Closure and Interior Structures in Relational Data Analysis and Their Morphisms

Doctoral Thesis

by

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ABSTRACT

We study relationships between compositions of fuzzy relations (formal fuzzy contexts) and morphisms of the structures (concept lattices) associated to the fuzzy relations. In particular, we study concept lattices of both, isotone and antitone concept-forming operators which are associated to the fuzzy relations. The presented theory brings new results on characterization, reduction, and similarity issues regarding concept lattices. Moreover, it brings a new insight to Boolean matrix theory generalizing some of its well-known results to fuzzy setting. In addition, we provide illustrative examples of applications of the presented theory, namely, conceptual scaling to fuzzy attributes and use of block relation to reduce size of a concept lattice.

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Contents

Chapter 1

Introduction

Morphisms represent a general way for modeling complex relationships between mathematical structures. We study morphisms between hierarchies of clusters contained in tabular data with truth degrees. The clusters—so-called formal concepts—are extracted by methods of formal concept analysis (FCA) and play an important role in several areas of relational data analysis.

Formal concept analysis [22] is a method of analysis of relational data invented by Rudolf Wille. In the 1980s, solid mathematical and computational foundations of FCA have been developed. In the past decade or so, FCA enjoyed a considerable interest in various communities and many papers on applications of FCA in various domains appeared, including papers in premier journals and conferences. The method is based on a formalization of a philosophical view of conceptual knowledge. The basic notion in FCA is that of a formal concept which consists of two sets: extent – a set of all objects sharing the same attributes, and intent – a set of all the shared attributes. This definition of formal concept comes from traditional (Port-Royal) logic [1, 29].

The basic input data for FCA, called a formal context, is a flat table in which rows represent objects and columns represent attributes. Entries of the table contain either 1 (or $\times$), which means that the corresponding object has the corresponding attribute, or 0 (blank) which means the opposite. One of the main outputs of FCA is a concept lattice – a hierarchy of formal concepts present in the formal context. The extents and intents of formal concepts are formed by particular operators induced by the formal context.

In everyday life we use concepts which are not sharply bounded (e.g. 'great dancer' or 'middle aged man'). In terms of FCA, objects and attributes do need not be divided sharply by a formal concept into those to which the formal concept applies and those to which it does not. That is to say, a formal concept applies to different objects to different, possibly intermediate degrees. For example, the concept 'middle aged man' may apply to a 45-year old person to degree 1, to a 55-year old person to degree 0.5, and to a 65-year old person to degree 0.2. There are several approaches to generalize formal concept analysis to be able to process such indeterminancy or uncertainty [5, 6, 39, 35, 27, 19]. Many of them are based on Zadeh's theory of fuzzy sets [42].

Boolean factor analysis (see e.g. [17] for the aim and references) concerns with a reduction of space dimension of Boolean (binary) data. Its goal is to decompose a table describing a relation between objects and attributes (in fact, a formal context) into two tables: one describing a relation between objects and factors, second describing a relation between factors and objects, such that the number of factors is as small as possible and the composition of these two relation (with standard relational product) yields the original relation. We can read the composition as: "object has an attribute if and only if there is a factor such that the factor applies to the object and the factor is one of the manifestations of the attribute." Belohlavek and Vychodil proved that formal concepts formed by antitone Galois connections serve as optimal and universal factors [17]. We can be interested in relational products with different meaning, for instance "object has an attribute iff for each factor we have, if the object has the factor, then the factor is a manifestation of the attribute." This kind of relational product is called triangular product and they was studied by Bandler and Kohout [30, 31]. In [8], Belohlavek proved that optimal and universal factors are formal concepts formed by isotone Galois connections.

This dissertation studies morphisms between concept lattices associated to different fuzzy relations and concept lattices formed by different operators. We show that several well-known methods in FCA are expressible as morphisms described here. We also show several useful applications of the presented theory.

For instance, one of the most important problems in FCA is the reduction of concept lattice. Even data which are not large can contain a large number of formal concepts. Large concept lattices are hard to read for a human user. To allow the human user obtain a useful information contained in the hierarchy of formal concepts, the number of formal concepts must be reduced. The morphisms of concept lattices described in this dissertation represent a natural way of reduction of concept lattices. We show that their application covers use of block relations and conceptual scaling and provides a natural way how to use these methods in fuzzy setting.

Outline of the thesis

The results presented in this thesis were published in the following papers (the numbers in square brackets are the numbers of the papers in the Bibliography).

[11] Radim Belohlavek and Jan Konecny. Scaling, granulation, and fuzzy attributes in formal concept analysis. In *FUZZ-IEEE*, pages 1–6, 2007.

[12] Radim Belohlavek and Jan Konecny. Closure spaces of isotone Galois connections and their morphisms. In *Proceedings of the 24th international conference on Advances in Artificial Intelligence*, AI'11, pages 182–191, Berlin, Heidelberg, 2011. Springer-Verlag.

[13] Radim Belohlavek and Jan Konecny. Concept lattices of isotone vs. antitone Galois connections in graded setting: Mutual reducibility revisited. *Information Sciences*, 199(0):133–137, 2012.

[14] Radim Belohlavek and Jan Konecny. Row and column spaces of matrices over residuated lattices. *Fundam. Inform.*, 115(4):279–295, 2012.

[34] Jan Konecny and Michal Krupka. Block relations in fuzzy setting. In *CLA 2011: Proceedings of the 8th International Conference on Concept Lattices and Their Applications*, page 115—130, INRIA Nancy – Grand Est and LORIA, 2011.

The dissertation is organized as follows. Chapter 2 describes basic notions of fuzzy sets, formal fuzzy concept analysis, fuzzy concept lattices, antitone and isotone fuzzy Galois connections, closure operators, and interior operators.

In Chapter 3 we develop the insight into the concept lattices of a given context: We study relationship between concept-forming operators induced by fuzzy relations in (de)composition $I = A * B$; we show a correspondence with notions from Boolean matrix theory; characterize closure spaces induced by isotone concept-forming operators; and show a relationship of isotone and antitone **L**-Galois connections.

Chapter 4 defines basic types of morphisms of structures associated to fuzzy relations (sets of extents/intents of a concept lattice) and shows the morphisms correspondence to fuzzy relations. Furthermore, we describe conditions under which two structures associated to fuzzy relations are isomorphic. Then we describe natural behavior of the morphisms with respect to similarities. The last part of this chapter is devoted to block relations which correspond to special morphisms of concept lattices.

In Chapter 5 we provide a practical application of the theory described in the previous chapters, namely, conceptual scaling to fuzzy attributes. In practice we need to process not only data containing truth-degrees but data containing numerical or categorical values. Conceptual scaling is a kind of transformation such data to a form appropriate for methods of formal concept analysis.

Chapter 2

Preliminaries

We recall basic facts of FCA, residuated lattices, fuzzy sets, and fuzzy relations.

2.1 Residuated Lattices and Fuzzy Sets

We use complete residuated lattices as basic structures of truth-degrees. A complete residuated lattice [5, 25, 40] is a structure $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that

(i) $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice, i.e. a partially ordered set in which arbitrary infima and suprema exist;

(ii) $\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e. $\otimes$ is a binary operation which is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$;

(iii) $\otimes$ and $\rightarrow$ satisfy adjointness, i.e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$.

0 and 1 denote the least and greatest elements. The partial order of $\mathbf{L}$ is denoted by $\leq$. Throughout this thesis, $\mathbf{L}$ denotes an arbitrary complete residuated lattice.

Elements a of L are called truth degrees. Operations $\otimes$ (multiplication) and $\rightarrow$ (residuum) play the role of a (truth functions of) "fuzzy conjunction" and "fuzzy implication". Furthermore, we define the complement of $a \in L$ as

$$\neg a = a \rightarrow 0 \tag{2.1}$$

Common examples of complete residuated lattices include those defined on the unit interval, (i.e. $L = [0,1]$) or on a finite chain in a unit interval, e.g. $L = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$, $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding residuum $\rightarrow$ given by $a \rightarrow b = \max\{c \mid a \times c \leq b\}$. The three most important pairs of adjoint operations on the unit interval are

- Łukasiewicz

$$a \otimes b = \max(a + b - 1, 0)$$
$$a \to b = \min(1 - a + b, 1)$$

- Gödel

$$a \otimes b = \min(a, b)$$
$$a \to b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$

- Goguen (product)

$$a \otimes b = a \cdot b$$
$$a \to b = \begin{cases} 1 & \text{if } a \leq b, \\ \frac{b}{a} & \text{otherwise.} \end{cases}$$

The following theorem summarizes some properties of residuated lattices used in this thesis.

Theorem 1 ([5])**.** *Every complete residuated lattice satisfies the following conditions:*

$$a \otimes (a \to b) \leq b, \tag{2.2}$$
$$b \leq a \to (a \otimes b), \tag{2.3}$$
$$a \leq (a \to b) \to b, \tag{2.4}$$
$$a \leq b \quad \textit{iff} \quad a \to b = 1, \tag{2.5}$$
$$a \to a = 1, \tag{2.6}$$
$$a \to 1 = 1, \tag{2.7}$$
$$0 \to a = 1, \tag{2.8}$$
$$1 \to a = a, \tag{2.9}$$
$$a \otimes 0 = 0, \tag{2.10}$$
$$a \otimes b \leq a, \tag{2.11}$$
$$a \leq b \to a, \tag{2.12}$$
$$a \otimes b \leq a \wedge b, \tag{2.13}$$
$$(a \otimes b) \to c = a \to (b \to c), \tag{2.14}$$
$$(a \to b) \otimes (b \to c) \leq a \to c, \tag{2.15}$$
$$a \to b \textit{ is the greatest element of } \{c \mid a \otimes c \leq b\}, \tag{2.16}$$
$$a \otimes b \textit{ is the least element of } \{c \mid a \leq b \to c\}, \tag{2.17}$$
$$b_1 \leq b_2 \textit{ implies } a \otimes b_1 \leq a \otimes b_2, \tag{2.18}$$
$$b_1 \leq b_2 \textit{ implies } a \to b_1 \leq a \to b_2, \tag{2.19}$$
$$a_1 \leq a_2 \textit{ implies } a_2 \otimes b \leq a_1 \otimes b, \tag{2.20}$$

$$a \otimes \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \otimes b_i), \tag{2.21}$$

$$a \otimes \bigwedge_{i \in I} b_i \leq \bigwedge_{i \in I} (a \otimes b_i), \tag{2.22}$$

$$a \to \bigwedge b_i = \bigwedge_{i \in I} (a \to b_i), \tag{2.23}$$

$$\bigvee_{i \in I} a_i \to b = \bigwedge_{i \in I} (a_i \to b), \tag{2.24}$$

$$\bigvee_{i \in I} (a \to b_i) \leq a \to \bigwedge_{i_i n I} b_i, \tag{2.25}$$

$$\bigvee_{i \in I} (a_i \to b) \leq \bigwedge_{i \in I} a_i \to b, \tag{2.26}$$

$$\bigwedge_{i \in I} (a_i \to b_i) \leq \bigwedge_{i \in I} a_i \to \bigwedge_{i \in I} b_i. \tag{2.27}$$

L-sets and L-relations An **L**-set (or fuzzy set) A in a universe set X is a mapping assigning to each $x \in X$ some truth degree $A(x) \in L$ where L is a support of a complete residuated lattice. The set of all **L**-sets in a universe X is denoted L^X, or $\mathbf{L}^X$ if the structure of **L**is to be emphasized.

The operations with **L**-sets are defined componentwise. For instance, the intersection of **L**-sets $A, B \in L^X$ is an **L**-set $A \cap B$ in X such that $(A \cap B)(x) = A(x) \wedge B(x)$ for each $x \in X$, etc. An **L**-set $A \in L^X$ is also denoted $\{^{A(x)}/x \mid x \in X\}$. If for all $y \in X$ distinct from $x_1, x_2, \dots, x_n$ we have $A(y) = 0$, we also write

$$\{^{A(x_1)}/x_1, {^{A(x_2)}}/x_1, \dots, {^{A(x_n)}}/x_n\}.$$

If there is exactly one $x \in X$ s.t. $A(x) > 0$ (i.e. $A = \{^{A(x)}/x\}$) we call A a singleton. We also use characteristic vector such as (such as $(A(x_1), A(x_2), \dots, A(x_n))$) to describe **L**-sets if there is no danger of confusion.

An **L**-set $A \in L^X$ is called crisp if $A(x) \in \{0, 1\}$ for each $x \in X$. Crisp **L**-sets can be identified with ordinary sets. For a crisp A, we also write $x \in A$ for $A(x) = 1$ and $x \notin A$ for $A(x) = 0$. An **L**-set $A \in L^X$ is called empty (denoted by $\emptyset$) if $A(x) = 0$ for each $x \in X$. For $a \in L$ and $A \in L^X$, the **L**-sets $a \otimes A \in \mathbf{L}^X$, $a \to A$, and $\neg A$ in X are defined by

$$(a \otimes A)(x) = a \otimes A(x), \tag{2.28}$$

$$(a \to A)(x) = a \to A(x), \tag{2.29}$$

$$\neg A(x) = A(x) \to 0. \tag{2.30}$$

Binary **L**-relations (binary fuzzy relations) between X and Y can be thought of as **L**-sets in the universe $X \times Y$. That is, a binary **L**-relation $I \in L^{X \times Y}$ between a set X and a set Y is a mapping assigning to each $x \in X$ and each $y \in Y$ a truth degree $I(x, y) \in L$ (a degree to which x and y are related by I).

For universe $A, B \in L^X$ we define the degree (*graded subsethood*) of inclusion of A in B by

$$S(A,B) = \bigwedge_{x \in X} A(x) \to B(x) \tag{2.31}$$

Graded inclusion generalizes the classical inclusion relation $\subseteq$ (note that unlike $\subseteq$, S is a binary $\mathbf{L}$-relation on $\mathbf{L}^X$. Described verbally, $S(A,B)$ represents a degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $S(A,B) = 1$. As a consequence, we have $A \subseteq B$ iff $A(x) \leq B(x)$ for each $x \in X$.

Further we set

$$A \approx^X B = S(A,B) \wedge S(B,A). \tag{2.32}$$

A binary $\mathbf{L}$-relation R on a set X is called *reflexive* if $R(x,x) = 1$ for any $x \in X$, *symmetric* if $R(x,y) = R(y,x)$ for any $x, y \in X$, and *transitive* if $R(x,y) \otimes R(y,z) \leq R(x,z)$ for any $x, y, z \in X$. R is called an $\mathbf{L}$-*tolerance*, if it is reflexive and symmetric, $\mathbf{L}$-*equivalence* if it is reflexive, symmetric and transitive. If R is an $\mathbf{L}$-equivalence such that for any $x, y \in X$ from $R(x,y) = 1$ it follows $x = y$, then R is called an $\mathbf{L}$-*equality* on X. $\mathbf{L}$-equalities are often denoted by $\approx$. The similarity $\approx^X$ of $\mathbf{L}$-sets (2.32) is an $\mathbf{L}$-equality on L^X.

Composition Operators We use three composition operators, $\circ$, $\triangleleft$, and $\triangleright$, and consider the corresponding compositions $I = A \circ B$, $I = A \triangleleft B$, and $I = A \triangleright B$ (for $I \in L^{X \times Y}, A \in L^{X \times F}, B \in L^{F \times Y}$). In the compositions, $I(x,y)$ is interpreted as the degree to which the object x has the attribute y; $A(x,f)$ as the degree to which the factor f applies to the object x; $B(f,y)$ as the degree to which the attribute y is a manifestation (one of possibly several manifestations) of the factor f. The composition operators are defined by

$$(A \circ B)(x,y) = \bigvee_{f \in F} A(x,f) \otimes B(f,y), \tag{2.33}$$

$$(A \triangleleft B)(x,y) = \bigwedge_{f \in F} A(x,f) \to B(f,y), \tag{2.34}$$

$$(A \triangleright B)(x,y) = \bigwedge_{f \in F} B(f,y) \to A(x,z). \tag{2.35}$$

Note that these operators were extensively studied by Bandler and Kohout, see e.g. [32]. They have natural verbal descriptions. For instance, $(A \circ B)(x,y)$ is the truth degree of the proposition "there is factor f such that f applies to object x and attribute y is a manifestation of f"; $(A \triangleleft B)(x,y)$ is the truth degree of "for every factor f, if f applies to object x then attribute y is a manifestation of f". Note also that for $L = \{0,1\}$, $A \circ B$ coincides with the well-known composition of binary relations.

Theorem 2 (associativity of composition operators)**.** *We have*

$$R \circ (S \circ T) = (R \circ S) \circ T, \tag{2.36}$$
$$R \triangleleft (S \triangleright T) = (R \triangleleft S) \triangleright T, \tag{2.37}$$
$$R \triangleleft (S \triangleleft T) = (R \circ S) \triangleleft T, \tag{2.38}$$
$$R \triangleright (S \circ T) = (R \triangleright S) \triangleright T. \tag{2.39}$$

Remark 1. *In [9] it is shown that* $\circ$, $\triangleright$, *and* $\triangleleft$ *can be considered to be the same composition as it can be covered by a general framework. We do not use the general framework in this thesis because most results contained here use specific properties of compositions defined by* (2.33),(2.34), *and* (2.35).

Isotone and antitone L-Galois connections, L-closure and L-interior operators An *antitone* **L**-*Galois connection* between the sets X and Y is a pair $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ of mappings $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ of mappings ${}^{\uparrow} : L^X \to L^Y$, ${}^{\downarrow} : L^Y \to L^X$, satisfying

$$S(C_1, C_2) \leq S(C_2^{\uparrow}, C_1^{\uparrow}) \qquad S(D_1, D_2) \leq S(D_2^{\downarrow}, D_1^{\downarrow}) \tag{2.40}$$
$$C \subseteq (C^{\uparrow})^{\downarrow} \qquad D \subseteq (D^{\downarrow})^{\uparrow} \tag{2.41}$$

for every $C, C_1, C_2 \in L^X, D, D_1, D_2 \in L^Y$.

An *isotone* **L**-*Galois connection* between the sets X and Y is a pair $\langle {}^{\cap}, {}^{\cup} \rangle$ of mappings $\langle {}^{\cap}, {}^{\cup} \rangle$ of mappings ${}^{\cap} : L^X \to L^Y$, ${}^{\cup} : L^Y \to L^X$, satisfying

$$S(C_1, C_2) \leq S(C_1^{\cap}, C_2^{\cap}) \qquad S(D_1, D_2) \leq S(D_1^{\cup}, D_2^{\cup}) \tag{2.42}$$
$$C \subseteq (C^{\cap})^{\cup} \qquad D \supseteq (D^{\cup})^{\cap} \tag{2.43}$$

for every $C, C_1, C_2 \in L^X, D, D_1, D_2 \in L^Y$.

