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NUMERICAL METHODS FOR FRACTIONAL DIFFERENTIAL EQUATIONS INITIAL VALUE PROBLEMS
NUMERICKÉ METODY PRO ŘEŠENÍ POČÁTEČNÍCH ÚLOH ZLOMKOVÝCH DIFERENCIÁLNÍCH ROVNIC

MASTER'S THESIS
DIPLOMOVÁ PRÁCE

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Assignment Master's Thesis

Institut: Institute of Mathematics
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As provided for by the Act No. 111/98 Coll. on higher education institutions and the BUT Study and Examination Regulations, the director of the Institute hereby assigns the following topic of Master's Thesis:

Numerical Methods for Fractional Differential Equations Initial Value Problems

Brief Description:

Fractional differential equations theory is one of the most studied areas in mathematical analysis. Many new problems and challenges were opened up by the introduction of non–integer derivatives. Numerical solution and analysis of fractional differential equation initial value problems is then one of them.

Master's Thesis goals:

1. Research of numerical methods and their properties for the fractional differential equation initial value problem with Caputo–type differential operators.
2. Analysis of selected methods, their comparison and implementation in the MATLAB system.

Recommended bibliography:

BUTCHER, John Charles. Numerical methods for ordinary differential equations. Chichester, West Sussex, England ; Hoboken, NJ: J. Wiley, 2003, xiv, 425 s. : il. ISBN 0-471-96758-0.

DIETHELM, Kai. The analysis of fractional differential equations: an application-oriented exposition using differential operators of Caputo type. Berlin: Springer, 2010, viii, 247 s. ISBN 978-3-642-14-73-5.

PODLUBNÝ, Igor. Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. San Diego: Academic Press, 1999, xxiv, 340 s. : il. ISBN 0-12-558840-2.

Deadline for submission Master's Thesis is given by the Schedule of the Academic year 2020/21

In Brno,

L. S.

prof. RNDr. Josef Šlapal, CSc. Director of the Institute	doc. Ing. Jaroslav Katolický, Ph.D. FME dean

Abstrakt
Tato diplomová práce se zabývá numerickými metodami pro řešení počátečních problémů zlomkových diferenciálních rovnic s Caputovou derivací. Jsou uvedeny dva numerické přístupy spolu s přehledem základních aproximačních formulí. Dvě verze Eulerovy metody jsou realizovány v Matlabu a porovnány na základě numerických experimentů.

Summary
The thesis deals with numerical methods for initial value problems of Caputo fractional differential equations. Particularly, two numerical approaches are introduced together with overview of fundamental approximation formulas. Two versions of Euler method are build in Matlab and they are compared by numerical experiments.

Klíčová slova
počáteční problém, zlomková diferenciální rovnice, numerické metody, Caputova derivace.

Keywords
initial value problem, fractional differential equation, numerical methods, Caputo derivatives.

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Declaration

I declare that I have written the diploma thesis “Numerical methods for fractional differential equations initial value problems” on my own under the guidance of my supervisor doc. Ing. Petr Tomášek, Ph.D., and that I used the sources listed in bibliography.

Brno ....

Vincent Bediako Oti

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Vincent Bediako Oti

Contents

1. Introduction

Fractional calculus and classical calculus are old areas of mathematics dated back to the seventeenth century [1]. Newton and Leibniz were the founding fathers of Differential and Integral Calculus. Leibniz invented a symbol for the *nth* derivative of a function f, i.e $\dfrac{d^n}{dx^n}f(x)$ and reported in a letter to L'Hopital with the implicit assumption that $n \in \mathbb{N}$. L'Hopital in a correspondence asked Leibniz on 30^{th} September 1695 that "what does $\dfrac{d^n}{dx^n}f(x)$ mean if $n = \frac{1}{2}$?."

Leibniz later wrote: "It will lead to a paradox, from which one day useful consequences will be drawn." The idea is that if we take two half-derivatives of a function, we should get back its first derivative. The letter of L'Hopital is recognized as the first occurrence of phenomenon which is now known as fractional derivative.

Many mathematicians have contributed to the growth of this branch of mathematics over the last three centuries, some of whom include: Laplace (1812), Lacroix (1812), Fourier (1822), Abel (1823-1826), Liouville (1832-1837), Riemann (1847), Grünwald (1867-1872), Letnikov (1868-1872), Senin (1869), Laurent (1884), Heaviside (1892-1912), Weyl (1917), Davis (1936), Erdelyi(1939-1965), Gelfand and Shilov (1959-1964), Dzherbarshian (1966), Caputo (1969) and others, see [2]. In particular, Caputo used his own definition of fractional differentiation to formulate and solve certain viscoelasticity problems.

When it comes to solving and simulating integer-order systems, numerical methods are important. Numerical integration is much more vital when dealing with fractional-order systems. As a result, the design of accurate and fast algorithms for the numerical integration of fractional-order differential equations becomes essential to the field. To solve single-term fractional differential equations, various computational approaches were used decades earlier. Diethelm et al. [3], Ford and Connolly [4] reviewed some of the existing methods and demonstrated their area of strengths and weaknesses. Several analytical ways and numerical schemes to solving fractional differential equations can be found in [5, 6].
An important work that maps fractional differential equations (FDE) from Caputo algebra to Riemann-Liouville algebra in order to preserve the additivity of base function powers under multiplication was published in [7]. The authors proposed a new method for constructing closed-form solutions using only 1/2 order derivative equations. They also stated that their method can be used in situations where the derivative order is a rational number. M. Pakdaman et. al [8] proposed a new approach for approximating the solution of fractional differential equations by using the fundamental properties of artificial neural networks for function approximation. A high-order algorithm for numerical estimation of fractional differential equations based on the Riemann-Liouville fractional derivative using polynomial interpolation was proposed in [9].
A predictor-corrector method and an iterative method for solving fractional differential equations have been proposed in [10] and [11]. In [12] the Riemann-Liouville integral for solving FDEs was approximated by piecewise quadratic polynomial interpolation. In [13] authors used non-equidistant step-sizes and in [14] they used a piecewise linear interpolation to obtain a discretization of a multi-term FDE. For Grünwald-Letnikov schemes,

linear optimization to obtain the discretization weights, updated in each iteration [15] and a matrix approach for numerically solving ordinary and partial fractional-order differential equations applied to the recursive fractional-order derivative [16] were presented.
In [17] Legendre wavelet method and Gauss-Legendre quadrature rule for evaluating the fractional integrals to solve nonlinear fractional mixed Volterra-Fredholm integro-differential equations along with mixed boundary conditions were used. [18, 19] demonstrated that a Caputo derivative can be approximated by higher integer order derivatives and used ordinary numerical methods to solve some special FDEs. Many other numerical methods with various approaches and descriptions of the derivative are still being researched in the literature.

In the literature, there are many definitions of fractional derivatives that do not coincide, making it more difficult in general. However, for fractional integrals, integer-order derivatives, and integrals, there is a unique definition, from which they can be determined in the classical sense [20, 21, 22, 23].

In 1823, Abel used fractional calculus to solve an integral equation that occurs in the formulation of the problem of finding the shape of a frictionless wire lying in a vertical plane such that the time of a bead placed on the wire slides to the lowest point of the wire in the same time regardless of where the bead is placed, a problem known as the tautochrone problem, see [20, 21].
Fractional derivatives are a powerful method for explaining memory and hereditary properties in a variety of materials and processes [24]. In certain cases, fractional order models of real systems are more suitable than integer order models. In general, the fractional calculus' superior performance is demonstrated by lower error levels generated during estimation, see e.g. [25, 26, 27, 28, 29]. The fact that fractional calculus has found various applications in Bio Chemistry (modelling of polymers and proteins), Mechanics (theory of viscoelasticity and viscoplasticity), Electrical Engineering (transmission of ultrasound waves), Medicine (modelling of human tissue under mechanical loads), Control Theory of Dynamical Systems, and Stochastic Analysis in the last four decades leads to its great theory development, see e.g. [2].
One explanation for this is that practical modelling of physical phenomena is not solely dependent on time, but rather on the background of previous time, which can be accomplished using fractional calculus. For more information on recent developments in fractional calculus see [30] . The continuous and discrete approaches to fractional calculus [2] are the two main approaches. The continuous approach is linked to the two most widely used definitions of fractional derivatives, the Riemann-Liouville and Caputo definitions [21]. The discrete approach dealt with Grünwald-Letnikov fractional derivatives. The Caputo operator will be the object of our attention as we attempt to implement these definitions in a continuous manner. The Riemann-Liouville definition is obviously very useful in the development of fractional derivatives and integrals theory. Even, for the sake of pure mathematics applications [22].
The Caputo fractional derivative is widely used because it offers initial conditions with clear meaning for fractional order differential equations in applied problems.

The aim of this thesis is to describe numerical methods for solving initial value problem of fractional differential equation in the Caputo sense. Much priority will be given to single-term equations. We assume that the reader has a basic knowledge of ordinary differential equations (ODEs) theory and numerical methods for initial value problems (IVPs) for ODEs.

This thesis consists of 4 sections. Section 1 consists of the introductory part. Section 2 introduces some useful special functions, Caputo fractional derivative and initial value problem for Caputo fractional differential equation. Section 3 deals with the numerical methods. Stability and convergence of the solution is also studied and some interesting new properties are discovered. Particular examples are calculated and illustrated by their graphs. Section 4 concludes the thesis by summarizing the results. The MATLAB codes developed for presented numerical simulations are enclosed in Appendix. Appendix A contains the algorithm for the forward Euler method and Appendix B contains the algorithm for the improved forward Euler method. Next is the list of abbreviations and symbols.

2. Theoretical Background

The branch of mathematics that deals with the properties of differential and integral operators of arbitrary order(non-integer order called fractional derivatives or integrals) is classified as fractional calculus see, [22]. This section introduces fundamentals of fractional calculus, which are related to the studied problems. It unifies and generalizes n-fold integration and integer-order differentiation. Fractional derivatives and integrals, in other words, can be thought of as an "interpolation" of the infinite sequence

$$\ldots, \quad \int_{t_0}^{t}\int_{t_0}^{\tau_1} f(\tau_2)\, d\tau_2 d\tau_1, \quad \int_{t_0}^{t} f(\tau_1)\, d\tau_1, \quad f(t), \quad \frac{df(t)}{dt}, \quad \frac{d^2 f(t)}{dt^2}, \quad \ldots$$

of the classical n -fold integrals and n -fold derivatives.

2.1. Special Functions

In this section a survey of several special functions which are of fundamental use in fractional calculus development is given.

• Gamma Function

Gamma function is an extension of the factorial function $n!$, thus $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$. Gamma function generalizes factorial for arguments $z \in \mathbb{C}\backslash\{0,-1,-2,\ldots\}$. Such a generalization exists and is well-known as Euler's Gamma function (or Euler's integral of the second kind) defined by

$$\Gamma(z) = \begin{cases} \int_0^\infty t^{z-1}\exp(-t)dt & \text{if } \operatorname{Re}(z) > 0 \\ \frac{\Gamma(z+1)}{z} & \text{if } \operatorname{Re}(z) < 0, z \neq 0,-1,-2,\ldots \end{cases}$$

Gamma function is defined for all points on the complex plane except at $0,-1,-2,\ldots$, where it has simple poles. Thus $\Gamma : \mathbb{C}\backslash\{0,-1,-2,\ldots\} \longrightarrow \mathbb{C}$ holds (see [31])

- $\Gamma(z+1) = z\Gamma(z)$ for $z \in \mathbb{C}\backslash\{0,-1,-2,\ldots\}$
- $\Gamma(n+1) = n!$ for $n \in \mathbb{N}_0$
- $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ for $z \in \mathbb{C}\backslash\{0,-1,-2,\ldots\}$
- For half integer arguments, $\Gamma(n/2), n \in \mathbb{N}$, we have

$$\Gamma(n/2) = \frac{(n-2)!!\sqrt{\pi}}{2^{(n-1)/2}},$$

where the double factorial is defined by;

$$n!! = \begin{cases} n\cdot(n-2)\ldots 5\cdot 3\cdot 1 & n>0, \text{ odd} \\ n\cdot(n-2)\ldots 6\cdot 4\cdot 2 & n>0, \text{ even} \\ 1 & n=0,-1. \end{cases}$$

Especially for $n = 1$, we have $\Gamma(1/2) = \sqrt{\pi}$.

• Beta Function

By Beta Function, we mean the function defined by the integral

$$B(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}dt, \quad \mathrm{Re}(p)>0, \mathrm{Re}(q)>0.$$

Substituting $t=1-\eta$, we see that the Beta function is symmetric

$$B(p,q)=B(q,p).$$

Beta function is sometimes introduced as combination of Gamma functions (see [31]), in the form;

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{2.1}$$

Beta function is defined for all points on the complex plane except $0,-1,-2,\ldots,$ where it has simple poles as a consequence of (2.1) .

• Mittag-Leffler Function

For $z\in\mathbb{C}$ the one parameter Mittag-Leffler function $E_\alpha(z)$ is defined, see eg. [22],p.16, as

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}, \quad \alpha>0, \alpha\in\mathbb{R}. \tag{2.2}$$

The two parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ is introduced by

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha,\beta>0, \alpha,\beta\in\mathbb{R}, z\in\mathbb{C}. \tag{2.3}$$

Remark 2.1.1. *Let $\beta=1$. Then (2.2) and (2.3) gives,*

$$E_{\alpha,1}(z)=E_\alpha(z).$$

Some special cases of Mittag-Leffler function are e.g.

$$E_{1,1}(z)=\exp(z),$$

$$E_{1,2}(z)=\frac{1}{z}(\exp(z)-1).$$

Mathematical induction then gives

$$E_{1,m}(z)=\frac{1}{z^{m-1}}\left(\exp(z)-\sum_{k=0}^{m-2}\frac{z^k}{k!}\right), \quad m=2,3,.$$

Mittag-Leffler function has particular cases in the hyperbolic sine and hyperbolic cosine.

$$E_{2,1}\left(z^2\right)=\sum_{k=0}^{\infty}\frac{z^{2k}}{\Gamma(2k+1)}=\sum_{k=0}^{\infty}\frac{z^{2k}}{2k!}=\cosh(z),$$

$$E_{2,2}\left(z^2\right)=\sum_{k=0}^{\infty}\frac{z^{2k}}{\Gamma(2k+2)}=\frac{1}{z}\sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k+1)!}=\frac{\sinh(z)}{z}.$$

Theorem 2.1.1. *[32] Mittag-Leffler function obeys the recurrence relation*

$$E_{\alpha,\beta}(z) = zE_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)}.$$

2.2. Integro-Differential Operators

Suppose the class of functions $f(t)$ such that $f(t)$ is continuous and integrable in every finite interval $(t_0, t)\,, t \in \mathbb{R}$, see ([22], p.63). Such functions $f(t)$ may have an integrable singularity of order $\alpha < 1$ at the point $t = t_0$;

$$\lim_{\tau \to t_0} (\tau - t_0)^{\alpha} f(t) = \text{ const. } (\neq 0).$$

On this class of functions are defined the next operators, considering eventual additional properties.

The Riemann-Liouville Operator

For detailed theoretical background we refer to [2] and [22]. First we introduce

Definition 2.2.1. *Let $J = [t_0, \infty) \subset \mathbb{R}_+$ and L^1 be Lebesgue space. Then the Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f \in L^1(J, \mathbb{R})$ is defined as:*

$$I^{\alpha}_{t_0} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad t > t_0 \in \mathbb{R}.$$

Definition 2.2.2. *The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f \in L^1(J, \mathbb{R})$ and $n = [\alpha] + 1$ with $[\alpha]$ being the integer part of α is given by:*

$$^{RL}D^{\alpha}_{t_0,t} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t} (t-\tau)^{n-\alpha-1} f(\tau) d\tau, \quad n-1 < \alpha < n, n \in \mathbb{N}, \tag{2.4}$$

provided the right hand side is pointwise defined on J. Relation (2.4) is also known as the Riemann-Liouville fractional differential operator of order α.

The Riemann-Liouville approach is based on the Cauchy formula for the $n-th$ integral, defined as ([21], p.64):

$$I^{n}_{t_0} f(t) = \int_{t_0}^{t} \int_{t_0}^{\tau_{n-1}} \cdots \int_{t_0}^{\tau_1} f(\tau) d\tau d\tau_1 \ldots d\tau_{n-1} = \frac{1}{(n-1)!} \int_{t_0}^{t} (t-\tau)^{n-1} f(\tau) d\tau. \tag{2.5}$$

It is clear how to get the integral of arbitrary order for instance α from the Cauchy formula above. We just have to make a generalization of the Cauchy formula (2.5),

$$I^{\alpha}_{t_0} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad \alpha \in \mathbb{R}_+, t > t_0. \tag{2.6}$$

Remark 2.2.1. *The integrand in (2.6) is still integrable since $\alpha - 1 > -1$, but in a case when $\alpha = 0$, under certain assumptions we have by convention the identity operator [22]*

$$I^{0}_{t_0} f(t) = f(t).$$

Unlike the case of integration, there is no explicit formula that defines an $n-th$ order derivative, so we have to define the fractional derivative of order $0<\alpha<1$ in terms of the fractional integral for $n=\lfloor\alpha\rfloor+1$ as,

$$\begin{aligned} D^{\alpha}_{t_0,t}f(t) &= \frac{d^n\left(I^{n-\alpha}_{t_0}f(t)\right)}{dt^n} \\ &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_{t_0}^{t}(t-\tau)^{n-\alpha-1}f(\tau)d\tau. \end{aligned} \tag{2.7}$$

Formula (2.7) also includes integer order derivatives. If $\alpha=k$ and $k\in\mathbb{N}_0$, then $n=k+1$ and (2.7) becomes:

$$D^{k}_{t_0,t}f(t)=\frac{d^{k+1}}{dt^{k+1}}\int_{t_0}^{t}f(\tau)d\tau=\frac{d^kf(t)}{dt^k}.$$

If we write $D^{-\alpha}_{t_0,t}f(t)=I^{\alpha}_{t_0,t}f(t)$ and $f^{(0)}(t)=f(t)$, we get the representation of the fractional derivative and integral by one single formula. We now state without proof the Laplace transform of the Riemann-Liouville fractional derivative.

$$\mathcal{L}\left\{{}^{RL}D^{\alpha}_{t_0,t}\right\}=s^{\alpha}F(s)-\sum_{k=0}^{m-1}s^k\left[\ {}^{RL}D^{\alpha-k-1}_{t_0,t}f(t)\ \right]_{t=t_0},\quad \alpha>0, n-1\le\alpha<n.$$

Caputo Operator

Another approach of fractional operator is due to Michele Caputo [33]:

Definition 2.2.3. *Let $\alpha\in\mathbb{R}_+$ such that $n-1<\alpha\le n$ and for $n\in\mathbb{N}$ and $t>t_0$. The Caputo fractional derivative of order α of a function f is defined as:*

$${}^{C}D^{\alpha}_{t_0,t}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}(t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau, \tag{2.8}$$

$${}^{C}D^{\alpha}_{t_0,t}f(t)=I^{n-\alpha}_{t_0,t}f^{(n)}(t),\quad 0<n-1<\alpha\le n, t\in[t_0,T]. \tag{2.9}$$

The Caputo fractional operator is denoted by ${}^{C}D^{\alpha}_{t_0,t}$. The fractional integral operator is as defined in (2.6) above. Hence considering $\alpha>0$ it holds

$${}^{C}D^{-\alpha}_{t_0,t}f(t)=D^{-\alpha}_{t_0,t}f(t).$$

The benefit of the Caputo definition is that the Caputo derivative of a constant is zero, seeing that before the integral is computed we first take an integer derivative of a constant. But, this adjustment alters the limiting case when the order of the fractional derivative is an integer.

