UNIVERZITA PALACKÉHO V OLOMOUCI
PŘÍRODOVĚDECKÁ FAKULTA

DIPLOMOVÁ PRÁCE

Metoda CLIL ve výuce vybraných kapitol středoškolské matematiky

Katedra algebry a geometrie
Vedoucí diplomové práce: **prof. RNDr. Josef Molnár, CSc.**
Vypracovala: **Bc. Adéla Cekulová**
Studijní program: N0114A170004 Učitelství matematiky pro střední školy
Studijní obor: Učitelství matematiky pro střední školy maior, Učitelství fyziky pro střední školy minor
Forma studia: Prezenční
Rok odevzdání: 2021

BIBLIOGRAFICKÁ IDENTIFIKACE

Autor: Bc. Adéla Cekulová

Název práce: Metoda CLIL ve výuce vybraných kapitol středoškolské matematiky

Typ práce: Diplomová práce

Pracoviště: Katedra algebry a geometrie

Vedoucí práce: prof. RNDr. Josef Molnár, CSc.

Rok obhajoby práce: 2021

Abstrakt: Tato diplomová práce se zabývá možnostmi využití metody CLIL ve vybraných kapitolách středoškolské matematiky. První část obsahuje krátké představení metody a jejích specifik. Ve druhé části práce jsou vytvořeny anglické texty vybraných kapitol k použití ve výuce, které obsahují řešené příklady s ukázkou využití anglické terminologie. V příloze jsou pak zařazeny slovníčky k jednotlivým kapitolám, pracovní listy pro žáky a také jejich řešení. Pracovní listy nebo jejich části je možno využít k procvičení, jako domácí úkol, nebo jako známkovanou písemnou práci.

Klíčová slova: CLIL; matematika; střední škola; gymnázium; funkce; rovnice; nerovnice; planimetrie; stereometrie; komplexní čísla; pracovní list; angličtina; výuka; cizí jazyk; geometrie; řešené příklady

Počet stran: 74

Počet příloh: 26

Jazyk: český, anglický

BIBLIOGRAPHICAL IDENTIFICATION

Author: Bc. Adéla Cekulová

Title: CLIL Method in Teaching Selected Chapters of Mathematics at Secondary Schools

Type of thesis: Diploma thesis

Department: Department of Algebra and Geometry

Supervisor: prof. RNDr. Josef Molnár, CSc.

The year of presentation: 2021

Abstract: This diploma thesis deals with possibilities of using the CLIL method in teaching selected chapters of Mathematics at secondary schools. The first part contains a short introduction of the method and its specifics. In the second part, there are English texts of the selected chapters that may be used in classes and that also include solved problems to show the use of English terminology. The appendices are vocabulary sheets for the chapters, worksheets for students and their solutions. The worksheets or their parts may be used as exercises, homework or assessed written exams.

Key words: CLIL; Maths; secondary school; grammar school; function; equation; inequality; plane geometry; stereometry; complex numbers; worksheet; English; teaching; foreign language; geometry; solved problems

Number of pages: 74

Number of appendices: 26

Language: Czech, English

Prohlášení

Prohlašuji, že jsem diplomovou práci vypracovala samostatně pod vedením pana prof. RNDr. Josefa Molnára, CSc. s vyznačením všech použitých pramenů a spoluautorství. Souhlasím se zveřejněním diplomové práce podle zákona č. 111/1998 Sb., o vysokých školách, ve znění pozdějších předpisů. Byla jsem seznámena s tím, že se na moji práci vztahují práva a povinnosti vyplývající ze zákona č. 121/2000 Sb., autorský zákon, ve znění pozdějších předpisů.

V Olomouci dne ....

podpis

Obsah

Poděkování

Ráda bych poděkovala panu prof. RNDr. Josefu Molnárovi, CSc. za čas, pomoc a odborné vedení při zpracování této práce. Dále bych chtěla poděkovat kolegovi Ing. Lukáši Matouškovi, MBA za doporučení materiálů, poskytnutí informací o výuce metodou CLIL na GJW Prostějov a konzultaci materiálů vytvořených v rámci této práce.

ÚVOD

Moje diplomová práce je zaměřena na možnosti využití metody CLIL ve výuce matematiky na střední škole. Tato metoda se v současné době stává stále populárnější ve výuce jak humanitních, tak i přírodovědných předmětů. V případě, že se rozhodneme do výuky matematiky zmíněnou metodu začlenit, budeme čelit výzvě, jaké materiály zvolit. To bylo motivací k napsání této práce.

Cílem práce je vytvořit text, který bude možné využít v anglické výuce matematiky na střední škole či gymnáziu, které se řídí českým RVP. Zaměřila jsem se převážně na kapitoly, které žáci již znají ze základní školy a které se ve středoškolské matematice pouze rozšiřují o nové poznatky. Výběr kapitol byl také částečně ovlivněn studijním programem English for the Future na Gymnáziu Jiřího Wolkera v Prostějově, proto je zařazena i kapitola komplexních čísel.

V první části práce jsou uvedeny základní informace o metodě CLIL a jejím využití ve výuce matematiky. V druhé části jsou pak zpracovány jednotlivé vybrané kapitoly pro výuku v angličtině včetně řešených příkladů. Přílohy obsahují k těmto kapitolám slovníčky a pracovní listy včetně řešení.

1 Metoda CLIL

1.1 Pojem CLIL

Zkratka **CLIL** pochází z anglického **Content and Language Integrated Learning**. Jde tedy o metodu, kdy se nejazykový předmět vyučuje v cizím jazyce (nejčastěji v angličtině, což je i náš případ). Žáci si tak rozvíjí slovní zásobu cizího jazyka v některé konkrétní odborné oblasti.

Pojem CLIL se začal užívat od roku 1994, byl určen pro popis metod výuky odborných předmětů v cizím jazyce. Pojetí tohoto označení se dále vyvíjelo a v současné době je **specifickým typem výuky** integrujícím postupy **didaktiky cizího jazyka** a **didaktiky nejazykového vyučovacího předmětu**.[1]

1.2 Formy CLILu

Specifickým rysem výuky metodou CLIL je stanovení tzv. **duálních cílů**, učitel si tedy stanoví cíl jak **obsahový**, tak i **jazykový**. Cíle by měly být vyvážené, v závislosti na formě CLILu se řídí vzájemné podřizování těchto dvou cílů.

Rozlišujeme dvě formy CLILu, tzv. **soft CLIL** a **hard CLIL**. U soft CLILu učitel vybírá tematický celek či jeho část nejazykového předmětu s ohledem na jazykové cíle. Oproti tomu hard CLIL je zaměřený na obsah nejazykového předmětu, vzdělávací program či jeho část je ve vybraných nejazykových předmětech plánován v cizím jazyce. V tomto případě se jazykové cíle podřizují cílům obsahovým.

1.3 Požadavky na učitele

Ideálními kandidáty na výuku metodou CLIL jsou učitelé s aprobací v cizím jazyce a v nejazykovém předmětu. V praxi však častěji tuto metodu využívají učitelé nejazykových předmětů s velmi dobrou znalostí cizího jazyka. Především

[1] CLIL ve výuce: Jak zapojit cizí jazyky do vyučování. NUV [online]. Praha: NUV, 2012 [cit. 2021-4-27]. Dostupné z: http://www.nuv.cz/uploads/Publikace/CLIL_ve_vyuce.pdf

u hard CLILu může být místo aprobace z cizího jazyka vyžadována jiná forma doložení jeho znalosti, a to například mezinárodní jazykovou zkouškou. Tito učitelé kladou důraz především na cíle obsahové.

Výuku mohou realizovat také učitelé cizích jazyků, kteří v rámci své výuky vyberou tematickou oblast, jíž se budou věnovat. V tomto případě však výuku podřizují jazykovým cílům.

1.4 Výhody a nevýhody CLILu

Jednou z výhod využití metody CLIL ve výuce je bezesporu rozšíření slovní zásoby v oblastech humanitních či přírodovědných předmětů. Žáci se naučí pracovat s reálnými informacemi a zvyšují svoje komunikativní kompetence v cizím jazyce. Pro žáky je použití této metody také přínosem do budoucna, a to jak pro studium na zahraničních vysokých školách, tak při uplatnění na trhu práce.

Využití této metody také klade vyšší nároky na žáky i učitele, což vede ke zvyšování jejich profesní kvalifikace. Tato metoda může být pozitivně přijata jazykově nadanými žáky, případně může motivovat žáky s nezájmem o daný předmět k jeho studiu.

Naopak nevýhodou této metody může být nedostatečná jazyková vybavenost žáků především na školách, kde není jazyková výuka posílená. Na stejný problém můžeme narazit také u pedagogů. Vyučující nejazykových předmětů nemusí mít vždy dostatečnou jazykovou výbavu, aby byli schopni v cizím jazyce vyučovat. Naopak vyučující s aprobací v cizím jazyce nemusí mít dostatečné odborné znalosti pro výuku nejazykového předmětu.

Příprava výuky metodou CLIL je velmi náročná ať už z hlediska času, tak z hlediska studijních materiálů a následného hodnocení žáků. Pokud není zavedení CLILu dostatečně systematicky naplánováno, může být výuka chaotická a pro žáky naopak nepřínosná.

2 CLIL v matematice

Matematika je jedním z náročnějších předmětů pro využití metody CLIL. Zpravidla bývá žáky označována za nejméně oblíbený a velmi náročný předmět, což není pro učitele příznivá startovní situace. Správně zavedená metoda CLIL však může alespoň část žáků přesvědčit, aby matematiku vzali na milost.

Při přípravě výuky metodou CLIL je prvním problémem, na který narazíme, výběr materiálů. České učebnice matematiky s doložkou MŠMT, které se běžně na středních školách a gymnáziích používají, nemají běžně cizojazyčný ekvivalent.

Nabízí se otázka, zda je možné využít učebnice používané v zahraničí. Tato cesta však není vhodná, jelikož vzdělávací programy různých zemí se liší. Zvláště v případech, kdy vyučujeme matematiku česky a v daném tématu zařadíme jen část výuky v cizím jazyce, by přechod mezi učebnicemi působil zbytečné zmatky. Tyto cizojazyčné učebnice by tedy měl využívat spíše vyučující pro přípravu vlastních materiálů, které budou vyhovovat jeho cílům a vzdělávacímu programu dané školy. Příprava těchto materiálů je tedy velmi časově náročná.

Výuka musí být také pečlivě plánována z hlediska témat, která zařadíme v cizím jazyce. Vhodné je zařazovat taková témata, která jsou ze zkušenosti pro žáky snazší a která se částečně probírají na základní škole a ve středoškolské matematice se rozšiřují.

Z těchto důvodů vznikla tato diplomová práce, ve které jsou vytvořeny texty některých kapitol středoškolské matematiky v angličtině tak, aby odpovídaly výuce podle českých učebnic z řady Matematika pro gymnázia od nakladatelství Prometheus. Záměrem bylo vytvoření textů, které bude možno využít ve výuce matematiky sext a septim v programu English for the Future na Gymnáziu Jiřího Wolkera v Prostějově.

3 Equations and Inequalities (Rovnice a nerovnice)

Kapitola rovnice a nerovnice je zařazována na počátek středoškolské matematiky, a to za zopakování a rozvinutí základních matematických poznatků. Jelikož se žáci s řešením některých typů rovnic a nerovnic setkali již na základní škole, je toto téma vhodným kandidátem pro využití metody CLIL.

3.1 Linear Equations (Lineární rovnice)

Nejprve vyslovíme definici.

Definition 3.1. A **linear equation** in **one variable** is an equation that can be written in the form

$$ax + b = 0$$

where a and b are **real numbers**, with $a \neq 0$.

A dále uvedeme, kdy je dané číslo řešením rovnice.

- A number is said to **satisfy** an equation if substituting the number for the variable produces an equation that is a true statement.
- To **solve** an equation means to find all values of the variable that satisfy the equation.
- The values that make the equation true are called **solutions** or **roots** of the equation.
- The **solution set** of an equation is the set of all solutions of the equation. We usually use capitals (K, P, M…) to mark the solution set of an equation.

Žáci také již ze základní školy znají **ekvivalentní úpravy** rovnic.

- **Equivalent equations** have the same solution set. The process of **solving** an equation is generally accomplished by **producing** simpler but **equivalent equations** until the solutions are easy to observe.
- We often apply Addition and Multiplication Properties of Equality:
  - For **real numbers** a, b, and c, $a = b$ and $a + c = b + c$ are **equivalent equations**.
  - If $c \neq 0$, then $a = b$ and $ac = bc$ are **equivalent equations**.

Při řešení lineárních rovnic obsahujících zlomky využíváme žákům již také dobře známý NSN (nejmenší společný násobek jmenovatelů).

- If an equation involves **fractions**, it is convenient to **multiply** each side of the equation by the **LCD** (lowest common denominator) of all the denominators to produce an equivalent equation that does not contain fractions.

Uveďme nyní řešení některých lineárních rovnic pomocí ekvivalentních úprav.

Example 3.1. Solve the equation $\frac{5}{6}x - 10 = 0$.

Solution: To isolate the *x* term on the left side of the equation, add 10 to each side of the equation.

$$\frac{5}{6}x - 10 + 10 = 0 + 10$$

$$\frac{5}{6}x = 10$$

To get the variable *x* alone on the left side of the equation, multiply each side by $\frac{6}{5}$ (the reciprocal of $\frac{5}{6}$).

$$\frac{6}{5} \cdot \frac{5}{6}x = \frac{6}{5} \cdot 10$$

$$x = 12$$

Check by substituting 12 for *x* in the original equation.

$$L(12) = \frac{5}{6} \cdot 12 - 10 = 10 - 10 = 0$$

$$P(12) = 0$$

$$L(12) = P(12)$$

which is a true statement. The proposed solution, 12, satisfies the original equation. The solution set is $\mathrm{K} = \{12\}$.

Example 3.2. Solve the equation $\frac{3}{4}x + 6 - \frac{2}{3}x = \frac{25}{6}$.

Solution: Multiply each side of the equation by the LCD of all denominators, which is 12.

$$12 \cdot \left(\frac{3}{4}x + 6 - \frac{2}{3}x\right) = 12 \cdot \frac{25}{6}$$

$$9x + 72 - 8x = 50$$

Combine like terms on each side of equation.

$$x + 72 = 50$$

To get the variable *x* alone on the left side of the equation, use the addition property of equality to add -72 to each side of the equation. Subtracting 72 is equivalent to adding -72.

$$x + 72 - 72 = 50 - 72$$

$$x = 22$$

It is not necessary to check the solution as we used properties of equality and equivalent equations have the same solution set. The solution set of this equation is $\mathrm{K} = \{22\}$.

3.2 Linear Inequalities (Lineární nerovnice)

Definice lineární nerovnice je obdobná definici lineární rovnice, kde znaménko rovnosti zaměníme za znaménko nerovnosti.

Definition 3.2. A linear inequality in one variable is an inequality that can be written in one of the forms

$$ax + b > 0, \qquad ax + b < 0,$$

$$ax + b \geq 0, \qquad ax + b \leq 0,$$

where a and b are real numbers, with $a \neq 0$.

Při řešení lineárních nerovnic opět využíváme ekvivalentní úpravy, na rozdíl od rovnic však nyní musíme brát ohled na polaritu čísla, jímž obě strany nerovnice násobíme.

To solve linear inequalities, we use Properties of Inequalities:

- **Adding** the same number to each side of an inequality **preserves** the order of the inequality.
- **Multiplying** each side of an inequality by the same **positive** number **preserves** the order of the inequality.
- **Multiplying** each side of an inequality by the same **negative** number **reverses** the order of the inequality.

Example 3.3. Solve the inequality $3(x-2) \geq x+10$.

Solution: Use the distributive property on the left side of the inequality.

$$3x-6 \geq x+10$$

Subtract x from each side of the inequality.

$$3x-6-x \geq x+10-x$$

$$2x-6 \geq 10$$

Add 6 to each side of the inequality.

$$2x-6+6 \geq 10+6$$

$$2x \geq 16$$

Multiply each side by the reciprocal of 2.

$$\frac{1}{2} \cdot 2x \geq \frac{1}{2} \cdot 16$$

$$x \geq 8$$

Thus, the original inequality is true for all real numbers greater than or equal to 8. The solution set is $< 8; \infty)$.

Example 3.4. Solve the inequality $5x-12 > 8x+7$.

Solution: Add $-8x$ and 12 to each side of the inequality.

$$5x-12-8x+12 > 8x+7-8x+12$$

$$-3x > 19$$

Divide each side of the inequality by -3 and reverse the inequality symbol.

$$x < \frac{19}{3}$$

The solution set is $\left(-\infty; \frac{19}{3}\right)$.

3.3 Quadratic Equations (Kvadratické rovnice)

Definition 3.3. A **quadratic equation** in one variable is an equation that can be written in the standard quadratic form

$$ax^2 + bx + c = 0$$

where *a, b* and *c* are real numbers with $a \neq 0$.

Names of the coefficients: ***a*** is called **quadratic coefficient**, ***b*** **linear coefficient** and ***c*** is called **constant** or **free term**.

Při řešení kvadratických rovnic opět používáme ekvivalentní úpravy, jak byly uvedeny v kapitolách o lineárních rovnicích a nerovnicích. V případě neúplné kvadratické rovnice jsou tyto úpravy společně s druhou odmocninou postačující k vyřešení dané rovnice, pro normovanou kvadratickou rovnici můžeme použít Vietovy vztahy a v nejobecnějším případě kvadratické rovnice využijeme vzorec pro výpočet kořenů.

3.3.1 Incomplete Quadratic Equations (Neúplné kvadratické rovnice)

A quadratic equation with $c = 0$ is an equation written in the form $\boldsymbol{ax^2 + bx = 0}$ with $a \neq 0$, it is a **quadratic equation with no constant term**.

To solve this type of equation we **factor out** the variable *x* and get $x(ax + b) = 0$. Using Zero Product Property we can see that one root is always zero, $x_1 = 0$, and we get the second root solving $ax + b = 0$. This is a linear equation with solution $x_2 = -\frac{b}{a}$.

A quadratic equation with $b = 0$ is an equation written in the form $\boldsymbol{ax^2 + c = 0}$ with $a \neq 0$, it is a **quadratic equation with no linear term** (or pure quadratic equation).

This equation is equivalent to $x^2 = -\frac{c}{a}$. We can establish $-\frac{c}{a} = d$. Now there are two possibilities:

- If $d < 0$, then the equation has **no roots** in real numbers.
- If $d \geq 0$, then the equation has **two roots**, $x_1 = -\sqrt{d}, x_2 = \sqrt{d}$.

  If $b = c = 0$ then the quadratic equation $\boldsymbol{ax^2 = 0}$ has **single root** because $x_1 = x_2 = 0$. This root is called **double root**.

Example 3.5. Solve the equation $25x^2 - 9 = 0$.

Solution: Use properties of equality.

$$x^2 = \frac{9}{25}$$

The number on the right side of the equation is greater than zero which means the equation has two roots, $x_1 = -\frac{3}{5}, x_2 = \frac{3}{5}$. The solution set is K = $\left\{\pm\frac{3}{5}\right\}$.

