\sqrt{\alpha c}\,(T-t_3)\right]\right|\right\} + ct_3.$$

Afterwards, we can compute the value of the time t_1 from the relation

$$t_1 = \frac{1}{\sqrt{\beta c}}\operatorname{arcsinh}\left\{\sqrt{\frac{\alpha}{\beta}}\,\mathrm{e}^{cL}\left|\sin\left[\sqrt{\alpha c}\,(T-t_3)\right]\right|\right\}$$

and the value of the maximum speed $v_{\max}$ from the relation

$$v_{\max} = \sqrt{\frac{\beta}{c}} \tanh \operatorname{arcsinh} \left\{ \sqrt{\frac{\alpha}{\beta}} \mathrm{e}^{cL} \left| \sin \left[\sqrt{\alpha c} \left(T - t_3 \right) \right] \right| \right\} .$$

The Figure 3.2 displays a sample speed profile for the case $t_1 = t_2$ and for quadratic type of resistance function r.

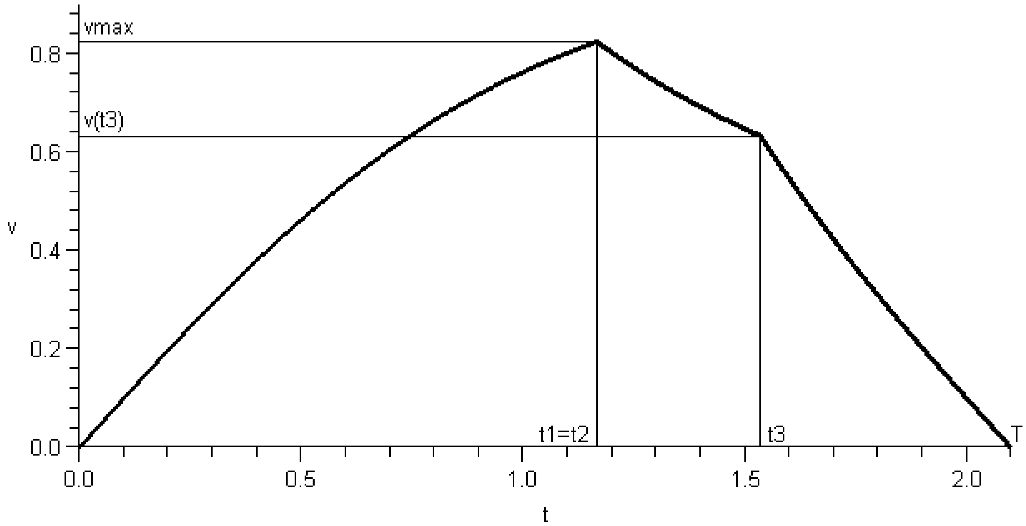

Figure 3.2: *A typical speed profile for* $t_1 = t_2$*, parameters* $\alpha = 1$*,* $\beta = 1$*,* $c = 1$*,* $L = 1$*,* $T = 2.1$ *and* $r = cv^2$

Let us assume the relation $t_1 < t_2$. In this case we need to determine the values of three unknown variables t_1, t_2 and t_3. However, position and speed with boundary conditions yield only two equations. Therefore, it is necessary to compare the values of the corresponding Hamilton function under a suitable choice of the variable t, make use of the property $H \equiv$ const. on $\langle 0, T \rangle$ and further use the continuity of the Lagrange multipliers on the interval $\langle 0, T \rangle$. To illustrate this, let us consider the linear resistance function $r(v) = bv$ (the following considerations can be performed for quadratic type of function r by analogy).

The Hamilton function is generally in the form

$$H = -u^+ v + \lambda_1 v + \lambda_2 \left(u - r(v) \right) .$$

Let us determine the values of the function H in the following points in time for the linear function r:

$$\begin{aligned}
&H(0) = \lambda_2(0)\,\beta\,, \\
&H\left(t_1^-\right) = -\beta v_{\max} + C_1 v_{\max} + \lambda_2\left(t_1^-\right)\left(\beta - b v_{\max}\right) , \\
&H\left(t_1^+\right) = -b v_{\max}^2 + C_1 v_{\max}\,, \\
&H\left(t_2^+\right) = C_1 v_{\max} - \lambda_2\left(t_2^+\right) b v_{\max}\,, \\
&H\left(t_3^-\right) = C_1 v\left(t_3\right) , \\
&H(T) = -\lambda_2(T)\,\alpha\,,
\end{aligned}$$

where $v_{\max}$ denotes the highest speed that the train reaches along its track (on the interval $\langle t_1, t_2 \rangle$) and $H\left(t_1^-\right)$ (respectively $H\left(t_1^+\right)$) denotes the corresponding one-sided limit (and similarly in the remaining cases). The constant C_1 corresponds to the Lagrange multiplier λ_1 as the solution of the adjoint system for multipliers λ_1 and λ_2 in the form:

$$\dot{\lambda}_1 = 0\,, \tag{3.9}$$

$$\dot{\lambda}_2 = \hat{u}^+ - \lambda_1 + b\lambda_2\,. \tag{3.10}$$

Let us recall that the variable λ_2 is continuous on the interval $\langle 0, T\rangle$. Therefore, it holds that $\lambda_2\left(t_1^-\right) = \lambda_2\left(t_1^+\right) = \lambda_2\left(t_1\right)$ and by analogy in other cases. Further, the relation $\lambda_2\left(t\right) = v_{\max}$ must be satisfied on (t_1, t_2) (this assertion follows directly from Pontryagin principle for this type of optimal control). Thus, λ_2 is constant here and therefore $\dot{\lambda}_2\left(t\right) = 0$ on (t_1, t_2). Consequently, the relations (3.9) and (3.10) imply that

$$\lambda_1\left(t\right) \equiv C_1 = 2bv_{\max}\,. \tag{3.11}$$

Now, we can use the relation for the Hamilton function in the point t_1^+ to derive the equation

$$H\left(t\right) \equiv b\left(v_{\max}\right)^2$$

for $t \in \langle 0, T\rangle$. The value of $H\left(t_3\right)$ and the Equation (3.11) lead us to conclude that

$$v\left(t_3\right) = \frac{v_{\max}}{2}\,. \tag{3.12}$$

A similar approach can be used for quadratic type of resistance function to derive the relation

$$v\left(t_3\right) = \frac{2}{3}v_{\max}\,. \tag{3.13}$$

The last two equations represent the required third equation that we need to derive the relations for calculation of the switching times for the case $t_1 < t_2$.

Thus, for linear type of resistance function r it is possible to derive (by analogy to the case $t_1 = t_2$ with use of the Equation (3.12)) the following equation for the unkwnown t_2

$$\begin{gathered}\left[\alpha\,\mathrm{e}^{b(T-t_2)} - 2\alpha - \beta\right]\ln\left[-\frac{\alpha}{\beta}\,\mathrm{e}^{b(T-t_2)} + \frac{2\alpha}{\beta} + 1\right] = \\ Lb^2 + \alpha bT + \alpha bt_2 - \alpha\ln 2 - \alpha bt_2\,\mathrm{e}^{b(T-t_2)}\end{gathered} \tag{3.14}$$

and the relations for remaining switching times t_1 and t_3 in the form

$$t_1 = -\frac{1}{b}\ln\left[-\frac{\alpha}{\beta}\,\mathrm{e}^{b(T-t_2)} + \frac{2\alpha}{\beta} + 1\right]\,, \tag{3.15}$$

$$t_3 = t_2 + \frac{1}{b}\ln 2\,. \tag{3.16}$$

The value of the maximum velocity $v_{\max}$ can be determined based on the following relation

$$v_{\max} = \frac{\alpha}{b}\,\mathrm{e}^{b(T-t_2)} - \frac{2\alpha}{b}\,. \tag{3.17}$$

Analogously as in the previous case of the linear resistance function r we can solve the case $t_1 < t_2$ under assumption of the quadratic form of the function r with use of the Equation (3.13). The value t_1 can be calculated from the equation

$$\begin{gathered}(T - t_1)\cdot\sqrt{\beta c}\tanh\left(\sqrt{\beta c}t_1\right) = \ln\left\{\frac{2}{3}\cos\arctan\left[\sqrt{\frac{\beta}{\alpha}}\frac{2}{3}\tanh\left(\sqrt{\beta c}t_1\right)\right]\right\} + \frac{1}{2} \\ -\ln\cosh\left(\sqrt{\beta c}t_1\right) + cL + \sqrt{\frac{\beta}{\alpha}}\tanh\left(\sqrt{\beta c}t_1\right)\cdot\arctan\left[\sqrt{\frac{\beta}{\alpha}}\frac{2}{3}\tanh\left(\sqrt{\beta c}t_1\right)\right]\end{gathered} \tag{3.18}$$

and consequently we can determine the values of the remaining switching times t_2 and t_3 according to the relations

$$t_3 = T - \frac{1}{\sqrt{\alpha c}} \cdot \arctan\left[\sqrt{\frac{\beta}{\alpha}}\frac{2}{3}\tanh\left(\sqrt{\beta c}t_1\right)\right], \tag{3.19}$$

$$t_2 = t_3 - \frac{1}{2\sqrt{\beta c}} \cdot \coth\left(\sqrt{\beta c}t_1\right). \tag{3.20}$$

The value of the maximum velocity of the train within the whole track can be easily calculated via the relation

$$v_{\max} = \sqrt{\frac{\beta}{c}}\tanh\left(\sqrt{\beta c}t_1\right).$$

A sample speed profile for the optimal control strategy with $t_1 < t_2$ is shown in the Figure 3.3 under assumption of the quadratic type of the function r.

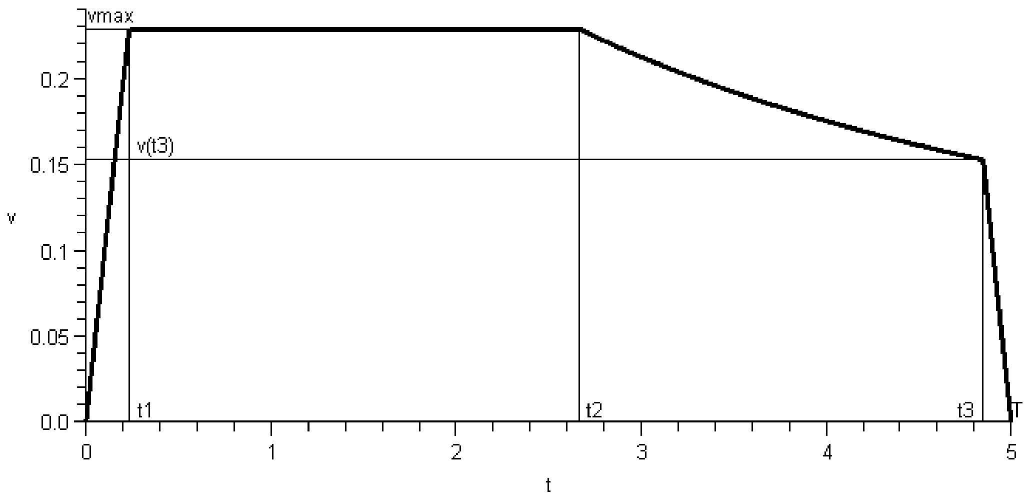

Figure 3.3: *A typical speed profile for $t_1 < t_2$, parameters $\alpha = 1$, $\beta = 1$, $c = 1$, $L = 1$, $T = 5$ and $r = cv^2$*

We have determined the values of the switching times for both possible control strategies which follow directly from the Pontryagin principle, i.e. including the speed holding phase or not. However, we still have not stated which of these two strategies is optimal for given entry parameters of the problem. We can choose the optimal case based on the value of the cost functional J. With use of the relation (4.1) we can derive the relations for calculation of J. In case $t_1 = t_2$ and under assumption of linear resistance function r we obtain the relation

$$J = -\frac{\beta^2}{b^2} + \frac{\beta^2}{b}t_1 + \frac{\beta^2}{b^2}\,\mathrm{e}^{-bt_1} \tag{3.21}$$

and for quadratic type of the function r we derive the relation

$$J = \frac{\beta}{c}\ln\cosh\left(\sqrt{\beta c}t_1\right).$$

For a control strategy containing the speed holding control mode, i.e. if $t_1 < t_2$, the value of the cost functional J can be determined according to the relation

$$J = -\frac{\beta^2}{b^2} + \frac{\beta^2}{b}t_1 + \frac{\beta^2}{b^2}\,\mathrm{e}^{-bt_1} + b\left(v_{\max}\right)^2\left(t_2 - t_1\right) \tag{3.22}$$

for linear type of function r and

$$J = \frac{\beta}{c} \ln \cosh \left(\sqrt{\beta c} t_1 \right) + c \left(v_{\max} \right)^3 (t_2 - t_1)$$

for quadratic resistance function.

We easily choose the lower value (of course, if more than one of the two possible strategies $t_1 = t_2$, resp. $t_1 < t_2$, is feasible). Some sample resulting values of the switching times t_1, t_2 and t_3 for quadratic and linear type of resistance function $r(v)$ can be found in the Table 3.1 and the Table 3.2, respectively.

T	t_1	t_2	t_3	$v_{\max}$	J
2.062	1.344	1.344	1.344	0.873	0.717
2.100	1.167	1.167	1.537	0.823	0.567
2.172	1.052	1.052	1.691	0.782	0.474
2.500	0.628	1.247	2.145	0.557	0.292
3.000	0.449	1.539	2.726	0.421	0.179
4.000	0.303	2.106	3.806	0.294	0.091
5.000	0.233	2.663	4.849	0.229	0.056
6.000	0.190	3.216	5.875	0.188	0.038
8.000	0.140	4.313	7.908	0.139	0.021
10.000	0.111	5.406	9.926	0.111	0.013

Table 3.1: *Sample values of the switching times t_1, t_2, t_3 and maximum velocity $v_{\max}$ for quadratic resistance function r and input parameters $\alpha = 1$, $\beta = 1$, $L = 1$ and $c = 1$ for various values of parameter T*

T	t_1	t_2	t_3	$v_{\max}$	J
2.170	1.585	1.585	1.585	0.795	0.790
2.200	1.445	1.445	1.755	0.764	0.681
2.300	1.323	1.323	1.977	0.734	0.590
2.316	1.311	1.311	2.005	0.731	0.581
2.500	0.846	1.556	2.249	0.571	0.506
3.000	0.533	2.119	2.812	0.413	0.390
4.000	0.331	3.175	3.868	0.282	0.275
5.000	0.244	4.204	4.897	0.217	0.214
6.000	0.195	5.222	5.915	0.177	0.175
8.000	0.139	7.244	7.937	0.130	0.129
10.000	0.109	9.257	9.950	0.103	0.103

Table 3.2: *Sample values of the switching times t_1, t_2, t_3 and maximum velocity $v_{\max}$ for linear resistance function r and input parameters $\alpha = 1$, $\beta = 1$, $L = 1$ and $c = 1$ for various values of parameter T*

The Figure 3.4 displays the values of the cost functional J for both types of resistance functions r for various values of the parameter T.

A different approach for determination of the optimal control strategy via the notion of critical time and theory of nonlinear parametric programming will be introduced in the Section 3.4.