Example 2.2.1. *Suppose $f(t)=t$, $n=1$, $t_0=0,\alpha=1/2$. Then, from (2.8) we have*

$${}^{C}D^{1/2}_{0,t}t=\frac{1}{\Gamma(1/2)}\int_{0}^{t}\frac{1}{(t-\tau)^{1/2}}d\tau,$$

By direct substitution, we define $u := (t-\tau)^{1/2}$. It follows that

$$
\begin{aligned}
{}^C D_{0,t}^{1/2} t &= -\frac{1}{\sqrt{\pi}} \int_0^t \frac{1}{(t-\tau)^{1/2}} d(t-\tau), \\
{}^C D_{0,t}^{1/2} t &= -\frac{1}{\sqrt{\pi}} \int_{\sqrt{t}}^0 \frac{1}{u} du^2, \\
{}^C D_{0,t}^{1/2} t &= \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{t}} \frac{2u}{u} du, \\
{}^C D_{0,t}^{1/2} t &= \frac{2}{\sqrt{\pi}} (\sqrt{t} - 0),
\end{aligned}
$$

therefore, the half order Caputo derivative of the function $f(t) = t$ is

$$
{}^C D_{0,t}^{1/2} t = \frac{2\sqrt{t}}{\sqrt{\pi}}.
$$

But unlike the Riemann-Liouville derivative, in the Caputo definition, the function is first differentiated in the classical sense before integrating fractionally to the required order. We shall now state without proof the Laplace transform of the Caputo fractional derivative

$$
\mathcal{L}\left\{{}^C D_{t_0,t}^{\alpha}\right\} = s^{\alpha} F(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(t_0), \quad \alpha > 0, n-1 < \alpha \leq n.
$$

Caputo operator has the following fundamental properties:

1. **Interpolation**

Lemma 2.2.1. *Let $f(t)$ be such that ${}^C D_{t_0,t}^{\alpha} f(t)$ exist, $n-1 < \alpha < n, \alpha \in \mathbb{R}, n \in \mathbb{N}$. Then the following holds*

$$
\lim_{\alpha \to n} {}^C D_{t_0,t}^{\alpha} f(t) = f^{(n)}(t).
$$

$$
\lim_{\alpha \to n-1} {}^C D_{t_0,t}^{\alpha} f(t) = f^{(n-1)}(t) - f^{(n-1)}(t_0).
$$

Proof. Via integration by parts ([22], p.79).

$$
\begin{aligned}
{}^C D_{t_0,t}^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \\
{}^C D_{t_0,t}^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \left(- f^{(n)}(\tau) \frac{(t-\tau)^{n-\alpha}}{n-\alpha} \bigg|_{\tau=t_0}^t - \int_{t_0}^t -f^{(n+1)}(\tau) \frac{(t-\tau)^{n-\alpha}}{n-\alpha} d\tau \right), \\
{}^C D_{t_0,t}^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha+1)} \left(f^{(n)}(t_0) t^{n-\alpha} + \int_{t_0}^t f^{(n+1)}(\tau)(t-\tau)^{n-\alpha} d\tau \right).
\end{aligned}
$$

Taking limits for $\alpha \to n$ and $\alpha \to n-1$, we have

$$
\lim_{\alpha \to n} {}^C D_{t_0,t}^{\alpha} f(t) = \left(f^{(n)}(t_0) + f^{(n)}(\tau) \right) \Big|_{\tau=t_0}^t = f^{(n)}(t),
$$

and

$$\lim_{\alpha\to n-1} {}^{C}D^{\alpha}_{t_0,t}f(t) = \left(f^{(n)}(t_0)t + f^{(n)}(\tau)(t-\tau)\right)\Big|_{\tau=t_0}^{t} - \int_{t_0}^{t} -f^{(n)}(\tau)d\tau {}^{C}D^{\alpha}_{t_0,t}f(t) \quad = f^{(n-1)}(\tau)\Big|_{\tau=t_0}^{t},$$

$${}^{C}D^{\alpha}_{t_0,t}f(t) = f^{(n-1)}(t) - f^{(n-1)}(t_0). \qquad \square$$

The corresponding interpolation property for the Riemann-Liouville fractional differential operator is given as

$$\lim_{\alpha\to n} {}^{RL}D^{\alpha}_{t_0,t}f(t) = f^{(n)}(t),$$

$$\lim_{\alpha\to n-1} {}^{RL}D^{\alpha}_{t_0,t}f(t) = f^{(n-1)}(t).$$

2. **Linearity**

Lemma 2.2.2. *Let $f(t)$ and $g(t)$ be functions such that ${}^{C}D^{\alpha}_{t_0,t}f(t)$ and ${}^{C}D^{\alpha}_{t_0,t}g(t)$ exist, then*

$${}^{C}D^{\alpha}_{t_0,t}(\mu f(t) + \lambda g(t)) = \mu {}^{C}D^{\alpha}_{t_0,t}f(t) + \lambda {}^{C}D^{\alpha}_{t_0,t}g(t), \quad n-1<\alpha<n, \quad n\in\mathbb{N}, \alpha, \lambda, \mu \in \mathbb{R}. \tag{2.10}$$

Proof. The proof follows from (2.8) and the fact that the integer order integration and differentiation is a linear operator. $\square$

As expected, the Riemann-Liouville operator is linear and also satisfies

$${}^{RL}D^{\alpha}_{t_0,t}(\mu f(t) + \lambda g(t)) = \mu {}^{RL}D^{\alpha}_{t_0,t}f(t) + \lambda {}^{C}D^{\alpha}_{t_0,t}g(t), \quad n-1<\alpha<n, \quad n\in\mathbb{N}, \alpha, \lambda, \mu \in \mathbb{R}.$$

3. **Non-Commutation**

Lemma 2.2.3. *Assume that ${}^{C}D^{\alpha}_{t_0,t}f(t)$ exist. Then*

$${}^{C}D^{\alpha}_{t_0,t}D^{m}f(t) = {}^{C}D^{\alpha+m}_{t_0,t}f(t) \neq D^{m}{}^{C}D^{\alpha}_{t_0,t}f(t), \quad n-1<\alpha<n \quad n,m\in\mathbb{N}, \alpha\in\mathbb{R}.$$

Proof.

$${}^{C}D^{\alpha}_{t_0,t}D^{m}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}(t-\tau)^{n-\alpha-1}\frac{d^m}{d\tau^m}f^{(n)}(\tau)d\tau.$$

By the property of integer order derivative

$${}^{C}D^{\alpha}_{t_0,t}D^{m}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}(t-\tau)^{n-\alpha-1}f^{(n+m)}(\tau)d\tau,$$

$${}^{C}D^{\alpha}_{t_0,t}D^{m}f(t) = {}^{C}D^{\alpha+m}_{t_0,t}f(t).$$

Similarly,

$$D^{m}{}^{C}D^{\alpha}_{t_0,t}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^m}{dt^m}\int_{t_0}^{t}(t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau.$$

To solve the above, we shall use the Leibniz formula

$$\frac{d}{dt}\int_{t_0}^{t} g(t,\tau)d\tau = g(t,t) + \int_{t_0}^{t} \frac{\partial g(t,\tau)}{\partial t}d\tau.$$

Let

$$g(t,\tau) = (t-\tau)^{n-\alpha-1}f^{(n)}(\tau),$$

it follows

$$= \tfrac{1}{\Gamma(n-\alpha)}\tfrac{d^{m-1}}{dt^{m-1}}\tfrac{d}{dt}\textstyle\int_{t_0}^{t}(t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau$$

$$= \tfrac{1}{\Gamma(n-\alpha)}\tfrac{d^{m-1}}{dt^{m-1}}\left[(n-\alpha-1)\textstyle\int_{t_0}^{t}(t-\tau)^{n-\alpha-2}f^{(n)}(\tau)d\tau\right]$$

$$= \tfrac{1}{\Gamma(n-\alpha)}\tfrac{d^{m-2}}{dt^{m-2}}\left[(n-\alpha-1)(n-\alpha-2)\textstyle\int_{t_0}^{t}(t-\tau)^{n-\alpha-3}f^{(n)}(\tau)d\tau\right]$$

continuing, we get

$$= \tfrac{1}{\Gamma(n-\alpha)}\left[(n-\alpha-1)(n-\alpha-2)\ldots(n-\alpha-m)\textstyle\int_{t_0}^{t}(t-\tau)^{n-\alpha-m-1}f^{(n)}(\tau)d\tau\right].$$

Using the property $(z-1)\Gamma(z-1) = \Gamma(z)$ of the Gamma function,

$${}^{C}D^{\alpha}_{t_0,t}D^{m}f(t) = \frac{1}{\Gamma(n-\alpha-m)}\int_{t_0}^{t}(t-\tau)^{n-\alpha-m-1}f^{(n)}(\tau)d\tau. \qquad \square$$

The Riemann-Liouville operator of order $n-1<\alpha<n$, $\alpha\in\mathbb{R}$ is also non-communicative

$${}^{RL}D^{\alpha}_{t_0,t}D^{m}f(t) = {}^{RL}D^{\alpha+m}_{t_0,t}f(t) \neq D^{m\,C}D^{\alpha}_{t_0,t}f(t),\ \ n,m\in\mathbb{N}.$$

Corollary 2.2.1. *Let $f(t)$ be a function such that ${}^{C}D^{\alpha}_{t_0,t}f(t)$ exist, with $n-1<\alpha<n, \beta=\alpha-(n-1)$ $[\implies (0<\beta<1)], n\in\mathbb{N}$, $\alpha,\beta\in\mathbb{R}$. Then*

$${}^{C}D^{\alpha}_{t_0,t}f(t) = {}^{C}D^{\beta}_{t_0,t}D^{n-1}f(t).$$

Proof. The proof follows from the above lemma 2.2.3

$${}^{C}D^{\beta}_{t_0,t}D^{n-1}f(t) = {}^{C}D^{\beta+(n-1)}_{t_0,t}f(t),$$

substituting the value of β, we have that

$$\begin{aligned}{}^{C}D^{\beta}_{t_0,t}D^{n-1}f(t) &= {}^{C}D^{\alpha-(n-1)+(n-1)}_{t_0,t}f(t),\\ {}^{C}D^{\beta}_{t_0,t}D^{n-1}f(t) &= {}^{C}D^{\alpha}_{t_0,t}f(t). \qquad \square\end{aligned}$$

Remark 2.2.2. *From corollary* 2.2.1, *we see that to find the Caputo derivative of arbitrary order $n-1<\alpha<n$ it is sufficient to find the Caputo derivative of order $\beta=\alpha-(n-1)$ of the $(n-1)$th derivative of the function. We also notice that $\beta\in(0,1)$.*

Proposition 2.2.1. *By (2.8) and (2.4), the Riemann-Liouville and Caputo operators do not coincide.*

$${}^{C}D^{\alpha}_{t_0,t}f(t) \neq {}^{RL}D^{\alpha}_{t_0,t}f(t). \tag{2.11}$$

We shall however give condition which will make (2.11) identical.

Caputo and Riemann-Liouville Operator Relations

In this paragraph a relationship between the Caputo and Riemann-Liouville differential operator is discussed, see [2].

Theorem 2.2.1. *Let $t > 0, \alpha \in \mathbb{R}, n \in \mathbb{N}$. Then the following relation between the Riemann-Liouville (2.4) and the Caputo differential operator (2.8) holds*

$$ {}^{C}D^{\alpha}_{t_0,t} f(t) = {}^{RL}D^{\alpha}_{t_0,t} f(t) - \sum_{k=0}^{n-1} \frac{(t-t_0)^{k-\alpha}}{\Gamma(k+1-\alpha)} f^{(k)}(t_0), \quad n-1 < \alpha < n. \tag{2.12} $$

Proof. The well known Taylor's series about t_0 is given by

$$ \begin{aligned} f(t) &= f(t_0) + (t-t_0) f'(t_0) + \tfrac{(t-t_0)^2}{2!} f''(t_0) + \cdots + \tfrac{(t-t_0)^{n-1}}{(n-1)!} f^{(n-1)}(t_0) + R_{n-1}, \\ f(t) &= \sum_{k=0}^{n-1} \frac{(t-t_0)^k}{\Gamma(k+1)} f^{(k)}(t_0) + R_{n-1}. \end{aligned} \tag{2.13} $$

Where R_{n-1} is the Lagrange Remainder. By (2.5)

$$ \begin{aligned} R_{n-1} &= \tfrac{1}{(n-1)!} \int_{t_0}^{t} (t-\tau)^{n-1} f^{(n)}(\tau) d\tau, \\ R_{n-1} &= \tfrac{1}{\Gamma(n)} \int_{t_0}^{t} (t-\tau)^{n-1} f^{(n)}(\tau) d\tau, \\ R_{n-1} &= I^{n}_{t_0} f^{(n)}(t). \end{aligned} $$

Now, taking the Riemann-Liouville derivative of (2.13) and using the linearity property of the Riemann-Liouville derivative

$$ \begin{aligned} {}^{RL}D^{\alpha}_{t_0,t} f(t) &= {}^{RL}D^{\alpha}_{t_0,t} \left(\sum_{k=0}^{n-1} \frac{(t-t_0)^k}{\Gamma(k+1)} f^{(k)}(t_0) + I^{n}_{t_0} f^{(n)}(t) \right), \\ {}^{RL}D^{\alpha}_{t_0,t} f(t) &= \sum_{k=0}^{n-1} \frac{{}^{RL}D^{\alpha}_{t_0,t}(t-t_0)^k}{\Gamma(k+1)} f^{(k)}(t_0) + {}^{RL}D^{\alpha}_{t_0,t} I^{n}_{t_0} f^{(n)}(t). \end{aligned} $$

taking the Riemann-Liouville fractional derivative of the power function

$$ {}^{RL}D^{\alpha}_{t_0,t} = \sum_{k=0}^{n-1} \frac{\Gamma(k+1)(t-t_0)^{k-\alpha}}{\Gamma(k+1-\alpha)\Gamma(k+1)} f^{(k)}(t_0) + I^{n-\alpha}_{t_0} f^{(n)}(t), $$

from (2.9)

$$ {}^{RL}D^{\alpha}_{t_0,t} f(t) = \sum_{k=0}^{n-1} \frac{(t-t_0)^{k-\alpha}}{\Gamma(k+1-\alpha)} f^{(k)}(t_0) + {}^{C}D^{\alpha}_{t_0,t} f(t), $$

which completes the proof. □

Remark 2.2.3. *The above theorem shows that the Riemann-Liouville differential operator and the Caputo differential operator coincide only if $f(t)$ together with its first $(n-1)$*

derivatives vanish at $t = t_0$ *i.e* $f(t_0) = 0, f'(t_0) = 0, \cdots, f^{(n-1)}(t_0) = 0.$
We state and prove, in composition of Riemann–Liouville integrals and Caputo differential operators, that the Caputo derivative is a left inverse of the Riemann–Liouville integral but not the right inverse of the Riemann–Liouville integral.

Corollary 2.2.2. *The following relation between the Riemann-Liouville and Caputo fractional derivatives holds*

$$ {}^{C}D^{\alpha}_{t_0,t}f(t) = {}^{RL}D^{\alpha}_{t_0,t}\left(f(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(t_0)\right) \tag{2.14}$$

Proof: See [2]. Using (2.12), the Riemann-Liouville fractional derivative of the power function (2.16) and the linearity property of the Riemann-Liouville operator, we obtain

$$\begin{aligned}
{}^{C}D^{\alpha}_{t_0,t}f(t) &= {}^{RL}D^{\alpha}_{t_0,t}f(t) - \sum_{k=0}^{n-1}\frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)}f^{(k)}(t_0),\\
{}^{C}D^{\alpha}_{t_0,t}f(t) &= {}^{RL}D^{\alpha}_{t_0,t}f(t) - \sum_{k=0}^{n-1}\frac{{}^{RL}D^{\alpha}_{t_0,t}t^k}{\Gamma(k+1)}f^{(k)}(t_0),\\
{}^{C}D^{\alpha}_{t_0,t}f(t) &= {}^{RL}D^{\alpha}_{t_0,t}\left(f(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(t_0)\right). \qquad \square
\end{aligned}$$

2.3. Fractional Differential Equation

Fractional differential equations (FDEs) involve fractional derivatives of the form $\frac{d^\alpha}{dt^\alpha}$ which are defined for $\alpha > 0$, where α is not necessarily an integer. They are generalizations of ordinary differential equations to a noninteger order. Particular case of FDE of the Riemann–Liouville type is given by the equation

$$ {}^{RL}D^{\alpha}_{t_0,t}f(t) = y(t, f(t)),$$

and the initial condition is of the form

$$\begin{cases} {}^{RL}D^{\alpha-k}_{t_0,t}f(t_0) = b_k \quad k = 0, 1, 2, \ldots, N-1 \\ I^{N-\alpha}f(t_0) = b_N. \end{cases}$$

Similarly, the corresponding FDE of the Caputo type with its initial conditions is of the form

$$\begin{cases} {}^{C}D^{\alpha}_{t_0,t}f(t) = y(t, f(t)), \\ f^{(k)}_{t_0} = b_k \quad k = 0, 1, 2, \ldots, N-1. \end{cases}$$

Consider the FDE based on the Riemann-Liouville derivative:

$$ {}^{RL}D^{\alpha}_{t_0,t}f = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_{t_0}^{t}\frac{f(\tau)}{(t-\tau)^\alpha}\mathrm{d}\tau.$$

Its Laplace transform is

$$\mathcal{L}\left[{}^{RL}D^{\alpha}_{t_0,t}f\right] = s^\alpha\tilde{f}(s) - \left[{}^{RL}D^{-(1-\alpha)}_{t_0,t}f\right](t_0).$$

The initial value of f is usually given in physical applications, and the Laplace transform is based on the initial value of the fractional integral of f. It is known that in order to produce a unique solution to classical and FDEs, additional conditions must be defined. These conditions are fractional derivatives and integrals of the unknown solution at the initial point $t_0 = 0$, and are functions of t for Riemann-Liouville FDEs. As a limitation of fractional derivatives of this kind, these initial conditions are not physical and cannot be measured in any way. The Caputo derivative of the fractional derivative provides a solution to this problem, where the additional conditions are essentially the traditional conditions that are identical to those of classical differential equations. In most cases, the equation of choice is based on the Caputo derivative, which contains the functions' initial values as well as its lower-order integer derivatives.

Fractional Derivatives of fundamental functions

Examples of fractional derivatives, such as the constant, power, and exponential functions, as well as the sine and cosine functions, are provided in this section. For more details, see [34].

• The Constant Function

It makes sense to have the fractional derivative of a constant equal to zero from a physical standpoint. For the Riemann-Liouville operator it holds [22]

$$^{RL}D^{\alpha}_{t_0,t}c = \frac{c}{\Gamma(1-\alpha)}t^{-\alpha} \neq 0, c = \text{ const.}$$

Lemma 2.3.1. *For the Caputo fractional derivative it holds*

$$^{C}D^{\alpha}_{t_0,t}c = 0, \quad c = \textit{ const.}$$

Proof. Let $0 < n-1 < \alpha < n, n \in \mathbb{N}, n \geq 1$. Using the definition of the Caputo derivative (2.3.1) and since the $n-th$ derivative $c^{(n)}$ of a constant equals zero it follows

$$^{C}D^{\alpha}_{t_0,t}c = \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}\frac{c^{(n)}}{(t-\tau)^{\alpha+1-n}}d\tau = 0. \quad \square$$

• The Power Function

The power function is of great importance. The Taylor expansion is given as ([35], p. 35)

$$f(t) = f(0) + f'(0)t + \frac{f''(0)}{2!}t^2 + \frac{f'''(0)}{3!}t^3 + \cdots$$

We know that the Caputo fractional derivative is linear (see (2.10)) . So if $^{C}D^{\alpha}_{t_0,t}t^p$ is known, then the Caputo fractional derivative for arbitrary function can be expressed as

$$^{C}D^{\alpha}_{t_0,t}f(t) = {}^{C}D^{\alpha}_{t_0,t}\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}t^k = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}{}^{C}D^{\alpha}_{t_0,t}t^k. \tag{2.15}$$

Theorem 2.3.1. *The Riemann-Liouville fractional derivative of the power function satisfies*

$$ {}^{RL}D^{\alpha}_{t_0,t}t^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}t^{p-\alpha}, \quad n-1<\alpha<n, p>-1, p\in\mathbb{R}. \tag{2.16}$$

Proof. See ([22], p. 72.)

Theorem 2.3.2. *The Caputo fractional derivative of the power function satisfies [34]*

$$ {}^{C}D^{\alpha}_{t_0,t}t^p = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}t^{p-\alpha} = {}^{RL}D^{\alpha}_{t_0,t}t^p, & n-1<\alpha<n, p>n-1, p\in\mathbb{R} \\ 0, & n-1<\alpha<n, p\le n-1, p\in\mathbb{N} \end{cases} \tag{2.17}$$

Proof: We use the relation between the Caputo and Riemann-Liouville derivatives (2.12) as well as the Riemann-Liouville fractional derivative of the power function (2.16).

Let $n-1<\alpha<n, p>n-1, p\in\mathbb{R}$,

$$ {}^{C}D^{\alpha}_{t_0,t}t^p = {}^{RL}D^{\alpha}_{t_0,t}t^p - \sum_{k=0}^{n-1}\frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)}\left.(t^p)^{(k)}\right|_{t=0},$$

and taking into account $\left.(t^p)^{(k)}\right|_{t=0} = 0$, for $k\le n-1<p$, we obtain

$$ {}^{C}D^{\alpha}_{t_0,t}t^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}t^{p-\alpha} - \sum_{k=0}^{n-1}\frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)}\cdot(0)$$

$$ {}^{C}D^{\alpha}_{t_0,t}t^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}t^{p-\alpha}. \qquad \square$$

The proof of the second case $\left({}^{C}D^{\alpha}_{t_0,t}t^p = 0, n-1<\alpha<n, p\le n-1, p\in\mathbb{N}\right)$ follows the pattern of the proof of the differentiation of the constant function, since $(t^p)^{(n)} = 0$ for $p\le n-1, p, n\in\mathbb{N}$.