Example 3.6. Solve the equation $4x^2 = -11x$.

Solution: Use properties of equality and factor out the variable *x*.

$$x(4x + 11) = 0$$

One root is $x_1 = 0$. For the second root solve $4x + 11 = 0$ using properties of equality.

$$x_2 = -\frac{11}{4}$$

The solution set is K = $\left\{-\frac{11}{4}; 0\right\}$.

3.3.2 The Quadratic Formula (Vzorec pro výpočet kořenů obecné kvadratické rovnice)

We use quadratic formula to find roots of general quadratic equations $ax^2 + bc + c = 0$ with $a \neq 0$.

If $ax^2 + bc + c = 0, a \neq 0$, then

$$x_{1,2} = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}.$$

The expression $b^2 - 4ac = D$ in the quadratic formula is called discriminant, its sign determines the number of roots in real numbers.

- If $\boldsymbol{D > 0}$, then the quadratic equation has **two distinct real roots**.
- If $\boldsymbol{D = 0}$, then the quadratic equation has a **real** root that is a **double root**.
- If $\boldsymbol{D < 0}$, then the quadratic equation has **no real root** (it has two nonreal complex roots).

Example 3.7. Solve the equation $3x = 4 - 2x^2$.

Solution: First write the equation in the standard quadratic form.

$$2x^2 + 3x - 4 = 0$$

Substitute in the quadratic formula.

$$x_{1,2} = \frac{-3 \pm \sqrt{9 + 32}}{4} = \frac{-3 \pm \sqrt{41}}{4}$$

The solution set is $K = \left\{\frac{-3 \pm \sqrt{41}}{4}\right\}$.

3.3.3 Vieta's Formulas (Vietovy vztahy)

If we divide each side of a quadratic equation by the coefficient a and establish $\frac{b}{a} = p, \frac{c}{a} = q$, we get an equation $x^2 + px + q = 0$ with roots x_1 and x_2. Then:

- The **sum** of the roots $x_1 + x_2 = -p$.
- The **product** of the roots $x_1 x_2 = q$.

3.4 Absolute Value Equations (Rovnice s absolutní hodnotou)

Definition 3.4. The **absolute value** of a real number x, denoted $|x|$, is the non-negative value of x without regard to its sign. $|x| = x$ if x is positive or zero and $|x| = -x$ if x is negative.

If a, b are **real numbers**, then:

- $|a| \geq 0$,
- $|a| = |-a|$,
- $|ab| = |a||b|$,
- $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, if $b \neq 0$.

Absolute value $|a|$ of a real number a is the **distance** between the number a and 0 on the real number line. Absolute value $|a - b| = |b - a|$ is the distance of a real number line between the graph of a and the graph of b.

Při výkladu tohoto učiva je vhodné postup řešení demonstrovat přímo na konkrétním příkladu, jelikož obecné řešení rovnice s absolutní hodnotou by nebylo dostatečně názorné. Uveďme tedy opět řešený příklad.

Example 3.8. Solve the equation $|x - 1| + 2x = 4$.

Solution: To use Definition 3.4 we need to know, when the expression in absolute value is negative and when is positive or zero.

$$x - 1 < 0 \text{ for } x \in (-\infty; 1)$$

$$x - 1 \geq 0 \text{ for } x \in \langle 1; \infty)$$

Solve the equation for $x \in (-\infty; 1)$: use Definition 3.4 for absolute value.

$$-x + 1 + 2x = 4$$

$$x = 3$$

Check if the solution belongs to the interval.

$$3 \notin (-\infty; 1)$$

The equation has no root in this interval.

Make the same steps for $x \in \langle 1; \infty)$.

$$x - 1 + 2x = 4$$

$$x = \frac{5}{3} = 1\frac{2}{3}$$

$$\frac{5}{3} \in \langle 1; \infty)$$

The solution set is $\mathrm{K} = \left\{\frac{5}{3}\right\}$.

4 Functions (Funkce)

Kapitola funkce následuje po kapitole rovnice a nerovnice (bezprostředně či s odmlkou v podobě jiného tématu). Pokud jsme tedy již metodu CLIL využili ve výuce rovnic a nerovnic, přirozeně se nabízí její využití i v učivu o funkcích, jelikož využijeme terminologii, se kterou jsme žáky již dříve seznámili. Samozřejmě předchozí zavedení anglické terminologie není podmínkou.

4.1 Introduction to Functions (Úvod k funkcím)

Začneme definicí funkce.

Definition 4.1. A **function** f from a set D to a set $\mathbb{R}$ is a **correspondence**, or **rule**, that **pairs** each element of D with **exactly one element** of $\mathbb{R}$. The set D is called the **domain** of f, and the set $\mathbb{R}$ is called the **range** of f.

A dále uvedeme vysvětlení pojmů, které se v definici vyskytují.

- The **domain** of a function is the **complete set** of **possible values** of the **independent variable**.
- The **range** of a function is the **complete set** of **all possible resulting values** of the **dependent variable** after we have substituted the domain.
- Functions are denoted by **letters** of a combination of letter, such as f, g or log.
- If x_0 is an **element** of the **domain** of a function f, then $f(x_0)$, read "f of x_0," is the element in the **range** of f that is associated with x_0, written $f(x_0) = y_0$.
- The process of **determining the value** of $f(x)$ is called **evaluating the function** f at x.

Example 4.1. For the function f defined by $f(x) = x^2 - 2x + 3$, evaluate $f(-1), f(3), f(5), f(10)$.

Solution: To evaluate the function we substitute the independent variable with given values from the domain.

$$f(-1) = (-1)^2 - 2(-1) + 3 = 1 + 2 + 3 = 6$$
$$f(3) = 3^2 - 2 \cdot 3 + 3 = 9 - 6 + 3 = 6$$
$$f(5) = 5^2 - 2 \cdot 5 + 3 = 25 - 10 + 3 = 18$$
$$f(10) = 10^2 - 2 \cdot 10 + 3 = 100 - 20 + 3 = 83$$

Example 4.2. Identify which of the following equations define y as a function of x:

a) $4y^2 + 8x = 16$,

b) $3y + 7x^3 + 9 = 2x$,

c) $x + 2y = 6$.

Solution: To identify which equation define y as a function of x we need to get the dependent variable y alone on the left side of the equation.

a) $4y^2 = 16 - 8x$

$y^2 = 4 - 2x$

$y = \pm\sqrt{4 - 2x}$

The right side of the equation is $\pm\sqrt{4 - 2x}$. That means it produces two different values for each value of x. Therefore, according to Definition 4.1, this equation does not define y as a function of x.

b) $3y = 2x - 7x^3 - 9$

$y = \frac{2}{3}x - \frac{7}{3}x^3 - 3$

The right side of the equation is $\frac{2}{3}x - \frac{7}{3}x^3 - 3$ which is a unique real number for each value of x. Therefore, this equation defines y as a function of x.

c) $2y = 6 - x$

$y = 3 - \frac{x}{2}$

Because the right side of the equation, $3 - \frac{x}{2}$, is a unique real number for each value of x, this equation defines y as a function of x.

4.2 Graphs of Functions (Graf funkce)

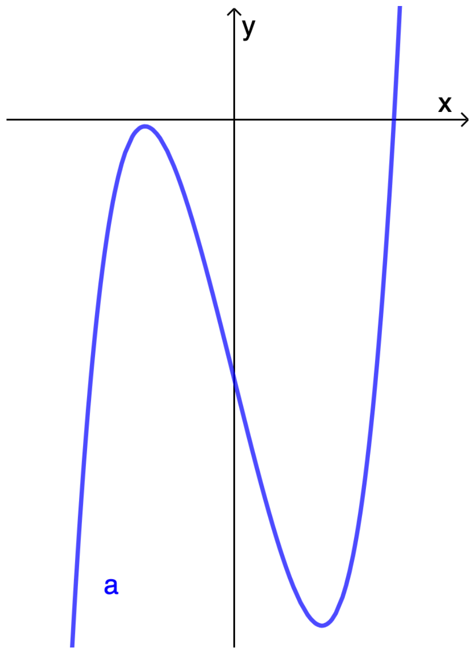
Definition 4.2. The **graph of a function** f in the coordinate plane Oxy is the **set of all points** $X[x, f(x)]$ where x is an element in the domain of f.

Jelikož jsme funkci v kapitole 4.1 definovali jako předpis, který přiřazuje každému x z definičního oboru funkce přiřazuje právě jedno $y = f(x)$ z oboru hodnot, uvedeme také pravidlo, jak poznat, zda je daný graf grafem funkce. V angličtině jej vyjadřuje tzv. The Vertical Line Test for Functions.

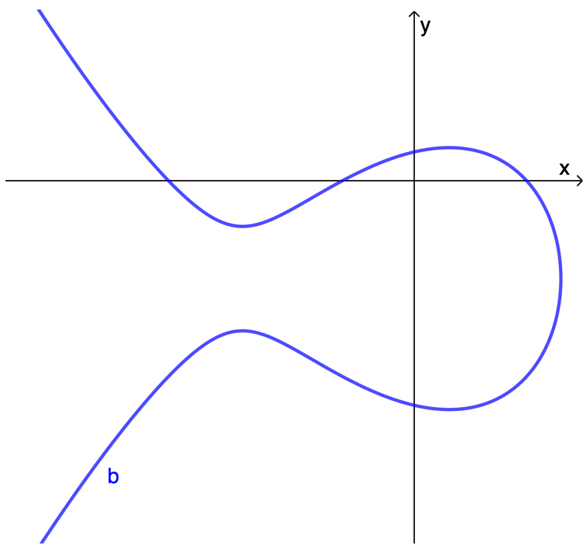
Theorem 4.3. A graph is the **graph of a function** if and only if **no vertical line intersects** the graph at **more than one point**.

Example 4.3. Use the Vertical Line Test to find out which of the following graphs are graphs of functions.

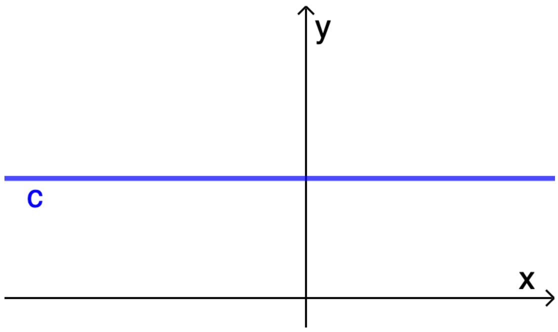
a)



b)



c)

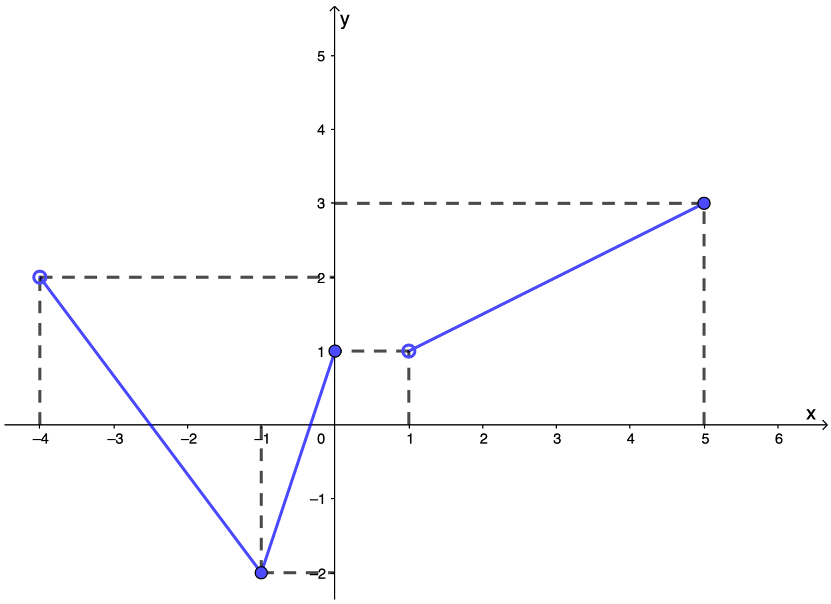


Solution: The graph a) is the graph of a function because every vertical line intersects the graph in at most one point.

The graph b) is not the graph of a function since we can find some vertical lines which intersect the graph in more than one point.

The graph c) is the graph of a constant function which is a special type of a function (we are talking about this function later in part 4.3).

Example 4.4. Determine the domain and the range of the function with the following graph:



Solution: The domain is the set of all possible values of the independent variable x. That means we read the values on the axis x. An empty point means the value is not an element of the domain, a full point, on the contrary, means the value is an element of the domain. The domain of the function is $x \in (-4; 0\rangle \cup (1; 5\rangle$.

The range is the set of all possible resulting values of the dependent variable after we have substituted the domain. That means we read the values on the axis y. The range of the function is $y \in \langle -2; 3\rangle$

Example 4.5. Determine the domain of the function $f: y = \frac{x-3}{x+2}$.

Solution: The domain is the set of all possible values that we can substitute x for. As for the given function f, it contains a fraction. We know that the denominator cannot be equal to zero which means:

$$x + 2 \neq 0$$

$$x \neq -2$$

The only value we cannot substitute x for is -2. Then the domain of the function is $D(f) = \mathbb{R} \setminus \{-2\}$.

4.3 Linear Function (Lineární funkce)

Definition 4.4. A **linear function** is a function that can by represented by an equation of the form

$$y = ax + b$$

where *a* and *b* are real constants.

A special type of linear functions is a linear function with $a = 0$, i.e., represented by an equation of the form

$$y = b$$

that is called a **constant function**.

A linear function with $b = 0$ is called **direct proportion** and is represented in the form

$$y = ax.$$

The **graph** of a **linear function** is a **nonvertical straight line**, the graph of a **constant function** is a **horizontal line**, the graph of a **direct proportion** is a **nonvertical straight line including the point** $O[0; 0]$.

The other way around, every nonvertical straight line is the graph of a linear function.

4.3.1 Increasing and Decreasing Function (Rostoucí a klesající funkce)

Nejprve definujeme funkci rostoucí a klesající na celém definičním oboru:

Definition 4.5. If a and b are elements of the domain of a function f, then f is **increasing** on $D(f)$ if $f(a) < f(b)$ whenever $a < b$, f is **decreasing** on $D(f)$ if $f(a) > f(b)$ whenever $a < b$.

A dále definujeme funkci rostoucí a klesající na intervalu:

Definition 4.6. If a and b are elements of an interval I that is a subset of the domain of a function f, then
f is **increasing** on I if $f(a) < f(b)$ whenever $a < b$,
f is **decreasing** on I if $f(a) > f(b)$ whenever $a < b$.

V souvislosti s funkcí klesající a rostoucí definujeme také funkci prostou:

Definition 4.7. A function f is a **one-to-one function** if no two elements of the domain of the function correspond to the same element of the range of the function.

This means that for every value of x, we get a unique value of $y = f(x)$.

Pravidlo, které určuje, zda je funkce prostá, se v angličtině nazývá <u>The Horizontal Line Test</u>.

Theorem 4.8. If any **horizontal line** intersects the graph of a function at most once, then the graph is the graph of a one-to-one function.

4.3.2 Properties of Linear Functions (Vlastnosti lineárních funkcí)

U lineárních funkcí můžeme snadno určit, zda je funkce rostoucí či klesající na celém svém definičním oboru v závislosti na koeficientu a.

Theorem 4.9. A linear function $y = ax + b$
is **increasing** if $a > 0$,
is **decreasing** if $a < 0$,
is not a one-to-one function if $a = 0$.

Pro směrnici přímky v grafu lineární funkce platí:

Theorem 4.10. The **slope** a of the line passing through the points $P_1[x_1; y_1]$ and $P_2[x_2; y_2]$ with $x_1 \neq x_2$ is given by

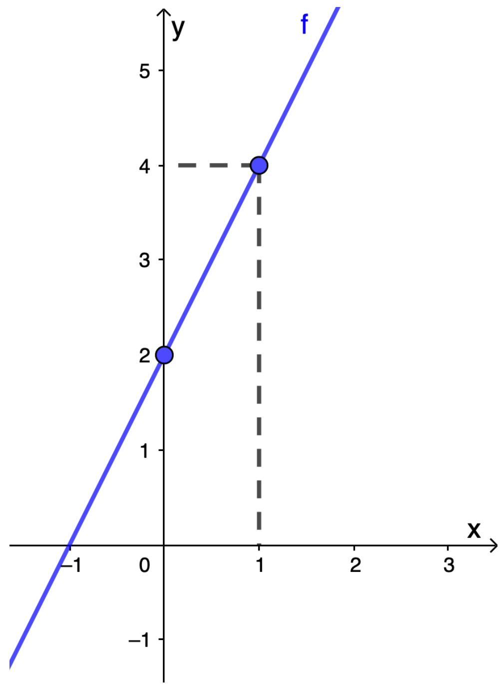
$$a = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Example 4.6. Sketch the graph of the function $f: y = 2x + 2$.

Solution: To sketch the graph of a linear function we need to know at least two points of the graph. The first step is to make a table with chosen values of the independent variable x and determined values of the dependent variable $f(x)$:

x	0	1
y	2	4

Since we know the graph of a linear function is a line and to draw a line two points are sufficient, we can sketch the graph in the coordinate plane:



Example 4.7. Decide whether the following functions are increasing or decreasing:

a) $f_1 : y = 2x - 3,$

b) $f_2 : y = -9x + 4,$

c) $f_3 : y = -4.$

Solution: Use Theorem 4.9.

a) $a > 0$ means the function f_1 is increasing,

b) $a < 0$ means the function f_2 is decreasing,

c) $a = 0$ means the function f_3 is a constant function. This type of function is neither increasing nor decreasing, it is not a one-to-one function.

4.4 Quadratic Functions (Kvadratické funkce)

Definition 4.11. A **quadratic function** is a function that can be represented by an equation of the form

$$y = ax^2 + bx + c$$

where a, b and c are real numbers and $a \neq 0$.

Theorem 4.12. The graph of a quadratic function is a **parabola**. The parabola is symmetric with respect to the axis of the parabola. The parabola opens up if $a > 0$, and it opens down if $a < 0$.

Example 4.8. Sketch the graph of the following functions $f: y = x^2$, $g: y = -x^2$, $h: y = 2x^2$, $i: y = -2x^2$, $j: y = \frac{1}{2}x^2$, $k: y = -\frac{1}{2}x^2$ in the same coordinate plane.

