# Temporal Attribute Implications

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**Abstract** — We deal with dependencies in object-attribute data which is recorded at separate points in time. The data is formalized by finitely many tables encoding the relationship between the objects and the attributes and each table can be seen as a single formal context observed at a separate point in time. Given such data, we are interested in concise ways of characterizing all if-then dependencies between the attributes that hold in the data and are preserved in all time points. In order to formalize the dependencies, we introduce if-then formulas called temporal attribute implications which can be seen as particular formulas of linear temporal logic.

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# Preface

We introduce a semantic entailment of temporal attribute implications, show its fixed-point characterization, investigate closure properties of model classes, present an axiomatization and prove its completeness, and investigate alternative axiomatizations and normalized proofs. We investigate decidability and complexity issues of the logic and prove that the entailment problem is NP-hard and belongs to EXPSPACE. We show that by restricting to predictive formulas, the entailment problem is decidable in pseudo-linear time. We introduce non-redundant bases of dependencies from data as non-redundant sets entailing exactly all the dependencies that hold in the data. In addition, we investigate minimality of bases as a stronger form of non-redundancy. For given data, we present a description of minimal bases using the notion of pseudo-intents generalized in the temporal setting. We further investigate properties of minimal sets of formulas and present sufficient and necessary conditions for their characterization. In addition to the characterization of minimality, we present an algorithm that can be used to minimize any finite set of temporal attribute implications. Particular parts of this document were published in the following articles:

[43] Jan Triska and Vilem Vychodil. “Logic of temporal attribute implications”. In: *Annals of Mathematics and Artificial Intelligence* 79.4 (Apr. 2017), pp. 307–335.

[44] Jan Triska and Vilem Vychodil. “Minimal bases of temporal attribute implications”. In: *Annals of Mathematics and Artificial Intelligence* 83.1 (May 2018), pp. 73–97.

[45] Jan Triska and Vilem Vychodil. “On minimal sets of temporal attribute implications”. submitted. 2018.

[46] Jan Triska and Vilem Vychodil. “Towards Armstrong-Style Inference System for Attribute Implications with Temporal Semantics”. In: *Modeling Decisions for Artificial Intelligence*. Ed. by Vicenç Torra, Yasuo Narukawa, and Yasunori Endo. Vol. 8825. LNCS. Springer International Publishing, 2014, pp. 84–95.

# 1 Introduction

Formulas describing if-then dependencies between attributes play fundamental role in reasoning about attributes in many disciplines including database systems [12, 34], formal concept analysis [23, 26], data mining [1, 51], logic programming [31, 40], and their applications. In these disciplines, the rules often appear under different names (e.g., attribute implications, functional dependencies, or simply "rules") with semantics defined in various structures (e.g., transactional data, Boolean matrices, or $n$-ary relations) but as it has been shown in [17], the rules may be seen as propositional formulas with the semantic entailment defined as in the propositional logic, possibly extended by additional measures of interestingness. The rules are popular because of their easy readability for non-expert users. In addition, the entailment problem related to a large family of the rules, including attribute implications used in formal concept analysis and functional dependencies used in database systems, is decidable in linear time [4] which also contributes to their popularity.

In this document, we introduce if-then formulas that express presence of attributes relatively in time and the formulas are evaluated in data where the presence or absence of attributes changes in time. In our approach, we adopt the notion of a discrete time, i.e., the data are observed at distinct points in time. Informally, the formulas can be seen as rules expressing dependencies between attributes (or features) in the following sense:

IF (a feature $y_1$ is present in time point $t_1$
and $\cdots$ and
a feature $y_m$ is present in time point $t_m$),

THEN (a feature $z_1$ is present in time point $s_1$
and $\cdots$ and
a feature $z_n$ is present in time point $s_n$).

As a formula, such dependency can be written as

$$\left(y_1^{t_1} \mathbin{\&} \cdots \mathbin{\&} y_m^{t_m}\right) \Rightarrow \left(z_1^{s_1} \mathbin{\&} \cdots \mathbin{\&} z_n^{s_n}\right), \tag{1.1}$$

where $y_i^{t_i}$ denotes an attribute/feature $y_i$ present in time point $t_i$ and analogously for $z_j^{s_j}$. As usual, $\&$ and $\Rightarrow$ used in (1.1) denote the usual logical connectives of conjunction and implication (logical conditional), see [36]. In the document, we exploit the fact that & is a logical connective that is interpreted by an idempotent, commutative, and associative truth function, allowing us to rewrite (1.1) in a set-theoretic notation as follows:

$$\{y_1^{t_1}, \ldots, y_m^{t_m}\} \Rightarrow \{z_1^{s_1}, \ldots, z_n^{s_n}\}. \tag{1.2}$$

In general, rules like (1.2) can be viewed as locally valid, that is, valid exactly in time points $t_1, \ldots, t_m$ and $s_1, \ldots, s_n$ that appear in the formula. This notion of validity is in a sense trivial because reasoning with such formulas is easily reducible to reasoning with classic attribute implications. In contrast, (1.2) can be viewed as globally valid, i.e., valid for time points $t_1 + k, \ldots, t_m + k$ and $s_1 + k, \ldots, s_n + k$ for an arbitrary $k$. In our approach, we consider the global validity since we want to capture dependencies that are preserved over all time points and thus endure in time.

We study the formulas from the point of view of temporal reasoning in formal concept analysis [23]. The classic (dyadic) formal concept analysis (FCA) is a method of analysis of object-attribute data formalized by binary incidence relations between a set of objects and a set of attributes. One of the typical outputs of FCA, given an input incidence data, is a set of if-then dependencies which entails exactly all if-then dependencies that hold in the data. Among the best known methods of determining such interesting sets of if-then rules is the method of Guigues and Duquenne based on computing pseudo-intents from data, see [21, 26]. In many situations, the object-attribute incidence data changes over time and one may be interested in if-then rules which are universaly valid in all time points. For instance, we may observe a mechanism which behaves as a transition system which makes transitions from a state to another one in discrete steps. Supposing that we do not know the internals of the system and we can only observe a set of Boolean attributes which are or are not satisfied at a given moment. This gives us a set of attributes (of a single object—the system) which changes in time, i.e., a series of object-attribute incidence data changing in time. Then, rules like (1.2) may be used to describe the behavior of the system during transitions in terms of the dependencies between the Boolean attributes. In this situation, an analog of the Guigues-Duquenne bases would be the most helpful because it would allow us to derive a set of if-then rules describing the system based on its observation during the transitions. The classic notions related to Guigues-Duquenne bases are closely related to the notion of entailment of if-then rules. Therefore,

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| 1 | | | × | |
| 2 | | × | | |
| 3 | × | × | | × |
| 4 | × | | | |

| $x$ | $y$ |
|---|---|
| 1 | $c$ |
| 2 | $b$ |
| 3 | $a$ |
| 3 | $b$ |
| 3 | $d$ |
| 4 | $a$ |

```
has(1,c).
has(2,b).
has(3,a).
has(3,b).
has(3,d).
has(4,a).
```

Figure 1: Example of object-attribute incidence data represented as a formal context $\langle X, Y, I\rangle$ with $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d\}$ depicted as a table (left), relation on relation scheme $\{x, y\}$ (middle), and PROLOG-style program consisting of facts (right).

before we show that a reasonable counterpart to the Guigues-Duquenne bases in the temporal setting indeed exists, we make a thorough investigation on the entailment.

The input data we consider consists of a finite set $X$ of objects, a finite set $Y$ of attributes (features), and a binary incidence relation $I \subseteq X \times Y$ with $\langle x, y\rangle \in I$ interpreted as "object $x$ has attribute $y$". In this setting, $I$ can be seen as a record of object-attribute data observed in a single time point and the triplet $\langle X, Y, I\rangle$ is called a (dyadic) formal context [23] in FCA. Let us note that despite the fact the document is written primarily from the FCA perspective, such data are the subject of study of many computer science disciplines. For instance, from the point of view of relational databases [34], $\langle X, Y, I\rangle$ can be understood as a (finite) relation on relation scheme with two attributes—an attribute whose domain is $X$ and an attribute whose domain is $Y$. From the point of view of logic programming [31], $\langle X, Y, I\rangle$ can be seen as a definite program consisting of facts, see Fig. 1 for illustration. In addition, the object-attribute incidence data is considered as the basic form of input data in most data mining disciplines, most notably the association rule mining [1, 51].

As we have outlined, we assume that $\langle X, Y, I\rangle$ can change in time. To be more specific, we assume that $X$ and $Y$ are fixed, i.e., the sets of observed objects and attributes do not change in time, and $I$ is subject to change. Therefore, instead of single $I$, we consider a sequence

$$I_l, I_{l+1}, \ldots, I_{r-1}, I_r \tag{1.3}$$

of incidence relations where $l, r$ are integers ($l \leq r$) denoting separate time points and $I_i \subseteq X \times Y$ for each $i = l, \ldots, r$. Alternatively, the input data can be understood as a (finite) database relation on relation scheme consisting of three attributes: objects (with domain $X$), attributes (with domain $Y$), and time (with domain $\mathbb{Z}$). Also, it can be seen as representing finitely many facts of the form `has(time_point,x,y)` as in Fig. 1.