Remark 2.3.1. *Theorem 2.3.2 can also be proved directly, using the definition of the Caputo fractional derivative (2.8) and the properties of the Gamma and Beta functions.*

Proposition 2.3.1. *The Caputo fractional derivative for an arbitrary function $f(t)$ can be computed by the formula*

$$ {}^{C}D^{\alpha}_{t_0,t}f(t) = \sum_{k=n}^{\infty}\frac{f^{(k)}(0)}{\Gamma(k-\alpha+1)}t^{k-\alpha}.$$

Proof. Taking into account Caputo arbitrary function (2.15) and (2.17), the following equalities hold

$$ {}^{C}D^{\alpha}_{t_0,t}f(t) = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}{}^{C}D^{\alpha}_{t_0,t}t^k = \sum_{k=n}^{\infty}\frac{f^{(k)}(0)}{k!}\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}t^{k-\alpha} = \sum_{k=n}^{\infty}\frac{f^{(k)}(0)}{\Gamma(k-\alpha+1)}t^{k-\alpha}. \square$$

Example 2.3.1. *Assume $\alpha\in\mathbb{R}\backslash\mathbb{N}$ and $n-1<\alpha<n<3$, $f(t)=t^2$, i.e $p=2$ is discussed for fixed values of the parameter α, in particular, $\alpha=1/3, \alpha=1/2$ and $\alpha=3/4$.*

Using (2.17) the fractional derivative is given as

$$^{C}D^{\alpha}_{t_0,t}t^2 = \frac{\Gamma(2+1)}{\Gamma(2-\alpha+1)}t^{2-\alpha} = \frac{2}{\Gamma(3-\alpha)}t^{2-\alpha}, n-1<\alpha<n<3,$$

$$\text{for } \alpha = \tfrac{1}{3}: \quad {}^{C}D^{1/3}_{t_0,t}t^2 = \tfrac{2}{\Gamma(3-1/3)}t^{2-1/3} = \tfrac{2}{\Gamma(8/3)}t^{5/3} \approx 1.3293t^{5/3},$$

$$\text{for } \alpha = \tfrac{1}{2}: \quad {}^{C}D^{1/2}_{t_0,t}t^2 = \tfrac{2}{\Gamma(3-1/2)}t^{2-1/2} = \tfrac{8}{3\sqrt{\pi}}\sqrt{t^3} \approx 1.5045\sqrt{t^3},$$

$$\text{for } \alpha = \tfrac{3}{4}: \quad {}^{C}D^{3/4}_{t_0,t}t^2 = \tfrac{2}{\Gamma(3-3/4)}t^{2-3/4} = \tfrac{2}{\Gamma(9/4)}t^{5/4} \approx 1.7652t^{5/4}.$$

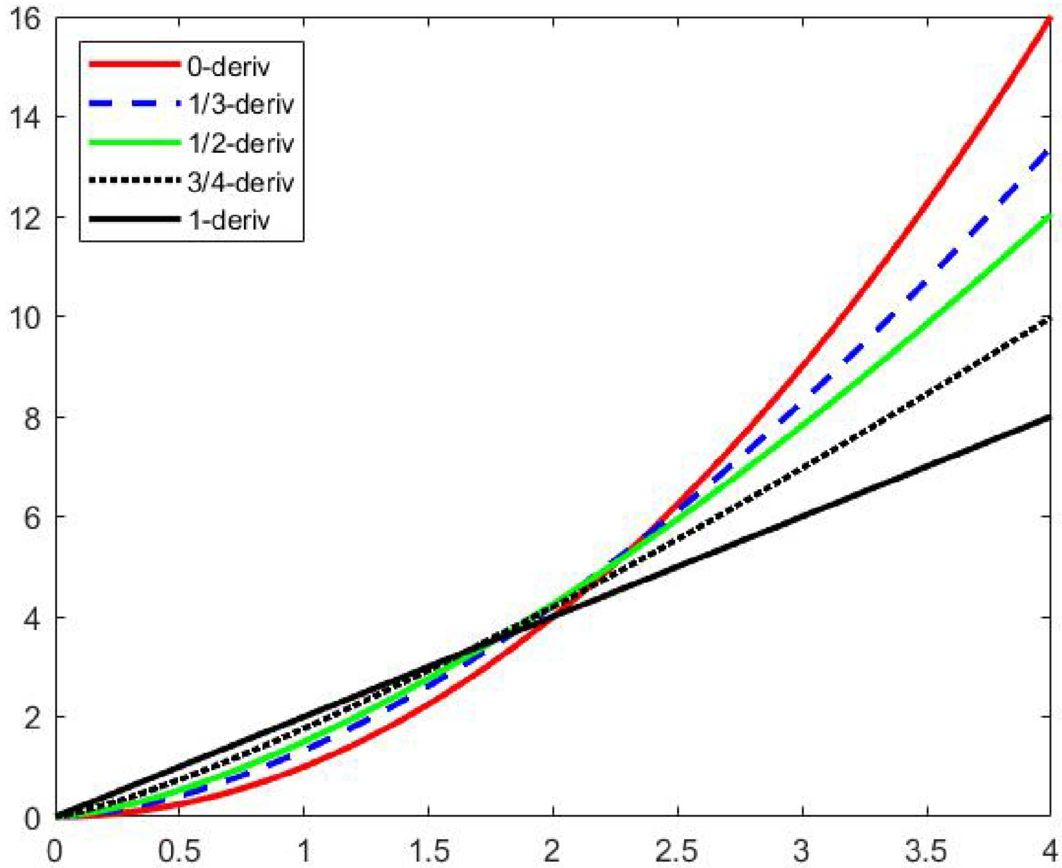

Figure 2.1: ${}^{C}D^{\alpha}_{t_0,t}t^2$ for $\alpha \in 0, \frac{1}{3}, \frac{1}{2}, \frac{3}{4}, 1$.

Except for a small interval, the graphs of the fractional derivatives are enclosed everywhere by the graphs of the classical integer-order derivatives. The greater the order $\alpha < 1$ of the derivative is, the closer is its graph to the graph of the 1$^{\text{st}}$ derivative of t^2 The smaller the order $\alpha > 0$ is, the closer is its graph to the graph original function.

Using the function $f(t) = t^p = t, i.e, p = 1$ for $\alpha = 1/2$ it follows

$$^{C}D^{\alpha}_{t_0,t}t = \frac{\Gamma(1+1)}{\Gamma(1-\alpha+1)}t^{1-\alpha} = \frac{1}{\Gamma(2-\alpha)}t^{1-\alpha}.$$

$$^{C}D_{t_0,t}^{1/2}t = \frac{1}{\Gamma(2-1/2)}t^{1-1/2} = \frac{2\sqrt{t}}{\sqrt{\pi}}.$$

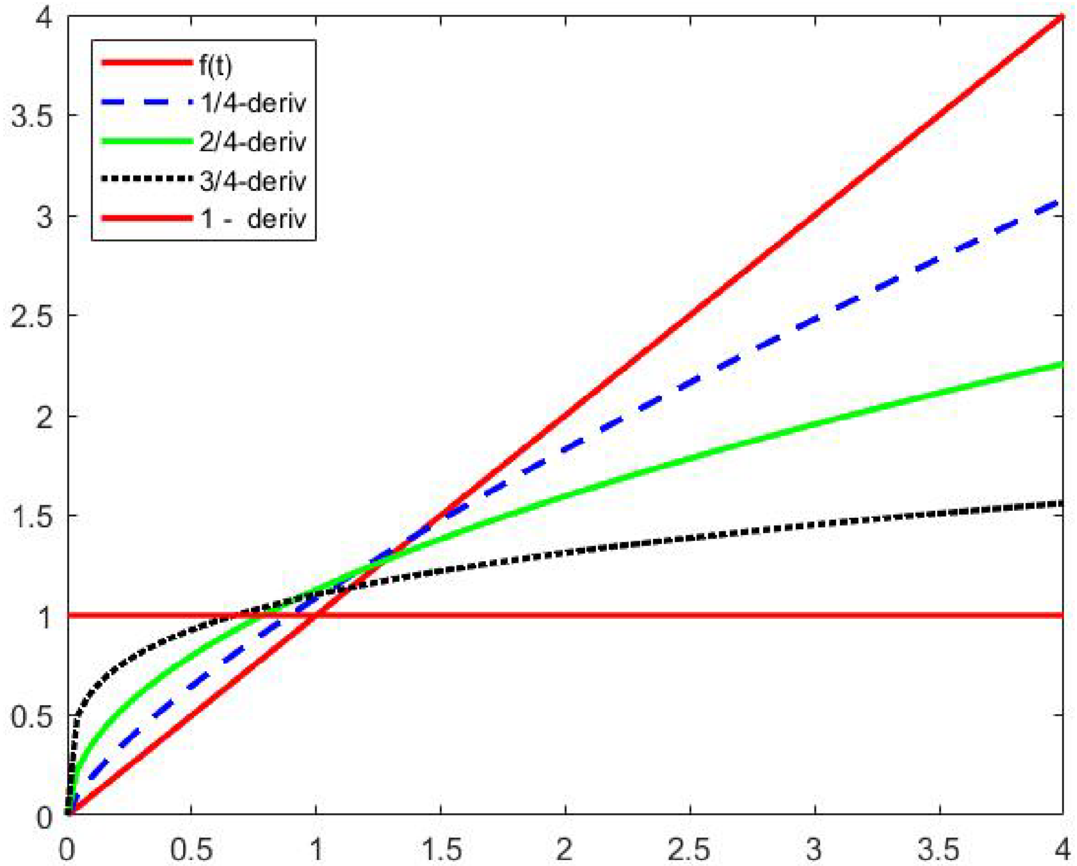

Figure 2.2: $^{C}D_{t_0,t}^{\alpha}t$ for $\alpha \in 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

From figure 2.2, the graphs of the fractional derivatives are enclosed between the graphs of the integer-order derivatives except for a small interval.

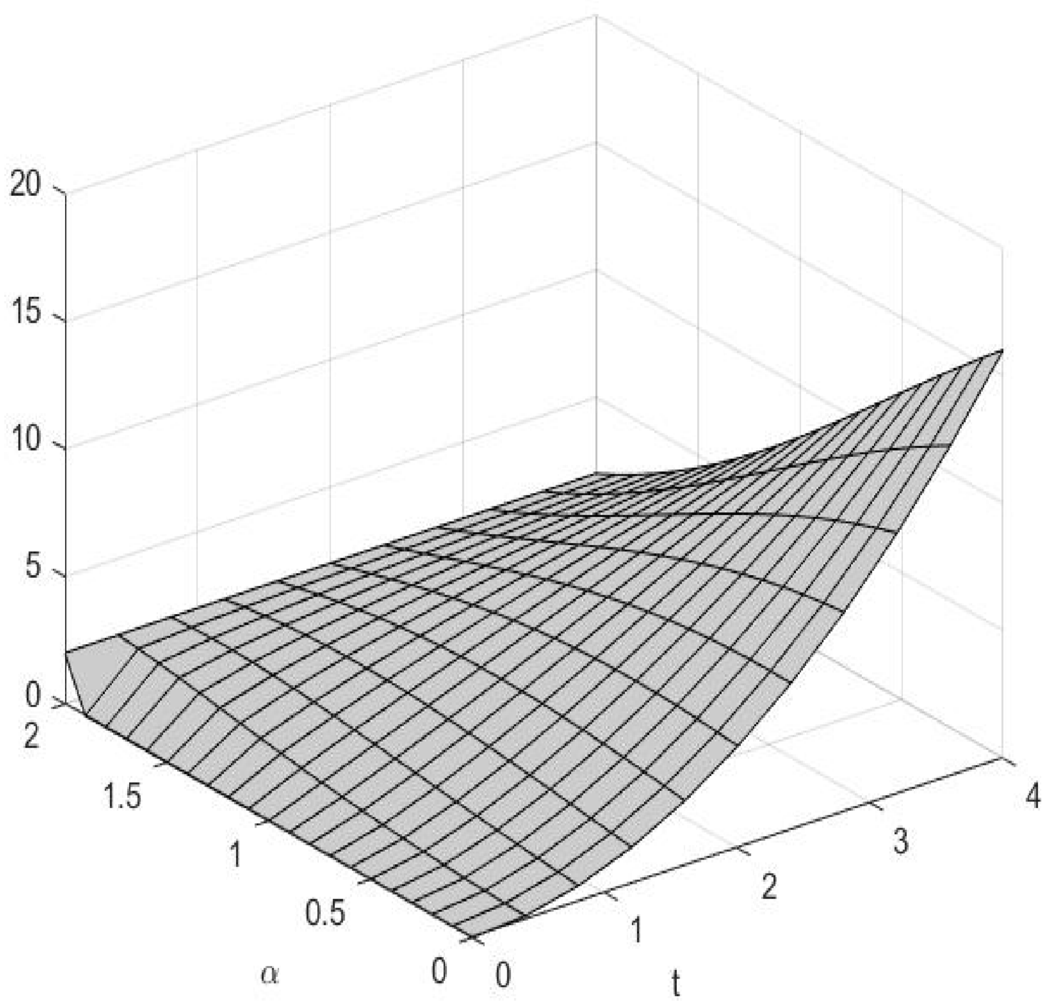

Figure 2.3: 3D ${}^{C}D^{\alpha}_{t_0,t}t^2$ for $\alpha \in$ [0,2]

Figure 2.3 gives the 3D representation of the function $f(t) = t^2$ with $0 < \alpha < 2$. Here the fractional derivatives interpolate the classical zero, first and second order derivatives $t^2, 2t$ and 2.

• The Exponential Function

The Caputo derivative of the exponential function $e^{\lambda t}$ is introduced by

Theorem 2.3.3. *Let $\alpha \in \mathbb{R}, n-1 < \alpha < n, n \in \mathbb{N}, \lambda \in \mathbb{C}$. Then the Caputo fractional derivative of the exponential function can be denoted as*

$$ {}^{C}D^{\alpha}_{t_0,t}e^{\lambda t} = \sum_{k=0}^{\infty} \frac{\lambda^{k+n}t^{k+n-\alpha}}{\Gamma(k+1+n-\alpha)} = \lambda^n t^{n-\alpha} E_{1,n-\alpha+1}(\lambda t), \tag{2.18}$$

where $E_{\alpha,\beta}(z)$ is the two-parameter function of Mittag-Leffler type.

Proof. The relation between Caputo and Riemann-Liouville fractional derivatives (2.12) and the Riemann-Liouville fractional derivative of the exponential equation,

$$ {}^{RL}D^{\alpha}_{t_0,t}e^{\lambda t} = t^{-\alpha}E_{1,1-\alpha}(\lambda t),$$

could be used to prove the theorem. It follows that,

$$
\begin{aligned}
{}^{C}D^{\alpha}_{t_0,t}e^{\lambda t} &= {}^{RL}D^{\alpha}_{t_0,t}e^{\lambda t} - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} \left(e^{\lambda t}\right)^{(k)}(t_0) \\
{}^{C}D^{\alpha}_{t_0,t}e^{\lambda t} &= t^{-\alpha}E_{1,1-\alpha}(\lambda t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} \cdot \lambda^k \\
{}^{C}D^{\alpha}_{t_0,t}e^{\lambda t} &= \sum_{k=0}^{\infty} \frac{(\lambda t)^k t^{-\alpha}}{\Gamma(k+1-\alpha)} - \sum_{k=0}^{n-1} \frac{\lambda^k t^{k-\alpha}}{\Gamma(k+1-\alpha)} \\
{}^{C}D^{\alpha}_{t_0,t}e^{\lambda t} &= \sum_{k=n}^{\infty} \frac{\lambda^k t^{k-\alpha}}{\Gamma(k+1-\alpha)} \\
{}^{C}D^{\alpha}_{t_0,t}e^{\lambda t} &= \sum_{k=0}^{\infty} \frac{\lambda^{k+n} t^{k+n-\alpha}}{\Gamma(k+n+1-\alpha)} \\
{}^{C}D^{\alpha}_{t_0,t}e^{\lambda t} &= \lambda^n t^{n-\alpha}E_{1,n-\alpha+1}(\lambda t). \qquad \square
\end{aligned}
$$

See [34] for a similar result of (2.18) without proof.

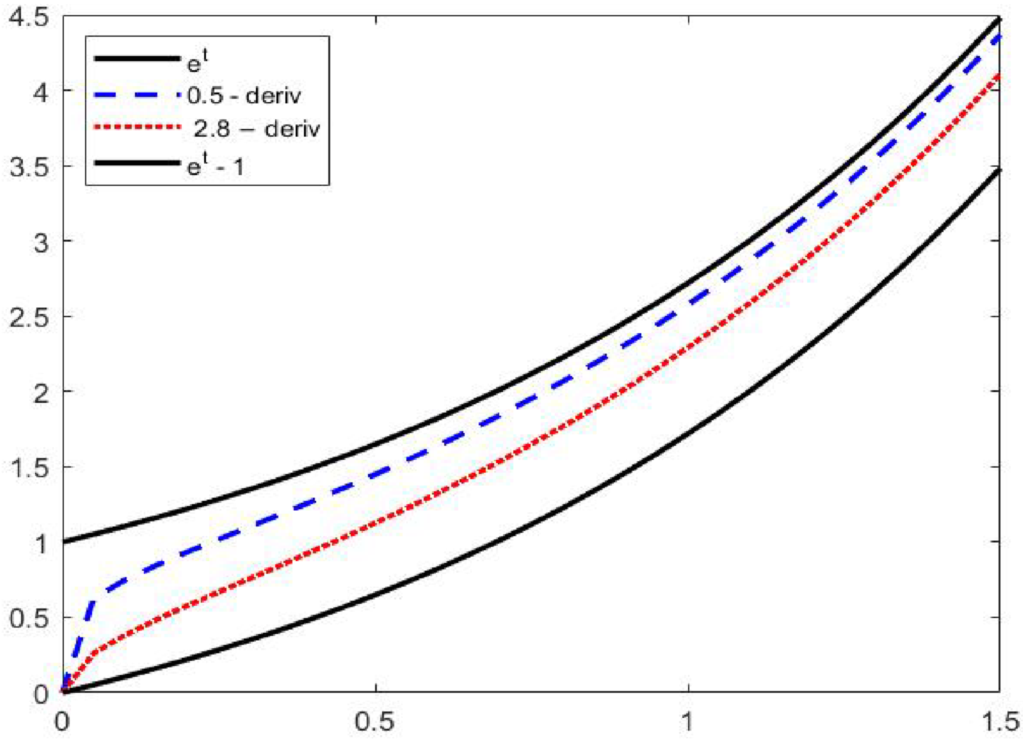

Figure 2.4: ${}^{C}D^{\alpha}_{t_0,t}e^{t}$ for 0.5-th and 2.8-th α in the interval (0,1.5]

In figure 2.4, graph of exponential fractional derivatives of the function e^t with $0.5-th$ and $2.8-th$ order in the interval $(0,1.5]$ at $\lambda=1$ is presented. The fractional derivatives are enclosed by the functions e^t and e^t-1. In general, graphs of the exponential function and its derivatives have the same shape.

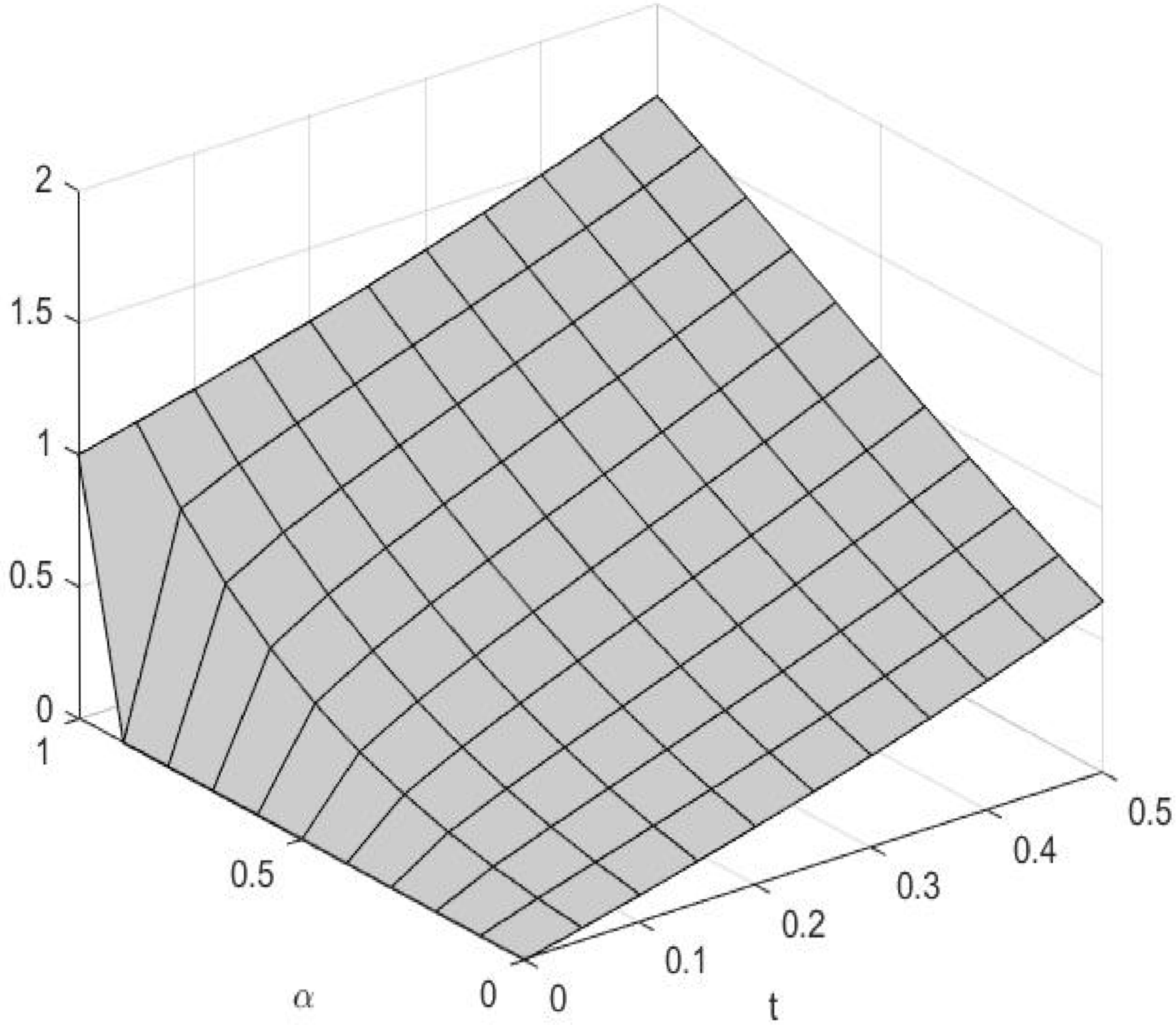

Figure 2.5: 3D ${}^{C}D^{\alpha}_{t_0,t}e^t$ for 0.5-th and 2.8-th α in the interval (0,1.5]

• The Trigonometric Functions

The Caputo fractional derivative of trigonometric functions i.e. sine and cosine is discussed in this section [34] .