Solution: First we start with making tables with several values of the independent and the dependent variables for each function.

f	x	-3	-2	-1	$-0{,}5$	0	$0{,}5$	1	2	3
	y	9	4	1	$0{,}25$	0	$0{,}25$	1	4	9

g	x	-3	-2	-1	$-0{,}5$	0	$0{,}5$	1	2	3
	y	-9	-4	-1	$-0{,}25$	0	$-0{,}25$	-1	-4	-9

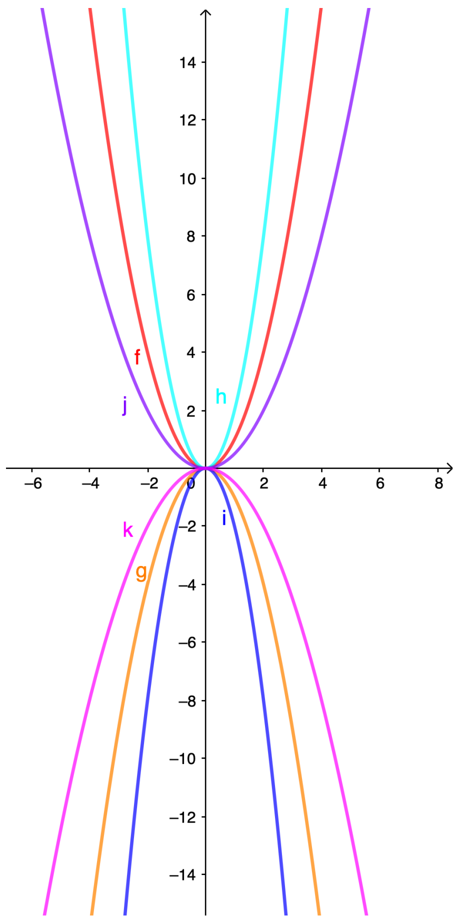
h	x	-3	-2	-1	$-0{,}5$	0	$0{,}5$	1	2	3
	y	18	8	2	$0{,}5$	0	$0{,}5$	2	8	18

i	x	-3	-2	-1	$-0{,}5$	0	$0{,}5$	1	2	3
	y	-18	-8	-2	$-0{,}5$	0	$-0{,}5$	-2	-8	-18

j	x	-3	-2	-1	$-0{,}5$	0	$0{,}5$	1	2	3
	y	$4{,}5$	2	$0{,}5$	$0{,}125$	0	$0{,}125$	$0{,}5$	2	$4{,}5$

k	x	-3	-2	-1	$-0{,}5$	0	$0{,}5$	1	2	3
	y	$-4{,}5$	-2	$-0{,}5$	$-0{,}125$	0	$-0{,}125$	$-0{,}5$	-2	$-4{,}5$

All the functions given are quadratic functions in the form $y = ax^2$. As we can see from the tables, all the graphs are parabolas symmetric with respect to the axis y with vertex $V[0; 0]$. According to the Theorem 4.12 the parabola opens up or down based on $a > 0$ or $a < 0$. Now we can sketch the graphs.



We can also say that the bigger the absolute value of the coefficient $|a|$ is the narrower the parabola is. The domain of all the functions is $\mathbb{R}$. The range is $\langle 0; \infty)$ for functions with $a > 0$ and $(-\infty; 0\rangle$ for functions with $a < 0$.

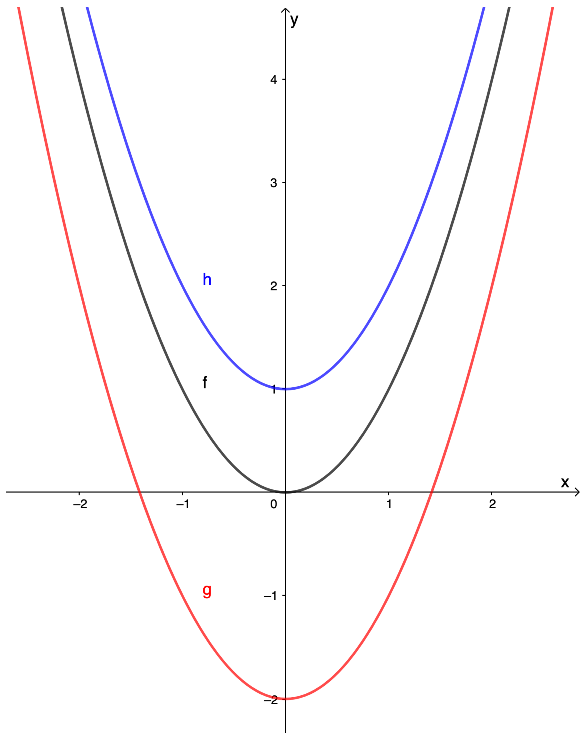
Example 4.9. Sketch the graph of the following functions: $f\colon y = x^2$, $g\colon y = x^2 - 2$, $h\colon y = x^2 + 1$ in the same coordinate plane.

Solution: From Example 4.8 we have already known the graph of the function $f\colon y = x^2$. To sketch the graphs of the other two functions we start with making tables for them.

g	x	-3	-2	-1	$-0,5$	0	$0,5$	1	2	3
	y	7	2	-1	$-1,75$	-2	$-1,75$	-1	2	7

h	x	-3	-2	-1	$-0,5$	0	$0,5$	1	2	3
	y	10	5	2	$1,25$	1	$1,25$	2	5	10

For every $x \in \mathbb{R}$ the function $g(x) = f(x) - 2$ and $h(x) = f(x) + 1$. That means the shape of all the graphs are exactly the same, only their position relative to the origin differs. The graphs of $g(x)$ and $h(x)$ are called vertical transitions of the graph of $f(x)$.
The graphs of the given functions are following.



We should mention the vertical transitions that we used in Example 4.9.

Theorem 4.13. If f is a function and c is a positive constant, then

$y = f(x) + c$ is the graph of $y = f(x)$ **shifted up** vertically c units.

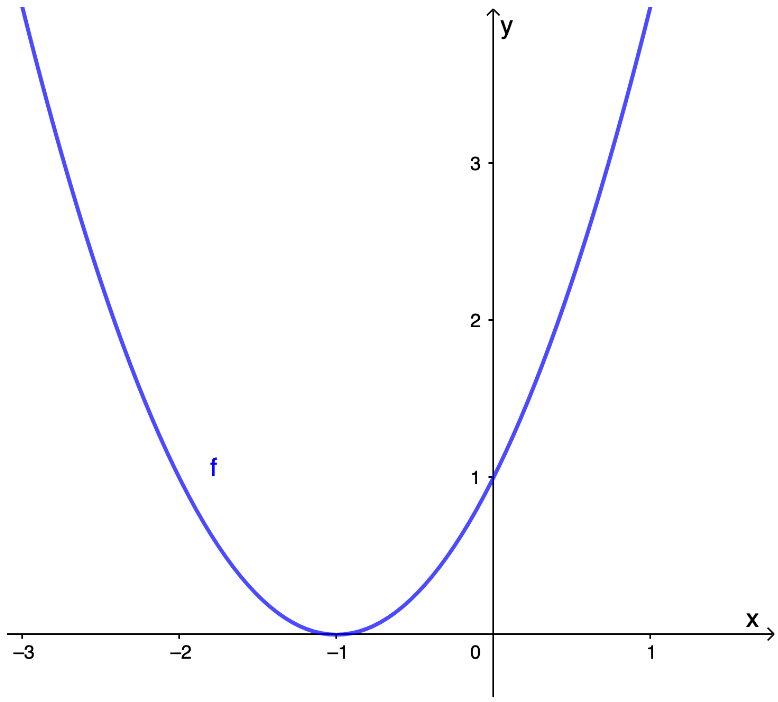
$y = f(x) - c$ is the graph of $y = f(x)$ **shifted down** vertically c units.

Example 4.10. Sketch the graph of the function $f_1 : y = (x+1)^2$.

Solution: First we start with a table for the function f_1.

f_1	x	-4	-3	-2	$-1{,}5$	-1	$-0{,}5$	0	1	2
	y	9	4	1	$0{,}25$	0	$0{,}25$	1	4	9

Then the graph is following.



As we can see, the shape of the graph of the function $f_1(x)$ in Example 4.10 is the same as the shape of the graph of the function f from the Examples 4.8 and 4.9, where each point is shifted horizontally to the left. This is called a horizontal translation of the graph of $f(x)$.

Theorem 4.14. If f is a function and c is a positive constant, then

$y = f(x + c)$ is the graph of $y = f(x)$ **shifted left** horizontally c units.

$y = f(x - c)$ is the graph of $y = f(x)$ **shifted right** horizontally c units.

To sketch the graph of a quadratic function in the general form (as defined in the Definition 4.11) we need to find the standard form of the quadratic function.

Theorem 4.15. Every quadratic function $f: y = ax^2 + bx + c$ can be written in the standard form

$$f: y = a(x - h)^2 + k \text{ with } a \neq 0.$$

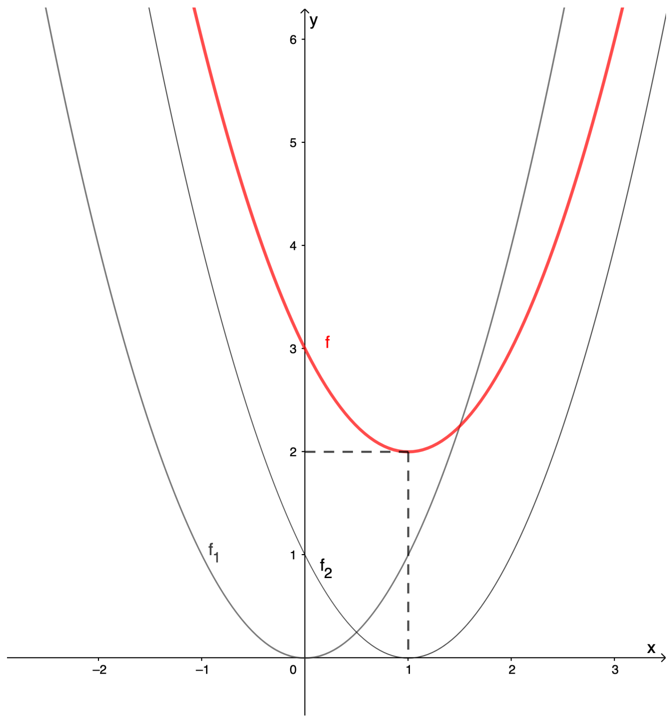
The graph of f is a parabola with vertex $V[h;k]$. The parabola is symmetric with respect to the vertical axis of the parabola $x = h$.

Example 4.11. Sketch the graph of the function $f: y = x^2 - 2x + 3$. Then determine the properties of the function f.

Solution: We need to find the standard form of the function:

$$f: y = (x - 1)^2 + 2.$$

Now we can sketch the graph step by step. At first, we sketch the graph $f_1: y = x^2$. Then we can use the horizontal transition to sketch the graph of $f_2: y = (x - 1)^2$. And finally, we use the vertical transition to sketch the graph of the given function $f: y = (x - 1)^2 + 2$.



From the graph of f we can determine the properties. The domain of the function is $\mathbb{R}$ as there is no value that the independent variable cannot be substituted for.

The range is $\langle 2;\infty)$. The function f is decreasing on $(-\infty;1\rangle$ and increasing on $\langle 1;\infty)$. According to The Horizontal Line Test the function f is not a one-to-one function because we can find a horizontal line that intersect the graph at more than one point.

5 Plane Geometry (Planimetrie)

Planimetrie bývá ve středoškolském učivu zařazována zpravidla do prvního nebo druhého ročníku. Vzhledem ke svojí názornosti je vhodným tématem pro výuku metodou CLIL.

5.1 Basic Geometric Figures (Základní geometrické útvary)

Výklad planimetrie začneme zavedením základních geometrických útvarů, jako je přímka, úsečka, úhel a další.

Definition 5.1. Two distinct points determine one, and only one, **straight line**.

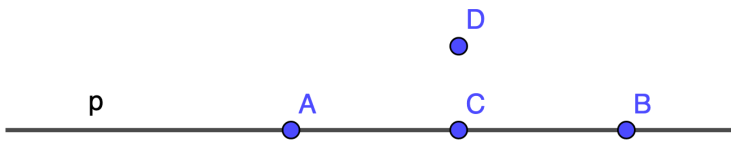


Figure 5.1: Straight Line

We usually label a point with a capital letter, a line with lowercase letter or two capital letters (representing two points on the line).

Dále zavedeme vzájemnou polohu bodu a přímky, polopřímku a úsečku.

- The points A and B **lie on the line** p. We also say the line p **passes through** points A and B, or that the line p contains points A and B. The notation is $A \in p, B \in p$.
- The point D **does not lie on the line** p. We also say that the line p does not contain the point D. The notation is $D \notin p$.
- Any point that lies on a straight line divides the line into two opposite **half lines** (also called rays).
  - For example, in the Figure 5.1, there we can see the point C between the points A and B. The point C is called the **common origin** of half lines $\longmapsto CA$ and $\longmapsto CB$.
- A **line segment** AB (where $A \neq B$) is the intersection of half lines AB and BA.

- The points A and B are called the **endpoints**.
- All the other points on the line segment AB are called the **interior points**.

○ The **length** of the line segment AB is the **distance** between the points A and B, which is a unique **positive real number** for each line segment. The notation is $|AB| = \mathrm{d}(A, B)$.

- Two line segments are **identical** if both of them are of the same length.
- The **sum** of two line segments with lengths a and b is a line segment with length $a + b$.
- The **subtraction** of two line segments with lengths a and b $(a > b)$ is a line segment with length $a - b$.

Definition 5.2. Any straight line in a plane divides the plane into two opposite **half planes** and the line is the common edge of the half planes.

5.1.1 Angles (Úhly)

Definition 5.3. An **angle** is a part of a plane formed by two half lines with a common origin.

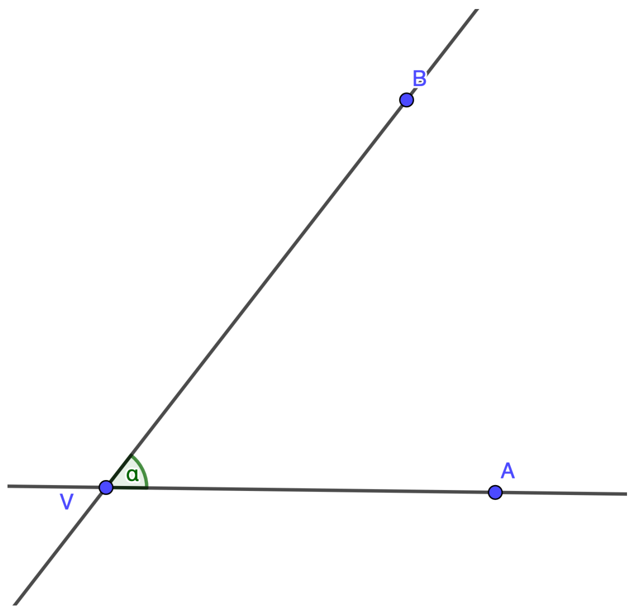


Figure 5.2: Angle AVB

- We usually label angles with lowercase letters of the Greek alphabet.
- The half lines VA and VB are called the **arms of the angle** AVB.
- The point V is called the **vertex of the angle** AVB.
- All the points that do not lie on the arms of the angle are called **interior points**.
- An angle greater than 0° and less than 180° is called a **convex angle**.
    - An **acute** angle is less than 90°.
    - The **right** angle is exactly 90°.
    - An **obtuse** angle is greater than 90° and less than 180°.
- A **straight angle** is exactly 180°.
- An angle greater than 180° and less than 360° is called a **reflex angle**.
- The **full rotation** is exactly 360°.

5.1.2 Relationship Among Two Lines (Vzájemná poloha dvou přímek)

- Two distinct lines are called **parallel** if they have no points of intersection.
- Two distinct lines are **nonparallel** lines if they have exactly one point of intersection.
- Two parallel lines are **identical** if they have infinitely many points in common.
- A line is said to be **perpendicular** to another line if the two lines intersect at a right angle.
    - The perpendicular to a given line through a given point is unique.

Figure 5.3: Parallel lines

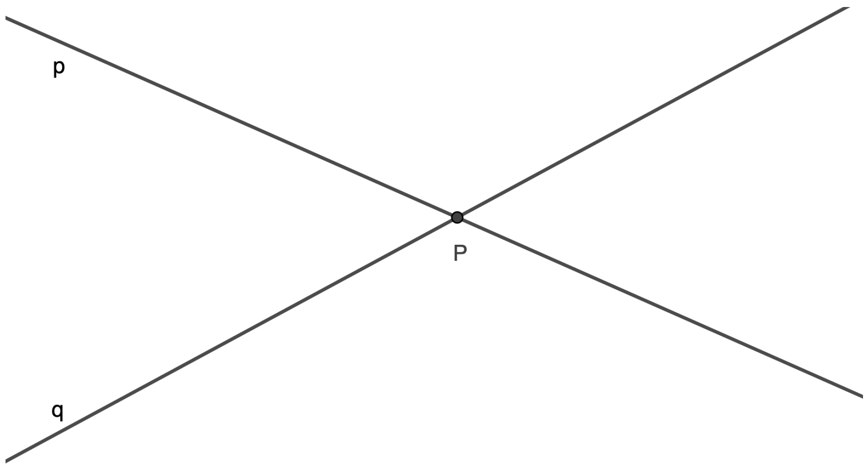


Figure 5.4: Nonparallel lines

Figure 5.5: Identical lines

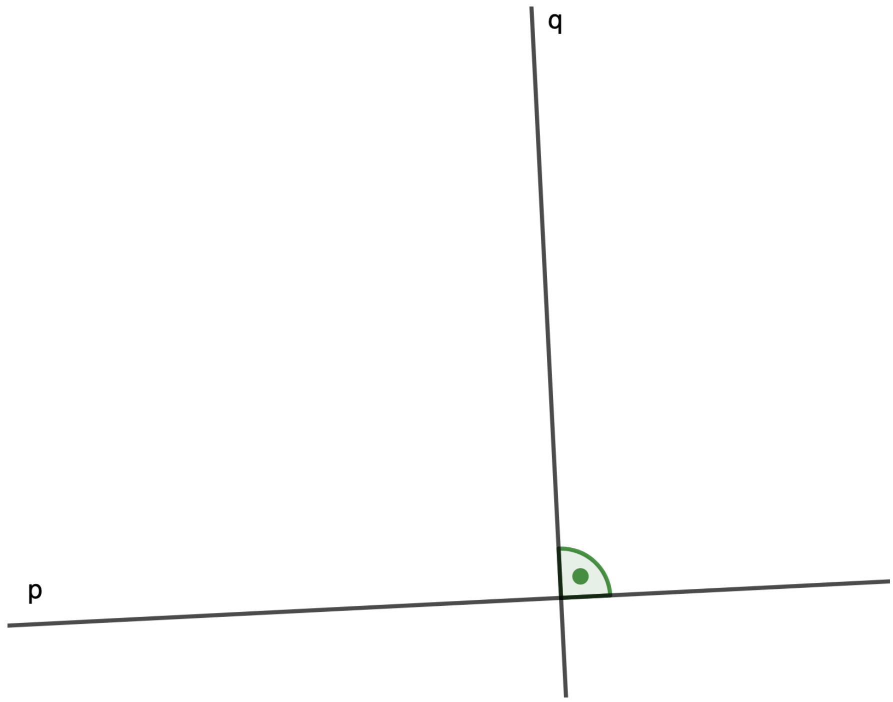


Figure 5.6: Perpendicular lines

5.2 Triangle (Trojúhelník)

Definition 5.4. A **triangle** ABC is the intersection of the half planes ABC, BCA and CAB, where the points A, B, C are distinct and noncollinear (do not lie on the same straight line).