The dependencies we identify in the data generalize the classic if-then dependencies called attribute implications. Recall that by an attribute implication [21, 23, 26] we mean a propositional formula of the form

$$(y_1 \mathbin{\&} \cdots \mathbin{\&} y_m) \Rightarrow (z_1 \mathbin{\&} \cdots \mathbin{\&} z_n), \tag{1.4}$$

where $y_i$, $z_j$ are propositional atoms. Both the semantic entailment (i.e., entailment defined in terms of validity in models) and syntactic entailment (i.e., entailment based on provability) of attribute implications are, in fact, notions inherited from the propositional logic [36]. Interestingly, attribute implications are closely related to functional dependencies. Although the interpretation of the formulas as attribute implications and functional dependencies are different, it follows from [15, 17, 41] that both the interpretations yield the same notion of the semantic entailment. As a consequence, common axiomatizations are used to characterize the semantic entailment which are typically based on the Armstrong inference rules [3].

If all $y_i$ and $z_j$, which appear in (1.4), denote attributes from $Y$, then (1.4) can be interpreted in $\langle X, Y, I\rangle$ based on its validity for each object $x \in X$. In more detail, each $x \in X$ induces a truth evaluation $e_x$ of propositional atoms such that $e_x(y) = 1$ (logical true) iff $\langle x, y\rangle \in I$ and $e_x(y) = 0$ (logical false) otherwise. Then, we may say that (1.4) holds in $I$ whenever (1.4) is true under $e_x$ for all $x \in X$ in the sense of propositional logic. According to our interpretation of $I$, the fact that (1.4) holds in $I$ means that each object $x \in X$ satisfies the following property: If the object has all attributes $y_1, \ldots, y_m$, then it has all attributes $z_1, \ldots, z_n$, i.e., the presence of all $z_1, \ldots, z_n$ in the data is implied by the presence of all $y_1, \ldots, y_m$. For instance, if we consider $I$ from Fig. 1 (left), then, e.g., $(a \mathbin{\&} b) \Rightarrow d$ and $(b \mathbin{\&} c) \Rightarrow d$ hold in $I$. On the other hand, neither of $a \Rightarrow d$, $b \Rightarrow d$, or $c \Rightarrow d$ holds in $I$.

The dependencies used in the document may be seen as extensions of formulas like (1.4) incorporating explicit time annotations for each $y_i$, $z_j$. That is, instead of (1.4), we consider formulas of the form (1.2) where $i_1, j_1, \ldots$ are integers interpreted as relative shifts in time, allowing us to express dependencies such as "if $y$ is currently present and $z$ was present yesterday, then $\chi$ will be present tomorrow" by formulas like $(y^0 \,\&\, z^{-1}) \Rightarrow \chi^1$ provided that the considered "unit of time" is a day. We call formulas of the form (1.2) *temporal attribute implications.*

We provide answers to several questions which emerge with temporal attribute implications. We define the notion of semantic entailment of the formulas, investigate closure structures of models of theories consisting of such formulas, and show that the problem of checking whether a formula is semantically entailed by a set of formulas can be reduced to checking its validity in a single model. We prove that the semantic entailment has a complete axiomatization. That is, we show a notion of provability of temporal attribute implications and show that it coincides with the semantic entailment. We discuss several possible axiomatizations, including ones that can be used to consider proofs in particular normal forms. Based on our insight into the properties of the semantic entailment and provability, we derive results on decidability and complexity of the entailment problem. Fourth, we include notes on the relationship of the formulas to formulas appearing in modal logics [8] and triadic formal concept analysis [29].

After the investigation of the properties of the semantic entailment, we focus on description of all temporal attribute implications that hold in given data. In particular, we seek sets of formulas entailing exactly all formulas that hold in given data. We call such sets *complete* (in given data). As in the classic setting, it is desirable to describe complete sets of formulas which are small. In the document, we introduce two notions which may be seen as two basic properties of "small sets of formulas": *non-redundancy* and *minimality.* Complete sets in data which are non-redundant (or minimal) are called non-redundant (or minimal) *bases* (of the data) and their structure and properties are investigated in the document. Unlike the classic case, where the time annotations are not present and minimal bases of finite incidence data are finite, minimal bases in the temporal setting are infinite in general. This is one of the aspects that makes the presented theory substantially different compared to the classic one [21, 23, 26].

Despite the fact that the bases of finite data are infinite, our observations show that each such base can be split into two parts based on the maximal difference of time points in the input data (so-called time range):

($i$) An interesting *finite part* which can be enumerated in finitely many steps. This part consists of formulas where the maximal difference of time points

$i = 0$

| | $a$ | $b$ | $c$ |
|---|---|---|---|
| $p$ | × | × | |
| $q$ | | | |
| $r$ | | | × |

$i = 1$

| | $a$ | $b$ | $c$ |
|---|---|---|---|
| $p$ | × | × | |
| $q$ | × | | × |
| $r$ | × | | × |

$i = 2$

| | $a$ | $b$ | $c$ |
|---|---|---|---|
| $p$ | | | |
| $q$ | × | × | |
| $r$ | × | | |

Figure 2: Input data depicted as formal contexts considered in separate time points for $i = 0$ (left), $i = 1$ (middle), and $i = 2$ (right).

in their antecedents are within the range of the input data.

(*ii*) An *infinite part* which consists of formulas whose antecedents contain time points which are outside the time range of the input data.

For illustration of the notions, let us consider the input data from Fig. 2. Using the introduced notation, the set of objects is $X = \{p, q, r\}$, the set of attributes is $Y = \{a, b, c\}$, and the tables in Fig. 2 encode the incidence relations $I_i \subseteq X \times Y$ in time points $i = 0, 1, 2$. Therefore, in this case, the time range of the input data is 2 (units) because 2 is the maximal distance of time points of any attributes in the data.

As an example of a particular minimal base of the data in Figure 2, we can consider the following set of formulas. We may split the base into three disjoint subsets. The first (and the most interesting) part of the base consists of formulas

$$b^0 \Rightarrow a^0,$$
$$c^0 \Rightarrow a^1,$$
$$(b^0 \mathbin{\&} a^1) \Rightarrow b^1,$$
$$(c^0 \mathbin{\&} b^1) \Rightarrow a^0,$$
$$(c^0 \mathbin{\&} a^2) \Rightarrow c^1.$$

For all antecedents (and consequents) of the previous formulas, we can consider their time range which is the maximal difference of time points of all used attributes. Clearly, the time ranges are 0 (first formula), 1 (next three formulas), and 2 (the last formula) which are all less than or equal to the time range of the data. For a data analyst, the formulas express that if there is an object which, in a certain time point, satisfies the condition given by the antecedent, then it must satisfy the condition given by the consequent. For instance, $(c^0 \mathbin{\&} b^1) \Rightarrow a^0$ says "if an object has the attribute $c$ in the current time point and it has $b$ in the next time point, then is also has $a$ in the current time point."

The second group consists of formulas with antecedents within the time range of the input data and having arbitrary conjunctions of attributes annotated by time points as their consequents. Namely, the group consists of

$$(b^0 \mathbin{\&} c^0) \Rightarrow \varphi,$$
$$(a^0 \mathbin{\&} a^2) \Rightarrow \varphi,$$
$$(c^0 \mathbin{\&} c^1 \mathbin{\&} b^2) \Rightarrow \varphi,$$

where $\varphi$ is an arbitrary conjunction of attributes annotated by time points. Technically, this already represents an infinite set of formulas but with only finitely many pairwise different antecedents. Intuitively, these formulas express that certain combinations of attributes in time are not possible in the input data. For example, $(b^0 \mathbin{\&} c^0) \Rightarrow b^5$ is a particular instance of the formula listed first. If there were an object $x \in X$ and a time point where $b$ is present and $c$ is present then the object would have $b$ present in five time points in the future which is absurd because the time range of the input data is 2. Therefore, considering the present data, the formula says that "in the input data, $b$ and $c$ are not present in the same time point for any object."

These first two groups of formulas are the most interesting for data analysts. Given input data encoded by tables as in Fig. 2, the document shows how such "interesting" formulas of a particular minimal base can be obtained based on systems of pseudo-intents [26] which we generalize in the temporal setting.

In order to conclude our example, the presented base of the data in Fig. 2 consists of formulas from which we infer all formulas with antecedents outside of the time range of the input data. In this particular case, it is sufficient to consider formulas

$$(a^0 \mathbin{\&} a^{2+n}) \Rightarrow \varphi,$$

where $n$ is any natural number and $\varphi$ is an arbitrary conjunction of attributes annotated by time points. This part of the base is infinite and, moreover, not so interesting for analysts (if one has a data that spans $n$ time units, there is no point in finding dependencies which go beyond $n$ units). It ensures that the base entails all formulas with antecedents outside of the time range of the input data. Such formulas trivially hold in the input data and one can easily verify that a formula is in such a form just by computing the time range of its antecedent.

The last question studied in the document is the problem of characterizing minimal sets of temporal attribute implications. Problems of finding minimal descriptions of various structures belong to classic problems in computer science as well as data analysis. It is well known that some minimality problems are easy (e.g., minimization of finite automata) and some are intractable or even undecidable (e.g., minimality of Turing machines). In data analysis, there is a natural need to find descriptions of dependencies, clusters, or patterns in data that are as small as possible in order to simplify further processing or to enable easier evaluation by human experts. In this document, we show a condition based on checking the presence of two formulas with special properties that is considerably simpler than checking the minimality by definition.