Theorem 2.3.4. *Let $\lambda \in \mathbb{C}, \alpha \in \mathbb{R}, n \in \mathbb{N}, n-1 < \alpha < n$. Then*

$$ {}^{C}D^{\alpha}_{t_0,t} \sin \lambda t = -\frac{1}{2} i (i\lambda)^n t^{n-\alpha} \left(E_{1,n-\alpha+1}(i\lambda t) - (-1)^n E_{1,n-\alpha+1}(-i\lambda t) \right). $$

Proof. Using the linearity property (2.10) of the Caputo fractional derivative, the exponential function (2.18) and the sine function

$$ \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad z \in \mathbb{C}, $$

it follows that

$$ \begin{aligned} {}^{C}D^{\alpha}_{t_0,t} \sin \lambda t &= {}^{C}D^{\alpha}_{t_0,t} \frac{e^{i\lambda t} - e^{-i\lambda t}}{2i} \\ {}^{C}D^{\alpha}_{t_0,t} \sin \lambda t &= \frac{1}{2i} \left({}^{C}D^{\alpha}_{t_0,t} e^{i\lambda t} - {}^{C}D^{\alpha}_{t_0,t} e^{-i\lambda t} \right) \\ {}^{C}D^{\alpha}_{t_0,t} \sin \lambda t &= \frac{1}{2i} \left((i\lambda)^n t^{n-\alpha} E_{1,n-\alpha+1}(i\lambda t) - (-i\lambda)^n t^{n-\alpha} E_{1,n-\alpha+1}(-i\lambda t) \right) \\ {}^{C}D^{\alpha}_{t_0,t} \sin \lambda t &= -\frac{1}{2} i (i\lambda)^n t^{n-\alpha} \left(E_{1,n-\alpha+1}(i\lambda t) - (-1)^n E_{1,n-\alpha+1}(-i\lambda t) \right). \qquad \square \end{aligned} $$

The Caputo derivative of the cosine function is derived in a similar way. Using the representation of cosine function

$$ \cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad z \in \mathbb{C}, $$

we state without proof a theorem for which the Caputo derivative of cosine holds,

Theorem 2.3.5. *Let$\lambda \in \mathbb{C}, \alpha \in \mathbb{R}, n \in \mathbb{N}, n-1 < \alpha < n$. Then*

$$ {}^{C}D^{\alpha}_{t_0,t} \cos \lambda t = \frac{1}{2} (i\lambda)^n t^{n-\alpha} \left(E_{1,n-\alpha+1}(i\lambda t) + (-1)^n E_{1,n-\alpha+1}(-i\lambda t) \right). $$

2.4. Initial value problem

A fractional initial value problem (FIVP) in the sense of Caputo's definition is given by

$$ \begin{cases} {}^{C}D^{\alpha}_{t_0,t} y(t) = f(t, y(t)) \\ y^{(k)}_{t_0} = b_k, b_k \in \mathbb{R}, \quad k = 0, \ldots, \lceil \alpha \rceil - 1. \end{cases} $$

The Riemann-Liouville and the Grünwald-Letnikov derivatives of a function f are equal when $f(t) \in C^{\lceil \alpha \rceil}[t_0, t]$, for $t > t_0$ [22]. From (2.12) the Caputo derivative of a function f

can also be determined using the Grünwald-Letnikov derivative for $f(t) \in C^{\lceil\alpha\rceil}[t_0, t]$ and $t > t_0$, that is

$$ {}^{C}D^{\alpha}_{t_0,t}f(t) = {}^{GL}D^{\alpha}_{t_0,t}f(t) - \sum_{k=0}^{n-1} \frac{(t-t_0)^{k-\alpha}}{\Gamma(k+1-\alpha)} f^{(k)}(t_0), \quad n-1 < \alpha < n. $$

In next we study IVP for particular case of linear FDE

$$ \begin{cases} {}^{C}D^{\alpha}_{t_0,t}y(t) - \lambda y(t) = 0 & t > 0, \quad n-1 < \alpha < n \\ y^{(k)}(0) = b_k, & b_k \in \mathbb{R}, \quad k = 0, \dots, n-1. \end{cases} \tag{2.19} $$

Theorem 2.4.1. *The solution of problem (2.19) is given by*

$$ y(t) = \sum_{k=0}^{n-1} b_k t^k E_{\alpha,k+1}(\lambda t^{\alpha}), $$

where $E_{\alpha,\beta}(z)$ is the two-parameter function of Mittag-Leffler type.

Proof. Applying the Laplace transform to the fractional differential equation in (2.19) it becomes

$$ s^{\alpha}Y(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}y^{(k)}(0) - \lambda Y(s) = 0, $$

where

$$ Y(s) = \sum_{k=0}^{n-1} \frac{s^{\alpha-k-1}}{s^{\alpha}-\lambda} y^{(k)}(0), \tag{2.20} $$

is the Laplace transform of $y(t)$ and $L\{-\lambda y(t); s\} = -\lambda Y(s)$.

Substituting the initial conditions from (2.19) into (2.20) we get

$$ Y(s) = \sum_{k=0}^{n-1} \frac{s^{\alpha-k-1}}{s^{\alpha}-\lambda} b_k. $$

Using the Laplace transform of the two-parameter function of Mittag-Leffler type, it follows

$$ Y(s) = \sum_{k=0}^{n-1} \frac{s^{\alpha-k-1}}{s^{\alpha}-\lambda} b_k = \sum_{k=0}^{n-1} L\left\{t^k E_{\alpha,k+1}(\lambda t^{\alpha}); s\right\} b_k = L\left\{\sum_{k=0}^{n-1} b_k t^k E_{\alpha,k+1}(\lambda t^{\alpha}); s\right\}. $$

Then using the inverse Laplace transform $y(t)$ can be found as

$$ y(t) = y(t,\alpha) = \sum_{k=0}^{n-1} b_k t^k E_{\alpha,k+1}(\lambda t^{\alpha}). \quad \square $$

3. Numerical Methods

We discuss methods often used to find the approximated solution to fractional differential equations. For information on numerical methods for ordinary differential equations see, [36]. In this thesis, emphasis is given to the single term Caputo fractional differential equations

$$^{C}D^{\alpha}_{t_0,t}y(t) = f(t, y(t)), \quad 0 < \alpha \leq 1, \tag{3.1}$$

with an initial condition

$$y(0) = y_0. \tag{3.2}$$

Many researchers have proposed several ways to discretize IVP (3.1), (3.2); the most often used two approaches are based on the following ideas:

- Direct discretization of the Caputo derivative in (3.1) to get the numerical schemes.
- Transformation of IVP (3.1), (3.2) into the fractional integral equation, then fractional integral discretization for getting numerical scheme.

In the following, we introduce the numerical methods for IVP (3.1), (3.2) based on the given ideas. First we introduce mesh on interval $[t_0, T]$ where the numerical solution will be considered. The mesh consists of $\{t_j\}_{j=0}^{N}$ such that $t_0 < t_1 < \cdots < t_N = T$.
In general, the stepsize $h_j = t_j - t_{j-1},\ j = 0, 1, \cdots, N$ is variable. In the following we restrict our consideration on equidistant mesh Δ with constant stepsize $h = (T - t_0)/N$, $\Delta :\, = \{t_j\}_{j=0}^{N} : t_j = t_0 + jh$.

3.1. Direct Methods

Direct methods are based on numerical approximation of the derivative term i.e. $^{C}\mathrm{D}^{\alpha}_{t_0,t}y(t)$ in case of IVP (3.1), (3.2).

• L1 and L2 Methods

• The Caputo derivative is mostly discretized by the $L1$ method given by:

$$\left[{}^{C}\mathrm{D}^{\alpha}_{t_0,t}f(t)\right]_{t=t_N} = \sum_{k=0}^{N-1} b_{N-k-1}\left(f\left(t_{k+1}\right) - f\left(t_k\right)\right) + O\left(h^{2-\alpha}\right), \quad 0 < \alpha < 1$$

where $b_k = \frac{h^{-\alpha}}{\Gamma(2-\alpha)}\left[(k+1)^{1-\alpha} - k^{1-\alpha}\right]$. The $L1$ method for the IVP (3.1), (3.2) is

$$\sum_{j=0}^{N-1} b_{N-j-1}\left(y_{j+1} - y_j\right) = f\left(t_N, y_N\right),$$

where y_N is the approximate solution of $y\left(t_N\right)$, and $b_j = \frac{h^{-\alpha}}{\Gamma(2-\alpha)}\left[(j+1)^{1-\alpha} - j^{1-\alpha}\right]$.

An improved version of the $L1$ method for the Caputo derivative is given by:

$$\left[{}^{C}D_{t_0,t}^{\alpha}f(t)\right]_{t=t_{N+1/2}} = \begin{cases} \frac{b_0}{2}\left(f\left(t_{N+1}\right)+f\left(t_N\right)\right)-\frac{1}{2}\sum_{j=1}^{N}\left(b_{N-j}-b_{N-j+1}\right)\left(f\left(t_{j-1}\right)+f\left(t_j\right)\right) \\ \\ -\frac{1}{2}\left(b_N-D_N\right)\left(f\left(t_0\right)+f\left(t_1\right)\right)-D_N f\left(t_0\right)+O\left(h^{2-\alpha}\right) \end{cases}$$

where

$$b_N = \frac{h^{-\alpha}}{\Gamma(2-\alpha)}\left[(N+1)^{1-\alpha}-N^{1-\alpha}\right]$$

and

$$D_N = \frac{2h^{-\alpha}}{\Gamma(2-\alpha)}\left[(N+1/2)^{1-\alpha}-N^{1-\alpha}\right].$$

- For the $L2$ method in the Caputo sense we have:

$$\left[{}^{C}D_{t_0,t}^{\alpha}f(t)\right]_{t=t_N} = \sum_{k=-1}^{N} W_k f\left(t_{N-k}\right)+O\left(h^{3-\alpha}\right), \quad 1<\alpha<2,$$

where

$$W_k = \frac{h^{-\alpha}}{\Gamma(3-\alpha)}\begin{cases} 1, & k=-1, \\ 2^{2-\alpha}-3, & k=0, \\ (k+2)^{2-\alpha}-3(k+1)^{2-\alpha}+3k^{2-\alpha}-(k-1)^{2-\alpha}, & 1\le k\le N-2, \\ -2N^{2-\alpha}+3(N-1)^{2-\alpha}-(N-2)^{2-\alpha}, & k=N-1, \\ N^{2-\alpha}-(N-1)^{2-\alpha}, & k=N. \end{cases}$$

• Product Trapezoidal Method

Since the Riemann-Liouville derivative is equivalent to the Hadamard finite-part integral [37, 38, 22], we have that,

$$^{RL}D_{t_0,t}^{\alpha}y(t) = \frac{1}{\Gamma(-\alpha)}\ \text{p.f.} \int_{t_0}^{t}\frac{y(s)}{(t-s)^{\alpha+1}}\,\mathrm{d}s, \quad \alpha\neq 0,1,2,\cdots \tag{3.3}$$

The equation (3.3) is approximated by the first-degree compound quadrature formula [39, 37, 40], which is given by

$$\frac{1}{\Gamma(-\alpha)}\ \text{p.f.} \int_{t_0}^{t_N}\frac{y(s)}{(t_N-s)^{\alpha+1}}\,\mathrm{d}s \approx \sum_{j=0}^{N} a_{j,N}\,y\left(t_{N-j}\right),$$

where

$$a_{j,N} = \frac{h^{-\alpha}}{\Gamma(2-\alpha)}\begin{cases} 1, & j=0, \\ (j+1)^{1-\alpha}-2j^{1-\alpha}+(j-1)^{1-\alpha}, & 0<j<N, \\ (1-\alpha)N^{-\alpha}-N^{1-\alpha}+(N-1)^{1-\alpha}, & j=N. \end{cases}$$

From the relationship

$$^{RL}D_{t_0,t}^{\alpha}[y(t)-y(0)] = {}^{C}D_{t_0,t}^{\alpha}y(t), \quad 0<\alpha<1,$$

we obtain the numerical scheme for IVP (3.1), (3.2) as follows

$$\sum_{j=0}^{N} a_{j,N}\left(y_{N-j}-y_0\right)=f\left(t_N,y_N\right), \tag{3.4}$$

where y_N is the approximate solution of $y\left(t_N\right)$. When $f(t,y)=\beta y(t)+f(t), \beta\leq 0$, (3.4) has the error estimate

$$\left|y\left(t_N\right)-y_N\right|\leq Ch^{2-\alpha}, \text{ (see [39]).}$$

Next, we introduce a modified trapezoidal rule , see [41]. The rule is used to approximate the fractional integral $I_{t_0}^{\alpha}f(t)$ by a weighted sum of functional values at specified points.

$$y(f,h,\alpha)=\begin{cases}\left((N-1)^{\alpha+1}-(N-\alpha-1)N^{\alpha}\right)\frac{h^{\alpha}f(t_0)}{\Gamma(\alpha+2)}+\frac{h^{\alpha}f(T)}{\Gamma(\alpha+2)}\\ +\sum_{j=1}^{N-1}\left((N-j+1)^{\alpha+1}-2(N-j)^{\alpha+1}+(N-j-1)^{\alpha+1}\right)\dfrac{h^{\alpha}f\left(t_j\right)}{\Gamma(\alpha+2)},\end{cases} \tag{3.5}$$

which is an approximation to the fractional integral

$$\left(I_{t_0}^{\alpha}f(t)\right)(T)=y(f,h,\alpha)-E_y(f,h,\alpha), \quad T>t_0, \quad \alpha>0.$$

There is a constant C_{α} depending only on α if $f(t)\in\mathbf{C}^2[t_0,T]$, so that the error term $E_y(f,h,\alpha)$ has the form

$$\left|E_y(f,h,\alpha)\right|\leq C_{\alpha}\left\|f''\right\|_{\infty}T^{\alpha}h^2=O\left(h^2\right).$$

• Grünwald–Letnikov Formula

Using the Grünwald–Letnikov formula to approximate the Riemann–Liouville derivative, we have

$${}^{RL}D_{t_0,t}^{\alpha}y(t)\Big|_{t=t_N}\approx\frac{1}{h^{\alpha}}\sum_{j=0}^{N}\omega_j^{(\alpha)}y\left(t_{N-j}\right), \quad \omega_j^{(\alpha)}=(-1)^j\begin{pmatrix}\alpha\\ j\end{pmatrix}. \tag{3.6}$$

It was proved (see [42]). That formula (3.6) is convergent and it is of order one for any α positive. By the relationship

$${}^{RL}D_{t_0,t}^{\alpha}\left[y(t)-\sum_{k=0}^{m}\frac{t^k}{k!}y^{(k)}(t_0)\right]={}^{C}D_{t_0,t}^{\alpha}y(t),$$

one gets the following method for IVP (3.1),(3.2) as

$$\frac{1}{h^{\alpha}}\sum_{j=0}^{N}\omega_{n-j}^{(\alpha)}\left[y_j-\sum_{k=0}^{m}\frac{y_0^k}{k!}t_j^k\right]=f\left(t_N,y_N\right), for \quad \alpha>0.$$

The right shifted Grünwald-Letnikov formula (3.7) is then used to approximate the Caputo derivative.

$$\left[{}^{RL}D_{t_0,t}^{\alpha}f(t)\right]_{t=t_N}\approx\frac{1}{h^{\alpha}}\sum_{j=0}^{N+p}\omega_j^{(\alpha)}f\left(t_{N-j+p}\right), (p\ shifts, p\in\mathbb{N}). \tag{3.7}$$

3.2. Integration Methods

Methods used to discretize the integral part of the IVP (3.1), (3.2) are presented. Our analysis is based on the fractional Euler method. Other methods are introduced as well.

• Fractional Euler methods

We introduce the derivation of Euler methods for the numerical solution of the IVP (3.1), (3.2) in the Caputo sense. This method is a generalization of the classical Eulers methods. All smooth functions $y(t)$ can be expanded in a Taylor series

$$y\left(t_{j+1}\right)=y\left(t_j\right)+h_j \frac{y^{\prime}\left(t_j\right)}{1!}+h_j^2 \frac{y^{(2)}\left(t_j\right)}{2!}+\ldots$$

If $y(t)$ is the local solution defined by

$$y^{\prime}=F(t, y), \quad y\left(t_j\right)=y_j,$$

this expansion suggests an approximation of $y\left(t_{j+1}\right)$ known as the forward Euler method

$$y_{j+1}=y_j+h_j F\left(t_j, y_j\right).$$

This formula is explicit and requires one evaluation of F per step. The series shows that the local error

$$l e_j=y\left(t_{j+1}\right)-y_{j+1},$$

is $O\left(h_j^2\right)$. When $y(t)$ in this definition is the solution $y(t)$ of the IVP (3.1), (3.2), the local error is called the truncation error or discretization error. The backward Euler method is introduced as

$$y_{j+1}=y_j+h_j F\left(t_{j+1}, y_{j+1}\right).$$

As we know from numerical theory for ODEs, backward Euler has better absolute stability property than its forward counterpart. Now we introduce several assertions which are necessary for fractional Euler methods formulation. Before we proceed, some theorems necessary in the derivation of the fractional Euler methods are presented.

Theorem 3.2.1. *(Generalized mean value theorem) Suppose that* $f(t) \in C[t_0, T]$ *and* ${}^C D_{t_0,t}^\alpha f(t) \in C(t_0, T]$, *for* $0<\alpha \leq 1$. *Then*

$$f(t)=f(0+)+\frac{1}{\Gamma(\alpha)}\left({}^C D_{t_0, t}^\alpha f\right)(\xi) \cdot t^\alpha, \tag{3.8}$$

with $0 \leq \xi \leq t, \forall t \in\left(t_0, T\right]$.

