Temporal Attribute Implications

Jan Tříska

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Author
Jan Tříska
Department of Computer Science
Faculty of Science, Palacký University
17. listopadu 12
CZ-771 46 Olomouc
Czech Republic
jan.triska@upol.cz

Supervisor
Vilém Vychodil

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Declaration
Hereby I declare that the thesis is my original work.
Parts of this thesis are based on outcomes of the joint scientific work with Vilém Vychodil. All authors have even share in the results.

Jan Tříska

To my family, with love.

Preface

We deal with dependencies in object-attribute data which is recorded at separate points in time. The data is formalized by finitely many tables encoding the relationship between the objects and the attributes and each table can be seen as a single formal context observed at a separate point in time. Given such data, we are interested in concise ways of characterizing all if-then dependencies between the attributes that hold in the data and are preserved in all time points. In order to formalize the dependencies, we introduce if-then formulas called temporal attribute implications which can be seen as particular formulas of linear temporal logic. We introduce a semantic entailment of the formulas, show its fixed-point characterization, investigate closure properties of model classes, present an axiomatization and prove its completeness, and investigate alternative axiomatizations and normalized proofs. We investigate decidability and complexity issues of the logic and prove that the entailment problem is NP-hard and belongs to EXPSPACE. We show that by restricting to predictive formulas, the entailment problem is decidable in pseudo-linear time. We introduce non-redundant bases of dependencies from data as non-redundant sets entailing exactly all the dependencies that hold in the data. In addition, we investigate minimality of bases as a stronger form of non-redundancy. For given data, we present a description of minimal bases using the notion of pseudo-intents generalized in the temporal setting. We further investigate properties of minimal sets of formulas and present sufficient and necessary conditions for their characterization. In addition to the characterization of minimality, we present an algorithm that can be used to minimize any finite set of temporal attribute implications. Particular parts of this thesis were published in the following articles:

[54] Jan Triska and Vilem Vychodil. “Logic of temporal attribute implications”. In: *Annals of Mathematics and Artificial Intelligence* 79.4 (Apr. 2017), pp. 307–335.

[55] Jan Triska and Vilem Vychodil. “Minimal bases of temporal attribute implications”. In: *Annals of Mathematics and Artificial Intelligence* 83.1 (May 2018), pp. 73–97.

[56] Jan Triska and Vilem Vychodil. “On minimal sets of temporal attribute implications”. submitted. 2018.

[57] Jan Triska and Vilem Vychodil. “Towards Armstrong-Style Inference System for Attribute Implications with Temporal Semantics”. In: *Modeling Decisions for Artificial Intelligence*. Ed. by Vicenç Torra, Yasuo Narukawa, and Yasunori Endo. Vol. 8825. LNCS. Springer International Publishing, 2014, pp. 84–95.

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The thesis is organized as follows. In Section 2, we present short preliminaries. We introduce the formulas and present the results on their semantic entailment in Section 3. In Section 4 we show complete axiomatizations and in Section 5 we deal with related computational issues related to the semantic entailment. In Section 6, we investigate finite representations of systems of models, introduce notions of completeness in data, and investigate its properties. In Section 7, we investigate the structure of non-redundant and minimal bases of dependencies in data. In Section 8, we provide a characterization of minimality and derived algorithms. Finally, in Section 9 and Section 10, we present a survey of related work and present a conclusion.

Contents

1 Introduction

Formulas describing if-then dependencies between attributes play fundamental role in reasoning about attributes in many disciplines including database systems [13, 41], formal concept analysis [27, 30], data mining [1, 63], logic programming [38, 50], and their applications. In these disciplines, the rules often appear under different names (e.g., attribute implications, functional dependencies, or simply "rules") with semantics defined in various structures (e.g., transactional data, Boolean matrices, or n-ary relations) but as it has been shown in [21], the rules may be seen as propositional formulas with the semantic entailment defined as in the propositional logic, possibly extended by additional measures of interestingness. The rules are popular because of their easy readability for non-expert users. In addition, the entailment problem related to a large family of the rules, including attribute implications used in formal concept analysis and functional dependencies used in database systems, is decidable in linear time [5] which also contributes to their popularity.

In this thesis, we introduce if-then formulas that express presence of attributes relatively in time and the formulas are evaluated in data where the presence or absence of attributes changes in time. In our approach, we adopt the notion of a discrete time, i.e., the data are observed at distinct points in time. Informally, the formulas can be seen as rules expressing dependencies between attributes (or features) in the following sense:

IF (a feature y_1 is present in time point t_1
and $\cdots$ and
a feature y_m is present in time point t_m),
THEN (a feature z_1 is present in time point s_1
and $\cdots$ and
a feature z_n is present in time point s_n).

As a formula, such dependency can be written as

$$\left(y_1^{t_1} \mathbin{\&} \cdots \mathbin{\&} y_m^{t_m}\right) \Rightarrow \left(z_1^{s_1} \mathbin{\&} \cdots \mathbin{\&} z_n^{s_n}\right), \tag{1.1}$$

where $y_i^{t_i}$ denotes an attribute/feature y_i present in time point t_i and analogously for $z_j^{s_j}$. As usual, $\&$ and $\Rightarrow$ used in (1.1) denote the usual logical connectives of conjunction and implication (logical conditional), see [43].

In the thesis, we exploit the fact that $\&$ is a logical connective that is interpreted by an idempotent, commutative, and associative truth function, allowing us to rewrite (1.1) in a set-theoretic notation as follows:

$$\{y_1^{t_1}, \ldots, y_m^{t_m}\} \Rightarrow \{z_1^{s_1}, \ldots, z_n^{s_n}\}. \tag{1.2}$$

In general, rules like (1.2) can be viewed as locally valid, that is, valid exactly in time points $t_1, \ldots, t_m$ and $s_1, \ldots, s_n$ that appear in the formula. This notion of validity is in a sense trivial because reasoning with such formulas is easily reducible to reasoning with classic attribute implications. In contrast, (1.2) can be viewed as globally valid, i.e., valid for time points $t_1 + k, \ldots, t_m + k$ and $s_1 + k, \ldots, s_n + k$ for an arbitrary k. In our approach, we consider the global validity since we want to capture dependencies that are preserved over all time points and thus endure in time.

We study the formulas from the point of view of temporal reasoning in formal concept analysis [27]. The classic (dyadic) formal concept analysis (FCA) is a method of analysis of object-attribute data formalized by binary incidence relations between a set of objects and a set of attributes. One of the typical outputs of FCA, given an input incidence data, is a set of if-then dependencies which entails exactly all if-then dependencies that hold in the data. Among the best known methods of determining such interesting sets of if-then rules is the method of Guigues and Duquenne based on computing pseudo-intents from data, see [25, 30]. In many situations, the object-attribute incidence data changes over time and one may be interested in if-then rules which are universaly valid in all time points. For instance, we may observe a mechanism which behaves as a transition system which makes transitions from a state to another one in discrete steps. Supposing that we do not know the internals of the system and we can only observe a set of Boolean attributes which are or are not satisfied at a given moment. This gives us a set of attributes (of a single object—the system) which changes in time, i.e., a series of object-attribute incidence data changing in time. Then, rules like (1.2) may be used to describe the behavior of the system during transitions in terms of the dependencies between the Boolean attributes. In this situation, an analog of the Guigues-Duquenne bases would be the most helpful because it would allow us to derive a set of if-then rules describing the system based on its observation during the transitions. The classic notions related to Guigues-Duquenne bases are closely related to the notion

	a	b	c	d
1			$\times$	
2		$\times$		
3	$\times$	$\times$		$\times$
4	$\times$			

x	y
1	c
2	b
3	a
3	b
3	d
4	a

```
has(1,c).
has(2,b).
has(3,a).
has(3,b).
has(3,d).
has(4,a).
```

Figure 1: Example of object-attribute incidence data represented as a formal context $\langle X, Y, I\rangle$ with $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d\}$ depicted as a table (left), relation on relation scheme $\{x, y\}$ (middle), and PROLOG-style program consisting of facts (right).

of entailment of if-then rules. Therefore, before we show that a reasonable counterpart to the Guigues-Duquenne bases in the temporal setting indeed exists, we make a thorough investigation on the entailment.

The input data we consider consists of a finite set X of objects, a finite set Y of attributes (features), and a binary incidence relation $I \subseteq X \times Y$ with $\langle x, y\rangle \in I$ interpreted as "object x has attribute y". In this setting, I can be seen as a record of object-attribute data observed in a single time point and the triplet $\langle X, Y, I\rangle$ is called a (dyadic) formal context [27] in FCA. Let us note that despite the fact the thesis is written primarily from the FCA perspective, such data are the subject of study of many computer science disciplines. For instance, from the point of view of relational databases [41], $\langle X, Y, I\rangle$ can be understood as a (finite) relation on relation scheme with two attributes—an attribute whose domain is X and an attribute whose domain is Y. From the point of view of logic programming [38], $\langle X, Y, I\rangle$ can be seen as a definite program consisting of facts, see Fig. 1 for illustration. In addition, the object-attribute incidence data is considered as the basic form of input data in most data mining disciplines, most notably the association rule mining [1, 63].

As we have outlined, we assume that $\langle X, Y, I\rangle$ can change in time. To be more specific, we assume that X and Y are fixed, i.e., the sets of observed objects and attributes do not change in time, and I is subject to change. Therefore, instead of single I, we consider a sequence

$$I_l, I_{l+1}, \ldots, I_{r-1}, I_r \tag{1.3}$$

of incidence relations where l, r are integers ($l \leq r$) denoting separate time points and $I_i \subseteq X \times Y$ for each $i = l, \ldots, r$. Alternatively, the input data can be understood as a (finite) database relation on relation scheme consisting of three attributes: objects (with domain X), attributes (with domain Y), and time (with domain $\mathbb{Z}$). Also, it can be seen as representing finitely many facts of the form `has(time_point,x,y)` as in Fig. 1.

The dependencies we identify in the data generalize the classic if-then dependencies called attribute implications. Recall that by an attribute implication [25, 27, 30] we mean a propositional formula of the form

$$(y_1 \,\&\, \cdots \,\&\, y_m) \Rightarrow (z_1 \,\&\, \cdots \,\&\, z_n), \tag{1.4}$$

where y_i, z_j are propositional atoms. Both the semantic entailment (i.e., entailment defined in terms of validity in models) and syntactic entailment (i.e., entailment based on provability) of attribute implications are, in fact, notions inherited from the propositional logic [43]. Interestingly, attribute implications are closely related to functional dependencies. Although the interpretation of the formulas as attribute implications and functional dependencies are different, it follows from [19, 21, 51] that both the interpretations yield the same notion of the semantic entailment. As a consequence, common axiomatizations are used to characterize the semantic entailment which are typically based on the Armstrong inference rules [3].

If all y_i and z_j, which appear in (1.4), denote attributes from Y, then (1.4) can be interpreted in $\langle X, Y, I\rangle$ based on its validity for each object $x \in X$. In more detail, each $x \in X$ induces a truth evaluation e_x of propositional atoms such that $e_x(y) = 1$ (logical true) iff $\langle x, y\rangle \in I$ and $e_x(y) = 0$ (logical false) otherwise. Then, we may say that (1.4) holds in I whenever (1.4) is true under e_x for all $x \in X$ in the sense of propositional logic. According to our interpretation of I, the fact that (1.4) holds in I means that each object $x \in X$ satisfies the following property: If the object has all attributes $y_1, \ldots, y_m$, then it has all attributes $z_1, \ldots, z_n$, i.e., the presence of all $z_1, \ldots, z_n$ in the data is implied by the presence of all $y_1, \ldots, y_m$. For instance, if we consider I from Fig. 1 (left), then, e.g., $(a \,\&\, b) \Rightarrow d$ and $(b \,\&\, c) \Rightarrow d$ hold in I. On the other hand, neither of $a \Rightarrow d$, $b \Rightarrow d$, or $c \Rightarrow d$ holds in I.

The dependencies used in the thesis may be seen as extensions of formulas like (1.4) incorporating explicit time annotations for each y_i, z_j. That is, instead of (1.4), we consider formulas of the form (1.2) where $i_1, j_1, \ldots$ are integers interpreted as relative shifts in time, allowing us to express dependencies such as "if y is currently present and z was present yesterday, then χ will be present tomorrow" by formulas like $(y^0 \mathbin{\&} z^{-1}) \Rightarrow \chi^1$ provided that the considered "unit of time" is a day. We call formulas of the form (1.2) *temporal attribute implications.*

We provide answers to several questions which emerge with temporal attribute implications. We define the notion of semantic entailment of the formulas, investigate closure structures of models of theories consisting of such formulas, and show that the problem of checking whether a formula is semantically entailed by a set of formulas can be reduced to checking its validity in a single model. We prove that the semantic entailment has a complete axiomatization. That is, we show a notion of provability of temporal attribute implications and show that it coincides with the semantic entailment. We discuss several possible axiomatizations, including ones that can be used to consider proofs in particular normal forms. Based on our insight into the properties of the semantic entailment and provability, we derive results on decidability and complexity of the entailment problem. Fourth, we include notes on the relationship of the formulas to formulas appearing in modal logics [9] and triadic formal concept analysis [35].

After the investigation of the properties of the semantic entailment, we focus on description of all temporal attribute implications that hold in given data. In particular, we seek sets of formulas entailing exactly all formulas that hold in given data. We call such sets *complete* (in given data). As in the classic setting, it is desirable to describe complete sets of formulas which are small. In the thesis, we introduce two notions which may be seen as two basic properties of "small sets of formulas": *non-redundancy* and *minimality.* Complete sets in data which are non-redundant (or minimal) are called non-redundant (or minimal) *bases* (of the data) and their structure and properties are investigated in the thesis. Unlike the classic case, where the time annotations are not present and minimal bases of finite incidence data are finite, minimal bases in the temporal setting are infinite in general. This is one of the aspects that makes the presented theory substantially different compared to the classic one [25, 27, 30].

Despite the fact that the bases of finite data are infinite, our observations show that each such base can be split into two parts based on the maximal difference of time points in the input data (so-called time range):

(i) An interesting *finite part* which can be enumerated in finitely many

$i = 0$

	a	b	c
p	×	×	
q			
r			×

$i = 1$

	a	b	c
p	×	×	
q	×		×
r	×		×

$i = 2$

	a	b	c
p			
q	×	×	
r	×		

Figure 2: Input data depicted as formal contexts considered in separate time points for $i = 0$ (left), $i = 1$ (middle), and $i = 2$ (right).

steps. This part consists of formulas where the maximal difference of time points in their antecedents are within the range of the input data.

(*ii*) An *infinite part* which consists of formulas whose antecedents contain time points which are outside the time range of the input data.

For illustration of the notions, let us consider the input data from Fig. 2. Using the introduced notation, the set of objects is $X = \{p, q, r\}$, the set of attributes is $Y = \{a, b, c\}$, and the tables in Fig. 2 encode the incidence relations $I_i \subseteq X \times Y$ in time points $i = 0, 1, 2$. Therefore, in this case, the time range of the input data is 2 (units) because 2 is the maximal distance of time points of any attributes in the data.

As an example of a particular minimal base of the data in Figure 2, we can consider the following set of formulas. We may split the base into three disjoint subsets. The first (and the most interesting) part of the base consists of formulas

$$b^0 \Rightarrow a^0,$$
$$c^0 \Rightarrow a^1,$$
$$(b^0 \,\&\, a^1) \Rightarrow b^1,$$
$$(c^0 \,\&\, b^1) \Rightarrow a^0,$$
$$(c^0 \,\&\, a^2) \Rightarrow c^1.$$

For all antecedents (and consequents) of the previous formulas, we can consider their time range which is the maximal difference of time points of all used attributes. Clearly, the time ranges are 0 (first formula), 1 (next three formulas), and 2 (the last formula) which are all less than or equal to the time range of the data. For a data analyst, the formulas express that if there is an object which, in a certain time point, satisfies the condition given by the antecedent, then it must satisfy the condition given by the consequent. For instance, $(c^0 \,\&\, b^1) \Rightarrow a^0$ says "if an object has the attribute c in the current time point and it has b in the next time point, then is also has a in the current time point."

The second group consists of formulas with antecedents within the time range of the input data and having arbitrary conjunctions of attributes annotated by time points as their consequents. Namely, the group consists of

$$
\begin{aligned}
&(b^0 \,\&\, c^0) \Rightarrow \varphi,\\
&(a^0 \,\&\, a^2) \Rightarrow \varphi,\\
&(c^0 \,\&\, c^1 \,\&\, b^2) \Rightarrow \varphi,
\end{aligned}
$$

where φ is an arbitrary conjunction of attributes annotated by time points. Technically, this already represents an infinite set of formulas but with only finitely many pairwise different antecedents. Intuitively, these formulas express that certain combinations of attributes in time are not possible in the input data. For example, $(b^0 \,\&\, c^0) \Rightarrow b^5$ is a particular instance of the formula listed first. If there were an object $x \in X$ and a time point where b is present and c is present then the object would have b present in five time points in the future which is absurd because the time range of the input data is 2. Therefore, considering the present data, the formula says that "in the input data, b and c are not present in the same time point for any object."

These first two groups of formulas are the most interesting for data analysts. Given input data encoded by tables as in Fig. 2, the thesis shows how such "interesting" formulas of a particular minimal base can be obtained based on systems of pseudo-intents [30] which we generalize in the temporal setting.

In order to conclude our example, the presented base of the data in Fig. 2 consists of formulas from which we infer all formulas with antecedents outside of the time range of the input data. In this particular case, it is sufficient to consider formulas

$$(a^0 \,\&\, a^{2+n}) \Rightarrow \varphi,$$

where n is any natural number and φ is an arbitrary conjunction of attributes annotated by time points. This part of the base is infinite and, moreover, not so interesting for analysts (if one has a data that spans n time units, there is no point in finding dependencies which go beyond n units). It ensures that the base entails all formulas with antecedents outside of the time range of the input data. Such formulas trivially hold in the input data and one can easily verify that a formula is in such a form just by computing the time range of its antecedent.

The last question studied in the thesis is the problem of characterizing minimal sets of temporal attribute implications. Problems of finding minimal descriptions of various structures belong to classic problems in computer science as well as data analysis. It is well known that some minimality problems are easy (e.g., minimization of finite automata) and some are intractable or even undecidable (e.g., minimality of Turing machines). In data analysis, there is a natural need to find descriptions of dependencies, clusters, or patterns in data that are as small as possible in order to simplify further processing or to enable easier evaluation by human experts. In this thesis, we show a condition based on checking the presence of two formulas with special properties that is considerably simpler than checking the minimality by definition.

The investigation of minimality of sets of if-then rules started with the seminal paper [40] where the author showed criteria for minimality of non-redundant sets of functional dependencies based on the notion of direct determination. The paper showed that transforming a set of rules into an equivalent and minimal one can be done in polynomial time using the standard tests of entailment [5]. Later, the result has been extended for a family of graded/fuzzy attribute implications in [60]. Since all classic if-then rules, graded/fuzzy attribute implications, and temporal attribute implications can be seen as general rules whose semantics is defined by particular systems of isotone Galois connections, see [61, 59], we investigate a minimality characterization for temporal attribute implications that is analogous to the classic one [40] and the one for the graded attribute implications [60].

2 Preliminaries

In this section, we present the basic notions of closure systems (also known as Moore families) and closure operators which are used further in the thesis. More details can be found in [8, 18].

If Y is a set, we denote by 2^Y its power set. A closure operator on Y is a map $c\colon 2^Y \to 2^Y$ such that

$$A \subseteq c(A), \tag{2.1}$$

$$A \subseteq B \text{ implies } c(A) \subseteq c(B), \tag{2.2}$$

$$c(c(A)) \subseteq c(A), \tag{2.3}$$

for all $A, B \subseteq Y$. The conditions (2.1)–(2.3) are called the extensivity, monotony, and idempotency of c, respectively. Note that (2.1) and (2.3) yield $c(A) = c(c(A))$ for all $A \subseteq Y$. A closure operator $c\colon 2^Y \to 2^Y$ is called an algebraic closure operator whenever

$$c(A) = \bigcup\{c(B) \mid B \subseteq A \text{ and } B \text{ is finite}\} \tag{2.4}$$

for all $A \subseteq Y$. Moreover, $A \subseteq Y$ is called a fixed point of c whenever $c(A) = A$.

A system $\mathcal{S} \subseteq 2^Y$ is called a closure system on Y if it is closed under arbitrary intersections, i.e., $\bigcap \mathcal{A} \in \mathcal{S}$ for any $\mathcal{A} \subseteq \mathcal{S}$. In the thesis, we utilize the well-known correspondence between closure systems and closure operators on Y [18, 8]. In particular, if c is an algebraic closure operator on Y, we call the closure system of all its fixed points the algebraic closure system induced by c.

The notion of minimality of theories that is being investigated further in the thesis relies on the standard notion of cardinality of sets. Recall that a set A is smaller than or equal to B (in terms of their size) whenever there is an injective map $f\colon A \to B$; we denote this fact by $|A| \leq |B|$. Moreover, A is strictly smaller than B if $|A| \leq |B|$ and it is not the case that $|B| \leq |A|$ in which case we write $|A| < |B|$. We put $|A| = |B|$ and say that A and B have the same size whenever $|B| \leq |A|$ and $|A| \leq |B|$.

3 Logic of temporal attribute implications

In this section, we present a formalization of the formulas, their interpretation, and semantic entailment. Let us assume that Y is a non-empty and finite set of symbols called attributes. Furthermore, we use integers in order to denote time points. We put

$$\mathcal{T}_Y = \left\{ y^i \,|\, y \in Y \text{ and } i \in \mathbb{Z} \right\} \tag{3.1}$$

and interpret each $y^i \in \mathcal{T}_Y$ as "attribute y observed in time i" (technically, $\mathcal{T}_Y$ can be seen as the Cartesian product $Y \times \mathbb{Z}$). It is easy to see that $\mathcal{T}_Y$ is countable. Furthermore, we introduce the following set:

$$\mathcal{F}_Y = \{ M \subseteq \mathcal{T}_Y \,|\, M \text{ is finite} \}, \tag{3.2}$$

and abbreviate the set by $\mathcal{F}$ if Y is clear from the context. Obviously, $\mathcal{F}$ is countable since $\mathcal{T}_Y$ is countable.

Under this notation, we may now formalize rules like (1.2) as follows:

Definition 3.1. A *temporal attribute implication over* Y is a formula of the form $A \Rightarrow B$, where $A, B \in \mathcal{F}$.

As we have outlined in the introduction, the purpose of time points encoded by integers which appear in antecedents and consequents of the considered formulas is to express points in time relatively to a current time point. Hence, the intended meaning of (1.2) abbreviated by $A \Rightarrow B$ is the following: "For all time points t, if an object has all attributes from A considering t as the current time point, then it must have all attributes from B considering t as the current time point". In what follows, we formalize the interpretation of $A \Rightarrow B$ in this sense.

Since we wish to define formulas being true in all time points (we are interested in formulas preserved over time), we need to shift relative times expressed in antecedents and consequents in formulas with respect to a changing time point. For that purpose, for each $M \subseteq \mathcal{T}_Y$ and $i \in \mathbb{Z}$, we may introduce a subset $M + j$ of $\mathcal{T}_Y$ by

$$M + j = \left\{ y^{i+j} \,|\, y^i \in M \right\} \tag{3.3}$$

and call it a *time shift of* M *by* j (shortly, a j-shift of M). In the thesis, we utilize the following properties of time shifts.

Proposition 3.2. *For all $M, N \subseteq \mathcal{T}_Y$, $\{N_k \subseteq \mathcal{T}_Y \,|\, k \in K\}$, and $i, j \in \mathbb{Z}$, we get*

$$\text{if } M \subseteq N \text{ then } M + i \subseteq N + i, \tag{3.4}$$

$$(M + i) + j = M + (i + j), \tag{3.5}$$

$$\textstyle\bigcup_{k\in K}(N_k + i) = \bigcup_{k\in K} N_k + i, \tag{3.6}$$

$$\textstyle\bigcap_{k\in K}(N_k + i) = \bigcap_{k\in K} N_k + i. \tag{3.7}$$

Proof. All (3.4)–(3.7) follow directly from (3.3). □

Based on (3.5), we may omit parentheses and write $M + j + i$ instead of $(M + i) + j$. Also, we write $M - i$ to denote $M + (-i)$.

Temporal attribute implications are formulas, i.e., syntactic notions for which we define their semantics (interpretation) as follows.

Definition 3.3. A formula $A \Rightarrow B$ *is true in* $M \subseteq \mathcal{T}_Y$ whenever, for each $i \in \mathbb{Z}$,

$$\text{if } A + i \subseteq M, \text{ then } B + i \subseteq M \tag{3.8}$$

and we denote the fact by $M \models A \Rightarrow B$.

Remark 1. (a) The value of i in the definition may be understood as a sliding time point. Moreover, $A + i$ and $B + i$ represent sets of attributes annotated by *absolute time points* considering i as the current time point. Note that using (3.3), the condition (3.8) can be equivalently restated as "$A \subseteq M - i$ implies $B \subseteq M - i$," i.e., instead of shifting the antecedents and consequents of the formula, we may shift the set M.

(b) Observe that $A \Rightarrow B$ is trivially true in M whenever $B \subseteq A$ because in that case (3.8) trivially holds for any i. By definition, $A \Rightarrow B$ is not true in M, written $M \not\models A \Rightarrow B$ iff there is i such that $A + i \subseteq M$ and $B + i \nsubseteq M$. In words, in the time point i, M has all attributes of A but does not have an attribute in B, i.e., the time point i serves as a counterexample.

Example 1. One particular example of a subset M of $\mathcal{T}_Y$ can be a daily weather observation from an airport station. For instance, we can consider Y as

$$Y = \{\mathtt{rn}, \mathtt{rl}, \mathtt{rm}, \mathtt{tv}, \mathtt{tc}, \mathtt{tm}, \mathtt{wl}, \mathtt{wm}, \mathtt{ws}\},$$

	rn	rl	rm	tv	tc	tm	wl	wm	ws
15	×				×		×		
16	×				×			×	
17		×		×			×		
18			×		×			×	
19	×				×				×
20	×				×		×		
21	×				×			×	
22	×				×			×	
23	×				×		×		
24		×				×		×	
25	×				×		×		
26			×		×			×	
27		×			×			×	
28	×				×			×	
29	×					×		×	

Figure 3: Daily weather observation from an airport station.

where the attributes have the following meaning: "no rainfall" (denoted $\mathtt{rn}$), "light rainfall" (denoted $\mathtt{rl}$), "moderate rainfall" (denoted $\mathtt{rm}$), "temperature is very cold", (denoted $\mathtt{tv}$), "temperature is cold", (denoted $\mathtt{tc}$) "temperate is mild", (denoted $\mathtt{tm}$) "light wind" (denoted $\mathtt{wl}$), "moderate wind" (denoted $\mathtt{wm}$), and "strong wind" (denoted $\mathtt{ws}$). A subset of $\mathcal{T}_Y$ may be depicted as a two-dimensional table with rows corresponding to time points, columns corresponding to attributes in Y, and crosses and blanks in the table, indicating whether attributes annotated by time points belong to the subset. For instance, if M is given by the table in Figure 3, then $\mathtt{rn}^{15} \in M$, $\mathtt{rl}^{15} \notin M$, etc[1]. In this case, we have $M \models \{\mathtt{wl}^0, \mathtt{wm}^1\} \Rightarrow \{\mathtt{tc}^3\}$. On the other hand, $M \not\models \{\mathtt{wm}^0, \mathtt{wl}^1\} \Rightarrow \{\mathtt{tc}^3, \mathtt{rm}^3, \mathtt{tc}^4\}$ because for $i = 22$, we have $\{\mathtt{wm}^0, \mathtt{wl}^1\} + 22 = \{\mathtt{wm}^{22}, \mathtt{wl}^{23}\} \subseteq M$ and $\{\mathtt{tc}^3, \mathtt{rm}^3, \mathtt{tc}^4\} + 22 = \{\mathtt{tc}^{25}, \mathtt{rm}^{25}, \mathtt{tc}^{26}\} \nsubseteq M$.

We consider the following notions of a theory and a model:

Definition 3.4. Let Σ be a set of formulas (called a *theory*). A subset $M \subseteq \mathcal{T}_Y$ is called a *model of* Σ if $M \models A \Rightarrow B$ for all $A \Rightarrow B \in \Sigma$. The system of all models of Σ is denoted by $\mathrm{Mod}(\Sigma)$, i.e.,

$$\mathrm{Mod}(\Sigma) = \big\{M \subseteq \mathcal{T}_Y \,|\, M \models A \Rightarrow B \text{ for all } A \Rightarrow B \in \Sigma\big\}. \tag{3.9}$$

In general, $\mathrm{Mod}(\Sigma)$ is infinite and there may be theories that do not have any finite model. For instance, consider a theory containing $\emptyset \Rightarrow \{y^0\}$.

[1] The data is based on discretization of real meteorological information for Aug 14 which can be found at `http://www.bom.gov.au/climate/dwo/IDCJDW0100.shtml`.

We now turn our attention to the structure of systems of all models of temporal attribute implications. In case of the ordinary attribute implications, it is well known that systems of their models are exactly closure systems in Y [27]. Interestingly, the systems of models in our case are exactly the algebraic closure systems that are closed under time shifts. This additional closure property is introduced by the following definition.