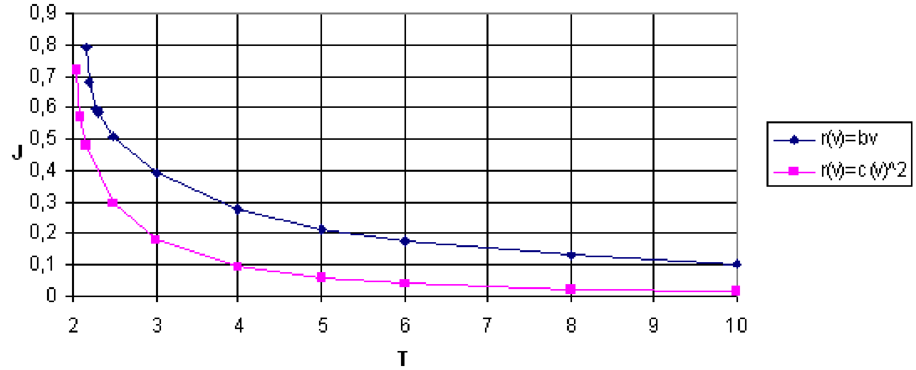

Figure 3.4: *Sample profile of values of the cost functional J for parameters $\alpha = 1$, $\beta = 1$, $c = 1$, $L = 1$ in dependence on parameter T for linear and quadratic type of resistance function r*

3.4. Analysis of the solution - critical time

Numerical calculations (based on algorithms from Bazaraa et al. [2]) show that the choice of the optimal control strategy depends only on the given value of the entry parameter T. The Figure 3.5 shows the dependence of the optimal control strategy on the input parameter T as well.

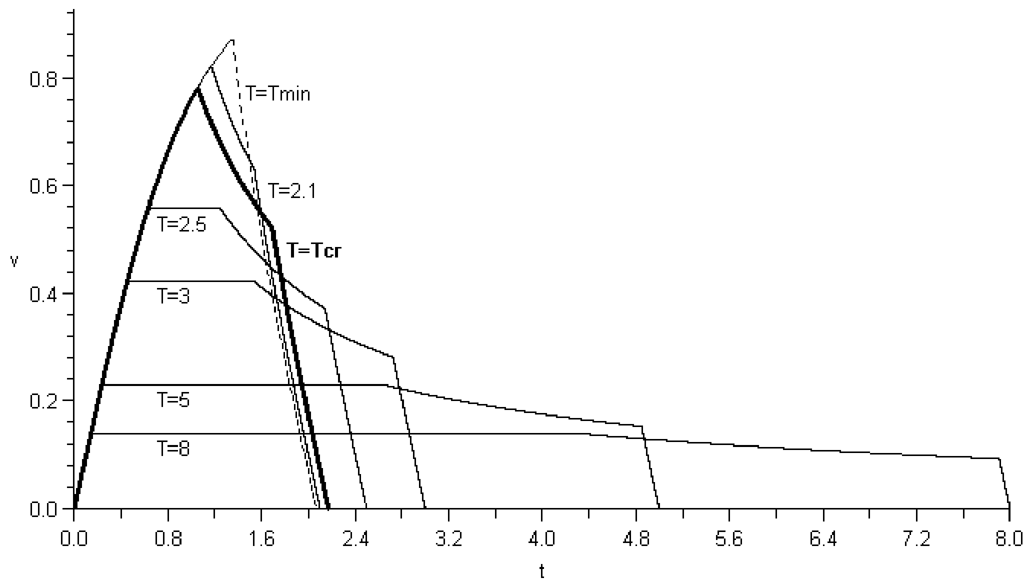

Figure 3.5: *Typical speed profiles for parameters $\alpha = 1$, $\beta = 1$, $c = 1$, $L = 1$, various values of parameter T and resistance function $r = cv^2$*

In order to analyse the properties of the solution of the problem (4.1)–(4.5) with respect to the value of the parameter T it is convenient to use the theory of nonlinear parametric programming with relating tools (see the Section 2.3). To simplify the analysis let us assume that there exists a certain value $T_{\max}$, sufficiently large, with the property $T_{min} \leq T \leq T_{\max}$ and consider only the case of the linear resistance function r (the quadratic case can be solved by analogy).

Using the Theorem 6 we can easily rewrite the problem (4.1)–(4.5) into the following form of the nonlinear programming problem. We wish to minimize the objective function

$$J = \frac{\beta^2}{b^2}\left(bt_1 + \mathrm{e}^{-bt_1} - 1\right) + \frac{\beta^2}{b}\left(t_2 - t_1\right)\left(1 - \mathrm{e}^{-bt_1}\right)^2 \quad \rightarrow \quad \min \tag{3.23}$$

with respect to the equations

$$\alpha\left(\mathrm{e}^{b(T-t_3)} - 1\right) = \beta\left(1 - \mathrm{e}^{-bt_1}\right)\mathrm{e}^{b(t_2-t_3)}\,, \tag{3.24}$$

$$\alpha\left(t_3 - T\right) + \beta\left(t_2 - t_2\,\mathrm{e}^{-bt_1} + t_1\,\mathrm{e}^{-bt_1}\right) = bL \tag{3.25}$$

and inequalities

$$0 \leq t_1 \leq t_2 \leq t_3 \leq T\,. \tag{3.26}$$

The constraints (3.24) and (3.25) can be derived with use of the boundary conditions (3.4) and (3.5). Since the set of all feasible solutions has to be closed for our future considerations, we assume the inequalities for t_1, t_2 and t_3 in the form (3.26) (let us note that the cases $t_1 = 0$, $t_2 = t_3$ and $t_3 = T$ cannot be optimal provided $T > T_{\min} > 0$).

Let us denote by symbol $M(T)$ the set of all feasible solutions of the given problem, i.e. the set of all (t_1, t_2, t_3) satisfying the relations (3.24)–(3.26) for a given parameter T. Let us further introduce the following assumption:

Hypothesis 1. The point-to-set mapping $M(T)$ is continuous in T for all $T \geq T_{\min}$.

Note that the validity of the Hypothesis 1 can be verified under specified values of the parameters α, β, b and L.

Lemma 2. *Let the Hypothesis 1 be fulfilled. Then the point-to-set mapping*

$$\psi(T) := \left\{(t_1, t_2, t_3) \in M(T) \,|\, J(t_1, t_2, t_3; T) = \varphi(T)\right\}\,,$$

where

$$\varphi(T) := \inf_{(t_1,t_2,t_3)\in M(T)} J(t_1, t_2, t_3; T)\,,$$

is u.s.c.-B for every $T_{\min} \leq T \leq T_{\max}$.

Proof. We apply the Theorem 3 and the Theorem 4 to our problem. The mapping $\varphi(T)$ represents now the optimal value of the cost functional J which is specified by the Equation (3.23) for a fixed value of the parameter T. The mapping $\psi(T)$ is a point-to-set mapping which to every fixed value $T \geq T_{\min}$ assigns a set of all optimal solutions of the given nonlinear programming problem, i.e. a set of all optimal (t_1, t_2, t_3). Under the Hypothesis 1 the mapping $M(T)$ is also l.s.c.-B for every $T \geq T_{\min}$. Moreover, $J = J(t_1, t_2, t_3; T)$ from the relation (3.23) is upper semicontinuous on $\mathbb{R}^3 \times \mathbb{R}$ (it is even continuous). Thus, $\varphi(T)$ is an upper semicontinuous mapping for every $T \geq T_{\min}$ according to the Theorem 4. Further, let us note that $M(T)$ is a non-empty set for every $T \geq T_{\min}$. Metric space X occuring in the Theorem 3 represents in our case the set of all (t_1, t_2, t_3) satisfying inequalities (3.26) and therefore X is compact because of $T \leq T_{\max}$. Further, we need the mapping $M(T)$ to be closed in T for every $T \geq T_{\min}$. This property follows from the Lemma 1 since $M(T)$ is (according to the Hypothesis 1) u.s.c.-H at T and the set of all (t_1, t_2, t_3) satisfying (3.24), (3.25) and (3.26) is closed. Therefore, by the Theorem 3 the mapping $\psi(T)$ is u.s.c.-B at T for every $T_{\min} \leq T \leq T_{\max}$.
□

The assertion of the Lemma 2 ensures that if we choose some fixed value $T = T^*$ and the corresponding optimal solution (t_1^*, t_2^*, t_3^*) of the problem (3.23)–(3.26), then considering T sufficiently close to T^* we obtain a solution (t_1, t_2, t_3) close to (t_1^*, t_2^*, t_3^*).

Now we shall introduce the concept of the critical time T_{cr} and present the way of its computation.

Definition 13. *A parameter T is said to be the* critical time *of the problem (3.23)–(3.26) (and we shall further denote it as T_{cr}), if there exists an $\epsilon > 0$ such that for $T = T_{cr}$ the nonlinear programming problem (3.23)–(3.26) has an optimal solution with the property $t_1 = t_2$ and for $T \in (T_{cr}, T_{cr} + \epsilon)$ the corresponding optimal solution satisfies $t_1 < t_2$.*

In other words, the value T_{cr} represents the critical driving time when the most interesting optimal driving mode, i.e. the speed holding phase, appears in the optimal control strategy.

Lemma 3. *Let T_{cr} be the critical time of the problem (3.23)–(3.26) and let the Hypothesis 1 be fulfilled. Then T_{cr} is the unique positive solution of the equation*

$$\alpha b T_{cr} + Lb^2 + (\alpha + \beta) \ln \left(\frac{2\alpha + \beta}{\beta + \alpha\, \mathrm{e}^{bT_{cr}}} \right) = \alpha \ln 2 \,. \tag{3.27}$$

Proof. In the Section 3.3 we determined the values of t_1, t_2 and t_3 under assumption $t_1 < t_2$. Due to the Lemma 2, $\psi(T)$ is u.s.c.-B for every $T \geq T_{\min}$. Hence, by letting $t_2 \to t_1^+$ and comparing both calculations performed for $t_1 < t_2$ and $t_1 = t_2$ we arrive at the determination of the relation for T_{cr}.

More precisely, the Equation (3.15) determines time t_1 provided $t_1 < t_2$. Now, let us use the relation $t_1 = t_2$ to obtain

$$t_2 = -\frac{1}{b} \ln \left[-\frac{\alpha}{\beta}\, \mathrm{e}^{b(T_{cr} - t_2)} + \frac{2\alpha}{\beta} + 1 \right] .$$

This relation leads us to the expression

$$t_2 = -\frac{1}{b} \ln \left[\frac{2\alpha + \beta}{\beta + \alpha \exp(bT_{cr})} \right] .$$

This value is substituted to the Equation (3.14) and after some simple modifications we arrive at the Equation (3.27).

To show that the Equation (3.27) admits a unique solution we put

$$F(T) := \alpha b T - \alpha \ln 2 + Lb^2 + (\alpha + \beta) \ln \left(\frac{2\alpha + \beta}{\beta + \alpha\, \mathrm{e}^{bT}} \right)$$

denoting the function which describes the left-hand side of the Equation (3.27). Thereafter

$$F\left(\frac{\ln 2}{b}\right) = Lb^2 > 0 \,.$$

Further,

$$\lim_{T\to\infty} F(T) = \lim_{T\to\infty} \left[\alpha b T - (\alpha + \beta) \ln \left(\beta + \alpha\, \mathrm{e}^{bT}\right)\right] - \alpha \ln 2 + Lb^2 +$$
$$(\alpha + \beta) \ln (2\alpha + \beta) = -\infty \,,$$

because $\ln\left(\beta + \alpha\,\mathrm{e}^{bT}\right) \approx \ln\left(\alpha\,\mathrm{e}^{bT}\right) = \ln\alpha + bT$ as $T \to \infty$. Moreover,

$$F'(T) = \alpha b - (\alpha+\beta)\,\frac{\alpha b\,\mathrm{e}^{bT}}{\beta + \alpha\,\mathrm{e}^{bT}} = \alpha b\left[1 - \frac{(\alpha+\beta)\,\mathrm{e}^{bT}}{\alpha\,\mathrm{e}^{bT} + \beta}\right] < 0$$

for $T > 0$ and this shows the uniqueness of the positive solution of the Equation (3.27). □

An easy consideration shows that if for some fixed $T = T^*$ the optimal solution of the problem (3.23)–(3.26) satisfies the relation $t_1 < t_2$, then for every $T \geq T^*$ the corresponding optimal solution of (3.23)–(3.26) has the same property. In other words, if the optimal trajectory contains the speed holding phase for some $T = T^*$, then the speed holding phase will be contained in every optimal strategy with $T > T^*$. The proof of this assertion can be performed by analogy to the proof of the Lemma 3. Indeed, let us assume that there exists a parameter T^{**} such that for $T \in (T^{**} - \epsilon, T^{**})$, $\epsilon > 0$ being sufficiently small, the problem (3.23)–(3.26) has an optimal solution with property $t_1 < t_2$ and for $T = T^{**}$ the corresponding optimal solution satisfies $t_1 = t_2$. Then the necessary condition for T^{**} is given by the Equation (3.27) (where T_{cr} is replaced by T^{**}). We have already shown that this equation admits only one positive solution, i.e. the existence of T^{**} implies that T_{cr} does not exist. Further, let us note that for $T = T_{\min}$ the corresponding optimal solution (t_1, t_2, t_3) of the problem (3.23)–(3.26) has the property $t_1 = t_2 = t_3$. Similarly, if for $T > T_{\min}$ this optimal solution satisfies the relation $t_1 < t_2$, then $t_3 = t_2 + \frac{1}{b}\ln 2$. However, the mapping $\psi(T)$ is u.s.c.-B for $T \geq T_{\min}$, thus for $T > T_{\min}$, T being sufficiently close to $T_{\min}$, the optimal solution must satisfy the relation $t_1 = t_2$. Therefore, T^{**} cannot exist without the appearance of T_{cr} and this is a contradiction.

Summarizing the previous considerations we can arrive at the following theorem.

Theorem 7. *Let (t_1, t_2, t_3) be the optimal solution of the problem (3.23)–(3.26) and let the Hypothesis 1 be fulfilled. Then either $t_1 = t_2$ for every $T \geq T_{\min}$ or there exists a unique value of T_{cr} with the property that for $T \in \langle T_{\min}, T_{cr}\rangle$ the optimal solution satisfies the relation $t_1 = t_2$ and for $T > T_{cr}$ the property $t_1 < t_2$ is fulfilled (moreover, the value T_{cr} can be determined as the unique positive solution of the Equation (3.27)).*

The numerical results show that considering the value of the parameter T large enough the optimal solution (t_1, t_2, t_3) of the problem (3.23)–(3.26) satisfies the relation $t_1 < t_2$ for given fixed parameters α, β, L and b. We can therefore introduce a conjecture that the first variant described in the Theorem 7 (i.e. $t_1 = t_2$ for every $T \geq T_{\min}$) does not actually occur.