Proof. From the definitions of the Riemann-Liouville fractional integral operator (2.2.1) and the Caputo fractional derivative operator (2.2.3), we have

$$\left(I_{t_0}^\alpha {}^C D_{t_0, t}^\alpha f\right)(t)=\frac{1}{\Gamma(\alpha)} \int_{t_0}^t(t-\tau)^{\alpha-1}\left({}^C D_{t_0, t}^\alpha f\right)(\tau) d \tau.$$

From the integral mean value theorem, we have

$$\left(I_{t_0}^{\alpha} {}^{C}D_{t_0,t}^{\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \left({}^{C}D_{t_0,t}^{\alpha} f\right)(\xi) \int_{t_0}^{t} (t-\tau)^{\alpha-1} = \frac{1}{\Gamma(\alpha)} \left({}^{C}D_{t_0,t}^{\alpha} f\right)(\xi) \cdot t^{\alpha}, \tag{3.9}$$

for $0 \leq \xi \leq t$. Also, from (2.14), we have

$$\left(I_{t_0}^{\alpha} {}^{C}D_{t_0,t}^{\alpha} f\right)(t) = f(t) - f(0+). \tag{3.10}$$

From (3.9) and (3.10), the generalized mean value theorem (3.8) is obtained. □

Remark 3.2.1. *In case of $\alpha = 1$, the generalized mean value theorem reduces to the classical mean value theorem.*

Theorem 3.2.2. *Suppose that ${}^{C}D_{t_0,t}^{j\alpha} f(t), {}^{C}D_{t_0,t}^{(j+1)\alpha} f(t) \in C(t_0, T]$, for $0 < \alpha \leq 1$. Then*

$$\left(I_{t_0}^{j\alpha C} D_{t_0,t}^{j\alpha} f\right)(t) - \left(I_{t_0}^{(j+1)\alpha C} D_{t_0,t}^{(j+1)\alpha} f\right)(t) = \frac{t^{j\alpha}}{\Gamma(j\alpha+1)} \left({}^{C}D_{t_0,t}^{j\alpha} f\right)(0+), \tag{3.11}$$

where

$${}^{C}D_{t_0,t}^{j\alpha} = {}^{C}D_{t_0,t}^{\alpha} {}^{C}D_{t_0,t}^{\alpha} \dots {}^{C}D_{t_0,t}^{\alpha} \quad (j- \text{ times }).$$

Proof. We can get the proof by using the properties of the Riemann-Liouville fractional integral operator and the Caputo fractional derivative operator and the relation:

$$\left(I_{t_0}^{j\alpha C} D_{t_0,t}^{j\alpha} f\right)(x) - \left(I_{t_0}^{(j+1)\alpha C} D_{t_0,t}^{(j+1)\alpha} f\right)(x) = I_{t_0}^{j\alpha} \left(\left({}^{C}D_{t_0,t}^{j\alpha} f\right)(t) - \left(I_{t_0}^{\alpha C} D_{t_0,t}^{\alpha}\right) \left({}^{C}D_{t_0,t}^{j\alpha} f\right)(t)\right)$$

$$\left(I_{t_0}^{j\alpha C} D_{t_0,t}^{j\alpha} f\right)(x) - \left(I_{t_0}^{(j+1)\alpha C} D_{t_0,t}^{(j+1)\alpha} f\right)(x) = I_{t_0}^{j\alpha} \left({}^{C}D_{t_0,t}^{j\alpha} f\right)(0+). \quad \square$$

Theorem 3.2.3. *(Generalized Taylor's formula) Suppose that ${}^{C}D_{t_0,t}^{k\alpha} f(t) \in C(t_0, T]$ for $j = 0, 1, \dots, N+1$, and $0 < \alpha \leq 1$. Then*

$$f(t) = \sum_{k=0}^{N} \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} \left({}^{C}D_{t_0,t}^{k\alpha}\right)(0+) + \frac{\left({}^{C}D_{t_0,t}^{(N+1)\alpha} f\right)(\xi)}{\Gamma((N+1)\alpha+1)} t^{(N+1)\alpha}, \tag{3.12}$$

with $0 \leq \xi \leq t, \forall t \in (t_0, T]$.

Proof. From (3.11), we get

$$\sum_{k=0}^{N} \left(I_{t_0}^{k\alpha C} D_{t_0,t}^{k\alpha} f\right)(t) - \left(I_{t_0}^{(k+1)\alpha C} D_{t_0,t}^{(k+1)\alpha} f\right)(t) = \sum_{k=0}^{N} \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} \left({}^{C}D_{t_0,t}^{k\alpha} f\right)(0+),$$

it follows,

$$f(t) - \left(I_{t_0}^{(N+1)\alpha C} D_{t_0,t}^{(N+1)\alpha} f\right)(t) = \sum_{k=0}^{N} \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} \left({}^{C}D_{t_0,t}^{k\alpha} f\right)(0+). \tag{3.13}$$

We apply the integral mean value theorem to (3.13), we have

$$\begin{aligned}\left(I_{t_0}^{(N+1)\alpha C}D_{t_0,t}^{(N+1)\alpha}f\right)(t) &= \frac{\left({}^{C}D_{t_0,t}^{(N+1)\alpha}f\right)(\xi)}{\Gamma((N+1)\alpha+1)}\int_{t_0}^{t}(t-\tau)^{(N+1)\alpha}d\tau,\\ \left(I_{t_0}^{(N+1)\alpha C}D_{t_0,t}^{(N+1)\alpha}f\right)(t) &= \frac{\left({}^{C}D_{t_0,t}^{(N+1)\alpha}f\right)(\xi)}{\Gamma((N+1)\alpha+1)}t^{(N+1)\alpha}.\end{aligned} \tag{3.14}$$

We obtain the generalized Taylor's formula (3.12) if we substitute (3.14) into (3.13) .

Remark 3.2.2. *When $\alpha = 1$, the generalized Taylor's formula (3.12) reduces to the classical Taylor's formula.*

Consider the IVP (3.1), (3.2). We denote $[t_0, T]$ as the interval on which solution to problem (3.1), (3.2) is found. A set of points $\{(t_j, y(t_j))\}$ is generated for our numerical approximation. The interval $[t_0, T]$ is subdivided into N subintervals $[t_j, t_{j+1}]$ of equal width $h = (T - t_0)/N$ and the nodes $t_j = t_0 + jh$, for $j = 0, 1, \ldots, N$. Assume that $y(t), {}^{C}D_{t_0,t}^{\alpha}y(t)$ and ${}^{C}D_{t_0,t}^{2\alpha}y(t)$ are continuous on $[t_0, T]$ and using the generalized Taylor's formula (3.12), we expand $y(t)$ about $t = t_0 = 0$. For all t values there exist a value c_1 such that

$$y(t) = y(t_0) + \left({}^{C}D_{t_0,t}^{\alpha}y(t)\right)(t_0)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \left({}^{C}D_{t_0,t}^{2\alpha}y(t)\right)(c_1)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}. \tag{3.15}$$

When $\left({}^{C}D_{t_0,t}^{\alpha}y(t)\right)(t_0) = f(t_0, y(t_0))$ and $h = t_1$ are substituted into equation (3.15), we get

$$y(t_1) = y(t_0) + f(t_0, y(t_0))\frac{h^{\alpha}}{\Gamma(\alpha+1)} + \left({}^{C}D_{t_0,t}^{2\alpha}y(t)\right)(c_1)\frac{h^{2\alpha}}{\Gamma(2\alpha+1)}.$$

We can neglect the second-order term involving $h^{2\alpha}$ if the step size h chosen is small enough, it follows

$$y(t_1) = y(t_0) + \frac{h^{\alpha}}{\Gamma(\alpha+1)}f(t_0, y(t_0)). \tag{3.16}$$

The process is repeated and generates a sequence of points that approximates the solution $y(t)$. The general formula for fractional Euler's method is

$$\begin{cases} t_{j+1} = t_j + h \\ y(t_{j+1}) = y(t_j) + \frac{h^{\alpha}}{\Gamma(\alpha+1)}f(t_j, y(t_j)). \end{cases} \tag{3.17}$$

for $j = 0, 1, \ldots, N-1$. A special case is shown if $\alpha = 1$, the fractional Euler's method (3.17) reduces to the classical Euler's method.

• Improved Fractional Euler method

The improved algorithm is based on the fractional Euler's method and the modified trapezoidal rule. The analytical property that the IVP (3.15) is identical to the integral equation

$$y(t)=J^{\alpha}f(t,y(t))+y(0), \tag{3.18}$$

underpins our approach. Let $[t_0,T]$ be the interval for the approximate solution. Assume that the interval $[t_0,T]$ is subdivided into N subintervals $[t_j,t_{j+1}]$ of equal width $h=(T-t_0)/N$ and the nodes $t_j=t_0+jh$, for $j=0,1,\ldots,N$. To obtain the solution point $(t_1,y(t_1))$, we substitute $t=t_1$ into (3.18), we have

$$y(t_1)=\left(J^{\alpha}f(t,y(t))\right)(t_1)+y(0).$$

We get

$$y(t_1)=\alpha\frac{h^{\alpha}f(t_0,y(t_0))}{\Gamma(\alpha+2)}+\frac{h^{\alpha}f(t_1,y(t_1))}{\Gamma(\alpha+2)}+y(0), \tag{3.19}$$

if the modified trapezoidal rule (3.5) is used to approximate $\left(I_{t_0}^{\alpha}f(t,y(t))\right)(t_1)$ with step size $h=t_1-t_0$. We can see that the formula on the right-hand side of (3.19) contains the term $y(t_1)$, therefore we use an estimate for $y(t_1)$. Fractional Euler's method is sufficient for this task. Substituting (3.16) into (3.19), gives

$$y(t_1)=\alpha\frac{h^{\alpha}f(t_0,y(t_0))}{\Gamma(\alpha+2)}+\frac{h^{\alpha}f\left(t_1,y(t_0)+\frac{h^{\alpha}}{\Gamma(\alpha+1)}f(t_0,y(t_0))\right)}{\Gamma(\alpha+2)}+y(0).$$

The process is repeated to generate a sequence of points that approximate the solution $y(t)$. At each step, the fractional Euler's method is used as a prediction, and then the modified trapezoidal rule is used to make a correction to obtain the finite value. The general formula for our algorithm is:

$$y(t_j)=\begin{cases}\frac{h^{\alpha}}{\Gamma(\alpha+2)}\left((j-1)^{\alpha+1}-(j-\alpha-1)j^{\alpha}\right)f(t_0,y(t_0))+y(0)\\ +\frac{h^{\alpha}}{\Gamma(\alpha+2)}\sum_{i=1}^{j-1}\left((j-i+1)^{\alpha+1}-2(j-i)^{\alpha+1}+(j-i-1)^{\alpha+1}\right)f(t_i,y(t_i))\\ +\frac{h^{\alpha}}{\Gamma(\alpha+2)}f\left(t_j,y(t_{j-1})+\frac{h^{\alpha}}{\Gamma(\alpha+1)}f(t_{j-1},y(t_{j-1}))\right).\end{cases}$$

The improved algorithm is simple for computational performance for all values of α and h. It is clear that the behavior of the method is independent of the parameter α and, as we will see in the next section, the accuracy of the approximation depends on the step size h.

The algorithm is used directly without applying linearization, perturbation or restrictive assumptions. The algorithm is shown to be reliable and useful for the numerical evaluation of functions such as the Mittag-Leffler function, (see [43]). Appendix A contains the forward Euler algorithm and Appendix B contains the improved forward Euler algorithm.

3.3. Numerical examples

In the tables and figures below are values and figures for the approximation of forward Euler and improved forward Euler method. To check the accuracy and stability of the algorithms, we evaluate them using example (3.3.1,3.3.2 and 3.3.3).

Example 3.3.1. *Take into account the following fractional differential equation*

$$^{C}D^{\alpha}_{t_0,t}y(t) = \frac{2}{\Gamma(3-\alpha)}t^{2-\alpha} - y(t) + t^2, \quad y(0) = 0, \quad 0 < \alpha \leq 1, and\ t > 0. \tag{3.20}$$

which has the exact solution

$$y(t) = t^2.$$

Considering $\alpha = 0.5$, 0.75 and $\alpha = 1.0$ with a fixed stepsize $h = 0.01$. *Table* 1 gives the numerical values of example 3.3.1. for the forward Euler method and *Table* 2 for the improved forward Euler method.

Table 1

	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1.0$	
t	Exact	Approx	Exact	Approx	Exact	Approx
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	0.0100	0.0175	0.0100	0.0133	0.0100	0.0090
0.2	0.0400	0.0871	0.0400	0.0642	0.0400	0.0382
0.3	0.0900	0.2077	0.0900	0.1550	0.0900	0.0874
0.4	0.1600	0.3733	0.1600	0.2847	0.1600	0.1567
0.5	0.2500	0.5799	0.2500	0.4513	0.2500	0.2461
0.6	0.3600	0.8246	0.3600	0.6528	0.3600	0.3555
0.7	0.4900	1.1053	0.4900	0.8874	0.4900	0.4849
0.8	0.6400	1.4206	0.6400	1.1535	0.6400	0.6345
0.9	0.8100	1.7694	0.8100	1.4497	0.8100	0.8040
1.0	1.0000	2.1509	1.0000	1.7749	1.0000	0.9937

Table 2

	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1.0$	
t	Exact	Approx	Exact	Approx	Exact	Approx
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	0.0100	0.0098	0.0100	0.0115	0.0100	0.0111
0.2	0.0400	0.0403	0.0400	0.0440	0.0400	0.0422
0.3	0.0900	0.0911	0.0900	0.0967	0.0900	0.0934
0.4	0.1600	0.1621	0.1600	0.1697	0.1600	0.1647
0.5	0.2500	0.2532	0.2500	0.2628	0.2500	0.2561
0.6	0.3600	0.3643	0.3600	0.3760	0.3600	0.3675
0.7	0.4900	0.4956	0.4900	0.5092	0.4900	0.4990
0.8	0.6400	0.6469	0.6400	0.6625	0.6400	0.6505
0.9	0.8100	0.8182	0.8100	0.8358	0.8100	0.8221
1.0	1.0000	1.0096	1.0000	1.0292	1.0000	1.0137

Table 3 and *Table* 4 give the absolute errors for forward Euler method and improved forward Euler method of example 3.3.1 at $\alpha = 0.5, 0.75$ and 1.0 with stepsizes $h = 0.01$ and $h = 0.001$ respectively.

Table 3

	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1.0$	
t	h=0.01	h=0.001	h=0.01	h=0.001	h=0.01	h=0.001
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	0.0075	0.0268	0.0033	0.0141	0.0010	0.0001
0.2	0.0471	0.0987	0.0242	0.0633	0.0018	0.0002
0.3	0.1177	0.1990	0.0650	0.1427	0.0026	0.0003
0.4	0.2133	0.3212	0.1247	0.2458	0.0033	0.0003
0.5	0.3299	0.4620	0.2013	0.3672	0.0039	0.0004
0.6	0.4646	0.6193	0.2928	0.5032	0.0045	0.0005
0.7	0.6153	0.7916	0.3974	0.6511	0.0051	0.0005
0.8	0.7806	0.9777	0.5135	0.8088	0.0055	0.0006
0.9	0.9594	1.1766	0.6397	0.9750	0.0060	0.0006
1.0	1.1509	1.3876	0.7749	1.1487	0.0063	0.0006

Table 4

	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1.0$	
t	h=0.01	h=0.001	h=0.01	h=0.001	h=0.01	h=0.001
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	0.0048	0.0014	0.0027	0.0005	0.0011	0.0001
0.2	0.0123	0.0037	0.0063	0.0012	0.0022	0.0002
0.3	0.0212	0.0064	0.0102	0.0019	0.0034	0.0003
0.4	0.0311	0.0095	0.0143	0.0026	0.0047	0.0005
0.5	0.0417	0.0128	0.0184	0.0033	0.0061	0.0006
0.6	0.0529	0.0163	0.0226	0.0040	0.0075	0.0007
0.7	0.0647	0.0200	0.0267	0.0047	0.0090	0.0009
0.8	0.0768	0.0238	0.0309	0.0054	0.0105	0.0010
0.9	0.0894	0.0277	0.0351	0.0061	0.0121	0.0012
1.0	0.1023	0.0318	0.0392	0.0068	0.0137	0.0014

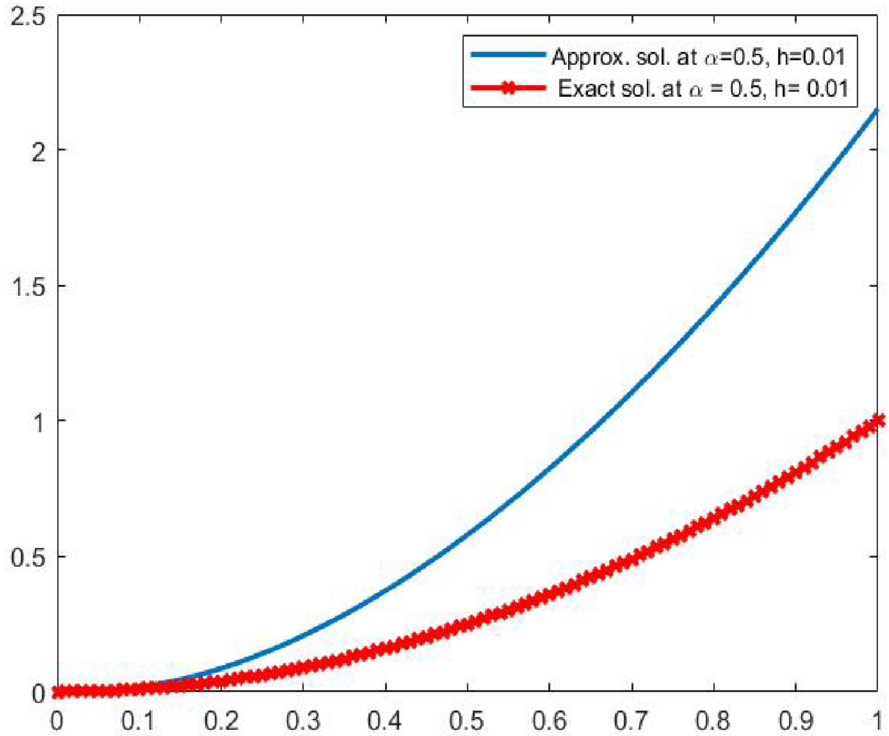

Figure 3.1: Forward Euler method to FIVP (3.20)

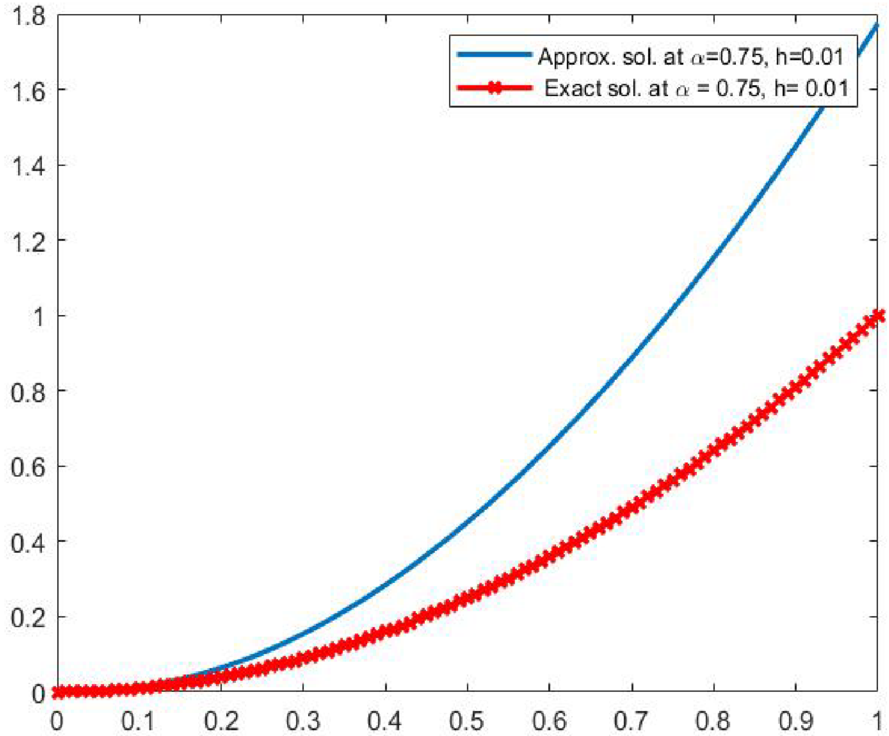

Figure 3.2: Forward Euler method to FIVP (3.20)

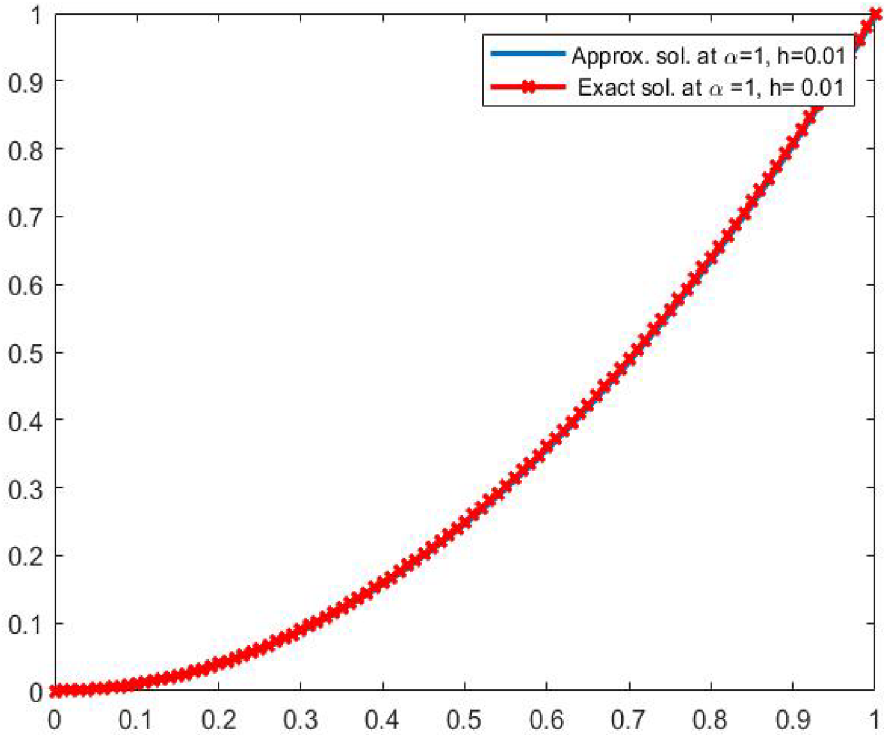

Figure 3.3: Forward Euler method to FIVP (3.20)