- The points A, B, C are called the **vertices**.
- The symbolic notation of a triangle is $\triangle\, ABC$.
- The line segments $AB = c, BC = a, CA = b$ are called the **sides**, a, b, c are also the lengths of the sides.
- α, β, γ are the values of the **angles** at vertices A, B, C. The sum of the interior angles is $\alpha + \beta + \gamma = 180°$.
- The sum $o = a + b + c$ is the **perimeter** of the triangle.

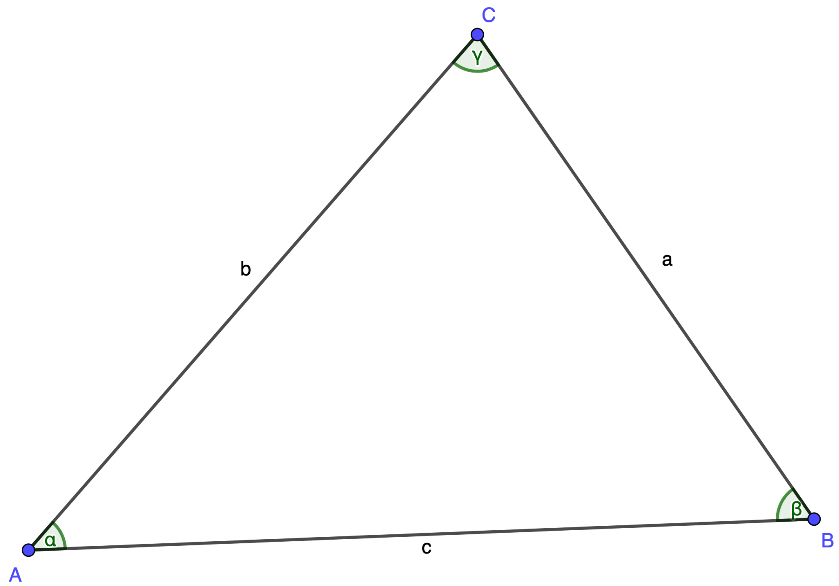


Figure 5.7: Triangle ABC

Triangles can be **classified** according to:

1) the **lengths of sides**:
   a) **Scalene** triangles have no equal sides and no equal angles.
   b) **Isosceles** triangles have two equal sides and two equal angles. The third side is called the base.

c) **Equilateral** triangles have three equal sides and three equal angles (always 60°). Equilateral triangles are a special type of isosceles triangles.

2) the **values of angles**:

a) **Acute-angled** triangles have all angles less than 90°.

b) **Right-angled** triangles have a right angle.

c) **Obtuse-angled** triangles have one angle greater than 90°.

For every triangle applies the triangle inequality:

Theorem 5.5. In any triangle, the sum of the lengths of any two sides is greater than the length of the remaining side.

Theorem 5.6. The line segments with lengths a, b, c are the sides of a triangle if and only if $|b - c| < a < b + c$.

Na závěr uvedeme důležité body a úsečky v trojúhelníku.

- Points A_1, B_1, C_1 are the **mid-points** of sides a, b, c (respectively), e.g., $|AB_1| = |B_1C| = \frac{1}{2}|AC|$.
- Line segments A_1B_1, B_1C_1, A_1C_1 are called the **midlines** of the triangle.
  - Each midline is parallel to the side of the triangle midpoints of which it does not connect. The length of the midline is half the length of the parallel side. E.g., $A_1B_1 \parallel AB$, $|A_1B_1| = \frac{1}{2}|AB|$.
- Line segments $AA_1 = t_\mathrm{a}, BB_1 = t_\mathrm{b}, CC_1 = t_\mathrm{c}$ are the **medians** of the triangle to the sides a, b, c (respectively).
  - The intersection of the medians is the **centroid** T.
- Points A_0, B_0, C_0 are the **feet of the altitudes** (or heights). They are the intersection of each side and the perpendicular to the side through the opposite vertex.
- Line segments $AA_0 = v_\mathrm{a}, BB_0 = v_\mathrm{b}, CC_0 = v_\mathrm{c}$ are the **altitudes** to the sides a, b, c (respectively).
  - The intersection of the altitudes is called the **orthocentre** O.

5.2.1 Circumscribed and Inscribed Circle (Kružnice opsaná a vepsaná)

Theorem 5.7. All three perpendicular bisectors o_a, o_b, o_c intersect in one point that is the **circumcentre** S_o.

A **circumscribed circle** k_o is a circle that passes through all three vertices of the triangle.

Theorem 5.8. All three interior angle bisectors $o_\alpha, o_\beta, o_\gamma$ intersect in one point that is the **incentre** S_v.

An **inscribed circle** k_v is a circle that touches all three sides of the triangle.

5.2.2 Congruence of Triangles (Shodnost trojúhelníků)

Definition 5.9. Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are called congruent if they can be transformed so that they fit exactly on each other. The notation is $\triangle ABC \cong \triangle A'B'C'$.

We usually do not transform triangles to find out if they are congruent. We use Triangle Congruence Postulates instead.

SSS Postulate. If the **three sides** of one triangle are equal to the three sides of another triangle, the triangles are congruent. The acronym SSS stands for side-side-side.

ASA Postulate. If **two angles** and the **included side** of one triangle are equal to two angles and the included side of another triangle, the triangles are congruent. The acronym ASA stands for angle-side-angle.

SAS Postulate. If **two sides** and the **included angle** of one triangle are equal to two sides and the included angle of another triangle, the triangles are congruent. The acronym SAS stands for side-angle-side.

SsA Postulate. If **two sides** and the **angle opposite the bigger side** of one triangle are equal to two sides and the angle opposite the bigger side of another triangle, the triangles are congruent. The acronym SsA stands for bigger side-smaller side-angle opposite the bigger side.

5.2.3 Similarity of Triangles (Podobnost trojúhelníků)

Theorem 5.10. Triangles ABC and $A'B'C'$ are called similar if, and only if there exists a positive real number k, so that, for the corresponding sides, $|A'B'| = k \cdot |AB|$, $|B'C'| = k \cdot |BC|$, $|C'A'| = k \cdot |CA|$, or $c' = k \cdot c$, $a' = k \cdot a$, $b' = k \cdot b$. The number k is called the length ratio.

To find out if two triangles are similar, we use Triangle Similarity Postulates.

AA Postulate. If **two angles** of one triangle are congruent to two angles of another triangle, the triangles are similar. The acronym AA stands for angle-angle.

SAS Postulate. If an **angle** of one triangle is congruent to an angle of another triangle and the lengths of the **sides including these angles** are proportional, the triangles are similar. The acronym SAS stands for side-angle-side.

SSS Postulate. If the **lengths of corresponding sides** of two triangles are proportional, the triangles are similar. The acronym SSS stands for side-side-side.

5.2.4 Euclidean Theorems, Pythagoras' Theorem (Eukleidovy věty, věta Pythagorova)

V této kapitole se budeme věnovat pravoúhlému trojúhelníku.

Definition 5.11. A **right-angled triangle** is a triangle with one right interior angle.

- The notation in a triangle ABC is usually following:
  - The right angle lies at the vertex C.
  - The sides CA and CB (i.e., sides including the right angle) are called **legs** of the triangle ABC.
  - The side AB (i.e., the side opposite the right angle) is called the **hypotenuse** of the triangle ABC.

There are two Euclidean Theorems for right-angled triangle. English translation of the Czech names of these theorems are "Euclidean Altitude Theorem" and "Euclidean Leg Theorem".

The notation of points and line segments is as in the following Figure 5.8.

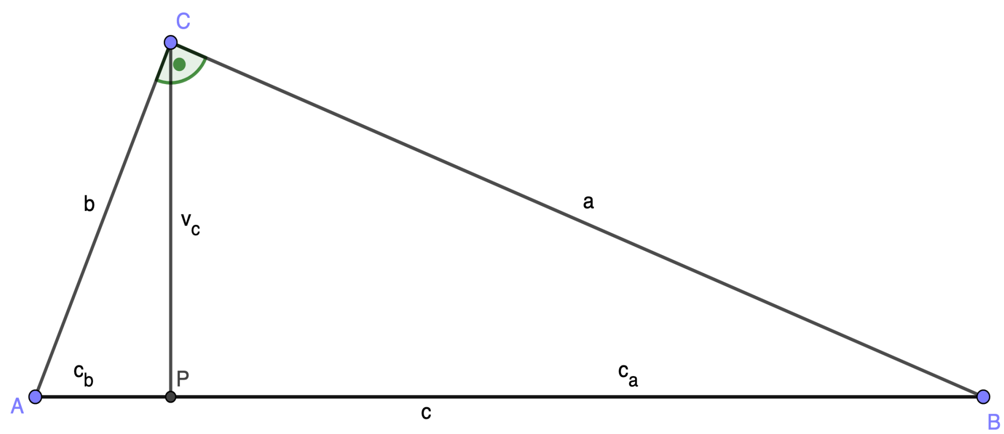


Figure 5.8: Right-angled triangle ABC

Euclidean altitude theorem. In every right-angled triangle ABC with the right angle at the vertex C, the square of the altitude v_c is equal to the product of the lengths of the line segments $AP = c_b$ and $PB = c_a$, where P is the foot of the altitude v_c:

$$v_c^2 = c_a \cdot c_b.$$

Euclidean leg theorem. In every right-angled triangle ABC with the right angle at the vertex C, the square of a leg is equal to the product the hypotenuse and the adjacent line segment of the hypotenuse:

$$a^2 = c \cdot c_a,$$

$$b^2 = c \cdot c_b.$$

Pythagoras' Theorem. In every right-angle triangle ABC with the right angle at the vertex C, the square of the hypotenuse is equal to the sum of the squares of both legs:

$$c^2 = a^2 + b^2.$$

5.3 Circle, Disc (Kružnice, kruh)

Definition 5.12. A **circle** $k(S; r)$ is the set of all points X in a plane that lie in the same distance r from a fixed point S.

- A circle is determined by its **centre** S and **radius** r.
- A line segment SX (X is any point of the circle) is also called the radius.

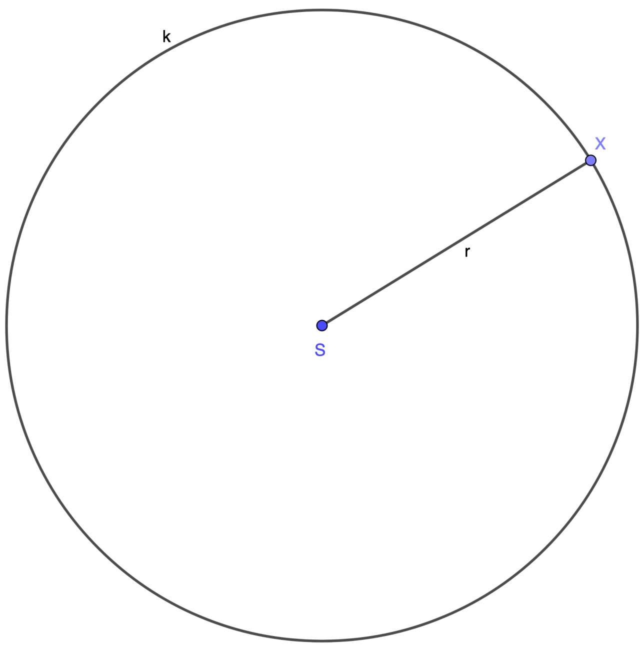


Figure 5.9: Circle $k(S; r)$

Definition 5.13. A **disc** $K(S; r)$ is the region in a plane bounded by a circle. A disc is said to be closed if it contains the circle that constitutes its boundary, and open if it does not.

5.3.1 A Circle and A Straight Line (Přímka a kružnice)

The relationship among a circle and a straight line is determined by number of their points of intersection.

- With **no point of intersection**, $p_1 \cap k = \emptyset$, the line is called an **external line**.
- With exactly **one point of intersection**, $t \cap k = \{T\}$, the line is called a **tangent** to the circle. The point T is called a **point of contact**.
- With **two points of intersection**, $p_2 \cap k = \{A, B\}$, the line is called a **secant**.
  - The line segment AB is called a **chord**.
  - A chord divides the circumference of the circle into two **arcs**. If an arc contains the endpoints A, B, it is called a closed arc, and it is called an open arc if it does not.
  - A chord that passes through the centre S is called the diameter d, $d = 2r$, and it divides the circle into two semicircles.

All three possibilities of the relationship among a circle and a line are shown in Figure 5.10.

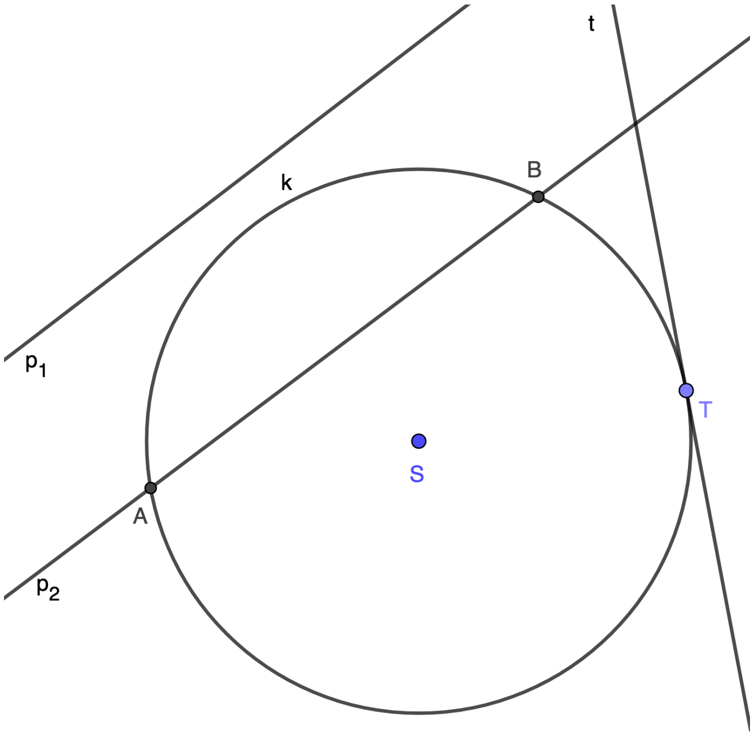


Figure 5.10: Relationship among a circle and lines

5.3.3 Two Circles (Dvě kružnice)

Two circles with different radius may have no, one or two points of intersection.

- Two circles with the **same centre** S are called **concentric** circles.
  - If they have the same radius S, the circles are **identical**.
  - Concentric circles with different radii r_1 and r_2 have no point of intersection. The part of a plane between two concentric circles is called an **annulus**.
- Two circles with **different centres** S_1 and S_2 are called **eccentric**. The relationship among two eccentric circles is one of the following.
  - The circles **lie outside each other** if $|S_1S_2| > r_1 + r_2$.
  - The circles **touch each other externally** if $|S_1S_2| = r_1 + r_2$.
  - The circles **intersect** each other at two points if $|r_1 - r_2| < |S_1S_2| < r_1 + r_2$.
  - The circles **touch each other internally** if $|S_1S_2| = |r_1 - r_2|$.
  - One circle **lies inside the other** circle if $|S_1S_2| < |r_1 - r_2|$.

5.3.4 Angles Subtended by An Arc (Úhly příslušné oblouku kružnice)

Definition 5.14. An angle whose vertex is the centre S of a circle and whose arms pass through endpoints A, B of an arc is called an **angle at the centre** of the circle subtended by the arc AB.

Definition 5.15. An angle whose vertex is a point on the circumference of a circle and whose arms pass through endpoints A, B of an arc is called an **angle at the circumference** of the circle subtended by the arc AB.

Theorem 5.16. The angle at the centre of a circle is twice the angle at the circumference when both are subtended by the same arc.

- All angles subtended at the circumference by the same arc are equal.
- An angle subtended at the circumference by the smaller arc is an acute angle.
- An angle subtended at the circumference by the greater arc is an obtuse angle.
- An angle subtended at the circumference by a semicircle is a right angle.

Thales' theorem. The diameter of a circle always subtends a right angle to any point on the circle.

5.4 Construction of a Triangle (Konstrukce trojúhelníku)

V této části kapitoly uvedeme pouze ilustrační příklady k nastínění užití anglické terminologie.

Example 5.1. Given the line segment $AB, |AB| = 6$ cm, complete the triangle ABC where $a = 5$ cm, $t_c = 5$ cm.

Solution:

1) Analysis

We draw a sketch of the triangle ABC as if it has been already solved and highlight what we know about the triangle.

We have the line segment AB, then we know $a = |BC| = 9$ cm, so the vertex C lies on a circle k with centre B and radius $r_k = 5$ cm. And $t_c = |CC_1| = 5$ cm, therefore the vertex C also lies on a circle l with centre C_1 and radius $r_l = 5$ cm.

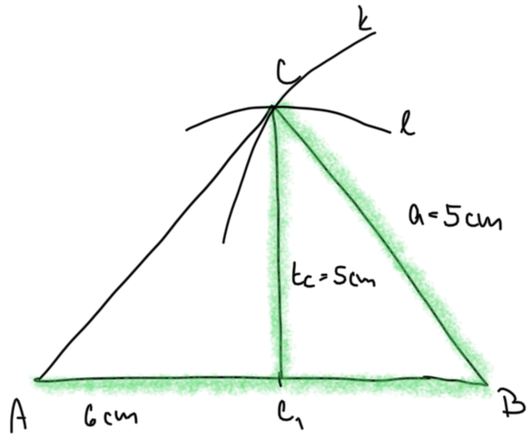


Figure 5.11: Triangle ABC (sketch)

2) Construction

1. construct C_1; $C_1 \in AB$, $|AC_1| = |C_1B|$
2. circle k; centre B, radius $r_k = 5$ cm
3. circle l; centre C_1, radius $r_l = 5$ cm
4. vertex C; $C \in k \cap l$
5. triangle ABC

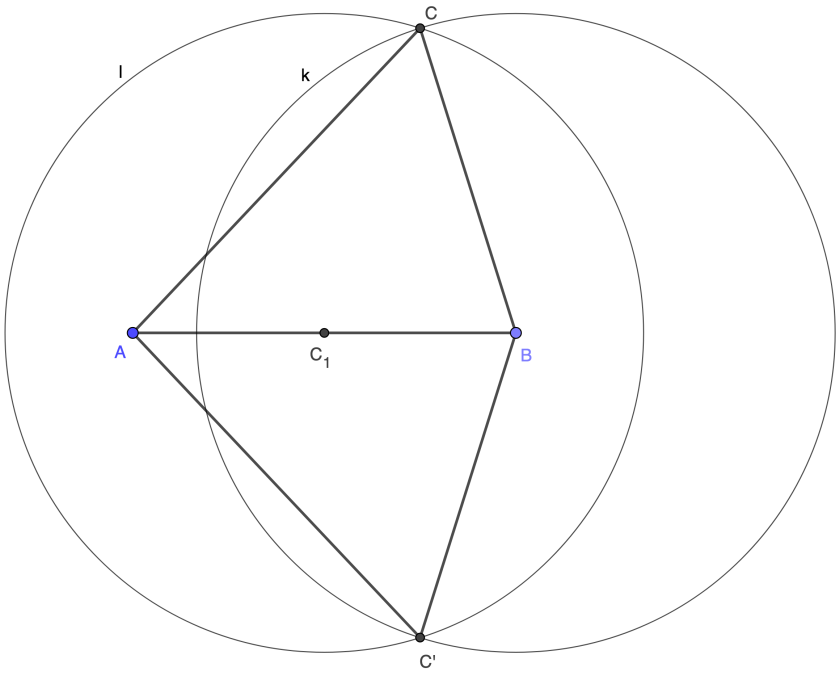


Figure 5.12: Triangle ABC (construction)

3) Check

The vertex C is the point of intersection of circles $k(B; 5\text{ cm})$ and $l(C_1; 5\text{ cm})$, therefore it is the required vertex of the triangle ABC.