The investigation of minimality of sets of if-then rules started with the seminal paper [33] where the author showed criteria for minimality of non-redundant sets of functional dependencies based on the notion of direct determination. The paper showed that transforming a set of rules into an equivalent and minimal one can be done in polynomial time using the standard tests of entailment [4]. Later, the result has been extended for a family of graded/fuzzy attribute implications in [49]. Since all classic if-then rules, graded/fuzzy attribute implications, and temporal attribute implications can be seen as general rules whose semantics is defined by particular systems of isotone Galois connections, see [50, 48], we investigate a minimality characterization for temporal attribute implications that is analogous to the classic one [33] and the one for the graded attribute implications [49].

# 2 Logic of temporal attribute implications

In this section, we present a formalization of the formulas, their interpretation, and semantic entailment. We provide its complete axiomatization and show normalized proofs. In addition, we show bounds on complexity of the semantic entailment and focus on a subproblem which typically appears in applications. For the subproblem we provide a pseudo-polynomial time [24] decision algorithm.

Let us assume that $Y$ is a non-empty and finite set of symbols called attributes.

Furthermore, we use integers in order to denote time points. We put

$$\mathcal{T}_Y = \left\{y^i \,|\, y \in Y \text{ and } i \in \mathbb{Z}\right\} \tag{2.1}$$

and interpret each $y^i \in \mathcal{T}_Y$ as "attribute $y$ observed in time $i$" (technically, $\mathcal{T}_Y$ can be seen as the Cartesian product $Y \times \mathbb{Z}$). It is easy to see that $\mathcal{T}_Y$ is countable. Furthermore, we introduce the following set:

$$\mathcal{F}_Y = \{M \subseteq \mathcal{T}_Y \,|\, M \text{ is finite}\}, \tag{2.2}$$

and abbreviate the set by $\mathcal{F}$ if $Y$ is clear from the context. Obviously, $\mathcal{F}$ is countable since $\mathcal{T}_Y$ is countable.

Under this notation, we may now formalize rules like (1.2) as follows:

**Definition 2.1.** A *temporal attribute implication over* $Y$ is a formula of the form $A \Rightarrow B$, where $A, B \in \mathcal{F}$.

As we have outlined in the introduction, the purpose of time points encoded by integers which appear in antecedents and consequents of the considered formulas is to express points in time relatively to a current time point. Hence, the intended meaning of (1.2) abbreviated by $A \Rightarrow B$ is the following: "For all time points $t$, if an object has all attributes from $A$ considering $t$ as the current time point, then it must have all attributes from $B$ considering $t$ as the current time point". In what follows, we formalize the interpretation of $A \Rightarrow B$ in this sense.

Since we wish to define formulas being true in all time points (we are interested in formulas preserved over time), we need to shift relative times expressed in antecedents and consequents in formulas with respect to a changing time point. For that purpose, for each $M \subseteq \mathcal{T}_Y$ and $i \in \mathbb{Z}$, we may introduce a subset $M + j$ of $\mathcal{T}_Y$ by

$$M + j = \left\{y^{i+j} \,|\, y^i \in M\right\} \tag{2.3}$$

and call it a *time shift of* $M$ *by* $j$ (shortly, a $j$-shift of $M$).

Temporal attribute implications are formulas, i.e., syntactic notions for which we define their semantics (interpretation) as follows.

**Definition 2.2.** A formula $A \Rightarrow B$ *is true in* $M \subseteq \mathcal{T}_Y$ whenever, for each $i \in \mathbb{Z}$,

$$\text{if } A + i \subseteq M, \text{ then } B + i \subseteq M \tag{2.4}$$

and we denote the fact by $M \models A \Rightarrow B$.

*Remark* 1. The value of $i$ in the definition may be understood as a sliding time point. Moreover, $A+i$ and $B+i$ represent sets of attributes annotated by *absolute time points* considering $i$ as the current time point.

We consider the following notions of a theory and a model:

**Definition 2.3.** Let $\Sigma$ be a set of formulas (called a *theory*). A subset $M \subseteq \mathcal{T}_Y$ is called a *model of* $\Sigma$ if $M \models A \Rightarrow B$ for all $A \Rightarrow B \in \Sigma$. The system of all models of $\Sigma$ is denoted by $\mathrm{Mod}(\Sigma)$, i.e.,

$$\mathrm{Mod}(\Sigma) = \bigl\{M \subseteq \mathcal{T}_Y \,|\, M \models A \Rightarrow B \text{ for all } A \Rightarrow B \in \Sigma\bigr\}. \tag{2.5}$$

In general, $\mathrm{Mod}(\Sigma)$ is infinite and there may be theories that do not have any finite model. For instance, consider a theory containing $\emptyset \Rightarrow \{y^0\}$.

We now turn our attention to the structure of systems of all models of temporal attribute implications. In case of the ordinary attribute implications, it is well known that systems of their models are exactly closure systems in $Y$ [23]. Interestingly, the systems of models in our case are exactly the algebraic closure systems that are closed under time shifts. This additional closure property is introduced by the following definition.

**Definition 2.4.** A system $\mathcal{S} \subseteq 2^{\mathcal{T}_Y}$ of subsets of $\mathcal{T}_Y$ is called *closed under time shifts* whenever $M + i \in \mathcal{S}$ for all $M \in \mathcal{S}$ and $i \in \mathbb{Z}$.

**Theorem 2.5.** *Let $\mathcal{S} \subseteq 2^{\mathcal{T}_Y}$ be an algebraic closure system that is closed under time shifts. Then, there is a theory $\Sigma$ such that $\mathcal{S} = \mathrm{Mod}(\Sigma)$.*

**Theorem 2.6.** *Let $\Sigma$ be a theory. Then, $\mathrm{Mod}(\Sigma)$ is an algebraic closure system closed under time shifts.*

Taking into account Theorem 2.6, for each theory $\Sigma$, we may consider a closure operator induced by $\mathrm{Mod}(\Sigma)$ which maps each $M \subseteq \mathcal{T}_Y$ to the least model of $\Sigma$ containing $M$.

**Definition 2.7.** Let $\Sigma$ be a theory. For each $M \subseteq \mathcal{T}_Y$, we put

$$[M]_\Sigma = \bigcap\{N \in \mathrm{Mod}(\Sigma) \,|\, M \subseteq N\} \tag{2.6}$$

and call $[M]_\Sigma$ the *semantic closure* of $M$ under $\Sigma$.

Using the well-known relationship between closure operators and closure systems [14, 7], $[\cdots]_\Sigma$ defined by (2.6) is indeed a closure operator.

We now define semantic entailment of formulas and explore its properties. The notion is defined the usual way using the notion of a model introduced before.

**Definition 2.8.** Let $\Sigma$ be a theory. Formula $A \Rightarrow B$ *is semantically entailed by* $\Sigma$ if $M \models A \Rightarrow B$ for each $M \in \mathrm{Mod}(\Sigma)$.

Analogously as for the classic attribute implications, the semantic entailment of $A \Rightarrow B$ by a theory $\Sigma$ can be checked using the least model of $\Sigma$ generated by $A$ as it is shown in the following theorem.

**Theorem 2.9.** *For any $\Sigma$ and $A \Rightarrow B$, the following conditions are equivalent:*

*(i)* $\Sigma \models A \Rightarrow B$,

*(ii)* $[A]_\Sigma \models A \Rightarrow B$,

*(iii)* $B \subseteq [A]_\Sigma$.

Next, we present a deduction system for our formulas and a related notion of provability which represents the syntactic entailment of formulas. The provability is based on an extension of the Armstrong axiomatic system [3] which is well known mainly in database systems [34]. The extension we propose accommodates the fact that time points in formulas are relative. The deductive system we use consists of the following deduction rules.

**Definition 2.10.** We introduce the following *deduction rules*:

(Ax) infer $A \cup B \Rightarrow A$,

(Cut) from $A \Rightarrow B$ and $B \cup C \Rightarrow D$ infer $A \cup C \Rightarrow D$,

(Shf) from $A \Rightarrow B$ infer $A + i \Rightarrow B + i$,

where $i \in \mathbb{Z}$ and $A, B, C, D$ are arbitrary finite subsets of $\mathcal{T}_Y$.

**Definition 2.11.** A proof of $A \Rightarrow B$ by $\Sigma$ is a finite sequence $\delta_1, \ldots, \delta_n$ such that $\delta_n$ equals $A \Rightarrow B$ and for each $i = 1, \ldots, n$ we have

*(i)* $\delta_i \in \Sigma$, or

*(ii)* $\delta_i$ is infered by (Ax), (Cut), or (Shf) from formulas $delta_j$ where $j < i$.

We say that $A \Rightarrow B$ *is provable by* $\Sigma$, denoted $\Sigma \vdash A \Rightarrow B$, if there is a proof of $A \Rightarrow B$ by $\Sigma$.

Our inference system is complete in the usual sense:

**Theorem 2.12** (completeness). *$\Sigma \vdash A \Rightarrow B$ iff $\Sigma \models A \Rightarrow B$.*

Now, we show bounds on the computational complexity of deciding whether a temporal attribute implication is provable by a finite set $\Sigma$ of other temporal attribute implications.

We formalize the *decision problem of entailment* as a language of encodings of finitely many formulas, i.e., we put

$$L_{\mathrm{ENT}} = \{\langle \Sigma, A \Rightarrow B \rangle \,|\, \Sigma \text{ is a finite theory and } \Sigma \vdash A \Rightarrow B\}, \tag{2.7}$$

considering a fixed $\mathcal{T}_Y$.