Definition 3.5. A system $\mathcal{S} \subseteq 2^{\mathcal{T}_Y}$ of subsets of $\mathcal{T}_Y$ is called *closed under time shifts* whenever $M + i \in \mathcal{S}$ for all $M \in \mathcal{S}$ and $i \in \mathbb{Z}$.

We first show that $\mathrm{Mod}(\Sigma)$ is a closure system closed under time shifts:

Theorem 3.6. *Let Σ be a theory. Then,* $\mathrm{Mod}(\Sigma)$ *is closed under arbitrary intersections and time shifts.*

Proof. The fact that $\mathrm{Mod}(\Sigma)$ is closed under arbitrary intersections follows by analogous arguments as in the case of ordinary attribute implications taking into account that (3.8) must hold for all $i \in \mathbb{Z}$. That is, for any $\mathcal{M} \subseteq \mathrm{Mod}(\Sigma)$ and arbitrary $A \Rightarrow B \in \Sigma$, we reason as follows. If $A + i \subseteq \bigcap \mathcal{M}$, then $A + i \subseteq M$ for all $M \in \mathcal{M}$ and thus $B + i \subseteq M$ for all $M \in \mathcal{M}$ because $\mathcal{M} \subseteq \mathrm{Mod}(\Sigma)$. Therefore, $B + i \subseteq \bigcap \mathcal{M}$, proving $\bigcap \mathcal{M} \models A \Rightarrow B$ which further gives $\bigcap \mathcal{M} \in \mathrm{Mod}(\Sigma)$ since $A \Rightarrow B \in \Sigma$ was arbitrary.

In order to show that $\mathrm{Mod}(\Sigma)$ is closed under time shifts, take $M \in \mathrm{Mod}(\Sigma)$ and $j \in \mathbb{Z}$. It suffices to prove that $M + j \in \mathrm{Mod}(\Sigma)$. In order to see that, take $A \Rightarrow B \in \Sigma$. If $A + i \subseteq M + j$, then $A + (i - j) \subseteq M$ and thus $B + (i - j) \subseteq M$ because $M \in \mathrm{Mod}(\Sigma)$ and $A \Rightarrow B \in \Sigma$. Therefore, $B + i \subseteq M + j$, i.e., $M + j \models A \Rightarrow B$ for arbitrary $A \Rightarrow B \in \Sigma$, showing $M + j \in \mathrm{Mod}(\Sigma)$. $\square$

Taking into account Theorem 3.6, for each theory Σ, we may consider a closure operator induced by $\mathrm{Mod}(\Sigma)$ which maps each $M \subseteq \mathcal{T}_Y$ to the least model of Σ containing M.

Definition 3.7. Let Σ be a theory. For each $M \subseteq \mathcal{T}_Y$, we put

$$[M]_\Sigma = \bigcap \{N \in \mathrm{Mod}(\Sigma) \mid M \subseteq N\} \tag{3.10}$$

and call $[M]_\Sigma$ the *semantic closure* of M under Σ.

Using the well-known relationship between closure operators and closure systems [18, 8], $[\cdots]_\Sigma$ defined by (3.10) is indeed a closure operator. Note that in general, $[M]_\Sigma$ can be infinite even if Y and M are finite. This is in contrast with the ordinary attribute implications using finite Y. Nevertheless, in our setting we can prove that even if $[M]_\Sigma$ is infinite, it can be obtained as a union of finitely generated elements of $\mathrm{Mod}(\Sigma)$, showing that $\mathrm{Mod}(\Sigma)$ is in fact an *algebraic* closure system.

Theorem 3.8. *Let Σ be a theory. For each $M \subseteq \mathcal{T}_Y$, we have*

$$[M]_\Sigma = \bigcup\{[N]_\Sigma \,|\, N \textit{ is finite subset of } M\}. \tag{3.11}$$

Proof. Observe that the monotony of $[\cdots]_\Sigma$ yields $[N]_\Sigma \subseteq [M]_\Sigma$ for any finite $N \subseteq M$ and thus the "$\supseteq$"-part of (3.11) is obvious.

For brevity, put $\mathcal{M} = \{[N]_\Sigma \,|\, N \text{ is finite subset of } M\}$. In order to prove the "$\subseteq$"-part of (3.11), it suffices to show that $\bigcup\mathcal{M}$ is a model of Σ that contains M because $[M]_\Sigma$ is the least model of Σ containing M. For any $y^i \in M$, we have $[\{y^i\}]_\Sigma \in \mathcal{M}$ and thus $y^i \in [\{y^i\}]_\Sigma \subseteq \bigcup\mathcal{M}$ by the extensivity of $[\cdots]_\Sigma$ which proves $M \subseteq \bigcup\mathcal{M}$.

Now, take any $A \Rightarrow B \in \Sigma$ and suppose that $A+i \subseteq \bigcup\mathcal{M}$. Observe that for every $y^j \in A+i$ there is $[N_{y^j}]_\Sigma \in \mathcal{M}$ such that $y^j \in [N_{y^j}]_\Sigma$. Moreover, the fact that $A+i$ is finite yields that $\bigcup\{N_{y^j} \,|\, y^j \in A+i\}$ is finite and we thus have $\left[\bigcup\{N_{y^j} \,|\, y^j \in A+i\}\right]_\Sigma \in \mathcal{M}$. Clearly, $A+i \subseteq \left[\bigcup\{N_{y^j} \,|\, y^j \in A+i\}\right]_\Sigma$ and thus it follows that $B+i \subseteq \left[\bigcup\{N_{y^j} \,|\, y^j \in A+i\}\right]_\Sigma \subseteq \bigcup\mathcal{M}$ because $A \Rightarrow B \in \Sigma$. Altogether, $\bigcup\mathcal{M} \models A \Rightarrow B$ and so $\bigcup\mathcal{M} \in \mathrm{Mod}(\Sigma)$. □

Using Theorem 3.8, we may establish that each algebraic closure system closed under time shifts is a system of models of some theory consisting of temporal attribute implications. Before we go to the proof, we show how the property of being closed under time shifts can be formulated in terms of closure operators.

Lemma 3.9. *Let $\mathcal{S}$ be a closure system that is closed under arbitrary time shifts and let $\mathrm{C}_\mathcal{S}$ be the induced closure operator. For each $M \subseteq \mathcal{T}_Y$ and $i \in \mathbb{Z}$,*

$$\mathrm{C}_\mathcal{S}(M+i) = \mathrm{C}_\mathcal{S}(M)+i. \tag{3.12}$$

Proof. "$\subseteq$": Since $\mathcal{S}$ is closed under time shifts, we get $\mathrm{C}_\mathcal{S}(M)+i \in \mathcal{S}$. In addition, $M+i \subseteq \mathrm{C}_\mathcal{S}(M)+i$ on account of the extensivity of $\mathrm{C}_\mathcal{S}$ and (3.4). Therefore, $\mathrm{C}_\mathcal{S}(M+i) \subseteq \mathrm{C}_\mathcal{S}(M)+i$ by monotony and idempotency of $\mathrm{C}_\mathcal{S}$.

"$\supseteq$": The extensivity of $\mathrm{C}_{\mathcal{S}}$ gives $M + i \subseteq \mathrm{C}_{\mathcal{S}}(M + i)$ and thus $M \subseteq \mathrm{C}_{\mathcal{S}}(M + i) - i$. Moreover, $\mathrm{C}_{\mathcal{S}}(M + i) - i \in \mathcal{S}$ because $\mathcal{S}$ is closed under time shifts and thus $\mathrm{C}_{\mathcal{S}}(M) \subseteq \mathrm{C}_{\mathcal{S}}(M + i) - i$ which gives $\mathrm{C}_{\mathcal{S}}(M) + i \subseteq \mathrm{C}_{\mathcal{S}}(M + i)$. □

Lemma 3.10. *Let* C *be a closure operator satisfying* $\mathrm{C}(M + i) = \mathrm{C}(M) + i$ *for each* $M \subseteq \mathcal{T}_Y$ *and* $i \in \mathbb{Z}$*. Then, the system* $\mathcal{S}_{\mathrm{C}}$ *of all fixed points of* C *is closed under arbitrary time shifts.*

Proof. Take $M \in \mathcal{S}_{\mathrm{C}}$ and any $i \in \mathbb{Z}$, i.e., $M \subseteq \mathcal{T}_Y$ such that $M = \mathrm{C}(M)$. Clearly, $M + i = \mathrm{C}(M) + i$ and since $\mathrm{C}(M) + i = \mathrm{C}(M + i)$, we get $M + i = \mathrm{C}(M + i)$, proving that $M + i \in \mathcal{S}_{\mathrm{C}}$. □

The previous two lemmas give the following consequence.

Corollary 3.11. *A closure system* $\mathcal{S}$ *is closed under arbitrary time shifts iff the corresponding closure operator* $\mathrm{C}_{\mathcal{S}}$ *satisfies* (3.12). □

Based on our previous observations, we may now establish the connection between systems of models of temporal attribute implications and algebraic closure systems closed under time shifts.

Theorem 3.12. *Let* $\mathcal{S} \subseteq 2^{\mathcal{T}_Y}$ *be an algebraic closure system that is closed under time shifts. Then, there is a theory* Σ *such that* $\mathcal{S} = \mathrm{Mod}(\Sigma)$*.*

Proof. Assume that $\mathrm{C}_{\mathcal{S}}$ is the closure operator induced by $\mathcal{S}$ and put

$$\Sigma = \{A \Rightarrow B \,|\, A, B \in \mathcal{F} \text{ and } B \subseteq \mathrm{C}_{\mathcal{S}}(A)\}.$$

We show that $\mathcal{S} = \mathrm{Mod}(\Sigma)$ by proving that both inclusions hold.

"$\subseteq$": Take $M \in \mathcal{S}$ and $A, B \in \mathcal{F}$ such that $B \subseteq \mathrm{C}_{\mathcal{S}}(A)$. We now check that $M \models A \Rightarrow B$. Assume that $A + i \subseteq M$. Then, $A \subseteq M - i$ and by the monotony of $\mathrm{C}_{\mathcal{S}}$ and utilizing (3.12), we have $\mathrm{C}_{\mathcal{S}}(A) \subseteq \mathrm{C}_{\mathcal{S}}(M - i) = \mathrm{C}_{\mathcal{S}}(M) - i = M - i$ which yields that $B \subseteq M - i$, i.e., $B + i \subseteq M$, showing $M \models A \Rightarrow B$. As a consequence, $\mathcal{S} \subseteq \mathrm{Mod}(\Sigma)$.

"$\supseteq$": We let $M \in \mathrm{Mod}(\Sigma)$ and prove that $M \in \mathcal{S}$ which means to prove that $\mathrm{C}_{\mathcal{S}}(M) = M$. Since $\mathcal{S}$ is an algebraic closure system, it suffices to check that $\mathrm{C}_{\mathcal{S}}(A) \subseteq M$ for each finite $A \subseteq M$. Assuming that $A \subseteq M$ and A is finite, take any finite $B \subseteq \mathrm{C}_{\mathcal{S}}(A)$. By definition, $A \Rightarrow B \in \Sigma$ and since $M \in \mathrm{Mod}(\Sigma)$, we get that for $i = 0$, $A + 0 \subseteq M$ implies $B + 0 \subseteq M$. Since $A + 0 = A$ and $A \subseteq M$, we therefore obtain $B = B + 0 \subseteq M$. Since B was an arbitrary finite subset of $\mathrm{C}_{\mathcal{S}}(A)$, we conclude that $\mathrm{C}_{\mathcal{S}}(A) \subseteq M$. □

We now define semantic entailment of formulas and explore its properties. The notion is defined the usual way using the notion of a model introduced before.

Definition 3.13. Let Σ be a theory. Formula $A \Rightarrow B$ *is semantically entailed by* Σ if $M \models A \Rightarrow B$ for each $M \in \mathrm{Mod}(\Sigma)$.

The following lemma justifies the description of time points in attribute implications as *relative* time points. Namely, it states that each $A \Rightarrow B$ semantically entails all formulas resulting by shifting the antecedent and consequent of $A \Rightarrow B$ by a constant factor.

Lemma 3.14. *$\{A \Rightarrow B\} \models A + i \Rightarrow B + i$.*

Proof. Take $M \in \mathrm{Mod}(\{A \Rightarrow B\})$ and let $(A+i)+j \subseteq M$. Then, $A+i \subseteq M-j$ and by Theorem 3.6, we get $M-j \in \mathrm{Mod}(\{A \Rightarrow B\})$ which yields $B+i \subseteq M-j$ and thus $(B+i)+j \subseteq M$, proving $M \models A+i \Rightarrow B+i$ □

Analogously as for the classic attribute implications, the semantic entailment of $A \Rightarrow B$ by a theory Σ can be checked using the least model of Σ generated by A as it is shown in the following theorem.

Theorem 3.15. *For any Σ and $A \Rightarrow B$, the following conditions are equivalent:*

(i) $\Sigma \models A \Rightarrow B$,

(ii) $[A]_\Sigma \models A \Rightarrow B$,

(iii) $B \subseteq [A]_\Sigma$.

Proof. Clearly, (i) implies (ii) since $[A]_\Sigma \in \mathrm{Mod}(\Sigma)$; (ii) implies (iii) because $A+0 \subseteq [A]_\Sigma$. Assume that (iii) holds and take $M \in \mathrm{Mod}(\Sigma)$ and $i \in \mathbb{Z}$ such that $A+i \subseteq M$. Then, $A \subseteq M-i$ and thus $B \subseteq [A]_\Sigma \subseteq [M-i]_\Sigma = [M]_\Sigma - i$ by (3.12) from which it follows that $B+i \subseteq [M]_\Sigma = M$, proving (i). □

We conclude this section by notes on the propositional semantics of our formulas. Recall from the introduction that the classic attribute implications on finite Y can be understood as propositional formulas. Namely, an attribute implication of the form

$$\{y_1, \ldots, y_m\} \Rightarrow \{z_1, \ldots, z_n\}, \tag{3.13}$$

can be seen as a propositional formula in the form (1.4). Thus, (1.4) may be called a propositional counterpart of (3.13). Obviously, there are in general several propositional counterparts of (3.13) since formulas equivalent to (1.4) in sense of the propositional logic result, e.g., by rearranging the propositional variables $y_1, \ldots, y_m, z_1, \ldots, z_n$ in a different order. We neglect this aspect and always consider a fixed propositional counterpart of each attribute implication. It can be shown that if one takes the propositional counterparts of attribute implications, then their semantic entailment in sense of the propositional logic coincides with the semantic entailment as it is defined for attribute implications. We now show that an analogous correspondence can also be established in our case.

We start by considering the following notation. For any finite $A, B \subseteq \mathcal{T}_Y$ and for any $M \subseteq \mathcal{T}_Y$, we put $M \models_{\mathrm{PL}} A \Rightarrow B$ whenever $A \nsubseteq M$ or $B \subseteq M$. That is, $M \models_{\mathrm{PL}} A \Rightarrow B$ means that $A \Rightarrow B$ is true in M as a classical attribute implication. Clearly, $M \models_{\mathrm{PL}} A \Rightarrow B$ does not imply that $M \models A \Rightarrow B$ in sense of Definition 3.3. Moreover, we may introduce the set of models of Σ in the classic sense:

$$\mathrm{Mod}^{\mathrm{PL}}(\Sigma) = \big\{M \subseteq \mathcal{T}_Y \,|\, M \models_{\mathrm{PL}} A \Rightarrow B \text{ for all } A \Rightarrow B \in \Sigma\big\} \tag{3.14}$$

and put $\Sigma \models_{\mathrm{PL}} A \Rightarrow B$ whenever $M \models_{\mathrm{PL}} A \Rightarrow B$ for all $M \in \mathrm{Mod}^{\mathrm{PL}}(\Sigma)$. Therefore, $\models_{\mathrm{PL}}$ denotes the semantic entailment of attribute implications in the classic sense. Again, $\models_{\mathrm{PL}}$ is in general different from $\models$ introduced in Definition 3.13 but we can establish the following characterization:

Theorem 3.16. *Let Σ be a theory and let*

$$\Sigma^{\mathrm{PL}} = \{A + i \Rightarrow B + i \,|\, A \Rightarrow B \in \Sigma \textit{ and } i \in \mathbb{Z}\}. \tag{3.15}$$

Then $\mathrm{Mod}(\Sigma) = \mathrm{Mod}^{\mathrm{PL}}(\Sigma^{\mathrm{PL}})$. As a consequence, for each $A \Rightarrow B$, we have $\Sigma \models A \Rightarrow B$ iff $\Sigma^{\mathrm{PL}} \models_{\mathrm{PL}} A \Rightarrow B$.

Proof. The first part of the claim is easy to see. Indeed, for each $A \Rightarrow B$ we have $M \in \mathrm{Mod}(\{A \Rightarrow B\})$ iff for each $i \in \mathbb{Z}$, we have $A + i \subseteq M$ implies $B + i \subseteq M$ which is true iff $M \in \mathrm{Mod}^{\mathrm{PL}}(\{A + i \Rightarrow B + i \,|\, i \in \mathbb{Z}\})$. Hence, it follows that $\mathrm{Mod}(\Sigma) = \mathrm{Mod}^{\mathrm{PL}}(\Sigma^{\mathrm{PL}})$.

Now, assume that $\Sigma \models A \Rightarrow B$ and take $M \in \mathrm{Mod}^{\mathrm{PL}}(\Sigma^{\mathrm{PL}})$ such that $A \subseteq M$. Then $A + 0 \subseteq M$ and $M \in \mathrm{Mod}(\Sigma)$ and thus $B = B + 0 \subseteq M$, proving that $\Sigma^{\mathrm{PL}} \models_{\mathrm{PL}} A \Rightarrow B$. Conversely, let $\Sigma^{\mathrm{PL}} \models_{\mathrm{PL}} A \Rightarrow B$ and

$A + i \subseteq M$ for $M \in \mathrm{Mod}(\Sigma)$. That is, we have $A \subseteq M - i$ and, owing to Theorem 3.6, $M - i \in \mathrm{Mod}(\Sigma) = \mathrm{Mod}^{\mathrm{PL}}(\Sigma^{\mathrm{PL}})$. As a consequence of $M - i \models_{\mathrm{PL}} A \Rightarrow B$, we get $B \subseteq M - i$ and thus $B + i \subseteq M$, showing $\Sigma \models A \Rightarrow B$. Altogether, $\Sigma \models A \Rightarrow B$ iff $\Sigma^{\mathrm{PL}} \models_{\mathrm{PL}} A \Rightarrow B$. $\square$

Now, based on Theorem 3.16, we may argue that for each Σ there is a set of propositional formulas Σ' such that the propositional counterpart of $A \Rightarrow B$ follows by Σ' in sense of the propositional logic. Indeed, Σ' can be taken as the set of propositional counterparts to all formulas in Σ^{PL}: Owing to Theorem 3.16, $A \Rightarrow B$ follows by Σ^{PL} as a classic attribute implication over (a denumerable set of attributes) $\mathcal{T}_Y$ and thus the propositional counterpart of $A \Rightarrow B$ follows by the propositional counterparts to all formulas in Σ^{PL}.

4 Complete axiomatizations

In this section, we present a deduction system for our formulas and a related notion of provability which represents the syntactic entailment of formulas. The provability is based on an extension of the Armstrong axiomatic system [3] which is well known mainly in database systems [41]. The extension we propose accommodates the fact that time points in formulas are relative. The deductive system we use consists of the following deduction rules.

Definition 4.1. We introduce the following *deduction rules*:

(Ax) infer $A \cup B \Rightarrow A$,

(Cut) from $A \Rightarrow B$ and $B \cup C \Rightarrow D$ infer $A \cup C \Rightarrow D$,

(Shf) from $A \Rightarrow B$ infer $A + i \Rightarrow B + i$,

where $i \in \mathbb{Z}$ and A, B, C, D are arbitrary finite subsets of $\mathcal{T}_Y$.

Remark 2. (a) Note that there are several equivalent systems which are called the Armstrong systems [41]. In our presentation, the rule (Ax) can be seen as a nullary deduction rule which is an axiom scheme, i.e., each $A \cup B \Rightarrow A$ may be called an axiom. (Cut) and (Shf) are binary and unary deduction rules, respectively. In the classic case, (Ax) and (Cut) form a system which is equivalent to that from [3]. We call the additional rule (Shf) the rule of "time shifts." Also note that in the database literature, (Cut) is also referred to as the rule of pseudo-transitivity [41].

(b) The rules in Definition 4.1 can be written as fractions with hypotheses (formulas preceding "infer") above the conclusion (formula following "infer") as

$$\frac{}{A \cup B \Rightarrow A}\,(\mathrm{Ax}), \qquad \frac{A \Rightarrow B,\ B \cup C \Rightarrow D}{A \cup C \Rightarrow D}\,(\mathrm{Cut}), \qquad \frac{A \Rightarrow B}{A + i \Rightarrow B + i}\,(\mathrm{Shf}).$$

Although we are going to use (Ax), (Cut), and (Shf) as the basic deduction rules in our system, we define the notion of provability relatively to a collection of deduction rules because we later investigate systems consisting of other rules. Thus, a general *deduction system* is a set $\mathcal{R}$ of n-ary rules of the form "from $\varphi_1, \ldots, \varphi_n$, infer ψ".

Definition 4.2. Let $\mathcal{R}$ be a deduction system. An $\mathcal{R}$-proof of $A \Rightarrow B$ by Σ is a finite sequence $\delta_1, \ldots, \delta_n$ such that δ_n equals $A \Rightarrow B$ and for each $i = 1, \ldots, n$ we have

(i) $\delta_i \in \Sigma$, or

(ii) $\mathcal{R}$ contains a rule "from $\varphi_1, \ldots, \varphi_n$ infer ψ" such that ψ is equal to δ_i and we have $\{\varphi_1, \ldots, \varphi_n\} \subseteq \{\delta_j \mid j < i\}$.

We say that $A \Rightarrow B$ *is* $\mathcal{R}$*-provable by* Σ, denoted $\Sigma \vdash_{\mathcal{R}} A \Rightarrow B$, if there is an $\mathcal{R}$-proof of $A \Rightarrow B$ by Σ.

If $\mathcal{R}$ consists solely of (Ax), (Cut), and (Shf), we write just $\Sigma \vdash A \Rightarrow B$ and call $A \Rightarrow B$ provable by Σ. Analogously, we use the term "proof" instead of "$\mathcal{R}$-proof". In the thesis, we use the following properties of provability.

Proposition 4.3. *For every $A, B, C, D \in \mathcal{F}$, we have*

(Ref) $\vdash A \Rightarrow A$,

(Wea) $\{A \Rightarrow C\} \vdash A \cup B \Rightarrow C$,

(Acc) $\{A \Rightarrow B \cup C, C \Rightarrow D \cup E\} \vdash A \Rightarrow B \cup C \cup D$,

(Add) $\{A \Rightarrow B, A \Rightarrow C\} \vdash A \Rightarrow B \cup C$,

(Aug) $\{B \Rightarrow C\} \vdash A \cup B \Rightarrow A \cup C$,

(Pro) $\{A \Rightarrow B \cup C\} \vdash A \Rightarrow B$,

(Tra) $\{A \Rightarrow B, B \Rightarrow C\} \vdash A \Rightarrow C$.

Proof. The laws hold because our system is an extension of the Armstrong system in which the laws hold as well, see [3, 41]. □

Our inference system is sound in the usual sense:

Theorem 4.4 (soundness)**.** *If $\Sigma \vdash A \Rightarrow B$ then $\Sigma \models A \Rightarrow B$.*

Proof. The proof goes by induction on the length of a proof, considering the facts that each axiom is true in all models, (Cut) is a sound deduction rule [41], and (Shf) is sound on account of Lemma 3.14. In more detail, let $\delta_1, \ldots, \delta_n$ be a proof by Σ and let $\Sigma \models \delta_i$ for all $i < j$. Then, if δ_j results by δ_i using (Shf) for some $i < j$, then $\Sigma \models \delta_i$ yields that $M \models \delta_i$ for all $M \in \mathrm{Mod}(\Sigma)$ and thus, using Lemma 3.14, $M \models \delta_j$ for all $M \in \mathrm{Mod}(\Sigma)$, showing $\Sigma \models \delta_j$. The rest follows as in the classic case. □

In the proof of completeness, we utilize the notion of a syntactic closure which is introduced as follows.

Definition 4.5. Let Σ be a theory. For each $M \subseteq \mathcal{T}_Y$, we put

$$M_\Sigma^0 = M, \tag{4.1}$$
$$M_\Sigma^{n+1} = M_\Sigma^n \cup \bigcup \{F + i \mid E \Rightarrow F \in \Sigma \text{ and } E + i \subseteq M_\Sigma^n\}, \tag{4.2}$$
$$M_\Sigma^\omega = \textstyle\bigcup_{n=0}^\infty M_\Sigma^n. \tag{4.3}$$

and call M_Σ^ω the *syntactic closure* of M under Σ.

By the Tarski fixpoint theorem [53], the operator that maps M to M_Σ^ω defined by (4.3) is indeed a closure operator, so the term "closure" in the name syntactic closure is appropriate. The following observation shows that the term "syntactic" is also appropriate since closures are directly related to provability.

Lemma 4.6. *Let $A, B \subseteq \mathcal{T}_Y$ be finite. Then, $B \subseteq A_\Sigma^\omega$ iff $\Sigma \vdash A \Rightarrow B$.*

Proof. Suppose that $B \subseteq A_\Sigma^\omega$. Since B is finite, there is m such that $B \subseteq A_\Sigma^m$. Thus, in order to show that $\Sigma \vdash A \Rightarrow B$, it suffices to check that for every n and every finite $D \subseteq A_\Sigma^n$, we have $\Sigma \vdash A \Rightarrow D$ since then the claim readily follows for $D = B$ and $n = m$. By induction, assume the claim holds for n and all finite $D \subseteq A_\Sigma^n$. Consider $n+1$ and take a finite $D \subseteq A_\Sigma^{n+1}$. Now, consider a finite

$$D' = \{\langle E \Rightarrow F, i\rangle \mid E \Rightarrow F \in \Sigma \text{ and } E + i \subseteq A_\Sigma^n\}$$

such that

$$D \subseteq A_\Sigma^n \cup \bigcup \{F + i \mid \langle E \Rightarrow F, i\rangle \in D'\} \subseteq A_\Sigma^{n+1}.$$

Notice that since we assume D finite, such finite D' always exists. Now, by induction hypothesis, for each $\langle E \Rightarrow F, i\rangle \in D'$, we have $\Sigma \vdash A \Rightarrow E + i$ owing to $E + i \subseteq A_\Sigma^n \subseteq A_\Sigma^\omega$. Furthermore, for $E \Rightarrow F \in \Sigma$, we have $\Sigma \vdash E + i \Rightarrow F + i$ using (Shf). Thus, (Tra) gives $\Sigma \vdash A \Rightarrow F + i$ for each $\langle E \Rightarrow F, i\rangle \in D'$. In addition to that, $D \cap A_\Sigma^n \subseteq A_\Sigma^n$ and thus $\Sigma \vdash A \Rightarrow D \cap A_\Sigma^n$. Since D' is finite and $D \subseteq (D \cap A_\Sigma^n) \cup \bigcup \{F + i \mid \langle E \Rightarrow F, i\rangle \in D'\}$, $\Sigma \vdash A \Rightarrow D$ follows by finitely many applications of (Add) and (Pro). As a consequence, $\Sigma \vdash A \Rightarrow B$.

Conversely, assume that $\Sigma \vdash A \Rightarrow B$. By Theorem 4.4, $\Sigma \models A \Rightarrow B$. We show that $A_\Sigma^\omega \in \mathrm{Mod}(\Sigma)$. Take $E \Rightarrow F \in \Sigma$, $i \in \mathbb{Z}$ and let $E + i \subseteq A_\Sigma^\omega$.

Since $E+i$ is finite, there must be n such that $E+i \subseteq A^n_\Sigma$ and thus $F+i \subseteq A^{n+1}_\Sigma \subseteq A^\omega_\Sigma$, proving that $A^\omega_\Sigma \in \mathrm{Mod}(\Sigma)$. Now, $\Sigma \models A \Rightarrow B$ and $A+0 = A \subseteq A^\omega_\Sigma$ yields that $B+0 = B \subseteq A^\omega_\Sigma$. $\square$

Note that Lemma 4.6 is in fact a syntactic counterpart of Theorem 3.15. Now, using previous observations, we derive that our logic is complete:

Theorem 4.7 (completeness)**.** *$\Sigma \vdash A \Rightarrow B$ iff $\Sigma \models A \Rightarrow B$.*

Proof. If $\Sigma \nvdash A \Rightarrow B$, we prove that there is $M \in \mathrm{Mod}(\Sigma)$ such that $M \not\models A \Rightarrow B$. Indeed, we show that one can take A^ω_Σ for M. By Lemma 4.6, $\Sigma \nvdash A \Rightarrow B$ yields $B \nsubseteq A^\omega_\Sigma$. So, for $i = 0$, we have that $A+i = A \subseteq A^\omega_\Sigma$ and $B+i = B \nsubseteq A^\omega_\Sigma$, i.e., $A^\omega_\Sigma \not\models A \Rightarrow B$. In addition to that, if $E+i \subseteq A^\omega_\Sigma$ for $E \Rightarrow F \in \Sigma$ and $i \in \mathbb{Z}$, then $\Sigma \vdash A \Rightarrow E+i$ by Lemma 4.6 and so $\Sigma \vdash A \Rightarrow F+i$ using (Shf) and (Tra). Using Lemma 4.6 again, $F+i \subseteq A^\omega_\Sigma$ which proves $A^\omega_\Sigma \in \mathrm{Mod}(\Sigma)$. The rest is a consequence of Theorem 4.4. $\square$

As a corollary of the previous observations, we get the following assertion showing that both the syntactic and semantic closures coincide.