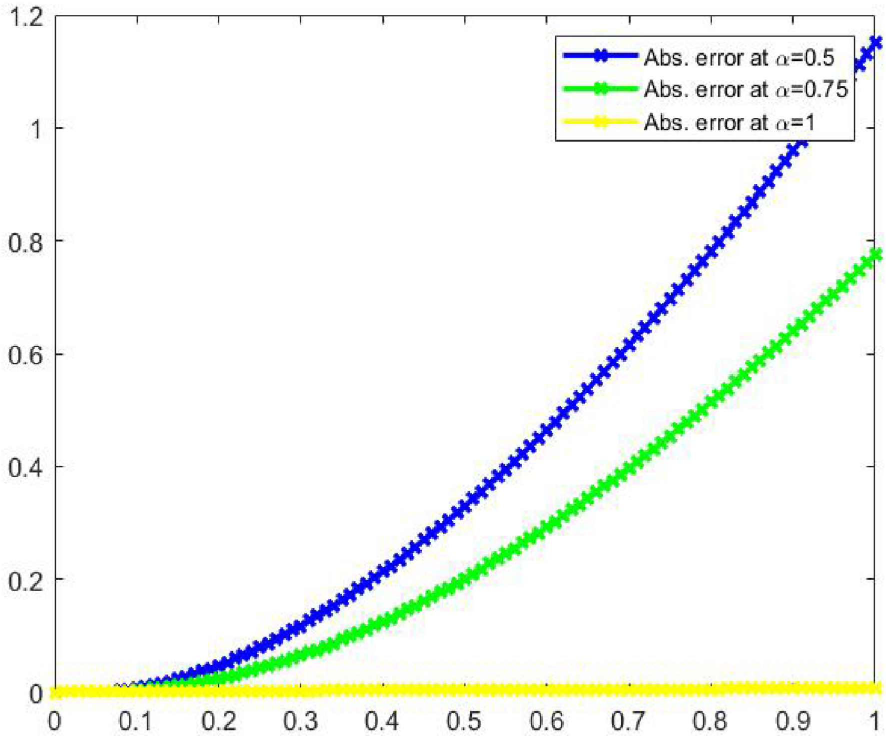

Figure 3.4: Absolute error of forward Euler method to FIVP (3.20) at $h = 0.01$

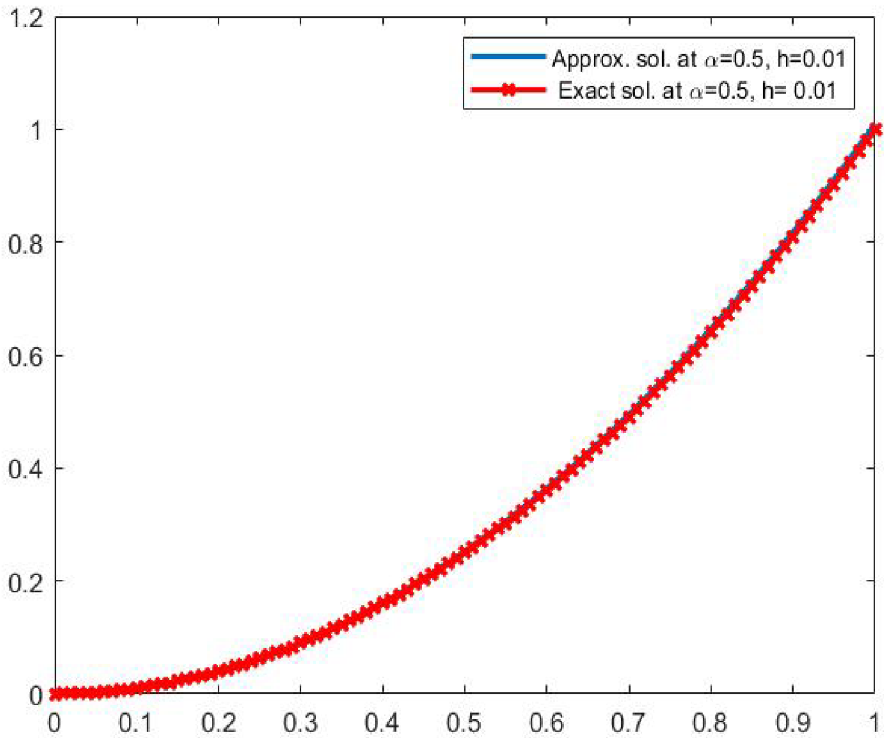

Figure 3.5: Improved forward Euler method to FIVP (3.20)

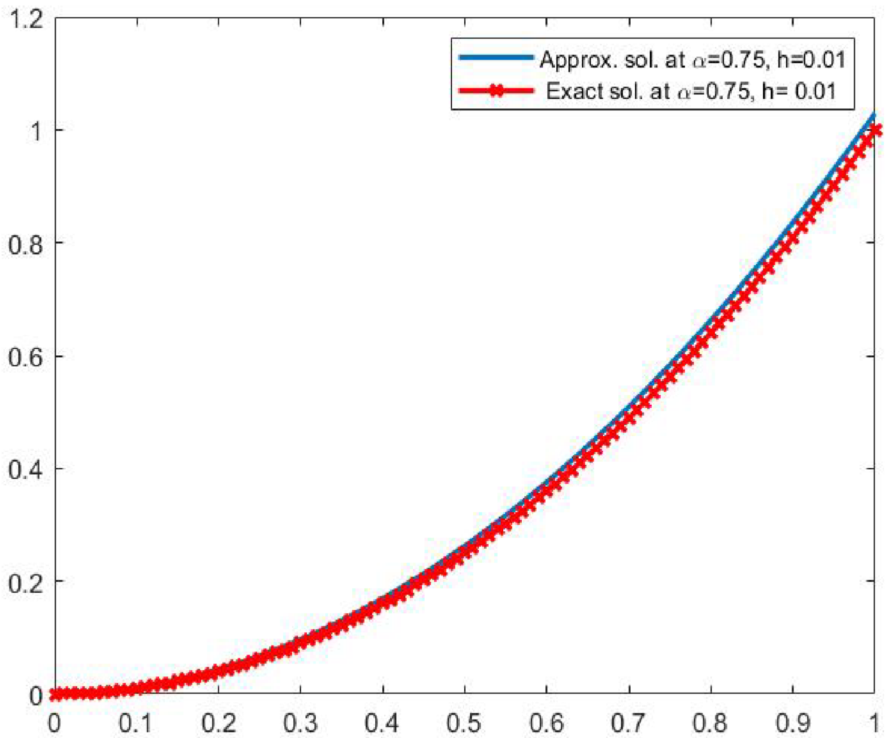

Figure 3.6: Improved forward Euler method to FIVP (3.20)

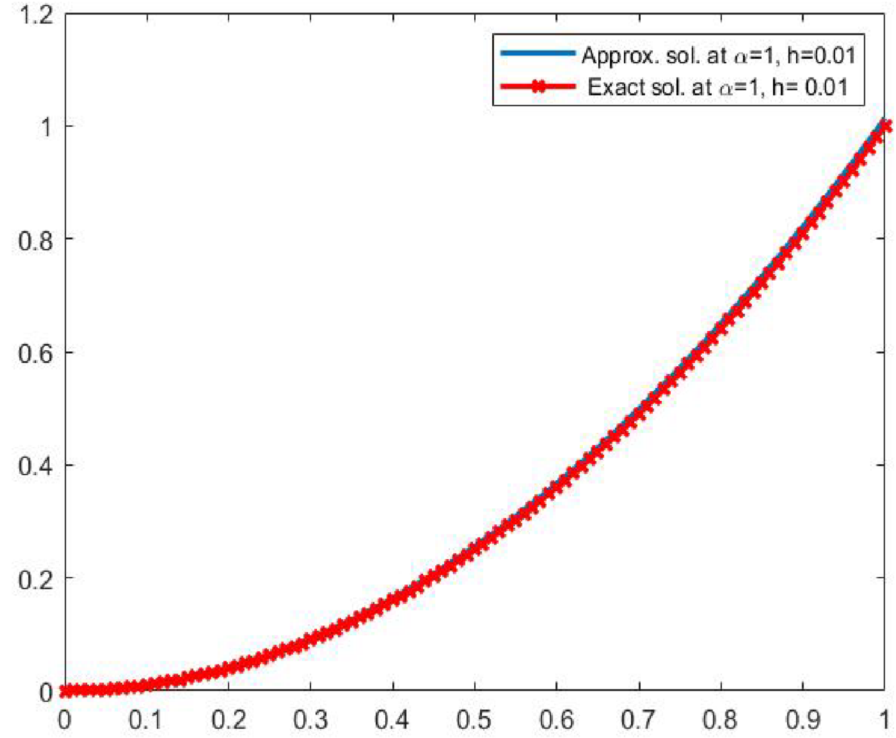

Figure 3.7: Improved forward Euler method to FIVP (3.20)

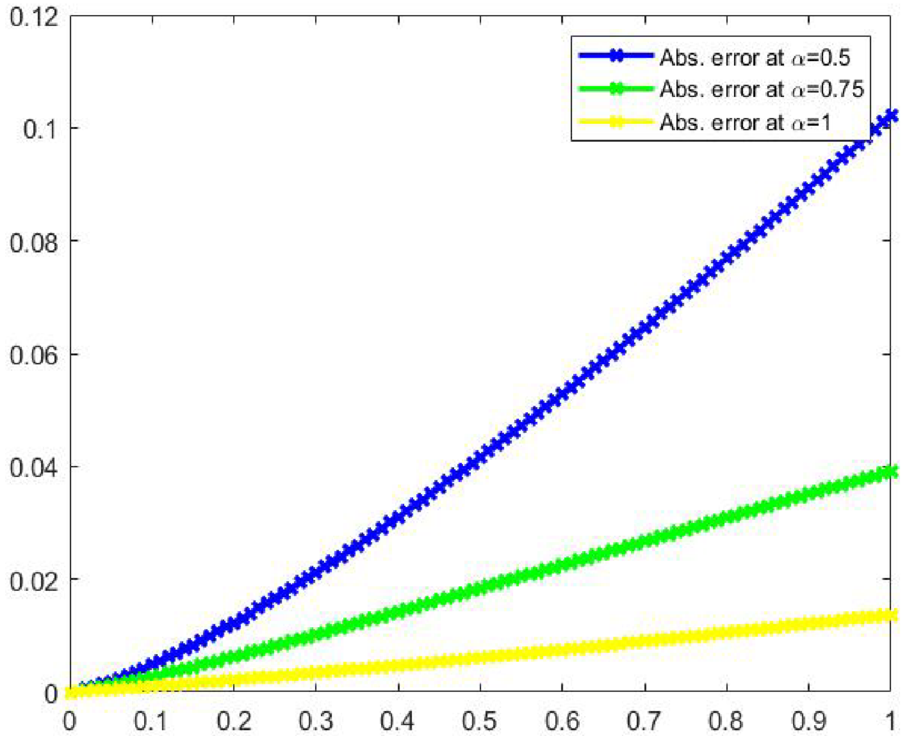

Figure 3.8: Absolute error of improved forward Euler method to FIVP (3.20) at $h = 0.01$

A numerical solution is obtained with 100 number of subintervals. In the figures for the forward Euler and improved forward Euler method, we observe that the approximated solutions approach closely to the exact solutions as the number of division increases.

Example 3.3.2. *This example covers the inhomogeneous linear equation*

$$ {}^{C}D^{\alpha}_{t_0,t}y(t) = \frac{2}{\Gamma(3-\alpha)}t^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)}t^{1-\alpha} - y(t) + t^2 - t, \quad y(0) = 0, \quad t > 0, \quad (3.21) $$

where $0 < \alpha \leq 1$.

The exact solution of equation (3.21) is given by

$$ y(t) = t^2 - t. $$

Considering $\alpha = 0.5$, 0.75 and $\alpha = 1.0$ with a fixed stepsize $h = 0.01$. *Table* 5 gives the numerical values of example 3.3.2. for the forward Euler method and *Table* 6 for the improved forward Euler method.

Table 5

	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1.0$	
t	Exact	Approx	Exact	Approx	Exact	Approx
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	−0.0900	−0.1942	−0.0900	−0.1391	−0.0900	−0.0910
0.2	−0.1600	−0.3931	−0.1600	−0.2883	−0.1600	−0.1618
0.3	−0.2100	−0.5120	−0.2100	−0.3962	−0.2100	−0.2126
0.4	−0.2400	−0.5602	−0.2400	−0.4566	−0.2400	−0.2433
0.5	−0.2500	−0.5501	−0.2500	−0.4699	−0.2500	−0.2539
0.6	−0.2400	−0.4902	−0.2400	−0.4384	−0.2400	−0.2445
0.7	−0.2100	−0.3858	−0.2100	−0.3652	−0.2100	−0.2151
0.8	−0.1600	−0.2405	−0.1600	−0.2531	−0.1600	−0.1655
0.9	−0.0900	−0.0568	−0.0900	−0.1045	−0.0900	−0.0960
1.0	0.0000	0.1635	0.0000	0.0782	0.0000	−0.0063

Table 6

	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1.0$	
t	Exact	Approx	Exact	Approx	Exact	Approx
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	−0.0900	−0.0916	−0.0900	−0.0980	−0.0900	−0.0899
0.2	−0.1600	−0.1629	−0.1600	−0.1683	−0.1600	−0.1596
0.3	−0.2100	−0.2132	−0.2100	−0.2170	−0.2100	−0.2092
0.4	−0.2400	−0.2429	−0.2400	−0.2449	−0.2400	−0.2386
0.5	−0.2500	−0.2524	−0.2500	−0.2524	−0.2500	−0.2479
0.6	−0.2400	−0.2416	−0.2400	−0.2397	−0.2400	−0.2370
0.7	−0.2100	−0.2107	−0.2100	−0.2067	−0.2100	−0.2061
0.8	−0.1600	−0.1597	−0.1600	−0.1537	−0.1600	−0.1550
0.9	−0.0900	−0.0886	−0.0900	−0.0805	−0.0900	−0.0838
1.0	0.0000	0.0026	0.0000	0.0127	0.0000	0.0074

Table 7 and *Table* 8 give the absolute errors for Euler method and improved Euler method of example 3.3.2 at $\alpha = 0.5, 0.75$ and 1.0 with stepsizes $h = 0.01$ and $h = 0.001$ respectively.

Table 7

	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1.0$	
t	h=0.01	h=0.001	h=0.01	h=0.001	h=0.01	h=0.001
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	0.1042	0.2414	0.0491	0.1447	0.0010	0.0001
0.2	0.2331	0.3392	0.1283	0.2639	0.0018	0.0002
0.3	0.3020	0.3606	0.1862	0.3185	0.0026	0.0003
0.4	0.3202	0.3385	0.2166	0.3192	0.0033	0.0003
0.5	0.3001	0.2848	0.2199	0.2790	0.0039	0.0004
0.6	0.2502	0.2058	0.1984	0.2082	0.0045	0.0005
0.7	0.1758	0.1052	0.1552	0.1142	0.0051	0.0005
0.8	0.0805	0.0144	0.0931	0.0021	0.0055	0.0006
0.9	0.0332	0.1511	0.0145	0.1245	0.0060	0.0006
1.0	0.1635	0.3033	0.0782	0.2631	0.0063	0.0006

Table 8

	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1.0$	
t	h=0.01	h=0.001	h=0.01	h=0.001	h=0.01	h=0.001
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	0.0280	0.0085	0.0134	0.0025	0.0001	0.0000
0.2	0.0292	0.0090	0.0122	0.0022	0.0004	0.0000
0.3	0.0260	0.0081	0.0094	0.0017	0.0008	0.0001
0.4	0.0203	0.0065	0.0060	0.0010	0.0014	0.0001
0.5	0.0131	0.0042	0.0025	0.0003	0.0021	0.0002
0.6	0.0047	0.0016	0.0025	0.0005	0.0030	0.0003
0.7	0.0049	0.0013	0.0063	0.0012	0.0039	0.0004
0.8	0.0150	0.0044	0.0104	0.0019	0.0050	0.0005
0.9	0.0257	0.0078	0.0146	0.0027	0.0062	0.0006
1.0	0.0370	0.0113	0.0188	0.0034	0.0074	0.0007

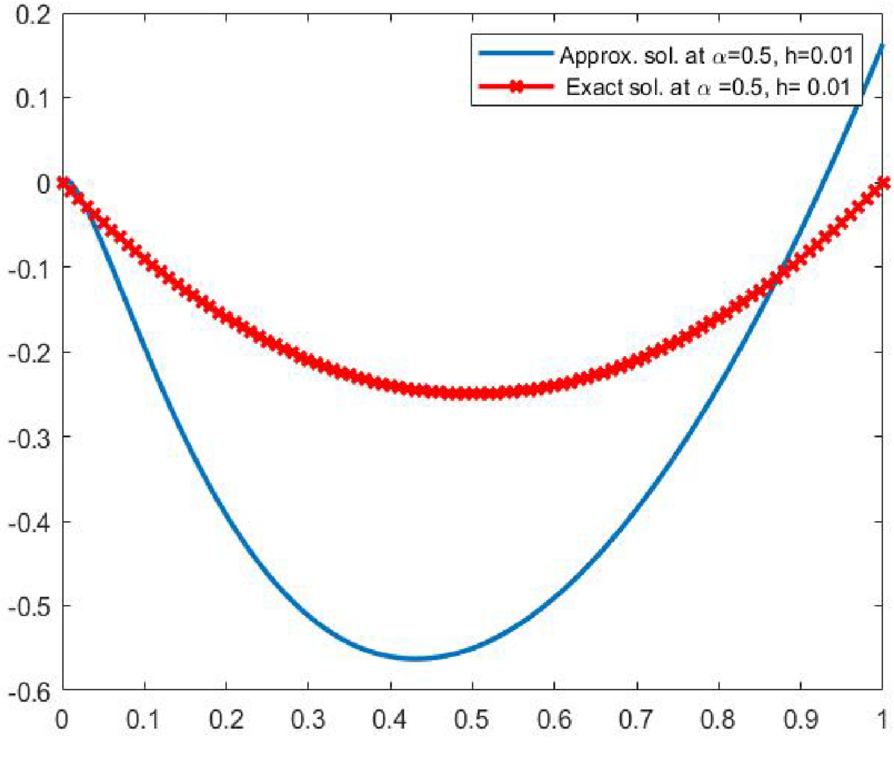

Figure 3.9: Forward Euler method to FIVP (3.21)

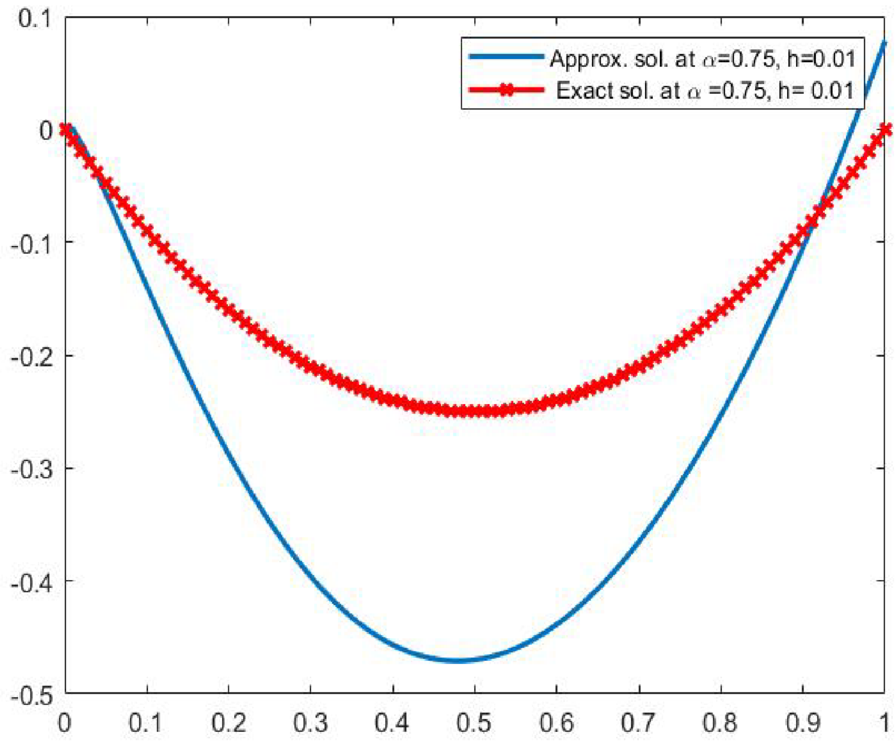

Figure 3.10: Forward Euler method to FIVP (3.21)