The following theorem summarizes properties of both antitone and isotone Galois connections.

Theorem 3. *An antitone* **L**-*Galois connection* $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ *satisfies the following properties:*

(i) $C_1 \subseteq C_2$ *implies* $C_2^{\uparrow} \subseteq C_1^{\uparrow}$ *and* $D_1 \subseteq D_2$ *implies* $D_2^{\downarrow} \subseteq D_1^{\downarrow}$

(ii) $S(C, D^{\downarrow}) = S(D, C^{\uparrow})$

(iii) $(\bigcup_{i \in I} C_i)^{\uparrow} = \bigcap_{i \in I} C_i^{\uparrow}$ *and* $(\bigcup_{i \in I} D_i)^{\downarrow} = \bigcap_{i \in I} D_i^{\downarrow}$

(iv) $C^{\uparrow\downarrow\uparrow} = C^{\uparrow}$ *and* $D^{\downarrow\uparrow\downarrow} = D^{\downarrow}$

for each $C, C_i \in L^X, D, D_i \in L^Y$.

An isotone **L**-*Galois connection* $\langle {}^{\cap}, {}^{\cup} \rangle$ *satisfies the following properties:*

(i) $C_1 \subseteq C_2$ *implies* $C_1^{\cap} \subseteq C_2^{\cap}$ *and* $D_1 \subseteq D_2$ *implies* $D_1^{\cup} \subseteq D_2^{\cup}$

(ii) $S(C, D^{\cup}) = S(C^{\cap}, D)$

(iii) $\left(\bigcup_{i\in I} C_i\right)^{\cap} = \bigcup_{i\in I} C_i^{\cap}$ *and* $\left(\bigcap_{i\in I} D_i\right)^{\cup} = \bigcap_{i\in I} D_i^{\cup}$

(iv) $C^{\cap\cup\cap} = C^{\cap}$ *and* $D^{\cup\cap\cup} = D^{\cup}$

for each $C, C_i \in L^X, D, D_i \in L^Y$.

System of **L**-sets $V \subseteq L^X$ is called an **L**-*interior system* if

- V is closed under $\otimes$-multiplication, i.e. for every $a \in L$ and $C \in V$ we have $a \otimes C \in V$ (here, $a \otimes C$ is defined by $(a \otimes C)(x) = a \otimes C(x)$ for $x \in X$);
- V is closed under union, i.e. for $C_j \in V$ $(j \in J)$ we have $\bigcup_{j\in J} C_j \in V$.

$V \subseteq L^X$ is called an **L**-*closure system* if

- V is closed under left $\rightarrow$-multiplication (or $\rightarrow$-shift), i.e. for every $a \in L$ and $C \in V$ we have $a \rightarrow C \in V$ (here, $a \rightarrow C$ is defined by $(a \rightarrow C)(i) = a \rightarrow C(i)$ for $i = 1, \ldots, n$);
- V is closed under intersection, i.e. for $C_j \in V$ $(j \in J)$ we have $\bigcap_{j\in J} C_j \in V$.

2.2 Formal Concept Analysis in the Fuzzy Setting

An **L**-context is a triplet $\langle X, Y, I\rangle$ where X and Y are (ordinary) sets and $I \in L^{X\times Y}$ is an **L**-relation between X and Y. Elements of X are called objects, elements of Y are called attributes, I is called an incidence relation. $I(x,y) = a$ is read: "The object x has the attribute y to degree a." An **L**-context is usually depicted as a table whose rows correspond to objects and whose columns correspond to attributes; entries of the table contain the degrees $I(x,y)$ (see Fig. 2.1 for example of an **L**-context).

Consider the following pairs of operators induced by an **L**-context $\langle X, Y, I\rangle$. First, the pair $\langle {}^{\uparrow}, {}^{\downarrow}\rangle$ of operators ${}^{\uparrow} : L^X \to L^Y$ and ${}^{\downarrow} : L^Y \to L^X$ is defined by

$$C^{\uparrow}(y) = \bigwedge_{x\in X} C(x) \rightarrow I(x,y), \quad D^{\downarrow}(x) = \bigwedge_{y\in Y} D(y) \rightarrow I(x,y). \tag{2.44}$$

Second, the pair $\langle {}^{\cap}, {}^{\cup}\rangle$ of operators ${}^{\cap} : L^X \to L^Y$ and ${}^{\cup} : L^Y \to L^X$ is defined by

$$C^{\cap}(y) = \bigvee_{x\in X} C(x) \otimes I(x,y), \quad D^{\cup}(x) = \bigwedge_{y\in Y} I(x,y) \rightarrow D(y), \tag{2.45}$$

Third, the pair $\langle {}^{\wedge}, {}^{\vee}\rangle$ of operators ${}^{\wedge} : L^X \to L^Y$ and ${}^{\vee} : L^Y \to L^X$ is defined by

$$C^{\wedge}(y) = \bigwedge_{x\in X} I(x,y) \rightarrow C(x), \quad D^{\vee}(x) = \bigvee_{y\in Y} D(y) \otimes I(x,y), \tag{2.46}$$

for $C \in L^X$, $D \in L^Y$.

$\langle^{\uparrow},^{\downarrow}\rangle$ forms an antitone **L**-Galois connection [2], $\langle^{\cap},^{\cup}\rangle$ and $\langle^{\vee},^{\wedge}\rangle$ each form an isotone **L**-Galois connection [23] (that is why we use the same symbols to for their notation). To emphasize that the operators are induced by I, we also denote the operators by $\langle^{\uparrow_I},^{\downarrow_I}\rangle$, $\langle^{\cap_I},^{\cup_I}\rangle$, and $\langle^{\wedge_I},^{\vee_I}\rangle$. Furthermore, denote the corresponding sets of fixpoints by $\mathcal{B}(X^{\uparrow},Y^{\downarrow},I)$, $\mathcal{B}(X^{\cap},Y^{\cup},I)$, and $\mathcal{B}(X^{\wedge},Y^{\vee},I)$, i.e.

$$\begin{aligned}
\mathcal{B}(X^{\uparrow},Y^{\downarrow},I) &= \{\langle C,D\rangle \in L^X \times L^Y \mid C^{\uparrow} = D,\, D^{\downarrow} = C\},\\
\mathcal{B}(X^{\cap},Y^{\cup},I) &= \{\langle C,D\rangle \in L^X \times L^Y \mid C^{\cap} = D,\, D^{\cup} = C\},\\
\mathcal{B}(X^{\wedge},Y^{\vee},I) &= \{\langle C,D\rangle \in L^X \times L^Y \mid C^{\wedge} = D,\, D^{\vee} = C\}.
\end{aligned}$$

The sets of fixpoints are complete lattices, called **L**-concept lattices associated to I, and their elements are called formal concepts.

For a concept lattice $\mathcal{B}(X^{\vartriangle},Y^{\triangledown},I)$, where $\langle^{\vartriangle},^{\triangledown}\rangle$ is either of $\langle^{\uparrow},^{\downarrow}\rangle$, $\langle^{\cap},^{\cup}\rangle$, or $\langle^{\wedge},^{\vee}\rangle$, denote the corresponding sets of extents and intents by $\mathrm{Ext}(X^{\vartriangle},Y^{\triangledown},I)$ and $\mathrm{Int}(X^{\vartriangle},Y^{\triangledown},I)$. That is,

$$\begin{aligned}
\mathrm{Ext}(X^{\vartriangle},Y^{\triangledown},I) &= \{C \in L^X \mid \langle C,D\rangle \in \mathcal{B}(X^{\vartriangle},Y^{\triangledown},I) \text{ for some } D\},\\
\mathrm{Int}(X^{\vartriangle},Y^{\triangledown},I) &= \{D \in L^Y \mid \langle C,D\rangle \in \mathcal{B}(X^{\vartriangle},Y^{\triangledown},I) \text{ for some } C\},
\end{aligned}$$

Note that the operators induced by an **L**-context and their sets of fixpoints have extensively been studied, see e.g. [2, 4, 6, 23, 39]. Clearly, $\langle C,D\rangle \in \mathcal{B}(X^{\cap},Y^{\cup},I)$ iff $\langle D,C\rangle \in \mathcal{B}(Y^{\wedge},X^{\vee},I^{\mathrm{T}})$, where I^{T} denotes the transpose of I; so one could consider only one pair, $\langle^{\cap_I},^{\cup_I}\rangle$ or $\langle^{\wedge_I},^{\vee_I}\rangle$, and obtain the properties of the other pair by a simple translation. Note also that if $L = \{0,1\}$, $\mathcal{B}(X^{\uparrow},Y^{\downarrow},I)$ coincides with the ordinary concept lattice of the formal context consisting of X, Y, and the binary relation (represented by) I.

It is well known that for $L = \{0,1\}$, each of the three operators is definable by any of the remaining two [21] and that, as a consequence, we have

$$\mathcal{B}(X^{\cap_I},Y^{\cup_I},I) \text{ and } \mathcal{B}(X^{\uparrow_{\neg I}},Y^{\downarrow_{\neg I}},\neg I) \text{ are isomorphic as lattices} \tag{2.47}$$

with $\langle C,D\rangle \mapsto \langle C,\neg D\rangle$ being an isomorphism ($\neg U$ denotes the complement of U).

Hence, in particular,

$$\mathrm{Ext}(X^{\cap_I},Y^{\cup_I},I) = \mathrm{Ext}(X^{\uparrow_{\neg I}},Y^{\downarrow_{\neg I}},\neg I), \tag{2.48}$$

i.e. the corresponding sets of extents are equal. Here, the concept lattices and the sets of extents of a binary relation $I \in \{0,1\}^{X\times Y}$ are defined by

$$\mathcal{B}(X^{\uparrow_I}, Y^{\downarrow_I}, I) = \{\langle C, D\rangle \in \{0,1\}^X \times \{0,1\}^Y \mid C^{\uparrow_I} = D,\, D^{\downarrow_I} = C\}, \tag{2.49}$$

$$\mathcal{B}(X^{\cap_I}, Y^{\cup_I}, I) = \{\langle C, D\rangle \in \{0,1\}^X \times \{0,1\}^Y \mid C^{\cap_I} = D,\, D^{\cup_I} = C\}, \tag{2.50}$$

$$\mathrm{Ext}(X^{\uparrow_I}, Y^{\downarrow_I}, I) = \{C \in \{0,1\}^X \mid \langle C, D\rangle \in \mathcal{B}(X^{\uparrow_I}, Y^{\downarrow_I}, I) \text{ for some } D\}, \tag{2.51}$$

$$\mathrm{Ext}(X^{\cap_I}, Y^{\cup_I}, I) = \{C \in \{0,1\}^X \mid \langle C, D\rangle \in \mathcal{B}(X^{\cap_I}, Y^{\cup_I}, I) \text{ for some } D\}. \tag{2.52}$$

The above reducibility results mean that, in a sense, one need not investigate the properties of the concept lattices of $\langle {}^{\uparrow_I}, {}^{\downarrow_I}\rangle$ and $\langle {}^{\cap_I}, {}^{\cup_I}\rangle$ separately because the properties of one are derivable from the properties of the other.

However, as shown in [23], when fuzzy relations instead of ordinary relations I are considered (i.e. graded attributes rather than yes/no attributes are considered), the above mutual reducibility results are no longer true.

L-Concept lattices The following theorem says that **L**-concept lattices are complete lattices (i.e. the name **L**-concept lattices is well justified).

Theorem 4. *(a) $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$ is a complete lattice with suprema and infima given by:*

$$\bigwedge_{j\in J} \langle C_j, D_j\rangle = \langle \bigcap_{j\in J} C_j, (\bigcup_{j\in J} D_j)^{\downarrow\uparrow}\rangle,$$
$$\bigvee_{j\in J} \langle C_j, D_j\rangle = \langle (\bigcup_{j\in J} D_j)^{\uparrow\downarrow}, \bigcap_{j\in J} D_j\rangle.$$

(b) $\mathcal{B}(X^{\cap}, Y^{\cup}, I)$ is a complete lattice with suprema and infima given by:

$$\bigwedge_{j\in J} \langle C_j, D_j\rangle = \langle \bigcap_{j\in J} C_j, (\bigcap_{j\in J} D_j)^{\cup\cap}\rangle,$$
$$\bigvee_{j\in J} \langle C_j, D_j\rangle = \langle (\bigcup_{j\in J} D_j)^{\cap\cup}, \bigcup_{j\in J} D_j\rangle.$$

(c) $\mathcal{B}(X^{\wedge}, Y^{\vee}, I)$ is a complete lattice with suprema and infima given by:

$$\bigwedge_{j\in J} \langle C_j, D_j\rangle = \langle (\bigcap_{j\in J} C_j)^{\vee\wedge}, \bigcap_{j\in J} D_j\rangle,$$
$$\bigvee_{j\in J} \langle C_j, D_j\rangle = \langle \bigcup_{j\in J} C_j, (\bigcup_{j\in J} D_j)^{\cap\cup}\rangle.$$

When displaying **L**-concept lattice $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$, we use labeled Hasse diagram to include all the information on extents and intents. For any $x \in X$, $y \in Y$ and formal **L**-concept $\langle A, B\rangle$ we have $A(x) \geq a$ and $B(y) \geq b$ if and only if there is a formal concept $\langle A_1, B_1\rangle \leq \langle A, B\rangle$, labeled by ${}^{a}\!/x$ and a formal concept $\langle A_2, B_2\rangle \geq \langle A, B\rangle$, labeled by ${}^{b}\!/y$. We use labels x resp. y instead of ${}^{1}\!/x$ resp. ${}^{1}\!/y$ and omit redundant labels (i.e., if a concept has both the labels ${}^{a}\!/x$ and ${}^{b}\!/x$ then we keep only that with the greater degree; dually for attributes). The whole

	y_1	y_2	y_3	y_4	y_5	y_6	y_7
x_1	0	0.5	1	0	0	0.5	1
x_2	0	1	0	0	0.5	1	1
x_3	0	1	1	0.5	1	1	1
x_4	0	1	1	0	0	0.5	1

Figure 2.1: Example of **L**-context with objects $x_1, \ldots, x_4$ and attributes $y_1, \ldots, y_7$ over 3-element Łukasievicz chain. The **L**-context is scaled fuzzy valued-context *cars* from [39].

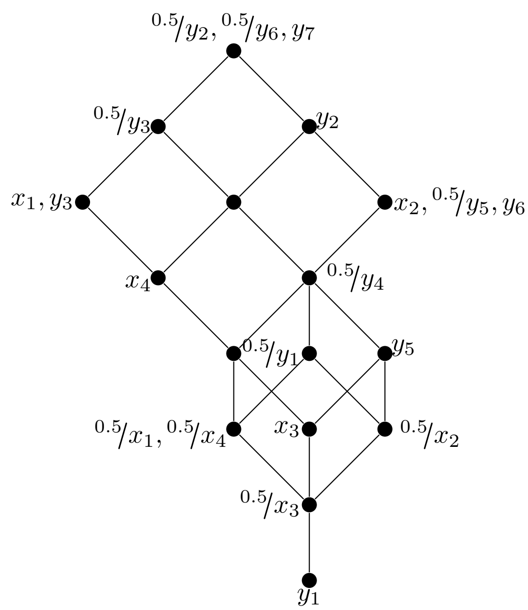


Figure 2.2: **L**-concept lattice $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$ of example **L**-context from Fig. 2.1.

structure of $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$ can be determined from the labeled diagram using the results from [6].

In $\mathcal{B}(X^{\cap}, Y^{\cup}, I)$, for any $x \in X$, $y \in Y$ and formal **L**-concept $\langle A, B\rangle$ we have $A(x) \leq a$ and $B(y) \geq b$ if and only if there is a formal concept $\langle A_1, B_1\rangle \leq \langle A, B\rangle$, labeled by $^{a}/x$ and a formal concept $\langle A_2, B_2\rangle \geq \langle A, B\rangle$, labeled by $^{b}/y$. Dually, in $\mathcal{B}(X^{\wedge}, Y^{\vee}, I)$. for any $x \in X$, $y \in Y$ and formal **L**-concept $\langle A, B\rangle$ we have $A(x) \geq a$ and $B(y) \leq b$ if and only if there is a formal concept $\langle A_1, B_1\rangle \geq \langle A, B\rangle$, labeled by $^{a}/x$ and a formal concept $\langle A_2, B_2\rangle \leq \langle A, B\rangle$, labeled by $^{b}/y$.

Example 1. Consider the **L**-context depicted in Fig. 2.1. Figure 2.2 shows its **L**-concept lattice $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$, Fig. 2.3 shows $\mathcal{B}(X^{\cap}, Y^{\cup}, I)$, and Fig. 2.4 shows $\mathcal{B}(X^{\wedge}, Y^{\vee}, I)$.

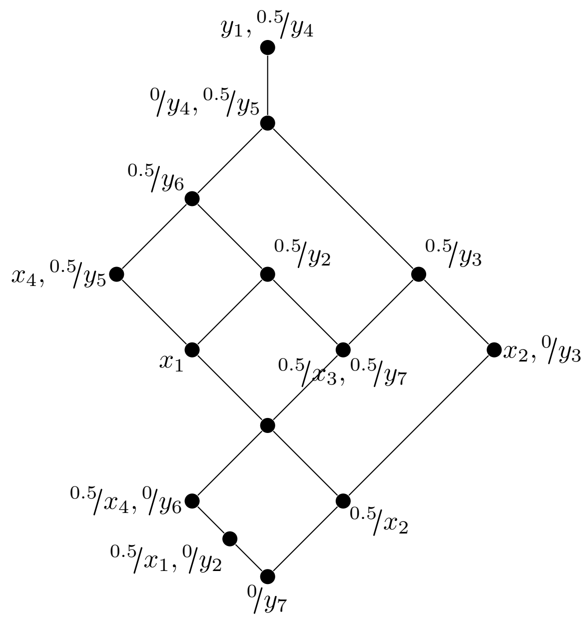


Figure 2.3: **L**-concept lattice $\mathcal{B}(X^{\cap}, Y^{\cup}, I)$ of example **L**-context from Fig. 2.1.