Theorem 4.8. *For every $M \subseteq \mathcal{T}_Y$, we have $[M]_\Sigma = M^\omega_\Sigma$.*

Proof. We get $[M]_\Sigma \subseteq M^\omega_\Sigma$ since $[M]_\Sigma$ is the least model of Σ containing M. Conversely, observe that for any $N \in \mathrm{Mod}(\Sigma)$ such that $M \subseteq N$, it follows that $M^\omega_\Sigma \subseteq N^\omega_\Sigma = N$. Hence, for N being $[M]_\Sigma$, we get $M^\omega_\Sigma \subseteq [M]_\Sigma$. $\square$

Remark 3. Let us stress that the notions of semantic and syntactic entailment we have considered in this thesis are different from their classic counterparts. Indeed, each temporal attribute implication can also be seen as a classic attribute implication *per se* because the sets A and B in $A \Rightarrow B$ are subsets of $\mathcal{T}_Y$. Therefore, in addition to the semantic entailment from Definition 3.13, we may consider the ordinary one which disregards the special role of time points. The same applies to the provability—the classic notion is obtained by omitting the rule (Shf). For instance, $\Sigma = \{\{x^1\} \Rightarrow \{y^2\}, \{y^5\} \Rightarrow \{z^2\}\}$ proves $\{x^4\} \Rightarrow \{y^5\}$ by (Shf) and thus $\{x^4\} \Rightarrow \{z^2\}$ by (Tra). On the other hand, Σ does not prove $\{x^4\} \Rightarrow \{z^2\}$ without (Shf).

Remark 4. (a) We can show that our system of deduction rules consisting of (Ax), (Cut), and (Shf) is non-redundant, i.e., all rules in the system are independent. Indeed, no formulas are provable by $\Sigma = \emptyset$ using only (Cut) and (Shf) and thus (Ax) is independent. Moreover, (Cut) is independent since all formulas provable by $\Sigma = \emptyset$ using only (Ax) and (Shf) are exactly all instances of (Ax). The independence of (Shf) follows by Remark 3.

(b) Let us note that the deductive system in Definition 4.1 is not minimal in terms of the number of deduction rules. Indeed, we may replace (Cut) and (Shf) by a single deduction rule

$$\frac{A \Rightarrow B + i,\ B \cup C \Rightarrow D}{A \cup (C + i) \Rightarrow D + i}\,(\mathrm{Cut}_i). \tag{4.4}$$

Indeed, observe that (Cut) is a particular case of (Cut$_i$) for $i = 0$ and (Shf) results by (Cut$_i$) and (Ax) for $A = B = \emptyset$. Conversely, $\{A \Rightarrow B+i, B \cup C \Rightarrow D\} \vdash A \cup (C + i) \Rightarrow D + i$ because using (3.6), the sequence

$$A \Rightarrow B + i, B \cup C \Rightarrow D, (B + i) \cup (C + i) \Rightarrow D + i, A \cup (C + i) \Rightarrow D + i$$

is a proof of $A \cup (C + i) \Rightarrow D + i$ using (Cut) and (Shf). As a consequence, the system consisting of (Ax), (Cut), and (Shf) is equivalent to (Ax) and (Cut$_i$).

(c) An alternative deduction system for our logic can be based on (Ref) instead of (Ax) and a single rule which is a modification of a simplification deduction rule [14]. First, it is easily seen that (Ax) and (Cut) may be equivalently replaced by the following rule and (Ref):

$$\frac{A \Rightarrow B,\ C \Rightarrow D}{A \cup (C \setminus B) \Rightarrow D}\,(\mathrm{Sim}). \tag{4.5}$$

Indeed, (Sim) is a rule derivable by (Ax) and (Cut) because the sequence

$$A \Rightarrow B, B \cup C \Rightarrow C, C \Rightarrow D, B \cup C \Rightarrow D, A \cup (C \setminus B) \Rightarrow D,$$

is a proof of $A \cup (C \setminus B) \Rightarrow D$ by $\{A \Rightarrow B, C \Rightarrow D\}$ using (Ax) and (Cut); apply the rule twice and observe that $B \cup C = B \cup (C \setminus B)$. Conversely, observe first that (Ax) is derivable by (Ref) and (Sim) because from $B \Rightarrow B$ and $A \Rightarrow A$ it follows that $B \cup (A \setminus B) \Rightarrow A$ that is, $A \cup B \Rightarrow A$. Moreover, (Cut) is derivable by (Ref) and (Sim) because the following sequence

$$C \Rightarrow C, \emptyset \Rightarrow \emptyset, C \Rightarrow \emptyset, A \Rightarrow B, B \cup C \Rightarrow D, A \cup ((B \cup C) \setminus B) \Rightarrow D, A \cup C \Rightarrow D,$$

is a proof of $A \cup C \Rightarrow D$ by $\{A \Rightarrow B, B \cup C \Rightarrow D\}$ in which we have used (Sim) three times and utilized the fact that $C \cup ((A \cup ((B \cup C) \setminus B)) \setminus \emptyset) = A \cup C$. Altogether, (Ax) and (Cut) can indeed be replaced by (Ref) and (Sim). Note that (Sim) may be perceived even more natural than (Cut) because it is applicable to any two input formulas. Note that a rule analogous to (Sim) with the inferred formula being $A \cup (C \setminus B) \Rightarrow B \cup D$ was first proposed by Darwen [16, page 140]. Now, we may consider an extension of (Sim) which involves time shifts:

$$\frac{A \Rightarrow B + i,\ C \Rightarrow D}{A \cup ((C \setminus B) + i) \Rightarrow D + i}\,(\mathrm{Sim}_i). \tag{4.6}$$

Analogously as in the case of (Cut$_i$), (Sim) is a particular case of (Sim$_i$) for $i = 0$ and (Shf) results by (Sim$_i$) and (Ref) for $A = B = \emptyset$. Therefore, the deductive system in Definition 4.1 can be replaced by (Ref) and (Sim$_i$).

As a more important corollary of the completeness, there is the following observation on compactness of the semantic entailment which is used in Section 8:

Corollary 4.9. *$\Sigma \models A \Rightarrow B$ iff there is a finite $\Sigma' \subseteq \Sigma$ such that $\Sigma' \models A \Rightarrow B$.* $\square$

We now focus on the order in which the deduction rules may be applied in proofs. We show that each proof may be transformed into a normalized proof which involves applications of deduction rules in a special order. First, we show that (Shf) commutes with the other rules. Formally, we introduce the property for a general deduction rule R as follows:

Let R be a deduction rule of the form "from $\varphi_1, \ldots, \varphi_n$ infer ψ". We say that (Shf) *commutes* with R if for any formula χ which results by ψ using (Shf) there are $\varphi'_1, \ldots, \varphi'_n$ which result by $\varphi_1, \ldots, \varphi_n$ using (Shf), respectively, such that χ is provable by $\{\varphi'_1, \ldots, \varphi'_n\}$ using R.

Lemma 4.10. (Shf) *commutes with* (Ax), (Cut)*, and* (Shf).

Proof. Clearly, (Shf) commutes with (Ax) because the result of application of (Shf) to an instance of (Ax) is again an instance of (Ax). Moreover, (Shf) commutes with itself since $(A+i)+j$ equals $A+(i+j)$ for any $A \subseteq \mathcal{T}_Y$ and $i, j \in \mathbb{Z}$. Therefore, it remains to check that (Shf) commutes with (Cut). Consider formulas $A \Rightarrow B$ and $B \cup C \Rightarrow D$ and the formula $A \cup C \Rightarrow D$

which results by (Cut) and formula $(A \cup C) + i \Rightarrow D + i$ which results by (Shf). Clearly, if we apply (Shf) to $A \Rightarrow B$ and $B \cup C \Rightarrow D$ for i, we obtain $A + i \Rightarrow B + i$ and $(B \cup C) + i \Rightarrow D + i$, respectively. The second formula equals $(B + i) \cup (C + i) \Rightarrow D + i$ and thus we may apply (Cut) to obtain $(A + i) \cup (C + i) \Rightarrow D + i$ which equals $(A \cup C) + i \Rightarrow D + i$, proving that (Shf) commutes with (Cut). $\square$

Theorem 4.11. *$\Sigma \vdash A \Rightarrow B$ iff there is a finite $\Sigma' \subseteq \Sigma^{\mathrm{PL}}$ such that $\Sigma' \vdash_{\mathcal{R}} A \Rightarrow B$ for $\mathcal{R}$ containing* (Ax) *and* (Cut).

Proof. In order to see the only-if part, assume that $\Sigma \vdash A \Rightarrow B$ which means there is a proof of $A \Rightarrow B$ by Σ. The proofs contains only finitely many formulas in Σ and thus, we may consider a finite $\Sigma'' \subseteq \Sigma$ such that $\Sigma'' \vdash A \Rightarrow B$. Moreover, the proof contains only finitely many applications of (Shf) and, using Lemma 4.10, there is a proof of $A \Rightarrow B$ by Σ'' which starts by formulas in Σ'', then continues with applications of (Shf), and terminates with formulas derived without using (Shf). Therefore, there is a finite $\Sigma' \subseteq (\Sigma'')^{\mathrm{PL}} \subseteq \Sigma^{\mathrm{PL}}$ such that $A \Rightarrow B$ is provable by Σ' using only (Ax) and (Cut). The if-part of the assertion is easy to see. $\square$

The previous observation allows us to introduce special derivation sequences which represent proofs in a normalized form in that all utilized deduction rules are applied in a particular order. The proofs are constructed using deduction rules (Ref), (Shf), (Acc), and (Pro), see Proposition 4.3.

Definition 4.12. A finite sequence of formulas $\varphi_1, \dots, \varphi_n$ is called a *normalized derivation sequence* of $A \Rightarrow B$ using formulas in Σ if the sequence

(*i*) starts with finitely many formulas in Σ;

(*ii*) continues by formulas obtained using (Shf) applied to formulas in (*i*);

(*iii*) continues by $A \Rightarrow A$;

(*iv*) continues by formulas obtained using (Acc) whose first argument is the preceding formula and the second argument is a formula in (*i*) or (*ii*);

(*v*) terminates with $A \Rightarrow B$ which results by the preceding formula by (Pro).

Normalized derivation sequences are sufficient and adequate means for determining provability of formulas:

Theorem 4.13. $\Sigma \vdash A \Rightarrow B$ *iff there is a normalized derivation sequence of* $A \Rightarrow B$ *using formulas in* Σ.

Proof. The if-part follows directly by the fact that a normalized derivation sequence of $A \Rightarrow B$ using formulas in Σ is a proof of $A \Rightarrow B$ by Σ using (Ref), (Shf), (Acc), and (Pro). Since all of them are rules derivable by (Ax), (Cut), and (Shf), see Proposition 4.3, we get $\Sigma \vdash A \Rightarrow B$.

Conversely, by Theorem 4.11 we get that $A \Rightarrow B$ is provable by a finite $\Sigma' \subseteq \Sigma^{\mathrm{PL}}$ using only (Ref) and (Cut). Therefore, we may form the (i) and (ii)-parts of the derivation sequence using the formulas in Σ' followed by $A \Rightarrow A$. Next, observe that there is a finite sequence $A_0, \ldots, A_n$ of subsets of $\mathcal{T}_Y$ such that $A_0 = A$, $A_i = A_{i-1} \cup F$ for some $E \Rightarrow F \in \Sigma'$ satisfying $E \subseteq A_{i-1}$, and $A_n \supseteq B$. In order to see that, consider (4.2) and the fact that $A \Rightarrow B$ is provable by Σ' without using (Shf). By moment's reflection, we can see that the (iv)-part of the derivation sequence is formed of formulas $A \Rightarrow A_i$ $(i = 0, \ldots, n)$, and the sequence is terminated by a single application of (Pro) to obtain $A \Rightarrow B$. $\square$

We conclude the section by showing further properties of provability. The next assertion may be viewed as a type of a deduction theorem.

Theorem 4.14. *Let* Σ *be a theory and* $A, B \subseteq \mathcal{T}_Y$ *be finite. Then the following statements are equivalent:*

(i) $\Sigma \cup \{\emptyset \Rightarrow A\} \vdash \emptyset \Rightarrow B$,

(ii) *there are* $i_1, \ldots, i_n \in \mathbb{Z}$ *such that* $\Sigma \vdash \bigcup_{m=1}^{n}(A + i_m) \Rightarrow B$.

Proof. "$(i) \Rightarrow (ii)$": Let $A_1 \Rightarrow B_1, \ldots, A_n \Rightarrow B_n$ be a proof of $\emptyset \Rightarrow B$ by $\Sigma \cup \{\emptyset \Rightarrow A\}$. For each $p = 1, \ldots, n$, we show that there are $i_1, \ldots, i_{p_n} \in \mathbb{Z}$ for which $\Sigma \vdash A_p \cup \bigcup_{m=1}^{p_n}(A + i_m) \Rightarrow B_p$. The proof goes by induction on p. Thus, take $p = 1, \ldots, n$ and assume the claim holds for all $q < p$. We distinguish the following cases:

- $A_p \Rightarrow B_p$ is an instance of (Ax). Then, we let $p_n = 1$, $i_1 = 0$, and thus $A_p \cup \bigcup_{m=1}^{p_n}(A + i_m)$ equals $A_p \cup A$, i.e., $A_p \cup A \Rightarrow B_p$ follows using (Ax).
- $A_p \Rightarrow B_p \in \Sigma$. As in the previous case, for $p_n = 1$ and $i_1 = 0$ using (Wea) we infer $A_p \cup A \Rightarrow B_p$, showing $\Sigma \vdash A_p \cup A \Rightarrow B_p$.

– Let $A_p \Rightarrow B_p$ result by $A_q \Rightarrow B_q$ and $A_r \Rightarrow B_r$ using (Cut). In this case, there is C such that $A_r = B_q \cup C$, $B_p = B_r$, and $A_p = A_q \cup C$. By induction hypothesis, there are $i_1, \ldots, i_{q_n} \in \mathbb{Z}$ and $i'_1, \ldots, i'_{q_r} \in \mathbb{Z}$ such that $\Sigma \vdash A_q \cup \bigcup_{m=1}^{q_n}(A + i_m) \Rightarrow B_q$ and $\Sigma \vdash B_q \cup C \cup \bigcup_{m=1}^{q_r}(A + i'_m) \Rightarrow B_r$. Therefore, using (Cut), $\Sigma \vdash A_q \cup \bigcup_{m=1}^{q_n}(A + i_m) \cup C \cup \bigcup_{m=1}^{q_r}(A + i'_m) \Rightarrow B_p$. Hence, for $i''_1 = i_1, \ldots, i''_{q_n} = i_{q_n}, i''_{q_{n+1}} = i'_1, \ldots, i''_{q_n+q_r} = i'_{q_r}$ it follows that $\Sigma \vdash A_q \cup C \cup \bigcup_{m=1}^{q_n+q_r}(A + i''_m) \Rightarrow B_p$, i.e., $\Sigma \vdash A_p \cup \bigcup_{m=1}^{q_n+q_r}(A + i''_m) \Rightarrow B_p$.

– Let $A_p \Rightarrow B_p$ result by $A_q \Rightarrow B_q$ using (Shf). Then, $A_p = A_q + i$ and $B_p = B_q + i$ for some $i \in \mathbb{Z}$. By induction hypotheses, there are $i_1, \ldots, i_{q_n}$ such that $\Sigma \vdash A_q \cup \bigcup_{m=1}^{q_n}(A + i_m) \Rightarrow B_q$. Using (Shf), we get $\Sigma \vdash \big(A_q \cup \bigcup_{m=1}^{q_n}(A + i_m)\big) + i \Rightarrow B_q + i$. Now, observe that $\big(A_q \cup \bigcup_{m=1}^{q_n}(A + i_m)\big) + i$ equals $(A_q + i) \cup \bigcup_{m=1}^{q_n}(A + i_m + i)$. Therefore, the claim holds for integers $i_1 + i, \ldots, i_{q_n} + i$.

As a special case for $p = n$, we get (ii) because $A_n = \emptyset$.

"$(ii) \Rightarrow (i)$": Let $\Sigma \vdash \bigcup_{m=1}^{n}(A + i_m) \Rightarrow B$ for some $i_1, \ldots, i_n \in \mathbb{Z}$. From the monotony of provability, we get that $\Sigma \cup \{\emptyset \Rightarrow A\} \vdash \bigcup_{m=1}^{n}(A + i_m) \Rightarrow B$. Moreover, for each $m = 1, \ldots, n$ we get $\Sigma \cup \{\emptyset \Rightarrow A\} \vdash \emptyset \Rightarrow A + i_m$ using (Shf). Hence, $\Sigma \cup \{\emptyset \Rightarrow A\} \vdash \emptyset \Rightarrow \bigcup_{m=1}^{n} A + i_m$ by finitely many applications of (Add) and (Tra) gives $\Sigma \cup \{\emptyset \Rightarrow A\} \vdash \emptyset \Rightarrow B$. □

Example 2. Let us observe that a direct counterpart of the classic deduction theorem does not hold in our system. For instance, we may take a theory $\Sigma = \{\emptyset \Rightarrow \{x^1\}\}$. Then, using (Shf) for $i = 1$, we easily see that $\Sigma \vdash \emptyset \Rightarrow \{x^2\}$. On the other hand, we have $\nvdash \{x^1\} \Rightarrow \{x^2\}$ and thus in general $\Sigma \cup \{\emptyset \Rightarrow A\} \vdash \emptyset \Rightarrow B$ does not imply that $\Sigma \vdash A \Rightarrow B$ which holds in the classic case.

Example 3. One of the classic laws about provability that apply to attribute implications and can be formulated in terms of attribute implications as formulas with limited expressive power compared to general propositional formulas is the principle of the proof by cases. Formally, if $\mathcal{R}$ consists only of (Ax) and (Cut), then the following are equivalent:

- $\Sigma \vdash_{\mathcal{R}} A \Rightarrow B$;
- $\Sigma \cup \{C \Rightarrow D\} \vdash_{\mathcal{R}} A \Rightarrow B$ and $\Sigma \cup \{D \Rightarrow C\} \vdash_{\mathcal{R}} A \Rightarrow B$.

This follows immediately by the fact that in this case, $\vdash_{\mathcal{R}}$ becomes the classic propositional provability. The law does not apply in our system where $\mathcal{R}$ contains the additional rule (Shf). For instance, consider the following theory

$$\Sigma = \{\{x^0\} \Rightarrow \{c^1\}, \{x^0\} \Rightarrow \{d^2\}, \{c^2\} \Rightarrow \{y^0\}, \{d^1\} \Rightarrow \{y^0\}\}.$$

Obviously, we have $\Sigma \cup \{\{c^0\} \Rightarrow \{d^0\}\} \vdash \{x^0\} \Rightarrow \{y^0\}$ using (Shf) and two applications of (Cut). Analogously, we get $\Sigma \cup \{\{d^0\} \Rightarrow \{c^0\}\} \vdash \{x^0\} \Rightarrow \{y^0\}$. On the other hand, we can show that $\Sigma \nvdash \{x^0\} \Rightarrow \{y^0\}$, i.e., the principle of the proof by cases does not hold. In order to see that $\Sigma \nvdash \{x^0\} \Rightarrow \{y^0\}$, observe that $[\{x^0\}]_\Sigma = \{y^{-1}, x^0, c^1, y^1, d^2\}$ for which $[\{x^0\}]_\Sigma \not\models \{x^0\} \Rightarrow \{y^0\}$. Thus, since our logic is sound and $[\{x^0\}]_\Sigma \in \mathrm{Mod}(\Sigma)$, we indeed have $\Sigma \nvdash \{x^0\} \Rightarrow \{y^0\}$.

Remark 5. We may say that Σ' is a *completion* of Σ if $\Sigma \subseteq \Sigma'$ and for any finite $C, D \subseteq \mathcal{T}_Y$, we have either $\Sigma' \vdash C \Rightarrow D$ or $\Sigma' \vdash D \Rightarrow C$. Let us note that analogous notions of completions play an important role in completeness proofs of various logics, cf. [31]. Namely, if a given theory does not prove a formula it is often desirable to find its completion that does not prove the formula as well. As a consequence of Example 3, we observe that this is not possible in our logic. Namely, the example shows a particular case where $\Sigma \nvdash \{x^0\} \Rightarrow \{y^0\}$ and there is no completion Σ' such that $\Sigma' \nvdash \{x^0\} \Rightarrow \{y^0\}$. Indeed, each completion Σ' proves either $\{c^0\} \Rightarrow \{d^0\}$ or $\{d^0\} \Rightarrow \{c^0\}$ and thus it also proves $\{x^0\} \Rightarrow \{y^0\}$. Nevertheless, we were able to prove Theorem 4.7 without having this property.

5 Complexity and algorithms for entailment

In this section, we show bounds on the computational complexity of deciding whether a temporal attribute implication is provable by a finite set Σ of other temporal attribute implications. Then, we focus on a subproblem which typically appears in applications. For the subproblem we provide a pseudo-polynomial time [28] decision algorithm.

We formalize the *decision problem of entailment* as a language of encodings of finitely many formulas, i.e., we put

$$L_{\mathrm{ENT}} = \{\langle \Sigma, A \Rightarrow B\rangle \mid \Sigma \text{ is a finite theory and } \Sigma \vdash A \Rightarrow B\}, \tag{5.1}$$

considering a fixed $\mathcal{T}_Y$. In order to show the lower bound of the time complexity of L_{ENT}, we utilize a reduction of decision problems [45] which involves the unbounded subset sum problem. The decision variant of the unbounded subset sum problem is formulated as follows: An instance of the problem is given by n non-negative integers $j_1, \ldots, j_n$ and a target value z; the answer to the instance is "yes" iff there are non-negative integers $c_1, \ldots, c_n$ such that

$$\textstyle\sum_{i=1}^{n} c_i j_i = z. \tag{5.2}$$

The unbounded subset sum decision problem is NP-complete, see [33, Proposition A.4.1].

Let us note that in the case of the ordinary attribute implications and functional dependencies, the problem of determining whether a given formula follows by a finite set of formulas is easy and there exist efficient linear time decision algorithms [5]. In contrast, the corresponding decision problem in our setting is hard:

Theorem 5.1 (lower bound)**.** L_{ENT} *is NP-hard.*

Proof. We prove the claim by showing that the unbounded subset sum problem is polynomial time reducible to L_{ENT}. Consider an instance of the unbounded subset sum problem given by non-negative integers $j_1, \ldots, j_n$ and z. For the integers we consider

$$\Sigma = \big\{\{y^0\} \Rightarrow \{y^{j_i}\} \mid i = 1, \ldots, n\big\} \tag{5.3}$$

and put $A = \{y^0\}$, $B = \{y^z\}$. We now prove that $\sum_{i=1}^{n} c_i j_i = z$ holds true for some non-negative integers $c_1, \ldots, c_n$ iff $\Sigma \vdash \{y^0\} \Rightarrow \{y^z\}$ by proving both implications.

In order to prove the if-part, assume that $\Sigma \vdash \{y^0\} \Rightarrow \{y^z\}$. Using Theorem 4.13, it follows there is a normalized derivation sequence $\varphi_1, \ldots, \varphi_k$ of $\{y^0\} \Rightarrow \{y^z\}$ using formulas in Σ. In the proof, we utilize a part of the sequence which results by applications of (Acc), see Definition 4.12 (*iv*). All formulas in this part of the sequence can be written as

$$\underbrace{\{y^0\} \Rightarrow A_i}_{\varphi_i}, \underbrace{\{y^0\} \Rightarrow A_{i+1}}_{\varphi_{i+1}}, \ldots, \underbrace{\{y^0\} \Rightarrow A_{k-1}}_{\varphi_{k-1}},$$

where $A_i, \ldots, A_{k-1}$ are finite subsets of $\mathcal{T}_Y$, $A_i = \{y^0\}$, and $y^z \in A_{k-1}$ because φ_k results from φ_{k-1} by (Pro), cf. Definition 4.12. By induction, we show for every A_l $(i \leq l \leq k-1)$ that the following property is satisfied:

> If $y^w \in A_l$, then there are non-negative integers $c_1, \ldots, c_n$
> such that $w = \sum_{i=1}^{n} c_i j_i$.

Notice the property is satisfied for $l = i$ since in that case we have $A_l = A_i = \{y^0\}$ and thus, we may put $c_1 = c_2 = \cdots = c_n = 0$. Assuming the claim holds for l, we prove it for $l+1$ as follows. Inspecting Definition 4.12 (*iv*), it follows that $\{y^0\} \Rightarrow A_{l+1}$ results from $\{y^0\} \Rightarrow A_l$ and $\{y^0\} + t \Rightarrow \{y^{j_m}\} + t$ using (Acc) where $t \in \mathbb{Z}$ and $1 \leq m \leq n$. As a consequence $\{y^0\} + t \subseteq A_l$ and thus, by induction hypothesis, there are non-negative integers $d_1, \ldots, d_n$ such that $t = 0 + t = \sum_{i=1}^{n} d_i j_i$. Then, $j_m + t = j_m + \sum_{i=1}^{n} d_i j_i$ and so $j_m + t = \sum_{i=1}^{n} c_i j_i$ for non-negative integers $c_1, \ldots, c_n$ defined by

$$c_i = \begin{cases} d_i + 1, & \text{if } i = m, \\ d_i, & \text{otherwise.} \end{cases}$$

Now, since we have $A_{l+1} \subseteq A_l \cup \{y^{j_m+t}\}$, the property holds for A_{l+1}. As a particular case, for $\{y^z\} \subseteq A_{k-1}$ we conclude there are non-negative integers $c_1, \ldots, c_n$ for which $\sum_{i=1}^{n} c_i j_i = z$ which concludes the first part of the proof of Theorem 5.1.

Conversely, let $\sum_{i=1}^{n} c_i j_i = z$ for some non-negative integers $c_1, \ldots, c_n$. By induction, we show that $\Sigma \vdash \{y^0\} \Rightarrow \{y^{z_k}\}$ for every $z_k = \sum_{i=1}^{k} c_i j_i$ where $k = 0, \ldots, n$. As a particular case for $k = n$, we obtain the desired fact that $\Sigma \vdash \{y^0\} \Rightarrow \{y^z\}$ because $z_n = z$.

Observe that for $k = 0$, the claim follows trivially by (Ax). Now, suppose the claim holds for $k < n$. By induction hypothesis, $\Sigma \vdash \{y^0\} \Rightarrow \{y^{z_k}\}$. Moreover, we have $\Sigma \vdash \{y^0\} \Rightarrow \{y^{j_{k+1}}\}$ because $\{y^0\} \Rightarrow \{y^{j_{k+1}}\} \in \Sigma$. Using (Shf), we also get $\Sigma \vdash \{y^0\} + j_{k+1} \Rightarrow \{y^{j_{k+1}}\} + j_{k+1}$, i.e., using (Cut), it follows that $\Sigma \vdash \{y^0\} \Rightarrow \{y^{2j_{k+1}}\}$. Repeating the last argument c_{k+1}-times, we obtain $\Sigma \vdash \{y^0\} \Rightarrow \{y^{c_{k+1}j_{k+1}}\}$. Now, using (Shf), we get $\Sigma \vdash \{y^0\} + z_k \Rightarrow \{y^{c_{k+1}j_{k+1}}\} + z_k$, i.e., $\Sigma \vdash \{y^{z_k}\} \Rightarrow \{y^{c_{k+1}j_{k+1}+z_k}\}$. Hence, $\Sigma \vdash \{y^0\} \Rightarrow \{y^{z_{k+1}}\}$ follows by (Cut) using the fact that $z_{k+1} = z_k + c_{k+1}j_{k+1}$, which finishes the proof. □

The reduction involved in Theorem 5.1 is illustrated in the following example.

Example 4. Let us show a particular instance of the unbounded subset sum problem and its reduction to L_{ENT}. We consider integers 5, 7, 11, and a target number 31 as an instance of the problem. The answer to this instance is "yes" because for numbers 4, 0, and 1, the sum $4 \cdot 5 + 0 \cdot 7 + 1 \cdot 11$ is equal to 31. The corresponding theory Σ, see the proof of Theorem 5.1, is

$$\Sigma = \{\{y^0\} \Rightarrow \{y^5\}, \{y^0\} \Rightarrow \{y^7\}, \{y^0\} \Rightarrow \{y^{11}\}\}.$$

In this case, $\{y^0\} \Rightarrow \{y^{31}\}$ is provable from Σ because we may chain four shifted instances of $\{y^0\} \Rightarrow \{y^5\}$ and a single shifted instance of $\{y^0\} \Rightarrow \{y^{11}\}$ by using (Cut). It corresponds with the sum $4 \cdot 5 + 0 \cdot 7 + 1 \cdot 11$. In a more detail, the corresponding proof of $\{y^0\} \Rightarrow \{y^{31}\}$ by Σ is the following sequence of formulas:

1. $\{y^0\} \Rightarrow \{y^5\}$ formula in Σ
2. $\{y^0\} + 5 \Rightarrow \{y^5\} + 5$ using (Shf) on 1.
3. $\{y^0\} \Rightarrow \{y^{10}\}$ using (Cut) on 1. and 2.
4. $\{y^0\} + 10 \Rightarrow \{y^5\} + 10$ using (Shf) on 1.
5. $\{y^0\} \Rightarrow \{y^{15}\}$ using (Cut) on 3. and 4.
6. $\{y^0\} + 15 \Rightarrow \{y^5\} + 15$ using (Shf) on 1.
7. $\{y^0\} \Rightarrow \{y^{20}\}$ using (Cut) on 5. and 6.
8. $\{y^0\} \Rightarrow \{y^{11}\}$ formula in Σ
9. $\{y^0\} + 20 \Rightarrow \{y^{11}\} + 20$ using (Shf) on 8.
10. $\{y^0\} \Rightarrow \{y^{31}\}$ using (Cut) on 7. and 9.