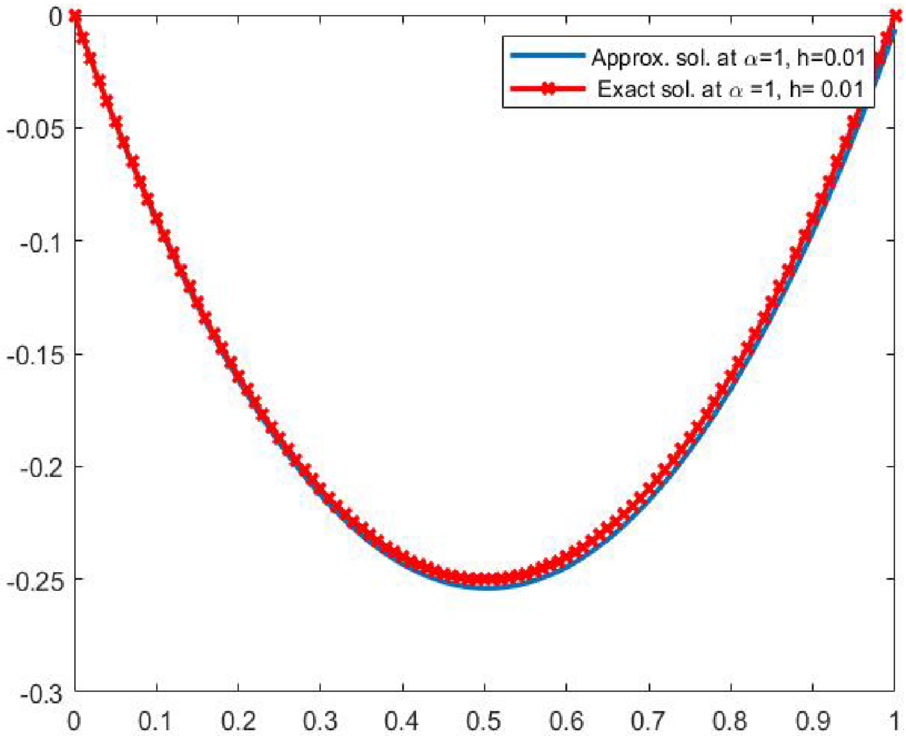

Figure 3.11: Forward Euler method to FIVP (3.21)

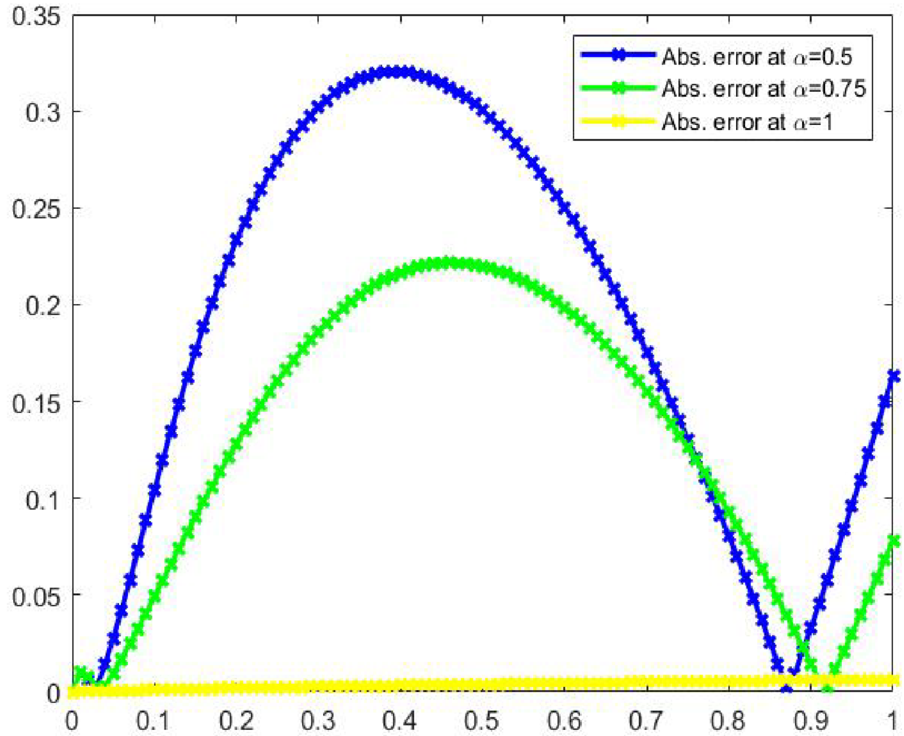

Figure 3.12: Absolute error of forward Euler method to FIVP (3.21) at $h = 0.01$

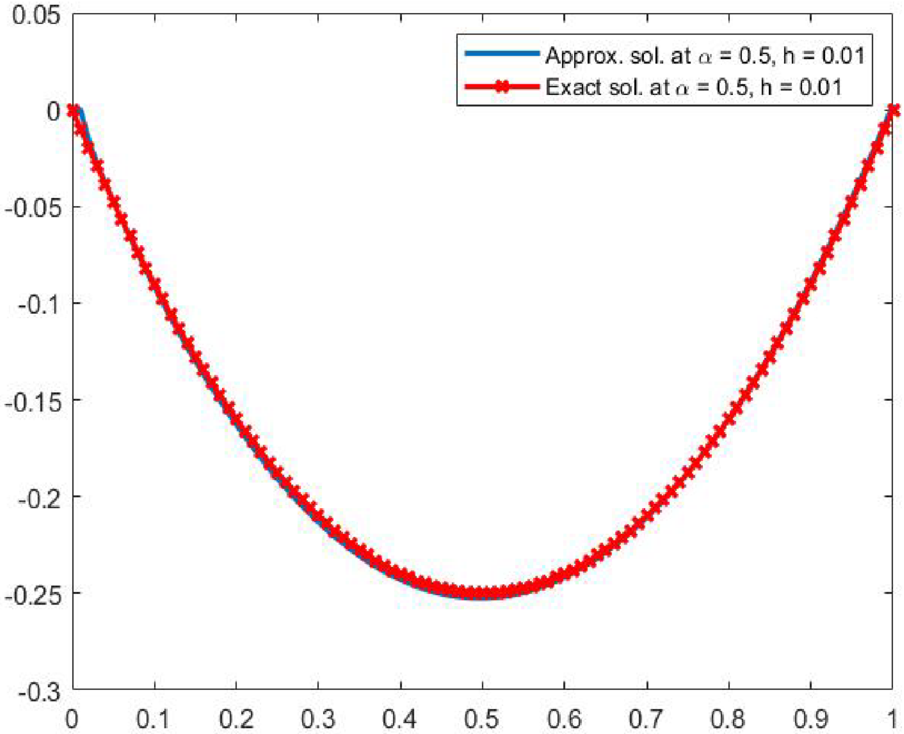

Figure 3.13: Improved forward Euler method to FIVP (3.21)

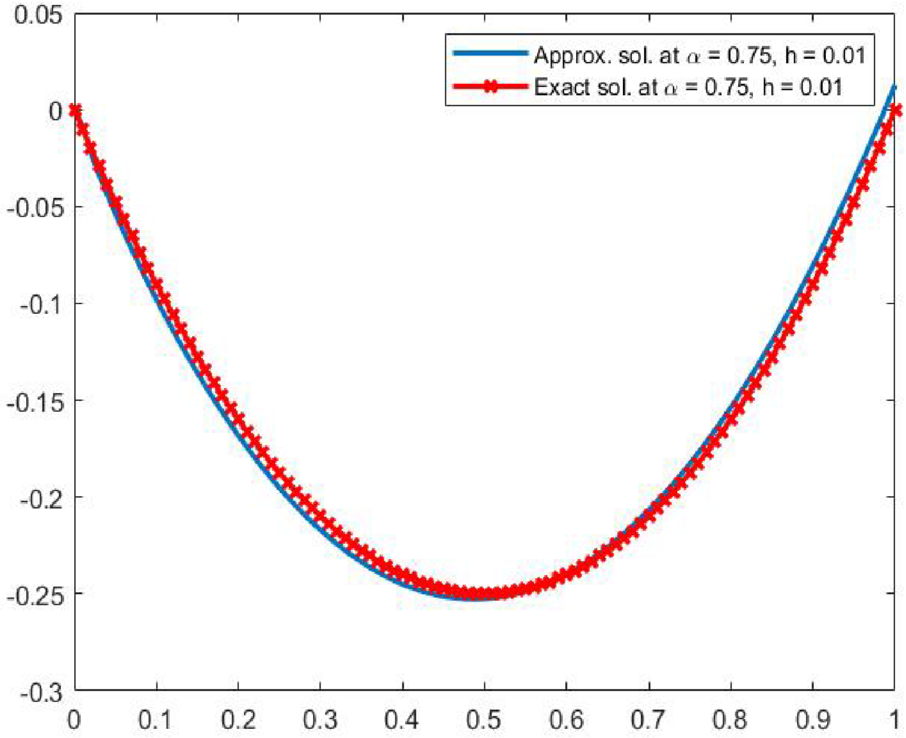

Figure 3.14: Improved forward Euler method to FIVP (3.21)

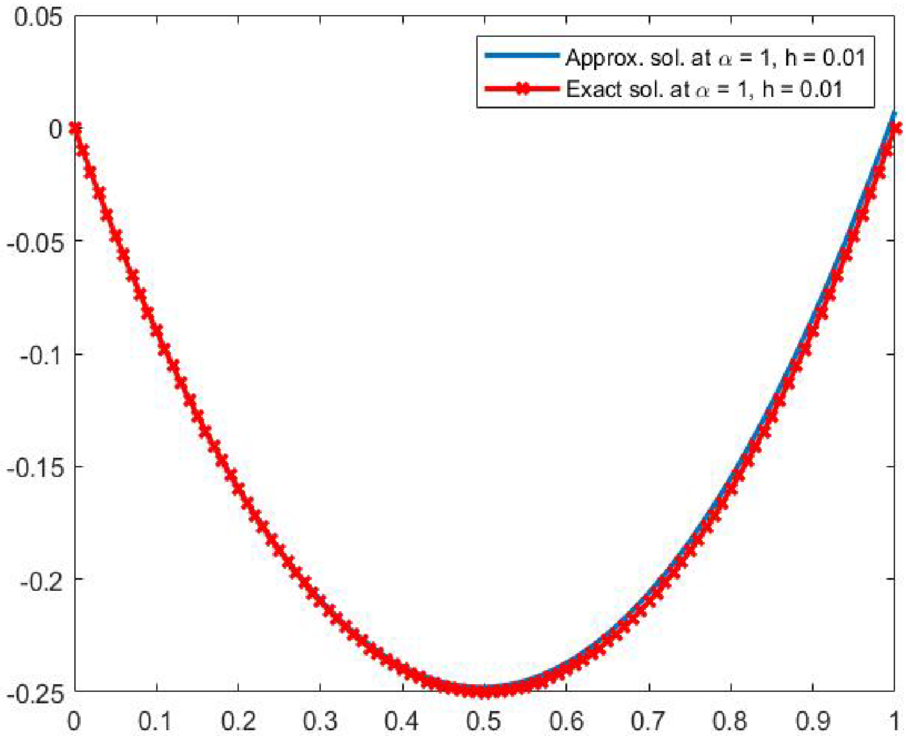

Figure 3.15: Improved forward Euler method to FIVP (3.21)

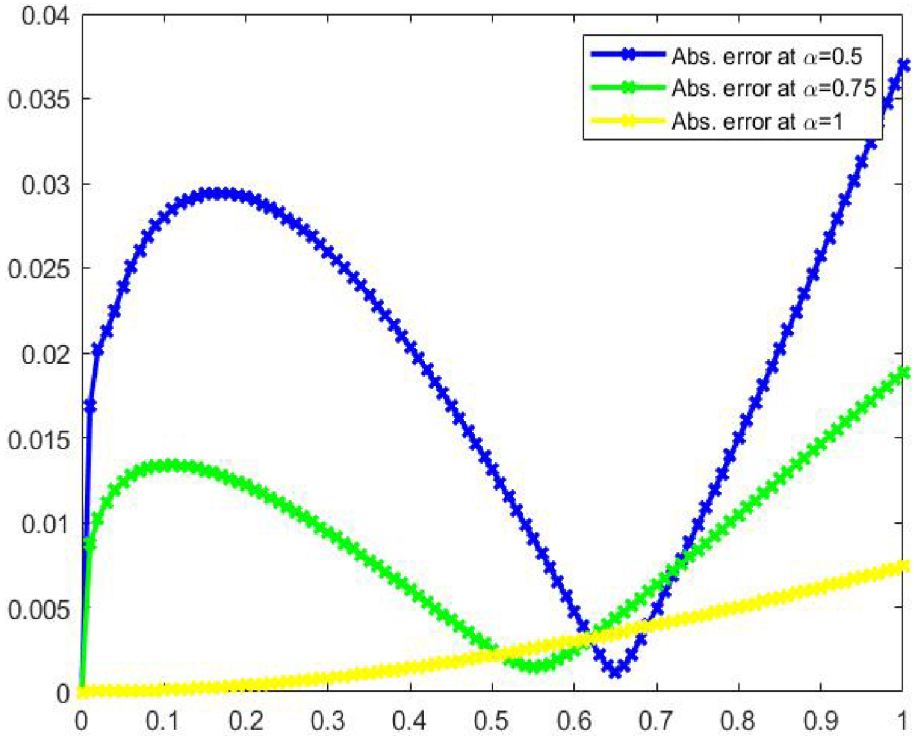

Figure 3.16: Absolute error of improved forward Euler method to FIVP (3.21) at $h = 0.01$

The linear equation (3.21) is solved by Diethelm et al. [44] using the fractional Adams-Bashforth-Moulton method. By the forward Euler method a numerical solution is obtained with $n = 100$ number of division. We observe that the numerical solution approaches more closely to the exact solution as the number of divisions increases. In real sense, the Euler methods is more stable and accurate. Also, an investigation is made by allowing the value of α to vary in the interval $(0, 1]$ with fixed step size $h = 0.01$ and it is clear that the approximate solutions are in high agreement with the exact solutions and the solution continuously depends on the time-fractional derivative. The results show that the accuracy can be improved using smaller values of h.

Example 3.3.3. *Consider the equation*

$$^{C}D^{\alpha}_{t_0,t}y(t) = \lambda y(t), \quad \lambda \in \mathbb{C} \quad \textit{with the initial condition} \quad y(0) = 1, \quad t > 0, \qquad (3.22)$$

where $0 < \alpha < 2$.

The exact solution of equation (3.22) in case of $\lambda \in \mathbb{R}$ is given by

$$y(t) = E_{\alpha,1}(\lambda t^{\alpha}).$$

We consider λ = -1, α = 0.5, 0.75 and α = 1.0 with a fixed stepsize $h = 0.01$. *Table* 9 gives the numerical values of example 3.3.3. for the forward Euler method and *Table* 10 for the improved forward Euler method.

Table 8

	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1.0$	
t	Exact	Approx	Exact	Approx	Exact	Approx
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	0.7236	0.3020	0.8283	0.7046	0.9048	0.9044
0.2	0.6438	0.0912	0.7326	0.4965	0.8187	0.8179
0.3	0.5920	0.0275	0.6603	0.3498	0.7408	0.7397
0.4	0.5536	0.0083	0.6021	0.2465	0.6703	0.6690
0.5	0.5232	0.0025	0.5536	0.1737	0.6065	0.6050
0.6	0.4980	0.0008	0.5123	0.1224	0.5488	0.5472
0.7	0.4767	0.0002	0.4766	0.0862	0.4966	0.4948
0.8	0.4582	0.0001	0.4453	0.0607	0.4493	0.4475
0.9	0.4420	0.0000	0.4177	0.0428	0.4066	0.4047
1.0	0.4276	0.0000	0.3931	0.0302	0.3679	0.3660

Table 9

	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1.0$	
t	Exact	Approx	Exact	Approx	Exact	Approx
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	0.7236	0.7212	0.8283	0.8187	0.9048	0.9044
0.2	0.6438	0.6407	0.7326	0.7231	0.8187	0.8179
0.3	0.5920	0.5891	0.6603	0.6518	0.7408	0.7397
0.4	0.5536	0.5510	0.6021	0.5945	0.6703	0.6690
0.5	0.5232	0.5208	0.5536	0.5469	0.6065	0.6050
0.6	0.4980	0.4959	0.5123	0.5063	0.5488	0.5472
0.7	0.4767	0.4748	0.4766	0.4713	0.4966	0.4948
0.8	0.4582	0.4565	0.4453	0.4407	0.4493	0.4475
0.9	0.4420	0.4405	0.4177	0.4136	0.4066	0.4047
1.0	0.4276	0.4262	0.3931	0.3895	0.3679	0.3660

Table 10 and *Table* 11 give the absolute errors for forward Euler method and improved forward Euler method of example 3.3.3 at $\alpha = 0.5, 0.75$ and 1.0 with stepsizes $h = 0.01$ and $h = 0.001$ respectively.

Table 10

	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1.0$	
t	h=0.01	h=0.001	h=0.01	h=0.001	h=0.01	h=0.001
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	0.4216	0.6972	0.1237	0.2869	0.0005	0.0000
0.2	0.5526	0.6431	0.2361	0.4396	0.0008	0.0001
0.3	0.5645	0.5920	0.3105	0.5017	0.0011	0.0001
0.4	0.5453	0.5536	0.3557	0.5163	0.0013	0.0001
0.5	0.5206	0.5232	0.3799	0.5071	0.0015	0.0002
0.6	0.4973	0.4980	0.3899	0.4871	0.0017	0.0002
0.7	0.4765	0.4767	0.3903	0.4629	0.0017	0.0002
0.8	0.4582	0.4582	0.3845	0.4379	0.0018	0.0002
0.9	0.4420	0.4420	0.3749	0.4137	0.0018	0.0002
1.0	0.4276	0.4276	0.3630	0.3909	0.0018	0.0002

Table 11

	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1.0$	
t	h=0.01	h=0.001	h=0.01	h=0.001	h=0.01	h=0.001
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	0.0589	0.0183	0.0191	0.0037	0.0005	0.0000
0.2	0.0451	0.0142	0.0156	0.0029	0.0008	0.0001
0.3	0.0372	0.0117	0.0130	0.0024	0.0011	0.0001
0.4	0.0319	0.0101	0.0109	0.0019	0.0013	0.0001
0.5	0.0280	0.0089	0.0091	0.0016	0.0015	0.0002
0.6	0.0250	0.0079	0.0078	0.0013	0.0017	0.0002
0.7	0.0225	0.0071	0.0066	0.0011	0.0017	0.0002
0.8	0.0205	0.0065	0.0056	0.0009	0.0018	0.0002
0.9	0.0188	0.0060	0.0048	0.0008	0.0018	0.0002
1.0	0.0174	0.0055	0.0042	0.0006	0.0018	0.0002

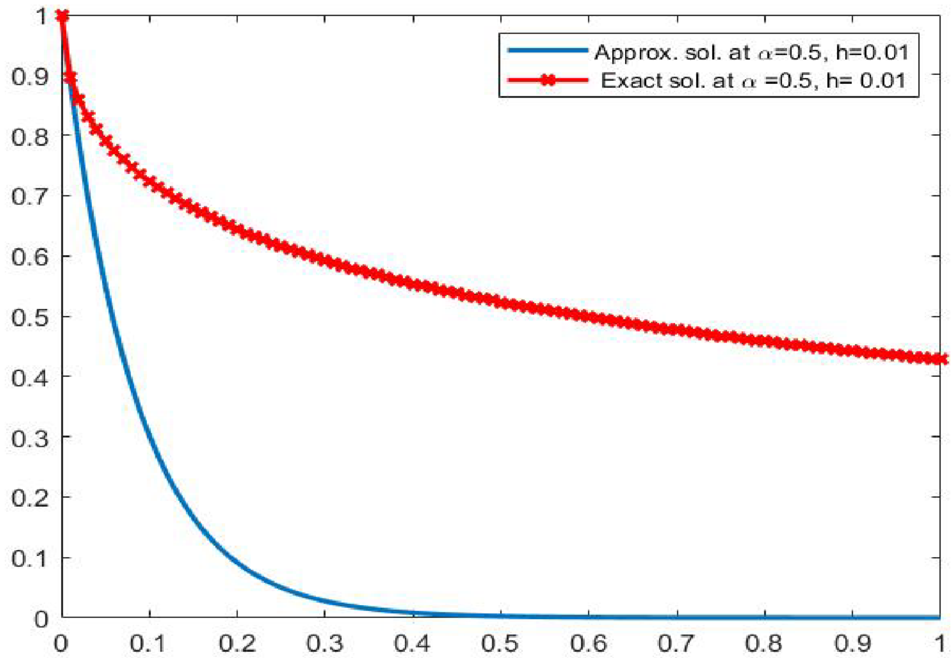

Figure 3.17: Forward Euler method to FIVP (3.22)