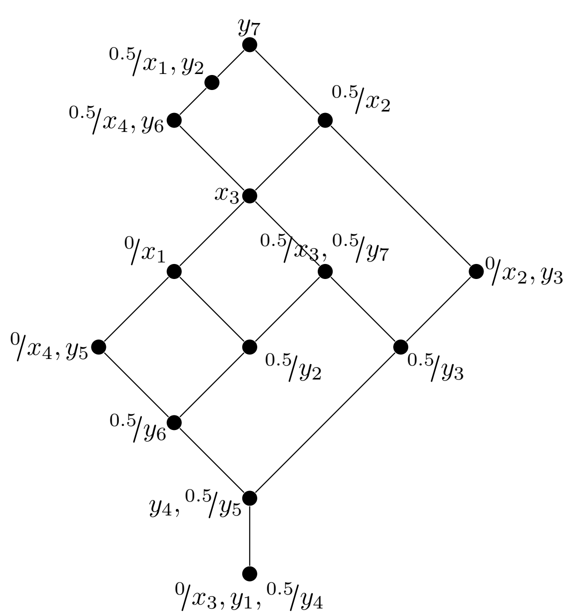


Figure 2.4: **L**-concept lattice $\mathcal{B}(X^{\wedge}, Y^{\vee}, I)$ of example **L**-context from Fig. 2.1.

2.3 Matrices and Vectors over Residuated Lattices

We use matrices whose degrees are elements of residuated lattices. For convenience and since there is no danger of misunderstanding, we take the advantage of identifying $n \times m$ matrices over residuated lattices (the set of all such matrices is denoted by $L^{n\times m}$) with **L**-relations between X and Y. Also, we identify vectors with n components over residuated lattices (the set of all such vectors is denoted by L^n) with fuzzy sets in X. As usual, we identify vectors with n components with $1 \times n$ matrices. We denote

$$X = \{1, \ldots, n\}, \quad Y = \{1, \ldots, m\}, \quad F = \{1, \ldots, k\},$$

and we use notation of **L**-sets and **L**-relations for matrices; for example we write $A(i,j)$ instead of A_{ij}.

Given an $n \times m$ matrix I and a composition operator $*$ (i.e., $\circ$, $\triangleleft$, or $\triangleright$), the decomposition problem consists in finding a decomposition $I = A * B$ of I into an $n \times k$ matrix A and a $k \times m$ matrix B with the number k (number of factors) as small as possible. The smallest k is called the *Schein rank* of I and is denoted by $\rho_{\mathrm{s}}(I)$ Looking for decompositions $I = A * B$ can be seen as looking for factors in data described by I. That is, decomposing I can be regarded as factor analysis in which the data as well as the operations used are different from the ordinary factor analysis [26].

The concept lattices associated to I play a fundamental role for decompositions of I. Namely, given a set

$$\mathcal{F} = \{\langle C_1, D_1\rangle, \ldots, \langle C_k, D_k\rangle\}$$

of **L**-sets $C_f \in L^X$ and $D_f \in L^Y$ define **L**-relations $A_{\mathcal{F}}$ and $B_{\mathcal{F}}$ by

$$A_{\mathcal{F}}(x,f) = C_f(x) \qquad \text{and} \qquad B_{\mathcal{F}}(f,y) = D_f(y). \tag{2.53}$$

This says: the l-th column of $A_{\mathcal{F}}$ is the transpose of the vector corresponding to C_l and the l-th row of $B_{\mathcal{F}}$ is the vector corresponding to D_l.

Then we have:

Theorem 5 (**universality, [7, 8]**)**.**

($\circ$) *For every $I \in L^{X,Y}$ there exists $\mathcal{F} \subseteq \mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$ such that $I = A_{\mathcal{F}} \circ B_{\mathcal{F}}$.*

($\triangleleft$) *For every $I \in L^{X,Y}$ there exists $\mathcal{F} \subseteq \mathcal{B}(X^{\cap}, Y^{\cup}, I)$ such that $I = A_{\mathcal{F}} \triangleleft B_{\mathcal{F}}$.*

($\triangleright$) *For every $I \in L^{X,Y}$ there exists $\mathcal{F} \subseteq \mathcal{B}(X^{\wedge}, Y^{\vee}, I)$ such that $I = A_{\mathcal{F}} \triangleright B_{\mathcal{F}}$.*

Theorem 6 (**optimality, [7, 8]**)**.**

($\circ$) *Let $I = A \circ B$, for $A \in L^{X\times Z}, B \in L^{Z\times Y}$. Then there exists $\mathcal{F} \subseteq \mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$ with $|\mathcal{F}| \leq |Z|$ such that for the **L**-relations $A_{\mathcal{F}}$ and $B_{\mathcal{F}}$, we have $I = A_{\mathcal{F}} \circ B_{\mathcal{F}}$.*

($\triangleleft$) *Let $I = A \triangleleft B$, for $A \in L^{X \times Z}, B \in L^{Z \times Y}$. Then there exists $\mathcal{F} \subseteq \mathcal{B}(X^{\cap}, Y^{\vee}, I)$ with $|\mathcal{F}| \leq |Z|$ such that for the* **L***-relations $A_{\mathcal{F}}$ and $B_{\mathcal{F}}$, we have $I = A_{\mathcal{F}} \triangleleft B_{\mathcal{F}}$.*

($\triangleright$) *Let $I = A \triangleright B$, for $A \in L^{X \times Z}, B \in L^{Z \times Y}$. Then there exists $\mathcal{F} \subseteq \mathcal{B}(X^{\wedge}, Y^{\vee}, I)$ with $|\mathcal{F}| \leq |Z|$ such that for the* **L***-relations $A_{\mathcal{F}}$ and $B_{\mathcal{F}}$, we have $I = A_{\mathcal{F}} \triangleright B_{\mathcal{F}}$.*

Theorem 5 and Theorem 6 state that formal concepts are universal and optimal factors for decompositions. In words, there always exists a decomposition $I = A_{\mathcal{F}} * B_F$ formed with formal concepts and for each decomposition $I = A * F$ there is a decomposition formed with formal concepts which is at least as good as this one. [7, 8] provide effective algorithms for decompositions based on this fact.

Chapter 3

Structures Associated to L-relations

In this chapter, we develop a new insight into the **L**-concept lattices of a given **L**-context: We study a relationship between concept-forming operators induced by **L**-relations involved in a (de)composition $I = A * B$; we show a correspondence with some notions from Boolean matrix theory; characterize closure spaces induced by isotone concept-forming operators; and show a relationship between isotone and antitone **L**-Galois connections.

3.1 L-concept Lattices Associated to Compositions of L-relations

We consider (de)compositions $I = A * B$ and the concept-forming operators and concept lattices associated to I, A, and B. We start with an assertion which is used later. Note that $C^{\wedge_A \wedge_B}$ denotes $(C^{\wedge_A})^{\wedge_B}$ and the like.

Theorem 7. *Let $\langle X, F, A \rangle$ and $\langle F, Y, B \rangle$ be **L**-contexts, let $C \in L^X$, $D \in L^Y$. We have*

$$C^{\cap_{A \circ B}} = C^{\cap_A \cap_B}, \qquad D^{\cup_{A \circ B}} = D^{\cup_B \cup_A}, \tag{3.1}$$

$$C^{\wedge_{A \circ B}} = C^{\wedge_A \wedge_B}, \qquad D^{\vee_{A \circ B}} = D^{\vee_B \vee_A}, \tag{3.2}$$

$$C^{\uparrow_{A \triangleleft B}} = C^{\cap_A \uparrow_B}, \qquad D^{\downarrow_{A \triangleleft B}} = D^{\downarrow_B \cup_A}, \tag{3.3}$$

$$C^{\uparrow_{A \triangleright B}} = C^{\uparrow_A \wedge_B}, \qquad D^{\downarrow_{A \triangleright B}} = D^{\vee_B \downarrow_A}. \tag{3.4}$$

Proof. (3.1)–(3.4) can be directly verified using properties of complete residuated lattices. Another way is to use Theorem 2. We only will demonstrate how (3.2) can be proved using (2.39). Consider a one-element universe $U = \{1\}$ and fuzzy relation R between U and X defined by $R(1,x) = C(x)$. Observe that

$$C^{\wedge_{A\circ B}}(y) = (R \triangleright (A \circ B))(1,y) \quad \text{and} \quad C^{\wedge_A \wedge_B}(y) = ((R \triangleright A) \triangleright B)(1,y).$$

This proves $C^{\wedge_{A\circ B}} = C^{\wedge_A \wedge_B}$.

In a similar way: (3.1) can be proved using (2.36); (3.3) can be proved using (2.38); (3.4) can be proved using (2.37) and (2.39). □

The following theorem describes basic relationships between the various concept lattices associated to decompositions $I = A * B$.

Theorem 8. *Let* $\langle X, F, A\rangle$ *and* $\langle F, Y, B\rangle$ *be* **L***-contexts. We have*

$$\begin{aligned}\mathrm{Ext}(X^{\cap_{A\circ B}}, Y^{\cup_{A\circ B}}, A\circ B) &\subseteq \mathrm{Ext}(X^{\cap_A}, F^{\cup_A}, A)\\ \mathrm{Int}(X^{\cap_{A\circ B}}, Y^{\cup_{A\circ B}}, A\circ B) &\subseteq \mathrm{Int}(F^{\cap_B}, Y^{\cup_B}, B)\end{aligned} \qquad (\circ 1)$$

$$\begin{aligned}\mathrm{Ext}(X^{\wedge_{A\circ B}}, Y^{\vee_{A\circ B}}, A\circ B) &\subseteq \mathrm{Ext}(X^{\wedge_A}, F^{\vee_A}, A)\\ \mathrm{Int}(X^{\wedge_{A\circ B}}, Y^{\vee_{A\circ B}}, A\circ B) &\subseteq \mathrm{Int}(F^{\wedge_B}, Y^{\vee_B}, B)\end{aligned} \qquad (\circ 2)$$

$$\begin{aligned}\mathrm{Ext}(X^{\uparrow_{A\triangleleft B}}, Y^{\downarrow_{A\triangleleft B}}, A\triangleleft B) &\subseteq \mathrm{Ext}(X^{\cap_A}, F^{\cup_A}, A)\\ \mathrm{Int}(X^{\uparrow_{A\triangleleft B}}, Y^{\downarrow_{A\triangleleft B}}, A\triangleleft B) &\subseteq \mathrm{Int}(F^{\uparrow_B}, Y^{\downarrow_B}, B)\end{aligned} \qquad (\triangleleft)$$

$$\begin{aligned}\mathrm{Ext}(X^{\uparrow_{A\triangleright B}}, Y^{\downarrow_{A\triangleright B}}, A\triangleright B) &\subseteq \mathrm{Ext}(X^{\uparrow_A}, F^{\downarrow_A}, A)\\ \mathrm{Int}(X^{\uparrow_{A\triangleright B}}, Y^{\downarrow_{A\triangleright B}}, A\triangleright B) &\subseteq \mathrm{Int}(F^{\wedge_B}, Y^{\vee_B}, B)\end{aligned} \qquad (\triangleright)$$

Proof. For ($\circ$1): Note first that C is an extent in $\mathrm{Ext}(X^{\cap}, Y^{\cup}, I)$ if and only if $C = D^{\cup}$ for some D. Let thus C be an extent of $\mathcal{B}(X^{\cap_{A\circ B}}, Y^{\cup_{A\circ B}}, A\circ B)$. Then $C = D^{\cup_{A\circ B}}$ for $\langle C, D\rangle \in \mathcal{B}(X^{\cap_{A\circ B}}, Y^{\cup_{A\circ B}}, A\circ B)$. Due to (3.1), $C = D^{\cup_{A\circ B}} = (D^{\cup_B})^{\cup_A}$, hence C is an extent of $\mathcal{B}(X^{\cap_A}, Y^{\cup_A}, A)$.

In a similar way, one can prove the second part of ($\circ$1) and also ($\circ$2), ($\triangleleft$), and ($\triangleright$). □

Remark 2. Note that the opposite inclusions in Theorem 8 are not true. For example, for $L = \{0,1\}$, consider $X = Y = F = \{1,2\}$, and matrices

$$A = \begin{pmatrix} 1 & 0\\ 1 & 1\end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}, \quad A\circ B = \begin{pmatrix} 1 & 1\\ 1 & 1\end{pmatrix}.$$

One can check that $(1,0)$ is (a characteristic vector of) an extent of $\mathcal{B}(X^{\cap}, F^{\cup}, A)$ and $(0,1)$ is (a characteristic vector of) an intent of $\mathcal{B}(F^{\cap}, Y^{\cup}, B)$. But neither $(1,0)$ is an extent nor $(0,1)$ is an intent of $\mathcal{B}(X^{\cap}, Y^{\cup}, A\circ B)$.

The next theorem shows that every formal concept $\langle C, D\rangle$ of a compound **L**-relation $A * B$ is generated by a collection of **L**-sets H of factors in such a way that C results by applying the extent-forming operator of A to H and D results by applying the intent-forming operator of B to H. The theorem also describes the collection of such Hs as a particular interval.

Theorem 9. *Let* $\langle X, F, A\rangle$ *and* $\langle F, Y, B\rangle$ *be formal* **L***-contexts.*

(∘1) *For every* $\langle C, D\rangle \in \mathcal{B}(X^{\cap_{A\circ B}}, Y^{\cup_{A\circ B}}, A \circ B)$*, denote*

$$\mathrm{pre}(E, G) = \{H \in L^F \mid H^{\cup_A} = C, H^{\cap_B} = D\}.$$

Then $\mathrm{pre}(E, G)$ *forms an interval in* L^F*;* $C^{\cap_A}$ *is its least element and* $D^{\cup_B}$ *its greatest element.*

(∘2) *For every* $\langle C, D\rangle \in \mathcal{B}(X^{\wedge_{A\circ B}}, Y^{\vee_{A\circ B}}, A \circ B)$*, denote*

$$\mathrm{pre}(C, D) = \{H \in L^F \mid H^{\vee_A} = C, H^{\wedge_B} = D\}.$$

Then $\mathrm{pre}(C, D)$ *forms an interval in* L^F*;* $D^{\vee_B}$ *is its least element and* $C^{\wedge_A}$ *its greatest element.*

(◁) *For every* $\langle C, D\rangle \in \mathcal{B}(X^{\uparrow_{A\triangleleft B}}, Y^{\downarrow_{A\triangleleft B}}, A \triangleleft B)$*, denote*

$$\mathrm{pre}(C, D) = \{H \in L^F \mid H^{\cup_A} = C, H^{\uparrow_B} = D\}.$$

Then $\mathrm{pre}(C, D)$ *forms an interval in* L^F*;* $C^{\cap_A}$ *is its least element and* $D^{\downarrow_B}$ *its greatest element.*

(▷) *For every* $\langle C, D\rangle \in \mathcal{B}(X^{\uparrow_{A\triangleright B}}, Y^{\downarrow_{A\triangleright B}}, A \triangleright B)$*, denote*

$$\mathrm{pre}(C, D) = \{H \in L^F \mid H^{\downarrow_A} = C, H^{\wedge_B} = D\}.$$

Then $\mathrm{pre}(C, D)$ *forms an interval in* L^F*;* $D^{\vee_B}$ *is its least element and* $C^{\uparrow_A}$ *its greatest element.*

Proof. We prove only (∘1). (∘2), (◁) and (▷) can be proved similarly.

Let $\langle C, D\rangle \in \mathcal{B}(X^{\cap_{A\circ B}}, Y^{\cup_{A\circ B}}, A\circ B)$. Observe that $C^{\cap_A} \in \mathrm{pre}(C, D)$. Indeed, we have $C^{\cap_A\cup_A} = C$ by Theorem 8 (∘1) and $C^{\cap_A\cap_B} = C^{\cap_{A\circ B}} = D$ by (3.1). Similarly, $D^{\cup_B} \in \mathrm{pre}(C, D)$.

Let $H \in \mathrm{pre}(C, D)$. We get $C^{\cap_A} \subseteq H$ directly from $H^{\cup_A} = C$ and $H^{\cup_A\cap_A} \subseteq H$ (see [23] for properties of $^{\cap}$ and $^{\cup}$). $H \subseteq D^{\cup_B}$ can be proved similarly. Therefore, $\mathrm{pre}(C, D)$ is contained in the interval $[C^{\cap_A}, D^{\cup_B}] = \{H \mid C^{\cap_A} \subseteq H \subseteq D^{\cup_B}\}$. On the other hand, if $H \in [C^{\cap_A}, D^{\cup_B}]$, i.e. $C^{\cap_A} \subseteq H \subseteq D^{\cup_B}$, then $C = C^{\cap_A\cup_A} \subseteq H^{\cup_A} \subseteq G^{\cup_B\cup_A} = D^{\cup_{A\circ B}} = C$, whence $H^{\cup_A} = C$. In a similar way, $H^{\cap_B} = D$. □

Theorem 9 is illustrated in Fig. 3.1.

The necessary and sufficient conditions for inclusions of sets of extents and intents of two **L**-contexts are the subject of the following theorem.