Remark 6. The entailment problem is closely related to the existence of non-negative solutions of linear Diophantine equations. Indeed, for a theory Σ which consists of formulas of the form $\{y^0\} \Rightarrow \{y^{j_i}\}$ for $i = 1, \ldots, n$, by inspecting the proof of Theorem 5.1, we can see that $\Sigma \vdash \{y^0\} \Rightarrow \{y^z\}$ iff the linear Diophantine equation $j_1x_1 + \cdots + j_nx_n = z$ has a non-negative solution.

Our observations on the upper bound of computational complexity involve additional classes of decision problems. In order to establish an upper bound, we utilize the fact that the satisfiability problem of temporal logic with "until" and "since" operators over a linear flow of time is decidable in polynomial space [49]. For the purpose of our proof, we use the linear temporal logic over $\langle \mathbb{Z}, < \rangle$ with the unary temporal operators $\circledast$ (always), $\circ_F$ (next time), and $\circ_P$ (previous time) because these operators are definable using operators "until" and "since", see [4] for details.

From now on, we consider Y (the set of attributes) as (a subset of) the set of propositional variables. Recall that formulas of the temporal logic with the above-mentioned operators are defined as follows: Each $y \in Y$ is a formula; if φ and ψ are formulas, then $\neg\varphi$, $\varphi \mathbin{\&} \psi$, $\varphi \Rightarrow \psi$, $\circledast\,\varphi$, $\circ_F\,\varphi$, and $\circ_P\,\varphi$ are formulas. In order to interpret the formulas we consider a standard structure $\mathbf{K} = \langle W, e, r \rangle$ where $W = \mathbb{Z}$, r is the genuine ordering $<$ on $\mathbb{Z}$, and e is an evaluation such that $e(w, y) \in \{0, 1\}$ for all $w \in \mathbb{Z}$ and $y \in Y$. Given $\mathbf{K}$ and $w \in \mathbb{Z}$, we interpret the formulas as usual: We put

(*i*) $\mathbf{K}, w \models y$ whenever $e(w, y) = 1$;

(*ii*) $\mathbf{K}, w \models \neg\varphi$ whenever $\mathbf{K}, w \not\models \varphi$;

(*iii*) $\mathbf{K}, w \models \varphi \mathbin{\&} \psi$ whenever $\mathbf{K}, w \models \varphi$ and $\mathbf{K}, w \models \psi$;

(*iv*) $\mathbf{K}, w \models \varphi \Rightarrow \psi$ whenever $\mathbf{K}, w \not\models \varphi$ or $\mathbf{K}, w \models \psi$;

(*v*) $\mathbf{K}, w \models \circledast\,\varphi$ whenever $\mathbf{K}, w' \models \varphi$ for all $w' \in \mathbb{Z}$;

(*vi*) $\mathbf{K}, w \models \circ_F\,\varphi$ whenever $\mathbf{K}, w' \models \varphi$ for $w' \in \mathbb{Z}$ such that $w < w'$ and there does not exist $z \in \mathbb{Z}$ such that $w < z < w'$;

(*vii*) $\mathbf{K}, w \models \circ_P\,\varphi$ whenever $\mathbf{K}, w' \models \varphi$ for $w' \in \mathbb{Z}$ such that $w' < w$ and there does not exist $z \in \mathbb{Z}$ such that $w' < z < w$.

We say that φ is *true in* $\mathbf{K}$ whenever $\mathbf{K}, w \models \varphi$ for all $w \in \mathbb{Z}$. Moreover, we say that φ is *satisfiable* whenever there is a structure $\mathbf{K}$ such that $\mathbf{K}, 0 \models \varphi$.

Moreover for each formula of the form (1.2), we consider its counterpart in the considered temporal logic

$$\circledast\Big(\big(\triangle^{i_1} y_1 \mathbin{\&} \cdots \mathbin{\&} \triangle^{i_m} y_m\big) \Rightarrow \big(\triangle^{j_1} z_1 \mathbin{\&} \cdots \mathbin{\&} \triangle^{j_n} z_n\big)\Big), \tag{5.4}$$

where $\triangle^i$ is defined as follows:

$$\triangle^i y = \begin{cases} y, & \text{if } i = 0, \\ \circ_F \triangle^{i-1} y, & \text{if } i > 0, \\ \circ_P \triangle^{i+1} y, & \text{if } i < 0. \end{cases} \tag{5.5}$$

Note that the construction of $\triangle^i y$ from y^i requires space that is linear in (the absolute value of) $i \in \mathbb{Z}$, i.e., it is exponential in the length of the encoding of i.

Theorem 5.2. *L_{ENT} is reducible in exponential space to the satisfiability problem of the linear temporal logic over $\langle \mathbb{Z}, < \rangle$ with unary temporal operators "always", "next time", and "previous time".*

Proof. First, observe that for each subset of $\mathcal{T}_Y$ we may consider a corresponding structure which makes the same formulas true—any $A \Rightarrow B$ is true in the subset of $\mathcal{T}_Y$ iff its counterpart given by (5.4) is true in the corresponding structure. Namely, for $M \subseteq \mathcal{T}_Y$, we may consider $\mathbf{K}_M = \langle W, e, r \rangle$, where $e(w, y) = 1$ if $y^w \in M$ and $e(w, y) = 0$ otherwise. Conversely, for $\mathbf{K} = \langle W, e, r \rangle$, we put $M_{\mathbf{K}} = \{y^w \mid e(w, y) = 1\}$. Now, for any $w \in W$, it is easy to see that $M \models A \Rightarrow B$ iff $\mathbf{K}_M, w \models \varphi$ where φ is the counterpart to $A \Rightarrow B$ given by (5.4). From now on, we tacitly identify attribute implications with their counterparts. Furthermore, we have $\mathbf{K}, w \models A \Rightarrow B$ iff $M_{\mathbf{K}} \models A \Rightarrow B$.

Now, for a given $\Sigma = \{A_1 \Rightarrow B_1, \ldots, A_m \Rightarrow B_m\}$ and $A \Rightarrow B$ we may consider formula $A_1 \Rightarrow B_1 \mathbin{\&} \cdots \mathbin{\&} A_m \Rightarrow B_m \mathbin{\&} \neg\big(A \Rightarrow B\big)$ whose construction requires exponential space. From the previous observation, it is obvious that the formula is satisfiable iff $\Sigma \not\models A \Rightarrow B$. □

Corollary 5.3 (upper bound)**.** *L_{ENT} belongs to EXPSPACE.*

Proof. The decision procedure reduces the input of L_{ENT} to the satisfiability problem of linear temporal logic over $\langle \mathbb{Z}, < \rangle$ with unary temporal operators "always", "next time", and "previous time" in exponential space, see Theorem 5.2. Then, the input is reduced to the satisfiability problem of the linear temporal logic over $\langle \mathbb{Z}, < \rangle$ with binary temporal operators "until" and "since" in linear space [4] which we can decide in polynomial space [49]. Altogether, the decision procedure decides L_{ENT} in exponential space. $\square$

Remark 7. Note that the results of Theorem 5.2 and Corollary 5.3 can also be interpreted so that L_{ENT} is decidable in a *pseudo-polynomial space* because we reduce an instance of L_{ENT} to an instance (of the satisfiability problem of the above-mentioned temporal logic) the length of which is bounded from above by the numeric value encoded in the original input. With respect to the new instance, the decision procedure works in polynomial space.

We now turn our attention to issues of entailment of formulas which typically appear in applications in prediction. The restriction on particular formulas allows us to improve the complexity of the entailment problem. Based on the time points present in antecedents and consequents of attribute implications, we may consider formulas that describe presence of attributes in future time points. That is, based on the presence of attributes in the past, the formulas indicate which attributes are present in future time points. Technically, such formulas can be seen as attribute implications where all time points in the antecedents are smaller (i.e., denote earlier time points) than all time points in the consequents which denote later time points. We call such formulas predictive and define the notion as follows.

Definition 5.4. A temporal attribute implication $A \Rightarrow B$ over Y is called *predictive* whenever $A, B \in \mathcal{F} \setminus \{\emptyset\}$ and for each $x^i \in A$ and $y^j \in B$, we have $i \leq j$. A theory Σ is called predictive whenever all its formulas are predictive.

Remark 8. Note that the deduction rules (Shf) and (Cut) preserve the property of being predictive. That is, if $A \Rightarrow B$ is provable by a predictive theory Σ without using (Ax), then $A \Rightarrow B$ is predictive. General instances of (Ax) are not predictive formulas.

In the next assertion, we utilize lower and upper time bounds of nonempty sets from $\mathcal{F}$: For any $M \in \mathcal{F} \setminus \{\emptyset\}$, put

$$l(M) = \min\{i \in \mathbb{Z} \mid y^i \in M \text{ for some } y \in Y\}, \tag{5.6}$$

$$u(M) = \max\{i \in \mathbb{Z} \mid y^i \in M \text{ for some } y \in Y\}. \tag{5.7}$$

Thus, $l(M)$ and $u(M)$ are the lowest and greatest time points which appear in M, respectively. Clearly, $A \Rightarrow B$ is *predictive* iff both A and B are non-empty and $u(A) \leq l(B)$.

Theorem 5.5. *Let Σ and $A \Rightarrow B$ be predictive. Then, for*

$$\Sigma_A^B = \{E + i \Rightarrow F + i \,|\, E \Rightarrow F \in \Sigma \text{ and } l(A) - l(E) \leq i \leq u(B) - l(F)\} \tag{5.8}$$

we have $\Sigma \vdash A \Rightarrow B$ iff $\Sigma_A^B \vdash_{\mathcal{R}} A \Rightarrow B$ for $\mathcal{R}$ containing (Ax) *and* (Cut).

Proof. Observe that the if-part of the claim is trivial. In order to prove the only-if part, assume that $\Sigma \vdash A \Rightarrow B$. That is, $B \subseteq [A]_\Sigma$ owing to Theorem 4.6 and Theorem 3.15. Note that $\Sigma_A^B \vdash_{\mathcal{R}} A \Rightarrow B$ for $\mathcal{R}$ containing (Ax) and (Cut) means that $A \Rightarrow B$ is provable by Σ_A^B as an ordinary attribute implication. Let A° denote the least subset of $\mathcal{T}_Y$ with the following properties:

(i) $A \subseteq A^\circ$, and

(ii) for each $E \Rightarrow F \in \Sigma_A^B$: if $E \subseteq A^\circ$ then $F \subseteq A^\circ$.

Since A° is in fact the syntactic closure of A with respect to $\mathcal{R}$, $\Sigma_A^B \vdash_{\mathcal{R}} A \Rightarrow B$ iff $B \subseteq A^\circ$. That is, in order to prove the desired claim, it suffices to show that $A^\circ \cap T = [A]_\Sigma \cap T$ for

$$T = \{y^i \in \mathcal{T}_Y \,|\, l(A) \leq i \leq u(B)\}.$$

Trivially, we get that $A^\circ \cap T \subseteq [A]_\Sigma \cap T$. In order to prove the converse inclusion, according to Theorem 4.8, it suffices to check that $A_\Sigma^n \cap T \subseteq A^\circ \cap T$ for each non-negative integer n. By induction, assume that $A_\Sigma^n \cap T \subseteq A^\circ \cap T$ and take $y^j \in (A_\Sigma^{n+1} \cap T) \setminus (A_\Sigma^n \cap T) = (A_\Sigma^{n+1} \setminus A_\Sigma^n) \cap T$. The fact $y^j \in A_\Sigma^{n+1} \setminus A_\Sigma^n$ yields there is $E \Rightarrow F \in \Sigma$ and $i \in \mathbb{Z}$ such that $E + i \subseteq A_\Sigma^n$ and $y^j \in F + i$. It can be shown that $E + i \Rightarrow F + i \in \Sigma_A^B$. Indeed, since Σ is predictive, observe that $l(E) + i = l(E + i) \geq l(A_\Sigma^n) = l(A)$ and thus $i \geq l(A) - l(E)$. Moreover, $y^j \in F + i$ yields $l(F + i) = l(F) + i \leq j$ and thus $i \leq j - l(F)$ which gives $i \leq u(B) - l(F)$ on account of $j \leq u(B)$ since $y^j \in T$. As a consequence, $E + i \Rightarrow F + i \in \Sigma_A^B$. Furthermore, $E + i \subseteq A_\Sigma^n$ and the fact that $E \Rightarrow F$ is predictive give $E + i = (E + i) \cap T \subseteq A_\Sigma^n \cap T$. By induction hypothesis, $E + i \subseteq A^\circ$ and thus $F + i \subseteq A^\circ$ by (ii). Hence, $y^j \in A^\circ$ and so $A_\Sigma^{n+1} \cap T \subseteq A^\circ \cap T$. $\square$

Let L_{PRE} be the language consisting of encodings of pairs of all finite predictive theories and predictive formulas, i.e.,

$$L_{\mathrm{PRE}} = \{\langle \Sigma, A \Rightarrow B \rangle \,|\, \Sigma \text{ is finite and } \Sigma \text{ and } A \Rightarrow B \text{ are predictive}\}. \quad (5.9)$$

Based on Theorem 5.5, we establish the following observation on the time complexity of deciding whether a predictive formula is provable by a finite predictive theory.

Theorem 5.6. $L_{\mathrm{ENT}} \cap L_{\mathrm{PRE}}$ *is decidable in a pseudo-polynomial time.*

Proof. Take a finite predictive Σ and a predictive formula $A \Rightarrow B$. The theory Σ_A^B given by (5.8) is finite. According to Theorem 5.5, the problem of deciding $\Sigma \vdash A \Rightarrow B$ is reducible to the problem of deciding whether Σ_A^B entails $A \Rightarrow B$ without using (Shf), i.e., in the sense of the entailment of ordinary attribute implications. Therefore, the problem is decidable in a time that is polynomial with respect to the size of Σ_A^B [5, 27, 41]. Now, observe that the size of (the encoding of) Σ_A^B may be bounded from above by the size of (the encoding of) Σ multiplied by

$$n = \max\{\max(0, u(B) + l(E) - l(A) - l(F) + 1) \,|\, E \Rightarrow F \in \Sigma\}, \quad (5.10)$$

i.e., the size of Σ_A^B is polynomial in the numeric value encoded in the input Σ and hence $L_{\mathrm{ENT}} \cap L_{\mathrm{PRE}}$ is decidable in a pseudo-polynomial time. □

Remark 9. (a) By considering only $L_{\mathrm{ENT}} \cap L_{\mathrm{PRE}}$, we have improved the upper bound since pseudo-polynomial time algorithms belong to EXPTIME [28] which is believed to be better than EXPSPACE. Observe that $L_{\mathrm{ENT}} \cap L_{\mathrm{PRE}}$ is also NP-hard because we can use the same reduction as in Theorem 5.1.

(b) Because of the complexity issues, in applications it is reasonable to consider temporal attribute implications with small difference between lower and upper time bounds (*maxspan* [22]) since $L_{\mathrm{ENT}} \cap L_{\mathrm{PRE}}$ is decidable in *pseudo-linear time* with respect to n given by (5.10).

An explicit procedure for deciding $L_{\mathrm{ENT}} \cap L_{\mathrm{PRE}}$ in a *pseudo-linear* time is described in Algorithm 1. It is a generalization of LinClosure [5], cf. also [41], which incorporates applicable time shifts of formulas in Σ. The algorithm accepts three arguments:

1. a finite predictive theory Σ,

Algorithm 1: PSEUDOLINCLOSURE (Σ, A, Max)

forall $E \Rightarrow F \in \Sigma$ **do**
    for i **from** $l(A) - l(E)$ **to** $Max - l(F)$ **do**
        set $count[E \Rightarrow F, i]$ **to** $|E|$;
        forall $y^j \in E$ **do**
            add $\langle E \Rightarrow F, i \rangle$ **to** $list[y^{i+j}]$;
        end
    end
end
set M **to** A;
set $update$ **to** A;
while $update \neq \emptyset$ **do**
    choose y^i **from** $update$;
    set $update$ **to** $update \setminus \{y^i\}$;
    forall $\langle E \Rightarrow F, j \rangle \in list[y^i]$ **do**
        set $count[E \Rightarrow F, j]$ **to** $count[E \Rightarrow F, j] - 1$;
        if $count[E \Rightarrow F, j] = 0$ **then**
            set new **to** $F + j \setminus M$;
            set M **to** $M \cup new$;
            set $update$ **to** $update \cup new$;
        end
    end
end
return M

2. a finite $A \subseteq \mathcal{T}_Y$, and

3. a non-negative number $Max \geq u(A)$,

and it returns a subset $M \subseteq [A]_\Sigma$ such that $M \cap T = [A]_\Sigma \cap T$ for

$$T = \{y^i \in \mathcal{T}_Y \mid l(A) \leq i \leq Max\}. \tag{5.11}$$

The soundness of the algorithm is justified by the following observation:

Theorem 5.7. *Let Σ and $A \Rightarrow B$ be predictive and let Σ be finite. Then, Algorithm 1 executed with arguments Σ, A, and $u(B)$, terminates after finitely many steps and for the returned value M we have $\Sigma \vdash A \Rightarrow B$ iff $B \subseteq M$.*

Proof. The arguments are fully analogous to those in case of the classic LinClosure, so we present here comments on issues arising only in the context of attributes annotated by time points. Technical details can be found in [5]. Notice that Algorithm 1 uses auxiliary structure *count* and *list* to store information about formulas. The structure *count* can be seen as an associative array indexed by (pointers to) formulas in Σ and integers i representing time shifts. The value of $count[E \Rightarrow F, i]$ is initially set to the number of attributes in the antecedent of $E \Rightarrow F$ (shifted by i). During the computation, $count[E \Rightarrow F, i]$ represents the number of remaining attributes in $E+i$ that have not been "updated." The structure *list* is an array indexed by attributes annotated by time points and the value of $list[y^i]$ is a list of records $\langle E \Rightarrow F, j\rangle$ representing (pointers to) formulas in Σ and their j-shifts such that y^i appears in the antecedent of $E \Rightarrow F$ shifted by j. An additional variable *update* is initialized at line 10 and maintains attributes annotated by time points that are waiting to be "updated." An update of y^i, see lines 13–21, consists in decrementing the counter of occurrences of attributes in shifted antecedents in all formulas where y^i appears. All such formulas (and their j-shifts) are found in $list[y^i]$, see line 14. If $count[E \Rightarrow F, j]$ reaches zero, see line 16, the antecedent of $E + j \Rightarrow F + j$ is already contained in M, and all new attributes in $F + j$ are prepared for update. Clearly, the procedure terminates after finitely many steps, and by Theorem 5.5, the attributes annotated by time points accumulated in M represent a subset of $[A]_\Sigma$. In addition, if $u(B) \leq Max$, then $B \subseteq M$ iff $B \subseteq [A]_\Sigma$ iff $\Sigma \vdash A \Rightarrow B$ as a consequence of our previous observations. □

Remark 10. The procedure in Algorithm 1 is called PseudoLinClosure because for given parameters, Σ, A, and Max, it computes a subset of the closure of $[A]_\Sigma$ in a linear time with respect to the numeric value of the encoding of its input arguments, i.e., its time complexity is *pseudo-linear.* Indeed, this is a consequence of the fact that each y^i where $l(A) \leq i \leq Max$ is updated during the computation at most once.

Example 5. Consider a set M given by the table in Figure 3. Since M can be regarded as transactional data over a set of items Y with a dimensional attribute $\mathfrak{d}$ the domain of which is $\mathbb{Z}$, we can utilize the algorithm proposed in [39]. The parameters for the algorithm are numbers *maxspan*, *minsupport*, and *minconfidence* for which we obtain a set Σ of all predictive $A \Rightarrow B$ where $u(A \cup B) - l(A \cup B) \leq \mathit{maxspan}$, $\mathit{minconfidence} \leq \mathit{confidence}(A \Rightarrow B)$, and $\mathit{minsupport} \leq \mathit{support}(A \Rightarrow B)$. For this particular example we consider $\mathit{maxspan} = 5$, $\mathit{minconfidence} = 1$ since we are interested in formulas true in M, and $\mathit{support} = 5$. In this setting, we obtain

$$\begin{aligned}\Sigma = \{&\{\mathtt{wm}^0\} \Rightarrow \{\mathtt{tc}^4\}, \{\mathtt{wl}^0\} \Rightarrow \{\mathtt{tc}^3\},\\ &\{\mathtt{wl}^0\} \Rightarrow \{\mathtt{wm}^1\}, \{\mathtt{wl}^0\} \Rightarrow \{\mathtt{wm}^1, \mathtt{tc}^3\},\\ &\{\mathtt{wl}^0, \mathtt{wm}^1\} \Rightarrow \{\mathtt{tc}^3\}, \{\mathtt{rn}^0, \mathtt{wl}^2\} \Rightarrow \{\mathtt{tc}^3\},\\ &\{\mathtt{rn}^0, \mathtt{rn}^3\} \Rightarrow \{\mathtt{tc}^3\}, \{\mathtt{tc}^0, \mathtt{rn}^5\} \Rightarrow \{\mathtt{tc}^5\},\\ &\{\mathtt{tc}^0, \mathtt{tc}^3, \mathtt{rn}^5\} \Rightarrow \{\mathtt{tc}^5\}, \{\mathtt{rn}^0, \mathtt{tc}^0, \mathtt{rn}^3\} \Rightarrow \{\mathtt{tc}^3\},\\ &\{\mathtt{rn}^0, \mathtt{tc}^0, \mathtt{wm}^2\} \Rightarrow \{\mathtt{tc}^3\}\}.\end{aligned}$$

Now, we may successively reduce the set Σ by removing formulas $A \Rightarrow B$ such that $\Sigma \setminus \{A \Rightarrow B\} \vdash A \Rightarrow B$, i.e., without loss of information. Since Σ is predictive we may use PseudoLinClosure and obtain the following set:

$$\begin{aligned}\Sigma' = \{&\{\mathtt{wm}^0\} \Rightarrow \{\mathtt{tc}^4\}, \{\mathtt{wl}^0\} \Rightarrow \{\mathtt{wm}^1, \mathtt{tc}^3\},\\ &\{\mathtt{rn}^0, \mathtt{rn}^3\} \Rightarrow \{\mathtt{tc}^3\}, \{\mathtt{rn}^0, \mathtt{wm}^2\} \Rightarrow \{\mathtt{tc}^3\},\\ &\{\mathtt{tc}^0, \mathtt{rn}^5\} \Rightarrow \{\mathtt{tc}^5\}\},\end{aligned}$$

i.e., the equivalent non-redundant set contains less than half of the formulas in Σ. For $\mathit{maxspan} = 5$ and $\mathit{support} = 2$, the reduction is much more significant. From the total number of $34,440$ generated formulas, PseudoLinClosure can be used to produce an equivalent set consisting of only 81 formulas.

6 Description of dependencies in data

In this section, we introduce and study properties of notions related to the containment and equivalence of sets of attributes annotated by time points with respect to possible time shifts. In addition, we show a finite representation of important subsets of $\mathcal{F} \cup \{\mathcal{T}_Y\}$ based on considering their finite quotient sets. The notions and their properties are extensively used in Section 7. Then we define complete theories as sets of formulas which semantically entail all formulas which hold in given data. We characterize complete theories in terms of their models which are closely related to fixed points of particular closure operators induced by data.

Definition 6.1. For $A, B \subseteq \mathcal{T}_Y$ we put $A \sqsubseteq B$ whenever there is $i \in \mathbb{Z}$ such that $A + i \subseteq B$; we put $A \not\sqsubseteq B$ if it is not the case that $A \sqsubseteq B$; we put $A \sqsubset B$ whenever $A \sqsubseteq B$ and $B \not\sqsubseteq A$; put $A \not\sqsubset B$ whenever $A \not\sqsubseteq B$ or $B \sqsubseteq A$; and put $A \equiv B$, whenever $A \sqsubseteq B$ and $B \sqsubseteq A$.

Remark 11. Following the definition, $A \sqsubseteq B$ can be read as "a shift of A is contained in B". It is easily seen that $\sqsubseteq$ is a quasi-order (reflexive and transitive) relation on any $\mathcal{S} \subseteq 2^{\mathcal{T}_Y}$ and, as a consequence, $\equiv$ is an equivalence relation. Trivially, if $A \subseteq B$ then $A \sqsubseteq B$ since $A + 0 = A$. As a consequence, $\emptyset \sqsubseteq B$ and $A \sqsubseteq \mathcal{T}_Y$ for any $A, B \subseteq \mathcal{T}_Y$. Furthermore, for any finite $A, B \subseteq \mathcal{T}_Y$, it follows that $A \sqsubset B$ iff there is $i \in \mathbb{Z}$ such that $A+i \subset B$. This observation cannot be extended for general (infinite) $A, B \subseteq \mathcal{T}_Y$: For $A = \{y^t \in \mathcal{T}_Y \,|\, 0 \leq t\}$, we have $A \sqsubseteq A$, i.e., $A \not\sqsubset A$ and $A + 1 \subset A$.

Proposition 6.2. *If $A \sqsubseteq B$ and $A \sqsubseteq C$ then $A \sqsubseteq (B + i) \cap C$ for some $i \in \mathbb{Z}$.*

Proof. By definition, $A \sqsubseteq B$ and $A \sqsubseteq C$ mean there are $j, k \in \mathbb{Z}$ such that $A + j \subseteq B$ and $A + k \subseteq C$. Thus, $A \subseteq B - j$ and $A \subseteq C - k$ and so $A \subseteq (B-j) \cap (C-k)$. Furthermore, we get $A+k \subseteq ((B-j) \cap (C-k))+k = (B - j + k) \cap C$. Thus, for $i = k - j$, we have $A \sqsubseteq (B + i) \cap C$. $\square$

Recall the set $\mathcal{F}$ defined by (3.2) and the values $l(M)$ and $u(M)$ defined by (5.6) and (5.7), respectively. For any $M \in \mathcal{F} \setminus \{\emptyset\}$, we put

$$\|M\| = u(M) - l(M). \tag{6.1}$$

The value $\|M\|$ is called the *time range* of M, respectively.

Remark 12. Note that the definition of time range can be more general if we define is as follows: For $M \in \mathcal{F}$

$$\|M\| = \sup\{\text{abs}(i - j) \mid y^i, z^j \in M\}$$

where $\text{abs}(i)$ denotes an absolute value of an integer i and $\sup\mathcal{S}$ is a supremum of the set $\mathcal{S}$. In this case we have included an emptyset for which we have $\|\emptyset\| = 0$. In this thesis, we present the results as they were published in [55] with one difference in notation. In [55], we denote $\mathcal{F}$ as a set of all non-empty and finite sets.

Lemma 6.3. *For any $A \in \mathcal{F} \cup \{\mathcal{T}_Y\}$ and $B \subseteq \mathcal{T}_Y$, $A \equiv B$ iff there is $i \in \mathbb{Z}$ such that $A + i = B$.*

Proof. The if-part is easy. Conversely, assume $A \equiv B$ and consider two cases: If $A \in \{\emptyset, \mathcal{T}_Y\}$ then clearly $A + i = B$ for every $i \in \mathbb{Z}$. Suppose that $A \notin \{\emptyset, \mathcal{T}_Y\}$. Since $A \equiv B$ then there are $i, j \in \mathbb{Z}$ such that $A + i \subseteq B$ and $B + j \subseteq A$. Then $B + j + i \subseteq A + i \subseteq B$. Thus, B is non-empty since $A + i \subseteq B$ and B is also finite since $B + j \subseteq A$ and $A \in \mathcal{F}$. Together with the observation $B + j + i \subseteq B$ we have $l(B) \leq l(B + j + i) = l(B) + j + i$, i.e., $0 \leq j + i$. Moreover, $u(B) + j + i = u(B + j + i) \leq u(B)$, i.e., $j + i \leq 0$. Finally, $j + i = 0$, i.e., $B = B + j + i \subseteq A + i \subseteq B$, which proves $A + i = B$. $\square$

Remark 13. The previous assertion cannot be extended to arbitrary $A \subseteq \mathcal{T}_Y$: For $A = \{y^t \in \mathcal{T}_Y \mid 0 \leq t\}$ and $B = \{y^t \in \mathcal{T}_Y \mid t = 0 \text{ or } 2 \leq t\}$, we have $A \equiv B$ and for every $i \in \mathbb{Z}$ we have $A + i \neq B$. Moreover, it is easily seen that $M, N \in \mathcal{F} \setminus \{\emptyset\}$ and $N \equiv M$ imply $\|N\| = \|M\|$.

In our representation of minimal bases, a key role will be played by subsets of $\mathcal{T}_Y$ which are in a canonical form in the following sense:

Definition 6.4. For $M \subseteq \mathcal{T}_Y$, we put

$$r(M) = \begin{cases} M - l(M), & \text{if } M \in \mathcal{F} \setminus \{\emptyset\}, \\ M, & \text{otherwise}, \end{cases} \tag{6.2}$$

and call $r(M)$ the canonical form of M. In addition, for any system $\mathcal{S} \subseteq 2^{\mathcal{T}_Y}$, we call $r(\mathcal{S}) = \{r(M) \mid M \in \mathcal{S}\}$ the canonical form of $\mathcal{S}$.