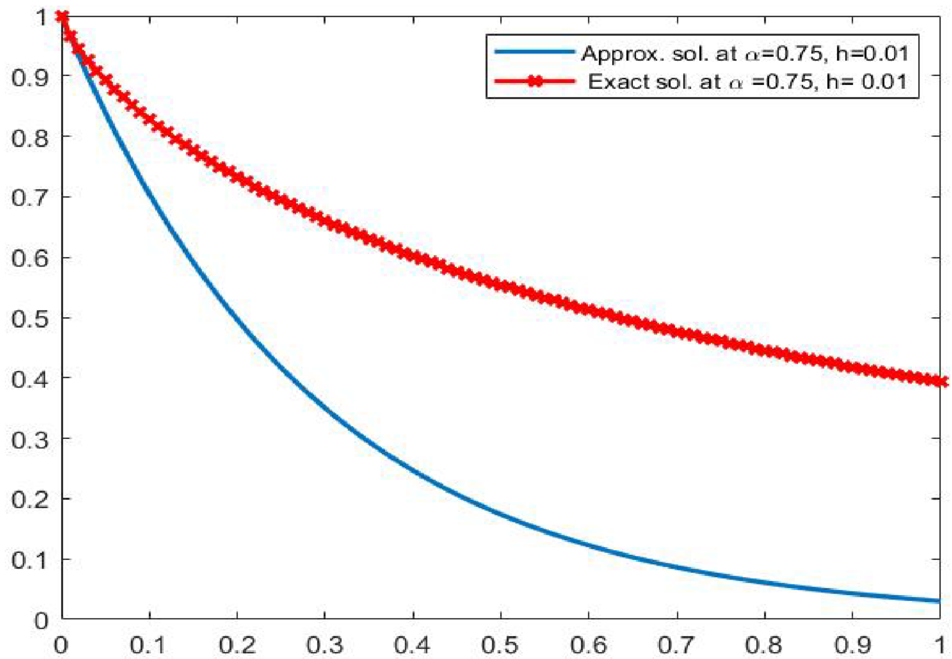

Figure 3.18: Forward Euler method to FIVP (3.22)

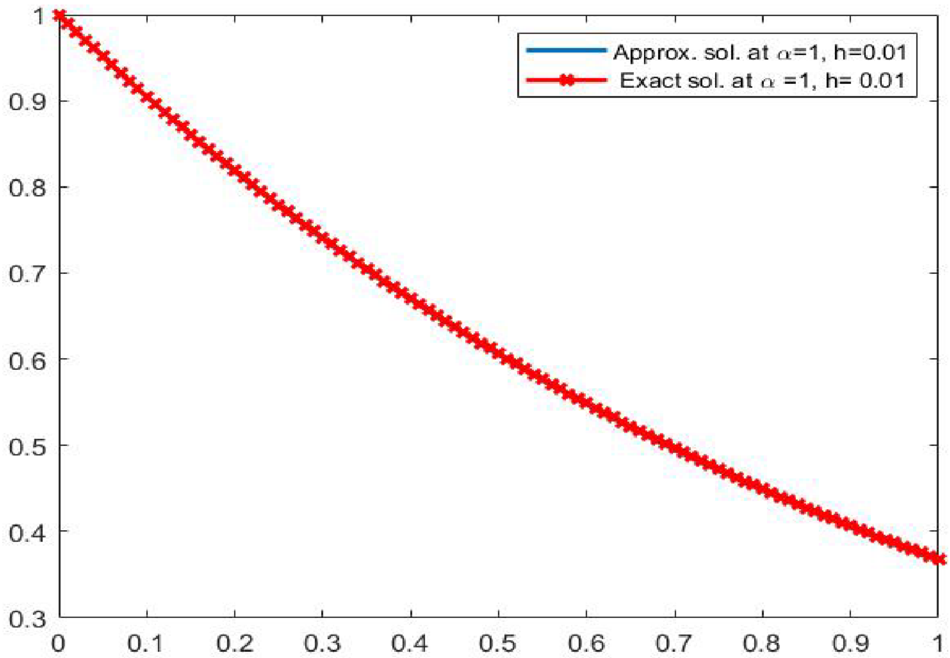

Figure 3.19: Forward Euler method to FIVP (3.22)

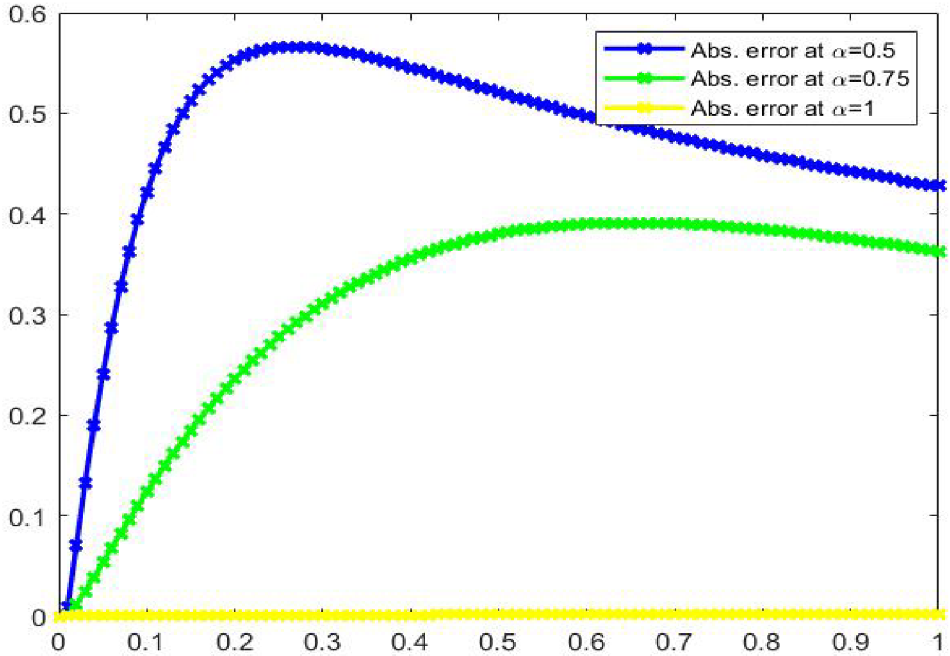

Figure 3.20: Absolute error to forward Euler method to FIVP (3.22) at $h = 0.01$

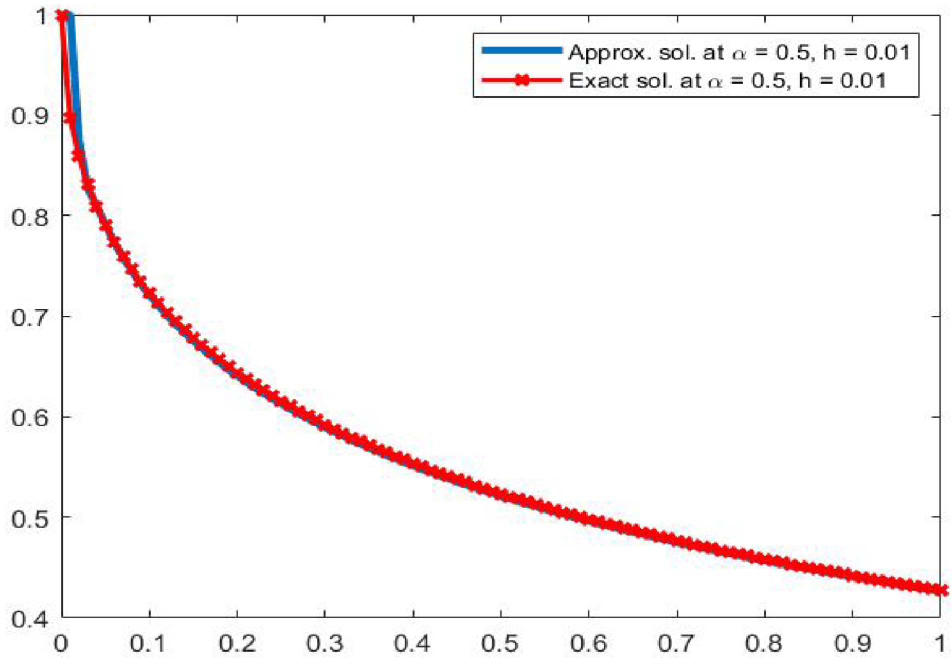

Figure 3.21: Improved forward Euler method to FIVP (3.22) at $h = 0.01$

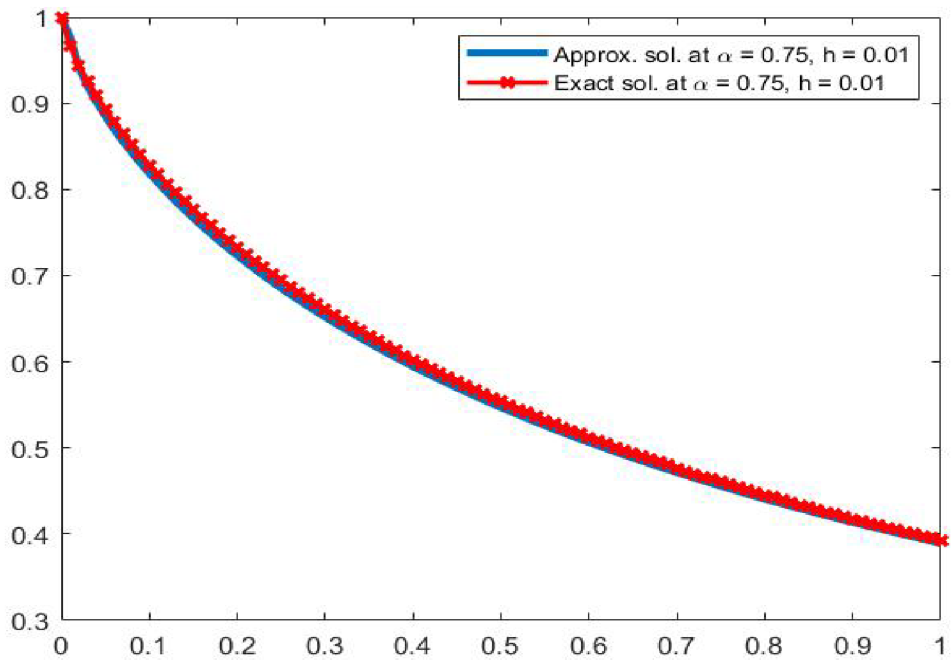

Figure 3.22: Improved forward Euler method to FIVP (3.22) at $h = 0.01$

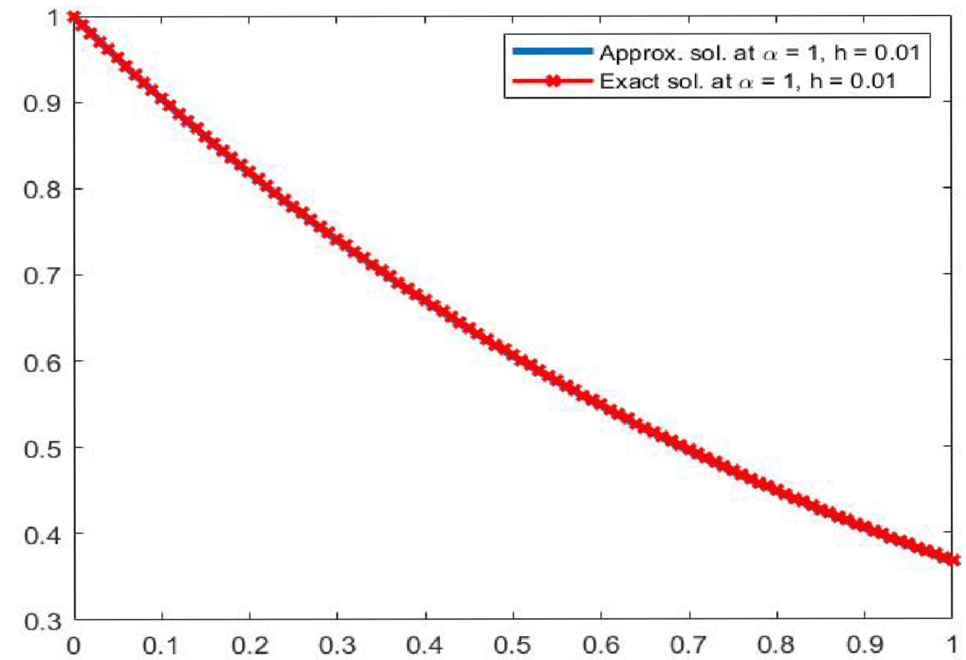

Figure 3.23: Improved forward Euler method to FIVP (3.22) at $h = 0.01$

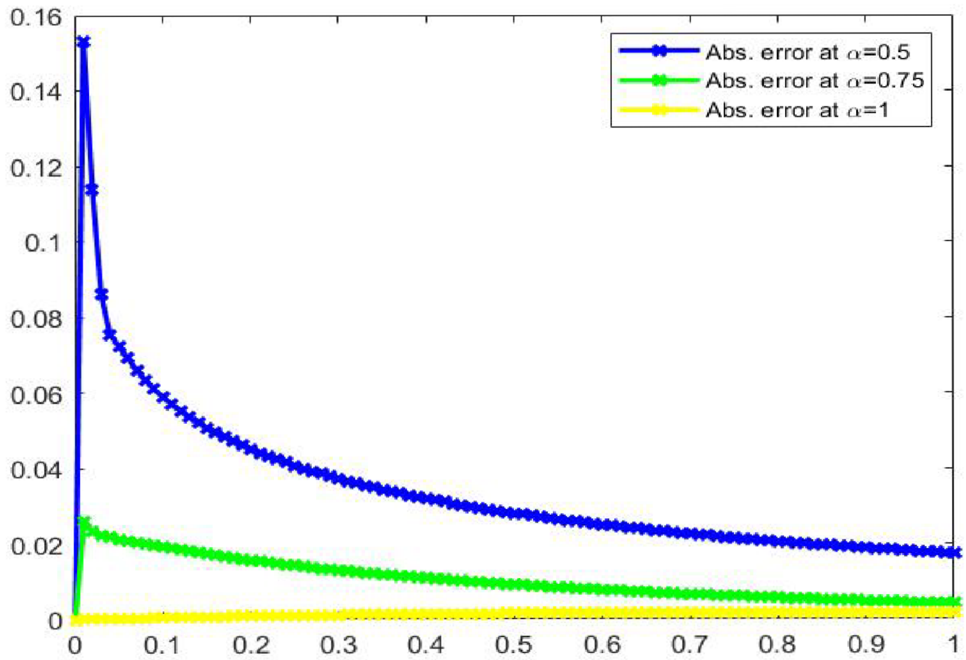

Figure 3.24: Absolute error of improved forward Euler method to FIVP (3.22) at $h = 0.01$.

4. Conclusion

It is not possible to solve most of the fractional differential equations in analytical way. The numerical analysis gives us a possibility to obtain at least numerical approximation of the solution (under certain conditions). The thesis focuses on Euler method for solving Caputo fractional differential equations. The method has intuitive geometric meaning. The convergence of the method is illustrated: the higher number of equal steps within fixed interval is the closer the numerical approximation is to the exact solution. The more effective improved Euler method was introduced too.
The stability of the forward Euler method and the improved forward Euler method are studied. As there are perturbations in the initial conditions of the experimental examples considered, the small changes did not cause large errors in the numerical solutions. These were compared by several numerical experiments. Another numerical methods are used such as homotopy analysis method [45],[46],[47], Chebyshev spectral method [48], [49], homotopy perturbation method[50],[51], variational iteration method[52],[53],[54],[55]. Finally, due to the various applications of fractional differential equations in mathematics and the real world. It becomes necessary to investigate the methods of solution i.e exact and numerical for such equations, this study serves as a manual for researches.

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Appendix

In the section that follows are the MATLAB code for the forward Euler method and the improved forward Euler method. Appendix A gives the algorithm for the forward Euler method and Appendix B gives the algorithm for the improved forward Euler method. These algorithms were used to evaluate three experimental examples. The value of α was varied with a fixed stepsize. In each case the absolute error between the exact and the numerical solutions were calculated. Graphical representations were also given.

Appendix A

```
% FORWARD EULER  METHOD
% alpha = 0.5, 0.75, 1, t0=0, tf=1, NN=100 were used in the thesis.
function [t,y] = FoFrEuler(alpha,f,t0,tf,NN,y0)
h=(tf-t0)/NN; % stepsize
t=[t0:h:tf];

% Initial conditions
y0=[0]; % Example 1
% y0=[0]; Example 2
% y0=[1]; Example 3
y=y0;
uj=[y0];

% This method was evaluated with the functions below
% Example 1 - Example 3.3.1 in thesis
f=@(t,y) 2./gamma(3-alpha)*power(t,2-alpha)- y + power(t,2);

% Example 2 - Example 3.3.2 in thesis
f=@(t,y) 2/gamma(3-alpha).*power(t,2-alpha)-1/gamma(2-alpha).*power(t,1-alpha)-y+power(t,2)-t;

% Example 3 - Example 3.3.3 in thesis
lambda=-1;
f=@(t,y) lambda*y;

for n=1:NN
    y(n+1)=y(n)+power(h,alpha)*f(t(n),y(n))/gamma(alpha+1);
    uj=[uj;y(n+1)];
end

 % Plot of approximated solutions
plot(t,uj,'LineWidth',2);
hold on;

% Example 1 exact solution
g=@(t)  power(t,2);

% Example 2 exact solution
g=@(t)  power(t,2)-t;

% Example 3 exact solution
g=@(t) mlf(alpha,1,lambda.*power(t,alpha),7);

% Plot of exact solutions
plot(t,g(t),'r-x','LineWidth',2)
axis tight;
legend('Approx. sol',' Exact sol')

% Plot of abolute errors
plot(t,abs(uj-g(t)),'LineWidth',2)
legend(" Abs. error")

uj=uj';
ER=[t(1:10:end);abs(uj(1:10:end)-g(t(1:10:end)))];
ER=ER';    %Absolute errors
FR=uj(1:10:end);
FR=FR';   %Approximated values
EV=g(t(1:10:end));
EV=EV';   %Exact solution

end
```

Appendix B

```
% IMPROVED FORWARD EULER METHOD
% alpha = 0.5, 0.75, 1, t0=0, tf=1, NN=100 were used in the thesis.
function [t,y] = FoFrEuler(alpha,f,t0,tf,NN,y0)
h=(tf-t0)/NN; % stepsize
t=[t0:h:tf];
% Initial conditions
y0=[0]; % Example 1
% y0=[0]; Example 2
% y0=[1]; Example 3
y=y0;
tj=[];
uj=[y0];

% This method was evaluated with the functions below
% Example 1 - Example 3.3.1 in thesis
f=@(t,y) 2./gamma(3-alpha)*power(t,2-alpha)- y + power(t,2);

% Example 2 - Example 3.3.2 in thesis
f=@(t,y) 2/gamma(3-alpha).*power(t,2-alpha)-1/gamma(2-alpha).*power(t,1-alpha)-y+power(t,2)-t;

% Example 3 - Example 3.3.3 in thesis
lambda=-1;
f=@(t,y) lambda*y;

for n=1:NN
    for j=1:n+1
    b(j,n+1)=1/gamma(alpha+1)*(power(n-j,alpha)-power(n-j-1,alpha));
    end
    tj=[tj;t(n+1)];
    y(n+1)=(power(t(n+1),0))*y0+power(h,alpha).*(b(1:n,n+1))'*f(tj,uj);
    uj=[uj;y(n+1)];
end
% Plot of approximated solutions
plot(t,uj,'LineWidth',2);
hold on;

 % Example 1 exact solution
g=@(t) power(t,2);
% Example 2 exact solution
g=@(t)  power(t,2)-t;
%  Example 3 exact solution
g=@(t) mlf(alpha,1,lambda.*power(t,alpha),7);

% Plot of exact solutions
plot(t,g(t),'r-x','LineWidth',2)
axis tight;
legend('Approx. sol',' Exact sol')

% Plot of absolute errors
plot(t,abs(uj-g(t)),'LineWidth',2)
legend(" Abs. error")

uj=uj';
ER=[t(1:10:end);abs(uj(1:10:end)-g(t(1:10:end)))];
ER=ER';            % Absolute errors
FR=uj(1:10:end);   % Approximated values
FR=FR';
EV=g(t(1:10:end));
EV=EV';            % Exact solution

end
```

List of Abbreviations and Symbols

SYMBOL	MEANING
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{Z}$	set of integers
$\mathbb{N}$	set of natural numbers
$\mathcal{L}$	Laplace operator
Γ	Gamma operator
${}^{RL}D^{\alpha}_{t_0,t}$	Riemann-Liouville operator
${}^{C}D^{\alpha}_{t_0,t}$	Caputo operator
IVP	Initial Value Problem
FIVP	Fractional initial value problem
ODE	Ordinary differential equation
FDE	Fractional differential equation
$\lceil \alpha \rceil$	Ceiling function of a real number α
h	Function spaces
Δ	Equidistant mesh
T	Final time
Approx. sol	Approximated solution
Abs. error	Absolute error