Let us recall that the values of the switching times t_1, t_2 and t_3 can be determined for $r(v) = bv$ via the relations (3.6)–(3.7) in case $t_1 = t_2$ and with use of the relations (3.14)–(3.16) for $t_1 < t_2$. The cost functional J is specified by the Equations (3.21) and (3.22) for $t_1 = t_2$ and $t_1 < t_2$, respectively. The value of the maximum velocity $v_{\max}$ can be determined from the Equation (3.8) if $t_1 = t_2$ and according to the Equation (3.17) under assumption $t_1 < t_2$.

So far the results of this section have been illustrated on the model with linear resistance function $r(v) = bv$. The extension to models with nonlinear type of resistance function consists only in more tedious computations and does not represent any qualitative advancement.

Let us consider now the quadratic resistance function r. We can introduce and discuss the problem of the critical time in a similar way to the case of the linear resistance.

However, a formal justification of the existence of T_{cr} is much more complicated. Therefore, we show at least the necessary condition for T_{cr}, i.e. an analogy of the Equation (3.27).

We use the above derived Equation (3.18) which we obtained under assumption $t_1 < t_2$. By letting $t_2 \to t_1^+$ we get $T \to T_{cr}$. We therefore put $t_1 = t_2 = t_{cr}$ in relevant formulas to obtain

$$t_3 = t_{cr} + \frac{1}{2\sqrt{\beta c}\tanh\left(\sqrt{\beta c}t_{cr}\right)}\,.$$

We compare this relation with Equation (3.19) and derive the relation

$$T_{cr} = \frac{1}{\sqrt{\alpha c}}\arctan\left[\sqrt{\frac{\beta}{\alpha}}\frac{2}{3}\tanh\left(\sqrt{\beta c}t_{cr}\right)\right] + t_{cr} + \frac{1}{2\sqrt{\beta c}\tanh\left(\sqrt{\beta c}t_{cr}\right)} \tag{3.28}$$

which can be substituted into Equation (3.18) and after some simple steps we arrive at the following equation

$$\frac{2}{3}\,\mathrm{e}^{cL}\left|\cos\arctan\left[\sqrt{\frac{\beta}{\alpha}}\frac{2}{3}\tanh\left(\sqrt{\beta c}t_{cr}\right)\right]\right| - \cosh\left(\sqrt{\beta c}t_{cr}\right) = 0$$

which can be used to determine the value of the time t_{cr}. Thereafter, the value T_{cr} is calculated via Equation (3.28).

4. Energy efficient train control with a speed constraint

This chapter is devoted to the description of the energy optimal driving strategy of an electric-powered train with a global speed constraint and describes the way of calculation of the switching times between the optimal driving regimes as well. We are going to use the theoretical background introduced in the Section 2.2. Most of the results discussed in this chapter were introduced in the paper [23].

4.1. Formulation of the problem

We are going to study the following optimal control problem:

$$J = \int_0^T u^+ v \,\mathrm{dt} \quad \rightarrow \quad \min \tag{4.1}$$

with respect to the system of differential equations

$$\dot{x}(t) = v(t) \,, \tag{4.2}$$

$$\dot{v}(t) = u(t) - r(v) \tag{4.3}$$

and boundary conditions

$$x(0) = 0 \,, \ v(0) = 0 \,, \tag{4.4}$$

$$x(T) = L \,, \ v(T) = 0 \,, \tag{4.5}$$

where function u^+ is defined as

$$u^+(t) := \begin{cases} u(t) & \text{for } u(t) > 0 \\ 0 & \text{for } u(t) \leq 0 \,. \end{cases}$$

We shall further assume a global speed constraint in the form

$$v(t) \leq v_m \,, \quad t \in \langle 0; T \rangle \ . \tag{4.6}$$

Now, we impose the same assumptions on functions u and r as in the Section 3.1. In particular, we further assume that u is a piecewise continuous function mapping $\langle 0, T \rangle$ into $\langle -\alpha, \beta \rangle$, where $\alpha, \beta > 0$ are given constants. Function $r = r(v)$ (which represents the frictional resistance) is a differentiable function (with respect to v) with the properties $r, r' \geq 0$ and $r'(v)\, v$ is nondecreasing for $v \geq 0$. The most usual type of resistance function r (which satisfies all these conditions) is a polynomial function

$$r(v) = bv + c(v)^2 \ . \tag{4.7}$$

To simplify the computations, we will consider the linear resistance function $r(v) = bv$ and the quadratic resistance function $r(v) = c(v)^2$. The possible generalization of our results to r given by (4.7) is only a technical matter.

The problem (4.1)-(4.6) describes the motion of a train along a straight level track of length $L > 0$ with minimal consumption of electric energy J. We assume that the mass of the train $m = 1$. Phase coordinates x and v correspond to position and speed of the train. Given parameter T represents the time that is available according to the timetable for the train to complete the track. The given constant v_m is the maximum allowed velocity of the train along the whole track.

4.2. Description of optimal control strategy

In this section we are going to determine the character of the optimal control strategy consisting of at most four successive driving modes (full power, speed-holding, coasting and full braking). First, we need to specify the value of the maximal speed $v_{\max}$ of the train within the whole track under assumption of the basic problem (4.1)–(4.5) without any further constraints so that we determine whether the global speed constraint (4.6) is active ($v_{\max} \geq v_m$) or not. In the latter case, we may easily apply the results of the Chapter 3 (optimal strategy and the values of switching times) also for the case of the global speed constraint. The relevant relations for calculation of the value of $v_{\max}$ were presented in the Chapter 3. Let us therefore further assume that the relation $v_{\max} \geq v_m$ holds.

First, we have to determine the value of the minimal time T^*_{min} that it is possible to complete the track within (involving the speed constraint (4.6)). Let $v_{\max} \geq v_m$. With use of Pontryagin principle and some further tools concerning the path constraints (for further details see e.g. Bryson et al. [4]) we can easily arrive at the following equation for calculation of T^*_{min}

$$T^*_{min} = \frac{1}{b^2 v_m} \ln \left[1 + \frac{b}{\alpha} v_m\right]^{\alpha} \cdot \left[1 - \frac{b}{\beta} v_m\right]^{\beta} + \frac{L}{v_m} + \frac{1}{b} \ln \left(1 + \frac{b}{\alpha} v_m\right) - \frac{1}{b} \ln \left(1 - \frac{b}{\beta} v_m\right)$$

in case of linear resistance function and similarly

$$T^*_{min} = \frac{1}{2cv_m} \left[\ln \left(1 - \frac{c}{\beta} v_m^2\right) - \ln \left(1 + \frac{c}{\alpha} v_m^2\right)\right] + \frac{L}{v_m} + \frac{1}{\sqrt{\beta c}} \operatorname{arctanh} \left(\sqrt{\frac{c}{\beta}} v_m\right) + \frac{1}{\sqrt{\alpha c}} \arctan \left(\sqrt{\frac{c}{\alpha}} v_m\right)$$

for quadratic resistance function r.

The Figure 4.1 displays a sample speed profile for a time optimal driving strategy under assumption of a speed constraint as well as the original speed profile for a trajectory without the speed constraint (for comparison).

In what follows, we assume that $T > T^*_{\min}$ and $v_{\max} > v_m$. Let us denote

$$S(x, v, t) := v(t) - v_m.$$

Then it holds for the first total time derivative of S that

$$S^{(1)}(x, v, t) = \dot{v}(t) = u(t) - r(v).$$

Thus, (4.6) is a first order state variable inequality constraint. Hamilton function is in the form

$$H = \lambda_0 u^+ v + \lambda_1 v + (\lambda_2 + \mu) \left[u - r(v)\right],$$

where λ_0, λ_1, λ_2 and μ denote the corresponding Lagrange multipliers (without loss of generality we put $\lambda_0 \equiv -1$, the case $\lambda_0 \equiv 0$ corresponds to time optimization). The variables λ_1 and λ_2 have to satisfy the adjoint system

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x} = 0$$
$$\dot{\lambda}_2 = -\frac{\partial H}{\partial v} = u^+ - \lambda_1 + \lambda_2 r'(v) + \mu r'(v).$$

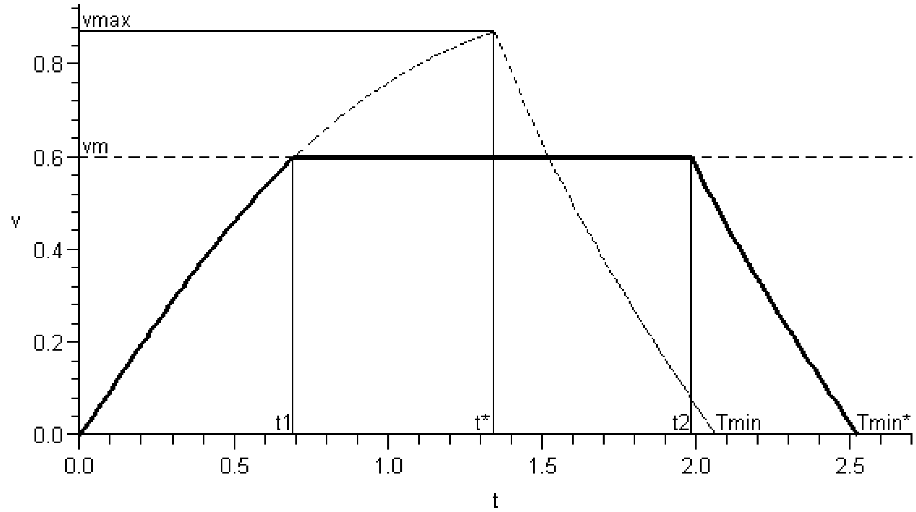

Figure 4.1: *A typical speed profile for time optimization with a speed constraint* $v(t) \leq 0.6$ *and parameters* $\alpha = 1$, $\beta = 1$, $c = 1$, $L = 1$ *and* $r = cv^2$

Further, $\mu \leq 0$ on the constraint boundary ($S = 0$) and $\mu = 0$ off the constraint boundary. The path entering onto the constraint boundary has to meet the tangency constraint $S = 0$ and if we denote t_1 as the entry point onto the boundary constraint, then the following jump conditions have to be satisfied:

$$\begin{aligned} \lambda_1\left(t_1^-\right) &= \lambda_1\left(t_1^+\right) \\ \lambda_2\left(t_1^-\right) &= \lambda_2\left(t_1^+\right) + \pi \quad (\pi \in \mathbf{R}) \\ H\left(t_1^-\right) &= H\left(t_1^+\right) , \end{aligned}$$

where t_1^- and t_1^+ denote the corresponding one-sided limits. Thus, $\lambda_1(t) \equiv C_1 = \text{const.}$ for $t \in \langle 0, T\rangle$ and λ_2 might be discontinuous at time t_1. Off the constraint boundary we may use the Pontryagin principle and derive the same four possible driving strategies as in the case without the speed constraints, i.e. full power, speed holding, coasting and full braking. Let us denote t_2 the time when the path is leaving the speed boundary. On the constraint boundary (if $t_1 < t_2$) it holds $u(t) = r(v_m)$ and $\frac{\partial H}{\partial u} = 0$. Thus, for $t \in \langle t_1, t_2)$ (with use of the relation $v(t) \equiv v_m$) it holds

$$\lambda_2(t) = v_m - \mu(t) \geq v_m \,.$$

As $\mu(t) \leq 0$ on the constraint boundary the relation $\lambda_2(t) \geq v_m$ must hold for $t \in \langle t_1, t_2)$. Further, let us assume the linear case $r(v) = bv$ (for quadratic resistance function r we can use analogical approach). With use of jump condition for Hamilton function in time t_1 it can be shown that $\lambda_2\left(t_1^-\right) = v_m$. Further, with respect to the continuity of λ_2 in t_2 it holds $\lambda_2\left(t_2^+\right) > 0$, thus $u\left(t_2^+\right) = 0$ and further $\lambda_2(t_2) = v_m$. Therefore,

$$H = C_1 v_m - b v_m^2 > 0\,, \quad \text{hence} \quad C_1 > b v_m \,.$$

Summarizing the previous ideas and analysing the properties of function $\lambda_2(t)$ based on previous results it is possible to prove the following theorem.

Theorem 8. *Let $(\hat{x}(t), \hat{v}(t); \hat{u}(t))$, $t \in \langle 0, T\rangle$ be the energy optimal solution of (4.1)-(4.5) and (4.6). Let $r(v) = bv$ $(r(v) = c(v)^2)$. Then there exist t_1, t_2, t_3 such that*

$$\hat{u}(t) = \begin{cases} \beta & \text{for } t \in \langle 0, t_1) \\ bv_m \quad (c(v_m)^2) & \text{for } t \in \langle t_1, t_2) \\ 0 & \text{for } t \in \langle t_2, t_3) \\ -\alpha & \text{for } t \in \langle t_3, T\rangle \end{cases},$$

where $0 < t_1 \le t_2 < t_3 < T$.

4.3. The calculation of switching times

The case $t_1 = t_2$ corresponds to the relation $v_{\max} = v_m$. By integration of the Equations (4.2) and (4.3) on separate time intervals and involving the boundary conditions (4.4) and (4.5) it is easy to find the equations for calculation of the switching times t_1, t_2 and t_3 for both linear and quadratic resistance functions. If $r(v) = bv$ then

$$t_1 = -\frac{1}{b}\ln\left(1 - \frac{bv_m}{\beta}\right).$$

Further, we can derive the equation for unknown t_3 in the form

$$\left(\frac{v_m}{b} - \frac{\beta}{b^2}\right)\ln\left(1 - \frac{bv_m}{\beta}\right) - \frac{\alpha}{b}(T - t_3) = L - \frac{v_m}{b}\ln\left[\frac{\alpha}{bv_m}\left(\mathrm{e}^{bT} - e^{bt_3}\right)\right]$$

and consequently calculate the value of t_2 via the relation

$$t_2 = \frac{1}{b}\ln\left[\frac{\alpha}{bv_m}\left(\mathrm{e}^{bT} - \mathrm{e}^{bt_3}\right)\right].$$

For $r(v) = c(v)^2$ we obtain the relation

$$t_1 = \frac{1}{\sqrt{\beta c}}\operatorname{arctanh}\left(\sqrt{\frac{c}{\beta}}v_m\right).$$

Thereafter we calculate the value of t_3 via the equation

$$\sqrt{\frac{c}{\alpha}}\, v_m \cot\left[\sqrt{\alpha c}\,(T - t_3)\right] - \ln v_m + \ln\sqrt{\frac{\alpha}{c}}\,\frac{\left|\cos\left[\sqrt{\alpha c}\,(T - t_3)\right]\right|}{\cot\left[\sqrt{\alpha c}\,(T - t_3)\right]} + cL$$
$$+ \sqrt{\frac{c}{\beta}}\, v_m \operatorname{arctanh}\left(\sqrt{\frac{c}{\beta}}\, v_m\right) = cv_m t_3 + 1 - \frac{1}{2}\ln\left(1 - \frac{c}{\beta}v_m^2\right)$$

and the value of t_2 from the relation

$$t_2 = t_3 + \frac{1}{cv_m} - \frac{1}{\sqrt{\alpha c}}\cot\left[\sqrt{\alpha c}\,(T - t_3)\right].$$

The equations for computation of the switching time t_3 usually yield two different possible values of t_3. However, only one of them satisfies the relations $0 < t_1 \le t_2 < t_3 < T$.