Remark 14. The letter "r" in $r(M)$ refers to "representation" because we are going to use $r(M)$ as a representation of M which has some desirable properties. Indeed, consider any $M \in \mathcal{F}$. Then, we clearly have $l(r(M)) = l(M - l(M)) = 0$ and $u(r(M)) = u(M - l(M)) = u(M) - l(M) = \|M\|$. Directly by the definiton of $\sqsubseteq$, we get $\|N\| \leq \|M\|$ provided that $N \sqsubseteq M$. Moreover, we have $N \not\sqsubseteq M$ provided that $\|N\| > \|M\|$. Since $r(M)$ is a shift of M we have $r(M) \equiv M$ for any $M \subseteq \mathcal{T}_Y$.

Lemma 6.5. *For any $\mathcal{S} \subseteq 2^{\mathcal{T}_Y}$, a map $h\colon r(\mathcal{S}) \to \mathcal{S}/_{\equiv}$ such that $h(r(M)) = [M]_{\equiv}$ for any $M \in \mathcal{S}$ is surjective.*

Proof. Notice that h is well defined since for every $N, M \subseteq \mathcal{T}_Y$ we have $r(N) \equiv N$ and $r(M) \equiv M$, i.e., if $r(N) = r(M)$ then $N \equiv M$ which is equivalent to $[N]_{\equiv} = [M]_{\equiv}$ since $\equiv$ is an equivalence. In addition, h is surjective: Take an arbitrary $\mathcal{M} \in \mathcal{S}/_{\equiv}$ and consider $r(M)$ where $M \in \mathcal{M}$. By the definition of h, we have $h(r(M)) = [M]_{\equiv} = \mathcal{M}$. $\square$

Note that the map h used in Lemma 6.5 is not injective in general. The following definition introduces the notion of finite representability that will be extensively used in this section.

Definition 6.6. A set $\mathcal{S} \subseteq \mathcal{F} \cup \{\mathcal{T}_Y\}$ is finitely representable whenever there is $t \in \mathbb{Z}$ such that for every $M \in \mathcal{S} \setminus \{\emptyset, \mathcal{T}_Y\}$ we have $\|M\| \leq t$.

Theorem 6.7. *$\mathcal{S} \subseteq \mathcal{F} \cup \{\mathcal{T}_Y\}$ is finitely representable iff $r(\mathcal{S})$ is a finite set.*

Proof. Assume that $\mathcal{S} \subseteq \mathcal{F} \cup \{\mathcal{T}_Y\}$ is finitely representable, i.e., there is $t \in \mathbb{Z}$ such that for every $M \in \mathcal{S} \setminus \{\emptyset, \mathcal{T}_Y\}$ we have $\|M\| \leq t$. For any $M \in \mathcal{S} \setminus \{\emptyset, \mathcal{T}_Y\}$, we have $l(r(M)) = 0$ and $u(r(M)) = \|M\|$. Hence, $\|r(M)\| = \|M\| \leq t$. Now, clearly $r(\mathcal{S}) \subseteq \mathcal{R}$ for $\mathcal{R} = \{N \in \mathcal{F} \setminus \{\emptyset\} \mid u(N) \leq t \text{ and } l(N) = 0\} \cup \{\emptyset, \mathcal{T}_Y\}$ which is finite because Y is finite, proving the finiteness of $r(\mathcal{S})$.

Conversely, let $\mathcal{S} \subseteq \mathcal{F} \cup \{\mathcal{T}_Y\}$ such that $r(\mathcal{S})$ is finite. If $\mathcal{S} \neq \emptyset$, then we can take t as the maximum of values $\|M\|$ for all $M \in \mathcal{S} \setminus \{\emptyset, \mathcal{T}_Y\}$. Again, utilizing the fact that $\|r(M)\| = \|M\|$ for each $M \in \mathcal{S} \setminus \{\emptyset, \mathcal{T}_Y\}$, we get that $\|M\| \leq t$ for all $M \in \mathcal{S} \setminus \{\emptyset, \mathcal{T}_Y\}$, i.e., $\mathcal{S}$ is finitely representable. $\square$

Corollary 6.8. *If $\mathcal{S} \subseteq \mathcal{F} \cup \{\mathcal{T}_Y\}$ is finitely representable then $\mathcal{S}/_{\equiv}$ is a finite set.*

Proof. Consequence of Theorem 6.7 and Lemma 6.5. $\square$

Following the motivation in the introduction, we are primarily interested in dependencies which hold not only for individual objects changing in time but for a general finite set of objects changing in time. Therefore, we formalize the input data and extend $\models$ accordingly to accomodate general sets of objects as follows. In addition to Y, we consider a finite non-empty set X of *objects* and, analogously as we have introduced $\mathcal{T}_Y$ for Y, see (3.1), we consider $\mathcal{T}_X$ for X as $\mathcal{T}_X = \{x^i \,|\, x \in X \text{ and } i \in \mathbb{Z}\}$. Then, each X-indexed system of non-empty sets in $\mathcal{F}$ is considered as input data. In other words, by *input data* we mean any $\mathcal{I}$ of the following form:

$$\mathcal{I} = \{I_x \in \mathcal{F} \setminus \{\emptyset\} \,|\, x \in X\}. \tag{6.3}$$

That is, each $I_x \in \mathcal{I}$ is a non-empty and finite subset of $\mathcal{T}_Y$. From the point of view of the interpretation of $\mathcal{I}$, each $I_x \in \mathcal{I}$ can be seen as a record of attributes (changing in time) of the object $x \in X$. Furthermore, we say that $A \Rightarrow B$ *is true in the input data* $\mathcal{I} = \{I_x \in \mathcal{F} \setminus \{\emptyset\} \,|\, x \in X\}$, written $\mathcal{I} \models A \Rightarrow B$, whenever $I_x \models A \Rightarrow B$ for all $x \in X$.

Clearly, each $\mathcal{I}$ of the form (6.3) can be represented by a $\mathbb{Z}$-indexed finite sequence of formal contexts as in Fig. 2 and, conversely, each $\mathbb{Z}$-indexed finite sequence of finite formal contexts (using fixed X and Y) can be represented by an $\mathcal{I}$ of the form (6.3).

Example 6. Let $X = \{p, q, r\}$, $Y = \{a, b, c, d\}$, and let $\mathcal{I} = \{I_p, I_q, I_r\}$ where

$$\begin{aligned} I_p &= \{a^0, b^0, a^1, b^1\}, \\ I_q &= \{a^1, c^1, a^2, b^2\}, \\ I_r &= \{c^0, a^1, c^1, a^2\}. \end{aligned}$$

Following the previous comment, the corresponding $\mathbb{Z}$-indexed sequence of contexts corresponding to this particular $\mathcal{I}$ is in fact the sequence depicted in Fig. 2. It is routine to check that $\mathcal{I} \models \{b^0\} \Rightarrow \{a^0\}$, $\mathcal{I} \models \{c^0\} \Rightarrow \{a^1\}$, and $\mathcal{I} \models \{a^0, b^0, a^1\} \Rightarrow \{b^1\}$. On the other hand, $\mathcal{I} \not\models \{a^0\} \Rightarrow \{b^0\}$ because $I_q \not\models \{a^0\} \Rightarrow \{b^0\}$ and $\mathcal{I} \not\models \{a^0, a^1\} \Rightarrow \{b^1\}$ because $I_r \not\models \{a^0, a^1\} \Rightarrow \{b^1\}$.

From now on, we assume we are given input data $\mathcal{I}$ of the form (6.3). For $A \subseteq \mathcal{T}_X$ and $B \subseteq \mathcal{T}_Y$, we put

$$A^{\uparrow_\mathcal{I}} = \bigcap\{I_x - i \,|\, x^i \in A\}, \tag{6.4}$$

$$B^{\downarrow_\mathcal{I}} = \{x^i \in \mathcal{T}_X \,|\, B \subseteq I_x - i\}. \tag{6.5}$$

If there is no danger of confusion, we write just $^{\uparrow}$ and $^{\downarrow}$ instead of $^{\uparrow_{\mathcal{I}}}$ and $^{\downarrow_{\mathcal{I}}}$. It is routine to check that $^{\uparrow}$ and $^{\downarrow}$ are a couple of operators which form an antitone Galois connection, see [18, 27]. That is, they are maps $^{\uparrow}\colon 2^{\mathcal{T}_X} \to 2^{\mathcal{T}_Y}$ and $^{\downarrow}\colon 2^{\mathcal{T}_Y} \to 2^{\mathcal{T}_X}$ such that (i) $A \subseteq A^{\uparrow\downarrow}$ and $B \subseteq B^{\downarrow\uparrow}$ for all $A \subseteq \mathcal{T}_X$ and $B \subseteq \mathcal{T}_Y$; and (ii) $A_1 \subseteq A_2$ implies $A_2^{\uparrow} \subseteq A_1^{\uparrow}$ and $B_1 \subseteq B_2$ implies $B_2^{\downarrow} \subseteq B_1^{\downarrow}$ for all $A_1, A_2 \subseteq \mathcal{T}_X$ and $B_1, B_2 \subseteq \mathcal{T}_Y$.

Remark 15. As an important consequence, the composed map $^{\downarrow\uparrow}\colon 2^{\mathcal{T}_Y} \to 2^{\mathcal{T}_Y}$ is a closure operator which plays an important role in characterization of minimal bases. Thus, we first investigate its properties. First, using (6.4) and (6.5), we get that

$$B^{\downarrow\uparrow} = \bigcap\{I_x - i \,|\, B \subseteq I_x - i \text{ and } x^i \in \mathcal{T}_X\}. \tag{6.6}$$

Furthermore, from (6.6) it follows that $B^{\downarrow\uparrow} + i = (B + i)^{\downarrow\uparrow}$ for every $i \in \mathbb{Z}$. We extend the notion of a time range introduced in (6.1) to subsets of $\mathcal{F}$. In particular, for $\emptyset \neq \mathcal{I} \subseteq \mathcal{F} \setminus \{\emptyset\}$, we put

$$\|\mathcal{I}\| = \max_{M \in \mathcal{I}} \|M\|. \tag{6.7}$$

Notice that there is always a non-empty $M \in \mathcal{F}$ such that $\|M^{\downarrow\uparrow}\| \leq \|\mathcal{I}\|$ since $\emptyset \neq \mathcal{I} \subseteq \mathcal{F} \setminus \{\emptyset\}$ and if we have a non-empty $M \in \mathcal{F}$ such that $\|M\| > \|\mathcal{I}\|$, then $M^{\downarrow} = \emptyset$ and thus $M^{\downarrow\uparrow} = \mathcal{T}_Y$. In addition, $\emptyset^{\downarrow\uparrow} = \emptyset$ because $\emptyset^{\downarrow\uparrow}$ is equal to the intersection of all sets $I_x - i$ for all $x^i \in \mathcal{T}_X$, see (6.6).

Lemma 6.9. *For $\mathcal{B} = \{A^{\uparrow} \,|\, A \subseteq \mathcal{T}_X\}$, we have $\mathcal{B} \subseteq \mathcal{F} \cup \{\mathcal{T}_Y\}$ and $\mathcal{B}$ is finitely representable.*

Proof. Take any $A \subseteq \mathcal{T}_X$ and denote the system $\{I_x - i \,|\, x^i \in A\}$ by $\mathcal{A}$. Thus $A^{\uparrow} = \bigcap \mathcal{A}$ on the account of (6.4) and $\mathcal{A} \subseteq \mathcal{F}$ since any I_x is a finite set. Then, for every $I_x - i \in \mathcal{A}$ we have $\bigcap \mathcal{A} \sqsubseteq I_x$, i.e., $\|\bigcap \mathcal{A}\| \leq \|I_x\| \leq \|\mathcal{I}\|$ provided that $\bigcap \mathcal{A} \notin \{\emptyset, \mathcal{T}_Y\}$. Altogether, $\mathcal{B} \subseteq \mathcal{F} \cup \{\mathcal{T}_Y\}$ and $\mathcal{B}$ is finitely representable. $\square$

Lemma 6.10. *The following statements are equivalent:*

(*i*) $A \not\sqsubseteq I_x$ *for every* $x \in X$,

(*ii*) $A^{\downarrow} = \emptyset$,

(*iii*) $A^{\downarrow\uparrow} = \mathcal{T}_Y$.

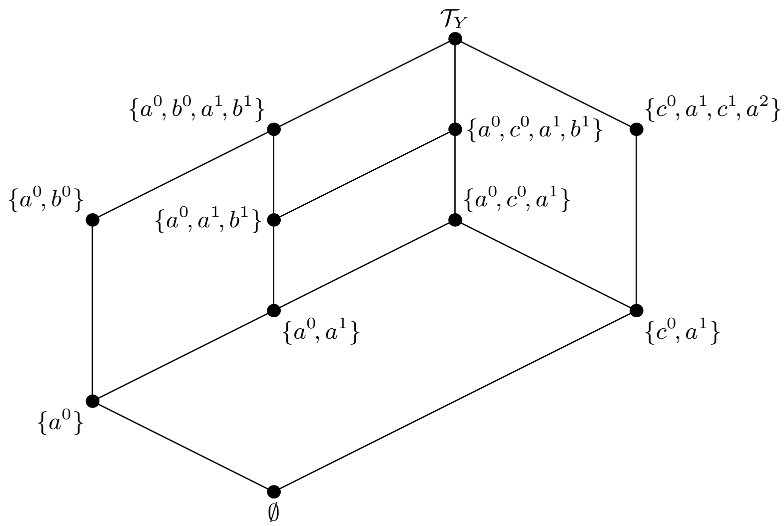

Figure 4: Lattice of canonical forms of all fixed points of the operator ${}^{\downarrow\uparrow}$ induced by $\mathcal{I}$ from Example 6.

Proof. Clearly, (*ii*) implies (*iii*) and (*i*) implies (*ii*). Assume that (*iii*) holds. Using (6.6), $A^{\downarrow\uparrow} = \bigcap \mathcal{A}$ where every $M \in \mathcal{A}$ is in the form $I_x - i$ for some $x \in X$ and $i \in \mathbb{Z}$. Then, by the assumption, we have either $\mathcal{A} = \emptyset$ or $\mathcal{A} = \{\mathcal{T}_Y\}$. Since every I_x is finite, the latter cannot be the case. Therefore, $\mathcal{A} = \emptyset$ yields $A \not\subseteq I_x$ for every $x \in X$. $\square$

The following assertion shows that the system of all fixed points of the closure operator ${}^{\downarrow\uparrow}$ given by (6.6) forms an algebraic (finitary) closure system which is in addition closed under time shifts in the following sense: If M is a fixed point of ${}^{\downarrow\uparrow}$, then so is any $M + i$.

Theorem 6.11. *$\mathcal{M}_{\mathcal{I}} = \{M^{\downarrow\uparrow} \mid M \subseteq \mathcal{T}_Y\}$ is an algebraic closure system which is closed under time shifts. Moreover, we have $\mathcal{M}_{\mathcal{I}} \subseteq \mathcal{F} \cup \{\mathcal{T}_Y\}$ and $\mathcal{M}_{\mathcal{I}}$ is finitely representable.*

Proof. Applying Lemma 6.9, $\mathcal{M}_{\mathcal{I}} \subseteq \mathcal{F} \cup \{\mathcal{T}_Y\}$ and $\mathcal{M}_{\mathcal{I}}$ is finitely representable. Since ${}^{\downarrow\uparrow}$ satisfies $M^{\downarrow\uparrow} + i = (M + i)^{\downarrow\uparrow}$ for any $M \subseteq \mathcal{T}_Y$ and $i \in \mathbb{Z}$, we get that $\mathcal{M}_{\mathcal{I}}$ is a closure system that is closed under time shifts. Therefore, it suffices to show that for every $M \subseteq \mathcal{T}_Y$ we have $M^{\downarrow\uparrow} = \bigcup \mathcal{A}$ where $\mathcal{A} = \{A^{\downarrow\uparrow} \mid A \text{ is a finite subset of } M\}$. The claim is trivial for M being finite. Assume that M is infinite. There is a finite $N \subseteq M$ such that $\|\mathcal{I}\| < \|N\|$ which implies that for every I_x we have $N \not\subseteq I_x$. Therefore, $N^{\downarrow\uparrow} = \mathcal{T}_Y$ by Lemma 6.10, i.e., $M^{\downarrow\uparrow} = N^{\downarrow\uparrow} = \mathcal{T}_Y = \bigcup \mathcal{A}$. $\square$

Example 7. Let us consider the input data from Example 6. The set $\mathcal{M}_\mathcal{I}$ of all fixed points of ${}^{\downarrow_\mathcal{I}\uparrow_\mathcal{I}}$ is obviously infinite. For instance, $\{a^0\} \in \mathcal{M}_\mathcal{I}$ and thus $\{a^i\} \in \mathcal{M}_\mathcal{I}$ for any $i \in \mathbb{Z}$. Applying Theorem 6.11, $\mathcal{M}_\mathcal{I}$ is finitely representable and thus according to Corollary 6.8, we can factorize $\mathcal{M}_\mathcal{I}$ by $\equiv$ to get a finite system of subsets of $\mathcal{M}_\mathcal{I}$. As a consequence of Theorem 6.7, the elements of such a factorization are in a one-to-one correspondence with the canonical form $r(\mathcal{M}_\mathcal{I})$ of $\mathcal{M}_\mathcal{I}$. For our particular $\mathcal{I}$, Fig. 4 depicts $r(\mathcal{M}_\mathcal{I})$ which is ordered by $\subseteq$. Therefore, each fixed point of ${}^{\downarrow_\mathcal{I}\uparrow_\mathcal{I}}$ results as a shift of a fixed point shown in the diagram in Fig. 4.

Furthermore, the closure operator ${}^{\downarrow\uparrow}$ is able to characterize the truthness of formulas in the input data. The following theorem states a similar result as Theorem 3.15.

Theorem 6.12. *For any input data $\mathcal{I}$ and formula $A \Rightarrow B$, the following conditions are equivalent:*

(i) $\mathcal{I} \models A \Rightarrow B$,

(ii) $A^\downarrow \subseteq B^\downarrow$,

(iii) $B \subseteq A^{\downarrow\uparrow}$.

Proof. By definition (i) whenever $I_x \models A \Rightarrow B$ for all $x \in X$ which holds iff $A + i \subseteq I_x$ implies $B + i \subseteq I_x$ for all $i \in \mathbb{Z}$. Moreover, the latter is equivalent to $A^\downarrow \subseteq B^\downarrow$ on the account of (6.5) which proves that (i) is equivalent to (ii). The equivalence of (ii) and (iii) follows by properties of antitone Galois connections. $\square$

We now introduce the notion of completeness of sets of temporal attribute implications with respect to given data.

Definition 6.13. Σ is called complete in $\mathcal{I}$ whenever for every $A \Rightarrow B$ we have $\mathcal{I} \models A \Rightarrow B$ iff $\Sigma \models A \Rightarrow B$.

Investigation of complete sets is interesting since they convey information about all discussed if-then dependencies which hold in given data. Theorem 3.12 show that each algebraic closure system which is closed under time shifts is a system of models of some set of temporal attribute implications. Therefore, for each $\mathcal{I}$ there is Σ such that $\mathrm{Mod}(\Sigma) = \mathcal{M}_\mathcal{I}$, i.e., $\mathrm{Mod}(\Sigma)$ is finitely representable by Theorem 6.11. As a further consequence of Theorem 3.15 and Theorem 6.12, each $\mathcal{I}$ admits Σ that is complete in $\mathcal{I}$. For instance, one can consider $\Sigma = \{A \Rightarrow B \mid \mathcal{I} \models A \Rightarrow B\}$ which is trivially complete in $\mathcal{I}$. Complete sets can be characterized in terms of their models as follows:

Theorem 6.14. *Σ is complete in $\mathcal{I}$ iff $\mathcal{M}_{\mathcal{I}} = \mathrm{Mod}(\Sigma)$.*

Proof. The assertion can be proved using the same arguments as for the classic attribute implications. Let Σ is complete in $\mathcal{I}$ and take any $A \in \mathcal{M}_{\mathcal{I}}$. Using Theorem 6.12, $\mathcal{I} \models A \Rightarrow A^{\downarrow\uparrow}$ and thus $\Sigma \models A \Rightarrow A^{\downarrow\uparrow}$ because Σ is complete in $\mathcal{I}$. As a consequence, we get $A^{\downarrow\uparrow} \subseteq [A]_{\Sigma}$ on the account of Theorem 3.15. Now, take any $A \in \mathrm{Mod}(\Sigma)$. Since, $\Sigma \models A \Rightarrow [A]_{\Sigma}$, we get $\mathcal{I} \models A \Rightarrow [A]_{\Sigma}$ and thus $[A]_{\Sigma} \subseteq A^{\downarrow\uparrow}$. Therefore, the operators ${}^{\downarrow\uparrow}$ and $[\cdots]_{\Sigma}$ have the same fixed points, i.e., $\mathcal{M}_{\mathcal{I}} = \mathrm{Mod}(\Sigma)$.

Conversely, let $\mathcal{M}_{\mathcal{I}} = \mathrm{Mod}(\Sigma)$. In that case, $[A]_{\Sigma} = A^{\downarrow\uparrow}$ for any $A \subseteq \mathcal{T}_Y$. Thus, using Theorem 3.15 and Theorem 6.12, $\Sigma \models A \Rightarrow B$ iff $B \subseteq [A]_{\Sigma}$ iff $B \subseteq A^{\downarrow\uparrow}$ iff $\mathcal{I} \models A \Rightarrow B$. □

As a consequence of Theorem 6.11 and Theorem 6.14, any Σ complete in some $\mathcal{I}$ has a finitely representable set of models. In order to characterize complete sets we utilize the following notion:

Definition 6.15. A theory Σ is finitely generated whenever $\mathrm{Mod}(\Sigma)$ is finitely representable, $\emptyset \in \mathrm{Mod}(\Sigma)$, and $\mathrm{Mod}(\Sigma) \cap (\mathcal{F} \setminus \{\emptyset\}) \neq \emptyset$.

As a direct consequence of our previous observations, we define the following property of theories that are complete in data:

Corollary 6.16. *Let $\mathcal{I}$ be in the form* (6.3)*. Then every Σ that is complete in $\mathcal{I}$ is finitely generated.* □

The name finitely generated is appropriate because by Theorem 6.7, the set $r(\mathrm{Mod}(\Sigma))$ of canonical forms of all models of Σ is finite provided that Σ is finitely generated. In addition, there is a tight connection between finitely generated theories and the input data we consider.

Theorem 6.17. *Let Σ be a finitely generated theory. Then there is $\mathcal{I}_{\Sigma}$ in the form* (6.3) *such that Σ is complete in $\mathcal{I}_{\Sigma}$.*

Proof. Put $X = r(\mathrm{Mod}(\Sigma)) \setminus \{\emptyset, \mathcal{T}_Y\}$ and let $\mathcal{I}_{\Sigma} = \{I_M \mid M \in X\}$ where $I_M = M$. From the fact that $\mathrm{Mod}(\Sigma)$ is finitely representable, we have that $r(\mathrm{Mod}(\Sigma))$ is a finite set, i.e., $\mathcal{I}_{\Sigma}$ is also finite. Furthermore, $\mathcal{I}_{\Sigma}$ is non-empty because $\mathrm{Mod}(\Sigma) \cap (\mathcal{F} \setminus \{\emptyset\}) \neq \emptyset$, i.e., $\mathcal{I}_\Sigma \subseteq \mathcal{F}$. Hence, $\mathcal{I}_\Sigma$ is in the form (6.3). Now, in order to show that Σ is complete in $\mathcal{I}_\Sigma$, it suffices to prove $\mathcal{M}_{\mathcal{I}_\Sigma} = \mathrm{Mod}(\Sigma)$, see Theorem 6.14. Let $M \in M_{\mathcal{I}_\Sigma}$. By Theorem 6.11, we have that $M = M^{\downarrow\uparrow}$. Therefore, by (6.6), we have

$$M = \bigcap\{N - i \,|\, N \in r(\mathrm{Mod}(\Sigma)) \setminus \{\emptyset, \mathcal{T}_Y\} \text{ and } M \subseteq N - i\}.$$

Hence, $M \in \mathrm{Mod}(\Sigma)$ since $\mathrm{Mod}(\Sigma)$ is a closure system closed under time shifts, i.e., $\mathcal{M}_{\mathcal{I}_\Sigma} \subseteq \mathrm{Mod}(\Sigma)$. Conversely, let $M \in \mathrm{Mod}(\Sigma)$. Obviously, $M \in \mathcal{M}_{\mathcal{I}_\Sigma}$ if $M = \emptyset$ and $M = \mathcal{T}_Y$. Suppose that $M \in \mathcal{F}$. In that case, $r(M) \in \mathcal{I}_\Sigma$ and thus $r(M) \in \mathcal{M}_{\mathcal{I}_\Sigma}$ owing to (6.6). Finally, $M \in \mathcal{M}_{\mathcal{I}_\Sigma}$ because $\mathcal{M}_{\mathcal{I}_\Sigma}$ is closed under time shifts, see Theorem 6.11. $\square$

To sum up the observations of Corollary 6.16 and Theorem 6.17, complete theories in data are exactly the theories which are finitely generated:

Corollary 6.18. *Σ is complete in some $\mathcal{I}$ iff Σ is finitely generated.* $\square$

Remark 16. It is possible to simplify the Definition 6.15 by excluding the condition $\mathrm{Mod}(\Sigma) \cap (\mathcal{F} \setminus \{\emptyset\}) \neq \emptyset$. However, the updated definition characterize different input data. Particularly, the data where we admit objects with no attributes, i.e., $\mathcal{I} = \{I_X \in \mathcal{F} \mid x \in X\}$. In this thesis, we present the definition and the result as in [55].

Our goal is to describe complete sets which are minimal in terms of their size. In the discourse, we utilize the following notion of equivalence of theories:

Definition 6.19. We put $\Sigma_1 \sqsubseteq \Sigma_2$ whenever, for every $A \Rightarrow B$, if $\Sigma_1 \models A \Rightarrow B$ then $\Sigma_2 \models A \Rightarrow B$; we put $\Sigma_1 \equiv \Sigma_2$ and say that Σ_1 and Σ_2 are equivalent whenever $\Sigma_1 \sqsubseteq \Sigma_2$ and $\Sigma_2 \sqsubseteq \Sigma_1$.

Remark 17. Obviously, if $\Sigma_1 \subseteq \Sigma_2$ then we have $\Sigma_1 \sqsubseteq \Sigma_2$. Also note that for any $A \Rightarrow B$ and $i, j \in \mathbb{Z}$, we have $\{A + i \Rightarrow B + i\} \equiv \{A + j \Rightarrow B + j\}$, see (Shf) in Proposition 4.3. By standard arguments, it can be shown that $\Sigma_1 \sqsubseteq \Sigma_2$ iff $\mathrm{Mod}(\Sigma_2) \subseteq \mathrm{Mod}(\Sigma_1)$ and, as a consequence, $\Sigma_1 \equiv \Sigma_2$ iff $\mathrm{Mod}(\Sigma_2) = \mathrm{Mod}(\Sigma_1)$.

As we shall see later in the thesis, the description of complete sets which are in addition minimal can be based on formulas whose consequents are based on closures of sets from $\mathcal{F}$ which can be infinite, namely, equal to $\mathcal{T}_Y$.

Since we consider formulas as implications between finite sets of attributes, we extend the notion of temporal attribute implications by allowing $\mathcal{T}_Y$ to appear as an antecedent or a consequent. By this, we are able to consider just a single formula $A \Rightarrow \mathcal{T}_Y$ which serves as a finite representation of an infinite theory of the form $\{A \Rightarrow B \mid B \in \mathcal{F}\}$.

Definition 6.20. An expression $A \Rightarrow B$ where $A, B \in \mathcal{F} \cup \{\mathcal{T}_Y\}$ is called an extended temporal attribute implication. We put $M \models A \Rightarrow B$ whenever, for every $i \in \mathbb{Z}$, $A + i \subseteq M$ implies $B + i \subseteq M$.

The notions of models and semantic entailment of extended temporal attribute implication are defined in much the same way as in the case of the original formulas, see Section 3. From now on, we are going to work with extended temporal attribute implications and we are not going to stress the term "extended."

Remark 18. By the previous definition and the fact that $\mathcal{T}_Y + i = \mathcal{T}_Y$ for any $i \in \mathbb{Z}$, $M \models A \Rightarrow \mathcal{T}_Y$ iff $M \models A + i \Rightarrow \mathcal{T}_Y$ for any $A, M \subseteq \mathcal{T}_Y$ and $i \in \mathbb{Z}$. Also, if $M \models A \Rightarrow \mathcal{T}_Y$ then either $A \not\sqsubseteq M$ or $M = \mathcal{T}_Y$. Hence, we have $M \models A \Rightarrow \mathcal{T}_Y$ iff $A \sqsubseteq M$ implies $M = \mathcal{T}_Y$.

Lemma 6.21. $\Sigma = \{A \Rightarrow A^{\downarrow\uparrow} \mid A \in \mathcal{F} \cup \{\mathcal{T}_Y\}\}$ *is complete in* $\mathcal{I}$.