4) Discussion

The task has two solutions.

Example 5.2. Given $a = 9\text{ cm},\ v_b = 4{,}5\text{ cm},\ t_a = 2{,}5\text{ cm},$ complete the triangle ABC.

Solution:

1) Analysis

We start with placing the altitude BB_0. The point B_0 is the foot of the altitude v_b, so vertices A and B lie on a perpendicular to the line segment BB_0 through B_0. Then we know $a = |BC| = 9\text{ cm}$ and therefore the vertex C lies on a circle k with centre B and radius $r_k = 9\text{ cm}$.

The last vertex, A, lies on a circle l with centre A_1 and radius $r_l = t_a = 2{,}5$ cm, where A_1 is the midpoint of the line segment BC.

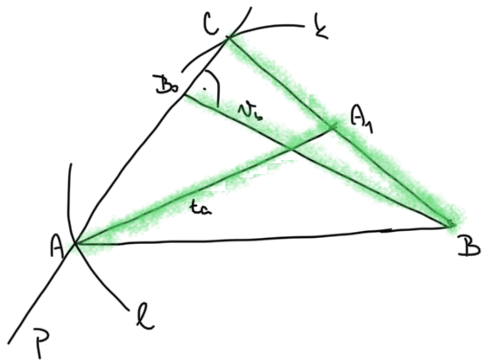


Figure 5.13: Triangle ABC (sketch)

2) Construction

1. place the altitude BB_0; $|BB_0| = v_b = 4{,}5$ cm
2. line p; p is perpendicular to BB_0 through the point B_0
3. circle k; centre B, radius $r_k = a = 9$ cm
4. vertex C; $C \in p \cap k$
5. point A_1; $A_1 \in BC$, $|A_1B| = |A_1C|$
6. circle l; centre A_1, radius $r_l = t_a = 2{,}5$ cm
7. vertex A; $A \in p \cap l$
8. triangle ABC

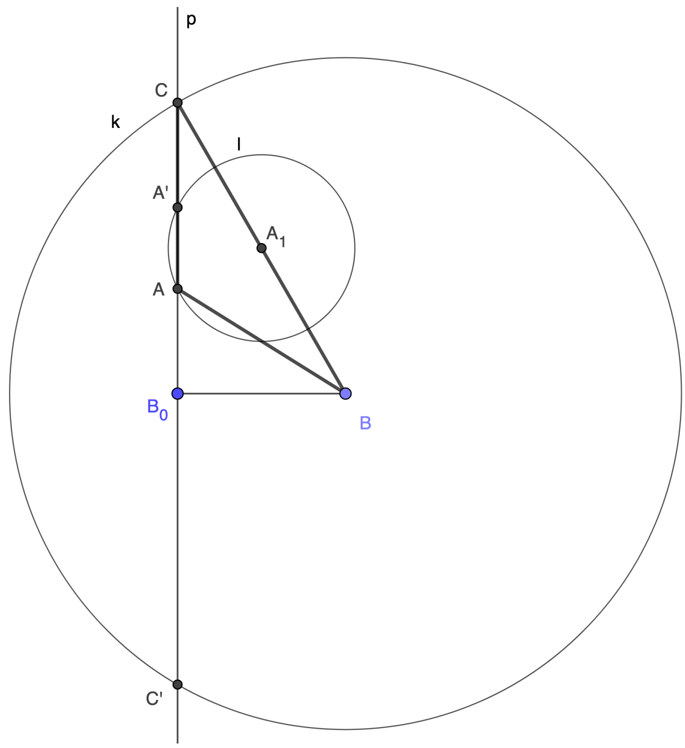


Figure 5.14: Triangle ABC (construction)

3) Check

The vertices A and C are points of intersection of the line p and circles l and k (respectively), therefore they are the required vertices of the triangle ABC.

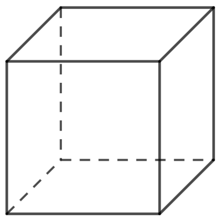
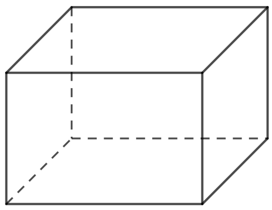
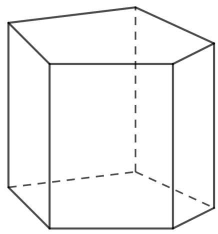
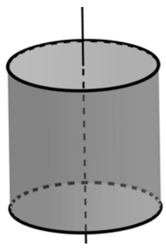
4) Discussion

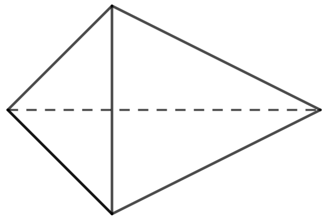
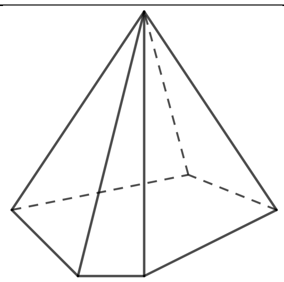
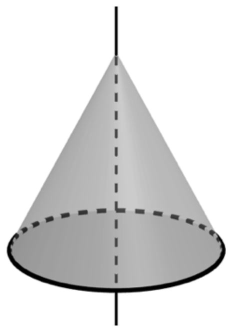
The task has four solutions.

6 Stereometry (Stereometrie)

6.1 Solids (Tělesa)

Nejprve uvedeme přehled základních geometrických těles, která žáci již znají. Jedná se tedy o opakování učiva ze základní školy, při kterém můžeme snadno zavést anglickou terminologii.

Name	Figure	Characteristics
CUBE		All six faces are identical squares.
CUBOID		It has six faces; the opposite faces are identical rectangles.
PRISM		Bases are identical polygons; other faces are parallelograms.
ROTATING CYLINDER		Bases are identical circles. It is the result of rotating a rectangle about one of its sides.

TETRAHEDRON		All four sides are triangles.
PYRAMID		The base is a polygon, other sides are triangles that meet at the apex.
ROTATING CONE		It is the result of rotating a right-angled triangle about one of its legs.

6.2 Parallel Projection (Volné rovnoběžné promítání)

When we want to draw a three-dimensional object into a plane (in high-school Maths), we usually use the **parallel projection**.

We draw projected objects into a projection plane and usually in a way that a part of the object (e.g., a face or an edge) lies in the projection plane. Lengths of line segments of the parts lying in the projection plane or parallel to it do not change. Lengths of line segments perpendicular to the projection plane are reduced to a half and these line segments are projected at an angle of 45°.

Parallel line segments with the same length are projected as parallel line segments with the same length.

A figure that lies in the projection plane or in a plane parallel to the projection plane is projected into an identical figure.

6.3 Points, Lines and Planes (Body, přímky a roviny)

Points, lines, and planes are subsets of the three-dimensional space.

- All points that lie in the same line are called **collinear** points.
- All points that lie in the same plane are called **coplanar** points.

Theorem 6.1. Through any two points there exists exactly **one line**.

Theorem 6.2. If a point A lies on a line p and the line p lies in a plane ρ, then the point A also lies in the plane ρ.

If two distinct points lie in a plane, then the line containing the points lies in the plane.

Theorem 6.3. Through any three non-collinear points there exists exactly **one plane**.

Through a line and a point not-lying on the line there exists exactly one plane.

Through two distinct lines there exists exactly one plane.

Theorem 6.4. A plane divides the three-dimensional space into two **half-spaces**.

6.3.1 Relationship Among Two Lines (Vzájemná poloha dvou přímek)

Vzájemná poloha přímek v prostoru je pro představu složitější, než tomu bylo v rovině, a proto je žádoucí použít alespoň náčrtek, nebo lépe prostorový model krychle.

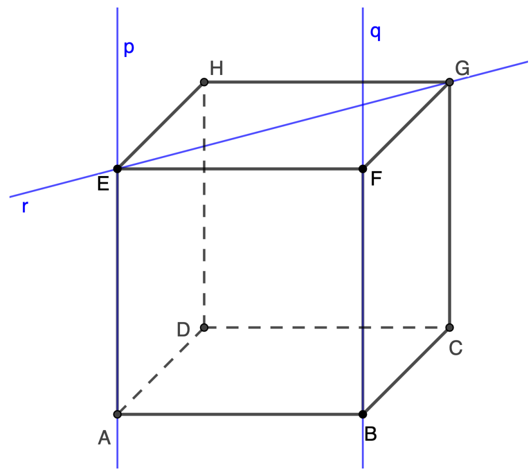


Figure 6.1: Relationship among two lines

- Two lines that lie in the same plane and do not intersect are called distinct **parallel** lines. Lines p and q in Figure 6.1 are distinct parallel lines.
  - Two parallel lines that have infinitely many points of intersection are identical lines.
- Two lines that lie in the same plane and have exactly one point of intersection, are called **concurrent** lines. Lines p and r in Figure 6.1 are concurrent lines; their point of intersection is the point E.
- Two lines that do not intersect and are not parallel (do not lie in the same plane) are called **skew** lines. Lines q and r in Figure 6.1 are skew lines.

6.3.2 Relationship Among A Line and A Plane (Vzájemná poloha přímky a roviny)

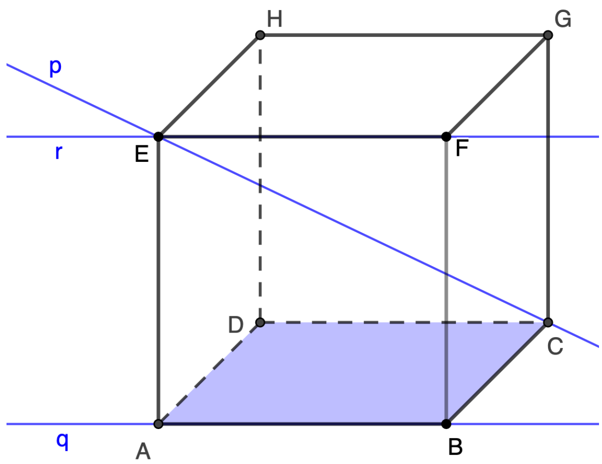


Figure 6.2: Relationship among a line and a plane

- If a line and a plane have exactly one point of intersection, they are called **intersecting**. The line p and the plane ABC in Figure 6.2 are intersecting, their point of intersection is the point C.
- If a line has no point of intersection with a plane, they are called **parallel**. The line r and the plane ABC in Figure 6.2 are parallel.

- If a line has at least two points in common with a plane, then they are parallel, and the **line lies in the plane**. The line q lies in the plane ABC in Figure 6.2.

6.3.3 Relationship Among Two Planes (Vzájemná poloha dvou rovin)

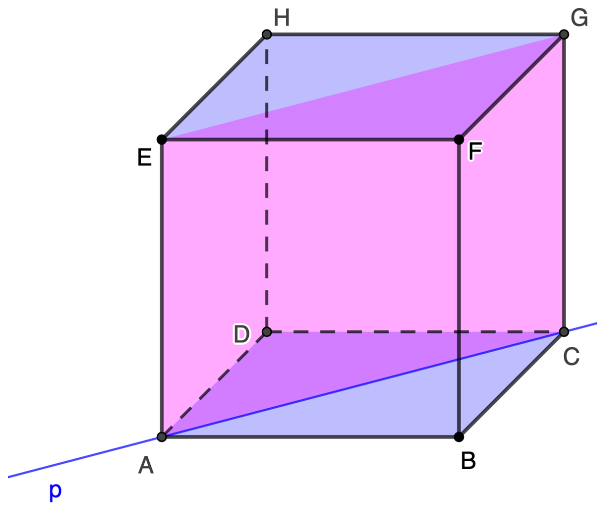


Figure 6.3: Relationship among two planes

- Two planes are called **parallel** if and only if they do not have any points of intersection or they have infinitely many points of intersection. In that case they are **identical**. The planes ABC and EFG in Figure 6.3 are parallel.
- Two planes are called **intersecting** if and only if they have exactly one line of intersection. The planes ABC and ACG in Figure 6.3 are intersecting, their line of intersection is the line $p = AC$.

6.3.4 Relationship Among Three Planes (Vzájemná poloha tří rovin)

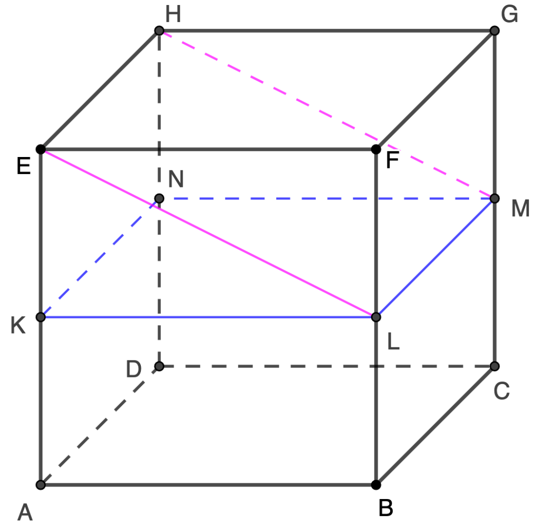


Figure 6.4: Relationship among three planes

Relationship among three distinct planes may be one of the following:

- If each two planes have no points of intersection, then all three planes together have no points of intersection; they are parallel planes. The planes ABC, KLM and EFG in Figure 6.4 are parallel.
- Two planes are parallel and the third one intersects with both – this plane is called a section plane. In Figure 6.4, the planes KLM and EFG are parallel and the plane EHL is the section plane with lines of intersection LM and EH (respectively).
- Each two planes have the same line of intersection – the three planes are intersecting with exactly one line of intersection. The planes KLM, BCG and EHL in the Figure 6.4 are intersecting planes with the line of intersection LM.
- Each two planes have distinct lines of intersection that are parallel lines. All three planes have no points in common. In Figure 6.4, the planes ADH, KLM and EHL have lines of intersection KN, LM and EH that are all parallel lines.
- Each two planes have distinct lines of intersection that intersect in one point – this point is a common point of all three planes. In Figure 6.4,

the planes ADH, DCG and EHL have exactly one point of intersection, the point H.

6.4 Sections of Solids (Řezy těles)

Definition 6.5. A **section** of a solid is its intersection with a plane.

When constructing sections of solid we use following theorems.

Theorem 6.6. A line segment that connects two points of a plane lies in the plane.

Theorem 6.7. Two parallel planes intersect the section plane in two parallel lines.

Theorem 6.8. If each two planes have a line of intersection and all three planes have exactly one point in common, then all three lines of intersection pass through the common point.

Example 6.1. Construct a section of a cube $ABCDEFGH$ by a plane CIJ; J is the midpoint of line segment AE and $|AI|:|IB| = 3:1$.

Solution:

1) Using Theorem 6.6, construct line segments IJ and IC (magenta line segments).
2) Using Theorem 6.7, construct a parallel line to the line segment IJ through the point C (blue line segment), and construct its intersection with the line segment DH (the point K).
3) Using Theorem 6.6 again, construct a line segment JK (green line segment).
4) The section is the polygon $CIJK$.

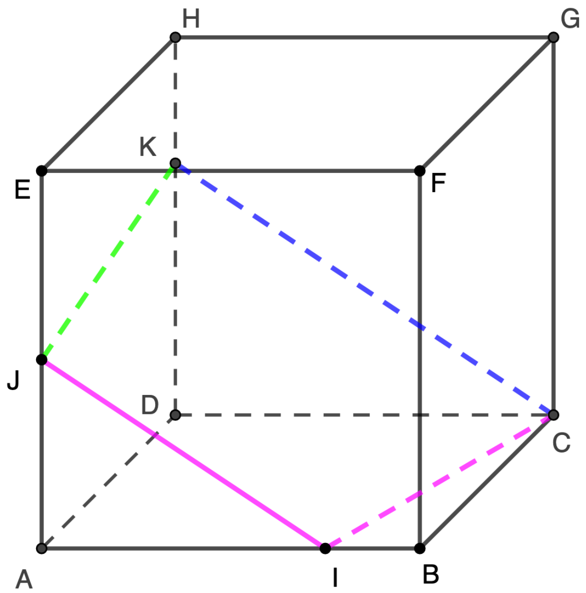


Figure 6.5: A section of a cube $ABCDEFGH$ by a plane CIJ

Example 6.2. Construct a section of a cube $ABCDEFGH$ by a plane XYZ; X is the midpoint of the edge AE, Y is the midpoint of the edge AB and $|CZ|:|ZG| = 2:1$.

Solution:

1) Construct line segment XY (magenta line segment).
2) Construct a parallel line to the line segment XY through the point Z. Construct its intersection with the line segment HG (the point I).
3) Construct the point 1 that is the point of intersection of lines DH and IZ. This point lies in the same plane as the point X.
4) Construct a line through points 1 and X, and then its intersection with the line segment EH (the point K).
5) Construct line segments KX, IK and IZ (red line segments).
6) Construct a parallel line to the line segment IK through the point Y and its intersection with the line segment BC (the point L).
7) Construct the line segments LY and LZ (blue line segments).
8) The section is the polygon $KIZLYX$.

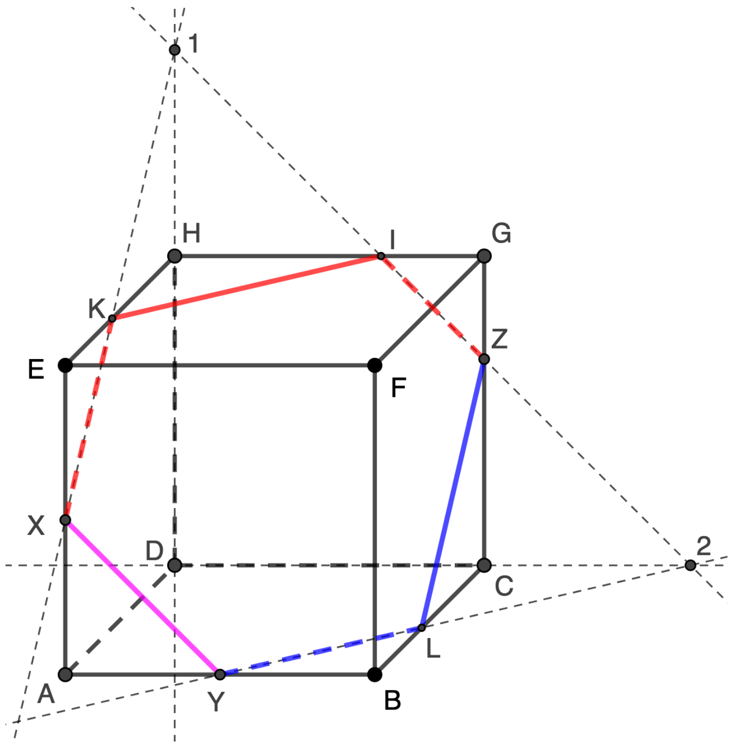


Figure 6.6: A section of a cube $ABCDEFGH$ by a plane XYZ.