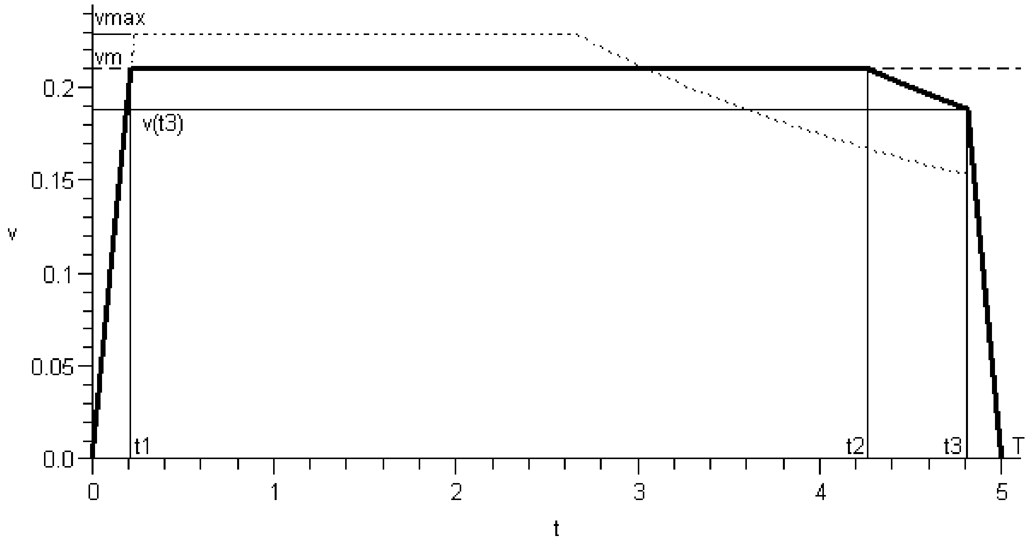

Figure 4.2: *A typical speed profile for constrained optimization and parameters* $\alpha = 1$, $\beta = 1$, $c = 1$, $L = 1$, $T = 5$, $v_m = 0.21$ *and* $r = cv^2$ *(the dotted line represents the case without speed constraint)*

The Figure 4.2 shows a typical speed profile for energy-efficient strategy with global speed constraint compared with the case without any constraints.

Let us note that a possible generalization of the problem with constrained velocity is the assumption of local speed constraints in the form

$$v \leq M_{j+1} \quad \text{for} \quad x \in (X_j, X_{j+1}) \ , \tag{4.8}$$

where $0 = X_0 < X_1 < \ldots < X_p = L$. The complex problem of speed constraints in the form (4.8) is much more complicated and it is going to be an object of author's further investigations. One way of solving this problem could be partitioning of the time interval $\langle 0, T \rangle$ on subintervals $\left\langle t_j^*, t_{j+1}^* \right\rangle, \quad j = 0, \ldots p-1$ with respect to the speed constraints (4.8), solving the corresponding energy-efficient train control problems on the separate intervals with global speed constraints (4.8) and with unknown values of the speed at the boundary points, comparing these values and solving the resulting nonlinear programming problem of minimization J according to the values of t_j^*. However, this leads to application of some numerical algorithms or methods of artificial intelligence and exceeds the aim of this thesis.

5. Time-energy efficient train control

This chapter deals with the time-energy efficient train control, i.e. a problem where both the time and energy consumption ought to be minimised with prescribed weight coefficients. We assume a partial reloading of energy into electrical circuit while braking. Some basic features of the problem were discussed by Kundrat et al. [17] by use of a numerical approach. The essential results discussed in this chapter were presented in the paper [21].

5.1. Formulation of the problem

We are going to investigate the following optimal control problem:

$$J = \int_0^T (pu_\gamma v + q)\,\mathrm{d}t \quad \rightarrow \quad \min \tag{5.1}$$

with respect to the system of differential equations

$$\dot{x}(t) = v(t)\,, \tag{5.2}$$
$$\dot{v}(t) = u(t) - r(v) \tag{5.3}$$

and boundary conditions

$$x(0) = 0\,,\ v(0) = 0\,, \tag{5.4}$$
$$x(T) = L\,,\ v(T) = 0\,, \tag{5.5}$$

where function u_γ satisfies

$$u_\gamma(t) := \begin{cases} u(t) & \text{for } u(t) \geq 0 \\ \gamma u(t) & \text{for } u(t) < 0\,. \end{cases}$$

Here $0 < \gamma < 1$, $p, q > 0$ and $p + q = 1$ are given real input parameters. A real constant $T > 0$ is to be determined. By analogy to the Section 3.1 we shall further assume that u is a piecewise continuous function mapping $\langle 0, T\rangle$ into $\langle -\alpha, \beta\rangle$, where $\alpha, \beta > 0$ are given real constants. Similarily, function $r = r(v)$ (which represents the frictional resistance) is a differentiable function (with respect to v) with the properties $r, r' \geq 0$ and $r'(v)\,v$ is nondecreasing for $v \geq 0$. A typical type of resistance function r (which satisfies these conditions) is again the quadratic function

$$r(v) = bv + cv^2\,. \tag{5.6}$$

To determine the values of the switching times in the Section 5.3 and to simplify the computations in the Section 5.4 we will consider the quadratic resistance function $r(v) = cv^2$ and the linear resistance function $r(v) = bv$. The possible generalization of our results to r given by (5.6) is only a technical matter.

Let us emphasize that the problem (5.1)–(5.5) describes the motion of a train along a straight level track of length $L > 0$ with the intention to minimize the consumption of electrical energy as well as time of the journey (represented by parameter T) with prescribed weight parameters p and q, respectively. We assume that the mass of the train $m = 1$. Phase coordinates x and v correspond to position and speed of the train. The real parameter γ represents the portion of the electrical energy that is being reloaded to the electrical circuit while braking.

5.2. Description of optimal control strategy

The following theorems determine the character of the optimal control strategy consisting of at most four successive driving modes (full power, speed-holding, coasting and full braking).

First, let us introduce the Hamilton function in the form

$$H = \lambda_0 \left(p u_\gamma v + q \right) + \lambda_1 v + \lambda_2 \left[u - r\left(v\right) \right] ,$$

where (without loss of generality) we consider $\lambda_0 = -1$ (the case $\lambda_0 = 0$ corresponds to time optimal problem). The other two Lagrange multipliers λ_1 and λ_2 are continuous solutions of the adjoint system

$$\begin{aligned} \dot{\lambda}_1 &= -\frac{\partial H}{\partial x} = 0 , \\ \dot{\lambda}_2 &= -\frac{\partial H}{\partial v} = p u_\gamma - \lambda_1 + \lambda_2 r'\left(v\right) . \end{aligned}$$

The Pontryagin maximum principle yields the following optimality condition (here $(\hat{x}, \hat{v}; \hat{u})$ denote the optimal controlled process):

$$\begin{aligned} & - p \hat{u}_\gamma \hat{v} - q + \lambda_1 \hat{v} + \lambda_2 \left[\hat{u} - r\left(\hat{v}\right) \right] = \\ & \max_{u \in \langle -\alpha, \beta \rangle} \left[- p u_\gamma \hat{v} - q + \lambda_1 \hat{v} + \lambda_2 \left(u - r\left(\hat{v}\right) \right) \right] \end{aligned}$$

This relation can be simplified for $\hat{u} \geq 0$ into the following form

$$\hat{u} \left(\lambda_2 - p \hat{v} \right) = \max_{u \in \langle 0, \beta \rangle} \left[u \left(\lambda_2 - p \hat{v} \right) \right]$$

and similarly for $\hat{u} < 0$ into form

$$\hat{u} \left(\lambda_2 - p \gamma \hat{v} \right) = \max_{u \in \langle -\alpha, 0 \rangle} \left[u \left(\lambda_2 - p \gamma \hat{v} \right) \right]$$

which after some simple calculations and standard steps (including singular mode determination) imply the following theorem.

Theorem 9. *Let* $\left(\hat{x}\left(t\right), \hat{v}\left(t\right); \hat{u}\left(t\right)\right)$, $t \in \left\langle 0, \hat{T} \right\rangle$ *be the time-energy optimal solution of (5.1)–(5.5). Then*

$$\hat{u}\left(t\right) = \begin{cases} \beta & \text{for } \lambda_2\left(t\right) - p\hat{v}\left(t\right) > 0 , \\ r\left(\hat{v}\right) \equiv \text{const.} & \text{for } \lambda_2\left(t\right) - p\hat{v}\left(t\right) = 0 , \\ 0 & \text{for } \lambda_2\left(t\right) - p\hat{v}\left(t\right) < 0 \quad \wedge \quad \lambda_2\left(t\right) - p\gamma\hat{v}\left(t\right) > 0 , \\ -\alpha & \text{for } \lambda_2\left(t\right) - p\gamma\hat{v}\left(t\right) < 0 . \end{cases}$$

The following theorem specifies the optimal order of the driving modes.

Theorem 10. *Let* $\left(\hat{x}\left(t\right), \hat{v}\left(t\right); \hat{u}\left(t\right)\right)$, $t \in \left\langle 0, \hat{T} \right\rangle$ *be the time-energy optimal solution of (5.1)–(5.5). Then there exist* t_1, t_2, t_3, *where* $0 < t_1 \leq t_2 < t_3 < \hat{T}$, *such that*

$$\hat{u}\left(t\right) = \begin{cases} \beta & \text{for } 0 \leq t < t_1 , \\ r\left(\hat{v}\right) \equiv \text{const.} & \text{for } t_1 \leq t < t_2 , \\ 0 & \text{for } t_2 \leq t < t_3 , \\ -\alpha & \text{for } t_3 \leq t \leq \hat{T} . \end{cases}$$

Proof. First, as $v(0) = v\left(\hat{T}\right) = 0$, there exists $t_1 \in \left(0, \hat{T}\right)$ such that

$$\lambda_2(t) - pv(t) > 0 \quad \text{for all} \quad t \in \langle 0, t_1) \quad \text{and} \quad \lambda_2(t_1) - pv(t_1) = 0\,.$$

Further, there exists $t_2 \in \left\langle t_1, \hat{T}\right)$ such that

$$\lambda_2(t) - pv(t) = 0 \quad \text{for all} \quad t \in \langle t_1, t_2)$$

(if $t_1 = t_2$, then the symbol $\langle t_1, t_2)$ denotes the singular interval consisting of t_1 only). Finally, we can show that there exists $t_3 \in \left(t_2, \hat{T}\right)$ such that

$$\lambda_2(t) - pv(t) < 0 \quad \wedge \quad \lambda_2(t) - p\gamma v(t) > 0 \quad \text{for all} \quad t \in (t_2, t_3)$$

and $\lambda_2(t_3) - p\gamma v(t_3) = 0$. This can be proved by contradiction. If $\lambda_2(t) - pv(t) > 0$ for every $t \in \left(t_2, \tilde{t}_2\right)$, where $\tilde{t}_2 > t_2$ is a real number, then $\dot{v}$ is a decreasing function and $\dot{\lambda}_2$ is an increasing function on $\left(t_2, \tilde{t}_2\right)$. Hence, $\lambda_2(t) - pv(t) > 0$ for all $t \in (t_2, \infty)$, which contradicts the condition $v\left(\hat{T}\right) = 0$. Since $\lambda_2(t) - pv(t) < 0$ and $\lambda_2(t) - p\gamma v(t) > 0$ for all $t > t_2$ sufficiently close to t_2, it holds $u(t) = 0$ for these t. Therefore, $\dot{\lambda}_2$ is decreasing and $\dot{v}$ is increasing and hence there exists t_3 such that $\lambda_2(t_3) = p\gamma v(t_3)$. Further, we wish to show that $\lambda_2(t) < p\gamma v(t)$ for all $t \in \left(t_3, \hat{T}\right)$. In a similar way as previously we can prove that v, λ_2 and $\dot{\lambda}_2$ are decreasing functions on $\left(t_3, \hat{T}\right)$ and function $\dot{v}$ is increasing. Thus, $\hat{u}(t) = -\alpha$ for $t \in \left(t_3, \hat{T}\right\rangle$. □

5.3. The calculation of switching times

Let us now determine the values of the switching times t_1, t_2 and t_3 and the value of the total driving time T. Of course, this determination is possible if the type of resistance function is specified. We emphasize that for unspecified driving time the Hamilton function satisfies the relation $H \equiv 0$ for $t \in \langle 0, T\rangle$. Further, $\lambda_1(t) \equiv C_1 = const.$ on $\langle 0, T\rangle$.

Suppose that the relation $t_1 < t_2$ holds. Then, the following condition is satisfied on (t_1, t_2):

$$\dot{\lambda}_2 = pr(v_{\max}) - C_1 + pv_{\max} r'(v_{\max}) \equiv 0\,, \tag{5.7}$$

where $v_{\max}$ denotes the speed-holding velocity. Further,

$$H\left(t_1^-\right) = -q + C_1 v_{\max} - pv_{\max} r(v_{\max}) = 0\,, \tag{5.8}$$

$$H\left(t_3^+\right) = -q + C_1 v(t_3) - p\gamma v(t_3)\, r\left[v(t_3)\right] = 0\,, \tag{5.9}$$

where $H\left(t_1^-\right)$ denotes the corresponding left-sided limit and $H\left(t_3^+\right)$ the corresponding right-sided limit of the Hamilton function. The Equations (5.7) and (5.8) yield the optimal value of the maximum velocity $v_{\max}$. For the resistance function $r = cv^2$ we obtain the relation

$$v_{\max} = \sqrt[3]{\frac{q}{2pc}}\,,$$

whereas for linear case we can derive the equation

$$v_{\max} = \sqrt{\frac{q}{pb}}\,.$$

By use of the Equation (5.9) we may easily arrive at the following cubic equation for calculation of the velocity $v(t_3)$ provided $r = cv^2$

$$-q + 3pcv_{\max}^2 v(t_3) - p\gamma c\,[v(t_3)]^3 = 0 \tag{5.10}$$

with a single feasible root and directly derive the relation

$$v(t_3) = v_{\max} \cdot \frac{1-\sqrt{1-\gamma}}{\gamma}$$

for $v(t_3)$ in case of linear type of resistance function r. Consequently, integrating the Equations (5.2) and (5.3) on corresponding time intervals, comparing the values of the variables x and v in the switching times and employing conditions (5.4) and (5.5) we obtain the following relations for calculation of the switching times t_1, t_2, t_3 and the total driving time T for quadratic type of resistance function r:

$$\begin{aligned}
t_1 &= \frac{1}{\sqrt{\beta c}} \operatorname{arctanh}\left(\sqrt{\frac{c}{\beta}} \cdot \sqrt[3]{\frac{q}{2pc}}\right), && (5.11)\\
t_2 &= t_1 + \frac{1}{cv_{\max}} \ln\left|\cos \arctan\left[\sqrt{\frac{c}{\alpha}} v(t_3)\right]\right| + \frac{L}{v_{\max}} \\
&\quad - \frac{1}{cv_{\max}} \ln\left[\frac{v_{\max}}{v(t_3)} \cosh\left(\sqrt{\beta c}\, t_1\right)\right], && (5.12)\\
t_3 &= t_2 + \frac{1}{c}\left[v^{-1}(t_3) - v_{\max}^{-1}\right], && (5.13)\\
T &= t_3 + \frac{1}{\sqrt{\alpha c}} \arctan\left[\sqrt{\frac{c}{\alpha}} v(t_3)\right] && (5.14)
\end{aligned}$$

and for linear type of resistance function we derive the following relations:

$$\begin{aligned}
t_1 &= -\frac{1}{b} \ln\left(1 - \frac{b}{\beta} v_{\max}\right), && (5.15)\\
t_2 &= \frac{\beta}{b^2 v_{\max}} \ln\left(1 - \frac{b}{\beta} v_{\max}\right) + \frac{\alpha}{b^2 v_{\max}} \ln\left[1 + \frac{b}{\alpha} v(t_3)\right] \\
&\quad - \frac{1}{b} \ln\left(1 - \frac{b}{\beta} v_{\max}\right) + \frac{1-\sqrt{1-\gamma}}{b\gamma} + \frac{L}{v_{\max}} - \frac{v(t_3)}{bv_{\max}}, && (5.16)\\
t_3 &= t_2 - \frac{1}{b} \ln\left(\frac{1-\sqrt{1-\gamma}}{\gamma}\right), && (5.17)\\
T &= t_3 + \frac{1}{b} \ln\left[1 + \frac{b}{\alpha} v(t_3)\right]. && (5.18)
\end{aligned}$$

The Figure 5.1 shows a typical sample speed profile for this type of optimal strategy.