Let us note that in the case of the ordinary attribute implications and functional dependencies, the problem of determining whether a given formula follows by a finite set of formulas is easy and there exist efficient linear time decision algorithms [4]. In contrast, the corresponding decision problem in our setting is hard:

**Theorem 2.13** (lower bound). *$L_{\mathrm{ENT}}$ is NP-hard.*

**Theorem 2.14** (upper bound). *$L_{\mathrm{ENT}}$ belongs to EXPSPACE.*

We now turn our attention to issues of entailment of formulas which typically appear in applications in prediction. The restriction on particular formulas allows us to improve the complexity of the entailment problem. Based on the time points present in antecedents and consequents of attribute implications, we may consider formulas that describe presence of attributes in future time points. That is, based on the presence of attributes in the past, the formulas indicate which attributes are present in future time points. Technically, such formulas can be seen as attribute implications where all time points in the antecedents are smaller (i.e., denote earlier time points) than all time points in the consequents which denote later time points. We call such formulas predictive and define the notion as follows.

**Definition 2.15.** A temporal attribute implication $A \Rightarrow B$ over $Y$ is called *predictive* whenever $A, B \in \mathcal{F} \setminus \{\emptyset\}$ and for each $x^i \in A$ and $y^j \in B$, we have $i \leq j$. A theory $\Sigma$ is called predictive whenever all its formulas are predictive.

In the next assertion, we utilize lower and upper time bounds of non-empty sets from $\mathcal{F}$: For any $M \in \mathcal{F} \setminus \{\emptyset\}$, put

$$l(M) = \min\{i \in \mathbb{Z} \,|\, y^i \in M \text{ for some } y \in Y\}, \tag{2.8}$$

$$u(M) = \max\{i \in \mathbb{Z} \,|\, y^i \in M \text{ for some } y \in Y\}. \tag{2.9}$$

Thus, $l(M)$ and $u(M)$ are the lowest and greatest time points which appear in $M$, respectively. Clearly, $A \Rightarrow B$ is *predictive* iff both $A$ and $B$ are non-empty and $u(A) \leq l(B)$.

Let $L_{\mathrm{PRE}}$ be the language consisting of encodings of pairs of all finite predictive theories and predictive formulas, i.e.,

$$L_{\mathrm{PRE}} = \{\langle \Sigma, A \Rightarrow B \rangle \,|\, \Sigma \text{ is finite and } \Sigma \text{ and } A \Rightarrow B \text{ are predictive}\}. \tag{2.10}$$

We establish the following observation on the time complexity of deciding whether a predictive formula is provable by a finite predictive theory.

**Theorem 2.16.** *$L_{\mathrm{ENT}} \cap L_{\mathrm{PRE}}$ is decidable in a pseudo-polynomial time.*

An explicit procedure for deciding $L_{\mathrm{ENT}} \cap L_{\mathrm{PRE}}$ in a *pseudo-linear* time is described in Algorithm 1. It is a generalization of LinClosure [4], cf. also [34], which incorporates applicable time shifts of formulas in $\Sigma$. The algorithm accepts three arguments:

1. a finite predictive theory $\Sigma$,
2. a finite $A \subseteq \mathcal{T}_Y$, and
3. a non-negative number $Max \geq u(A)$,

and it returns a subset $M \subseteq [A]_\Sigma$ such that $M \cap T = [A]_\Sigma \cap T$ for

$$T = \{y^i \in \mathcal{T}_Y \,|\, l(A) \leq i \leq Max\}. \tag{2.11}$$

*Remark* 2. The procedure in Algorithm 1 is called PseudoLinClosure because for given parameters, $\Sigma$, $A$, and $Max$, it computes a subset of the closure of $[A]_\Sigma$ in a linear time with respect to the numeric value of the encoding of its input arguments, i.e., its time complexity is *pseudo-linear*. Indeed, this is a consequence of the fact that each $y^i$ where $l(A) \leq i \leq Max$ is updated during the computation at most once.

**Algorithm 1:** PseudoLinClosure ($\Sigma, A, Max$)

```
forall E ⇒ F ∈ Σ do
    for i from l(A) − l(E) to Max − l(F) do
        set count[E ⇒ F, i] to |E|;
        forall y^j ∈ E do
            add ⟨E ⇒ F, i⟩ to list[y^{i+j}];
        end
    end
end
set M to A;
set update to A;
while update ≠ ∅ do
    choose y^i from update;
    set update to update \ {y^i};
    forall ⟨E ⇒ F, j⟩ ∈ list[y^i] do
        set count[E ⇒ F, j] to count[E ⇒ F, j] − 1;
        if count[E ⇒ F, j] = 0 then
            set new to F + j \ M;
            set M to M ∪ new;
            set update to update ∪ new;
        end
    end
end
return M
```

# 3 Concise descriptions of dependencies

In this section, we define input data and complete theories as sets of formulas which semantically entail all formulas which hold in given data. We introduce non-redundancy and minimality as properties of "small theories" and show that for each data there is a minimal complete set of formulas. In addition, we provide characterization of minimal theories and an algorithm for minimalization.

Recall the set $\mathcal{F}$ defined by (2.2) and the values $l(M)$ and $u(M)$ defined by (2.8) and (2.9), respectively. For any $M \in \mathcal{F} \setminus \{\emptyset\}$, we put

$$\|M\| = u(M) - l(M). \tag{3.1}$$

The value $\|M\|$ is called the *time range* of $M$, respectively.

In our representation of minimal bases, a key role will be played by subsets of $\mathcal{T}_Y$ which are in a canonical form in the following sense:

**Definition 3.1.** For $M \subseteq \mathcal{T}_Y$, we put

$$r(M) = \begin{cases} M - l(M), & \text{if } M \in \mathcal{F} \setminus \{\emptyset\}, \\ M, & \text{otherwise,} \end{cases} \tag{3.2}$$

and call $r(M)$ the canonical form of $M$. In addition, for any system $\mathcal{S} \subseteq 2^{\mathcal{T}_Y}$, we call $r(\mathcal{S}) = \{r(M) \,|\, M \in \mathcal{S}\}$ the canonical form of $\mathcal{S}$.

Following the motivation in the introduction, we are primarily interested in dependencies which hold not only for individual objects changing in time but for a general finite set of objects changing in time. Therefore, we formalize the input data and extend $\models$ accordingly to accomodate general sets of objects as follows. In addition to $Y$, we consider a finite non-empty set $X$ of *objects* and, analogously as we have introduced $\mathcal{T}_Y$ for $Y$, see (2.1), we consider $\mathcal{T}_X$ for $X$ as $\mathcal{T}_X = \{x^i \,|\, x \in X \text{ and } i \in \mathbb{Z}\}$. Then, each $X$-indexed system of non-empty sets in $\mathcal{F}$ is considered as input data. In other words, by *input data* we mean any $\mathcal{I}$ of the following form:

$$\mathcal{I} = \{I_x \in \mathcal{F} \setminus \{\emptyset\} \,|\, x \in X\}. \tag{3.3}$$

That is, each $I_x \in \mathcal{I}$ is a non-empty and finite subset of $\mathcal{T}_Y$. From the point of view of the interpretation of $\mathcal{I}$, each $I_x \in \mathcal{I}$ can be seen as a record of attributes (changing in time) of the object $x \in X$. Furthermore, we say that $A \Rightarrow B$ *is true in the input data* $\mathcal{I} = \{I_x \in \mathcal{F} \setminus \{\emptyset\} \,|\, x \in X\}$, written $\mathcal{I} \models A \Rightarrow B$, whenever $I_x \models A \Rightarrow B$ for all $x \in X$.

Clearly, each $\mathcal{I}$ of the form (3.3) can be represented by a $\mathbb{Z}$-indexed finite sequence of formal contexts as in Fig. 2 and, conversely, each $\mathbb{Z}$-indexed finite sequence of finite formal contexts (using fixed $X$ and $Y$) can be represented by an $\mathcal{I}$ of the form (3.3).

*Example* 1. Let $X = \{p, q, r\}$, $Y = \{a, b, c, d\}$, and let $\mathcal{I} = \{I_p, I_q, I_r\}$ where

$$\begin{aligned} I_p &= \{a^0, b^0, a^1, b^1\}, \\ I_q &= \{a^1, c^1, a^2, b^2\}, \\ I_r &= \{c^0, a^1, c^1, a^2\}. \end{aligned}$$

Following the previous comment, the corresponding $\mathbb{Z}$-indexed sequence of contexts corresponding to this particular $\mathcal{I}$ is in fact the sequence depicted in Fig. 2.

From now on, we assume we are given input data $\mathcal{I}$ of the form (3.3). For $A \subseteq \mathcal{T}_X$ and $B \subseteq \mathcal{T}_Y$, we put

$$A^{\uparrow_{\mathcal{I}}} = \bigcap\{I_x - i \,|\, x^i \in A\}, \tag{3.4}$$

$$B^{\downarrow_{\mathcal{I}}} = \{x^i \in \mathcal{T}_X \,|\, B \subseteq I_x - i\}. \tag{3.5}$$

If there is no danger of confusion, we write just $^{\uparrow}$ and $^{\downarrow}$ instead of $^{\uparrow_{\mathcal{I}}}$ and $^{\downarrow_{\mathcal{I}}}$. It is routine to check that $^{\uparrow}$ and $^{\downarrow}$ are a couple of operators which form an antitone Galois connection, see [14, 23].

We now introduce the notion of completeness of sets of temporal attribute implications with respect to given data.