Example 6.3. Construct a section of a regular quadrilateral pyramid $ABCDV$ by a plane XYZ; X is the midpoint of the edge AV, $|BY|:|YV| = 7:1$, $|CZ|:|ZV| = 1:3$.

Solution:

1) Construct points of intersection of lines AB, XY (the point 1), and BC, YZ (the point 2).
2) Construct a point of intersection of lines 12 and CD (the point 3).
3) Construct a line $3Z$ and its intersection with the line segment DV (the point L).
4) The section is the polygon $LXYZ$.

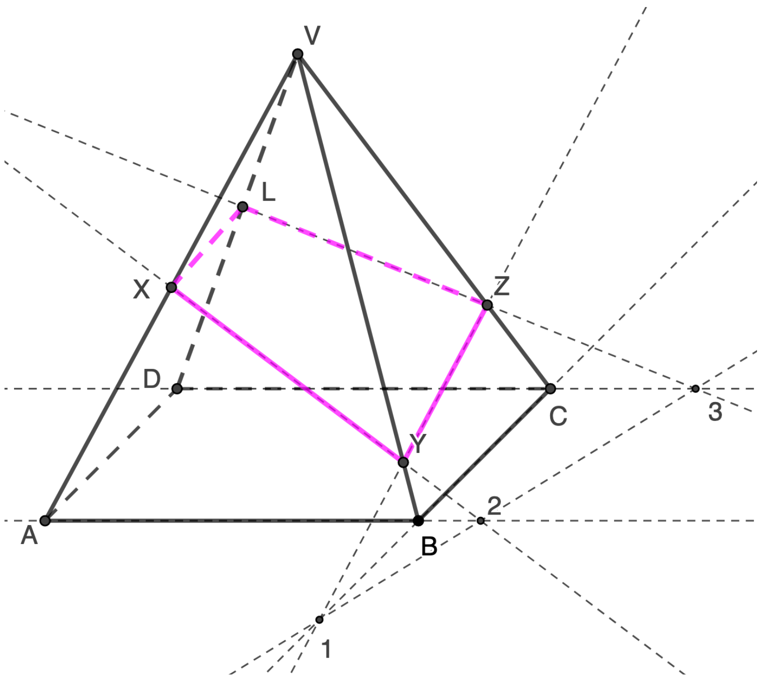


Figure 6.7: A section of a regular quadrilateral pyramid $ABCDV$ by a plane XYZ

7 Complex Numbers (Komplexní čísla)

Učivo komplexních čísel není zahrnuto v RVP pro střední školy. Nápad zařadit toto téma do této diplomové práce vzešel ze spolupráce s Gymnáziem Jiřího Wolkera v Prostějově, kde je kapitola komplexních čísel zařazena do tematického plánu třetího ročníku čtyřletého a sedmého ročníku osmiletého studia. V posledních letech je navíc tato kapitola v septimě vyučována v anglickém jazyce.

7.1 Complex Numbers and Their Properties (Komplexní čísla a jejich vlastnosti)

Výklad komplexních čísel začínáme motivací k zavedení odmocniny ze záporného čísla. Budeme předpokládat, že slovní zásoba pro tento úvod je v rámci této práce dostatečná, a přejdeme rovnou k definici komplexního čísla.

Definition 7.1. A **complex number** can be written in the form

$$a + b\mathrm{i}$$

where a and b are real numbers (including 0) and i is an imaginary unit; $\mathrm{i}^2 = -1$. The number a is called the real part, the number b the imaginary part.

Dále potřebujeme zavést sčítání a násobení komplexních čísel.

Definition 7.2. The **sum** of two complex numbers $a + b\mathrm{i}$ and $c + d\mathrm{i}$ is

$$(a + b\mathrm{i}) + (c + d\mathrm{i}) = (a + c) + (b + d)\mathrm{i}.$$

Definition 7.3. The **product** of two complex numbers $a + b\mathrm{i}$ and $c + d\mathrm{i}$ is

$$(a + b\mathrm{i})(c + d\mathrm{i}) = (ac - bd) + (ad + bc)\mathrm{i}.$$

- The product of zero and any complex number is equal to zero.

Definition 7.4. Two complex numbers $a + b\mathrm{i}$ and $c + d\mathrm{i}$ are **equal** if and only if their real parts are equal, $a = c$, and their imaginary parts are equal, $b = d$.

- The notation of the complex domain is the letter $\mathbb{C}$.

7.1.1 Addition and Multiplication of Complex Numbers (Sčítání a násobení komplexních čísel)

The notation of a complex number z in the form $a + b\mathrm{i}$ is called the algebraic form of a complex number z.

A number $z = a + b\mathrm{i}$, where $b \neq 0$, is called an imaginary number. If $a = 0$ then the number z is called a pure imaginary number.

Example 7.1. Write the algebraic form of the following number:

$(3 + 2\mathrm{i})(1 + \mathrm{i}) + (4 - \mathrm{i})(1 + 3\mathrm{i})$.

Solution: When adding and multiplying complex numbers in the algebraic form we use the same method as when working with polynomials.

Start with multiplying the brackets.

$$(3 + 2\mathrm{i})(1 + \mathrm{i}) + (4 - \mathrm{i})(1 + 3\mathrm{i}) = (3 + 3\mathrm{i} + 2\mathrm{i} + 2\mathrm{i}^2) + (4 + 12\mathrm{i} - \mathrm{i} - 3\mathrm{i}^2)$$

Use $\mathrm{i}^2 = -1$.

$$(3 + 3\mathrm{i} + 2\mathrm{i} + 2\mathrm{i}^2) + (4 + 12\mathrm{i} - \mathrm{i} - 3\mathrm{i}^2) = (3 + 5\mathrm{i} - 2) + (4 + 11\mathrm{i} + 3) =$$
$$= (1 + 5\mathrm{i}) + (7 + 11\mathrm{i})$$

Add the real parts and the imaginary parts as in Definition 7.2.

$(1 + 5\mathrm{i}) + (7 + 11\mathrm{i}) = 8 + 16\mathrm{i}$

The algebraic form of the number given is $8 + 16\mathrm{i}$.

Definition 7.5. The **subtraction** $z_1 - z_2$ of complex numbers z_1 and z_2 is the sum of the number z_1 and the opposite of the number z_2:

$$z_1 - z_2 = z_1 + (-z_2).$$

- The opposite of a complex number $z = a + b\mathrm{i}$ is the number $z' = -a - b\mathrm{i}$.

Theorem 7.6. If the product of two complex numbers is equal to zero, then at least one of the numbers is equal to zero.

Theorem 7.7. For any complex numbers z, z_1, z_2

$$z^m \cdot z^n = z^{m+n}, \qquad (z_1 z_2)^n = z_1^n z_2^n, \qquad (z^m)^n = z^{mn}.$$

- **Powers** of the complex unit are the following ($k \in \mathbb{R}$):

$$\mathrm{i}^2 = -1, \qquad \mathrm{i}^3 = -\mathrm{i}, \qquad \mathrm{i}^4 = 1$$

$$\mathrm{i}^{4k+1} = \mathrm{i}, \qquad \mathrm{i}^{4k+2} = -1, \qquad \mathrm{i}^{4k+3} = -\mathrm{i}, \qquad \mathrm{i}^{4k} = 1.$$

Example 7.2. Solve $\mathrm{i}^2 + \mathrm{i}^6 + \mathrm{i}^{10} + \mathrm{i}^{12} + \mathrm{i}^{14} + \mathrm{i}^{18} + \mathrm{i}^{20}$.

Solution: Use the knowledge of powers of the complex unit.

$$\mathrm{i}^2 + \mathrm{i}^6 + \mathrm{i}^{10} + \mathrm{i}^{12} + \mathrm{i}^{14} + \mathrm{i}^{18} + \mathrm{i}^{20} =$$

$$= \mathrm{i}^2 + \mathrm{i}^{4+2} + \mathrm{i}^{4\cdot2+2} + \mathrm{i}^{4\cdot3} + \mathrm{i}^{4\cdot3+2} + \mathrm{i}^{4\cdot4+2} + \mathrm{i}^{4\cdot5} =$$

$$= -1 - 1 - 1 + 1 - 1 - 1 + 1 = -3.$$

Example 7.3. Solve $\mathrm{i}^3 \cdot \mathrm{i}^7 \cdot \mathrm{i}^9 \cdot \mathrm{i}^{11} \cdot \mathrm{i}^{15} \cdot \mathrm{i}^{19} \cdot \mathrm{i}^{21}$.

Solution: Use the knowledge of powers of the complex unit.

$$\mathrm{i}^3 \cdot \mathrm{i}^7 \cdot \mathrm{i}^9 \cdot \mathrm{i}^{11} \cdot \mathrm{i}^{15} \cdot \mathrm{i}^{19} \cdot \mathrm{i}^{21} =$$

$$= \mathrm{i}^{3+7+9+11+15+19+21} = \mathrm{i}^{85} = \mathrm{i}^{4\cdot21+1} = \mathrm{i}.$$

7.1.2 Dividing Complex Numbers (Dělení komplexních čísel)

Pro zavedení dělení komplexních čísel potřebujeme nejprve zavést pojem číslo komplexně sdružené.

Definition 7.8. The complex **conjugate** of a complex number $z = a + b\mathrm{i}$ is the complex number $\overline{z} = a - b\mathrm{i}$.

- We also call the numbers z and $\overline{z}$ **associated**.

Theorem 7.9. The **product** of a complex number z and its conjugate $\overline{z}$ is a non-negative real number. The product equals zero only if $z = 0$.

Definition 7.10. **Dividing** a complex number z_1 by a complex number $z_2 \neq 0$ is the product of the number z_1 and the inverse of the number z_2, i.e., $\frac{z_1}{z_2} = z_1 \cdot \frac{1}{z_2}$.

Postup dělení komplexních čísel demonstrujeme na příkladu.

Example 7.4. Divide $z_1 = 4 + 3\mathrm{i}$ by $z_2 = 2 + \mathrm{i}$.

Solution: Expand the fraction $\frac{4+3\mathrm{i}}{2+\mathrm{i}}$ by the complex conjugate of the denominator.

$$\frac{4+3\mathrm{i}}{2+\mathrm{i}} = \frac{(4+3\mathrm{i})(2-\mathrm{i})}{(2+\mathrm{i})(2-\mathrm{i})}$$

Multiply the brackets. According to Theorem 7.9, the product in the denominator is a positive real number.

$$\frac{(4+3\mathrm{i})(2-\mathrm{i})}{(2+\mathrm{i})(2-\mathrm{i})} = \frac{8+2\mathrm{i}-3\mathrm{i}^2}{4-\mathrm{i}^2} = \frac{11+2\mathrm{i}}{5}$$

The solution of the task is $\frac{11}{5} + \frac{2}{5}\mathrm{i}$.

7.1.3 Absolute Value of a Complex Number (Absolutní hodnota komplexního čísla)

Definition 7.11. The **absolute value** of a complex number z is a number $\sqrt{z\bar{z}}$, i.e., $|z| = \sqrt{z\bar{z}}$.

- If $z \neq 0$, then $|z| > 0$.
- If $z = 0$, then $|z| = 0$.
- If $z = a + b\mathrm{i}$, then $|z| = \sqrt{a^2 + b^2}$.

Theorem 7.12. For any complex numbers z_1, z_2 ($z_2 \neq 0$ for dividing):

$$|z_1 z_2| = |z_1| \cdot |z_2|,$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

Definition 7.13. A **complex unit** is a complex number whose absolute value is 1.

Example 7.5. Determine the absolute value of $z = 2 - 3\mathrm{i}$.

Solution: Use Definition 7.11.

$$|z| = \sqrt{(2-3\mathrm{i})(2+3\mathrm{i})} = \sqrt{4+9} = \sqrt{13}.$$

The absolute value of z is $\sqrt{13}$.

Example 7.6. Solve $||3+\mathrm{i}|^2+(3+\mathrm{i})^2|$.

Solution: First determine the absolute value and the second power of $3+\mathrm{i}$.

$$||3+\mathrm{i}|^2+(3+\mathrm{i})^2| = \left|\left(\sqrt{9+1}\right)^2+9+6\mathrm{i}+\mathrm{i}^2\right| = |9+1+9+6\mathrm{i}-1|$$

Determine the absolute value of $18+6\mathrm{i}$.

$$|18+6\mathrm{i}| = \sqrt{(18+6\mathrm{i})(18-6\mathrm{i})} = \sqrt{18^2+6^2} = 6\sqrt{10}.$$

The solution of this task is $6\sqrt{10}$.

Definition 7.14. A **complex plane** (or Gauss plane) is a plane whose points are identified by means of complex numbers.

Theorem 7.15. The absolute value of a complex number is the distance of its point in the complex plane from the origin.

7.2 Equations on The Complex Domain (Rovnice v oboru komplexních čísel)

7.2.1 Linear Equations (Lineární rovnice)

Lineárním rovnicím se věnuje kapitola 3.1 této práce. Při řešení lineárních rovnic v oboru komplexních čísel užíváme stejná pravidla a postupy, pouze přidáme pravidla pro počítání s komplexními čísly, která jsou uvedena v této kapitole. Uvedeme tedy pouze příklad řešení lineární rovnice na množině komplexních čísel.

Example 7.7. Solve the equation $\frac{1-2\mathrm{i}}{2+\mathrm{i}}x+\frac{1}{1-\mathrm{i}}=1$.

Solution: Multiply the equation by $(2+\mathrm{i})(1-\mathrm{i})$.

$$(1-2\mathrm{i})(1-\mathrm{i})x+1(2+\mathrm{i}) = 1(2+\mathrm{i})(1-\mathrm{i})$$

Multiply the numbers in brackets.

$$(-1-3\mathrm{i})x+2+\mathrm{i} = 2-2\mathrm{i}+\mathrm{i}+1$$

Subtract $2+\mathrm{i}$ from each side of the equation.

$$(-1-3\mathrm{i})x = 1-2\mathrm{i}$$

Divide each side of the equation by $(-1-3\mathrm{i})$.

$$x = \frac{1 - 2\mathrm{i}}{-1 - 3\mathrm{i}}$$

Determine $\frac{1-2\mathrm{i}}{-1-3\mathrm{i}}$ using the method shown in Example 7.4.

$$x = \frac{(1 - 2\mathrm{i})(-1 + 3\mathrm{i})}{(-1 - 3\mathrm{i})(-1 + 3\mathrm{i})} = \frac{5 + 5\mathrm{i}}{10} = \frac{1}{2} + \frac{1}{2}\mathrm{i}$$

The solution set of this equation is $K = \left\{\frac{1}{2} + \frac{1}{2}\mathrm{i}\right\}$.

7.2.2 Quadratic Equations (Kvadratické rovnice)

V tomto textu se omezíme pouze na kvadratické rovnice s reálnými koeficienty řešené v oboru komplexních čísel. Motivací pro jejich řešení v tomto oboru je fakt, že kvadratická rovnice nemá v oboru reálných čísel řešení, pokud je její diskriminant záporný (kapitola 3.3.2).

While solving quadratic equations with real coefficients on the complex domain, we look for roots among the set of complex numbers. On the real domain, we get two, one or no roots depending on the discriminant, whereas on complex domain, we **always get roots** (one real double root, two real roots or two associated complex roots).

Example 7.8. Solve the equation $5x^2 - 2x + 1 = 0$ on the real domain and on the complex domain.

Solution: The discriminant is $D = 4 - 20 = -16 < 0$. That means the equation has no real roots, but it has two complex roots (that are associated).

Using $\mathrm{i}^2 = -1$ we get $-16 = 16 \cdot \mathrm{i}^2$. Its square root is $\sqrt{16\mathrm{i}^2} = \pm 4\mathrm{i}$. This means we can use the quadratic formula.

$$x_{1,2} = \frac{2 \pm 4\mathrm{i}}{10} = \frac{1}{5} \pm \frac{2}{5}\mathrm{i}$$

The solution set is $K = \left\{\frac{1}{5} - \frac{2}{5}\mathrm{i}; \frac{1}{5} + \frac{2}{5}\mathrm{i}\right\}$.

Example 7.9. Solve the equation $x^2 + 4 = 0$ on the complex domain.

Solution: This equation is an incomplete quadratic equation without linear term. There are two possible ways to solve this type of equation.

The first way is by subtracting 4 from both sides of the equation.

$$x^2 = -4 = 4\text{i}^2$$
$$x_{1,2} = \pm 2\text{i}$$

The other way is by using the quadratic formula.

$$D = 0 - 16 = 16\text{i}^2$$
$$\sqrt{D} = \pm 4\text{i}$$
$$x_{1,2} = \frac{0 \pm 4\text{i}}{2} = \pm 2\text{i}$$

The solution set is $K = \{-2\text{i}; 2\text{i}\}$.

Let us show another way of solving a quadratic equation – by substitution.

Example 7.10. Solve the equation $x^2 - 4x + 8 = 0$ on the complex domain.

Solution: Let us solve this equation by substitution.

$$(x-2)^2 - 4 + 8 = 0$$
$$(x-2)^2 = -4$$

Substitute $x - 2 = a$.

$$a^2 = -4$$
$$a_{1,2} = \pm 2\text{i}$$

Substitute back for $a = x - 2$.

$$x_{1,2} - 2 = \pm 2\text{i}$$
$$x_{1,2} = 2 \pm 2\text{i}$$

The solution set is $K = \{2 - 2\text{i}; 2 + 2\text{i}\}$.

8 Hodnocení

Vzhledem k nastalým nečekaným okolnostem a pandemické situaci nebylo možné realizovat praxi v plném rozsahu, a proto neproběhla plánovaná výuka podle vytvořených materiálů.

Před uzavřením škol ve školním roce 2019/2020 proběhla pouze jedna vyučovací hodina s využitím textů vytvořených v této práci, jejímž tématem bylo řešení kvadratických rovnic s reálnými koeficienty na množině komplexních čísel (kapitola 7.2.2). Text se ukázal jako dostačující pro vysvětlení látky a ilustraci řešení příkladů, zbylá část hodiny byla věnována procvičení příkladů. Přestože hodnocení hospitujícího kolegy bylo kladné, nelze z jedné vyučovací hodiny vyvozovat závěry o celé této práci.

Ve školním roce 2020/2021 nebylo z časových a personálních důvodů možné realizovat další výuku, a tak byly vytvořené materiály konzultovány alespoň distanční formou s kolegou Ing. Lukášem Matouškem, MBA z Gymnázia Jiřího Wolkera v Prostějově. Podle jeho názoru je práce vytvořená logicky, obsahuje stěžejní kapitoly, které jsou na GJW v anglické matematice probírány (především funkce a komplexní čísla), i některé kapitoly navíc.

Materiály vytvořené v této práci budou zařazeny do výuky sext a septim od školního roku 2021/2022 celoročně.

9 Závěr

Cílem diplomové práce bylo vypracovat texty pro výuku středoškolské matematiky metodou CLIL. Podařilo se mi vytvořit materiály, které budu od příštího školního roku využívat ve své práci na Gymnáziu Jiřího Wolkera v Prostějově.