In the case $t_1 = t_2$ we need to determine the values of three unknown parameters $t_1 = t_2$, t_3 and T. We cannot use the Equation (5.7), whereas the Equation (5.8) is still applicable. With use of the Equation (5.8), integrating the variables x and v on separate time intervals, comparing the values of these variables at switching points t_1 and t_3 and

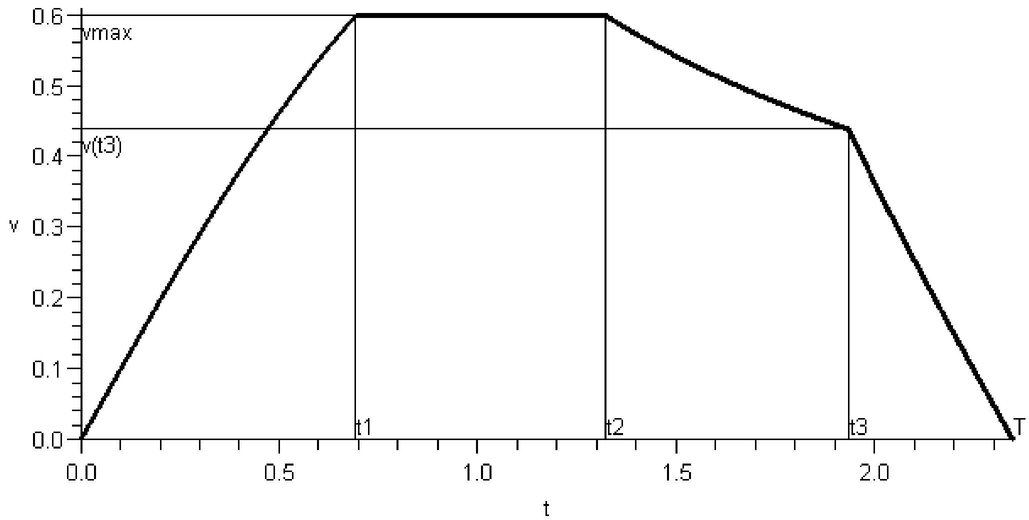

Figure 5.1: *A typical speed profile for* $t_1 < t_2$*, parameters* $\alpha = 1$*,* $\beta = 1$*,* $c = 1$*,* $L = 1$*,* $\gamma = 0.5$*,* $p = 0.7$ *and* $r = cv^2$

employing the conditions (5.4) and (5.5) we arrive at the following relation for calculation of the value $v(t_3)$ in case of the resistance function $r = cv^2$:

$$\begin{gathered} q\left[1+\left(\frac{c}{\alpha}+\frac{c}{\beta}\,\mathrm{e}^{2cL}\right)v^2(t_3)\right]^{\frac{3}{2}}+pcv^3(t_3)\,\mathrm{e}^{3cL}- \\ \left[1+\left(\frac{c}{\alpha}+\frac{c}{\beta}\,\mathrm{e}^{2cL}\right)v^2(t_3)\right]\cdot\left[q\,\mathrm{e}^{cL}+p\gamma cv^3(t_3)\,\mathrm{e}^{cL}\right]=0\,. \end{gathered} \tag{5.19}$$

The value of the maximal velocity $v_{\max} = v(t_1)$ can be calculated afterwards from the following relation

$$v_{\max} = \frac{v(t_3)\,\mathrm{e}^{cL}}{\sqrt{1+\left(\frac{c}{\alpha}+\frac{c}{\beta}\,\mathrm{e}^{2cL}\right)v^2(t_3)}} > v(t_3)\,.$$

The last inequality determines which of the roots of the Equation (5.19) it is necessary to choose in order to obtain a feasible solution of the problem. Equations for the determination of the values of switching times for the resistance function $r = cv^2$ in case $t_1 = t_2$ are as follows:

$$t_1 = t_2 = \frac{1}{\sqrt{\beta c}}\,\mathrm{arctanh}\left(\sqrt{\frac{c}{\beta}}\cdot v_{\max}\right), \tag{5.20}$$

$$t_3 = t_1 + \frac{1}{c}\left[v^{-1}(t_3) - v_{\max}^{-1}\right], \tag{5.21}$$

$$T = t_3 + \frac{1}{\sqrt{\alpha c}}\arctan\left[\sqrt{\frac{c}{\alpha}}v(t_3)\right]. \tag{5.22}$$

For linear case we can use a similar approach and derive the following equation for calculation of the value $v(t_3)$:

$$\begin{gathered} qbv(t_3)+pb^2\left\{\frac{\beta}{b}\cdot\left[1-\mathrm{e}^{-\frac{b^2L}{\beta}}\left[1+\frac{b}{\alpha}v(t_3)\right]^{-\frac{\alpha}{\beta}}\right]-\gamma v(t_3)\right\}\cdot v(t_3)\cdot \\ \left\{1-\mathrm{e}^{-\frac{b^2L}{\beta}}\left[1+\frac{b}{\alpha}v(t_3)\right]^{-\frac{\alpha}{\beta}}\right\}=q\beta\left\{1-\mathrm{e}^{-\frac{b^2L}{\beta}}\left[1+\frac{b}{\alpha}v(t_3)\right]^{-\frac{\alpha}{\beta}}\right\}. \end{gathered}$$

Thereafter, we can determine the value of $v_{\max} = v(t_1)$ via the relation

$$v_{\max} = \frac{\beta}{b} \cdot \left\{ 1 - \mathrm{e}^{-\frac{b^2 L}{\beta}} \left[1 + \frac{b}{\alpha} v(t_3) \right]^{-\frac{\alpha}{\beta}} \right\} > v(t_3) \ .$$

The values of the switching times for the linear type of resistance function r can be calculated by use of the following relations:

$$t_1 = t_2 = -\frac{1}{b} \ln \left(1 - \frac{b}{\beta} v_{\max} \right) , \tag{5.23}$$

$$t_3 = t_1 + \frac{1}{b} \ln \left(\frac{v_{\max}}{v(t_3)} \right) , \tag{5.24}$$

$$T = t_3 + \frac{1}{b} \ln \left[1 + \frac{b}{\alpha} v(t_3) \right] . \tag{5.25}$$

The Figure 5.2 shows a typical sample speed profile for this type of optimal strategy (i.e. $t_1 = t_2$).

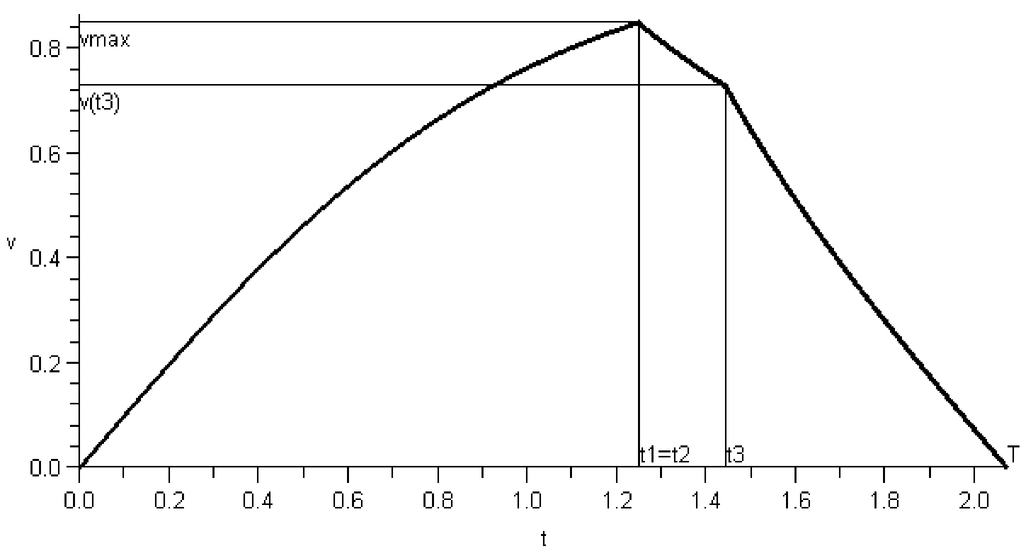

Figure 5.2: *A typical speed profile for* $t_1 = t_2$*, parameters* $\alpha = 1$*,* $\beta = 1$*,* $c = 1$*,* $L = 1$*,* $\gamma = 0.5$*,* $p = 0.3$ *and* $r = cv^2$

We have determined the values of the switching times t_1, t_2, t_3 and the total driving time T for both possible types of driving strategy which follow directly from the Pontryagin principle. We can choose the optimal case based on the value of the cost functional J. This value can be calculated according to the following relation for quadratic resistance function r

$$\begin{aligned} J = & \frac{p\beta}{c} \ln \cosh \left(\sqrt{\beta c} t_1 \right) + pc \left[\sqrt{\frac{\beta}{c}} \tanh \left(\sqrt{\beta c} t_1 \right) \right]^3 \cdot (t_2 - t_1) + \\ & \frac{p\gamma\alpha}{c} \ln \left| \cos \left[\sqrt{\alpha c} \, (T - t_3) \right] \right| + qT \end{aligned} \tag{5.26}$$

and for linear type of resistance function r

$$J = \frac{p\beta^2}{b^2}\left[e^{-bt_1} + bt_1 - 1 + b\left(1 - e^{-bt_1}\right)^2 \cdot (t_2 - t_1)\right] + \frac{p\gamma\alpha^2}{b^2}\left[1 + b\,(T - t_3) - e^{b(T-t_3)}\right] + qT\,.$$

We easily choose the lower value (of course, if more than one of the two possible strategies $t_1 = t_2$, resp. $t_1 < t_2$, is feasible). Some sample resulting values of the switching times t_1, t_2 and t_3 and total driving time T for quadratic and linear type of resistance function $r(v)$ can be found in the Table 5.1 and the Table 5.2, respectively.

p	t_1	t_2	t_3	T	J
0	1.344	1.344	1.344	2.062	2.062
0.1	1.322	1.322	1.367	2.062	1.900
0.2	1.291	1.291	1.398	2.065	1.742
0.3	1.249	1.249	1.443	2.073	1.588
0.4	1.192	1.192	1.508	2.090	1.439
0.5	1.081	1.139	1.600	2.126	1.290
0.6	0.854	1.209	1.737	2.206	1.116
0.7	0.691	1.323	1.935	2.348	0.924
0.8	0.549	1.512	2.244	2.595	0.705
0.9	0.402	1.901	2.860	3.132	0.442
0.99	0.173	4.051	6.185	6.309	0.093

Table 5.1: *Sample values of the switching times t_1, t_2 and t_3 and total driving time T for quadratic resistance function r and input parameters $\alpha = 1$, $\beta = 1$, $\gamma = 0.5$, $L = 1$ and $c = 1$ for various values of parameter p*

p	t_1	t_2	t_3	T	J
0	1.585	1.585	1.585	2.170	2.170
0.1	1.568	1.568	1.603	2.170	2.015
0.2	1.546	1.546	1.626	2.172	1.861
0.3	1.519	1.519	1.657	2.176	1.710
0.4	1.485	1.485	1.699	2.184	1.562
0.5	1.443	1.443	1.758	2.201	1.417
0.6	1.389	1.389	1.845	2.234	1.276
0.7	1.063	1.463	1.997	2.322	1.082
0.8	0.693	1.821	2.355	2.612	0.888
0.9	0.405	2.724	3.259	3.437	0.633
0.99	0.106	9.571	10.106	10.163	0.201

Table 5.2: *Sample values of the switching times t_1, t_2 and t_3 and total driving time T for linear resistance function r and input parameters $\alpha = 1$, $\beta = 1$, $\gamma = 0.5$, $L = 1$ and $c = 1$ for various values of parameter p*

The Figure 5.3 displays the values of the cost functional J for both types of resistance functions r for various values of the parameter p.

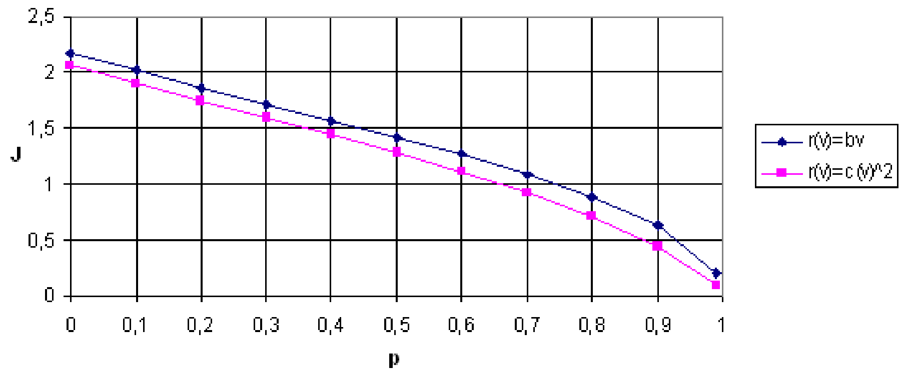

Figure 5.3: *Sample profile of values of the cost functional J for parameters $\alpha = 1$, $\beta = 1$, $c = 1$, $L = 1$, $\gamma = 0.5$ in dependence on parameter p for linear and quadratic type of resistance function r*

A different approach for determination of the optimal control strategy via the notion of critical parameter and theory of nonlinear parametric programming will be introduced in the Section 5.4.