Proof. First, observe that Σ has no infinite model except for $\mathcal{T}_Y$. Indeed, if $M \in \mathrm{Mod}(\Sigma)$ is infinite, then owing to the fact that Y is finite, there is a finite $A \subseteq M$ such that $\|A\| > \|\mathcal{I}\|$ and so $A^{\downarrow\uparrow} = \mathcal{T}_Y$, see Lemma 6.10. Therefore, the fact that $M \models A \Rightarrow A^{\downarrow\uparrow}$ yields $M = \mathcal{T}_Y$. Now, using Theorem 6.11 and Theorem 6.14, it suffices to show that for any $A \in \mathcal{F} \cup \{\mathcal{T}_Y\}$ we have $A = A^{\downarrow\uparrow}$ iff $A \in \mathrm{Mod}(\Sigma)$. For any $A \in \mathcal{F} \cup \{\mathcal{T}_Y\}$ we have $A \Rightarrow A^{\downarrow\uparrow} \in \Sigma$. Therefore, if $A \in \mathrm{Mod}(\Sigma)$ then $A = A^{\downarrow\uparrow}$. Conversely, assume that $A = A^{\downarrow\uparrow}$ and take an arbitrary $C \Rightarrow C^{\downarrow\uparrow} \in \Sigma$. The properties of ${}^{\downarrow\uparrow}$ together with the assumption $C + i \subseteq A$ yield $C^{\downarrow\uparrow} + i = (C + i)^{\downarrow\uparrow} \subseteq A^{\downarrow\uparrow}$. Finally, $C^{\downarrow\uparrow} + i \subseteq A$ using $A = A^{\downarrow\uparrow}$, showing $A \models C \Rightarrow C^{\downarrow\uparrow}$. $\square$

7 Non-redundant and minimal bases

Lemma 6.21 shows a particular set of formulas that is complete in data. The set is large and not very interesting because many of the contained formulas are entailed by other formulas. We therefore look for complete sets which are at least non-redundant in the following sense:

Definition 7.1. Σ is called non-redundant whenever for any $\Sigma' \subset \Sigma$ we have $\Sigma' \not\equiv \Sigma$. If Σ is non-redundant and complete in $\mathcal{I}$ then Σ is called a (non-redundant) base of $\mathcal{I}$.

Lemma 7.2. *Σ is non-redundant iff for any $A \Rightarrow B \in \Sigma$ we have $\Sigma \setminus \{A \Rightarrow B\} \not\models A \Rightarrow B$.*

Proof. To prove the if-part, take $A \Rightarrow B \in \Sigma$ and put $\Sigma' = \Sigma \setminus \{A \Rightarrow B\}$. Clearly, $\Sigma' \subset \Sigma$. Furthermore, $\Sigma' \not\equiv \Sigma$ since $\Sigma' \not\models A \Rightarrow B$ and $\Sigma \models A \Rightarrow B$. Since $A \Rightarrow B$ was taken arbitrarily, for each $\Sigma'' \subset \Sigma$ we have $\Sigma'' \not\equiv \Sigma$, i.e., Σ is non-redundant. Conversely, let Σ be non-redundant, take $A \Rightarrow B \in \Sigma$, and put $\Sigma' = \Sigma \setminus \{A \Rightarrow B\}$, i.e., $\Sigma' \not\equiv \Sigma$. By contradiction, let $\Sigma' \models A \Rightarrow B$. Thus, for every $M \in \mathrm{Mod}(\Sigma')$ we have $M \models A \Rightarrow B$. In addition, $M \models C \Rightarrow D$ for every $C \Rightarrow D \in \Sigma'$ since $M \in \mathrm{Mod}(\Sigma')$. Altogether we have $\mathrm{Mod}(\Sigma') \subseteq \mathrm{Mod}(\Sigma)$. Moreover, $\Sigma' \subseteq \Sigma$ yeilds $\mathrm{Mod}(\Sigma) \subseteq \mathrm{Mod}(\Sigma')$, i.e., $\mathrm{Mod}(\Sigma') = \mathrm{Mod}(\Sigma)$. Thus, $\Sigma' \equiv \Sigma$ which is a contradiction. $\square$

Remark 19. Clearly, a non-redundant theory Σ does not contain formulas $\emptyset \Rightarrow \emptyset$ and $\mathcal{T}_Y \Rightarrow B$ for any $B \in \mathcal{F} \cup \{\mathcal{T}_Y\}$. In addition, if Σ is finitely generated, it does not contain formulas $\emptyset \Rightarrow B$ for any $B \in \mathcal{F} \cup \{\mathcal{T}_Y\}$ since $\emptyset$ is a model of Σ. Hence, formulas of non-redundant and finitely generated theories have antecedents from $\mathcal{F}$.

In the rest of the section, we express particular non-redundant sets of formulas which are given by special systems that are subsets of $\mathcal{F}$. The systems are introduced in the following definition and generalize the classic notion of pseudo-intents proposed in [30].

Definition 7.3. A set $P \in \mathcal{F}$ is a pseudo-intent of $\mathcal{I}$ if $P \neq P^{\downarrow\uparrow}$ and for any pseudo-intent Q of $\mathcal{I}$ such that $Q \subset P$ we have $Q^{\downarrow\uparrow} \subseteq P$. The set of all pseudo-intents of $\mathcal{I}$ is denoted by $\mathcal{P}_{\mathcal{I}}$.

Remark 20. If $\mathcal{I}$ is clear from the context, we write $\mathcal{P}$ to denote $\mathcal{P}_{\mathcal{I}}$. Note that under the conditions we require for $\mathcal{I}$, $\mathcal{P}_{\mathcal{I}}$ always exists and is given

#	$r(P)$	$\|P\|$	$r(P)^{\downarrow\uparrow}$
1	$\{b^0\}$	0	$\{a^0, b^0\}$,
2	$\{c^0\}$	0	$\{c^0, a^1\}$,
3	$\{a^0, b^0, a^1\}$	1	$\{a^0, b^0, a^1, b^1\}$,
4	$\{c^0, a^1, b^1\}$	1	$\{a^0, c^0, a^1, b^1\}$,
5	$\{a^0, b^0, c^0, a^1, b^1\}$	1	$\mathcal{T}_Y$,
6	$\{a^0, a^2\}$	2	$\mathcal{T}_Y$,
7	$\{c^0, a^1, a^2\}$	2	$\{c^0, a^1, c^1, a^2\}$,
8	$\{c^0, a^1, c^1, a^2, b^2\}$	2	$\mathcal{T}_Y$

Figure 5: Example of canonical forms of all pseudo-intents of the input data from Example 6 limited to pseudo-intents with time range up to 2.

uniquely which can be verified by Noetherian induction [62] using the fact that $\subset$ is a well-founded relation on $\mathcal{S} = \{P \in \mathcal{F} \,|\, P \neq P^{\downarrow\uparrow}\}$. Let us note that the notion of a pseudo-intent in our temporal setting behaves differently as in the classic case. For instance, unlike the classic case, $\mathcal{P}_{\mathcal{I}}$ is infinite and $\emptyset$, which may be a pseudo-intent in the classic case, is never a pseudo-intent in our setting because $\emptyset^{\downarrow\uparrow} = \emptyset$ as we have already observed. Moreover, $\mathcal{T}_Y$ is not a pseudo-intent because $\mathcal{T}_Y$ is always equal to its closure.

Lemma 7.4. *$\mathcal{P}_{\mathcal{I}}$ is closed under time shifts.*

Proof. Recall $\sqsubset$ from Definition 6.1 and Remark 11. It is easily seen that $\sqsubset$ restricted to $\mathcal{P}_{\mathcal{I}} \times \mathcal{P}_{\mathcal{I}}$ is a well-founded relation. By Noetherian induction, we prove that any $P \in \mathcal{P}_{\mathcal{I}}$ satisfies the following property: "For any $i \in \mathbb{Z}$, we have $P + i \in \mathcal{P}_{\mathcal{I}}$." Assume that the property holds for any $Q \in \mathcal{P}_{\mathcal{I}}$ such that $Q \sqsubset P$. Take any $i \in \mathbb{Z}$. It suffices to show that $P + i \in \mathcal{P}_{\mathcal{I}}$. Following Definition 7.3, take $Q \in \mathcal{P}_{\mathcal{I}}$ such that $Q \subset P + i$. The last strict inclusion means $Q \sqsubset P$ and so $Q - i \in \mathcal{P}_{\mathcal{I}}$. Furthermore, $Q - i \subset P$ yields $(Q - i)^{\downarrow\uparrow} \subseteq P$ owing to Definition 7.3. Using Theorem 6.11, $Q^{\downarrow\uparrow} - i \subseteq P$, i.e., $Q^{\downarrow\uparrow} \subseteq P + i$. Hence, $P + i \in \mathcal{P}_{\mathcal{I}}$. $\square$

Example 8. The fact that $\mathcal{P}_{\mathcal{I}}$ is closed under time shift means, among other things, that $\mathcal{P}_{\mathcal{I}}$ is infinite. However, if we restrict ourselves to the pseudo-intents in the canonical form and, in addition, we limit ourselves only to those with time range within the time range of the input data, there are only finitely many of such pseudo-intents. Following our preliminary discussion in the introduction, such pseudo-intents turn out to be the most interesting ones. Going back to the data in Example 6, see also Figure 2, there are exactly eight pseudo-intents with these properties. They are listed together with their time ranges and closures in Figure 5.

The previous observation allows us to give an equivalent description of the sets in $\mathcal{P}_{\mathcal{I}}$:

Corollary 7.5. *For any $P \in \mathcal{F}$, we have that P is a pseudo-intent of $\mathcal{I}$ iff for any $Q \in r(\mathcal{P}_{\mathcal{I}})$ and $i \in \mathbb{Z}$ such that $Q + i \subset P$ we have $Q^{\downarrow\uparrow} + i \subseteq P$.* □

For any system $\mathcal{S} \subseteq \mathcal{F}$ and $\mathcal{I}$ of the form (6.3), we put

$$\Sigma_{\mathcal{S}} = \big\{M \Rightarrow M^{\downarrow\uparrow} \,|\, M \in r(\mathcal{S})\big\}. \tag{7.1}$$

By Theorem 6.11, the theory $\Sigma_{\mathcal{S}}$ given by (7.1) is well defined because $M^{\downarrow\uparrow}$ is either finite or equal to $\mathcal{T}_Y$. The following observations show that systems of pseudo-intents define non-redundant bases of the form (7.1). Note that for brevity, in the rest of the section we denote $\mathcal{P}_{\mathcal{I}}$ just by $\mathcal{P}$.

Lemma 7.6. *For every $P \in \mathcal{P}$ we have $\Sigma_{\mathcal{P}} \models P \Rightarrow P^{\downarrow\uparrow}$.*

Proof. Take an arbitrary $P \in \mathcal{P}$. Then there is $Q \Rightarrow Q^{\downarrow\uparrow} \in \Sigma_{\mathcal{P}}$ such that $Q = r(P)$. Such Q indeed exists since $\mathcal{P}$ is closed under time shifts. In addition, $Q, P \in \mathcal{F}$, i.e., $Q = P - l(P)$ and so $Q^{\downarrow\uparrow} = (P - l(P))^{\downarrow\uparrow} = P^{\downarrow\uparrow} - l(P)$. Therefore, using Proposition 4.3, $\Sigma_{\mathcal{P}} \models P \Rightarrow P^{\downarrow\uparrow}$ since $\Sigma_{\mathcal{P}} \models Q \Rightarrow Q^{\downarrow\uparrow}$ holds trivially. □

Theorem 7.7. *$\Sigma_{\mathcal{P}}$ is a non-redundant base of $\mathcal{I}$.*

Proof. Utilizing Theorem 6.14, we prove that $\Sigma_{\mathcal{P}}$ is complete in $\mathcal{I}$ by showing $\mathcal{M}_{\mathcal{I}} = \mathrm{Mod}(\Sigma_{\mathcal{P}})$. Take an arbitrary $A \in \mathcal{M}_{\mathcal{I}}$, i.e., $A = A^{\downarrow\uparrow}$. Moreover, take $P \Rightarrow P^{\downarrow\uparrow} \in \Sigma_{\mathcal{P}}$ and $i \in \mathbb{Z}$ such that $P + i \subseteq A$. Then using the properties of the operator ${}^{\downarrow\uparrow}$ we have $P^{\downarrow\uparrow} + i = (P + i)^{\downarrow\uparrow} \subseteq A^{\downarrow\uparrow}$. Thus $P^{\downarrow\uparrow} + i \subseteq A^{\downarrow\uparrow} = A$ using the assumption. Conversely, assume that $A \in \mathrm{Mod}(\Sigma_{\mathcal{P}})$. Then we have $A \models P \Rightarrow P^{\downarrow\uparrow}$ for every $P \in r(\mathcal{P})$, i.e., for every $P \in r(\mathcal{P})$ and $i \in \mathbb{Z}$ such that $P + i \subseteq A$ we have $P^{\downarrow\uparrow} + i \subseteq A$. Moreover, assume that $A \neq A^{\downarrow\uparrow}$. Then $A \in \mathcal{P}$ by Corollary 7.5. Therefore, $A \models A \Rightarrow A^{\downarrow\uparrow}$ owing to (7.1) and Proposition 4.3, i.e., $A^{\downarrow\uparrow} \subseteq A$ since we have $A + 0 \subseteq A$. In addition, the extensivity of ${}^{\downarrow\uparrow}$ yields $A^{\downarrow\uparrow} = A$ which is a contradiction, i.e., $A \in \mathcal{M}_{\mathcal{I}}$.

In order to prove the non-redundancy, we take an arbitrary $\Sigma' \subset \Sigma_{\mathcal{P}}$ and show $\Sigma' \not\equiv \Sigma_{\mathcal{P}}$, i.e., $\mathrm{Mod}(\Sigma_{\mathcal{P}}) \neq \mathrm{Mod}(\Sigma')$. Since $\Sigma' \subset \Sigma_{\mathcal{P}}$, there is

$P \Rightarrow P^{\downarrow\uparrow} \in \Sigma_{\mathcal{P}}$ such that $P \Rightarrow P^{\downarrow\uparrow} \notin \Sigma'$. Take $Q \Rightarrow Q^{\downarrow\uparrow} \in \Sigma'$ and $i \in \mathbb{Z}$ such that $Q + i \subseteq P$. Observe that by the definition of $\Sigma_{\mathcal{P}}$ it cannot be the case that $Q + i = P$ for any $i \in \mathbb{Z}$ since $Q \neq P$ and $Q, P \in r(\mathcal{P})$, i.e., $l(Q) = l(P) = 0$. Therefore, by Corollary 7.5, we have $Q^{\downarrow\uparrow} + i \subseteq P$, i.e., $P \in \mathrm{Mod}(\Sigma')$. On the other hand, we have $\Sigma_{\mathcal{P}} \models P \Rightarrow P^{\downarrow\uparrow}$ and $P \neq P^{\downarrow\uparrow}$. Therefore, $P \notin \mathrm{Mod}(\Sigma_{\mathcal{P}})$ since $P + 0 \subseteq P$ and $P^{\downarrow\uparrow} + 0 \nsubseteq P$. $\square$

Example 9. Again, consider the data in Example 6 that are also depicted in Figure 2. A non-redundant base of the data given by its system of pseudo-intents that is described in Theorem 7.7 can be written as

$$
\begin{aligned}
\Sigma_{\mathcal{P}} = \{&\{b^0\} \Rightarrow \{a^0, b^0\},\\
&\{c^0\} \Rightarrow \{c^0, a^1\},\\
&\{a^0, b^0, a^1\} \Rightarrow \{a^0, b^0, a^1, b^1\},\\
&\{c^0, a^1, b^1\} \Rightarrow \{a^0, c^0, a^1, b^1\},\\
&\{a^0, b^0, c^0, a^1, b^1\} \Rightarrow \mathcal{T}_Y,\\
&\{a^0, a^2\} \Rightarrow \mathcal{T}_Y,\\
&\{c^0, a^1, a^2\} \Rightarrow \{c^0, a^1, c^1, a^2\},\\
&\{c^0, a^1, c^1, a^2, b^2\} \Rightarrow \mathcal{T}_Y, \ldots\}.
\end{aligned}
$$

The ellipsis in the previous expression indicates that $\Sigma_{\mathcal{P}}$ consists of other formulas of the form $P \Rightarrow P^{\downarrow\uparrow}$ where $P \in r(\mathcal{P})$ and $\|P\| > 2$, i.e., formulas whose antecedents fall outside the time range of the input data. As we shall see in a moment, there are infinitely many pseudo-intents with this property, i.e., the non-redundant base $\Sigma_{\mathcal{P}_{\mathcal{I}}}$ is always infinite for any $\mathcal{I}$ of the form (6.3).

In the rest of this section, we prove that the non-redundant theory $\Sigma_{\mathcal{P}}$ is satisfying a stronger condition of minimality. Technically, the minimality is defined in a different way than in the classical setting as we shall see in a moment. The main reason behind this is that finitely generated theories are always infinite which is a direct consequence of the following observation.

Theorem 7.8. *Let Σ be a finitely generated theory. Then,*

$$\mathcal{S} = \{A \subseteq \mathcal{T}_Y \mid A \Rightarrow B \in \Sigma\}$$

is not finitely representable.

Proof. Note that $\mathcal{S}$ is the system of all antecedents of all formulas in Σ. By contradiction, assume we have $t \in \mathbb{Z}$ such that $\|A\| \leq t$ for every $A \in \mathcal{S}$. Since Σ is finitely generated, there is at least one non-empty and finite $M \in \mathrm{Mod}(\Sigma)$. Therefore, we can consider an infinite set

$$C = \bigcup \bigl\{ M + (\|M\| + t + 1) \cdot i \,|\, i \in \mathbb{Z} \bigr\}$$

It can be shown that C is a model of Σ. Indeed, observe that for each $E \Rightarrow F \in \Sigma$ such that $E + i \subseteq C$ for some $i \in \mathbb{Z}$ we have $E + i \sqsubseteq M$ because $\|E\| \leq t$. Thus, there is $j \in \mathbb{Z}$ such that $(E + i) + j \subseteq M$. Since $M \models E + i \Rightarrow F + i$, we get that $(F + i) + j \subseteq M$ and so $F + i \subseteq C$. Hence, C is an infinite model of Σ such that $C \neq \mathcal{T}_Y$ which contradicts the fact that $\mathrm{Mod}(\Sigma)$ is finitely representable. □

Hence, Theorem 7.8 yields that finitely generated theories are not finite. As a consequence, no $\mathcal{I}$ of the form (6.3) admits a finite non-redundant base. This is in contrast with the classic non-redundant bases of finite formal contexts which are always finite [27, 30]. Also note that taking into account the fact that $\mathcal{F}$ is countable, we have that any theory is at most countable, i.e., finitely generated theories are countable, i.e., of the same size. Therefore, it would be worthless to define minimal theories in our setting the same way as in the classic case [41] as theories with the least size among all equivalent theories since all finitely generated theories would be minimal. Instead, we introduce the following notion of minimality:

Definition 7.9. A finitely generated theory Σ is minimal whenever for each $\Sigma' \subseteq \Sigma$ and Γ' such that $\Sigma' \equiv \Gamma'$ we have $|\Sigma'| \leq |\Gamma'|$.

Remark 21. Note that if Σ in Definition 7.9 were finite then the condition of minimality would coincide with the classic one but as we have seen in Theorem 7.8, the considered theories are always infinite. In such setting, Definition 7.9 introduces a stronger notion of minimality of Σ than just requiring that $|\Sigma| \leq |\Gamma|$ for any Γ satisfying $\Sigma \equiv \Gamma$.

Before we prove the minimality of $\Sigma_{\mathcal{P}}$ where $\mathcal{P}$ is the system of pseudo-intents of $\mathcal{I}$, we show properties of minimality that will be further used.

Proposition 7.10. *Let Σ be a theory and Σ' its subset. Then for each Γ' equivalent to Σ' we have that $\Sigma \equiv (\Sigma \setminus \Sigma') \cup \Gamma'$.*

Proof. The claim follows from the fact that Σ and $(\Sigma \setminus \Sigma') \cup \Gamma'$ have the same models which is easy to see. Indeed, let $M \in \mathrm{Mod}(\Sigma)$. In that case, we have $M \in \mathrm{Mod}(\Sigma')$. Since for any $A \Rightarrow B \in \Gamma'$, we have $\Sigma' \models A \Rightarrow B$, it follows that $M \models A \Rightarrow B$, i.e., $M \in \mathrm{Mod}(\Gamma')$. The rest is clear. Conversely, let $M \in \mathrm{Mod}((\Sigma \setminus \Sigma') \cup \Gamma')$. For any $A \Rightarrow B \in \Sigma'$, we have $\Gamma' \models A \Rightarrow B$ and thus $M \models A \Rightarrow B$, i.e., $M \in \mathrm{Mod}(\Sigma')$ from which we get $M \in \mathrm{Mod}(\Sigma)$. $\square$

Lemma 7.11. *A finitely generated theory Σ is minimal iff for each $\Sigma' \subseteq \Sigma$ and Γ' such that $(\Sigma \setminus \Sigma') \cup \Gamma' \equiv \Sigma$ we have $|\Sigma'| \leq |\Gamma'|$.*

Proof. First assume that Σ is minimal and take $\Sigma' \subseteq \Sigma$ and Γ' such that $\Sigma \equiv (\Sigma \setminus \Sigma') \cup \Gamma'$. Obviously, $|\Sigma'| = |\Sigma \setminus (\Sigma \setminus \Sigma')|$ because $\Sigma \setminus \Sigma' \subseteq \Sigma$. Now, applying the minimality of Σ for $\Sigma \subseteq \Sigma$ we get $|\Sigma| \leq |(\Sigma \setminus \Sigma') \cup \Gamma'|$ and together with the previous observation, it follows that

$$|\Sigma'| = |\Sigma \setminus (\Sigma \setminus \Sigma')| \leq |((\Sigma \setminus \Sigma') \cup \Gamma') \setminus (\Sigma \setminus \Sigma')|.$$

Utilizing $\Sigma \setminus \Sigma' \subseteq (\Sigma \setminus \Sigma') \cup \Gamma'$, the previous inequality gives $|\Sigma'| \leq |\Gamma'|$.

Conversely, the assertion follows by Proposition 7.10. Indeed, assume that for each $\Sigma' \subseteq \Sigma$ and Γ' such that $\Sigma \equiv (\Sigma \setminus \Sigma') \cup \Gamma'$, we have $|\Sigma'| \leq |\Gamma'|$. Take $\Sigma' \subseteq \Sigma$ and Γ' such that $\Sigma' \equiv \Gamma'$, i.e., $\Sigma \equiv (\Sigma \setminus \Sigma') \cup \Gamma'$ by Proposition 7.10, i.e., $|\Sigma'| \leq |\Gamma'|$ using the assumption. $\square$

Put in words, the observation in Lemma 7.11 says that no subset of a minimal theory can be equivalently replaced by a smaller theory. Therefore, minimal theories are non-redundant according to Definition 7.1, see Theorem 7.2. We now elaborate the proof of the minimality of $\Sigma_{\mathcal{P}}$. Note that in the rest of the section, $\mathcal{P}$ denotes the system of all pseudo-intents of $\mathcal{I}$.

Lemma 7.12. *Let $P \in \mathcal{P}$ and $A \subseteq \mathcal{T}_Y$ be such that $P \nsubseteq A^{\downarrow\uparrow}$. Then $A^{\downarrow\uparrow} \cap P = (A^{\downarrow\uparrow} \cap P)^{\downarrow\uparrow}$.*

Proof. Assume that $P \nsubseteq A^{\downarrow\uparrow}$ which means that $P \nsubseteq A^{\downarrow\uparrow} \cap P$. It suffices to show that $A^{\downarrow\uparrow} \cap P$ is a model of $\Sigma_{\mathcal{P}}$ because the rest is a consequence of Theorem 7.7 and Theorem 6.14. Take $Q \Rightarrow Q^{\downarrow\uparrow} \in \Sigma_{\mathcal{P}}$ and $i \in \mathbb{Z}$ such that $Q + i \subseteq A^{\downarrow\uparrow} \cap P$. Then $Q + i \subset P$ since $P \nsubseteq A^{\downarrow\uparrow} \cap P$. Therefore, $Q^{\downarrow\uparrow} + i \subseteq P$. We also have $Q + i \subseteq A^{\downarrow\uparrow}$ thus $Q^{\downarrow\uparrow} + i = (Q + i)^{\downarrow\uparrow} \subseteq (A^{\downarrow\uparrow})^{\downarrow\uparrow} = A^{\downarrow\uparrow}$ by the properties of ${}^{\downarrow\uparrow}$. As a consequence, $Q^{\downarrow\uparrow} + i \subseteq A^{\downarrow\uparrow} \cap P$. $\square$

Lemma 7.13. *Let Σ be equivalent to $\Sigma_{\mathcal{P}}$. Then for each $P \in \mathcal{P}$ there is $A \Rightarrow B \in \Sigma$ such that $A^{\downarrow\uparrow} \equiv P^{\downarrow\uparrow}$ and $P \not\models A \Rightarrow B$.*

Proof. Let $P \in \mathcal{P}_{\mathcal{I}}$ which means that $P \neq P^{\downarrow\uparrow}$. Clearly, $P \not\models P \Rightarrow P^{\downarrow\uparrow}$ and thus $P \notin \mathrm{Mod}(\Sigma_{\mathcal{P}}) = \mathrm{Mod}(\Sigma)$. Therefore, there is $A \Rightarrow B \in \Sigma$ and $i \in \mathbb{Z}$ such that $A + i \subseteq P$ and $B + i \nsubseteq P$, meaning that $P \not\models A \Rightarrow B$. Now, we prove $A^{\downarrow\uparrow} + i = P^{\downarrow\uparrow}$. Using $A + i \subseteq P$, it follows that $A^{\downarrow\uparrow} + i \subseteq P^{\downarrow\uparrow}$. For the converse inclusion, it suffices to show that $P \subseteq A^{\downarrow\uparrow} + i$. By contradiction, assume $P \nsubseteq A^{\downarrow\uparrow} + i$. Since $A \Rightarrow B \in \Sigma$ and Σ is complete in $\mathcal{I}$, we have $B \subseteq A^{\downarrow\uparrow}$ and so $B + i \subseteq A^{\downarrow\uparrow} + i$. Therefore, $A^{\downarrow\uparrow} + i \nsubseteq P$ since $B + i \nsubseteq P$. Then, it follows that $(A^{\downarrow\uparrow} + i) \cap P \subset A^{\downarrow\uparrow} + i$. From $P \nsubseteq (A + i)^{\downarrow\uparrow}$ and Lemma 7.12, we have $(A + i)^{\downarrow\uparrow} \cap P = ((A + i)^{\downarrow\uparrow} \cap P)^{\downarrow\uparrow}$. Since $A + i \subseteq P$ and $A + i \subseteq (A + i)^{\downarrow\uparrow}$ we have $A + i \subseteq (A + i)^{\downarrow\uparrow} \cap P$. Altogether $A^{\downarrow\uparrow} + i = (A+i)^{\downarrow\uparrow} \subseteq ((A+i)^{\downarrow\uparrow} \cap P)^{\downarrow\uparrow} = (A^{\downarrow\uparrow} + i) \cap P \subset A^{\downarrow\uparrow} + i$, a contradiction. Thus $P \subseteq A^{\downarrow\uparrow} + i$, i.e., $P^{\downarrow\uparrow} \subseteq A^{\downarrow\uparrow} + i$. Altogether $A^{\downarrow\uparrow} + i = P^{\downarrow\uparrow}$, i.e., $A^{\downarrow\uparrow} \equiv P^{\downarrow\uparrow}$ by Lemma 6.3 since $A^{\downarrow\uparrow} \subseteq \mathcal{F} \cup \{\mathcal{T}_Y\}$ by Theorem 6.11. $\square$

Lemma 7.14. *If $P_1 \not\sqsubseteq P_2$ and $P_2 \not\sqsubseteq P_1$ for $P_1, P_2 \in r(\mathcal{P})$, then $((P_1 + k) \cap P_2)^{\downarrow\uparrow} = (P_1 + k) \cap P_2$ for any $k \in \mathbb{Z}$.*

Proof. Take $\Sigma_1 = \Sigma_{\mathcal{P}} \setminus \{P_1 \Rightarrow P_1^{\downarrow\uparrow}\}$ and $\Sigma_2 = \Sigma_{\mathcal{P}} \setminus \{P_2 \Rightarrow P_2^{\downarrow\uparrow}\}$. Under this notation, $P_1 \in \mathrm{Mod}(\Sigma_1) \subseteq \mathrm{Mod}(\Sigma_1 \cap \Sigma_2)$ and $P_2 \in \mathrm{Mod}(\Sigma_2) \subseteq \mathrm{Mod}(\Sigma_1 \cap \Sigma_2)$. Moreover, $\mathrm{Mod}(\Sigma_1 \cap \Sigma_2)$ is closed under time shifts, i.e., $P_1 + k \in \mathrm{Mod}(\Sigma_1 \cap \Sigma_2)$ for any $k \in \mathbb{Z}$. Therefore, $(P_1 + k) \cap P_2 \in \mathrm{Mod}(\Sigma_1 \cap \Sigma_2)$. In addition, from $P_1 \not\sqsubseteq P_2$ it follows that $P_1 \not\sqsubseteq P_2 \cap (P_1 + k)$. As a consequence, $P_2 \cap (P_1 + k) \models P_1 \Rightarrow P_1^{\downarrow\uparrow}$ and so $(P_1 + k) \cap P_2 \in \mathrm{Mod}(\Sigma_2)$. In much the same way, $P_2 \not\sqsubseteq P_1$ gives $(P_1 + k) \cap P_2 \in \mathrm{Mod}(\Sigma_1)$. Then $(P_1 + k) \cap P_2 \in \mathrm{Mod}(\Sigma_1 \cup \Sigma_2) = \mathrm{Mod}(\Sigma_{\mathcal{P}})$. Hence, $(P_1 + k) \cap P_2 = ((P_1 + k) \cap P_2)^{\downarrow\uparrow}$ owing to the fact that $\Sigma_{\mathcal{P}}$ is complete in $\mathcal{I}$, see Theorem 6.14. $\square$

Theorem 7.15. *$\Sigma_{\mathcal{P}}$ is minimal.*

Proof. By contradiction, assume that $\Sigma_{\mathcal{P}}$ is not minimal. Therefore, by Lemma 7.11, there is $\Sigma' \subseteq \Sigma_{\mathcal{P}}$ and Γ' such that $\Sigma_{\mathcal{P}}$ is equivalent to $\Sigma = (\Sigma_{\mathcal{P}} \setminus \Sigma') \cup \Gamma'$ and $|\Gamma'| < |\Sigma'|$. Therefore, Γ' is finite since any subset of a finitely generated theory is at most countable. Also note that Γ' can always be taken so that Γ' and $\Sigma_{\mathcal{P}} \setminus \Sigma'$ are disjoint. We use such Γ' and $\Sigma = (\Sigma_{\mathcal{P}} \setminus \Sigma') \cup \Gamma'$ further in the proof.