V první části textu je metoda CLIL představena. Mým záměrem nebylo provést detailní rozbor metody a jejího využití, proto je tato část velmi stručná.

Hlavním obsahem práce jsou kapitoly 3 až 7, jejichž účelem je zavedení matematické terminologie, rozšíření slovní zásoby a předvedení jejich užití při řešení příkladů. S ohledem na zamýšlené využití textů ve výuce není předmětem ani dokazování matematických vět a tvrzení, a proto důkazy nejsou uvedeny.

Přílohy pak zahrnují slovníčky k jednotlivým kapitolám, stejně tak jako pracovní listy pro žáky a jejich řešení. Texty v celé práci jsou tvořeny tak, aby mohly být využity jak vyučujícími pro přípravu výuky, tak i žáky pro samostudium.

Literatura

[1] CHARVÁT, Jura, ZHOUF, Jaroslav, BOČEK, Leo, 2013. *Matematika pro gymnázia – Rovnice a nerovnice.* Dotisk 4. vydání. Praha: Nakladatelství Prometheus, spol. s r.o. 223 s. ISBN 978-80-7196-362-2.

[2] ODVÁRKO, Oldřich, Leo, 2013. *Matematika pro gymnázia –Funkce.* Dotisk 4. vydání. Praha: Nakladatelství Prometheus, spol. s r.o. 168 s. ISBN 978-80-7196-357-8.

[3] POMYKALOVÁ, Eva, 2013. *Matematika pro gymnázia – Planimetrie.* Dotisk 5. vydání. Praha: Nakladatelství Prometheus, spol. s r.o. 206 s. ISBN 978-80-7196-358-5.

[4] POMYKALOVÁ, Eva, 2013. *Matematika pro gymnázia – Stereometrie.* Dotisk 4. vydání. Praha: Nakladatelství Prometheus, spol. s r.o. 223 s. ISBN 978-80-7196-389-9.

[5] CALDA, Emil, 2013. *Matematika pro gymnázia – Komplexní čísla.* Dotisk 4. vydání. Praha: Nakladatelství Prometheus, spol. s r.o. 134 s. ISBN 978-80-7196-364-6.

[6] MOLNÁR, Josef, 2001. *Planimetrie.* 1. vydání. Olomouc: Univerzita Palackého v Olomouci. 128 s. ISBN 80-244-0370-6.

[7] MOLNÁR, Josef, 1999. *Plane Geometry.* 1. vydání. Olomouc: Univerzita Palackého v Olomouci, VUP. 126 s. ISBN 80-244-0002-2.

[8] FÜRST, Tomáš, MOLNÁR, Josef, POHANĚL, Karel, 2005. *A Guide to Three-Dimensional Space.* 1. vydání. Olomouc: Univerzita Palackého v Olomouci, VUP. 173 s. ISBN 80-244-1157-1.

[9] FÜRST, Tomáš, MOLNÁR, Josef, POHANĚL, Karel, 2004. *Průvodce trojrozměrným prostorem.* 1. vydání. Olomouc: Univerzita Palackého v Olomouci, VUP. 175 s. ISBN 80-244-0817-1.

[10] BILLSTEIN, Rick, LIBESKIND, Shlomo, LOTT, Johnny W., 1990. *A Problem Solving Approach to Mathematics for Elementary School Teachers.* Fourth edition. Redwood City, California: The Benjamin/Cummings Publishing Company, Inc. 945 s. ISBN 0-8053-0390-1.

[11] AUFMANN, Richard N., BARKER, Vernon C., NATION, Jr., Richard D., 1990. *College Algebra*. Boston, Massachusetts: Houghton Mifflin Company. 481 s. ISBN 0-395-38097-9.

[12] CLIL ve výuce: Jak zapojit cizí jazyky do vyučování. NUV [online]. Praha: NUV, 2012 [cit. 2021-4-27]. Dostupné z: http://www.nuv.cz/uploads/Publikace/CLIL_ve_vyuce.pdf

[13] BALADOVÁ, Gabriela. Výuka metodou CLIL. Metodický portál RVP.CZ [online]. 2009. [cit. 2021-4-27]. Dostupné z: https://clanky.rvp.cz/clanek/o/z/2965/vyuka-metodou-clil.html/

Přílohy

Slovní zásoba – Matematická terminologie

addition	[əˈdɪʃən]	sčítání
axes	[ˈæksɪz]	osy
axis	[ˈæksɪs]	osa
coordinate plane	[kəʊˈɔːdɪnət pleɪn]	dvourozměrná soustava souřadnic
coordinate system	[kəʊˈɔːdɪnət ˈsɪstəm]	soustava souřadnic
definition	[ˌdefɪˈnɪʃən]	definice
denote	[dɪˈnəʊt]	označovat
divide	[dɪˈvaɪd]	dělit
if and only if	[ɪf ænd ˈəʊnlɪ ɪf]	právě tehdy, když
multiplication	[ˌmʌltɪplɪˈkeɪʃən]	násobení
multiply	[ˈmʌltɪˌplaɪ]	(vy)násobit
notation	[nəʊˈteɪʃən]	zápis
origin	[ˈɒrɪdʒɪn]	počátek
product	[ˈprɒdʌkt]	součin
subtract	[səbˈtrækt]	odečíst
subtraction	[səbˈtrækʃən]	odčítání
sum	[sʌm]	součet
theorem	[ˈθɪərəm]	věta (matematická)

Slovní zásoba – Rovnice a nerovnice

absolute value	[ˈæbsəˌluːt ˈvæljuː]	absolutní hodnota
constant	[ˈkɒnstənt]	konstanta
denominator	[dɪˈnɒmɪˌneɪtə]	jmenovatel
distinct	[dɪˈstɪŋkt]	odlišný
distributive	[dɪˈstrɪbjʊtɪv]	distributivní
equation	[ɪˈkweɪʒən or -ʃən]	rovnice
equivalent	[ɪˈkwɪvələnt]	ekvivalentní
factor out	[ˈfæktə aʊt]	vytknout
fraction	[ˈfrækʃən]	zlomek
check	[tʃek]	zkouška, kontrola
inequality	[ˌɪnɪˈkwɒlɪtɪ]	nerovnice
linear	[ˈlɪnɪə]	lineární
numerator	[ˈnjuːməreɪtə(r)]	čitatel
preserve	[prɪˈzɜːv]	zachovat
property	[ˈprɒpətɪ]	vlastnost
quadratic	[kwäˈdradik]	kvadratický
reciprocal	[rɪˈsɪprəkəl]	převrácená hodnota
reverse	[rɪˈvɜːs]	obrátit
root	[ruːt]	kořen
satisfy	[ˈsætɪsˌfaɪ]	splňovat
solution	[səˈluːʃən]	řešení
solution set	[səˈluːʃən set]	množina řešení
solve	[sɒlv]	(vy)řešit
statement	[ˈsteɪtmənt]	výrok
substitute	[ˈsʌbstɪˌtjuːt]	nahradit
variable	[ˈveərɪəbəl]	proměnná

Slovní zásoba – Funkce

decreasing	[dɪˈkriːsɪŋ]	klesající
dependent	[dɪˈpendənt]	závislý
determine	[dɪˈtɜːmɪn]	určit
direct proportion	[dɪˈrekt or daɪ- prəˈpɔːʃən]	přímá úměra
domain	[dəˈmeɪn]	definiční obor
element	[ˈelɪmənt]	prvek
evaluate	[ɪˈvæljʊˌeɪt]	vyhodnotit
function	[ˈfʌŋkʃən]	funkce
graph	[grɑːf or græf]	graf
horizontal	[ˌhɒrɪˈzɒntəl]	vodorovný
increasing	[ɪnˈkriːsɪŋ]	rostoucí
independent	[ˌɪndɪˈpendənt]	nezávislý
intersect	[ˌɪntəˈsekt]	protnout
parabola	[pəˈræbələ]	parabola
range	[reɪndʒ]	obor hodnot
result	[rɪˈzʌlt]	výsledek
slope	[sləʊp]	směrnice přímky
value	[ˈvæljuː]	hodnota
variable	[ˈveərɪəbəl]	proměnná
vertex	[ˈvɜːteks]	vrchol
vertical	[ˈvɜːtɪkəl]	svislý

Slovní zásoba – Planimetrie

acute	[əˈkjuːt]	ostrý (úhel)
altitude	[ˈæltɪˌtjuːd]	výška
analysis	[əˈnælɪsɪs]	analýza, rozbor
angle	[ˈæŋgəl]	úhel
angle bisector	[ˈæŋgəl baɪˌsektə(r)]	osa úhlu
annulus	[ˈænjʊləs]	mezikruží
arc	[ɑːk]	oblouk
arm	[ɑːm]	rameno (úhlu)
centre	[ˈsentə]	střed
centroid	[ˈsentrɔɪd]	těžiště
circle	[ˈsɜːkəl]	kružnice
circumcentre	[səˈkʌmfərəns]	střed kružnice opsané
circumscribed circle	[ˌsɜːkəmˈskraɪbəd ˈsɜːkəl]	kružnice opsaná
concentric	[kənˈsentrɪk]	soustředné
congruence	[ˈkɒŋgrʊəns]	shodnost
construction	[kənˈstrʌkʃən]	konstrukce
convex	[ˈkɒnveks]	konvexní
diameter	[daɪˈæmɪtə]	průměr
disc	[dɪsk]	kruh
discussion	[dɪˈskʌʃən]	diskuze
distinct	[dɪˈstɪŋkt]	odlišný
eccentric	[ɪkˈsentrɪk]	nesoustředné
edge	[edʒ]	hranice
endpoint	[endpɔɪnt]	koncový bod
equilateral	[ˌiːkwɪˈlætərəl]	rovnostranný
external line	[ɪkˈstɜːnəl laɪn]	vnější přímka
foot	[fʊt]	pata (kolmice)
half line	[hɑːf laɪn]	polopřímka
half plane	[hɑːf pleɪn]	polorovina
hypotenuse	[haɪˈpɒtɪˌnjuːz]	přepona
chord	[kɔːd]	tětiva
identical	[aɪˈdentɪkəl]	shodný, totožný
incentre	[ɪnˈsentə(r)]	střed kružnice vepsané
inscribed circle	[ɪnˈskraɪbəd ˈsɜːkəl]	vepsaná kružnice
interior	[ɪnˈtɪərɪə]	vnitřní
intersection	[ˌɪntəˈsekʃən]	průsečík

isosceles	[aɪˌsɒsəli:z]	rovnoramenný (trojúhelník)
leg	[leg]	odvěsna (v trojúhelníku)
length	[leŋθ]	délka
line segment	[laɪn ˈsɛgmənt]	úsečka
median	[ˈmi:dɪən]	těžnice
mid-point	[ˈmɪdˌpɔɪnt]	střed
midline	[ˈmɪdˌlaɪn]	střední příčka
obtuse	[əbˈtju:s]	tupý (úhel)
origin	[ˈɒrɪdʒɪn]	počátek
orthocentre	[ˈɔ:θəʊˌsentə(r)]	ortocentrum
parallel	[ˈpærəˌlel]	rovnoběžný
perimeter	[pəˈrɪmɪtə]	obvod
perpendicular	[ˌpɜ:pənˈdɪkjʊlə]	kolmý
perpendicular bisector	[ˌpɜ:pənˈdɪkjʊlə baɪˌsektə(r)]	osa strany
point	[pɔɪnt]	bod
radius	[ˈreɪdɪəs]	poloměr
reflex	[rēˌfleks]	nekonvexní (úhel)
respectively	[rɪˈspektɪvlɪ]	v tomto pořadí
scalene	[skeɪli:n]	různostranný (trojúhelník)
secant	[ˈsi:kənt]	sečna
similarity	[ˌsɪmɪˈlærɪtɪ]	podobnost
straight angle	[streɪt ˈæŋgəl]	přímý úhel
straight line	[streɪt laɪn]	přímka
tangent	[ˈtændʒənt]	tečna
triangle	[ˈtraɪˌæŋgəl]	trojúhelník
vertex	[ˈvɜ:teks]	vrchol

Slovní zásoba – Stereometrie

collinear	[kəʊˈlɪnɪə(r)]	ležící v jedné přímce
concurrent	[kənˈkʌrənt]	různoběžné
cone	[kəʊn]	kužel
coplanar	[kəʊˈpleɪnə(r)]	ležící v jedné rovině
cube	[kjuːb]	krychle
cuboid	[kjuːbɒɪd]	kvádr
cylinder	[ˈsɪlɪndə]	válec
half-space	[hɑːfspeɪs]	poloprostor
line of intersection	[laɪn ɒv ˌɪntəˈsekʃən]	průsečnice
prism	[ˈprɪzəm]	hranol
pyramid	[ˈpɪrəmɪd]	jehlan
rotating	[rəʊˈteɪtɪŋ]	rotační
section	[ˈsekʃən]	řez
skew	[skjuː]	mimoběžné
solid	[ˈsɒlɪd]	těleso
tetrahedron	[ˌtetrəˈhiːdrən]	čtyřstěn

Slovní zásoba – Komplexní čísla

algebraic	[ˌældʒɪˈbreɪɪk]	algebraický
associated	[əˈsəʊʃɪˌeɪtɪd]	sdružené
complex	[ˈkɒmpleks]	komplexní
conjugate	[ˈkɒndʒʊˌgeɪt]	sdružené
expand	[ɪkˈspænd]	rozšířit
imaginary	[ɪˈmædʒɪnərɪ]	imaginární
inverse	[ɪnˈvɜːs]	inverzní
polynomial	[ˌpɒlɪˈnəʊmɪəl]	polynom, mnohočlen
power	[ˈpaʊə]	mocnina
unit	[ˈjuːnɪt]	jednotka

Equations and Inequalities – Worksheet A

1) Fill in the missing parts.
   a) A ________________________ in one variable is an _____________ that can be written in one of the forms $ax + b > 0$, $ax + b < 0$, $ax + b \geq 0$, $ax + b \leq 0$, where *a* and *b* are _____ numbers, with $a \neq 0$.
   b) Multiplying each side of an inequality by the same positive number ________ the order of the inequality.
   c) The _________________ of a real number x, denoted ________, is the non-negative value of x without regard to its _______.
   d) If $\boldsymbol{D < 0}$, then the quadratic equation has ____________________________.
2) Translate the words into Czech language as used in Maths.
   addition – _________________________
   fraction – ______________________
   variable – _________________
   negative – __________________
   quadratic – _________________
   product – ________________
   divide – _______________
3) Translate the words into English language as used in Maths.
   množina řešení – ________________________
   násobení – ______________________
   kladný – _______________
   koeficient – ________________________
   součet – ____________________
   jmenovatel – _______________________
4) Solve on the real domain:
   a) $\frac{4}{3}x + 10 - \frac{10}{3}x = \frac{15}{2}$
   b) $4(3x + 5) > 12 - 8x$
   c) $4(x + 2) \geq 8(x - 3)$
   d) $42x^2 - 4x - 6 = 0$
   e) $4x^2 - 12x - 40 = 0$ (use Vieta's formulas)
   f) $4x + |x - 3| = 1$

Equations and Inequalities – Worksheet B

1) Fill in the missing parts.
   a) A linear equation in one variable is an equation that can be written in the form ______________ where __ and __ are _____ numbers, with ______.
   b) Multiplying each side of an inequality by the same negative number ________ the order of the inequality.
   c) A ______________________ in one variable is an equation that can be written in the standard quadratic form $ax^2 + bc + c = 0$ where *a, b* and *c* are _______ numbers with ________.
   d) If $\boldsymbol{D > 0}$, then the quadratic equation has __________________________.
2) Translate the words into Czech language as used in Maths.
   root – _________________
   property – ____________________
   isolate – __________________
   inequality – ___________________
   check – ________________
   double root – _________________________
   expression – _________________
3) Translate the words into English language as used in Maths.
   řešit – ___________
   nejmenší společný násobek – ______________________________
   rovnice – _________________
   vytknout – ___________________
   absolutní hodnota – _____________________________
   převrácená hodnota – _____________________
4) Solve on the real domain:
   a) $-\frac{7}{4}x - 8 + \frac{15}{4}x = \frac{1}{2}$
   b) $5(2x + 3) \leq 10 + 5x$
   c) $3(x - 2) < 6(x + 4)$
   d) $20x^2 - 6x - 8 = 0$
   e) $3x^2 + 9x - 54 = 0$ (use Vieta's formulas)
   f) $3x + |x - 1| = 4$

Equations and Inequalities – Worksheet A (Solution)

1)

a) A linear inequality in one variable is an inequality that can be written in one of the forms $ax + b > 0$, $ax + b < 0$, $ax + b \geq 0$, $ax + b \leq 0$, where a and b are real numbers, with $a \neq 0$.

b) Multiplying each side of an inequality by the same positive number preserves the order of the inequality.

c) The absolute value of a real number x, denoted $|x|$, is the non-negative value of x without regard to its sign.

d) If $\boldsymbol{D} < \mathbf{0}$, then the quadratic equation has no real root.

2)

addition – sčítání
fraction – zlomek
variable – proměnná
negative – záporný
quadratic – kvadratický
product – součin
divide – dělit

3)

množina řešení – solution set
násobení – multiplication
kladný – positive
koeficient – coefficient
součet – sum
jmenovatel – denominator

4)

a) $K = \left\{\frac{5}{4}\right\}$

b) $x \in (-\frac{2}{5}; +\infty)$

c) $x \in (-\infty; 8\rangle$

d) $K = \left\{-\frac{1}{3}; \frac{3}{7}\right\}$

e) $K = \{-2; 5\}$

f) $K = \left\{-\frac{2}{3}\right\}$

Equations and Inequalities – Worksheet B (Solution)

1)

a) A linear equation in one variable is an equation that can be written in the form $ax + b = 0$ where a and b are real numbers, with $a \neq 0$.

b) Multiplying each side of an inequality by the same negative number reverses the order of the inequality.

c) A quadratic equation in one variable is an equation that can be written in the standard quadratic form $ax^2 + bc + c = 0$ where *a*, *b* and *c* are real numbers with $a \neq 0$.

d) If $\boldsymbol{D} > \mathbf{0}$, then the quadratic equation has two distinct real roots.