Theorem 10. *Consider contexts* $\langle X, Y, I\rangle$*,* $\langle X, F, A\rangle$*, and* $\langle F, Y, B\rangle$*.*

(a) $\mathrm{Int}(X^{\cap}, Y^{\cup}, I) \subseteq \mathrm{Int}(F^{\cap}, Y^{\cup}, B)$ *if and only if there exists* $A' \in L^{X\times F}$ *such that* $I = A' \circ B$*,*

(b) $\mathrm{Ext}(X^{\wedge}, Y^{\vee}, I) \subseteq \mathrm{Ext}(X^{\wedge}, F^{\vee}, A)$ *if and only if there exists* $B' \in L^{F\times Y}$ *such that* $I = A \circ B'$*,*

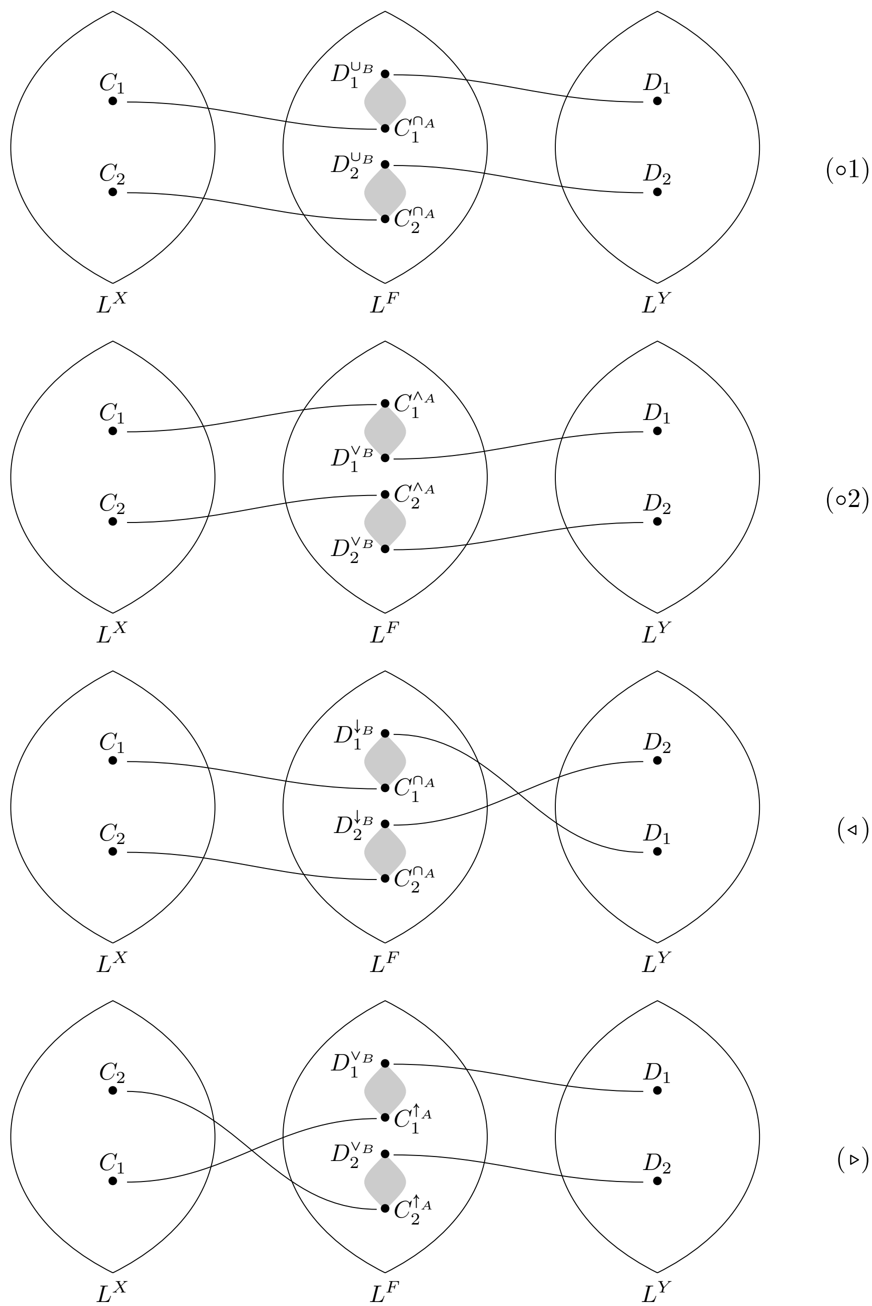


Figure 3.1: Illustration of Theorem 9.

(c) $\mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I) \subseteq \mathrm{Int}(F^{\uparrow}, Y^{\downarrow}, B)$ *if and only if there exists* $A' \in L^{X \times F}$ *such that* $I = A' \triangleleft B$*,*

(d) $\mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, I) \subseteq \mathrm{Ext}(X^{\uparrow}, F^{\downarrow}, A)$ *if and only if there exists* $B' \in L^{F \times Y}$ *such that* $I = A \triangleright B'$*.*

In addition,

(e) $\mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, I) \subseteq \mathrm{Ext}(X^{\cap}, F^{\cup}, A)$ *if and only if there exists* $B' \in L^{F \times Y}$ *such that* $I = A \triangleleft B'$*,*

(f) $\mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I) \subseteq \mathrm{Int}(F^{\wedge}, Y^{\vee}, B)$ *if and only if there exists* $A' \in L^{X \times Y}$ *such that* $I = A' \triangleright B$*.*

Proof. (a) "⇒": Let $\mathrm{Int}(X^{\cap}, Y^{\cup}, I) \subseteq \mathrm{Int}(F^{\cap}, Y^{\cup}, B)$. Every $H \in \mathrm{Int}(F^{\cap}, Y^{\cup}, B)$ can be written as

$$H = \bigvee_{f \in F} c_f \otimes B_{f\_}.$$

Thus every $H \in \mathrm{Int}(X^{\cap}, Y^{\cup}, I)$ can be written as $\bigvee_{f \in F} c_f \otimes B_{f\_}$. Therefore, since every row $I_{x\_}$ of I belongs to $\mathrm{Int}(X^{\cap}, Y^{\cup}, I)$, $I_{x\_}$ can be written as

$$I_{x\_} = \bigvee_{f \in F} c_{xf} \otimes B_{f\_}.$$

Now, we get the required **L**-relation A by putting $A(x, f) = c_{xf}$.

"⇐" is established in Theorem 8.

(b) follows from (a) and the fact that $(C \circ D)^{\mathrm{T}} = D^{\mathrm{T}} \circ C^{\mathrm{T}}$.

(c) "⇒": Let $\mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I) \subseteq \mathrm{Int}(F^{\uparrow}, Y^{\downarrow}, B)$. Every $H \in \mathrm{Int}(F^{\uparrow}, Y^{\downarrow}, B)$ can be written as

$$H = \bigwedge_{f \in F} c_f \to B_{f\_}.$$

Thus every $H \in \mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I)$ can be written as $\bigwedge_{f \in F} c_f \to B_{f\_}$. Therefore, since every row $I_{i\_}$ of I belongs to $\mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I)$, $I_{x\_}$ can be written as

$$I_{x\_} = \bigwedge_{f \in F} c_{xf} \to B_{f\_}.$$

Now, we get the required **L**-relation A by putting $A(x, f) = c_{xf}$.

"⇐" is established in Theorem 8.

(d) follows from (c) and the fact that $(C \triangleleft D)^{\mathrm{T}} = D^{\mathrm{T}} \triangleright C^{\mathrm{T}}$.

(e) "⇒": Let $\mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, I) \subseteq \mathrm{Ext}(X^{\cap}, F^{\cup}, B)$. Every $H \in \mathrm{Ext}(X^{\cap}, Y^{\cup}, B)$ can be written as

$$H = \bigwedge_{f \in F} B_{\_l} \to c_f.$$

Thus every $H \in \mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, I)$ can be written as $\bigwedge_{f\in F} A_{\text{-}f} \to c_f$. Therefore, since every column $I_{\text{-}y}$ of I belongs to $\mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, I)$, $I_{\text{-}y}$ can be written as

$$I_{\text{-}y} = \bigwedge_{f\in F} A_{\text{-}f} \to c_{fy}$$

Now, we get the required **L**-relation B by putting $B(f, y) = c_{fy}$.

Again, "$\Leftarrow$" is established in Theorem 8.

(f) follows from (e) and the fact that $(C \triangleleft D)^{\mathrm{T}} = D^{\mathrm{T}} \triangleright C^{\mathrm{T}}$. $\square$

Remark 3. *Reducing size of concept lattice is one of the most important problem in FCA. Even data which are not large can contain large number of formal concepts. Large concept lattices are then not readable by human user. There is several approaches to reduction of the size of concept lattice (e.g. [18, 33] enrich concept-forming operators with additional parameters – truth stressing (or truth relaxing) hedges, [3] considers approach based on a choice of a threshold $a \in L$ and using an a-cut ${}^{a}\!\approx$ of the **L**-equality $\approx$ on $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$ for factorization, etc.). Notice, that Theorem 10 provides crucial insight to that problem. It says that if we reduce concept lattice in such a way that it preserves original extents (or intents) then it can be expressed as composition of **L**-relations. We show its applications in Chapter 4 (block relations) and Chapter 5.*

Remark 4. *Note that from Theorem 10(a) we have:*

$\mathrm{Int}(X^{\cap}, Y^{\cup}, I) \subseteq \mathrm{Int}(F^{\cap}, Y^{\cup}, B)$ *iff there exists* $A' \in L^{X\times F}$ *such that* $I = A' \circ B$.

None the less, $\mathrm{Ext}(X^{\cap}, Y^{\cup}, I) \subseteq \mathrm{Ext}(X^{\cap}, F^{\cup}, A)$, *does not imply existence of* **L***-relation* $B' \in L^{F\times Y}$ *such that* $I = A \circ B'$. *As an counterexample, consider L being a finite chain containing $a < b$ with $\otimes$ defined as follows:*

$$x \otimes y = \begin{cases} x \wedge y & \textit{if } x = 1 \textit{ or } y = 1, \\ 0 & \textit{otherwise,} \end{cases}$$

for each $x, y \in L$. One can easily see that $x \otimes \bigvee_j y_j = \bigvee_j (x \otimes y_j)$ and thus an adjoint operation $\to$ exists such that $\langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ is a complete residuated lattice (see e.g. [24]). Namely, $\to$ is given as follows:

$$x \to y = \begin{cases} 1 & \textit{if } x \le y, \\ y & \textit{if } x = 1, \\ b & \textit{otherwise,} \end{cases}$$

for each $x, y \in L$. Consider $I = \big(a\big)$ and $B = \big(b\big)$. One can check that, we have $\mathrm{Ext}(\{x\}^{\cap}, \{y\}^{\cup}, I) = \mathrm{Ext}(\{x\}^{\cap}, \{f\}^{\cup}, A) = \{\{{}^{b}\!/x\}, x\}$, *but there is no* **L***-relation* $B' \in L^{\{f\}\times\{y\}}$ *such that* $I = A \circ B'$.

As a corollary of Theorem 10, we obtain the following theorem.

Theorem 11. *Let I and J be* **L***-relations between X and Y.*

(a) *If* $\mathrm{Int}(X^{\cap}, Y^{\cup}, I) = \mathrm{Int}(X^{\cap}, Y^{\cup}, J)$ *and* $I = A \circ B$ *for some* $A \in L^{X\times F}, B \in L^{F\times Y}$ *then there exists* $A' \in L^{X\times F}$ *such that* $J = A' \circ B$.

(b) *If* $\mathrm{Ext}(X^{\wedge}, Y^{\vee}, I) = \mathrm{Ext}(X^{\wedge}, Y^{\vee}, J)$ *and* $I = A \circ B$ *for some* $A \in L^{X\times F}, B \in L^{F\times Y}$ *then there exists* $B' \in L^{F\times Y}$ *such that* $J = A \circ B'$.

(c) *If* $\mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I) = \mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, J)$ *and* $I = A \triangleleft B$ *for some* $A \in L^{X\times F}, B \in L^{F\times Y}$ *then there exists* $A' \in L^{X\times F}$ *such that* $J = A' \triangleleft B$.

(d) *If* $\mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, I) = \mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, J)$ *and* $I = A \triangleright B$ *for some* $A \in L^{X\times F}, B \in L^{F\times Y}$ *then there exists* $B' \in L^{F\times Y}$ *such that* $J = A \triangleright B'$.

(e) *If* $\mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, I) = \mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, J)$ *and* $I = A \triangleleft B$ *for some* $A \in L^{X\times F}, B \in L^{F\times Y}$ *then there exists* $B' \in L^{Y\times F}$ *such that* $J = A \triangleleft B'$.

(f) *If* $\mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I) = \mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, J)$ *for some* $A \in L^{X\times F}, B \in L^{F\times Y}$ *then there exists* $A' \in L^{X\times F}$ *such that* $J = A' \triangleright B$.

Proof. (a): If $I = A \circ B$ then, due to Theorem 10 (a), we have $\mathrm{Int}(X^{\cap}, Y^{\cup}, I) \subseteq \mathrm{Int}(X^{\cap}, Y^{\cup}, B)$. Since $\mathrm{Int}(X^{\cap}, Y^{\cup}, J) = \mathrm{Int}(X^{\cap}, Y^{\cup}, I)$, we have $\mathrm{Int}(X^{\cap}, Y^{\cup}, J) \subseteq \mathrm{Int}(X^{\cap}, Y^{\cup}, B)$. Another application of Theorem 10(a) yields A' for which $J = A' \circ B$.

The proof for (b)–(f) is similar. □

L-concept Lattices Associated to (de)compositions with Concepts as Factors

Now, we consider (de)compositions $I = A_{\mathcal{F}} * B_{\mathcal{F}}$ which use formal concepts as factors as described by (2.53). Since column of $A_{\mathcal{F}}$ are extents of I and rows of $B_{\mathcal{F}}$ are intents I their concept-forming operators have additional properties to those presented in Section 3.1.

Theorem 12. ($\circ$) *Let* $\langle X, Y, I\rangle$ *be an* **L**-*context and let* $\mathcal{F} \subseteq \mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$. *Consider* $A_{\mathcal{F}} \in L^{X\times F}$ *and* $B_{\mathcal{F}} \in L^{F\times Y}$ *as in* (2.53). *We have*

$$C^{\uparrow_{A_{\mathcal{F}}}} = C^{\uparrow_I \cup_{B_{\mathcal{F}}}} \qquad H^{\downarrow_{A_{\mathcal{F}}}} = H^{\cap_{B_{\mathcal{F}}} \downarrow_I}$$
$$H^{\uparrow_{B_{\mathcal{F}}}} = H^{\vee_{A_{\mathcal{F}}} \uparrow_I} \qquad D^{\downarrow_{B_{\mathcal{F}}}} = D^{\downarrow_I \wedge_{A_{\mathcal{F}}}}$$

for each $C \in L^X, D \in L^Y, H \in L^F$.

($\triangleleft$) *Let* $\langle X, Y, I\rangle$ *be an* **L**-*context and let* $\mathcal{F} \subseteq \mathcal{B}(X^{\cap}, Y^{\cup}, I)$. *Consider* $A_{\mathcal{F}} \in L^{X\times F}$ *and* $B_{\mathcal{F}} \in L^{F\times Y}$ *as in* (2.53). *We have*

$$C^{\uparrow_{A_{\mathcal{F}}}} = C^{\cap_I \downarrow_{B_{\mathcal{F}}}} \qquad H^{\downarrow_{A_{\mathcal{F}}}} = H^{\uparrow_{B_{\mathcal{F}}} \cup_I}$$
$$H^{\cap_{B_{\mathcal{F}}}} = H^{\vee_{A_{\mathcal{F}}} \cap_I} \qquad D^{\cup_{B_{\mathcal{F}}}} = D^{\cup_I \wedge_{A_{\mathcal{F}}}}$$

for each $C \in L^X, D \in L^Y, H \in L^F$.

($\triangleright$) *Let $\langle X,Y,I\rangle$ be an* **L**-*context and let $\mathcal{F}\subseteq\mathcal{B}(X^{\uparrow},Y^{\downarrow},I)$. Consider $A_{\mathcal{F}}\in L^{X\times F}$ and $B_{\mathcal{F}}\in L^{F\times Y}$ as in* (2.53).

$$C^{\cap_{A_{\mathcal{F}}}}=C^{\cap_I\vee_{B_{\mathcal{F}}}}\qquad H^{\cup_{A_{\mathcal{F}}}}=H^{\cap_{B_{\mathcal{F}}}\cup_I}$$
$$H^{\uparrow_{B_{\mathcal{F}}}}=H^{\downarrow_{A_{\mathcal{F}}}\wedge_I}\qquad D^{\downarrow_{B_{\mathcal{F}}}}=D^{\vee_I\uparrow_{A_{\mathcal{F}}}}$$

for each $C\in L^X, D\in L^Y, H\in L^F$.

Proof. We prove only ($\circ$); parts ($\triangleleft$) and ($\triangleright$) can be proved similarly. By (2.53) we have $A(x,f)=A_f(x)=B_f^{\downarrow_I}(x)=\bigwedge_{y\in Y}B_f(y)\to I(x,y)=\bigwedge_{y\in Y}B(f,y)\to I(x,y)=(I\triangleright B^{\mathrm{T}})(x,f)$. Whence, we have $A_{\mathcal{F}}=I\triangleright B^{T}_{\mathcal{F}}$. Now, $C^{\uparrow_{A_{\mathcal{F}}}}=C^{\cap_I\downarrow_{B_{\mathcal{F}}}}$ and $H^{\downarrow_{A_{\mathcal{F}}}}=H^{\uparrow_{B_{\mathcal{F}}}\cup_I}$ follow from Theorem 7(3.4) and from $\downarrow_{B_{\mathcal{F}}}=\uparrow_{B^{\mathrm{T}}_{\mathcal{F}}}$ and $\uparrow_{B_{\mathcal{F}}}=\downarrow_{B^{\mathrm{T}}_{\mathcal{F}}}$ ($B^{\mathrm{T}}_{\mathcal{F}}$ denotes transpose of $B_{\mathcal{F}}$). Similarly, we can obtain $B_{\mathcal{F}}=A^{\mathrm{T}}_{\mathcal{F}}\triangleleft I$ and apply Theorem 7(3.3) to obtain the other two assertions. □

Theorem 13. ($\circ$) *Let $\langle X,Y,I\rangle$ be an* **L**-*context and let $\mathcal{F}\subseteq\mathcal{B}(X^{\uparrow},Y^{\downarrow},I)$. Then we have*

$$\mathrm{Ext}(X^{\uparrow},Y^{\downarrow},A_{\mathcal{F}})\subseteq\mathrm{Ext}(X^{\uparrow},Y^{\downarrow},I),$$
$$\mathrm{Int}(X^{\uparrow},Y^{\downarrow},B_{\mathcal{F}})\subseteq\mathrm{Int}(X^{\uparrow},Y^{\downarrow},I).$$

($\triangleleft$) *Let $\langle X,Y,I\rangle$ be an* **L**-*context and let $\mathcal{F}\subseteq\mathcal{B}(X^{\uparrow},Y^{\downarrow},I)$. Then we have*

$$\mathrm{Ext}(X^{\uparrow},Y^{\downarrow},A_{\mathcal{F}})\subseteq\mathrm{Ext}(X^{\cap},Y^{\cup},I),$$
$$\mathrm{Int}(X^{\cap},Y^{\cup},B_{\mathcal{F}})\subseteq\mathrm{Int}(X^{\cap},Y^{\cup},I).$$

($\triangleright$) *Let $\langle X,Y,I\rangle$ be an* **L**-*context and let $\mathcal{F}\subseteq\mathcal{B}(X^{\wedge},Y^{\vee},I)$. Then we have*

$$\mathrm{Ext}(X^{\wedge},Y^{\vee},A_{\mathcal{F}})\subseteq\mathrm{Ext}(X^{\wedge},Y^{\vee},I),$$
$$\mathrm{Int}(X^{\uparrow},Y^{\downarrow},B_{\mathcal{F}})\subseteq\mathrm{Int}(X^{\wedge},Y^{\vee},I).$$

Proof. The theorem can be proved similarly as Theorem 8 using Theorem 12. □

Remark 5. *Notice that the propositions in Theorem 12 and Theorem 13 do not assume that $I=A_{\mathcal{F}}*B_{\mathcal{F}}$. Indeed, only preposition is that columns of A are extents of I and rows of B are the corresponding intents. Thus, the two theorems can be applied even to approximate decompositions, i.e. solutions that do not assure equality $I=A_{\mathcal{F}}*B_{\mathcal{F}}$.*

3.2 Row and Column Spaces of Graded Matrices

In this section, we define the notions of row and column spaces for matrices over residuated lattices and establish their properties and connections to concept lattices.