Take $P \Rightarrow P^{\downarrow\uparrow} \in \Sigma'$ and observe that by Lemma 7.13 there is $A \Rightarrow B \in \Sigma$ such that $A^{\downarrow\uparrow} \equiv P^{\downarrow\uparrow}$ and $P \not\models A \Rightarrow B$. It can be shown that such $A \Rightarrow B$ belongs to Γ'. Indeed, by contradiction, suppose that $A \Rightarrow B \notin \Gamma'$, i.e., $A \Rightarrow B \in \Sigma_{\mathcal{P}} \setminus \Sigma'$. In that case, $A \in r(\mathcal{P})$, $B = A^{\downarrow\uparrow}$, and also $A \neq P$ because

$P \Rightarrow P^{\downarrow\uparrow} \notin \Sigma_{\mathcal{P}} \setminus \Sigma'$. Now, suppose that $A + i \subseteq P$ for some $i \in \mathbb{Z}$. We cannot have $A+i = P$ since $A, P \in r(\mathcal{P})$ and $A \neq P$. Therefore, $A+i \subset P$. Now, using Corollary 7.5, it follows that $B + i = A^{\downarrow\uparrow} + i \subseteq P$. Since $i \in \mathbb{Z}$ was taken arbitrarily, the latter fact shows $P \models A \Rightarrow B$ which contradicts $P \not\models A \Rightarrow B$. Hence, for each $P \Rightarrow P^{\downarrow\uparrow} \in \Sigma'$ there is $A \Rightarrow B \in \Gamma'$ such that $A^{\downarrow\uparrow} \equiv P^{\downarrow\uparrow}$ and $P \not\models A \Rightarrow B$.

Now, by the pigeonhole principle and using the previous claim, the facts that Γ' is finite and $|\Gamma'| < |\Sigma'|$ yield there are distinct $P_1 \Rightarrow P_1^{\downarrow\uparrow} \in \Sigma'$ and $P_2 \Rightarrow P_2^{\downarrow\uparrow} \in \Sigma'$ for which there is $A \Rightarrow B \in \Gamma'$ such that $P_1^{\downarrow\uparrow} \equiv A^{\downarrow\uparrow} \equiv P_2^{\downarrow\uparrow}$, $P_1 \not\models A \Rightarrow B$, and $P_2 \not\models A \Rightarrow B$.

In order to finish the proof, it suffices to show that $P_1 = P_2$ which contradicts the fact that $P_1 \Rightarrow P_1^{\downarrow\uparrow}$ and $P_2 \Rightarrow P_2^{\downarrow\uparrow}$ are distinct. It cannot be the case that $P_2 \sqsubset P_1$. Indeed, then by the definition of pseudo-intent $P_2^{\downarrow\uparrow} \sqsubseteq P_1 \subset P_1^{\downarrow\uparrow}$ which would contradict $P_1^{\downarrow\uparrow} \sqsubseteq P_2^{\downarrow\uparrow}$. In much the same way, we argue that it cannot be the case that $P_1 \sqsubset P_2$ either.

Now, assume $P_1 \not\equiv P_2$. Directly from Definition 6.1, $P_1 \not\equiv P_2$ together with our previous observations $P_1 \not\sqsubset P_2$ and $P_2 \not\sqsubset P_1$ yield $P_1 \not\sqsubseteq P_2$ and $P_2 \not\sqsubseteq P_1$. Therefore, by Lemma 7.14, $(P_1 + k) \cap P_2 = ((P_1 + k) \cap P_2)^{\downarrow\uparrow}$ for any $k \in \mathbb{Z}$. Utilizing $A \sqsubseteq P_1$ and $A \sqsubseteq P_2$, Proposition 6.2 yields $A \sqsubseteq (P_1 + j) \cap P_2$ for some $j \in \mathbb{Z}$. Then $A^{\downarrow\uparrow} \sqsubseteq ((P_1 + j) \cap P_2)^{\downarrow\uparrow} = (P_1 + j) \cap P_2 \subseteq P_2 \subset P_2^{\downarrow\uparrow}$, meaning that $A^{\downarrow\uparrow} \sqsubset P_2^{\downarrow\uparrow}$. This would be absurd if both $A^{\downarrow\uparrow}$ and $P_2^{\downarrow\uparrow}$ were infinite and thus equal to $\mathcal{T}_Y$. So, both $A^{\downarrow\uparrow}$ and $P_2^{\downarrow\uparrow}$ must be finite. Then, $A^{\downarrow\uparrow} \sqsubset P_2^{\downarrow\uparrow}$ contradicts the fact that $A^{\downarrow\uparrow} \equiv P_2^{\downarrow\uparrow}$ owing to Lemma 6.3. Therefore, $P_1 \equiv P_2$. Since each pseudo-intent is finite, see Definition 7.3, there is $i \in \mathbb{Z}$ such that $P_1 + i = P_2$ owing once again to Lemma 6.3. Then $i = 0$ since $P_1, P_2 \in r(\mathcal{P})$, i.e., $P_1 = P_2$. □

Based on the observations in this section, we argue that in the temporal setting we use in this thesis, there is a reasonable notion of a pseudo-intent which can be used to determine bases of input data which are minimal. The notion of minimality has been introduced to accommodate the fact that all bases of input data in our setting are infinite. Nevertheless, the observed minimality of the obtained bases has some implications for the finite "interesting part" of bases that was discussed in the introduction. Namely, in any base given by pseudo-intents, the interesting part cannot be replaced by smaller and equivalent set of formulas. This is a direct consequence of the previous observations and the notion of minimality from

Definition 7.9.

Example 10. We conclude this section by presenting a minimal base which is equivalent to that in the introduction. Consider again the input data from Fig. 2. The base given by the system of pseudo-intents of the input data has been presented in Example 9. Theorem 7.15 yields that the base is in addition minimal in sense of Definition 7.9. Following our discussion on "interesting rules" in the introduction, we can split the base into disjoint subsets Σ_1, Σ_2, and Σ_3. Namely, Σ_1 consists of all formulas where the antecedents and consequents are withing the time range of the input data:

$$\begin{aligned}\Sigma_1 = \{&\{b^0\} \Rightarrow \{a^0, b^0\},\\ &\{c^0\} \Rightarrow \{c^0, a^1\},\\ &\{a^0, b^0, a^1\} \Rightarrow \{a^0, b^0, a^1, b^1\},\\ &\{c^0, a^1, b^1\} \Rightarrow \{a^0, c^0, a^1, b^1\},\\ &\{c^0, a^1, a^2\} \Rightarrow \{c^0, a^1, c^1, a^2\}\}.\end{aligned}$$

The formulas in Σ_1 can be further simplified by reducing the attributes in their antecedents and consequents. For instance, we can use the fact that $\{A \Rightarrow B\} \equiv \{A \Rightarrow B \backslash A\}$, see Definition 6.19, and take Σ'_1 where all consequents and antecedents of all formulas are disjoint:

$$\begin{aligned}\Sigma'_1 = \{&\{b^0\} \Rightarrow \{a^0\},\\ &\{c^0\} \Rightarrow \{a^1\},\\ &\{a^0, b^0, a^1\} \Rightarrow \{b^1\},\\ &\{c^0, a^1, b^1\} \Rightarrow \{a^0\},\\ &\{c^0, a^1, a^2\} \Rightarrow \{c^1\}\}.\end{aligned}$$

Furthermore, the antecedents of some of the formulas in Σ'_1 can further be simplified. For instance, we can easily see that $\{A \Rightarrow B, A \cup B \cup C \Rightarrow D\}$ is equivalent to $\{A \Rightarrow B, A \cup C \Rightarrow D\}$. Applying this rule three times on Σ'_1, we obtain Σ''_1 of the following form:

$$\begin{aligned}\Sigma''_1 = \{&\{b^0\} \Rightarrow \{a^0\},\\ &\{c^0\} \Rightarrow \{a^1\},\\ &\{b^0, a^1\} \Rightarrow \{b^1\},\\ &\{c^0, b^1\} \Rightarrow \{a^0\},\\ &\{c^0, a^2\} \Rightarrow \{c^1\}\}.\end{aligned}$$

By moment's reflection, we can see that the formulas in Σ_1'' correspond to the formulas in the first part of the base in the introduction.

The second important subset of the base is Σ_2 which consists of formulas whose antecedents fall withing the time range of the input data and consequents do not:

$$\begin{aligned}\Sigma_2 = \{&\{a^0, b^0, c^0, a^1, b^1\} \Rightarrow \mathcal{T}_Y,\\ &\{a^0, a^2\} \Rightarrow \mathcal{T}_Y,\\ &\{c^0, a^1, c^1, a^2, b^2\} \Rightarrow \mathcal{T}_Y\}.\end{aligned}$$

Again, the formulas in Σ_2 can be simplified by considering formulas in Σ_1'' and the above-mentioned equivalence:

$$\begin{aligned}\Sigma_2' = \{&\{b^0, c^0\} \Rightarrow \mathcal{T}_Y,\\ &\{a^0, a^2\} \Rightarrow \mathcal{T}_Y,\\ &\{c^0, c^1, b^2\} \Rightarrow \mathcal{T}_Y\}.\end{aligned}$$

The formulas in Σ_2' correspond to the formulas in the second part of the base presented in the introduction. Let us note that since we consider extended formulas (with $\mathcal{T}_Y$ allowed as a consequent), Σ_2' is in fact a finite representation of infinitely many formulas from the second part of the base from the introduction.

Finally, the base consists of infinitely many formulas whose antecedents are $P \in r(\mathcal{P})$ such that $\|P\| > \|\mathcal{I}\| = 2$. That is, the final part can be written as $\Sigma_3 = \Sigma_{\mathcal{P}} \setminus (\Sigma_1 \cup \Sigma_2)$ and we have $\Sigma_{\mathcal{P}} \equiv \Sigma_1'' \cup \Sigma_2' \cup \Sigma_3$. Applying the minimality of $\Sigma_{\mathcal{P}}$, see Theorem 7.15, there is no $\Gamma = \Gamma' \cup \Sigma_3$ such that $\Gamma \equiv \Sigma_{\mathcal{P}}$. From this point of view, the interesting part of $\Sigma_{\mathcal{P}}$ is the smallest possible in terms of the number of its formulas.

Let us conclude the example by outlining how to show that in this particular case, the infinite subset Σ_3 of the base is in the form

$$\Sigma_3 = \big\{\{a^0, a^{2+n}\} \Rightarrow \mathcal{T}_Y \,|\, n \in \mathbb{N}\big\}$$

which corresponds with the third part of the base presented in the introduction. It is easy to see that each $\{a^0, a^{2+n}\}$ belongs to $r(\mathcal{P})$ for any natural n. In order to see that all $P \in r(\mathcal{P})$ with $\|P\| > 2$ are of the form $\{a^0, a^{2+n}\}$ for some natural n, we first assume that $P = \{a^0, \ldots\}$ and $b^i \in P$ for some $i > 2$. Using the fact that $\{b^0\} \in r(\mathcal{P})$ and $\{b^0\}^{\downarrow\uparrow} = \{a^0, b^0\}$, we conclude that $a^i \in P$. Furthermore, $\{a^0, a^i\} \in r(\mathcal{P})$ and $\{a^0, a^i\}^{\downarrow\uparrow} = \mathcal{T}_Y$. Hence, $\{a^0, a^i\} \subset P$ yields $\mathcal{T}_Y \subseteq P$ which is absurd because P is finite. Therefore, $b^i \notin P$. Analogously, we get that $c^i \notin P$ for $i > 2$. Using similar arguments, we can show that (i) if $P = \{b_0, \ldots\}$, then $a^i \notin P$ and $c^i \notin P$ for any $i > 2$; (ii) if $P = \{c_0, \ldots\}$, then $a^i \notin P$ and $b^i \notin P$ for any $i > 2$, which concludes the argument.

8 Structure of minimal sets

We start this section by introducing a notation for expressing that considering a theory Σ, an antecedent A implies a shift of another finite subset of $\mathcal{T}_Y$ and prove essential properties of the new notion. This property will later be used to define equivalence of antecedents of formulas and will be crucial for our investigation of minimality.

Definition 8.1. For a theory Σ and $A, B \in \mathcal{F} \cup \{\emptyset\}$, we put $\Sigma \models A \Rightarrow^* B$ whenever there is $i \in \mathbb{Z}$ such that $\Sigma \models A \Rightarrow B + i$.

Example 11. Consider a theory $\Sigma = \{\{a^0, c^1\} \Rightarrow \{b^1\}\}$. Then we have $[\{a^0, c^1\}]_\Sigma = \{a^0, b^1, c^1\}$ which means $\Sigma \models \{a^0, c^1\} \Rightarrow \{a^0, b^1, c^1\}$. Moreover, it is easy to see that $\{a^0, b^1, c^1\} = \{a^3, b^4, c^4\} - 3$, i.e., we have also $\Sigma \models \{a^0, c^1\} \Rightarrow \{a^3, b^4, c^4\} - 3$. Using the notation introduced in Definition 8.1, we can write $\Sigma \models \{a^0, c^1\} \Rightarrow^* \{a^3, b^4, c^4\}$.

Using Definition 8.1 and Definition 6.1, we can establish a characterization of $\Sigma \models A \Rightarrow^* B$ similar to that of Theorem 3.15:

Lemma 8.2. *For any theory Σ and any $A, B \in \mathcal{F}$, the following conditions are equivalent:*

(*i*) $\Sigma \models A \Rightarrow^* B$,

(*ii*) $B \sqsubseteq [A]_\Sigma$,

(*iii*) $[B]_\Sigma \sqsubseteq [A]_\Sigma$.

Proof. By definition, $\Sigma \models A \Rightarrow^* B$ means that there is $i \in \mathbb{Z}$ such that $\Sigma \models A \Rightarrow B + i$. By Theorem 3.15, we get $B + i \subseteq [A]_\Sigma$ and thus $B \sqsubseteq [A]_\Sigma$ by Definition 6.1. This shows that (*i*) implies (*ii*). Now, assume that (*ii*) holds, i.e., $B \sqsubseteq [A]_\Sigma$, meaning that $B + i \subseteq [A]_\Sigma$ for some $i \in \mathbb{Z}$. By isotony and idempotency of the semantic closure, we get $[B + i]_\Sigma \subseteq [[A]_\Sigma]_\Sigma = [A]_\Sigma$. Finally, using Lemma 3.9, we get $[B]_\Sigma + i = [B + i]_\Sigma \subseteq [A]_\Sigma$, showing that (*iii*) holds. Now, assume that (*iii*) holds. In order to prove $\Sigma \models A \Rightarrow^* B$, it suffices to check that $B + i \subseteq [A]_\Sigma$ for some $i \in \mathbb{Z}$, see Theorem 3.15. Observe that $[B]_\Sigma \sqsubseteq [A]_\Sigma$ means $[B]_\Sigma + i \subseteq [A]_\Sigma$ for some $i \in \mathbb{Z}$ and, applying Lemma 3.9 together with the extensivity of $[\cdots]_\Sigma$, we get $B + i \subseteq [B + i]_\Sigma \subseteq [A]_\Sigma$, finishing the proof. $\square$

It is easily seen that the law of transitivity of implication extends to $\Rightarrow^*$ as it is shown in the next lemma.

Lemma 8.3. *If* $\Sigma \models A \Rightarrow^* B$ *and* $\Sigma \models B \Rightarrow^* C$ *then* $\Sigma \models A \Rightarrow^* C$.

Proof. From $\Sigma \models A \Rightarrow^* B$ and $\Sigma \models B \Rightarrow^* C$, it follows that $\Sigma \models A \Rightarrow B + i$ and $\Sigma \models B \Rightarrow C + j$ for some $i, j \in \mathbb{Z}$. Using Lemma 3.14, we get $\Sigma \models B + i \Rightarrow C + (i + j)$ and thus $\Sigma \models A \Rightarrow C + (i + j)$ by the transitivity of implication (see Proposition 4.3), showing $\Sigma \models A \Rightarrow^* C$. □

The following assertion is used as one of the core arguments in proofs of the subsequent observations. It shows that in a situation where a formula $A \Rightarrow B$ is entailed by Σ and its entailment is dependent on a presence of other formula $C \Rightarrow D$ in Σ, then there is an important relationship between the antecedents A and C:

Theorem 8.4. *If* $\Sigma \models A \Rightarrow B$ *and* $\Sigma \setminus \{C \Rightarrow D\} \not\models A \Rightarrow B$ *then we have* $\Sigma \setminus \{C \Rightarrow D\} \models A \Rightarrow^* C$.

Proof. Let $\Sigma \models A \Rightarrow B$ and $\Sigma \setminus \{C \Rightarrow D\} \not\models A \Rightarrow B$ and observe that by Theorem 3.15, we get $B \subseteq [A]_\Sigma$ and $B \nsubseteq [A]_{\Sigma\setminus\{C\Rightarrow D\}}$. The last two facts together with the isotony of $[\cdots]_\Sigma$ yield $A \subseteq [A]_{\Sigma\setminus\{C\Rightarrow D\}} \subset [A]_\Sigma$. As an immediate consequence, we get that $[A]_{\Sigma\setminus\{C\Rightarrow D\}}$ is not a model of Σ. Since $[A]_{\Sigma\setminus\{C\Rightarrow D\}}$ is a model of $\Sigma \setminus \{C \Rightarrow D\}$, there must be $i \in \mathbb{Z}$ such that $C+i \subseteq [A]_{\Sigma\setminus\{C\Rightarrow D\}}$ (and $D+i \nsubseteq [A]_{\Sigma\setminus\{C\Rightarrow D\}}$), which means $C \sqsubseteq [A]_{\Sigma\setminus\{C\Rightarrow D\}}$, i.e., $\Sigma \setminus \{C \Rightarrow D\} \models A \Rightarrow^* C$ by Lemma 8.2. □

We now turn our attention to a particular equivalence relation defined on antecendents of formulas in a theory Σ.

Definition 8.5. Let Σ be a theory and $A, C \in \mathcal{F}$. We say that A and C are equivalent under Σ, written $A \equiv_\Sigma C$, whenever $\Sigma \models A \Rightarrow^* C$ and $\Sigma \models C \Rightarrow^* A$. Furthermore, we define $\mathrm{E}_\Sigma(A)$ as the set of all $C \Rightarrow D \in \Sigma$ such that $A \equiv_\Sigma C$.

It can be easily checked that $\equiv_\Sigma$ is reflexive, symmetric, and transitive, see Lemma 8.3. Hence, $\equiv_\Sigma$ can be seen as an equivalence on Σ. In this sense, each $\mathrm{E}_\Sigma(A) \neq \emptyset$ acts as an equivalence class modulo $\equiv_\Sigma$.

Example 12. Consider a theory

$$\begin{aligned}\Sigma = \{&\{a^0, b^1, c^1\} \Rightarrow \{b^0\},\\ &\{b^0, c^1\} \Rightarrow \{a^{-1}\},\\ &\{a^0\} \Rightarrow \{c^2\},\\ &\{b^0\} \Rightarrow \{c^0\},\\ &\{b^0, c^0\} \Rightarrow \{c^{-1}\}\}.\end{aligned}$$

Then we have $[\{a^0, b^1, c^1\}]_\Sigma = \{c^{-1}, a^0, b^0, c^0, b^1, c^1, c^2\}$. Therefore, using Theorem 3.15, we have $\Sigma \models \{a^0, b^1, c^1\} \Rightarrow \{b^0, c^1\} + 0$ from which we conclude that $\Sigma \models \{a^0, b^1, c^1\} \Rightarrow^* \{b^0, c^1\}$. On the other hand, $[\{b^0, c^1\}]_\Sigma = \{c^{-2}, a^{-1}, b^{-1}, c^{-1}, b^0, c^0, c^1\}$ from which we can deduce $\Sigma \models \{b^0, c^1\} \Rightarrow \{a^0, b^1, c^1\} - 1$ using the same argument as before, i.e., $\Sigma \models \{b^0, c^1\} \Rightarrow^* \{a^0, b^1, c^1\}$. Thus, under the notation of Definition 8.5, $\{b^0, c^1\} \equiv_\Sigma \{a^0, b^1, c^1\}$. Moreover, we have

$$\mathrm{E}_\Sigma(\{a^0, b^1\}) = \{\{a^0, b^1, c^1\} \Rightarrow \{b^0\}, \{b^0, c^1\} \Rightarrow \{a^{-1}\}\}$$

since we can show $\{a^0, b^1\} \equiv_\Sigma \{a^0, b^1, c^1\}$ similarly as above using the fact that $[\{a^0, b^1\}]_\Sigma = \{c^{-1}, a^0, b^0, c^0, b^1, c^1, c^2\}$. The other non-empty sets $\mathrm{E}_\Sigma(\cdots)$ are

$$\begin{aligned}&\mathrm{E}_\Sigma(\{a^0\}) = \{\{a^0\} \Rightarrow \{c^2\}\} \text{ and}\\ &\mathrm{E}_\Sigma(\{b^0\}) = \{\{b^0\} \Rightarrow \{c^0\}, \{b^0, c^0\} \Rightarrow \{c^{-1}\}\}\end{aligned}$$

and together with $\mathrm{E}_\Sigma(\{a^0, b^1\})$ they form a partition on Σ induced by the equivalence $\equiv_\Sigma$ of antecedents.

Theorem 8.6. *Let Σ and Γ be equivalent theories and $H \in \mathcal{F}$ be such that $\mathrm{E}_\Sigma(H) \neq \emptyset$. Then for every $A \Rightarrow B \in \mathrm{E}_\Sigma(H)$ such that $\Sigma \setminus \{A \Rightarrow B\} \not\models A \Rightarrow B$ there is $C \Rightarrow D \in \mathrm{E}_\Gamma(H)$.*

Proof. Take any $A \Rightarrow B \in \mathrm{E}_\Sigma(H)$ such that $\Sigma \setminus \{A \Rightarrow B\} \not\models A \Rightarrow B$. Since Σ is equivalent to Γ we have $\Gamma \models A \Rightarrow B$. Moreover, using Propostion 4.9, we can take a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \models A \Rightarrow B$ and for every $\Gamma'' \subset \Gamma'$ we have $\Gamma'' \not\models A \Rightarrow B$. Observe that Γ' is not empty since $B \not\subseteq A$ which follows from the assumption $\Sigma \setminus \{A \Rightarrow B\} \not\models A \Rightarrow B$.

Now, we claim that there is $C \Rightarrow D \in \Gamma'$ such that $\Sigma \setminus \{A \Rightarrow B\} \not\models C \Rightarrow D$. We prove it by contradiction. Assume that for every $C \Rightarrow D \in \Gamma'$

we have $\Sigma \setminus \{A \Rightarrow B\} \models C \Rightarrow D$. Then, we would have $\Sigma \setminus \{A \Rightarrow B\} \models A \Rightarrow B$ because $\Gamma' \models A \Rightarrow B$. This, however, contradicts the assumption $\Sigma \setminus \{A \Rightarrow B\} \not\models A \Rightarrow B$.

Obviously, $H \equiv_\Gamma A$. In order to prove that $H \equiv_\Gamma C$, it is sufficient to prove that $A \equiv_\Gamma C$ since $\equiv_\Gamma$ is transitive. Since $\Sigma \setminus \{A \Rightarrow B\} \not\models C \Rightarrow D$ and $\Sigma \models C \Rightarrow D$ we have $\Sigma \models C \Rightarrow^* A$ using Theorem 8.4. Therefore, $\Gamma \models C \Rightarrow^* A$ since Γ and Σ are equivalent. On the other hand, since $\Gamma' \setminus \{C \Rightarrow D\} \not\models A \Rightarrow B$ and $\Gamma' \models A \Rightarrow B$ we have $\Gamma' \models A \Rightarrow^* C$ using Theorem 8.4 and so $\Gamma \models A \Rightarrow^* C$. Altogether, $A \equiv_\Gamma C$. $\square$

In the following definition, we introduce a notion capturing a stronger form of semantic entailment of temporal attribute implications. The notion plays a central role in the characterization of minimal sets of formulas.

Definition 8.7. Let Σ be a theory, $A, B \in \mathcal{F}$. We say that $A \Rightarrow B$ is directly entailed by Σ, written $\Sigma \Vdash A \Rightarrow B$, whenever $\Sigma \setminus \mathrm{E}_\Sigma(A) \models A \Rightarrow B$.

Note that the direct entailment introduced in Definition 8.7 generalizes the notion of direct determination known from the classic setting [40]. There are basically two main differences between the notions. First, direct entailment refers to formulas in our temporal setting whereas the classic notion does not. Second, direct entailment is defined in terms of the semantic entailment whereas the classic direct determination was defined in terms of derivation DAGs [40, 41] that can be seen as graphical proof system that is equivalent to the system of Armstrong inference rules [3]. Let us also note that [60] introduces a notion of direct provability that utilizes graded attribute implications and is based on an Armstrong-style inference system parameterized by globalization [52].

Example 13. Consider the same theory Σ as in Example 12 and put $\Gamma = \Sigma \setminus \mathrm{E}_\Sigma(\{a^0, b^1, c^1\})$, i.e.,

$$\begin{aligned}\Gamma = \{&\{a^0\} \Rightarrow \{c^2\},\\ &\{b^0\} \Rightarrow \{c^0\},\\ &\{b^0, c^0\} \Rightarrow \{c^{-1}\}\}.\end{aligned}$$

Then we have $[\{a^0, b^1, c^1\}]_\Gamma = \{a^0, c^0, b^1, c^1, c^2\}$ from which we deduce $\Gamma \models \{a^0, b^1, c^1\} \Rightarrow \{b^1, c^2\}$ using Theorem 3.15. Using the notation introduced in Definition 8.7, we have $\Sigma \Vdash \{a^0, b^1, c^1\} \Rightarrow \{b^1, c^2\}$.

Lemma 8.8. *For each $C \in \mathcal{F}$ satisfying $\mathrm{E}_\Sigma(C) \neq \emptyset$ there is $A \Rightarrow B \in \mathrm{E}_\Sigma(C)$ such that $\Sigma \Vdash C \Rightarrow^* A$.*

Proof. Take any $C \in \mathcal{F}$ satisfying $\mathrm{E}_\Sigma(C) \neq \emptyset$ and put

$$\mathcal{S} = \{\Sigma' \subseteq \Sigma \,|\, \Sigma' \models C \Rightarrow^* A \text{ for some } A \Rightarrow B \in \mathrm{E}_\Sigma(C)\}.$$

On the account of the assumption $\mathrm{E}_\Sigma(C) \neq \emptyset$, we have $\mathcal{S} \neq \emptyset$ because $\Sigma \in \mathcal{S}$. Moreover, applying Proposition 4.9, there are finite sets in $\mathcal{S}$. Hence, we can fix $\Sigma' \in \mathcal{S}$ as any of the finite sets in $\mathcal{S}$ that have the least number of formulas.

Now, it suffices to prove that $\Sigma' \cap \mathrm{E}_\Sigma(C) = \emptyset$ because in that case we would obtain $\Sigma' \subseteq \Sigma \setminus \mathrm{E}_\Sigma(C)$. By contradiction, assume that there is $E \Rightarrow F \in \Sigma' \cap \mathrm{E}_\Sigma(C)$. Then, $\Sigma' \setminus \{E \Rightarrow F\} \not\models C \Rightarrow^* A$ because Σ' has the least number of formulas. By Theorem 8.4, we have $\Sigma' \setminus \{E \Rightarrow F\} \models C \Rightarrow^* E$. Since $E \Rightarrow F \in \mathrm{E}_\Sigma(C)$, we have found a finite theory in $\mathcal{S}$ that contains less formulas than Σ', a contradiction. □

Theorem 8.9. *Let Σ and Γ be equivalent. Then for each $A \Rightarrow B \in \mathrm{E}_\Sigma(H)$ satisfying $\Sigma \setminus \{A \Rightarrow B\} \not\models A \Rightarrow B$ there is $C \Rightarrow D \in \mathrm{E}_\Gamma(H)$ such that $\Gamma \Vdash A \Rightarrow^* C$.*

Proof. Acording to Theorem 8.6, for each $A \Rightarrow B \in \mathrm{E}_\Sigma(H)$ satisfying $\Sigma \setminus \{A \Rightarrow B\} \not\models A \Rightarrow B$ there is $C' \Rightarrow D' \in \mathrm{E}_\Gamma(H)$, i.e., $\mathrm{E}_\Gamma(H) \neq \emptyset$. Moreover, $\mathrm{E}_\Gamma(H) = \mathrm{E}_\Gamma(A)$, i.e., by Lemma 8.8, there is $C \Rightarrow D \in \mathrm{E}_\Gamma(A)$ such that $\Gamma \Vdash A \Rightarrow^* C$. □

The following theorem shows a natural property that equivalent theories induce the same direct entailment relations.