**Definition 3.2.** $\Sigma$ is called complete in $\mathcal{I}$ whenever for every $A \Rightarrow B$ we have $\mathcal{I} \models A \Rightarrow B$ iff $\Sigma \models A \Rightarrow B$.

Investigation of complete sets is interesting since they convey information about all discussed if-then dependencies which hold in given data. In order to characterize complete sets we utilize the following notion:

**Definition 3.3.** A theory $\Sigma$ is finitely generated whenever there is $t \in \mathbb{Z}$ such that for every $M \in \mathrm{Mod}(\Sigma) \setminus \{\emptyset, \mathcal{T}_Y\}$ we have $M \in \mathcal{F}$ and $\|M\| \leq t$, $\emptyset \in \mathrm{Mod}(\Sigma)$, and $\mathrm{Mod}(\Sigma) \cap (\mathcal{F} \setminus \{\emptyset\}) \neq \emptyset$.

**Theorem 3.4.** *$\Sigma$ is complete in some $\mathcal{I}$ iff $\Sigma$ is finitely generated.*

Our goal is to describe complete sets which are minimal in terms of their size. In the discourse, we utilize the following notion of equivalence of theories:

**Definition 3.5.** We put $\Sigma_1 \sqsubseteq \Sigma_2$ whenever, for every $A \Rightarrow B$, if $\Sigma_1 \models A \Rightarrow B$ then $\Sigma_2 \models A \Rightarrow B$; we put $\Sigma_1 \equiv \Sigma_2$ and say that $\Sigma_1$ and $\Sigma_2$ are equivalent whenever $\Sigma_1 \sqsubseteq \Sigma_2$ and $\Sigma_2 \sqsubseteq \Sigma_1$.

The description of complete sets which are in addition minimal can be based on formulas whose consequents are based on closures of sets from $\mathcal{F}$ which can be infinite, namely, equal to $\mathcal{T}_Y$. Since we consider formulas as implications between finite sets of attributes, we extend the notion of temporal attribute implications by allowing $\mathcal{T}_Y$ to appear as an antecedent or a consequent. By this, we are able to consider just a single formula $A \Rightarrow \mathcal{T}_Y$ which serves as a finite representation of an infinite theory of the form $\{A \Rightarrow B \,|\, B \in \mathcal{F}\}$.

**Definition 3.6.** An expression $A \Rightarrow B$ where $A, B \in \mathcal{F} \cup \{\mathcal{T}_Y\}$ is called an extended temporal attribute implication. We put $M \models A \Rightarrow B$ whenever, for every $i \in \mathbb{Z}$, $A + i \subseteq M$ implies $B + i \subseteq M$.

The notions of models and semantic entailment of extended temporal attribute implication are defined in much the same way as in the case of the original formulas, see Section 2. From now on, we are going to work with extended temporal attribute implications and we are not going to stress the term "extended."

Complete sets can be large and not very interesting because many of the contained formulas can be entailed by other formulas. We therefore look for complete sets which are at least non-redundant in the following sense:

**Definition 3.7.** $\Sigma$ is called non-redundant whenever for any $\Sigma' \subset \Sigma$ we have $\Sigma' \not\equiv \Sigma$. If $\Sigma$ is non-redundant and complete in $\mathcal{I}$ then $\Sigma$ is called a (non-redundant) base of $\mathcal{I}$.

Next, we express particular non-redundant sets of formulas which are given by special systems that are subsets of $\mathcal{F}$. The systems are introduced in the following definition and generalize the classic notion of pseudo-intents proposed in [26].

**Definition 3.8.** A set $P \in \mathcal{F}$ is a pseudo-intent of $\mathcal{I}$ if $P \neq P^{\downarrow\uparrow}$ and for any pseudo-intent $Q$ of $\mathcal{I}$ such that $Q \subset P$ we have $Q^{\downarrow\uparrow} \subseteq P$. The set of all pseudo-intents of $\mathcal{I}$ is denoted by $\mathcal{P}_{\mathcal{I}}$.

| # | $r(P)$ | $\lVert P\rVert$ | $r(P)^{\downarrow\uparrow}$ |
|---|---|---|---|
| 1 | $\{b^0\}$ | 0 | $\{a^0, b^0\}$, |
| 2 | $\{c^0\}$ | 0 | $\{c^0, a^1\}$, |
| 3 | $\{a^0, b^0, a^1\}$ | 1 | $\{a^0, b^0, a^1, b^1\}$, |
| 4 | $\{c^0, a^1, b^1\}$ | 1 | $\{a^0, c^0, a^1, b^1\}$, |
| 5 | $\{a^0, b^0, c^0, a^1, b^1\}$ | 1 | $\mathcal{T}_Y$, |
| 6 | $\{a^0, a^2\}$ | 2 | $\mathcal{T}_Y$, |
| 7 | $\{c^0, a^1, a^2\}$ | 2 | $\{c^0, a^1, c^1, a^2\}$, |
| 8 | $\{c^0, a^1, c^1, a^2, b^2\}$ | 2 | $\mathcal{T}_Y$ |

Figure 3: Example of canonical forms of all pseudo-intents of the input data from Example 1 limited to pseudo-intents with time range up to 2.

*Example* 2. The fact that $\mathcal{P}_{\mathcal{I}}$ is closed under time shift means, among other things, that $\mathcal{P}_{\mathcal{I}}$ is infinite. However, if we restrict ourselves to the pseudo-intents in the canonical form and, in addition, we limit ourselves only to those with time range within the time range of the input data, there are only finitely many of such pseudo-intents. Following our preliminary discussion in the introduction, such pseudo-intents turn out to be the most interesting ones. Going back to the data in Example 1, see also Figure 2, there are exactly eight pseudo-intents with these properties. They are listed together with their time ranges and closures in Figure 3.

For any system $\mathcal{S} \subseteq \mathcal{F}$ and $\mathcal{I}$ of the form (3.3), we put

$$\Sigma_{\mathcal{S}} = \big\{ M \Rightarrow M^{\downarrow\uparrow} \,|\, M \in r(\mathcal{S}) \big\}. \tag{3.6}$$

The following observations show that systems of pseudo-intents define non-redundant bases of the form (3.6). Note that for brevity, in the rest of the section we denote $\mathcal{P}_{\mathcal{I}}$ just by $\mathcal{P}$.

**Theorem 3.9.** *$\Sigma_{\mathcal{P}}$ is a non-redundant base of $\mathcal{I}$.*

The non-redundant theory $\Sigma_{\mathcal{P}}$ is satisfying a stronger condition of minimality. Technically, the minimality is defined in a different way than in the classical setting as we shall see in a moment. The main reason behind this is that no $\mathcal{I}$ of the form (3.3) admits a finite non-redundant base. This is in contrast with the classic non-redundant bases of finite formal contexts which are always finite [23, 26]. Also note that taking into account the fact that $\mathcal{F}$ is countable, we have that any theory is at most countable, i.e., finitely generated theories are countable, i.e., of the same size. Therefore, it would be worthless to define minimal theories in our setting the same way as in the classic case [34] as theories with the least size among all equivalent theories since all finitely generated theories would be minimal. Instead, we introduce the following notion of minimality:

**Definition 3.10.** A finitely generated theory $\Sigma$ is minimal whenever for each $\Sigma' \subseteq \Sigma$ and $\Gamma'$ such that $\Sigma' \equiv \Gamma'$ we have $|\Sigma'| \leq |\Gamma'|$.

Before we prove the minimality of $\Sigma_{\mathcal{P}}$ where $\mathcal{P}$ is the system of pseudo-intents of $\mathcal{I}$, we show properties of minimality that will be further used.

**Lemma 3.11.** *A finitely generated theory* $\Sigma$ *is minimal iff for each* $\Sigma' \subseteq \Sigma$ *and* $\Gamma'$ *such that* $(\Sigma \setminus \Sigma') \cup \Gamma' \equiv \Sigma$ *we have* $|\Sigma'| \leq |\Gamma'|$*.*

Put in words, the observation in Lemma 3.11 says that no subset of a minimal theory can be equivalently replaced by a smaller theory. Therefore, minimal theories are non-redundant according to Definition 3.7.

**Theorem 3.12.** $\Sigma_{\mathcal{P}}$ *is minimal.*

Based on the observations in this section, we argue that in the temporal setting we use in this document, there is a reasonable notion of a pseudo-intent which can be used to determine bases of input data which are minimal. The notion of minimality has been introduced to accommodate the fact that all bases of input data in our setting are infinite. Nevertheless, the observed minimality of the obtained bases has some implications for the finite "interesting part" of bases that was discussed in the introduction. Namely, in any base given by pseudo-intents, the interesting part cannot be replaced by smaller and equivalent set of formulas. This is a direct consequence of the previous observations and the notion of minimality from Definition 3.10.

In order to characterize minimal sets of formulas, we start by introducing a notation for expressing that considering a theory $\Sigma$, an antecedent $A$ implies a shift of another finite subset of $\mathcal{T}_Y$. This property will later be used to define equivalence of antecedents of formulas and will be crucial for the investigation of minimality.

**Definition 3.13.** For a theory $\Sigma$ and $A, B \in \mathcal{F} \cup \{\emptyset\}$, we put $\Sigma \models A \Rightarrow^* B$ whenever there is $i \in \mathbb{Z}$ such that $\Sigma \models A \Rightarrow B + i$.