Using the terminology known from Boolean matrices [28], we define the following notions as follows.

Definition 1. *$V \subseteq L^n$ is called an* i-subspace *if*

- *V is closed under $\otimes$-multiplication, i.e. for every $a \in L$ and $C \in V$ we have $a \otimes C \in V$ (here, $a \otimes C$ is defined by $(a \otimes C)(i) = a \otimes C(i)$ for $i = 1, \ldots, n$);*

- *V is closed under $\bigvee$-union, i.e. for $C_j \in V$ $(j \in J)$ we have $\bigvee_{j \in J} C_j \in V$ (here, $\bigvee_{j \in J} C_j$ is defined by $(\bigvee_{j \in J} C_j)(i) = \bigvee_{j \in J} C_j(i)$).*

$V \subseteq L^n$ is called a c-subspace *if*

- *V is closed under left $\rightarrow$-multiplication (or $\rightarrow$-shift), i.e. for every $a \in L$ and $C \in V$ we have $a \rightarrow C \in V$ (here, $a \rightarrow C$ is defined by $(a \rightarrow C)(i) = a \rightarrow C(i)$ for $i = 1, \ldots, n$);*

- *V is closed under $\bigwedge$-intersection, i.e. for $C_j \in V$ $(j \in J)$ we have $\bigwedge_{j \in J} C_j \in V$ (here, $\bigwedge_{j \in J} C_j$ is defined by $(\bigwedge_{j \in J} C_j)(i) = \bigwedge_{j \in J} C_j(i)$).*

If elements of V are regarded as fuzzy sets, the concepts of an i-subspace and a c-subspace coincide with the concept of a **L**-interior system and a **L**-closure system.

Remark 6. For $L = \{0, 1\}$ the concept of an i-subspace coincides with the concept of a subspace from the theory of Boolean matrices [28]. In fact, closedness under $\otimes$-multiplication is satisfied for free in the case of Boolean matrices. Note also that for Boolean matrices, V forms a c-subspace iff $\overline{V} = \{\overline{C} \mid C \in V\}$ forms an i-subspace (with $\neg C$ defined by $\neg C(i) = \neg C(i)$ where $\neg a = a \rightarrow 0$, i.e. $\neg 0 = 1$ and $\neg 1 = 0$), and vice versa. However, such a reducibility among the concepts of i-subspace and c-subspace is not available in general because in residuated lattices, the law of double negation (saying that $(a \rightarrow 0) \rightarrow 0 = a$) does not hold.

Definition 2. The *i-span* (*c-span*) of $V \subseteq L^n$ is the intersection of all i-subspaces (c-subspaces) of L^n that contain V, hence itself an i-subspace (c-subspace) of L^n.

The *row i-space* (*row c-space*) of matrix $I \in L^{n \times m}$ is the i-span (c-span) of the set of all rows of I (considered as vectors from L^n). The *column i-space* (*column c-space*) is defined analogously as the i-span (c-span) of the set of columns of I. The row i-space, row c-space, column i-space, and column c-space of matrix I is denoted by $R_{\mathrm{i}}(I)$, $R_{\mathrm{c}}(I)$, $C_{\mathrm{i}}(I)$, $C_{\mathrm{c}}(I)$.

A fundamental connection between the row and column spaces on one hand, and the concept lattices on the other hand, is described in the following theorem (I^{T} denotes the transpose of I).

Theorem 14. *For a matrix $I \in L^{n\times m}$, $X = \{1,\dots,n\}$, $Y = \{1,\dots,m\}$, we have*

$$R_{\mathrm{i}}(I) = \mathrm{Int}(X^{\cap}, Y^{\cup}, I) = \mathrm{Ext}(Y^{\wedge}, X^{\vee}, I^{\mathrm{T}}), \tag{3.5}$$

$$R_{\mathrm{c}}(I) = \mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I) = \mathrm{Ext}(Y^{\uparrow}, X^{\downarrow}, I^{\mathrm{T}}), \tag{3.6}$$

$$C_{\mathrm{i}}(I) = \mathrm{Ext}(X^{\wedge}, Y^{\vee}, I) = \mathrm{Int}(Y^{\cap}, X^{\cup}, I^{\mathrm{T}}), \tag{3.7}$$

$$C_{\mathrm{c}}(I) = \mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, I) = \mathrm{Int}(Y^{\uparrow}, X^{\downarrow}, I^{\mathrm{T}}). \tag{3.8}$$

Proof. (3.5): To establish $R_{\mathrm{i}}(I) = \mathrm{Int}(X^{\cap}, Y^{\cup}, I)$, notice that $\mathrm{Int}(X^{\cap}, Y^{\cup}, I)$ is just the set of all fixpoints of the fuzzy interior operator ${}^{\cup\cap}$ (see e.g. [10, 23]), i.e. a fuzzy interior system. To see that this fuzzy interior system is the least one that contains all rows of I, it is sufficient to observe that every intent $D \in \mathrm{Int}(X^{\cap}, Y^{\cup}, I)$ is a $\bigvee$-union of $\otimes$-multiplications of rows of I and that $\mathrm{Int}(X^{\cap}, Y^{\cup}, I)$ contains every row of I. To observe this fact, consider the corresponding formal concept $\langle C, D\rangle \in \mathcal{B}(X^{\cap}, Y^{\cup}, I)$. It follows from the description of suprema in $\mathcal{B}(X^{\cap}, Y^{\cup}, I)$ that

$$\begin{aligned}\langle C, D\rangle = \bigvee_{x\in X} \langle \{{}^{C(x)}\!/x\}^{\cap\cup}, \{{}^{C(x)}\!/x\}^{\cap}\rangle = \\ \langle (\bigvee_{x\in X} \{{}^{C(x)}\!/x\})^{\cap\cup}, \bigvee_{x\in X} \{{}^{C(x)}\!/x\}^{\cap}\rangle,\end{aligned}$$

(note that $\{{}^{a}\!/x\}$ denotes a singleton fuzzy set A defined by $A(u) = a$ for $u = x$ and $A(u) = 0$ for $u \neq x$) and hence

$$\begin{aligned}D &= \bigvee_{x\in X} \{{}^{C(x)}\!/x\}^{\cap} = \\ &= \bigvee_{x\in X} C(x) \otimes \{{}^{1}\!/x\}^{\cap}.\end{aligned}$$

In addition, $\langle \{{}^{1}\!/x\}^{\cap\cup}, \{{}^{1}\!/x\}^{\cap}\rangle$ is a particular formal concept from $\mathcal{B}(X^{\cap}, Y^{\cup}, I)$. It is now sufficient to realize that $\{{}^{1}\!/x\}^{\cap}$ is just the x-th row of I.

The second equality of (3.5) is immediate. (3.7) is a consequence of (3.5) when taking a transpose of I. Namely, in such case extents and intents switch their roles.

(3.6): Similarly, to establish $R_{\mathrm{c}}(I) = \mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I)$, notice that $\mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I)$ is just the set of all fixpoints of the fuzzy closure operator ${}^{\downarrow\uparrow}$ (see e.g. [2, 4]), i.e. a fuzzy closure system. To see that $\mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I)$ is the least fuzzy closure system which contains all rows of I, it is sufficient to observe that every intent $D \in \mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I)$ is an $\bigwedge$-intersection of $\rightarrow$-shifts of rows of I and that $\mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I)$ contains every row of I. To observe this fact, consider the corresponding formal concept $\langle C, D\rangle \in \mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$. Then it follows from the description of suprema in $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$ that

$$\langle C, D\rangle = \bigvee_{x\in X} \langle \{{}^{C(x)}\!/x\}^{\uparrow\downarrow}, \{{}^{C(x)}\!/x\}^{\uparrow}\rangle = \langle (\bigvee_{x\in X} \{{}^{C(x)}\!/x\})^{\uparrow\downarrow}, \bigwedge_{x\in X} \{{}^{C(x)}\!/x\}^{\uparrow}\rangle,$$

and hence

$$D = \bigwedge_{x\in X}\{^{C(x)}/x\}^{\uparrow} = \bigwedge_{x\in X} C(x) \to \{^1/x\}^{\uparrow}.$$

In addition, $\langle\{^1/x\}^{\uparrow\downarrow},\{^1/x\}^{\uparrow}\rangle$ is a particular formal concept from $\mathcal{B}(X^{\uparrow},Y^{\downarrow},I)$. It is now sufficient to realize that $\{^1/x\}^{\uparrow}$ is just the x-th row of I.

Again, (3.8) is a consequence of (3.6) when taking the transpose of I. □

Remark 7. From the point of view of concept lattices, as developed within formal concept analysis, the row space of a Boolean matrix I, i.e. $R_{\rm i}(I)$, is dually isomorphic as a lattice to the lattice of all intents of the ordinary concept lattice of the complement of I, i.e. to $\mathrm{Int}(X^{\uparrow},Y^{\downarrow},\neg I)$. Namely, according to Theorem 14, $R_{\rm i}(I) = \mathrm{Int}(X^{\cap},Y^{\cup},I)$ and it is well known that for $L = \{0,1\}$, $\mathrm{Int}(X^{\cap},Y^{\cup},I)$ is dually isomorphic to $\mathrm{Int}(X^{\uparrow},Y^{\downarrow},\neg I)$ with $D \mapsto \neg D$ being the dual isomorphism. Lattices $\mathrm{Int}(X^{\cap},Y^{\cup},I)$ have been studied by Markowsky, see e.g. [37].

Corollary 15. (1) *For Boolean matrices A and B, the row space of $A \circ B$ is a subset of the row space of B.*

(2) *For a Boolean matrix A, the row space of A has the same number of elements as the columns space of A.*

Proof. (1) is a particular case of (3.5) for $L = \{0,1\}$.

(2): By Theorem 14, $R_{\rm i}(A) = \mathrm{Int}(X^{\cap},Y^{\cup},A)$ and $C_{\rm i}(A) = \mathrm{Int}(Y^{\cap},X^{\cup},A^{\rm T})$. As is mentioned in Remark 7, system $\mathrm{Int}(X^{\cap},Y^{\cup},A)$ is dually isomorphic to $\mathrm{Int}(X^{\uparrow},Y^{\downarrow},\overline{A})$ and hence isomorphic to $\mathcal{B}(X^{\uparrow},Y^{\downarrow},\overline{A})$. Thus, $\mathrm{Int}(Y^{\cap},X^{\cup},A^{\rm T})$ is isomorphic to $\mathcal{B}(Y^{\uparrow},X^{\downarrow},\overline{A}^{\rm T})$. As is well-known from FCA [22], $\mathcal{B}(X^{\uparrow},Y^{\downarrow},\overline{A})$ is dually isomorphic to $\mathcal{B}(Y^{\uparrow},X^{\downarrow},\overline{A}^{\rm T})$, proving the claim. □

Remark 8. (1) From Theorem 14 we have $|R_{\rm c}(I)| = |C_{\rm c}(I)|$ for any $I \in L^{n\times m}$ since $C_{\rm c}(I) = \mathrm{Ext}(X^{\uparrow},Y^{\downarrow},I)$ and, as is well known, $\mathrm{Ext}(X^{\uparrow},Y^{\downarrow},I)$ is dually isomorphic to $R_{\rm c}(I) = \mathrm{Int}(X^{\uparrow},Y^{\downarrow},I)$.

(2) Contrary to Corollary 15 (2), $|R_{\rm i}(I)| = |C_{\rm i}(I)|$ does not hold for general L. As an example, consider L from Remark 4. For the matrix $I = \begin{pmatrix} a & b \end{pmatrix}$, we have $R_{\rm i}(I) = \{(a,b),(0,0)\}$ and $C_{\rm i}(I) = \{(0),(a),(b)\}$.

3.3 Closure Spaces Induced by $\langle^\wedge,^\vee\rangle$

It is known that $\langle^\wedge,^\vee\rangle$ forms an isotone **L**-Galois connection [23], $^{\wedge\vee}$ and $^{\vee\wedge}$ are **L**-interior and **L**-closure operators in X and Y, and $\mathrm{Ext}(X^{\wedge},Y^{\vee},I)$ and $\mathrm{Int}(X^{\wedge},Y^{\vee},I)$ are **L**-interior and **L**-closure systems in X and Y, respectively. For antitone **L**-Galois connection $\langle^{\uparrow},^{\downarrow}\rangle$ any **L**-closure system in Y is in the form of $\mathrm{Ext}(X^{\uparrow},Y^{\downarrow},I)$ (same for X). The situation for $\langle^\wedge,^\vee\rangle$ might seem completely dual to that of $\langle^{\uparrow},^{\downarrow}\rangle$ (which is the case when $L = \{0,1\}$). However, as the next example shows, it is not. Namely, there exist **L**-closure systems that are not of the form $\mathrm{Int}(X^{\wedge},Y^{\vee},I)$.

Example 2. Let $\mathbf{L}$ be the standard Gödel algebra, $U = \{u\}$, $\mathcal{S} = \{\{^{0.5}/u\}, \{^{1}/u\}\}$. Therefore, $L = [0,1]$ and $a \to b = 1$ if $a \le b$ and $a \to b = b$ of $a > b$. Clearly, $\mathcal{S}$ is closed under intersections and $\to$-shifts, hence it is an $\mathbf{L}$-closure system. However, $\mathcal{S}$ is not of the form $\mathcal{S} = \mathrm{Int}(X^{\wedge}, Y^{\vee}, I)$. (This claim is justified at the end of this section.)

Therefore, $\mathbf{L}$-closure systems that are of the form $\mathrm{Int}(X^{\wedge}, Y^{\vee}, I)$ are just particular $\mathbf{L}$-closure systems. Below, we provide their characterization. For a system $\mathcal{S} \subseteq L^U$, put

$$[\mathcal{S}]_{\wedge} = \{\bigwedge \mathcal{T} \mid \mathcal{T} \subseteq \mathcal{S}\},$$
$$[\mathcal{S}]_{\to} = \{a \to A \mid a \in L,\ A \in \mathcal{S}\},$$
$$[\mathcal{S}]^{\to} = \{A \to a \mid a \in L,\ A \in \mathcal{S}\}.$$

Note that $A \to a$ is defined by $(A \to a)(u) = A(u) \to a$ and call $A \to a$ the *right $\to$-multiple* of A by a. Therefore, $[\mathcal{S}]_{\wedge}$ is the system of all intersections of fuzzy sets from $\mathcal{S}$, $[\mathcal{S}]_{\to}$ is the system of all left $\to$-multiplications of fuzzy sets from $\mathcal{S}$, and $[\mathcal{S}]^{\to}$ is the system of all right $\to$-multiplications of fuzzy sets from $\mathcal{S}$. It is known that for any $\mathcal{S} \subseteq L^U$, $[[\mathcal{S}]_{\to}]_{\wedge}$ is the least, w.r.t. inclusion, $\mathbf{L}$-closure system containing $\mathcal{S}$. $[[\mathcal{S}]_{\to}]_{\wedge}$ is called the $\mathbf{L}$-closure system generated by $\mathcal{S}$, or the *c-span* of $\mathcal{S}$.

Note that in fuzzy logic, $b \to 0$ is called the negation of the truth degree b. Correspondingly, the fuzzy set $A \to 0$ is called the complement of A. Clearly, in the above terms, $A \to 0$ is the right multiple of A by 0. From this point of view, the right multiples $A \to a$ generalize the concept of a complement of a fuzzy set. $A \to a$ could naturally be called the *a-complement* of A.

In the classical case ($L = \{0,1\}$), every A is a complement of some B; namely, of $B = A \to 0$. This is no longer true for the general setting of residuated lattices (not even for $a = 0$). We only have:

Lemma 16. *A is an a-complement of some fuzzy set if and only if $A = (A \to a) \to a$.*

Proof. Easy, follows from $((b \to a) \to a) \to a = b \to a$. □

This lemma is, in a sense, the key observation in characterizing the $\mathbf{L}$-closure systems $\mathrm{Int}(X^{\wedge}, Y^{\vee}, I)$. We are going to show that $\mathrm{Int}(X^{\wedge}, Y^{\vee}, I)$ are just the $\mathbf{L}$-closure systems that are generated by a-complements of some collection $\mathcal{T}$ of fuzzy sets. Such systems are conveniently characterized by the following theorem.

Theorem 17. *For any $\mathcal{T} \subseteq L^U$, $[[\mathcal{T}]^{\to}]_{\wedge}$ is an $\mathbf{L}$-closure system. It is the least, w.r.t. inclusion, $\mathbf{L}$-closure system containing all a-complements (i.e., right $\to$-multiplications) of fuzzy sets from $\mathcal{T}$.*

Proof. Clearly, $[[\mathcal{T}]^{\to}]_{\wedge}$ contains all a-complements of fuzzy sets from $\mathcal{T}$.

The closedness of $[[\mathcal{T}]^{\to}]_{\wedge}$ under $\wedge$ is obvious form definition of $[\cdot]_{\wedge}$. Now, we show that $[[\mathcal{T}]^{\to}]_{\wedge}$ is closed under left $\to$-multiplications. Let $A \in [[\mathcal{T}]^{\to}]_{\wedge}$.

Then A is in the form

$$A = \bigwedge_{T_i \in \mathcal{T}} T_i \to a_i.$$

The closedness under $\to$ follow from $a \to (b \to c) = b \to (a \to c)$ and $a \to \bigwedge b_i = \bigwedge(a \to b_i)$. The rest is by standard arguments. □

The following theorem provides our characterization.