Theorem 8.10. *If Σ is equivalent to Γ, then $\Sigma \Vdash A \Rightarrow B$ iff $\Gamma \Vdash A \Rightarrow B$.*

Proof. Assume that $\Sigma \Vdash A \Rightarrow B$ and take a finite $\Sigma' \subseteq \Sigma \setminus \mathrm{E}_\Sigma(A)$ such that $\Sigma' \models A \Rightarrow B$ and, for each $\Sigma'' \subset \Sigma'$, we have $\Sigma'' \not\models A \Rightarrow B$. Clearly, such Σ' exists owing to Proposition 4.9. Now, it suffices to show that for each $C \Rightarrow D \in \Sigma'$ we have $\Gamma \setminus \mathrm{E}_\Gamma(A) \models C \Rightarrow D$. In that case, we would obtain $\Gamma \Vdash A \Rightarrow B$ as a consequence of $\Sigma' \models A \Rightarrow B$.

By contradiction, let there be $C \Rightarrow D \in \Sigma'$ so that $\Gamma \setminus \mathrm{E}_\Gamma(A) \not\models C \Rightarrow D$. Since Γ and Σ are equivalent, we have $\Gamma \models C \Rightarrow D$. Using Proposition 4.9, there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \models C \Rightarrow D$ and $\Gamma' \setminus \{E \Rightarrow F\} \not\models C \Rightarrow D$

for some $E \Rightarrow F \in \mathrm{E}_{\Gamma}(A)$. By Theorem 8.4, $\Gamma' \models C \Rightarrow^* E$ and so $\Gamma \models C \Rightarrow^* E$. Moreover, we have $\Gamma \models E \Rightarrow^* A$ since $E \equiv_{\Gamma} A$, i.e., $\Gamma \models C \Rightarrow^* A$ and so $\Sigma \models C \Rightarrow^* A$ by Lemma 8.3 and utilizing the fact that Σ and Γ are equivalent. In addition to that, we have $\Sigma' \setminus \{C \Rightarrow D\} \models A \Rightarrow^* C$ using Theorem 8.4 and the fact that $\Sigma' \setminus \{C \Rightarrow D\} \not\models A \Rightarrow B$, i.e., $\Sigma \models A \Rightarrow^* C$. Altogether, $A \equiv_{\Sigma} C$ which contradicts the fact that $\Sigma' \subseteq \Sigma \setminus \mathrm{E}_{\Sigma}(A)$. $\square$

The following assertion shows that the property of transitivity of implication, i.e., from $\Sigma \models A \Rightarrow B$ and $\Sigma \models B \Rightarrow C$, one derives $\Sigma \models A \Rightarrow C$, holds for the direct entailment.

Theorem 8.11. *If $\Sigma \Vdash A \Rightarrow B$ and $\Sigma \Vdash B \Rightarrow C$ then $\Sigma \Vdash A \Rightarrow C$.*

Proof. Clearly, the claim is trivial if $C \subseteq B$. Thus, we inspect the situation when $C \nsubseteq B$. It suffices to show that $\Sigma \setminus \mathrm{E}_{\Sigma}(A) \models B \Rightarrow C$. Indeed, the rest follows by the transitivity of implication, see Proposition 4.3. We proceed by contradiction. Let $\Sigma \setminus \mathrm{E}_{\Sigma}(A) \not\models B \Rightarrow C$. Since $\Sigma \models B \Rightarrow C$, utilizing Proposition 4.9, there is a finite $\Sigma' \subseteq \Sigma$ such that $\Sigma' \models B \Rightarrow C$ and $\Sigma' \setminus \{E \Rightarrow F\} \not\models B \Rightarrow C$ for some $E \Rightarrow F \in \mathrm{E}_{\Sigma}(A)$. Using Theorem 8.4, we have $\Sigma' \setminus \{E \Rightarrow F\} \models B \Rightarrow^* E$, so $\Sigma \models B \Rightarrow^* A$ using $\Sigma \models E \Rightarrow^* A$ and Lemma 8.3. In addition, using $\Sigma \models A \Rightarrow B$, we get $A \equiv_{\Sigma} B$, i.e., $\mathrm{E}_{\Sigma}(A) = \mathrm{E}_{\Sigma}(B)$ which contradicts the assumption $\Sigma \setminus \mathrm{E}_{\Sigma}(A) \not\models B \Rightarrow C$ since $\Sigma \Vdash B \Rightarrow C$. $\square$

The following assertion presents a sufficient condition for a non-redundant theory to be minimal. The condition is based on checking the non-existence of a pair of formulas with particular properties.

Theorem 8.12. *Let Σ be a non-redundant theory which is not minimal. Then there are distinct $A \Rightarrow B, C \Rightarrow D \in \Sigma$ such that $A \equiv_{\Sigma} C$ and $\Sigma \Vdash A \Rightarrow^* C$.*

Proof. It is easy to see that the claim is trivial if Σ contains distinct formulas $A \Rightarrow B$ and $C \Rightarrow D$ such that $A = C$. Thus, we focus on the case when Σ contains no distinct formulas with equal antecedents.

We prove the theorem by contradiction. Therefore, take a non-redundant theory Σ which is not minimal and satisfies the following condition: For every distinct $A \Rightarrow B, C \Rightarrow D \in \Sigma$ such that $A \equiv_{\Sigma} C$ we have $\Sigma \nVdash A \Rightarrow^* C$. According to Lemma 7.11, there are Γ' and $\Sigma' \subseteq \Sigma$ such that $|\Gamma'| < |\Sigma'|$ and

$\Sigma \equiv (\Sigma \setminus \Sigma') \cup \Gamma'$. Moreover, Γ' is finite because $|\Sigma'|$ is at most denumerable, i.e., we can take Γ' such that for each $\Gamma'' \subset \Gamma'$ we have $\Sigma \not\equiv (\Sigma \setminus \Sigma') \cup \Gamma''$.

In the following we denote $(\Sigma \setminus \Sigma') \cup \Gamma'$ by Γ. Taking into account the non-redundancy of Σ and applying Theorem 8.9, for each $A \Rightarrow B \in \mathrm{E}_{\Sigma}(H)$ there is $C \Rightarrow D \in \mathrm{E}_{\Gamma}(H)$ such that $\Gamma \Vdash A \Rightarrow^{*} C$. In other words, for each $A \Rightarrow B \in \Sigma$ there is $C \Rightarrow D \in \Gamma$ such that $A \equiv_{\Gamma} C$ and $\Gamma \Vdash A \Rightarrow^{*} C$. If in addition $A \Rightarrow B \in \Sigma'$ then $C \Rightarrow D \in \Gamma'$ because $C \Rightarrow D \in \Sigma \setminus \Sigma' \subseteq \Sigma$ would contradict the assumption that $\Sigma \nVdash A \Rightarrow^{*} C$.

Hence, using the fact that $|\Gamma'| < |\Sigma'|$ and the pigeonhole principle, there are two distinct $A \Rightarrow B, A' \Rightarrow B' \in \Sigma'$ for which there is a single $C \Rightarrow D \in \Gamma'$ such that $A \equiv_{\Gamma} C$, $A' \equiv_{\Gamma} C$, $\Gamma \Vdash A \Rightarrow^{*} C$, and $\Gamma \Vdash A' \Rightarrow^{*} C$. Moreover, we have $\Gamma \setminus \{C \Rightarrow D\} \not\models C \Rightarrow D$ since for each $\Gamma'' \subset \Gamma'$ we have $\Sigma \not\equiv (\Sigma \setminus \Sigma') \cup \Gamma''$, i.e., using Theorem 8.9, for $C \Rightarrow D$ there is $E \Rightarrow F \in \Sigma$ such that $C \equiv_{\Sigma} E$ and $\Sigma \Vdash C \Rightarrow^{*} E$. Therefore, by equivalency of Σ and Γ and the transitivity of $\equiv_{\Sigma}$, we have $A \equiv_{\Sigma} E$ and $A' \equiv_{\Sigma} E$. Furthermore, by Theorem 8.11 and Theorem 8.10, we have $\Sigma \Vdash A \Rightarrow^{*} E$ and $\Sigma \Vdash A' \Rightarrow^{*} E$. Since $A \neq A'$ (formulas $A \Rightarrow B$ and $A' \Rightarrow B'$ do not have equal antecedents) we have either $A' \neq E$ or $A \neq E$. In both cases the assumed property of Σ is violated. Indeed, in the first case there are distinct $A' \Rightarrow B', E \Rightarrow F \in \Sigma$ such that $A' \equiv_{\Sigma} E$ and $\Sigma \Vdash A' \Rightarrow^{*} E$. The other case is similar. $\square$

The following assertion presents a necessary condition of minimality.

Theorem 8.13. *Let Σ be a theory and let $A \Rightarrow B, C \Rightarrow D \in \Sigma$ be distinct formulas such that $\Sigma \models C \Rightarrow A - i$ and $\Sigma \Vdash A \Rightarrow C + i$ for some $i \in \mathbb{Z}$. Then Σ is not minimal.*

Proof. We prove the theorem by showing that there is $\Sigma' \subseteq \Sigma$ and Γ' such that $|\Sigma'| > |\Gamma'|$ and $\Sigma \equiv (\Sigma \setminus \Sigma') \cup \Gamma'$ which is a sufficient condition to show that Σ is not minimal, see Lemma 7.11. We put $\Sigma' = \{A \Rightarrow B, C \Rightarrow D\}$ and $\Gamma' = \{C \Rightarrow D \cup (B - i)\}$. Moreover, we denote $(\Sigma \setminus \Sigma') \cup \Gamma'$ by Γ. Clearly $|\Sigma'| > |\Gamma'|$. Therefore, it suffices to prove $\Sigma \equiv \Gamma$.

It is easy to see that $\Gamma \models C \Rightarrow D$ since by $C \Rightarrow D \cup (B - i) \in \Gamma$ we get $\Gamma \models C \Rightarrow D \cup (B - i)$ and so $\Gamma \models C \Rightarrow D$, cf. Proposition 4.3. Furthermore, $\Sigma \Vdash A \Rightarrow C + i$ means $\Sigma \setminus \mathrm{E}_{\Sigma}(A) \models A \Rightarrow C + i$ which together with $\Sigma \models C \Rightarrow A - i$ yields $A \equiv_{\Sigma} C$. Hence, $\Sigma \setminus \mathrm{E}_{\Sigma}(A) \subseteq \Gamma$ and so $\Gamma \models A \Rightarrow C + i$. From $C \Rightarrow D \cup (B - i) \in \Gamma$, we get $\Gamma \models C \Rightarrow D \cup (B - i)$ and so $\Gamma \models C + i \Rightarrow (D \cup (B - i)) + i$ using Lemma 3.14. Then, by transitivity of implication, we get $\Gamma \models A \Rightarrow (D+i) \cup B$ and so $\Gamma \models A \Rightarrow B$ by Proposition 4.3.

Conversely, $A \Rightarrow B \in \Sigma$ yields $\Sigma \models A-i \Rightarrow B-i$ owing to Lemma 3.14. Hence, using $\Sigma \models C \Rightarrow A-i$, it follows that $\Sigma \models C \Rightarrow B-i$. Combining the last observation with the fact that $C \Rightarrow D \in \Sigma$, we obtain $\Sigma \models C \Rightarrow D \cup (B-i)$, see Proposition 4.3. Altogether, $\Sigma \equiv \Gamma$. $\square$

Putting the observations of Theorem 8.12 and Theorem 8.13 together, we obtain the following conclusion:

Corollary 8.14 (Characterization of Minimality)**.** *Let Σ be a non-redundant theory such that for each $A \Rightarrow B, C \Rightarrow D \in \mathrm{E}_\Sigma(H)$ we have $\Sigma \Vdash A \Rightarrow^* C$ iff $\Sigma \models C \Rightarrow A-i$ and $\Sigma \Vdash A \Rightarrow C+i$ for some $i \in \mathbb{Z}$. Then Σ is minimal iff there are no distinct $A \Rightarrow B, C \Rightarrow D \in \Sigma$ such that $A \equiv_\Sigma C$ and $\Sigma \Vdash A \Rightarrow^* C$.* $\square$

Remark 22. (a) As an example of theories for which the assumption in Corollary 8.14 holds, consider theories where the semantic closures of finite sets are finite. Indeed, assume that for Σ we have that $A \in \mathcal{F}$ implies $[A]_\Sigma \in \mathcal{F}$ and take $A \Rightarrow B, C \Rightarrow D \in \mathrm{E}_\Sigma(H)$ such that $\Sigma \Vdash A \Rightarrow^* C$ and $A \neq \emptyset$ (a non-trivial case). Then we have $\Sigma \models C \Rightarrow A+j$ and $\Sigma \Vdash A \Rightarrow C+i$ for some $i, j \in \mathbb{Z}$. As a consequence, $\Sigma \models A \Rightarrow A+(j+i)$ and so $\Sigma \models A \Rightarrow A+(j+i)\cdot k$ for any $k \in \mathbb{N}$ which holds iff $A+(j+i)\cdot k \subseteq [A]_\Sigma$ for any $k \in \mathbb{N}$. Hence, $j = -i$ because $[A]_\Sigma$ cannot be infinite. The converse implication is trivial.

(b) Another important example of theories that fullfill the conditions of Corollary 8.14 are finitely generated theories Definition 6.15. A theory Σ is finitely generated whenever (i) $\emptyset$ is its model, (ii) it has another non-trivial model, and (iii) there is $t \in \mathbb{N}$ such that in each model of Σ except for $\mathcal{T}_Y$ the largest difference between times points of attributes is at most t. In other words, the time span of each model (except for $\mathcal{T}_Y$) is bounded from above. In that case, $A \in \mathcal{F}$ implies $[A]_\Sigma \in \mathcal{F} \cup \{\mathcal{T}_Y\}$. Hence, we can use the same arguments as in (a) and handle the case when $[A]_\Sigma = \mathcal{T}_Y$. Using the assumption $\Sigma \models C \Rightarrow A+j$ we have $A+j \subseteq [C]_\Sigma$ which means $\mathcal{T}_Y = \mathcal{T}_Y + j = [A]_\Sigma + j \subseteq [C]_\Sigma$, i.e., $[C]_\Sigma = \mathcal{T}_Y$. Therefore, $\Sigma \models C \Rightarrow A-i$ holds.

Let us stress that the finitely generated theories used in Remark 22 (b) represent a wide family of theories that are natural from users' point of view.

Indeed, as it has been shown in Corollary 6.18, finitely generated theories are exactly theories entailing all if-then dependencies that hold in finite data sets. Therefore, Theorem 8.13 can be applied to any set of temporal attribute implications that is derived from a finite data set and entails all temporal attribute implications that hold in the data set.

We can summarize our observations by the following two algorithms the soundness of which follows from Theorem 8.12, Theorem 8.13, and Theorem 3.15.

Algorithm 1 (Test of Minimality)**.**

input: Σ *satisfying the assumptions of Corollary 8.14 (see Remark 22)*

output: *YES (is minimal) / NO (is not minimal)*

> *If there are distinct* $A \Rightarrow B, C \Rightarrow D \in \Sigma$ *such that* $A - i \subseteq [C]_\Sigma$ *and* $C+i \subseteq [A]_{\Sigma \setminus \mathrm{E}_\Sigma(A)}$ *for some* $i \in \mathbb{Z}$*, then return NO, otherwise return YES.*

■

Algorithm 2 (Minimization Step)**.**

input: Σ *satisfying the assumptions of Corollary 8.14 (see Remark 22)*

output: *a theory that is equivalent to* Σ

> *If there are distinct* $A \Rightarrow B, C \Rightarrow D \in \Sigma$ *such that* $A - i \subseteq [C]_\Sigma$ *and* $C + i \subseteq [A]_{\Sigma \setminus \mathrm{E}_\Sigma(A)}$ *for some* $i \in \mathbb{Z}$*, then return* $(\Sigma \setminus \{A \Rightarrow B, C \Rightarrow D\}) \cup \{C \Rightarrow D \cup (B - i)\}$*, otherwise return* Σ*.*

■

Remark 23. Observe that if Σ is finite, then both Algorithm 1 and Algorithm 2 terminate after finitely many steps and the total number of computed closures is polynomial in the number of formulas in Σ. Indeed, the tests involve distinct pairs of formulas from Σ and, clearly, $\Sigma \setminus \mathrm{E}_\Sigma(A)$ can also be determined based on computing closures the number of which is polynomial in the size of Σ. From this point of view, the complexity of our procedure is no worse than for the classic test of minimality [40].

The algorithms are demontrated in the following example.

Example 14. Consider the same theory as in Example 12, i.e.,

$$\begin{aligned}\Sigma = \{&\{a^0, b^1, c^1\} \Rightarrow \{b^0\},\\ &\{b^0, c^1\} \Rightarrow \{a^{-1}\},\\ &\{a^0\} \Rightarrow \{c^2\},\\ &\{b^0\} \Rightarrow \{c^0\},\\ &\{b^0, c^0\} \Rightarrow \{c^{-1}\}\}.\end{aligned}$$

It is easy to check that Σ is non-redundant. In order to check minimality, we have to check whether there are distinct antecedents satisfying the conditions of Theorem 8.13 that are also summarized in Algorithm 1. Take $\{a^0, b^1, c^1\}$ and $\{b^0, c^1\}$ for which we have $\{a^0, b^1, c^1\} - 1 \subseteq \{c^{-2}, a^{-1}, b^{-1}, c^{-1}, b^0, c^0, c^1\} = [\{b^0, c^1\}]_\Sigma$, i.e., $\Sigma \models \{b^0, c^1\} \Rightarrow \{a^0, b^1, c^1\} - 1$. Moreover, in Example 13, we have showed $\Sigma \Vdash \{a^0, b^1, c^1\} \Rightarrow \{b^0, c^1\} + 1$. Therefore, Σ is not minimal by Theorem 8.13.

We can use a size reduction introduced in the proof of Theorem 8.13, which is also present in Algorithm 2, and transform Σ into an equivalent theory

$$\begin{aligned}\Sigma' = \{&\{b^0, c^1\} \Rightarrow \{a^{-1}, b^1\},\\ &\{a^0\} \Rightarrow \{c^2\},\\ &\{b^0\} \Rightarrow \{c^0\},\\ &\{b^0, c^0\} \Rightarrow \{c^{-1}\}\}.\end{aligned}$$

However, Σ' is not minimal since there are antecedents $\{b^0\}$, $\{b^0, c^0\}$ for which we have $\{b^0, c^0\} \subseteq \{c^{-1}, b^0, c^0\} = [\{b^0\}]_\Sigma$, i.e., $\Sigma \models \{b^0\} \Rightarrow \{b^0, c^0\}$. Moreover, $\emptyset \models \{b^0, c^0\} \Rightarrow \{b^0\}$, i.e., $\Sigma' \Vdash \{b^0, c^0\} \Rightarrow \{b^0\}$. Therefore, using Algorithm 2 again, we transform Σ' into

$$\begin{aligned}\Sigma'' = \{&\{b^0, c^1\} \Rightarrow \{a^{-1}, b^1\},\\ &\{a^0\} \Rightarrow \{c^2\},\\ &\{b^0\} \Rightarrow \{c^{-1}, c^0\}\}.\end{aligned}$$

In order to check that Σ'' is minimal it suffices to investigate equivalent antecedents by means of $\equiv_{\Sigma''}$, see Theorem 8.12. It is easy to see that there are no equivalent antecedents. Hence, Σ'' is minimal.

Note that minimal theories can contain formulas with equivalent antecedents. For instance, consider a theory

$$\Gamma = \{\{a^0\} \Rightarrow \{a^0, b^0\}, \{b^0\} \Rightarrow \{a^0, b^0\}\}$$

for which we have $[\{a^0\}]_\Gamma = \{a^0, b^0\} = [\{b^0\}]_\Gamma$. Therefore, $\Gamma \models \{a^0\} \Rightarrow^* \{b^0\}$ and $\Gamma \models \{b^0\} \Rightarrow^* \{a^0\}$, i.e., $\{a^0\} \equiv_\Gamma \{b^0\}$. However, it is not the case that either $\Gamma \Vdash \{a^0\} \Rightarrow^* \{b^0\}$ or $\Gamma \models \{b^0\} \Rightarrow^* \{a^0\}$. Hence, Γ is minimal.

9 Related work

In database systems and knowledge engineering, there appeared isolated approaches which propose temporal semantics of if-then rules. We present here a short survey of the approaches and highlight the differences between our approach and the existing ones.

Formulas called temporal functional dependencies emerged in databases with time granularities [6]. In this approach, a time granularity is a general partition of time like seconds, weeks, years, etc., and a time granularity is associated to each relational schema. In addition, each tuple in a relation is associated with a part (so-called granule) of granularity. In this setting, temporal functional dependencies are like the ordinary functional dependencies [21, 41] with a time granularity as an additional component. The concept of validity of temporal functional dependencies is defined in much the same way as its classic counterpart and includes an additional condition that granules of tuples need to be covered by any granule from granularity of the temporal functional dependency. Thus, [6] uses an ordinary notion of validity of functional dependencies which is restricted to some time segments. This is conceptually very different from the problem we deal with in this thesis since in our approach, each attribute appearing in a rule is annotated by a relative time point and our rules are considered true in data whenever they hold in all time points.

Several approaches to temporal if-then rules, which are conceptually similar to [6], appeared in the field of association rules [1, 63] as the so-called temporal association rules [2, 36, 47]. In these approaches, the input data is in the form of transactions (i.e., subsets of items) where each transaction occurred at some point in time and the interest of the papers lies in extracting association rules from data which occur during a specified time cycle. For instance, one may be interested in extracting rules which are valid in "every spring month of a year", "every Monday in every year", etc. As in the case of the temporal functional dependencies, the temporal association rules may be understood as classic association rules occurring during specified time cycles.

Other results motivated by temporal semantics of association rules includes the so-called inter-transaction association rules [22, 23, 34, 58], see [39] for a survey of approaches. The papers propose algorithms to extract, given an input transactional data and a measure of interestingness (based on levels of minimal support and confidence), if-then rules which are preserved over a given period of time. From this point of view, the rules can be seen as formulas studied in this thesis restricted to predictive rules (see Definition 5.4 in Section 5) whose validity is considered with respect to the additional parameter of interestingness. As a consequence, the inter-transaction association rules are related to the rules in our approach in the same way as the ordinary association rules [1] are related to the ordinary attribute implications [27]. The results in [22, 23, 34, 39, 58] are focused almost exclusively on algorithms for mining the inter-transaction association rules and are not concerned with problems of entailment of the rules and the underlying logic. In contrast, the problems of entailment of rules are investigated in this thesis and we show there is reasonably strong logic for reasoning with such rules. Furthermore, we deal with a problem of extracting sets of rules satisfying a condition–minimality and the ability to describe all dependencies which hold in the data, instead of extracting rules from data satisfying a condition (given by the interestingness measure). Our observations may stimulate further development in the field of inter-transaction association rules and similar formulas and their applications in various domains [22, 32].

The formulas studied in this thesis are also related to particular program rules which appear in *Datalog* extensions dealing with flow of time and related phenomena [12, 11, 10] such as $Datalog_{nS}$ (*Datalog* with n successors). The formulas we consider correspond to a fragment of rules which appear in such *Datalog* extensions. Despite the similar form of our formulas and the program rules, there does not seem to be a direct relationship (or a reduction) of the entailment problem of our formulas and the recognition problem of $Datalog_{nS}$ programs. As we have outlined in the introduction our formulas can also be seen as particular PROLOG rules. Despite the possibility to consider our rules in these (and other) database and logic programming languages, we aim at different goals. Most importantly, we have provided an Armstrong-style axiomatization which is strong-complete, i.e., complete over arbitrary Σ, and focuses on the inference of formulas (rules) from (finite or infinite) sets of rules. In contrast, PROLOG uses definite programs (finite sets of formulas) and its inference system is based on the resolution principle. Our development of the topic is primarily motivated by temporal extensions of rules which are used in FCA [27] where the Armstrong-style systems are extensively used and, therefore, our approach is a natural direction to go in that matter.

Note that predictive formulas, as they were introduced in Definition 5.4, can be translated into further existing languages. For instance, the formulas can be represented by TeDiLog [24] rules—a recent temporal logic programming language whose semantics is defined using structures with a beginning and a linear flow of time. Thus, the semantics of TeDiLog differs from our because of the existence of the beginning of time and it includes a modality "always in future". In contrast, our rules are interpreted as if they contained a hidden modality "always (including points in the past)". With analogous conceptual differences, the predictive formulas can also be translated into plans of the planning domain definition language (PDDL, see [15, 29]) or expressed in situation calculus [42, 46, 48]. An open question is whether such transformations can be used to get further insight into the entailment problem of our formulas.

Temporal attribute implications can be seen as extensions of attribute implications studied in FCA and functional dependencies in relational databases [41]. Interestingly, in both the FCA and database communities there appeared results characterizing minimal sets of if-then formulas with different motivations. The minimality of sets of functional dependencies was thoroughly examined in the seminal paper [40] where the author gives criteria for minimality of non-redundant sets of functional dependencies based on the notion of direct determination. In this thesis, we present a similar result for temporal attribute implications. Interestingly, [40] shows that transforming a set of functional dependencies into an equivalent and minimal one can be done in polynomial time and the algorithm exploits the standard tests of entailment of functional dependencies [5].

In FCA, the seminal paper [30] shows a description of minimal sets of attribute implications based on the notion of pseudo-intents. Unlike the results on functional dependencies where the input for minimization is a set of formulas, [30] computes the minimal bases directly from object-attribute incidence data which turns out to be a hard problem as it is shown in [20]. In this thesis, we generalize the results of [30] in the temporal setting.

The form of data we consider as "input data" in our approach is closely related to triadic formal contexts [35]. Although there appeared approaches to attribute implications from the point of view of the triadic FCA [7, 26], they do not annotate attributes by conditions (such as time points as in our case). Our formulas are syntactically different and have a different interpretation than if-then dependencies which were introduced in triadic FCA. The initial approach to if-then rules in triadic FCA [7] considers formulas written as $(A \Rightarrow B)_C$ where A, B are subsets of attributes and C is a set of conditions. A formula of this form is considered true in a triadic context if the following condition is satisfied:

> If an object has all attributes from A under all conditions from C, then it also has all attributes from B under all conditions from C.

Clearly, our formulas represent different dependencies since the approach in [7] annotate whole formulas by conditions (such as time points as in our case) whereas in our case is annotated each particular attribute. Hence, different attributes appearing in a formula can be annotated by different conditions. Later, stronger formulas were proposed in [26] which are considered true in a triadic context if the following condition is satisfied:

> For each condition $c \in C$: If an object has all attributes in A (under c) then it also has all attributes in B (under c).

Again, our formulas are different in that the annotations appear in antecedents and consequents of the formulas.

10 Conclusion and future work

We have presented logic for reasoning with if-then rules expressing dependencies between attributes changing in time. The logic extends the classic logic for dealing with if-then rules by considering discrete time points as an additional component. We have studied both the semantic entailment based on preserving validity in models in all time points and syntactic entailment represented by a provability relation. We have shown a characterization of the semantic entailment based on least models and syntactico-semantical completeness of the logic. We have shown the problem of entailment is NP-hard, decidable in exponential space, and its simplified variant which involves only predictive formulas is decidable in pseudo-linear time. We have studied the notions of completeness in data changing in time, non-redundancy, and minimality of theories which are derived from finite sequences of object-attribute incidence data recorded in separate points in time. We have shown a generalization of the notion of a pseudo-intent which fits well into our model and proved that important non-redundant and minimal bases are determined by systems of pseudo-intents. Unlike the classic case, the bases are always infinite but contain finitely many formulas which constitute a part which is most relevant to data analysts. We have paid attention to properties inherent to minimal theories. We have introduced and investigated the notion of equivalence of antecedents of formulas and the notion of direct entailment that has been introduced as a stronger form of semantic entailment. Using the notions, we have presented necessary and sufficient conditions of minimality and presented families of theories for which such conditions can be applied. In the special case of finite theories, our criteria of minimality yield algorithms that can be used to minimize theories in finitely many steps. The minimization procedure relies on computing semantic closures whose number is polynomial in the size of the input—in this sense, the algorithm behaves as the classic minimization algorithm for attribute implications (or functional dependencies).

Future research directions we consider interesting include utilization of generalized quantifiers [37, 44] to capture notions like "validity in all time points with possible exceptions", connections to rules which may emerge in temporal databases [17], further analysis of algorithms related to the entailment, and finding connections to various types of logic programing schemes and formalisms supporting temporal extensions or modalities [10, 24, 46], adaptation of classic algorithms for computing bases, detailed complexity analysis and experimental evaluation of the proposed algorithms, and applications of bases in domain-specific areas of data mining concerned with input data changing in time.

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