We now turn our attention to a particular equivalence relation defined on antecendents of formulas in a theory $\Sigma$.

**Definition 3.14.** Let $\Sigma$ be a theory and $A, C \in \mathcal{F}$. We say that $A$ and $C$ are equivalent under $\Sigma$, written $A \equiv_\Sigma C$, whenever $\Sigma \models A \Rightarrow^* C$ and $\Sigma \models C \Rightarrow^* A$. Furthermore, we define $\mathrm{E}_\Sigma(A)$ as the set of all $C \Rightarrow D \in \Sigma$ such that $A \equiv_\Sigma C$.

In the following definition, we introduce a notion capturing a stronger form of semantic entailment of temporal attribute implications. The notion plays a central role in the characterization of minimal sets of formulas.

**Definition 3.15.** Let $\Sigma$ be a theory, $A, B \in \mathcal{F}$. We say that $A \Rightarrow B$ is directly entailed by $\Sigma$, written $\Sigma \Vdash A \Rightarrow B$, whenever $\Sigma \setminus \mathrm{E}_\Sigma(A) \models A \Rightarrow B$.

Note that the direct entailment introduced in Definition 3.15 generalizes the notion of direct determination known from the classic setting [33]. There are basically two main differences between the notions. First, direct entailment refers to formulas in our temporal setting whereas the classic notion does not. Second, direct entailment is defined in terms of the semantic entailment whereas the classic direct determination was defined in terms of derivation DAGs [33, 34] that can be seen as graphical proof system that is equivalent to the system of Armstrong inference rules [3]. Let us also note that [49] introduces a notion of direct provability that utilizes graded attribute implications and is based on an Armstrong-style inference system parameterized by globalization [42].

The following assertion presents a necessary and sufficient condition for a non-redundant theory to be minimal. The condition is based on checking the non-existence of a pair of formulas with particular properties.

**Theorem 3.16** (Characterization of Minimality)**.** *Let $\Sigma$ be a non-redundant theory such that for each $A \Rightarrow B, C \Rightarrow D \in \mathrm{E}_\Sigma(H)$ we have $\Sigma \Vdash A \Rightarrow^* C$ iff $\Sigma \models C \Rightarrow A - i$ and $\Sigma \Vdash A \Rightarrow C + i$ for some $i \in \mathbb{Z}$. Then $\Sigma$ is minimal iff there are no distinct $A \Rightarrow B, C \Rightarrow D \in \Sigma$ such that $A \equiv_\Sigma C$ and $\Sigma \Vdash A \Rightarrow^* C$.*

*Remark* 3. (a) As an example of theories for which the assumption in Corollary 3.16 holds, consider theories where the semantic closures of finite sets are finite. Indeed, assume that for $\Sigma$ we have that $A \in \mathcal{F}$ implies $[A]_\Sigma \in \mathcal{F}$ and take $A \Rightarrow B, C \Rightarrow D \in \mathrm{E}_\Sigma(H)$ such that $\Sigma \Vdash A \Rightarrow^* C$ and $A \neq \emptyset$ (a non-trivial case). Then we have $\Sigma \models C \Rightarrow A + j$ and $\Sigma \Vdash A \Rightarrow C + i$ for some $i, j \in \mathbb{Z}$. As a consequence, $\Sigma \models A \Rightarrow A + (j + i)$ and so $\Sigma \models A \Rightarrow A + (j + i) \cdot k$ for any $k \in \mathbb{N}$ which holds iff $A + (j + i) \cdot k \subseteq [A]_\Sigma$ for any $k \in \mathbb{N}$. Hence, $j = -i$ because $[A]_\Sigma$ cannot be infinite. The converse implication is trivial.

(b) Another important example of theories that fullfill the conditions of Corollary 3.16 are finitely generated theories, see Definition 3.3. For such theory it holds $A \in \mathcal{F}$ implies $[A]_\Sigma \in \mathcal{F} \cup \{\mathcal{T}_Y\}$. Hence, we can use the same arguments as in (a) and handle the case when $[A]_\Sigma = \mathcal{T}_Y$. Using the assumption $\Sigma \models C \Rightarrow A + j$ we have $A + j \subseteq [C]_\Sigma$ which means $\mathcal{T}_Y = \mathcal{T}_Y + j = [A]_\Sigma + j \subseteq [C]_\Sigma$, i.e., $[C]_\Sigma = \mathcal{T}_Y$. Therefore, $\Sigma \models C \Rightarrow A - i$ holds.

Let us stress that the finitely generated theories used in Remark 3 (b) represent a wide family of theories that are natural from users' point of view. Indeed, as it has been shown in Corollary 3.4, finitely generated theories are exactly theories entailing all if-then dependencies that hold in finite data sets. Therefore, Theorem 3.16 can be applied to any set of temporal attribute implications that is derived from a finite data set and entails all temporal attribute implications that hold in the data set.

We can summarize our observations by the following two algorithms the soundness of which follows from Theorem 3.16 and Theorem 2.9.

**Algorithm 1** (Test of Minimality)**.**

**input:** $\Sigma$ *satisfying the assumptions of Corollary 3.16 (see Remark 3)*

**output:** *YES (is minimal) / NO (is not minimal)*

> *If there are distinct* $A \Rightarrow B, C \Rightarrow D \in \Sigma$ *such that* $A - i \subseteq [C]_\Sigma$ *and* $C + i \subseteq [A]_{\Sigma \setminus \mathrm{E}_\Sigma(A)}$ *for some* $i \in \mathbb{Z}$*, then return NO, otherwise return YES.*

■

**Algorithm 2** (Minimization Step)**.**

**input:** $\Sigma$ *satisfying the assumptions of Corollary 3.16 (see Remark 3)*

**output:** *a theory that is equivalent to* $\Sigma$

> *If there are distinct* $A \Rightarrow B, C \Rightarrow D \in \Sigma$ *such that* $A - i \subseteq [C]_\Sigma$ *and* $C + i \subseteq [A]_{\Sigma \setminus \mathrm{E}_\Sigma(A)}$ *for some* $i \in \mathbb{Z}$*, then return* $(\Sigma \setminus \{A \Rightarrow B, C \Rightarrow D\}) \cup \{C \Rightarrow D \cup (B - i)\}$*, otherwise return* $\Sigma$*.*

■

*Remark* 4. Observe that if $\Sigma$ is finite, then both Algorithm 1 and Algorithm 2 terminate after finitely many steps and the total number of computed closures is polynomial in the number of formulas in $\Sigma$. Indeed, the tests involve distinct pairs of formulas from $\Sigma$ and, clearly, $\Sigma \setminus \mathrm{E}_{\Sigma}(A)$ can also be determined based on computing closures the number of which is polynomial in the size of $\Sigma$. From this point of view, the complexity of our procedure is no worse than for the classic test of minimality [33].

# 4 Related work

In database systems and knowledge engineering, there appeared isolated approaches which propose temporal semantics of if-then rules. We present here a short survey of the approaches and highlight the differences between our approach and the existing ones.

Formulas called temporal functional dependencies emerged in databases with time granularities [5]. In this approach, a time granularity is a general partition of time like seconds, weeks, years, etc., and a time granularity is associated to each relational schema. In addition, each tuple in a relation is associated with a part (so-called granule) of granularity. In this setting, temporal functional dependencies are like the ordinary functional dependencies [17, 34] with a time granularity as an additional component. The concept of validity of temporal functional dependencies is defined in much the same way as its classic counterpart and includes an additional condition that granules of tuples need to be covered by any granule from granularity of the temporal functional dependency. Thus, [5] uses an ordinary notion of validity of functional dependencies which is restricted to some time segments. This is conceptually very different from the problem we deal with in this document since in our approach, each attribute appearing in a rule is annotated by a relative time point and our rules are considered true in data whenever they hold in all time points.

Several approaches to temporal if-then rules, which are conceptually similar to [5], appeared in the field of association rules [1, 51] as the so-called temporal association rules [2, 30, 38]. In these approaches, the input data is in the form of transactions (i.e., subsets of items) where each transaction occurred at some point in time and the interest of the papers lies in extracting association rules from data which occur during a specified time cycle. For instance, one may be interested in extracting rules which are valid in "every spring month of a year", "every Monday in every year", etc. As in the case of the temporal functional dependencies, the temporal association rules may be understood as classic association rules occurring during specified time cycles.

Other results motivated by temporal semantics of association rules includes the so-called inter-transaction association rules [18, 19, 28, 47], see [32] for a survey of approaches. The papers propose algorithms to extract, given an input transactional data and a measure of interestingness (based on levels of minimal support and confidence), if-then rules which are preserved over a given period of time. From this point of view, the rules can be seen as formulas studied in this document restricted to predictive rules (see Definition 2.15) whose validity is considered with respect to the additional parameter of interestingness. As a consequence, the inter-transaction association rules are related to the rules in our approach in the same way as the ordinary association rules [1] are related to the ordinary attribute implications [23]. The results in [18, 19, 28, 32, 47] are focused almost exclusively on algorithms for mining the inter-transaction association rules and are not concerned with problems of entailment of the rules and the underlying logic. In contrast, the problems of entailment of rules are investigated in this document and we show there is reasonably strong logic for reasoning with such rules. Furthermore, we deal with a problem of extracting sets of rules satisfying a condition–minimality and the ability to describe all dependencies which hold in the data, instead of extracting rules from data satisfying a condition (given by the interestingness measure). Our observations may stimulate further development in the field of inter-transaction association rules and similar formulas and their applications in various domains [18, 27].