Theorem 18. *For any $\mathcal{S} \subseteq L^U$, $\mathcal{S} = \mathrm{Int}(X^\wedge, Y^\vee, I)$ for some I if and only if $\mathcal{S} = [[\mathcal{T}]^\to]_\wedge$ for some $\mathcal{T} \subseteq L^U$, i.e. $\mathcal{S}$ is an* **L***-closure system generated by a system of all a-complements of fuzzy sets from $\mathcal{T}$.*

Proof. "⇒": Let $\mathcal{T}$ be set of rows I_x of I. Since

$$C^\wedge(y) = \bigwedge_{x \in X} I(x,y) \to C(x)$$

we have

$$C^\wedge = \bigwedge_{x \in X} I_x \to c_x$$

and

$$\mathrm{Int}(X^\wedge, Y^\vee, I) = [[\mathcal{T}]^\to]_\wedge.$$

"⇐": Let $X = \mathcal{T}$, $Y = U$, $I(A,u) = A(u)$ for $A \in \mathcal{S}$, $u \in U$. One can show that $\mathcal{S} = \mathrm{Int}(X^\wedge, Y^\vee, I)$.

□

Definition 3. *We call the systems $\mathcal{S}$ satisfying the condition of Theorem 18* c-closure systems *("c" for "complement").*

Example 2 (continued). Suppose, by contradiction, that $\mathcal{S} = \mathrm{Int}(X^\wedge, Y^\vee, I)$. Then $U = X$ and by Theorem 18, $\mathcal{S}$ is a system generated by a system of all a-complements of fuzzy sets from some $\mathcal{T}$. According to Theorem 17, $[[\mathcal{T}]^\to]_\wedge = \{\{^{0.5}/u\}, \{^1/u\}\}$. Then, $\{^{0.5}/u\}$ needs to be an intersection of other fuzzy sets from $[\mathcal{T}]^\to$ or $\{^{0.5}/u\} \in [\mathcal{T}]^\to$. Clearly, $\{^{0.5}/u\} \in [\mathcal{T}]^\to$ must be the case. Therefore, $\{^{0.5}/u\} = \{^a/u\} \to b$ for some b. Clearly, $a > b = 0.5$ must be the case. But then, we also have $\{^a/u\} \to 0.4 = \{^{0.4}/u\} \in [\mathcal{T}]^\to$, a contradiction to $[\mathcal{T}]^\to \subseteq [[\mathcal{T}]^\to]_\wedge = \{\{^{0.5}/u\}, \{^1/u\}\}$.

3.4 Concept Lattices of Isotone vs. Antitone Galois Connections

The classical notion of a complement $\neg I$ of a fuzzy relation may be looked at the following way. Each attribute $y \in Y$ in the data table representing I is replaced by its complement. That it, each fuzzy set $I_y \in L^X$, representing attribute y, defined by $I_y(x) = I(x,y)$ is replaced in the table by its complement $\neg I_y$ defined by

$$(\neg I_y)(x) = \neg(I_y(x)), \qquad \text{i.e. } (\neg I_y)(x) = I_y(x) \to 0.$$

The complement (2.1) is in fact the residuum of a w.r.t. 0. However, one may also consider a residuum of $a \in L$ w.r.t. to an arbitrary element $b \in L$, i.e. one may consider

$$\neg_b a = a \to b, \tag{3.9}$$

of which $\neg a$ is a particular case because $\neg a = \neg_0 a$. In addition to $\neg I_y$, the "negation relative to 0" one may therefore also consider $\neg_b I_y$, the "negation relative to b", for other degrees b, defined by

$$(\neg_b I_y)(x) = \neg_b(I_y(x)), \qquad \text{i.e. } (\neg_b I_y)(x) = I_y(x) \to b.$$

For every original attribute y, I_y may therefore be replaced not just by the complement $\neg_0 I_y$ w.r.t. 0 but by several complements $\neg_b I_y$ w.r.t. $b \in K$ with $K \subseteq L$ being a set of selected values, bringing us the following definition.

Definition 4. *For a set $K \subseteq L$, the K-complement of a fuzzy relation I between X and Y is a fuzzy relation $\neg_K I$ between X and $Y \times K$ defined by*

$$(\neg_K I)(x, \langle y, b\rangle) = \neg_b I(x, y) \tag{3.10}$$

for every $x \in X$, $y \in Y$, and $b \in K$.

Remark 9. (a) Going from I to $\neg_K I$ may be seen as replacing every attribute $y \in Y$, represented by I_y in I, by a collection of new attributes $\langle y, b\rangle \in Y \times K$, represented by $\neg_b I_y$ in $\neg_K I$ for $b \in K$.

(b) Clearly, for $K = \{0\}$, $\neg_K I$ may be identified with $\neg I$, because $Y \times \{0\}$ may be identified with Y and $\neg_K I(x, \langle y, \{0\}\rangle) = \neg I(x, y)$.

(c) In what follows, we use $\neg_K I$ for $K = L - \{1\}$. Observe that for $L = \{0, 1\}$ (the ordinary case), $\neg_{L-\{1\}} I = \neg_{\{0\}} I$, i.e. in view of (b) of this Remark, $\neg_{L-\{1\}} I$ may be identified with the classical complement $\neg I$ of I.

In view of Remark 9(c), there are two ways to generalize the notion of a complement of an ordinary relation I between X and Y to a fuzzy setting:

(i) First, a complement of I may be defined as a fuzzy relation between X and Y by (2.30).

(ii) Second, a complement of I may be defined as a fuzzy relation between X and $Y \times K$ by (3.10) with $K = L - \{1\}$.

While $\mathcal{B}(X^{\cap_I}, Y^{\cup_I}, I)$ and $\mathcal{B}(X^{\uparrow_{\neg I}}, Y^{\downarrow_{\neg I}}, \neg I)$ are isomorphic as lattices holds true in the ordinary setting (see (2.47) and (2.48)), it fails to hold in a fuzzy setting for (i), they hold in a fuzzy setting with the complement understood according to (ii):

Theorem 19. *For a fuzzy relation I between X and Y, let $\neg I$ denote $\neg_{L-\{1\}} I$. Then $\mathcal{B}(X^{\cap_I}, Y^{\cup_I}, I)$ and $\mathcal{B}(X^{\uparrow_{\neg I}}, Y \times (L-\{1\})^{\downarrow_{\neg I}}, \neg I)$ are isomorphic as lattices, with the mappings $\langle A, B\rangle \mapsto \langle A, D\rangle$, where*

$$D(y, b) = \neg_b B(y) \tag{3.11}$$

for $y \in Y$, $b \in L - \{1\}$, *and* $\langle A, D\rangle \mapsto \langle A, B\rangle$, *where*

$$B(y) = \bigwedge_{b \in L - \{1\}} \neg_b D(y, b) \tag{3.12}$$

for $y \in Y$, *being the isomorphism and its inverse. Hence, in particular,*

$$\mathrm{Ext}(X^{\cap_I}, Y^{\cup_I}, I) = \mathrm{Ext}(X^{\uparrow_{\neg I}}, Y \times (L - \{1\})^{\downarrow_{\neg I}}, \neg I). \tag{3.13}$$

Proof. We first prove (3.13). Since ${}^{\uparrow_{\neg I}\downarrow_{\neg I}}$ is an **L**-closure operator in X [5], it follows that $\mathrm{Ext}(X^{\uparrow_{\neg I}}, Y \times (L - \{1\})^{\downarrow_{\neg I}}, \neg I)$ is an **L**-closure system in X, i.e. it is closed under arbitrary $\bigwedge$-intersections and left $\rightarrow$-multiplications. This means that for all $A_j \in \mathrm{Ext}(X^{\uparrow_{\neg I}}, Y \times (L - \{1\})^{\downarrow_{\neg I}}, \neg I)$, $j \in J$, we have $\bigwedge_{j \in J} A_j \in \mathrm{Ext}(X^{\uparrow_{\neg I}}, Y \times (L - \{1\})^{\downarrow_{\neg I}}, \neg I)$ and for each $a \in L$ and $A \in \mathrm{Ext}(X^{\uparrow_{\neg I}}, Y \times (L - \{1\})^{\downarrow_{\neg I}}, \neg I)$ we have $a \rightarrow A \in \mathrm{Ext}(X^{\uparrow_{\neg I}}, Y \times (L - \{1\})^{\downarrow_{\neg I}}, \neg I)$ with $a \rightarrow A \in L^X$ defined by $(a \rightarrow A)(x) = a \rightarrow A(x)$ for each $x \in X$. Moreover, [14, Theorem 2 (10)] implies that $\mathrm{Ext}(X^{\uparrow_{\neg I}}, Y \times (L - \{1\})^{\downarrow_{\neg I}}, \neg I)$ is the least **L**-closure system in X containing every column of $\neg I$, i.e. every $\neg_b I_y$ for each $b \in L - \{1\}$.

To prove (3.13), it is therefore sufficient to show that $\mathrm{Ext}(X^{\cap_I}, Y^{\cup_I}, I)$ is the least **L**-closure system in X containing every column of $\neg I$. This assertion follows from the fact that $\mathrm{Ext}(X^{\cap_I}, Y^{\cup_I}, I)$ is always an **L**-closure system and from the following claim.

Claim. $\mathrm{Ext}(X^{\cap_I}, Y^{\cup_I}, I)$ consists of all possible $\bigwedge$-intersections of fuzzy sets $\neg_b I_y$ ($y \in Y$, $b \in L - \{1\}$).

Namely, if $\mathcal{S}$ is an **L**-closure system that contains every column of $\neg I$, it contains all intersections of the columns of $\neg I$ and, due to Claim, it contains $\mathrm{Ext}(X^{\cap_I}, Y^{\cup_I}, I)$. Therefore, to prove (3.13), it remains to prove Claim.

Proof of Claim. Since ${}^{\cap_I}$ and ${}^{\cup_I}$ form an isotone Galois connection, we have

$$\mathrm{Ext}(X^{\cap_I}, Y^{\cup_I}, I) = \{B^{\cup_I} \mid B \in L^Y\}. \tag{3.14}$$

On one hand, every $B^{\cup_I}$ is an intersection of fuzzy sets of the form $\neg_b I_y$ because

$$B^{\cup_I}(x) = \bigwedge_{y \in Y} (I(x, y) \rightarrow B(y)) = \bigwedge_{y \in Y} \neg_{B(y)} I_y. \tag{3.15}$$

On the other hand, consider an arbitrary intersection A of $\neg_b I_y$s, i.e. $A = \bigwedge_{\langle y,b\rangle \in P} \neg_b I_y$ for some $P \subseteq Y \times (L - \{1\})$. Define $B(y) = \bigwedge_{\langle y,b\rangle \in P} b$. Then

$$\begin{aligned} A(x) &= \bigwedge_{y \in Y} \bigwedge_{\langle y,b\rangle \in P} (I(x, y) \rightarrow b) = \bigwedge_{y \in Y} I(x, y) \rightarrow \bigwedge_{\langle y,b\rangle \in P} b = \\ &= \bigwedge_{y \in Y} I(x, y) \rightarrow B(y) = B^{\cup_I}(x), \end{aligned}$$

hence $A \in \mathrm{Ext}(X^{\cap_I}, Y^{\cup_I}, I)$, finishing the proof of Claim and hence also the proof of (3.13).

Now, since $\mathrm{Ext}(X^{\cap_I}, Y^{\cup_I}, I)$ and $\mathrm{Ext}(X^{\uparrow_{\neg I}}, Y\times(L-\{1\})^{\downarrow_{\neg I}}, \neg I)$ are isomorphic as lattices to $\mathcal{B}(X^{\cap_I}, Y^{\cup_I}, I)$ and $\mathcal{B}(X^{\uparrow_{\neg I}}, Y\times(L-\{1\})^{\downarrow_{\neg I}}, \neg I)$, respectively, it follows that $\mathcal{B}(X^{\cap_I}, Y^{\cup_I}, I)$ and $\mathcal{B}(X^{\uparrow_{\neg I}}, Y\times(L-\{1\})^{\downarrow_{\neg I}}, \neg I)$ are isomorphic as lattices.

Take any $\langle A, B\rangle \in \mathcal{B}(X^{\cap_I}, Y^{\cup_I}, I)$ and the corresponding $\langle A, D\rangle \in \mathcal{B}(X^{\uparrow_{\neg I}}, Y\times (L-\{1\})^{\downarrow_{\neg I}}, \neg I)$. Then

$$\begin{aligned} D(y,b) &= A^{\uparrow_{\neg I}}(y,b) = \bigwedge_{x\in X} A(x) \to \neg I(x,\langle y,b\rangle) = \\ &= \bigwedge_{x\in X} A(x) \to (I(x,y)\to b) = \bigwedge_{x\in X} ((A(x)\otimes I(x,y))\to b) = \\ &= [\bigvee_{x\in X}(A(x)\otimes I(x,y)] \to b = A^{\cap_I}(y)\to b = B(y)\to b = \neg_b B(y), \end{aligned}$$

verifying (3.11). To check (3.12), consider any $A\in L^X$ and the corresponding $B = A^{\cap_I}$ and $D = A^{\uparrow_{\neg I}}$. Observe first that

$$B(y) \le \neg_b D(y,b) \tag{3.16}$$

for each $b\in L-\{1\}$. Indeed, taking into account $a \le (a\to b)\to b = \neg_b\neg_b a$ for any $a\in L$ and (3.11), we have $B(y)\le \neg_b\neg_b B(y) = \neg_b D(y,b)$. This verifies the "$\le$" part of (3.12). Let now $c = B(y)$. If $c < 1$, then c is one of the degrees from $L-\{1\}$ over which the infimum in (3.12) is taken and since $\neg_c D(y,c) = \neg_c\neg_c B(y) = \neg_c\neg_c c = c = B(y)$ in this case, the infimum in (3.12) is indeed equal to $B(y)$. If $c = 1$ then due to (3.16), $\neg_b D(y,b) = 1$ for each $b\in L-\{1\}$, hence also the infimum in (3.12) is equal to 1, i.e. equal to $B(y)$. □

Remark 10. (a) Observe, that $\neg I$ is possible to express as composition $I \triangleleft B$ of **L**-relations I and $B\in L^{Y, Y\times(L-\{1\})}$ with

$$B(y_1,\langle y_2,a\rangle) = \begin{cases} a & \text{if } y_1 = y_2 \\ 1 & \text{otherwise.}\end{cases}$$

Thus we have $A^{\uparrow_{\neg I}} = A^{\uparrow_{I\triangleleft B}} = A^{\cap_I\uparrow_B}$.

(b) One easily checks that since $\neg_1 I_y(x) = 1$ for each $x\in X$, one may replace $L-\{1\}$ by L in Theorem 19.

(c) A converse statement to Theorem 19 does not hold. That is, there is no notion of a complement $\sim$ such that for any fuzzy relation I, $\mathrm{Ext}(X^{\uparrow_I}, Y^{\downarrow_I}, I)$ is equal to $\mathrm{Ext}(X^{\cap_{\sim I}}, Z^{\cup_{\sim I}}, \sim I)$ for any suitable Z. This is because for some fuzzy relations I, $\mathrm{Ext}(X^{\uparrow_I}, Y^{\downarrow_I}, I)$ is not a system of extents of any fuzzy relation J w.r.t. the operators $^{\cap_J}$ and $^{\cup_J}$ as it is shown in Section 3.3.

(d) In view of Remark 9 (c), Theorem 19 generalizes (2.47) and (2.48) and its proof does not use the law of double negation.

(e) Theorem 19 uses a new notion of complement. Unlike the usual notion of complement, the new one does not result by a straightforward replacement of truth functions of classical logic by the truth functions of fuzzy logic. It is an interesting question to explore to what extent this notion may be used in other areas of fuzzy set theory to replace the usual notion of complement in such a way that the resulting concepts behave as in the classical, bivalent case even without the law of double negation.

(f) One of the main result in [33] is description of scaling of a fuzzy relation $I \in L^{X \times Y}$ to crisp relation $I_c \subseteq (X \times L) \times (Y \times L)$ s.t. $\mathcal{B}(X^{\cap_I}, Y^{\cup_I}, I)$ and $\mathcal{B}(X \times L^{\uparrow_{I_c}}, Y \times L^{\downarrow_{I_c}}, I_c)$ are isomorphic as lattices. The result can be considered to be a consequence of Theorem 19; using the new notion of complement can simplify several proofs in that paper.

3.5 Summary and Future Research

This chapter provided a study of relationship between concept-forming operators induced by fuzzy relations in (de)composition $I = A * B$; a correspondence with notions from Boolean matrix theory, characterization of closure spaces induced by isotone concept-forming operators, and a relationship of isotone and antitone **L**-Galois connections via a new notion of a complement.

Our future research includes:

- Theorem 10 says that reduction of a concept lattice can be obtained by using relational compositions. We want to study **L**-relations which are reasonable to be used for such a reduction. Some initial results on this are given in Section 4.4.
- Possible applications of the new notion of a complement in wider scope. For example fuzzification of attribute dependency formulas and failure dependency formulas [20] seem to be appropriate goal since in their crisp setting law of double negation is frequently used.
- Several results in this section are known to hold true in the general framework (mentioned in Remark 1); it is our plan to further develop the framework.
- The second main output of FCA are attribute implications, our future research includes a study of a relationship between the attribute implications and the (de)compositions of **L**-relations.
- Study of matrices over more general structures than residuated lattices.

Chapter 4

Morphisms of Structures Associated to L-relations

In this chapter we define notions of i-morphisms, c-morphisms, and a-morphisms and show their basic properties.

Definition 5. *A mapping $h : V \to W$ from an i-subspace $V \subseteq L^p$ into an i-subspace $W \subseteq L^q$ is called an* i-morphism *if it is a $\otimes$- and $\bigvee$-morphism, i.e. if*

- $h(a \otimes C) = a \otimes h(C)$ *for each $a \in L$ and $C \in V$;*
- $h(\bigvee_{k \in K} C_k) = \bigvee_{k \in K} h(C_k)$ *for every collection of $C_k \in V$ $(k \in K)$.*

An i-morphism $h : V \to W$ is called an extendable i-morphism *if h can be extended to an i-morphism of L^p to L^q, i.e. if there exists an i-morphism $h' : L^p \to L^q$ such that for every $C \in V$ we have $h'(C) = h(C)$;*

A mapping $h : V \to W$ from a c-subspace $V \subseteq L^p$ into a c-subspace $W \subseteq L^q$ is called a c-morphism *if it is a $\to$- and $\bigwedge$-morphism, i.e. if*

- $h(a \to C) = a \to h(C)$ *for each $a \in L$ and $C \in V$;*
- $h(\bigwedge_{k \in K} C_k) = \bigwedge_{k \in K} h(C_k)$ *for every collection of $C_k \in V$ $(k \in K)$;*
- *if C is an a-complement then $h(C)$ is an a-complement.*

A mapping $h : V \to W$ from an i-subspace $V \subseteq L^p$ into an c-subspace $W \subseteq L^q$ is called an a-morphism *if*

- $h(a \otimes C) = a \to h(C)$ *for each $a \in L$ and $C \in V$;*
- $h(\bigvee_{k \in K} C_k) = \bigwedge_{k \in K} h(C_k)$ *for every collection of $C_k \in V$.*

The notions of extendable c-morphism, extendable a-morphism *are defined similarly as in the case of i-morphisms.*

In what follows we consider only extendable morphisms.