5.4. Analysis of the solution - critical parameter

We have determined the way of calculation of the switching times and the total driving time for both possible control strategies following from the Pontryagin principle, i.e. with $t_1 = t_2$ or $t_1 < t_2$. As it is obvious from the numerical results shown in the Section 5.3, we can conjecture that there exists a certain value of the input parameter p (that we will further call critical parameter and denote as p_{cr}) such that for $p \leq p_{cr}$ the optimal solution satisfies the relation $t_1 = t_2$ whereas for $p > p_{cr}$ it holds $t_1 < t_2$ (if the remaining input parameters α, β, γ, L and c are fixed). The Figure 5.4 shows the dependence of the optimal control strategy on the input parameter p as well. Let us verify this conjecture and determine the value of p_{cr} with use of the theory of nonlinear parametric programming (for corresponding concepts, exact formulations and proofs of the theorems see Bank [1]). We will further assume the resistance function $r = cv^2$ again. For linear resistance function $r = bv$ we only derive the necessary condition for critical parameter p_{cr}.

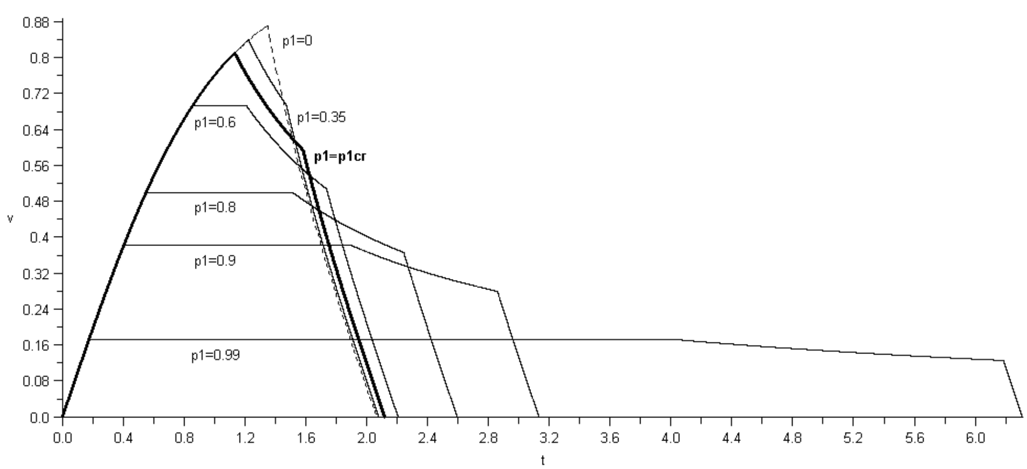

Figure 5.4: *Typical speed profiles for parameters $\alpha = 1$, $\beta = 1$, $c = 1$, $L = 1$, $\gamma = 0.5$, various values of parameter p and resistance function $r = cv^2$*

First, we rewrite the original optimal control problem (5.1)–(5.5) by use of the Theorem 10 into the form of a nonlinear programming problem. We wish to minimize the objective function (5.26) with respect to the equalities

$$\begin{aligned} &\ln\left|\sqrt{\beta c}\,(t_3-t_2)\,\frac{\sinh\left(\sqrt{\beta c}t_1\right)}{\cos\left[\sqrt{\alpha c}\,(T-t_3)\right]}+\frac{\cosh\left(\sqrt{\beta c}t_1\right)}{\cos\left[\sqrt{\alpha c}\,(T-t_3)\right]}\right| \\ &+\sqrt{\beta c}\,(t_2-t_1)\tanh\left(\sqrt{\beta c}t_1\right)-cL=0\,, \end{aligned} \tag{5.27}$$

$$\sqrt{\alpha}\tan\left[\sqrt{\alpha c}\,(t_3-T)\right]\cdot\left[\sqrt{\beta c}\,(t_3-t_2)+\coth\left(\sqrt{\beta c}t_1\right)\right]+\beta=0 \tag{5.28}$$

and inequalities

$$0\le t_1\le t_2\le t_3\le T\,. \tag{5.29}$$

We shall denote by $M\,(p)$ the set of all feasible solutions of the specified nonlinear programming problem, i.e. the set of all (t_1,t_2,t_3,T) satisfying (5.27)–(5.29) for a given p. It is easy to see that the point-to-set mapping $M\,(p)$ is continuous in p for all $p\in\langle 0,1)$ (the set of feasible solutions of the problem actually does not depend on p).

Lemma 4. *The point-to-set mapping*

$$\psi\,(p):=\{(t_1,t_2,t_3,T)\in M\,(p)\,|\,J\,(t_1,t_2,t_3,T;p)=\varphi\,(p)\}\ ,$$

where

$$\varphi\,(p):=\inf_{(t_1,t_2,t_3,T)\in M(p)}J\,(t_1,t_2,t_3,T;p)\ ,$$

is upper semicontinuous (according to Berge - see Bank [1]) for every $0\le p\le p_{\max}<1$, $p_{\max}\in(0,1)$.

Proof. The mapping φ represents the optimal value of the cost functional J specified in the Equation (5.26) for a fixed value of p. The mapping ψ is a point-to-set mapping which assigns to every fixed value of $p\in\langle 0,p_{\max}\rangle$ a set of all optimal solutions of the given nonlinear programming problem, i.e. the set of all optimal (t_1,t_2,t_3,T). The mapping M is also lower semicontinuous (according to Berge) on $\langle 0,1)$. Further, $J=J\,(t_1,t_2,t_3,T;p)$ is upper semicontinuous on $M\,(p)\times p$ for a fixed parameter p. Thus, φ is an upper semicontinuous mapping for every $p\in\langle 0,p_{\max}\rangle$ (see Bank [1]).

Let us note that M is a non-empty set for every $p\in\langle 0,p_{\max}\rangle$. Further, there exists a value $T_{\max}$ such that all $(t_1,t_2,t_3,T)\in\psi\,(p)$ satisfy the relation $T\le T_{\max}$ for every $p\in\langle 0,p_{\max}\rangle$ (the value $T_{\max}$ obviously depends on $p_{\max}$). Therefore, we may restrict (without loss of generality) the set of all feasible solutions $M\,(p)$ of the problem (5.26)–(5.29) on those satisfying $T\le T_{\max}$ for arbitrary $p\in\langle 0,p_{\max}\rangle$ (assuming $T_{\max}$ large enough). The set of all (t_1,t_2,t_3,T) satisfying (5.29) as well as the relation $T\le T_{\max}$ is a compact metric space. Further, mapping M is closed in p for every $p\in\langle 0,p_{\max}\rangle$ since M is continuous in p and the set of all (t_1,t_2,t_3,T) satisfying (5.27)–(5.29) is closed. Therefore, the mapping ψ is upper semicontinuous (according to Berge) for every $0\le p\le p_{\max}$ (see Bank [1]). □

The assertion of the Lemma 4 ensures that if we choose some fixed p^* and the corresponding optimal solution $\left(\hat{t}_1^*,\hat{t}_2^*,\hat{t}_3^*,\hat{T}^*\right)$ of (5.26)–(5.29), then considering p sufficiently close to p^* we obtain optimal solution $\left(\hat{t}_1,\hat{t}_2,\hat{t}_3,\hat{T}\right)$ close to $\left(\hat{t}_1^*,\hat{t}_2^*,\hat{t}_3^*,\hat{T}^*\right)$.

Now, let us introduce the notion of the critical parameter p_{cr} and describe its calculation.

Definition 14. *A parameter p is said to be the critical parameter of the problem (5.26)–(5.29) (and we shall further denote it as p_{cr}) if there exists an $\epsilon > 0$ such that for $p = p_{cr}$ the nonlinear programming problem (5.26)–(5.29) has an optimal solution with property $\hat{t}_1 = \hat{t}_2$ and for $p \in (p_{cr}, p_{cr} + \epsilon)$ the corresponding optimal solution satisfies $\hat{t}_1 < \hat{t}_2$.*

Lemma 5. *Let p_{cr} be the critical parameter of the problem (5.26)–(5.29). Then*

$$p_{cr} = \frac{1}{2cv_{cr}^3 + 1}, \qquad \textit{where} \qquad v_{cr} = \sqrt{\frac{1-\eta^2}{\frac{c}{\beta} + \frac{c}{\alpha}\,\mathrm{e}^{-2cL}}} \tag{5.30}$$

and η is the unique solution, satisfying the relation $\eta > \mathrm{e}^{-cL}$, of the equation

$$2\eta^3\,\mathrm{e}^{3cL} - 3\eta^2\,\mathrm{e}^{2cL} + \gamma = 0\,. \tag{5.31}$$

Proof. In the Section 5.2 we derived the values t_1, t_2, t_3 and T under the assumption $t_1 < t_2$ (the Equations (5.11)–(5.14)). According to the Lemma 4, mapping $\psi(p)$ is upper semicontinuous (according to Berge) for every $0 \le p \le p_{\max} < 1$, $p_{\max} \in \langle 0, 1)$. Thus, we may use the Equations (5.11)–(5.14) and letting $t_2 \to t_1^+$ (right-sided limit) we arrive after some modifications at the Equation (5.31).

Let us show the uniqueness of the solution to the Equation (5.31) for $\eta > \mathrm{e}^{-cL}$. This can be proved by setting

$$F(\eta) := 2\eta^3\,\mathrm{e}^{3cL} - 3\eta^2\,\mathrm{e}^{2cL} + \gamma\,.$$

Then the following relations can be easily verified:

$$F\left(\mathrm{e}^{-cL}\right) = \gamma - 1 < 0\,, \quad \lim_{\eta\to\infty} F(\eta) = \infty\,, \quad F'(\eta) > 0 \,\mathrm{on} \left(\mathrm{e}^{-cL}, \infty\right)\,.$$

Therefore, $F(\eta)$ is a strictly increasing function on $\left(\mathrm{e}^{-cL}, \infty\right)$ with values of opposite sign in boundary points of the interval. Hence, the Equation (5.31) has a unique solution on $\left(\mathrm{e}^{-cL}, \infty\right)$. The interval $\left(\mathrm{e}^{-cL}, \infty\right)$ corresponds to all feasible values of the velocity $v > 0$ under assumption $u \in \langle -\alpha, \beta \rangle$. □

Theorem 11. *Let $\left(\hat{t}_1, \hat{t}_2, \hat{t}_3, \hat{T}\right)$ be the optimal solution of the problem (5.26)–(5.29). Then either $\hat{t}_1 = \hat{t}_2$ for every $p \in (0, 1)$ or there exists a unique value p_{cr} with the property that for $p \in (0, p_{cr})$ the optimal solution satisfies $\hat{t}_1 = \hat{t}_2$ and for $p \in (p_{cr}, 1)$ the relation $\hat{t}_1 < \hat{t}_2$ is fulfilled. Moreover, the value p_{cr} can be found via the Equation (5.30).*

Proof. Let us assume that there exists a parameter p^* such that for $p \in (p^* - \epsilon, p^*)$, $\epsilon > 0$ being sufficiently small, the problem (5.26)–(5.29) has an optimal solution with property $\hat{t}_1 < \hat{t}_2$ and for $p = p^*$ the corresponding optimal solution satisfies $\hat{t}_1 = \hat{t}_2$. Then the necessary condition for p^* is given by (5.30) (with p_{cr} replaced by p^*). We have shown previously that there exists a unique such value p^*, i.e. the existence of p^* implies that p_{cr} does not exist. Further note that for $p = 0$ (time optimal control) the corresponding optimal solution $\left(\hat{t}_1, \hat{t}_2, \hat{t}_3, \hat{T}\right)$ satisfies the relation $\hat{t}_1 = \hat{t}_2 = \hat{t}_3$. If for $p > 0$ the corresponding optimal solution satisfies $\hat{t}_1 < \hat{t}_2$, then the relations (5.10) and (5.13) have to be fulfilled. However, the mapping ψ is upper semicontinuos (according to Berge) for every $0 \le p \le p_{\max} < 1$, $p_{\max} \in \langle 0, 1)$. Hence, for $p > 0$ (p being sufficiently close to 0) the optimal solution has to satisfy $\hat{t}_1 = \hat{t}_2$. Consequently, p^* cannot exist without appearance of p_{cr} and that is a contradiction. □

Numerical results show that for p large enough the optimal solution always satisfies $\hat{t}_1 < \hat{t}_2$ and therefore we may introduce a conjecture that the first variant described in the Theorem 11 (i.e. $\hat{t}_1 = \hat{t}_2$ for all $p \in (0,1)$) actually does not occur.

For linear type of resistance function r we may use a similar approach and derive the following necessary condition for p_{cr}:

$$p_{cr} = \frac{1}{1 + b v_{cr}^2},$$

where v_{cr} is the unique solution of the following equation on $\left\langle 0, \frac{\beta}{b} \right\rangle$:

$$\left(1 - \frac{b}{\beta} \cdot v_{cr}\right)^{\beta} \cdot \left(1 + \frac{b}{\alpha} \cdot \frac{1 - \sqrt{1-\gamma}}{\gamma} \cdot v_{cr}\right)^{\alpha} - \mathrm{e}^{-b^2 L} = 0\,.$$

Let us note that for $\alpha = 1$, $\beta = 1$, $L = 1$, $\gamma = 0.5$ and under assumption of the resistance function $r = v^2$ (resp. $r = v$) we obtain the value of critical parameter $p_{cr} = 0.48347$ (resp. $p_{cr} = 0.64384$).

6. Energy efficient train control on a track with non-zero gradient

This chapter deals with the energy efficient train control under additional assumption of a non-zero track gradient. We shall discuss the case of the uphill and downhill drive with prescribed gradient of the track. We introduce the optimal driving strategy as well as calculation of the switching times for all types of optimal control strategy. Further, we will discuss the concept of the critical time and explain its significance for the choice of the optimal type of control strategy. For downhill drive there are another two characteristic values of the driving time that determine the optimal strategy which we are going to investigate. For this case, the Pontryagin principle admits also a completely different optimal driving strategy for certain values of input parameters of the problem in comparison to the basic energy efficient train control problem. Let us note that most of the results discussed in this chapter have not been published yet and will be a subject of author's further investigation.

6.1. Formulation of the problem

Throughout this chapter we are going to deal with the following optimal control problem:

$$J = \int_0^T u^+ v \,\mathrm{dt} \;\; \rightarrow \;\; \min \tag{6.1}$$

with respect to the system of differential equations

$$\dot{x}(t) = v(t)\,, \tag{6.2}$$
$$\dot{v}(t) = u(t) - r(v) + g \tag{6.3}$$

and boundary conditions

$$x(0) = 0\,,\;\; v(0) = 0\,, \tag{6.4}$$
$$x(T) = L\,,\;\; v(T) = 0\,, \tag{6.5}$$

where function u^+ fulfills the relation

$$u^+(t) := \begin{cases} u(t) & \text{for } u(t) > 0 \\ 0 & \text{for } u(t) \le 0\,. \end{cases}$$

Similar properties of the relevant functions to those presented in the Section 3.1 will be applied for this type of optimal control problem as well. We assume that the control variable u is a piecewise continuous function mapping the interval $[0, T]$ into $[-\alpha, \beta]$, where $\alpha, \beta > 0$ are given constants and $r = r(v)$ is a differentiable function (with respect to v) with the properties $r, r' \ge 0$, $r(0) = 0$ and $r'(v)\,v$ is a nondecreasing function for $v \ge 0$. We shall illustrate our considerations in this chapter utilizing the linear resistance function $r = bv$ (satisfying the required properties). A generalization to other common types of resistance function is only a technical matter. To simplify our future considerations, we shall further assume that the constant g satisfies $g \in (-\alpha, \beta)$. The general case $g = g(x)$ will be briefly discussed in the next section (in such a case g is assumed to be at least continuous and (obviously) constrained).