The formulas studied in this document are also related to particular program rules which appear in *Datalog* extensions dealing with flow of time and related phenomena [11, 10, 9] such as $Datalog_{nS}$ (*Datalog* with $n$ successors). The formulas we consider correspond to a fragment of rules which appear in such *Datalog* extensions. Despite the similar form of our formulas and the program rules, there does not seem to be a direct relationship (or a reduction) of the entailment problem of our formulas and the recognition problem of $Datalog_{nS}$ programs. As we have outlined in the introduction our formulas can also be seen as particular PROLOG rules. Despite the possibility to consider our rules in these (and other) database and logic programming languages, we aim at different goals. Most importantly, we have provided an Armstrong-style axiomatization which is strong-complete, i.e., complete over arbitrary $\Sigma$, and focuses on the inference of formulas (rules) from (finite or infinite) sets of rules. In contrast, PROLOG uses definite programs (finite sets of formulas) and its inference system is based on the resolution principle. Our development of the topic is primarily motivated by temporal extensions of rules which are used in FCA [23] where the Armstrong-style systems are extensively used and, therefore, our approach is a natural direction to go in that matter.

Note that predictive formulas, as they were introduced in Definition 2.15, can be translated into further existing languages. For instance, the formulas can be represented by TeDiLog [20] rules—a recent temporal logic programming language whose semantics is defined using structures with a beginning and a linear flow of time. Thus, the semantics of TeDiLog differs from our because of the existence of the beginning of time and it includes a modality "always in future". In contrast, our rules are interpreted as if they contained a hidden modality "always (including points in the past)". With analogous conceptual differences, the predictive formulas can also be translated into plans of the planning domain definition language (PDDL, see [13, 25]) or expressed in situation calculus [35, 37, 39]. An open question is whether such transformations can be used to get further insight into the entailment problem of our formulas.

Temporal attribute implications can be seen as extensions of attribute implications studied in FCA and functional dependencies in relational databases [34]. Interestingly, in both the FCA and database communities there appeared results characterizing minimal sets of if-then formulas with different motivations. The minimality of sets of functional dependencies was thoroughly examined in the seminal paper [33] where the author gives criteria for minimality of non-redundant sets of functional dependencies based on the notion of direct determination. In this document, we present a similar result for temporal attribute implications. Interestingly, [33] shows that transforming a set of functional dependencies into an equivalent and minimal one can be done in polynomial time and the algorithm exploits the standard tests of entailment of functional dependencies [4].

In FCA, the seminal paper [26] shows a description of minimal sets of attribute implications based on the notion of pseudo-intents. Unlike the results on functional dependencies where the input for minimization is a set of formulas, [26] computes the minimal bases directly from object-attribute incidence data which turns out to be a hard problem as it is shown in [16]. In this document, we generalize the results of [26] in the temporal setting.

The form of data we consider as "input data" in our approach is closely related to triadic formal contexts [29]. Although there appeared approaches to attribute implications from the point of view of the triadic FCA [6, 22], they do not annotate attributes by conditions (such as time points as in our case). Our formulas are syntactically different and have a different interpretation than if-then dependencies which were introduced in triadic FCA. The initial approach to if-then rules in triadic FCA [6] considers formulas written as $(A \Rightarrow B)_C$ where $A, B$ are subsets of attributes and $C$ is a set of conditions. A formula of this form is considered true in a triadic context if the following condition is satisfied:

> If an object has all attributes from $A$ under all conditions from $C$, then it also has all attributes from $B$ under all conditions from $C$.

Clearly, our formulas represent different dependencies since the approach in [6] annotate whole formulas by conditions (such as time points as in our case) whereas in our case is annotated each particular attribute. Hence, different attributes appearing in a formula can be annotated by different conditions. Later, stronger formulas were proposed in [22] which are considered true in a triadic context if the following condition is satisfied:

> For each condition $c \in C$: If an object has all attributes in $A$ (under $c$) then it also has all attributes in $B$ (under $c$).

Again, our formulas are different in that the annotations appear in antecedents and consequents of the formulas.

# 5 Conclusion

We have presented logic for reasoning with if-then rules expressing dependencies between attributes changing in time. The logic extends the classic logic for dealing with if-then rules by considering discrete time points as an additional component. We have studied both the semantic entailment based on preserving validity in models in all time points and syntactic entailment represented by a provability relation. We have shown a characterization of the semantic entailment based on least models and syntactico-semantical completeness of the logic. We have shown the problem of entailment is NP-hard, decidable in exponential space, and its simplified variant which involves only predictive formulas is decidable in pseudo-linear time. We have studied the notions of completeness in data changing in time, non-redundancy, and minimality of theories which are derived from finite sequences of object-attribute incidence data recorded in separate points in time. We have shown a generalization of the notion of a pseudo-intent which fits well into our model and proved that important non-redundant and minimal bases are determined by systems of pseudo-intents. Unlike the classic case, the bases are always infinite but contain finitely many formulas which constitute a part which is most relevant to data analysts. We have paid attention to properties inherent to minimal theories. We have introduced and investigated the notion of equivalence of antecedents of formulas and the notion of direct entailment that has been introduced as a stronger form of semantic entailment. Using the notions, we have presented necessary and sufficient conditions of minimality and presented families of theories for which such conditions can be applied. In the special case of finite theories, our criteria of minimality yield algorithms that can be used to minimize theories in finitely many steps. The minimization procedure relies on computing semantic closures whose number is polynomial in the size of the input—in this sense, the algorithm behaves as the classic minimization algorithm for attribute implications (or functional dependencies).

# Souhrn

Představili jsme logiku pro vyvozování pomocí if-then pravidel vyjadřujícími závislosti mezi atributy měnícimi se v čase. Tato logika rozšiřuje klasickou logiku zabývající se if-then pravidly, kde přidáváme diskrétní čas jako další složku. Zkoumali jsme semantické vyplývání založené na zachování platnosti v modelech ve všech okamžicích v průběhu času a syntaktické vyplývání reprezentované relací dokazatelnosti. Ukázali jsme charakterizaci semantického vyplývání založeném na nejmenším modelu a syntakticko-semantickou úplnost logiky. Dokázali jsme, že problém semantického vyplývání je NP-těžký, rozhodnutelný v exponenciálním prosotoru a jeho zjednodušená varianta zahrnující prediktivní formule je rozhodnutelná v pseudo-lineárním čase. Zkoumali jsme pojem úplnosti na datech měnících se v čase, neredundantnost a minimalitu teorií, které jsou odvozené z konečných posloupností objekt-atributových incidenčních datech zaznamenaných v oddělených bodech v čase. Ukázali jsme zobecnění pojmu pseudo-intent, který se dobře hodí do našeho modelu a prokázali, že důležité neredundantní a minimální báze jsou odvozeny ze systémů pseudo-intentů. Na rozdíl od klasického případu jsou báze vždy neokonečné, ale obsahují konečné množství formulí, které obsahují část, která je nejvíce zajímavá pro datové analytiky. Zajímaly nás základní

vlastnosti minimálních teorií. Zavedli jsme a prozkoumali pojem ekvivalence antecedentů formulí a pojem přímého vyplývání byl zaveden jako silnější forma sémantického vyplývání. Pomocí těchto pojmů jsme ukázali nutnou a postačující podmínku minimality a představili skupiny teorií, pro které tato podmínka může být použita. Ve speciálním případě, kdy máme konečnou teorii, naše podmínka minimality indukuje algoritmus, kterým lze teorii minimalizovat v konečně mnoha krocích. Minimalizační procedura je založená na výpočtu sémantických uzávěrů, jejichž počet je polynomický ve velikosti vstupu. V tomto smyslu se algoritmus chová jako klasický minimalizační algoritmus pro atributové implikace (nebo funkční závislosti).

# References


[1] Rakesh Agrawal, Tomasz Imieliński, and Arun Swami. "Mining Association Rules Between Sets of Items in Large Databases". In: *Proceedings of the 1993 ACM SIGMOD International Conference on Management of Data*. SIGMOD '93. Washington, D.C., USA: ACM, 1993, pp. 207–216.

[2] Juan M. Ale and Gustavo H. Rossi. "An Approach to Discovering Temporal Association Rules". In: *Proceedings of the 2000 ACM Symposium on Applied Computing*. Vol. 1. SAC '00. Como, Italy: ACM, 2000, pp. 294–300.

[3] William Ward Armstrong. "Dependency structures of data base relationships". In: *Information Processing 74: Proceedings of IFIP Congress*. Ed. by J. L. Rosenfeld and H. Freeman. Amsterdam: North Holland, 1974, pp. 580–583.

[4] Catriel Beeri and Philip A. Bernstein. "Computational problems related to the design of normal form relational schemas". In: *ACM Trans. Database Syst.* 4 (1 Mar. 1979), pp. 30–59.

[5] Claudio Bettini, Sushil Jajodia, and Xiaoyang Sean Wang. *Time Granularities in Databases, Data Mining, and Temporal Reasoning*. Springer, 2000.

[6] Klaus Biedermann. "A foundation of the theory of trilattices". Dissertation. Aachen: TU Darmstadt, 1998.

[7] Garrett Birkhoff. *Lattice theory*. 1st. Providence: American Mathematical Society, 1940.

[8] Patick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic*. Secaucus, NJ, USA: Cambridge University Press, 2002.