2)

root – kořen
property – vlastnost
isolate – izolovat (osamostatnit)
inequality – nerovnice
check – zkouška
double root – dvojnásobný kořen
expression – výraz

3)

řešit – solve
nejmenší společný násobek jmenovatelů – lowest common denominator
rovnice – equation
vytknout – factor out
absolutní hodnota – absolute value
převrácená hodnota – reciprocal

4)

a) $K = \left\{\frac{17}{4}\right\}$

b) $x \in (-\infty; -1\rangle$

c) $x \in (-10; +\infty)$

d) $K = \left\{-\frac{1}{2}; \frac{4}{5}\right\}$

e) $K = \{-6; 3\}$

f) $K = \left\{\frac{5}{4}\right\}$

Functions – Worksheet A

1) Fill in the missing parts.
   a) The range of a function is the __________ set of the ______________________________________ of the __________________________ after we have substituted the domain.
   b) A graph is the graph of a function if and only if no _______________ intersects the graph at _________________ point.
   c) If a and b are elements of the domain of a function f, then f is _____________ on $D(f)$ if $f(a) > f(b)$ whenever $a < b$.
2) Translate the words into Czech language as used in Maths.

   range – _________________________

   direct proportion – ______________________

   coordinate plane – _________________

   vertex – ___________________

   shift down vertically – ________________________________
3) Translate the words into English language as used in Maths.

   nezávisle proměnná – ________________________

   vyhodnotit funkci – ______________________

   konstantní – _______________

   prostá funkce – ________________________

   posunout vpravo – ____________________
4) Identify whether the equation $3(y+3) = \frac{6}{y-3}$ defines y as a function of x.
5) Determine the domain of the function $f\colon y = \frac{3x+7}{x(x+8)}$.
6) Decide whether the function $g\colon y = -\frac{6}{5}x + 7$ is increasing or decreasing.
7) Sketch the graph of a function $h\colon y = 2x^2 - 4x + 3$. Then determine the properties of the function h.

Functions – Worksheet B

1) Fill in the missing parts.
   a) The domain of a function is the __________ set of __________________ of the _____________________________.
   b) The _______________________ in the coordinate plane Oxy is the set of all points $X[x, f(x)]$ where x is an element in the ___________ of f.
   c) If a and b are elements of the domain of a function f, then$f$ is ______________ on $D(f)$ if $f(a) < f(b)$ whenever $a < b$.
2) Translate the words into Czech language as used in Maths.

   domain – _________________

   straight line – ____________________

   axis – __________________

   narrow – ___________________

   shift left horizontally – ________________
3) Translate the words into English language as used in Maths.

   závisle proměnná – ___________

   definovat – ___________________________________

   lineární – _________________

   směrnice – ___________________

   posunout nahoru – _____________________________
4) Identify whether the equation $4(y-2) = \frac{2}{y+2}$ defines y as a function of x.
5) Determine the domain of the function $f\colon y = \frac{2x-3}{x(x-12)}$.
6) Decide whether the function $g\colon y = \frac{4}{5}x - 6$ is increasing or decreasing.
7) Sketch the graph of a function $h\colon y = 3x^2 + 6x - 5$. Then determine the properties of the function h.

Functions – Worksheet A (Solution)

1)

a) The range of a function is the complete set of all possible resulting values of the dependent variable after we have substituted the domain.

b) A graph is the graph of a function if and only if no vertical line intersects the graph at more than one point.

c) If a and b are elements of the domain of a function f, then f is decreasing on $D(f)$ if $f(a) > f(b)$ whenever $a < b$.

2)

range – obor hodnot

direct proportion – přímá úměra

coordinate plane – dvourozměrná soustava souřadnic

vertex – vrchol

shift down vertically – posunout dolů

3)

nezávisle proměnná – independent variable

vyhodnotit funkci – evaluate a function

konstantní – constant

prostá funkce – one-to-one function

posunout vpravo – shift right horizontally

4) $y = \pm\sqrt{11}$, this equation does not define y as a function of x.

5) $D(f) = \mathbb{R}\setminus\{-8; 0\}$.

6) $-\frac{6}{5} < 0$, the function g is decreasing.

7) $D(f) = \mathbb{R}$, $H(f) = \langle 1; \infty)$ The function f is decreasing on $(-\infty; 1\rangle$ and increasing on $\langle 1; \infty)$. The function f is not a one-to-one function.

Functions – Worksheet B (Solution)

1)

a) The domain of a function is the complete set of possible values of the independent variable.

b) The graph of a function f in the coordinate plane Oxy is the set of all points $X[x, f(x)]$ where x is an element in the domain of f.

c) If a and b are elements of the domain of a function f, then f is increasing on $D(f)$ if $f(a) < f(b)$ whenever $a < b$.

2)

domain – definiční obor

straight line – přímka

axis – osa

narrow – úzký

shift left horizontally – posunout vlevo

3)

závisle proměnná – dependent variable

definovat – define

lineární – linear

směrnice – slope

posunout nahoru – shift up vertically

4) $y = \pm\sqrt{\frac{9}{2}}$, this equation does not define y as a function of x.

5) $D(f) = \mathbb{R}\setminus\{0; 12\}$

6) $\frac{4}{5} > 0$, the function g is increasing.

7) $D(f) = \mathbb{R}$, $H(f) = \langle -8; \infty)$ The function f is decreasing on $(-\infty; -1\rangle$ and increasing on $\langle -1; \infty)$. The function f is not a one-to-one function.

Plane Geometry – Worksheet A

1) Fill in the missing parts.
   a) Any straight line in a plane divides the plane into ________________.
   b) The full rotation is _____________________.
   c) All three interior angle bisectors $o_\alpha, o_\beta, o_\gamma$ intersect in ____________________ that is the __________________________.
2) Translate the words into Czech language as used in Maths.

   acute angle – ________________________

   perimeter – _____________________

   equilateral triangle – _________________

   vertex – __________________

   perpendicular bisector – _________________
3) Translate the words into English language as used in Maths.

   polopřímka – _______________________

   koncový bod – _____________________

   kolmé – _______________

   tupoúhlý trojúhelník – _______________________

   těžnice – ___________________
4) What may be the relationship between two circles with different centres? How many intersections do they have?
5) Given $b = 7$ cm, $\gamma = 30°$, $v_b = 5$ cm, complete the triangle ABC.
6) Given the line segment AB, $|AB| = 4$ cm, complete the triangle ABC where $v_c = 3$ cm and $t_c = 5$ cm.

Plane Geometry – Worksheet B

1) Fill in the missing parts.
   a) Two distinct points determine ______________________________.
   b) A straight angle is _____________________.
   c) All three perpendicular bisectors o_a, o_b, o_c intersect in _______________ that is the _________________________.
2) Translate the words into Czech language as used in Maths.
   convex angle – __________________
   straight line – _____________________
   perimeter – ___________________
   isosceles triangle – ____________________
   altitude – _________________
3) Translate the words into English language as used in Maths.
   úhel – ____________
   úsečka – ____________________________________
   rovnoběžné – __________________
   pravoúhlý trojúhelník – ____________________
   střed strany – ______________________________
4) What may be the relationship between a line and a circle? How many intersections do they have?
5) Given $c = 5$ cm, $\alpha = 55°$, $v_c = 3$ cm, complete the triangle ABC.
6) Given the line segment AB, $|AB| = 5$ cm, complete the triangle ABC where $v_c = 3$ cm and $v_a = 4$ cm.

Plane Geometry – Worksheet A (Solution)

1)

a) Any straight line in a plane divides the plane into two opposite half planes.

b) The full rotation is exactly 360°.

c) All three interior angle bisectors $o_\alpha, o_\beta, o_\gamma$ intersect in one point that is the incentre S_v.

2)

acute angle – ostrý úhel

perimeter – obvod

equilateral triangle – rovnostranný trojúhelník

vertex – vrchol

perpendicular bisector – osa strany

3)

polopřímka – half line, ray

koncový bod – endpoint

kolmé – perpendicular

tupoúhlý trojúhelník – obtuse-angled triangle

těžnice – median

4) Circles lie outside each other (no point of intersection), external touch, two points of intersection, internal touch, one circle lies inside the other circle with no point of intersection.

5) 1) place the side $AC; |AC| = 7$ cm, 2) $\sphericalangle ACX; |\sphericalangle ACX| = 30°$, 3) $p; p||AC; d(p, AC) = 5$ cm, 4) $B; B \in CX \cap p$, 5) $\triangle ABC$.

6) 1) $p; p||AB; d(p, AB) = 3$ cm, 2) $C_1; |AC_1| = |C_1B|$, 3) $k; k(C_1; 5$ cm$)$, 4) $C; C \in k \cap p$, 5) $\triangle ABC$.

Plane Geometry – Worksheet B (Solution)

1)

a) Two distinct points determine one, and only one, straight line.

b) A straight angle is exactly 180°.

c) All three perpendicular bisectors o_a, o_b, o_c intersect in one point that is the circumcentre S_o.

2)

convex angle – konvexní úhel

straight line – přímka

perimeter – obvod

isosceles triangle – rovnoramenný trojúhelník

altitude – výška

3)

úhel – angle

úsečka – line segment

rovnoběžné – parallel

pravoúhlý trojúhelník – right-angled triangle

střed strany – mid-point of a side

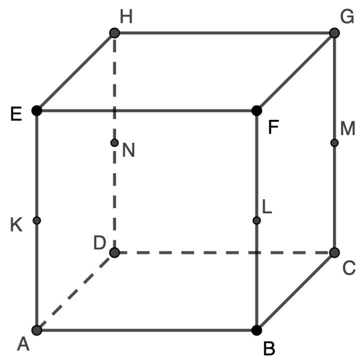
4) External line (no point of intersection), tangent (one point of intersection – a contact point), secant (two points of intersection).

5) 1) place the side $AB; |AB| = 5$ cm, 2) $\sphericalangle BAX; |\sphericalangle BAX| = 55°$, 3) $p; p||AB; d(p, AB) = 3$ cm, 4) $C; C \in AX \cap p$, 5) $\triangle ABC$.

6) 1) $C_1; |AC_1| = |C_1B|$, 2) $k; k(C_1; r = |AC_1|)$, 3) $l; l(A; 4$ cm), 4) $A_0; A_0 \in k \cap l$, 5) $m; m(C_1; 3$ cm), 6) $C; C \in m \cap BA_0$, 7) $\triangle ABC$.

Stereometry – Worksheet A

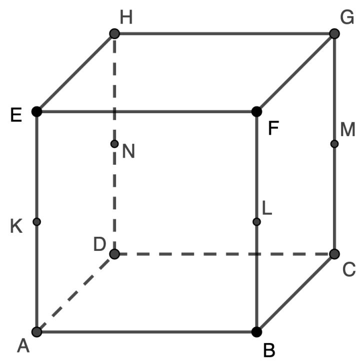
1) Fill in the missing parts.
    a) All points that lie in the same plane are called ________________.
    b) Through a line and a point not-lying on the line there exists ________________.
    c) Two planes are called intersecting if and only if they have _______ ______________________________.
2) Translate the words into Czech language as used in Maths.
    cylinder – ________________________
    pyramid – _____________________
    skew lines – _________________
    intersecting planes – __________________
    face of a cube – _________________
3) Translate the words into English language as used in Maths.
    kvádr – _______________________
    promítání – _____________________
    totožné přímky – _______________
    řez tělesa – _______________________
    rovnoběžné roviny – ___________________
4) Given the cube $ABCDEFGH$, determine the relationship among:



    a) lines CD and EH,
    b) lines CN and DG,
    c) lines GM and DH,
    d) the line KL and the plane AFG,
    e) the line DM and the plane FGN,
    f) planes ADK and MNH.
5) Construct a section of a cube $ABCDEFGH$ by a plane XYZ; X is the midpoint of the edge GH, Y is the midpoint of the edge CG and Z is the mid-point of the edge AE.

Stereometry – Worksheet B

1) Fill in the missing parts.
   a) All points that lie in the same line are called ____________________.
   b) Through any three non-collinear points there exists ______________________.
   c) If a line has at least two points in common with a plane, then they are ________, and the line __________________________________.
2) Translate the words into Czech language as used in Maths.
   prism – __________________
   cone – _____________________
   concurrent lines – ___________________
   identical planes – ____________________
   polygon – _________________
3) Translate the words into English language as used in Maths.
   krychle – ____________
   čtyřstěn – _____________________________________
   rovnoběžné různé přímky – __________________
   čtyřboký jehlan – ____________________
   průsečnice – ______________________________
4) Given the cube $ABCDEFGH$, determine the relationship among:



   a) lines FH and BD,
   b) lines CE and DG,
   c) lines AF and BK,
   d) the line LN and the plane KLM,
   e) the line AM and the plane KFG,
   f) planes BEG and CFH.
5) Construct a section of a cube $ABCDEFGH$ by a plane XYZ; X is the midpoint of the edge AD, Y is the midpoint of the edge DC and Z is the mid-point of the edge FG.

Stereometry – Worksheet A (Solution)

1)

a) All points that lie in the same plane are called coplanar points.

b) Through a line and a point not-lying on the line there exists exactly one plane.

c) Two planes are called intersecting if and only if they have exactly one line of intersection.

2)

cylinder – válec

pyramid – jehlan

skew lines – mimoběžné přímky

intersecting planes – různoběžné roviny

face of a cube – stěna krychle

3)

kvádr – cuboid

promítání – projection

totožné přímky – identical lines

řez tělesa – section of a solid

rovnoběžné roviny – parallel planes

4)

a) skew lines,

b) intersecting lines,

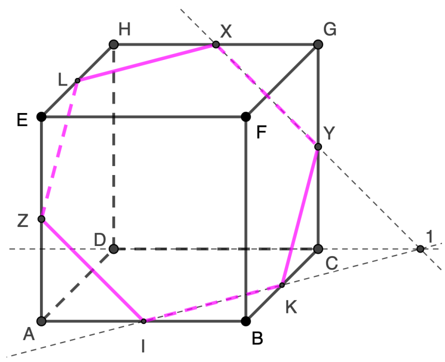
c) distinct parallel lines,

d) the line and the plane are intersecting,

e) the line and the plane are parallel,

f) intersecting planes.

5)



Stereometry – Worksheet B (Solution)

1)

a) All points that lie in the same line are called collinear points.
b) Through any three non-collinear points there exists exactly one plane.
c) If a line has at least two points in common with a plane, then they are parallel, and the line lies in the plane.

2)

prism – hranol
cone – kužel
concurrent lines – různoběžné přímky
identical planes – totožné roviny
polygon – mnohoúhelník

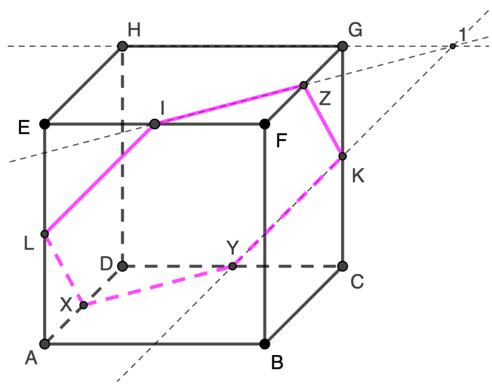
3)

krychle – cube
čtyřstěn – tetrahedron
rovnoběžné různé přímky – distinct parallel lines
čtyřboký jehlan – quadrilateral pyramid
průsečnice – line of intersection

4)

a) distinct parallel lines,
b) skew lines,
c) intersecting lines,
d) the line lies in the plane,
e) the line and the plane are parallel,
f) intersecting planes.

5)



Complex Numbers – Worksheet A

1) Write the algebraic form of the following numbers:
   a) $(8 + \mathrm{i})(3 - 2\mathrm{i}) - (4 + 5\mathrm{i})(1 - 3\mathrm{i})$,
   b) $(6 + 3\mathrm{i})(2 + 2\mathrm{i}) + (3 - 4\mathrm{i})(7 - 5\mathrm{i})$.
2) Solve:
   a) $\mathrm{i}^5 + \mathrm{i}^7 + \mathrm{i}^{11} + \mathrm{i}^{13} + \mathrm{i}^{17} + \mathrm{i}^{21} + \mathrm{i}^{29} + \mathrm{i}^{35}$,
   b) $\mathrm{i}^6 + \mathrm{i}^{10} + \mathrm{i}^{14} + \mathrm{i}^{16} + \mathrm{i}^{24} + \mathrm{i}^{28} + \mathrm{i}^{30} + \mathrm{i}^{38} + \mathrm{i}^{42}$,
   c) $\mathrm{i}^2 \cdot \mathrm{i}^7 \cdot \mathrm{i}^9 \cdot \mathrm{i}^{15} \cdot \mathrm{i}^{17} \cdot \mathrm{i}^{27} \cdot \mathrm{i}^{30} \cdot \mathrm{i}^{33} \cdot \mathrm{i}^{42} \cdot \mathrm{i}^{44}$.
3) Divide $z_1 = 6 + 2\mathrm{i}$ by $z_2 = 3 - 4\mathrm{i}$.
4) Determine the absolute value of $z = 4 + 7\mathrm{i}$.
5) Solve $\left||4 - 2\mathrm{i}|^2 - (4 - 2\mathrm{i})^2\right|$.
6) Solve the equation $\frac{2+\mathrm{i}}{3-2\mathrm{i}}x + \frac{5-4\mathrm{i}}{3-\mathrm{i}} = 5$ on the complex domain.
7) Solve the equation $6x^2 + 14x + 10 = 0$ on the complex domain.

Complex Numbers – Worksheet B

1) Write the algebraic form of the following numbers:
   a) $(3 + 4i)(2 - 5i) - (5 - 3i)(6 + 7i)$,
   b) $(4 + 5i)(3 + i) + (1 - i)(2 - 7i)$.
2) Solve:
   a) $i^7 + i^9 + i^{15} + i^{21} + i^{27} + i^{31} + i^{35} + i^{43}$
   b) $i^4 + i^8 + i^{12} + i^{14} + i^{20} + i^{22} + i^{28} + i^{32} + i^{38}$
   c) $i^3 \cdot i^5 \cdot i^6 \cdot i^{10} \cdot i^{18} \cdot i^{21} \cdot i^{26} \cdot i^{33} \cdot i^{37} \cdot i^{40}$
3) Divide $z_1 = 3 + 4i$ by $z_2 = 4 - 6i$.
4) Determine the absolute value of $z = 3 + 8i$.
5) Solve $||2 - 5i|^2 - (2 - 5i)^2|$.
6) Solve the equation $\frac{3+i}{2-2i}x + \frac{1+3i}{4+5i} = 3$ on the complex domain.
7) Solve the equation $4x^2 + 12x + 15 = 0$ on the complex domain.

Complex Numbers – Worksheet A (Solution)

1)
   a) $7 - 6i$
   b) $7 - 25\mathrm{i}$
2)
   a) $2\mathrm{i}$
   b) -3
   c) -1
3) $\frac{2}{5} + \frac{6}{5}\mathrm{i}$
4) $\sqrt{65}$
5) Solve $8\sqrt{5}$.
6) $K = \left\{\frac{173}{50} - \frac{189}{50}\mathrm{i}\right\}$
7) $K = \left\{-\frac{7}{6} - \frac{\sqrt{11}}{6}\mathrm{i}; -\frac{7}{6} + \frac{\sqrt{11}}{6}\mathrm{i}\right\}$

Complex Numbers – Worksheet B (Solution)

1)

   a) $-25 - 24\mathrm{i}$

   b) $2 + 10\mathrm{i}$

2)

   a) $-4\mathrm{i}$

   b) 3

   c) $-\mathrm{i}$

3) $-\frac{3}{13} + \frac{17}{26}\mathrm{i}$

4) $\sqrt{73}$

5) $10\sqrt{29}$

6) $K = \left\{\frac{36}{41} - \frac{86}{41}\mathrm{i}\right\}$

7) $K = \left\{-\frac{3}{2} - \frac{\sqrt{6}}{2}\mathrm{i}; -\frac{3}{2} + \frac{\sqrt{6}}{2}\mathrm{i}\right\}$