4.1 Basic Properties of Morphisms

The following two lemmas show that i-morphisms, c-morphisms, and a-morphisms are in a correspondence with **L**-relations.

Lemma 20. *For $V \subseteq L^X$, (a) if $h : V \to L^Y$ is an i-morphism then there exists an **L**-relation $A \in L^{X\times Y}$ such that $h(C) = C \circ A$ for every $C \in L^p$.*

*(b) if $h : V \to L^Y$ is an c-morphism then there exists an **L**-relation $A \in L^{X\times Y}$ such that $h(C) = C \triangleright A$ for every $C \in L^Y$.*

*(c) if $h : V \to L^Y$ is an a-morphism then there exists an **L**-relation $A \in L^{X\times Y}$ such that $h(C) = C \triangleleft A$ for every $C \in L^X$.*

Proof. (a) Since h is extendable, we may safely assume that $h : L^X \to L^Y$, i.e. that h is defined for every $C \in L^X$. Let $A \in L^{X\times F}$ be defined by

$$A(x, y) = \bigwedge_{C\in L^X} (C(x) \to (h(C))(y)).$$

That is, $A(x, \_) = \bigwedge_{C\in L^X}(C(i) \to h(C))$, i.e. the row $A_{x\_}$ contains a vector of degrees that can be interpreted as the intersection of images of those vectors C from V for which the corresponding fuzzy set contains x.

To establish the equation $(h(C))(y) = (C \circ A)(y)$, we first show

$$(h(E_k))(y) = (E_k \circ A)(y) \tag{4.1}$$

for every $k \in X$, where E_k is defined by

$$E_k(x) = \begin{cases} 0 & \text{for } x \neq k, \\ 1 & \text{for } x = k, \end{cases}$$

for every $x \in X$.

Notice that for any $C \in V$, as $C(k) \otimes E_k \leq C$, we have $C(k) \otimes h(E_k) = h(C(k) \otimes E_k) \leq h(C)$, whence $E_k(k) \to h(E_k) \leq C(k) \to h(C)$.

Using this inequality, we get

$$\begin{aligned}(E_x \circ A)(y) &= \bigvee_{x\in X} [E_k(x) \otimes (\bigwedge_{C\in L^X} (C(x) \to (h(C))(y)))] = \\ &= \bigwedge_{C\in L^X} (C(k) \to (h(C))(y)) = \bigwedge_{k\in X} (E_k(k) \to (h(E_k))(y)) = \\ &= (h(E_k))(j).\end{aligned}$$

Using (4.2), we now get

$$\begin{aligned}(h(C))(y) &= (h(\bigvee_{x\in X} (C(x) \otimes E_x)))(j) = \bigvee_{x\in X} (C(x) \otimes h(E_x)(y)) \leq \\ &\leq \bigvee_{x\in X} (C(x) \otimes (E_x \circ A)(y)) = (\bigvee_{x\in X} (C(x) \otimes E_x) \circ A)(y)) = (C \circ A)(y),\end{aligned}$$

finishing the proof of (a).

(b) Let $A \in L^{X\times Y}$ be defined by

$$A_(x,y) = \bigwedge_{C\in V}((h(C))(y) \to C(x)).$$

That is, $A(x,\_) = \bigwedge_{C\in V}(h(C) \to C(x))$, i.e. the row $A_{x\_}$ contains a vector of degrees that can be interpreted as the intersection of images of those vectors C from V for which the corresponding fuzzy set contains i.

We now check $h(C) = C \triangleright A$ for every $C \in L^X$. First,

$$\begin{aligned}(C \triangleright A)(y) &= \bigwedge_{x\in X}[A(x,y) \to C(x)] = \\ &= \bigwedge_{x\in X}[(\bigwedge_{C'\in V}(h(C'))(y) \to C'(x))) \to C(x)] \geq (h(C))(y).\end{aligned}$$

Second, to establish $(h(C))(y) \geq (C \triangleright A)(y)$, we first show

$$(h(\bigcap_{a\in L} E_{k,a}))(j) \geq \bigcap_{a\in L}(E_{k,a} \triangleright A)(j) \tag{4.2}$$

for every $k \in X$, where $E_{k,a}$ is a-complement of E_k, i.e.

$$E_{k,a}(x) = E_k \to a = \begin{cases} 1 & \text{for } x \neq k, \\ a & \text{for } x = k, \end{cases}$$

for every $x \in X$.

Indeed,

$$\begin{aligned}(E_{k,a} \triangleright A)(j) &= \bigwedge_{i=1}^{p}[(\bigwedge_{C\in V}((h(C))(j) \to C(i))) \to E_{k,a}(i)] \leq \\ &\leq ((h(E_{k,a}))(j) \to E_{k,a}(k)) \to E_{k,a}(k) = \\ &= ((h(E_{k,a}))(j) \to a) \to a = \\ &= (h(E_{k,a}))(j).\end{aligned}$$

Using (4.2), we now get

$$\begin{aligned}h(C) = h(\bigwedge_i(E_k \to c(i))) &= \bigwedge_i h(E_k \to c(i)) \geq \\ \geq \bigwedge_i((E_k \to c(i)) \triangleright A) &= \bigwedge_i(E_k \to c(i)) \triangleright A = C \triangleright A,\end{aligned}$$

finishing the proof of (b).

(c) Since h is extendable, we may safely assume that $h : L^X \to L^Y$, i.e. that h is defined for every $C \in L^X$. Let $A \in L^{X\times F}$ be defined by

$$A(x,y) = \bigwedge_{C\in L^X}(C(x) \to (h(C))(y)).$$

That is, $A(x,\_) = \bigwedge_{C\in L^X}(C(i) \to h(C))$, i.e. the row $A_{x\_}$ contains a vector of degrees that can be interpreted as the intersection of images of those vectors C

from V for which the corresponding fuzzy set contains x.

Let $A(x,y) = \bigvee_{C \in L^X} C(x) \circ h(C)(y)$. First, we prove that $h(E_k)(y) = (E_k \triangleleft A)(y)$ for each $y \in Y$, where $E_k(x)$ is defined as in proof of part (a).

$\geq$:

$$\begin{aligned}(E_k \triangleleft A)(x) &= \bigwedge_{x \in X} E_k(x) \to \bigvee_{C \in L^X} (C(x) \circ h(C)(y)) \\ &\geq \bigwedge_{x \in X} E_k(x) \to (E_k(x) \circ h(E_k)(y)) \geq h(E_k)(y).\end{aligned}$$

$\leq$:

$$\begin{aligned}(E_k \triangleleft A)(y) &= \bigwedge_{x \in X} E_k(x) \to \bigvee_{C} (C(x) \circ h(C)(y)) = (\bigvee_{C} C(k) \circ h(C)(y)) \\ &= \bigvee_{C} C(k) \circ h(\bigvee_{i} (C(x) \otimes E_x))(y) = \bigvee_{C} C(k) \circ \bigwedge_{i} (C(x) \to h(E_x)(j)) \\ &\leq \bigvee_{C} C(k) \circ (C(k) \to h(E_k)(y)) \leq h(E_k)(y).\end{aligned}$$

Finally, we show that $h(C) = (C \triangleleft A)$ for each $C \in L^X$:

$$\begin{aligned}h(C) &= h(\bigvee_{x \in X} C(x) \otimes E_k) = \bigwedge_{x \in X} C(x) \to h(E_k) = \bigwedge_{x \in X} C(x) \to (E_k \triangleleft A) \\ &= C \triangleleft (E_k \triangleleft A) = (C \circ E_k) \triangleleft A) = C \triangleleft A,\end{aligned}$$

finishing the proof of (c). ☐

Lemma 21. *Let $A \in L^{X \times Y}$,*

(a) the mapping $h_A : L^X \to L^Y$ defined by $h_A(C) = C \circ A \quad (= C^{\cap_A})$ is an i-morphism.

(b) the mapping $h_A : L^X \to L^Y$ defined by $h_A(C) = C \triangleright A \quad (= C^{\wedge_A})$ is a c-morphism.

(c) the mapping $h_A : L^X \to L^Y$ defined by $h_A(C) = C \triangleleft A \quad (= C^{\uparrow_A})$ is an a-morphism.

Proof. (a) – (c) follow from properties of residuated lattices. ☐

Remark 11. (1) As a result of Lemma 21 and Lemma 20, extendable i-morphisms may be represented by **L**-relations by means of $\circ$-composition, extendable c-morphisms may be represented by **L**-relations by means of $\triangleright$-composition and extendable a-morphisms may be represented by means of $\triangleleft$-composition.

(2) In the crisp case, every i-morphism is extendable. Namely, due to [28, Lemma 1.3.2], for every i-morphism $h : V \to \{0,1\}^q$ there exists a Boolean matrix $A \in \{0,1\}^{p \times q}$ such that $h(C) = C \circ A$ for every $C \in V$. Clearly, $h' : \{0,1\}^p \to \{0,1\}^q$ defined by $h'(C) = C \circ A$ for any $C \in \{0,1\}^p$ is the required extension of h

which is an i-morphism. Analogously every c-morphism and every a-morphism is extendable.

(3) For general residuated lattices, however, there exist i-morphisms that are not extendable. Consider any finite chain L with $a < b$ being two elements of L. Let $\otimes$ be defined as in Remark 8 (2). For $p = q = 1$, put $V = \{(0), (a)\}$, $W = \{(0), (b)\}$. Clearly, both V and W are i-subspaces for which $h((0)) = (0)$ and $h((a)) = (b)$ defines an i-morphism h. If h was extendable, there would exist a matrix $A = (c)$ for which $h(C) = C \circ A$ (Lemma 20). In particular, this would mean $(b) = h((a)) = (a) \circ (c)$, i.e. $b = a \otimes c$ which is impossible because $b > a$. Therefore, h is not extendable.

Lemmas 20 and 21 say that i-morphisms, a-morphisms, and c-morphisms are in a correspondence with **L**-relations. From this correspondence we can easily derive some properties of the morphisms like the one in the following theorem.

Theorem 22. *(a) Let $f : L^X \to L^F, g : L^F \to L^Y$ be an i-morphisms, then $g \circ f : L^X \to L^Y$ is an i-morphism.*

(b) Let $f : L^X \to L^F, g : L^F \to L^Y$ be c-morphisms, then $g \circ f : L^X \to L^Y$ is an c-morphism.

(c) Let $f : L^X \to L^F$ be an i-morphism and $g : L^F \to L^Y$ be an a-morphism, then $g \circ f : L^X \to L^Y$ is an a-morphism.

(d) Let $f : L^X \to L^F$ be an a-morphism and $g : L^F \to L^Y$ be a c-morphism, then $g \circ f : L^X \to L^Y$ is an a-morphism.

Proof. Directly from Lemma 20, Lemma 21 and Theorem 7. □

Notice, that different **L**-relations can define the same morphism. In other words, the **L**-relation A_h that characterizes an i-morphism (a-morphism, c-morphism) h is not generally unique. As an example, consider the same residuated lattice as in Remark 4, i.e. a finite chain containing $a < b$ with $\otimes$ defined as follows:

$$x \otimes y = \begin{cases} x \wedge y & \text{if } x = 1 \text{ or } y = 1, \\ 0 & \text{otherwise,} \end{cases} \qquad x \to y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{if } x = 1, \\ b & \text{otherwise,} \end{cases}$$

for each $x, y \in L$, and **L**-interior system $\{(a), (0)\}$ over an one-element universe. It is easy to observe that i-relations given by **L**-relations $(0), (a), (b)$ turn both **L**-sets of the **L**-interior system to (0). The following theorem explains structure of all such **L**-relations.

Theorem 23. *(a) Let h be an i-morphism $V \to W$ and let $\mathbf{A}_h$ be set of all* **L**-*relations, s.t. $C^{\cap_{A_h}} = h(C)$ for each $C \in U$. Then $\mathbf{A}_h$ is closed under $\vee$-union.*

(b) Let h be an c-morphism $V \to W$ and let $\mathbf{A}_h$ be set of all **L**-*relations, s.t. $C^{\wedge_{A_h}} = h(C)$ for each $C \in U$. Then $\mathbf{A}_h$ is closed under $\wedge$-intersection.*

(c) Let h be an a-morphism $V \to W$ and let $\mathbf{A}_h$ be set of all **L**-*relations, s.t. $C^{\uparrow_{A_h}} = h(C)$ for each $C \in U$. Then $\mathbf{A}_h$ is closed under $\vee$-union.*

Proof. Follows from properties of residuated lattices; for instance, consider $\mathbf{K} \subseteq \mathbf{A}_h$. Thus we have

$$\begin{aligned} h(C)(y) &= \bigvee_{A \in \mathbf{K}} C^{\cap_{A_h}}(y) \\ &= \bigvee_{A \in \mathbf{K}} \bigvee_{x \in X} C(x) \otimes A_h(x,y) \\ &= \bigvee_{x \in X} C(x) \otimes \bigvee_{A \in \mathbf{K}} A_h(x,y) \\ &= C^{\cap_{\bigvee_{A \in \mathbf{A}_h} A_h(x,y)}}(y). \end{aligned}$$

which proves (a). □

The next theorem says that morphisms which are defined for entire L^X have a unique **L**-relation A_h.

Theorem 24. *Let $h : L^X \to W$ be an i-morphism (c-morphism, a-morphism). Then there is a unique **L**-relation A_h s.t. $C^{\cap_{A_h}} = h(C)$ ($C^{\wedge_{A_h}} = h(C), C^{\uparrow_{A_h}} = h(C)$)*

Proof. From properties of residuated lattices. □

4.2 Isomorphisms of Concept Lattices

In this section, we study the i-isomorphisms and c-isomorphisms of concept lattices.

Definition 6. *An i-morphism $h : V \to W$ is called i-isomorphism if h is bijective, its inverse h^{-1} is i-morphism. If an i-isomorphism $h : V \to W$ exists we also say that V is i-isomorphic to W.*

A c-morphism $h : V \to W$ is called c-isomorphism if h is bijective, its inverse h^{-1} is c-morphism. If a c-isomorphism $h : V \to W$ exists we also say that V is c-isomorphic to W.

The following theorem provides the sufficient and necessary condition under which two systems of intents are isomorphic.

Theorem 25. *Let $\langle X_1, Y_1, I_1\rangle$ and $\langle X_2, Y_2, I_2\rangle$ be **L**-contexts. Then system of intents $\mathrm{Int}(X_1^{\cap}, Y_1^{\cup}, I_1)$ is i-isomorphic to $\mathrm{Int}(X_2^{\cap}, Y_2^{\cup}, I_2)$ if and only if there exists an **L**-relation $K \in L^{X_2 \times Y_1}$ such that*

$\mathrm{Int}(X_1^{\cap}, Y_1^{\cup}, I_1) = \mathrm{Int}(X_2^{\cap}, Y_1^{\cup}, K)$ *and* $\mathrm{Ext}(X_2^{\wedge}, Y_2^{\vee}, I_2) = \mathrm{Ext}(X_2^{\wedge}, Y_1^{\vee}, K)$.

Proof. "⇒": Let $h\colon \mathrm{Int}(X_2^{\cap}, Y_2^{\cup}, I_2) \to \mathrm{Int}(X_1^{\cap}, Y_1^{\cup}, I_1)$ be the i-isomorphism. According to Lemma 20, there exist **L**-relations $J_{2\to1} \in L^{Y_2 \times Y_1}$ and $J_{1\to2} :\in L^{Y_1 \times Y_2}$ such that

$$h(D_2) = D_2 \circ J_{2\to1} \text{ and } h^{-1}(D_1) = D_1 \circ J_{1\to2}$$

for every $D_1 \in \mathrm{Int}(X_1^{\cap}, Y_1^{\cup}, I_1)$ and $D_2 \in \mathrm{Int}(X_2^{\cap}, Y_2^{\cup}, I_2)$. Because every row of I_2 is intent of $\mathrm{Int}(X_2^{\cap}, Y_2^{\cup}, I_2)$, it follows that

$$I_2 \circ J_{2\to1} \circ J_{1\to2} = I_2.$$

Therefore, according to Theorem 8 ($\circ$2),

$$\mathrm{Ext}(X_2^{\wedge}, Y_2^{\vee}, I_2) \subseteq \mathrm{Ext}(X_2^{\wedge}, Y_2^{\vee}, I_2 \circ J_{2\to1}).$$

Since, according to Theorem 8 ($\circ$1) again,

$$\mathrm{Ext}(X_2^{\wedge}, Y_2^{\vee}, I_2) \supseteq \mathrm{Ext}(X_2^{\wedge}, Y_2^{\vee}, I_2 \circ J_{2\to1}),$$

we conclude

$$\mathrm{Ext}(X_2^{\wedge}, Y_2^{\vee}, I_2) = \mathrm{Ext}(X_2^{\wedge}, Y_1^{\vee}, I_2 \circ J_{2\to1}).$$

Furthermore, if $D_1 \in \mathrm{Int}(X_1^{\cap}, Y_1^{\cup}, I_1)$, then $D_1 \circ J_{1\to2} = h^{-1}(D_1) \in \mathrm{Int}(Y_1^{\cap}, Y_2^{\cup}, J_{1\to2})$, hence $D_1 \circ J_{1\to2} = C_2 \circ I_2$ for some $C_2 \in L^{X_2}$. Since $D_1 = (D_1 \circ J_{1\to2}) \circ J_{2\to1}$, we get $D_1 = (C_2 \circ I_2) \circ J_{2\to1} = C_2 \circ (I_2 \circ J_{2\to1})$, showing $D_1 \in \mathrm{Int}(X_2^{\cap}, Y_1^{\cup}, I_2 \circ J_{2\to1})$. We established $\mathrm{Int}(X_1^{\cap}, Y_1^{\cup}, I_1) \subseteq \mathrm{Int}(X_2^{\cap}, Y_1^{\cup}, I_2 \circ J_{2\to1})$.