[9] Jan Chomicki and Tomasz Imieliński. “Finite Representation of Infinite Query Answers”. In: *ACM Trans. Database Syst.* 18.2 (June 1993), pp. 181–223.

[10] Jan Chomicki and Tomasz Imieliński. “Relational Specifications of Infinite Query Answers”. In: *Proceedings of the 1989 ACM SIGMOD International Conference on Management of Data*. SIGMOD ’89. Portland, Oregon, USA: ACM, 1989, pp. 174–183.

[11] Jan Chomicki and Tomasz Imieliński. “Temporal Deductive Databases and Infinite Objects”. In: *Proceedings of the 7th ACM SIGACT-SIGMOD-SIGART Symposium on Principles of Database Systems*. PODS ’88. Austin, Texas, USA: ACM, 1988, pp. 61–73.

[12] Edgar F. Codd. “A relational model of data for large shared data banks”. In: *Commun. ACM* 13 (6 1970), pp. 377–387.

[13] Stephen Cresswell and Alexandra M. Coddington. “Compilation of LTL Goal Formulas into PDDL”. In: *Proc. 16th Eureopean Conference on Artificial Intelligence, ECAI’2004, including Prestigious Applicants of Intelligent Systems, PAIS 2004, Valencia, Spain, August 22-27, 2004*. Ed. by Ramon López de Mántaras and Lorenza Saitta. IOS Press, 2004, pp. 985–986.

[14] Brian A. Davey and Hilary A. Priestley. *Introduction to Lattices and Order*. Cambridge: Cambridge University Press, 1990.

[15] Claude Delobel and Richard G. Casey. “Decomposition of a Data Base and the Theory of Boolean Switching Functions”. In: *IBM Journal of Research and Development* 17.5 (1973), pp. 374–386.

[16] Felix Distel and Barış Sertkaya. “On the Complexity of Enumerating Pseudo-intents”. In: *Discrete Appl. Math.* 159.6 (Mar. 2011), pp. 450–466.

[17] Ronald Fagin. “Functional dependencies in a relational database and propositional logic”. In: *IBM Journal of Research and Development* 21.6 (Nov. 1977), pp. 534–544.

[18] Ling Feng, Tharam Dillon, and James Liu. “Inter-transactional Association Rules for Multi-dimensional Contexts for Prediction and Their Application to Studying Meterological Data”. In: *Data Knowl. Eng.* 37.1 (Apr. 2001), pp. 85–115.

[19] Ling Feng et al. “A Template Model for Multidimensional Inter-transactional Association Rules”. In: *The VLDB Journal* 11.2 (Oct. 2002), pp. 153–175.

[20] Jose Gaintzarain and Paqui Lucio. “Logical Foundations for More Expressive Declarative Temporal Logic Programming Languages”. In: *ACM Trans. Comput. Logic* 14.4 (Nov. 2013), 28:1–28:41.

[21] Bernhard Ganter. “Two Basic Algorithms in Concept Analysis”. In: *Proceedings of the 8th International Conference on Formal Concept Analysis.* ICFCA’10. Agadir, Morocco: Springer-Verlag, 2010, pp. 312–340.

[22] Bernhard Ganter and Sergei Obiedkov. “Implications in Triadic Formal Contexts”. In: *Conceptual Structures at Work.* Ed. by Karl Erich Wolff, Heather D. Pfeiffer, and Harry S. Delugach. Vol. 3127. Lecture Notes in Computer Science. Springer Berlin Heidelberg, 2004, pp. 186–195.

[23] Bernhard Ganter and Rudolf Wille. *Formal Concept Analysis: Mathematical Foundations.* 1st. Secaucus, NJ, USA: Springer-Verlag New York, Inc., 1997.

[24] Michael R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* New York, NY, USA: W. H. Freeman & Co., 1979.

[25] Malik Ghallab et al. *PDDL—the Planning Domain Definition Language, version 1.2. Technical Report CVC TR-98-003/DCS TR-1165 (Oct.)* Tech. rep. Yale Center for Computational Vision and Control, Yale University, 1998.

[26] Jean-Louis Guigues and Vincent Duquenne. “Familles minimales d’implications informatives resultant d’un tableau de données binaires”. In: *Math. Sci. Humaines* 95 (1986), pp. 5–18.

[27] Yo-Ping Huang, Li-Jen Kao, and Frode-Eika Sandnes. “Efficient Mining of Salinity and Temperature Association Rules from ARGO Data”. In: *Expert Syst. Appl.* 35.1–2 (July 2008), pp. 59–68.

[28] Anthony J. T. Lee et al. “An Efficient Algorithm for Mining Closed Intertransaction Itemsets”. In: *Data Knowl. Eng.* 66.1 (July 2008), pp. 68–91.

[29] Fritz Lehmann and Rudolf Wille. “A triadic approach to formal concept analysis”. In: *Conceptual Structures: Applications, Implementation and Theory.* Ed. by Gerard Ellis et al. Vol. 954. Lecture Notes in Computer Science. Springer Berlin Heidelberg, 1995, pp. 32–43.

[30] Yingjiu Li et al. "Discovering calendar-based temporal association rules". In: *Data and Knowledge Engineering* 44.2 (2003), pp. 193–218.

[31] John W. Lloyd. *Foundations of Logic Programming*. New York, NY, USA: Springer-Verlag New York, Inc., 1984.

[32] Hongjun Lu, Ling Feng, and Jiawei Han. "Beyond Intratransaction Association Analysis: Mining Multidimensional Intertransaction Association Rules". In: *ACM Trans. Inf. Syst.* 18.4 (Oct. 2000), pp. 423–454.

[33] David Maier. "Minimum Covers in Relational Database Model". In: *J. ACM* 27.4 (Oct. 1980), pp. 664–674.

[34] David Maier. *Theory of Relational Databases*. Rockville, MD, USA: Computer Science Pr, 1983.

[35] John McCarthy and Phillip J. Hayes. "Readings in Nonmonotonic Reasoning". In: ed. by Matthew L. Ginsberg. San Francisco, CA, USA: Morgan Kaufmann Publishers Inc., 1987. Chap. Some Philosophical Problems from the Standpoint of Artificial Intelligence, pp. 26–45.

[36] Elliott Mendelson. *Introduction to Mathematical Logic*. Chapman and Hall, 1987.

[37] Javier Pinto and Raymond Reiter. "Reasoning about time in the situation calculus". In: *Annals of Mathematics and Artificial Intelligence* 14.2 (1995), pp. 251–268.

[38] Chris P. Rainsford and John F. Roddick. "Adding Temporal Semantics to Association Rules". English. In: *Principles of Data Mining and Knowledge Discovery*. Ed. by Jan M. Żytkow and Jan Rauch. Vol. 1704. Lecture Notes in Computer Science. Springer Berlin Heidelberg, 1999, pp. 504–509.

[39] Raymond Reiter. "Artificial Intelligence and Mathematical Theory of Computation". In: ed. by Vladimir Lifschitz. San Diego, CA, USA: Academic Press Professional, Inc., 1991. Chap. The Frame Problem in the Situation Calculus: A Simple Solution (Sometimes) and a Completeness Result for Goal Regression, pp. 359–380.

[40] John A. Robinson. "A Machine-Oriented Logic Based on the Resolution Principle". In: *J. ACM* 12.1 (1965), pp. 23–41.

[41] Yehoshua Sagiv et al. "An equivalence between relational database dependencies and a fragment of propositional logic". In: *J. ACM* 28.3 (July 1981), pp. 435–453.

[42] Gaisi Takeuti and Satoko Titani. "Globalization of intuitionistic set theory". In: *Annals of Pure and Applied Logic* 33 (1987), pp. 195–211.

[43] Jan Triska and Vilem Vychodil. "Logic of temporal attribute implications". In: *Annals of Mathematics and Artificial Intelligence* 79.4 (Apr. 2017), pp. 307–335.

[44] Jan Triska and Vilem Vychodil. "Minimal bases of temporal attribute implications". In: *Annals of Mathematics and Artificial Intelligence* 83.1 (May 2018), pp. 73–97.

[45] Jan Triska and Vilem Vychodil. "On minimal sets of temporal attribute implications". submitted. 2018.

[46] Jan Triska and Vilem Vychodil. "Towards Armstrong-Style Inference System for Attribute Implications with Temporal Semantics". In: *Modeling Decisions for Artificial Intelligence*. Ed. by Vicenç Torra, Yasuo Narukawa, and Yasunori Endo. Vol. 8825. LNCS. Springer International Publishing, 2014, pp. 84–95.

[47] Anthony K. H. Tung et al. "Breaking the Barrier of Transactions: Mining Inter-transaction Association Rules". In: *Proceedings of the Fifth ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*. KDD '99. San Diego, California, USA: ACM, 1999, pp. 297–301.

[48] Vilem Vychodil. "Closure structures parameterized by systems of isotone Galois connections". In: *International Journal of Approximate Reasoning* 91 (2017), pp. 1–21.

[49] Vilem Vychodil. "On minimal sets of graded attribute implications". In: *Information Sciences* 294 (2015), pp. 478–488.

[50] Vilem Vychodil. "Parameterizing the semantics of fuzzy attribute implications by systems of isotone Galois connections". In: *IEEE Trans. on Fuzzy Systems* 24 (3 2016), pp. 645–660.

[51] Mohammed J. Zaki. "Mining non-redundant association rules". In: *Data Mining and Knowledge Discovery* 9 (2004), pp. 223–248.