Let us recall that the problem (6.1)–(6.5) describes the motion of a train along a straight track of length $L > 0$ with minimal consumption of electric energy J and with a constant gradient. Parameter g represents the gravitational acceleration caused by the track gradient (obviously, $g > 0$ corresponds to downhill drive whereas $g < 0$ describes an uphill drive with a constant gradient). Without loss of generality let us further assume that the mass of the train $m = 1$. The phase coordinates x and v correspond to position and speed of the train, respectively. The given parameter T represents the time that is available according to the timetable for the train to complete the track. The function r represents the frictional resistance.

6.2. Description of optimal control strategy

In this section we are going to develop the optimal control strategy for the problem (6.1)–(6.5). First, we need to determine the value of the minimum time $T_{\min}$ again that it is possible to complete the track within. Solving the corresponding minimum time problem (i.e. $J = T \quad\rightarrow\quad \min.$) we easily arrive at the standard "bang-bang" control.

As it is obvious, the value of the time $T_{\min}$ can be exactly determined if we specify the form of the resistance function r. Under assumption $r\left(v\right) = bv$ $(b > 0)$ we obtain the following relation

$$T_{\min} = \frac{1}{b} \ln \eta \, ,$$

where η has to satisfy the equation

$$\left(\alpha + \beta\right) \mathrm{e}^{Lb^2/(\alpha+\beta)} \cdot \eta^{(\alpha - g)/(\alpha+\beta)} - \left(\alpha - g\right) \cdot \eta - \beta - g = 0 \, .$$

Let us further assume that the relation $T > T_{\min}$ is satisfied for the given time T.

The following considerations determine the character of the optimal control strategy. First, let us introduce the Hamilton function in the form

$$H = \lambda_0 u^+ v + \lambda_1 v + \lambda_2 \left[u - r\left(v\right) + g\right] \, ,$$

where (without loss of generality) we consider $\lambda_0 = -1$. The other two Lagrange multipliers λ_1 and λ_2 are continuous solutions of the adjoint system

$$\begin{aligned} \dot{\lambda}_1 &= -\frac{\partial H}{\partial x} = 0 \, , \\ \dot{\lambda}_2 &= -\frac{\partial H}{\partial v} = u^+ - \lambda_1 + \lambda_2 r'\left(v\right) \, . \end{aligned}$$

The Pontryagin maximum principle yields the following optimality condition (here $(\hat{x}, \hat{v}; \hat{u})$ denote the optimal controlled process):

$$-u^+ \hat{v} + \lambda_1 \hat{v} + \lambda_2 \left[\hat{u} - r\left(\hat{v}\right) + g\right] = \max_{u \in \langle -\alpha, \beta \rangle} \left[-u^+ \hat{v} + \lambda_1 \hat{v} + \lambda_2 \left(u - r\left(\hat{v}\right) + g\right)\right] \, .$$

This relation can be simplified for $\hat{u} \geq 0$ into the following form

$$\hat{u}\left(\lambda_2 - \hat{v}\right) = \max_{u \in \langle 0, \beta \rangle} \left[u\left(\lambda_2 - \hat{v}\right)\right]$$

and similarly for $\hat{u} < 0$ into form

$$\hat{u}\lambda_2 = \max_{u \in \langle -\alpha, 0 \rangle} [u\lambda_2] \, .$$

Summarizing this we obtain the relation

$$\hat{u}(t) = \begin{cases} \beta & \text{for } \lambda_2(t) - \hat{v}(t) > 0\,, \\ \text{not specified in } \langle 0, \beta \rangle & \text{for } \lambda_2(t) - \hat{v}(t) = 0\,, \\ 0 & \text{for } \lambda_2(t) - \hat{v}(t) < 0 \quad \wedge \quad \lambda_2(t) > 0\,, \\ \text{not specified in } \langle -\alpha, 0 \rangle & \text{for } \lambda_2(t) = 0\,, \\ -\alpha & \text{for } \lambda_2(t) < 0\,. \end{cases} \tag{6.6}$$

Further, let us assume that $\lambda_2(t) = \hat{v}(t)$ on a nontrivial interval $I \subset \langle 0, T \rangle$. Then, $\dot{\lambda}_2(t) = \dot{\hat{v}}(t)$ on I. Hence, it holds the following relation

$$\hat{u} - r(\hat{v}) + g = \hat{u} - \lambda_1 + \hat{v}r'(\hat{v}) \qquad \text{on } I\,.$$

Therefore,

$$\lambda_1 = r(\hat{v}) + \hat{v}r'(\hat{v}) - g \qquad \text{on } I.$$

Thus, with respect to the relation $\dot{\lambda}_1 \equiv 0$ on $\langle 0, T \rangle$ and by utilizing properties of the function r (namely, $r, r' \geq 0$ and $r'(v)\,v$ is nondecreasing for $v \geq 0$) the previous equation implies the relation $\dot{\hat{v}} \equiv 0$ on I which yields the optimal singular control $\hat{u} = r(\hat{v}) - g \geq 0$ on I (speed holding driving mode).

The second singular case can be easily excluded for $g \leq 0$ by differentiating the relation $\lambda_2(t) = 0$ on a nontrivial interval $\bar{I} \subset \langle 0, T \rangle$ and utilizing the properties of the function r. However, for $g > 0$ (downhill drive) it cannot be generally excluded and occurs for certain values of input parameters of the problem. In such a case, by differentiating the relation $\lambda_2(t) = 0$ we obtain $\dot{\lambda}_2(t) = 0$ which results in the relation $\lambda_1(t) \equiv 0$ on $\bar{I}$. However, $\dot{\lambda}_1(t) \equiv 0$ on $\langle 0, T \rangle$. Thus, $\lambda_1(t) \equiv 0$ on $\langle 0, T \rangle$. Hence

$$\dot{\lambda}_2 = \hat{u}^+ + \lambda_2 r'(\hat{v}) \tag{6.7}$$

on $\langle 0, T \rangle$. By (6.7) the function λ_2 is nondecreasing provided $\lambda_2 > 0$ and nonincreasing if $\lambda_2 < 0$. Let us denote by t_l the left endpoint of the interval $\bar{I}$. If $\lambda_2\left(\tilde{t}\right) > 0$ for $\tilde{t} \in \langle 0, t_l)$, then $\lambda_2(t) > 0$ for all $t \geq \tilde{t}$. That is a contradiction with the assumption $\lambda_2 \equiv 0$ on $\bar{I}$. Hence, $\lambda_2(t) \leq 0$ for $t \in \langle 0, t_l) \cup \bar{I}$. The previous considerations along with the condition (6.5) yield the relation $\lambda_2(t) \leq 0$ for $t \in \langle 0, T \rangle$. Therefore, $\hat{u}(t) \leq 0$ on $\langle 0, T \rangle$.

Such a control yields the value of the cost functional $J = 0$ and therefore would be optimal. However, to ensure a feasible control of this type the value of the time T has to be sufficiently large to complete the track only with coasting and braking. The minimum value of the time T (that we shall further denote as T_c) that yields a feasible control of this type can be determined as the solution of the minimum time problem under assumption $u \in \langle -\alpha, 0 \rangle$. Therefore,

$$T_c = \frac{1}{b} \ln \omega \,,$$

where ω satisfies the equation

$$\alpha \, \mathrm{e}^{Lb^2/\alpha} \cdot \omega^{(\alpha - g)/\alpha} - (\alpha - g) \cdot \omega - g = 0\,.$$

Summarizing the previous considerations we can prove the following theorem.

Theorem 12. *Let $(\hat{x}(t), \hat{v}(t); \hat{u}(t))$, $t \in \langle 0, T \rangle$ be the energy optimal solution of (6.1)–(6.5). Then for $g \leq 0$ it holds*

$$\hat{u}(t) = \begin{cases} \beta & \text{for } \lambda_2(t) - \hat{v}(t) > 0\,, \\ r(\hat{v}) - g \equiv const. & \text{for } \lambda_2(t) - \hat{v}(t) = 0\,, \\ 0 & \text{for } \lambda_2(t) - \hat{v}(t) < 0 \quad \wedge \quad \lambda_2(t) > 0\,, \\ -\alpha & \text{for } \lambda_2(t) < 0\,, \end{cases} \tag{6.8}$$

where λ_2 is defined by the corresponding adjoint system. For $g > 0$ there exists a certain value $T > T_{\min}$ (which we shall further denote as T_c) that for $T < T_c$ the previous relation (6.8) is fulfilled, whereas for $T \geq T_c$ the optimal solution satisfies

$$\hat{u}(t) = \begin{cases} 0 & \text{for } 0 \leq t < t_c \text{ (coasting)}, \\ -\alpha & \text{for } t_c \leq t < T_c \text{ (full braking)}, \\ -g & \text{for } T_c \leq t < T \text{ (standstill)}, \end{cases}$$

where $0 < t_c < T_c \leq T$.

Let us note that for $g > 0$ and $T \geq T_c$ the optimal solution described in the previous theorem satisfies $J = 0$ and for $T > T_c$ is not unique.

It can be easily shown that the value of the switching time t_c can be determined via the following relation for $r(v) = bv$

$$t_c = T \cdot \left(1 - \frac{g}{\alpha}\right) + \frac{bL}{\alpha}\,.$$

The following theorem specifies the optimal order of the driving modes for all values of the input parameters except for the case $g > 0$ and $T \geq T_c$. It can be proved with use of the properties of the Lagrange multipliers (especially their continuity) and involving the conditions (6.4) and (6.5).

Theorem 13. *Let $(\hat{x}(t), \hat{v}(t); \hat{u}(t))$, $t \in \langle 0, T \rangle$ be the energy optimal solution of (6.1)–(6.5). Then for $g \leq 0$ there exist t_1, t_2, t_3, where $0 < t_1 \leq t_2 < t_3 < T$, such that*

$$\hat{u}(t) = \begin{cases} \beta & \text{for } 0 \leq t < t_1\,, \\ r(\hat{v}) - g \equiv const. & \text{for } t_1 \leq t < t_2\,, \\ 0 & \text{for } t_2 \leq t < t_3\,, \\ -\alpha & \text{for } t_3 \leq t \leq T\,. \end{cases}$$

The assertion of this theorem is valid for $g > 0$ and $T < T_c$ as well (where the value T_c was specified in the Theorem 12).

Let us note that the general case $g = g(x)$ yields the same optimal driving modes as specified in (6.6). The first singular case (i.e. $\lambda_2(t) = \hat{v}(t)$ on a nontrivial interval I) results again in the speed-holding control mode (i.e. $\hat{u} = r(\hat{v}) - g(\hat{x}) \geq 0$ on I). However, the second singular case (i.e. $\lambda_2(t) = 0$ on a nontrivial interval $\bar{I}$) cannot be generally easily excluded or described.

6.3. The calculation of switching times

Let us now determine the values of the switching times t_1, t_2 and t_3. Of course, this determination is possible if the type of resistance function is specified. We shall further assume that $r(v) = bv$.

First, let us suppose that the relation $t_1 < t_2$ holds. The adjoint variable λ_1 satisifies $\lambda_1(t) \equiv C_1 = const.$ on $\langle 0, T\rangle$. Then, the following condition is satisfied on (t_1, t_2):

$$\dot{\lambda}_2 = r(v_{\max}) - g - C_1 + v_{\max} \cdot r'(v_{\max}) \equiv 0\,, \tag{6.9}$$

where $v_{\max}$ is the speed-holding velocity. Further,

$$H\left(t_1^-\right) = -\beta v_{\max} + C_1 v_{\max} + v_{\max} \cdot [\beta - r(v_{\max}) + g]\,, \tag{6.10}$$

$$H\left(t_3^-\right) = C_1 v(t_3)\,, \tag{6.11}$$

where $H\left(t_1^-\right)$ and $H\left(t_3^-\right)$ denote the corresponding left-sided limits of the Hamilton function. The Equations (6.9)–(6.11) yield optimal value of the speed $v(t_3)$ for $r(v) = bv$

$$v(t_3) = v_{\max} \cdot \frac{bv_{\max}}{2bv_{\max} - g}\,.$$

Consequently, integrating the Equations (6.2)–(6.3) on corresponding time intervals, comparing the values of the variables x and v in the switching times and employing conditions (6.4)–(6.5) we obtain the following relation for calculation of the velocity $v_{\max}$:

$$\begin{aligned}&(bv_{\max} - g + \alpha) \cdot \ln \frac{(\alpha - g)\cdot(bv_{\max} - g)}{(\alpha - g)\cdot(2bv_{\max} - g) + b^2 v_{\max}^2} - \alpha \cdot \ln \frac{bv_{\max} - g}{2bv_{\max} - g}\\ &= (\beta + g - bv_{\max}) \cdot \ln \frac{\beta + g - bv_{\max}}{\beta + g} + bL^2 - b^2 v_{\max} T\end{aligned}$$

and derive the following relations for calculation of the switching times:

$$\begin{aligned}t_1 &= -\frac{1}{b} \ln \left(1 - \frac{b}{\beta + g} v_{\max}\right)\,,\\ t_2 &= T + \frac{1}{b} \ln \frac{(\alpha - g)\cdot(bv_{\max} - g)}{(\alpha - g)\cdot(2bv_{\max} - g) + b^2 v_{\max}^2}\,,\\ t_3 &= T - \frac{1}{b} \ln \left[1 + \frac{b^2 v_{\max}^2}{(\alpha - g)\cdot(2bv_{\max} - g)}\right].\end{aligned}$$

In the case $t_1 = t_2$ we need to determine the values of two unknown parameters $t_1 = t_2$ and t_3. Integrating the variables x and v on separate time intervals, comparing the values of these variables at switching points t_1 and t_3 and employing the conditions (6.4) and (6.5) we arrive at the following relation for calculation of the switching time t_1 in case of the resistance function $r = bv$:

$$\alpha^\alpha \,\mathrm{e}^{Lb^2 + \alpha bT - bgT} = \left[(\alpha - g)\,\mathrm{e}^{bT} - \beta\,\mathrm{e}^{bt_1} + \beta + g\right]^\alpha \cdot \mathrm{e}^{\beta b t_1}$$

The equation for determination of the value of the remaining switching time t_3 in case $t_1 = t_2$ is as follows:

$$t_3 = \frac{1}{b} \ln \left[(\alpha - g)\,\mathrm{e}^{bT} - \beta\,\mathrm{e}^{bt_1} + \beta + g\right] - \frac{1}{b} \ln \alpha\,.$$

We have determined the values of the switching times t_1, t_2 and t_3 for both possible types of driving strategy which follow directly from the Pontryagin principle. We can choose the optimal case based on the value of the cost functional J. This value can be calculated according to the following relation

$$J = \beta \left(\frac{\beta + g}{b^2} \, \mathrm{e}^{-bt_1} + \frac{\beta + g}{b} t_1 - \frac{\beta + g}{b^2} \right) + (bv_{\max} - g) \, v_{\max} \, (t_2 - t_1) \ .$$

We easily choose the lower value (of course, if more than one of the control strategies $t_1 = t_2$, resp. $t_1 < t_2$, is feasible). Some sample resulting values of the switching times t_1, t_2 and t_3 for linear type of resistance function $r(v)$ and parameters $g = 0.1$ (downhill drive) and $g = -0.1$ (uphill drive) can be found in the Table 6.1 and the Table 6.2, respectively.

T	t_1	t_2	t_3	J
2.110	1.449	1.449	1.449	0.752
2.200	1.211	1.211	1.769	0.560
2.277	1.141	1.141	1.908	0.506
2.400	0.842	1.295	2.079	0.450
3.000	0.475	1.927	2.767	0.297
4.000	0.298	2.891	3.825	0.180
5.000	0.221	3.808	4.854	0.118
7.000	0.147	5.501	6.882	0.052
10.000	0.097	5.684	9.895	0.006
10.101	0.095	0.095	9.995	0.005
10.500	0.055	0.055	10.395	0.002
11.054	0.000	0.000	10.949	0.000

Table 6.1: *Sample values of the switching times t_1, t_2, t_3 for linear resistance function r and input parameters $\alpha = 1$, $\beta = 1$, $L = 1$, $c = 1$ and $g = 1$ for various values of parameter T*

T	t_1	t_2	t_3	J
2.257	1.741	1.741	1.741	0.825
2.300	1.586	1.586	1.944	0.711
2.386	1.498	1.498	2.127	0.650
2.500	1.046	1.664	2.281	0.604
3.000	0.613	2.255	2.845	0.486
4.000	0.373	3.345	3.897	0.372
5.000	0.274	4.403	4.923	0.312
6.000	0.218	5.446	5.940	0.274
8.000	0.155	7.511	7.959	0.228
10.000	0.121	9.560	9.969	0.202

Table 6.2: *Sample values of the switching times t_1, t_2, t_3 and maximum velocity $v_{\max}$ for linear resistance function r and input parameters $\alpha = 1$, $\beta = 1$, $L = 1$, $c = 1$ and $g = -1$ for various values of parameter T*

The Figure 6.1 displays the values of the cost functional J for linear type of resistance function r for various values of the parameter T for both uphill and downhill drive.

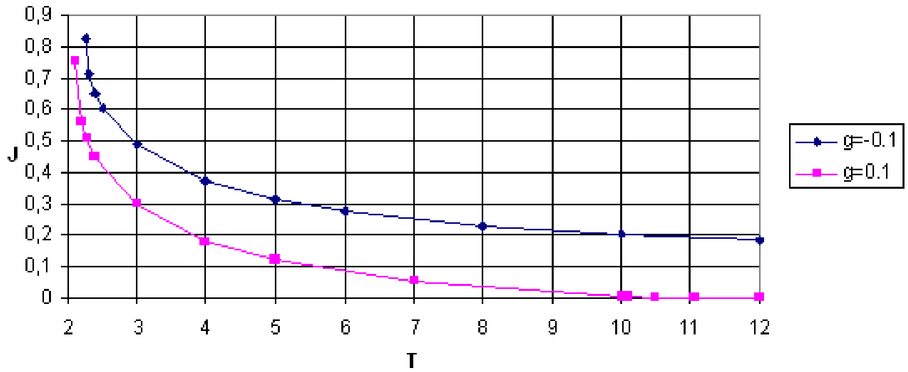

Figure 6.1: *Sample profile of values of the cost functional J for parameters $\alpha = 1$, $\beta = 1$, $c = 1$, $L = 1$ in dependence on parameter T for linear type of resistance function r and downill ($g = 1$), resp. uphill ($g = -1$), drive*

A different approach for determination of the optimal control strategy with use of the notion of the critical time and nonlinear parametric programming will be introduced in the Section 6.4.

6.4. Analysis of the solution - critical time

Numerical calculations (based on algorithms from Bazaraa et al. [2]) show that the choice of the optimal control strategy depends only on the given value of the entry parameter T. The Figure 6.2 and the Figure 6.3 show the dependence of the optimal control strategy on the input parameter T as well.

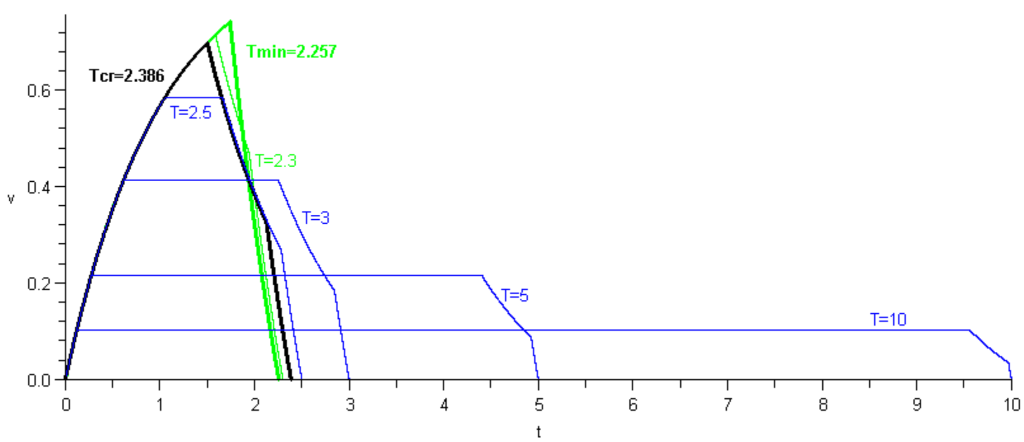

Figure 6.2: *Typical speed profiles for parameters $\alpha = 1$, $\beta = 1$, $c = 1$, $L = 1$, $g = -0.1$, various values of parameter T and resistance function $r = bv$*

A similar analysis to that introduced for the basic energy efficient train control problem in the Section 3.4 can be performed in this case as well. The resulting relation for calculation of the critical time under assumption of analogical condition to the Hypothesis 1 is as follows

$$T_{cr} = \frac{1}{b} \ln \frac{(\beta + g)}{(\alpha - g)} \cdot \frac{(bv_{cr} - g)^2 + \alpha\,(2bv_{cr} - g)}{(bv_{cr} - g) \cdot (\beta + g - bv_{cr})}\,,$$

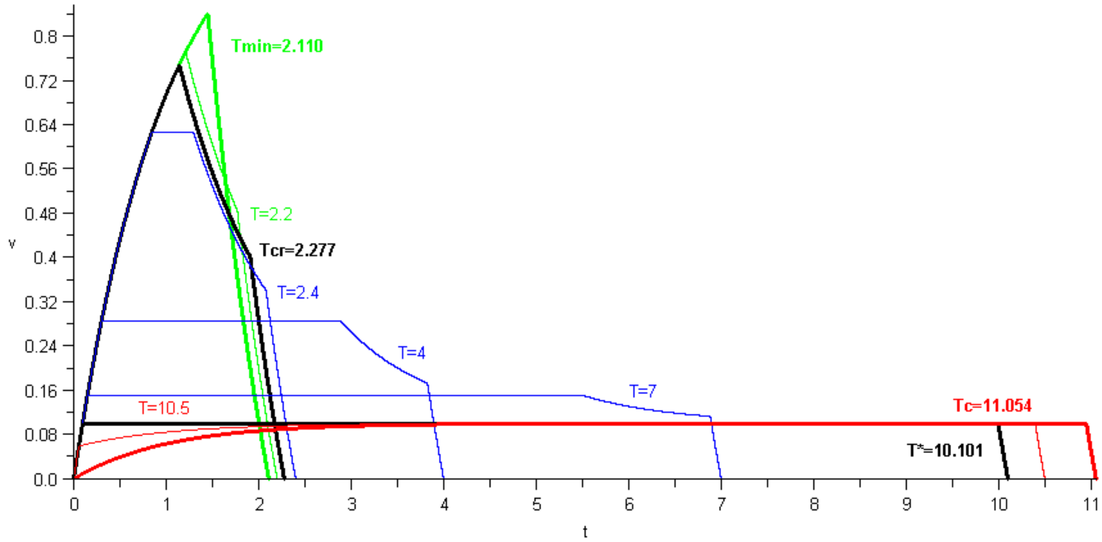

Figure 6.3: *Typical speed profiles for parameters $\alpha = 1$, $\beta = 1$, $c = 1$, $L = 1$, $g = 0.1$, various values of parameter T and resistance function $r = bv$*

where v_{cr} can be determined according to the following equation

$$\left(\frac{bv_{cr} - g}{2bv_{cr} - g}\right)^{\alpha} \cdot \left(\frac{\beta + g - bv_{cr}}{\beta + g}\right)^{\beta + g} \cdot \left[\frac{(\alpha - g) \cdot (bv_{cr} - g)}{(\alpha - g) \cdot (2bv_{cr} - g) + b^2 v_{cr}^2}\right]^{g - \alpha} = \mathrm{e}^{-b^2 L} \ . \quad (6.12)$$

This relation yields for $g < 0$ (according to numerical results) one positive solution satisfying $T_{cr} > T_{\min}$ and thus the change of the optimal control strategy (from $t_1 = t_2$ to $t_1 < t_2$) in relation to the value of the input parameter T can occur for at most one value of T (analogy to the basic energy optimal problem).

An interesting behaviour of the optimal solution can be observed for $g > 0$ (downhill drive). In such a case the Equation (6.12) results in two distinct values of T_{cr} (this can be well understood in the Figure 6.3). One of them corresponds to the notion of the critical time as was defined for the basic energy-efficient train control problem (in the Definition 13). The other one (let us denote it further as T^*) represents the reverse case where the optimal solution of the problem (6.1)–(6.5) satisfies the relation $t_1 < t_2$ for $T \in (T^* - \epsilon, T^*)$, where $\epsilon > 0$ is sufficiently small, and $t_1 = t_2$ for $T = T^*$. Thus, the optimal solution satisfies the relation $t_1 < t_2$ for $T \in (T_{cr}, T^*)$. The value T^* can be also found as the transition value where for $T > T^*$ the coasting phase (i.e. $\hat{u} = 0$) leads to accelerating of the train (which excludes the speed-holding phase). It can be therefore determined per the relation

$$T^* = \frac{1}{bg} \ln \left[\frac{\alpha^{\alpha} \beta^{\beta} \, \mathrm{e}^{Lb^2}}{(\alpha - g)^{\alpha} \cdot (\beta + g)^{\beta}}\right] .$$

This behaviour will be a subject of author's further investigation and will be introduced in a prospective paper.

7. Conclusion

7.1. Summary of obtained results

The thesis described the character of the optimal control strategy and the way of calculation of the switching times for the energy-efficient train control problem and its modifications. We performed an analysis of the solution for the presented mathematical models with use of nonlinear parametric programming. We introduced the concept of the critical time (or critical parameter) and explained its significance as the deciding factor for developing of the optimal control strategy.

We presented the basic energy-efficient train control problem under assumption of standard types of resistance function as well as some of the natural generalizations of the problem. We introduced and analysed the problem with speed constraint and discussed the problem with a non-zero track gradient. We formulated and completely solved the time-energy efficient train control problem which represents a different view on this area.

The emphasis was put mainly on exact form of solutions where the application of numerical methods is restricted only on solving algebraic equations. Let us note that most of the results presented in this thesis represent a different approach towards solving this problem than introduced in previous papers. This approach enabled a detailed analysis of the solution with use of analytical means.

7.2. Future directions

The energy-efficient train control problem can be generalized or modified in several ways. The enhanced models can be more or less complicated than those presented in this thesis. However, the general behaviour of the solution of such problems will remain similar. The introduced optimal driving modes will be present in most of the models what was proved by implementation of the results on real railway or suburban traffic with positive results. The critical time (or critical parameter) and the relating analysis with use of nonlinear parametric programming can be applied on several models as well.

The natural generalizations and extensions to our results can be achieved especially for the speed constraints or track gradient. We may assume local speed constraints represented by the Equation (4.8) as it was introduced in the Section 4.3. The general form of the track gradient can be represented by a function $g(x)$ describing varying profile of the track as it was mentioned in the Section 6.2. There will be performed a further investigation of the behaviour of the problem with constant track gradient as well. We may also further investigate steep inclines (declines) as it was discussed by Cheng et al. [6] or Howlett et al. [11]. Moreover, a combination of the restrictions and further assumptions may be applied. Further, there might be used another types of resistance functions, e.g. exponential form of the function $r(v)$.

Most of the input parameters presented in this thesis are not constant in real situations. Usually, we may observe stochastical behaviour with a mean value and a certain standard deviation based on the corresponding probability distribution. This can be applied e.g. for the maximum allowable accelaration of the train, for resistance function r or constant γ and results in a completely different approach to the problem.

The main aim of this thesis was to present an exact form of the solution for the energy efficient train control problem and its modifications where it is applicable. However, most of the problems mentioned in this section lead us to use some more or less sophisticated numerical methods or methods of artificial intelligence which was out of the scope of this thesis and will be a subject of author's future investigation.

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8. List of the used abbreviations and symbols

a, b, c	coefficients in the resistance functions
α, β	minimum and maximum allowed acceleration of the train
$\varphi(\lambda)$	optimal value of the cost functional for a nonlinear parametric programming problem
γ	coefficient in the cost functional relating to return of electrical energy while braking
$g(x)$	gravitational acceleration caused by the track gradient
H	Hamilton function
J	cost functional
L	length of the track
l.s.c.-B	lower semicontinuous mapping (according to Berge)
l.s.c.-H	lower semicontinuous mapping (according to Hausdorff)
$\lambda_0, \lambda_1, \lambda_2, \mu$	Lagrange multipliers
m	mass of the train
$M(\lambda)$	set of all feasible solutions of a nonlinear parametric programming problem for fixed value λ
$N(\mathbf{x}, t)$	function representing tangency conditions
p, q	parameters in the cost functional for time-energy efficient train control
p_{cr}	critical parameter
$r = r(v)$	frictional resistance
$S(\mathbf{x}, t)$	function representing constraint on the state variable
$S^{(q)}$	q-th derivative of the function S
t	time variable
T	time available according to timetable for the train to complete the track
T_{cr}	critical time
$T_{\min}$	minimum time that it is possible to complete the track within

t_1, t_2, t_3	switching times
t_1^+, t_1^-	right-sided (left-sided) limit of the corresponding function
u	control variable
U	control space
u.s.c.-B	upper semicontinuous mapping (according to Berge)
u.s.c.-H	upper semicontinuous mapping (according to Hausdorff)
$v = v(t)$	velocity of the train along the track
v_{cr}	critical velocity
$v_{\max}$	maximum velocity achieved by the train along the whole track
v_m	maximum allowed velocity
$x = x(t)$	position of the train along the track
$\dot{x}$	derivative of the function x according to time
$\mathbf{x} = (x_1, \ldots, x_n)$	state variables
X	state space of the investigated object
$(\hat{\mathbf{x}}(t); \hat{\mathbf{u}}(t))$	optimal control process
$\psi(\lambda)$	point to set mapping assigning to parameter λ the set of all optimal solutions of the given nonlinear parametric programming problem