If $D_1 \in \mathrm{Int}(X_2, Y_1, I_2 \circ J_{2\to1})$ then $D_1 = C_2 \circ (I_2 \circ J_{2\to1}) = (C_2 \circ I_2) \circ J_{2\to1}$ for some $C_2 \in L^{X_2}$. Since $C_2 \circ I_2 \in \mathrm{Int}(X_2^{\cap}, Y_2^{\cup}, I_2)$, we get

$$D_1 = (C_2 \circ I_2) \circ J_{2\to1} = h(C_2 \circ I_2) \in \mathrm{Int}(X_1^{\cap}, Y_1^{\cup}, I_1),$$

proving $\mathrm{Int}(X_2^{\cap}, Y_1^{\cup}, I_2 \circ J_{2\to1}) \subseteq \mathrm{Int}(X_1^{\cap}, Y_1^{\cup}, I_1)$.

Summing up, we proved

$$\mathrm{Int}(X_2^{\cap}, Y_1^{\cup}, I_2 \circ J_{2\to1}) = \mathrm{Int}(X_1^{\cap}, Y_1^{\cup}, I_1).$$

Now, $I_2 \circ J_{2\to1}$ yields the required **L**-relation K.

"$\Leftarrow$": Since $\mathrm{Ext}(X_2^{\wedge}, Y_1^{\vee}, K) = \mathrm{Ext}(X_2^{\wedge}, Y_2^{\vee}, I_2)$, an application of Theorem 10 (b) to inclusions $\mathrm{Ext}(X_2^{\wedge}, Y_1^{\vee}, K) \subseteq \mathrm{Ext}(X_2^{\wedge}, Y_2^{\vee}, I)$ and $\mathrm{Ext}(X_2^{\wedge}, Y_1^{\vee}, K) \supseteq \mathrm{Ext}(X_2^{\wedge}, Y_2^{\vee}, I)$, respectively, yields **L**-relations $J_{1\to2} \in L^{Y_1 \times Y_2}$ and $J_{2\to1} \in L^{Y_2 \times Y_1}$ for which $I_2 \circ J_{1\to2} = K$ and $K \circ J_{1\to2} = I_2$.

Define mappings $f\colon \mathrm{Int}(X_2, Y_2, I_2) \to \mathrm{Int}(X_2, Y_1, K)$ and $g\colon \mathrm{Int}(X_2, Y_1, K) \to \mathrm{Int}(X_2, Y_2, I_2)$ as follows

$$f(D_2) = D_2 \circ J_{2\to1} \qquad \textit{and} \qquad g(D_1) = D_1 \circ J_{1\to2} \tag{4.3}$$

for $D_2 \in \mathrm{Int}(X_2^{\cap}, Y_2^{\cup}, I_2)$ and $D_1 \in \mathrm{Int}(X_1^{\cap}, Y_1^{\cup}, I_1)$.
Notice that every $D_1 \in \mathrm{Int}(X_2^{\cap}, Y_1^{\cup}, K)$ is in the form $D_1 = C_2 \circ K$ for some $C_2 \in L^{X_2}$ and that every $D_2 \in \mathrm{Int}(X_2^{\cap}, Y_2^{\cup}, I_2)$ is in the form $D_2 = C_2 \circ I_2$ for some $C_2 \in L^{X_2}$. The mappings f and g are defined correctly. Indeed,

$$f(D_2) = D_2 \circ J_{2\to1} = (C_2 \circ I_2) \circ J_{2\to1} = C_2 \circ (I_2 \circ J_{2\to1}) = C_2 \circ K$$

for some C_2, and because we have $C_2 \circ K \in \mathrm{Int}(X_2^{\cap}, Y_1^{\cup}, K)$, we also have $f(D_2) \in$

$\mathrm{Int}(X_2^{\cap}, Y_1^{\cup}, K)$. In a similar way one obtains $g(D_1) \in \mathrm{Int}(X_2^{\cap}, Y_2^{\cup}, I_2)$. Next, since D_1 is in the form $D_1 = C_2 \circ K$ for some $C_2 \in L^{X_2}$, we have

$$g(f(D_2)) = ((C_2 \circ K) \circ J_{1\to 2}) \circ J_{2\to 1} = (C_2 \circ (K \circ J_{1\to 2})) \circ J_{2\to 1} =$$
$$= (C_2 \circ I_2) \circ J_{2\to 1} = C_2 \circ (I_2 \circ J_{2\to 1}) = C_2 \circ K = D_2$$

and, similarly, $f(g(D_1)) = D_1$, proving that f and g are mutually inverse bijections. Finally, due to (4.3), Lemma 21 implies that f and g are extendable i-morphisms. This shows that $\mathrm{Int}(X_2^{\cap}, Y_1^{\cup}, K)$ is i-isomorphic to $\mathrm{Int}(X_2^{\cap}, Y_2^{\cup}, I_2)$, and hence $\mathrm{Int}(X_1^{\cap}, Y_1^{\cup}, I_1)$ is i-isomorphic to $\mathrm{Int}(X_2^{\cap}, Y_2^{\cup}, I_2)$. □

Remark 12. *Note, that Theorem 25 generalizes a well-known result on Green's relations on Boolean matrices. Namely, by Theorem 14 system of intents w.r.t.* $\langle^{\cap}, ^{\cup}\rangle$ *and system of extents w.r.t.* $\langle^{\wedge}, ^{\vee}\rangle$ *correspond to row and column space, respectively. The existence of K in Theorem 25 is then equivalent to the condition under which are two Boolean matrices D-related. The Theorem 25 generalizes [28, Theorem 1.3.3] which says that two Boolean matrices are D-related if and only if their row spaces are isomorphic.*

Next, we show how Theorem 25 may be used to prove a characterization of isomorphism of concept lattices induced by the $^{\cap}$ and $^{\cup}$ operators. We consider mappings of concept lattices. Since every extent of a formal concept is uniquely determined by the corresponding intent and vice versa (using operators $^{\cap}$ and $^{\cup}$), a mapping $h : \mathcal{B}(X_1^{\cap}, Y_1^{\cup}, I_1) \to \mathcal{B}(X_2^{\cap}, Y_2^{\cup}, I_2)$ may be thought of as consisting of a pair $\langle h_{\mathrm{Ext}}, h_{\mathrm{Int}}\rangle$ of mappings, such that $h(A, B) = \langle h_{\mathrm{Ext}}(A), h_{\mathrm{Int}}(B)\rangle$. That is, h consists of

$$h_{\mathrm{Ext}}\colon \mathrm{Ext}(X_1^{\cap}, Y_1^{\cup}, I_1) \to \mathrm{Ext}(X_2^{\cap}, Y_2^{\cup}, I_2)$$

and

$$h_{\mathrm{Int}}\colon \mathrm{Int}(X_1^{\cap}, Y_1^{\cup}, I_1) \to \mathrm{Int}(X_2^{\cap}, Y_2^{\cup}, I_2).$$

Since $\mathrm{Ext}(X_i^{\cap}, Y_i^{\cup}, I_i)$ are **L**-closure systems and $\mathrm{Int}(X_i^{\cap}, Y_i^{\cup}, I_i)$ are **L**-interior systems, the following definition provides natural requirements for h to be a morphism.

Definition 7. *A mapping* $h = \langle h_{\mathrm{Ext}}, h_{\mathrm{Int}}\rangle : \mathcal{B}(X_1^{\cap}, Y_1^{\cup}, I_1) \to \mathcal{B}(X_2^{\cap}, Y_2^{\cup}, I_2)$ *is called an* morphism *if* h_{Ext} *is an c-morphism and* h_{Int} *is an i-morphism (cf. Definition 5).* h *is called an* isomorphism *if* h_{Ext} *is a c-isomorphism and* h_{Int} *is an i-isomorphism; if such* h *exists, we write* $\mathcal{B}(X_1^{\cap}, Y_1^{\cup}, I_1) \cong \mathcal{B}(X_2^{\cap}, Y_2^{\cup}, I_2)$.

Lemma 26. *Let* $\langle X_1, Y_1, I_1\rangle$ *and* $\langle X_2, Y_2, I_2\rangle$ *be* **L***-contexts. Let* h *be a lattice isomorphism* $h : \mathcal{B}(X_1^{\cap}, Y_1^{\cup}, I_1) \to \mathcal{B}(X_2^{\cap}, Y_2^{\cup}, I_2)$. *If its* Int*-component* h_{Int} *is an i-morphism then its* Ext*-component* h_{Ext} *and its inverse* h_{Ext}^{-1} *are a c-morphisms.*

Proof. Due to Lemma 20, there exists an **L**-relation A_h such that

$$h_{\mathrm{Int}}(D_1) = D_1 \circ A_h,$$

i.e. $h_{\mathrm{Int}}(D_1) = D_1^{\cap A_h}$ for every $D_1 \in \mathrm{Int}(X_1^{\cap}, Y_1^{\cup}, I_1)$. As a result,

$$h_{\mathrm{Ext}}(C_1) = (h_{\mathrm{Int}}(C_1^{\cap I_1}))^{\cup I_2} = C_1^{\cap I_1 \cap A_h \cup I_2} \tag{4.4}$$

for every $C_1 \in \mathrm{Ext}(X_1^{\cap}, Y_1^{\cup}, I_1)$. Theorem 25 (a) and its proof imply that the **L**-relation $K = I_1 \circ A_h$ satisfies $\mathrm{Int}(X_1^{\cap}, Y_2^{\cup}, K) = \mathrm{Int}(X_2^{\cap}, Y_2^{\cup}, I_2)$.

Since $\mathrm{Int}(X_1^{\cap}, Y_2^{\cup}, K) \subseteq \mathrm{Int}(X_2^{\cap}, Y_2^{\cup}, I_2)$, there is a **L**-relation J such that $K = J \circ I_2$ (Theorem 10 (a)). Note that due to Theorem 7(3.1), ${}^{\cap J \circ I_2} = {}^{\cap J \cap I_2}$ and ${}^{\cup J \circ I_2} = {}^{\cup I_2 \cup J}$. As a result, (4.4) implies

$$h_{\mathrm{Ext}}(C_1) = C_1^{\cap I_1 \cap A_h \cup I_2} = C_1^{\cap I_1 \circ A_h \cup I_2} = C_1^{\cap K \cup I_2} = C_1^{\cap J \circ I_2 \cup I_2} = C_1^{\cap J \cap I_2 \cup I_2}. \tag{4.5}$$

Observe now that since h_{Ext} is a bijection, we have

$$C_1^{\cap J \cap I_2 \cup I_2 \cup J} = C_1 \tag{4.6}$$

for every $C_1 \in \mathrm{Ext}(X_1^{\cap}, Y_1^{\cup}, I_1)$. Indeed, since $C_1^{\cap J \cap I_2 \cup I_2 \cup J} = C_1^{\cap J \circ I_2 \cup J \circ I_2}$, it follows from the general properties of isotone Galois connections that

$$C_1^{\cap J \circ I_2 \cup J \circ I_2} \supseteq C_1. \tag{4.7}$$

If in (4.7), $C_1^{\cap J \circ I_2 \cup J \circ I_2} \supset C_1$, i.e. $C_1^{\cap J \circ I_2 \cup J \circ I_2} \neq C_1$ then applying ${}^{\cap J \circ I_2 \cup I_2}$ to both sides of the inequality and taking into account that ${}^{\cap J \circ I_2 \cup I_2} = h_{\mathrm{Ext}}$ is a bijection, we get

$$C_1^{\cap J \circ I_2 \cup J \circ I_2 \cap J \circ I_2 \cup I_2} \neq C_1^{\cap J \circ I_2 \cup I_2}, \tag{4.8}$$

which yields a contradiction because using ${}^{\cap J \circ I_2 \cup J \circ I_2 \cap J \circ I_2} = {}^{\cap J \circ I_2}$, both sides of (4.8) are equal.

We established (4.13) and (4.6) from which it follows that ${}^{\cup J}$ is inverse to h_{Ext}, i.e.

$$h_{\mathrm{Ext}}^{-1}(C_2) = C_2^{\cup J} \tag{4.9}$$

for each $C_2 \in \mathrm{Ext}(X_2^{\cap}, Y_2^{\cup}, I_2)$.

Now, in a similar way, one may show that there exists a matrix J' such that

$$h_{\mathrm{Ext}}(C_1) = C_1^{\cup J'} \tag{4.10}$$

for each $C_1 \in \mathrm{Ext}(X_1^{\cap}, Y_1^{\cup}, I_1)$. Namely, just start as in the beginning of this proof with h_{Int}^{-1} instead of h_{Int}, i.e. start by claiming the existence of A'_h for which $h_{\mathrm{Int}}^{-1}(D) = D \circ A'_h$ and proceed dually to how we have proceeded above.

Finally, h_{Ext}^{-1} and h_{Ext} are c-morphisms by Lemma 21. □

Figure 4.2 illustrates Theorem 29.

Remark 13. *Note that opposite direction of the proposition in Theorem 26 does not hold. That is, having lattice isomorphism $h : \mathcal{B}(X_1, Y_1, I_1) \to \mathcal{B}(X_2, Y_2, I_2)$ with* Ext*-component being c-morphism does not implies* Int*-component (or its inverse) to be extendable i-morphism. As a counterexample consider the same*

$$
\begin{tikzcd}
\mathrm{Int}(X_1^{\cap}, Y_1^{\cup}, I_1) \arrow[rr, "h_{\mathrm{Int}}", "\text{i-morphism}"'] \arrow[d, shift right, "{}^{\cup_{I_1}}"'] & & \mathrm{Int}(X_2^{\cap}, Y_2^{\cup}, I_2) \arrow[d, shift right, "{}^{\cup_{I_2}}"'] \\
\mathrm{Ext}(X_1^{\cap}, Y_1^{\cup}, I_1) \arrow[rr, "h_{\mathrm{Ext}}", "\text{c-morphism}"'] \arrow[u, shift right, "{}^{\cap_{I_1}}"'] & & \mathrm{Ext}(X_2^{\cap}, Y_2^{\cup}, I_2) \arrow[u, shift right, "{}^{\cap_{I_2}}"']
\end{tikzcd}
$$

Figure 4.1: Illustration of Lemma 26.

residuated lattice as in Remark 4, i.e. a finite chain containing $a < b$ *with* $\otimes$ *defined as follows:*

$$
x \otimes y = \begin{cases} x \wedge y & \textit{if } x = 1 \textit{ or } y = 1, \\ 0 & \textit{otherwise,} \end{cases} \qquad x \to y = \begin{cases} 1 & \textit{if } x \leq y, \\ y & \textit{if } x = 1, \\ b & \textit{otherwise,} \end{cases}
$$

for each $x, y \in L$. *Now,* **L***-contexts* I, J *given by matrices* $\big(a\big), \big(b\big) \in L^{1\times 1}$, *respectively, induce following sets of concepts:*

$$
\mathcal{B}(\{x\}^{\cap}, \{y\}^{\cup}, I) = \{\langle\{{}^{b}\!/x\}, \varnothing\rangle, \{y, \{{}^{b}\!/y\}\}\},
$$
$$
\mathcal{B}(\{x\}^{\cap}, \{y\}^{\cup}, J) = \{\langle\{{}^{b}\!/x\}, \varnothing\rangle, \{y, \{{}^{a}\!/y\}\}\}.
$$

It is easy to check, that the identity on $L^{\{x\}}$ – Ext*-component of lattice isomorphism* $\mathcal{B}(\{x\}^{\cap}, \{y\}^{\cup}, I) \to \mathcal{B}(\{x\}^{\cap}, \{y\}^{\cup}, J)$ – *is complement-preserving c-isomorphism (as well as its inverse). On the other hand, the* Int*-component is i-morphism which fails to be extendable (as well as its inverse).*

Theorem 27. *Let* $I_1 \in L^{X_1 \times X_1}$ *and* $I \in L^{X_2 \times Y_2}$ *be* **L***-relations.* $\mathcal{B}(X_1^{\cap}, Y_1^{\cup}, I_1) \cong \mathcal{B}(X_2^{\cap}, Y_2^{\cup}, I_2)$ *if and only if there exists an* **L***-relation* $K \in L^{X_2 \times Y_1}$ *such that* $\mathrm{Int}(X_1^{\cap}, Y_1^{\cup}, I_1) = \mathrm{Int}(X_2^{\cap}, Y_1^{\cup}, K)$ *and* $\mathrm{Ext}(X_1^{\wedge}, Y_1^{\vee}, I_2) = \mathrm{Ext}(X_2^{\wedge}, Y_1^{\vee}, K)$.

Proof. Follows directly from Theorem 25 and Lemma 26. □

The Antitone Case

We can get analogous results c-isomorphisms and concept lattices formed by antitone Galois connections.

Definition 8. *A mapping* $h = \langle h_{\mathrm{Ext}}, h_{\mathrm{Int}}\rangle : \mathcal{B}(X_1^{\uparrow}, Y_1^{\downarrow}, I_1) \to \mathcal{B}(X_2^{\uparrow}, Y_2^{\downarrow}, I_2)$ *is called an* morphism *if both* h_{Ext} *and* h_{Int} *are c-morphisms.* h *is called an* isomorphism *if* h_{Ext} *and* h_{Int} *are c-isomorphisms; if such* h *exists, we write* $\mathcal{B}(X_1^{\uparrow}, Y_1^{\downarrow}, I_1) \cong_c \mathcal{B}(X_2^{\uparrow}, Y_2^{\downarrow}, I_2)$.

Theorem 28. *Let* $\langle X_1, Y_1, I_1\rangle$ *and* $\langle X_2, Y_2, I_2\rangle$ *be* **L**-*contexts. We have system of extents* $\mathrm{Ext}(X_1^{\uparrow}, Y_1^{\downarrow}, I_1)$ *is isomorphic to* $\mathrm{Ext}(X_2^{\uparrow}, Y_2^{\downarrow}, I_2)$ *if and only if there exists an* **L**-*relation* $K \in L^{X_2\times Y_1}$ *s.t.* $\mathrm{Int}(X_1^{\uparrow}, Y_1^{\downarrow}, I_1) = \mathrm{Int}(X_2^{\uparrow}, Y_1^{\downarrow}, K)$ *and* $\mathrm{Ext}(X_2^{\uparrow}, Y_2^{\downarrow}, I_2) = \mathrm{Ext}(X_2^{\uparrow}, Y_1^{\downarrow}, K)$.

Proof. "⇒": Let $h\colon \mathrm{Int}(X_2^{\uparrow}, Y_2^{\downarrow}, I_2) \to \mathrm{Int}(X_1^{\uparrow}, Y_1^{\downarrow}, I_1)$ be the c-isomorphism. According to Lemma 20, there exist **L**-relations $J_{2\to1} \in L^{Y_2\times Y_1}$ and $J_{1\to2} :\in L^{Y_1\times Y_2}$ such that

$$h(D_2) = D_2 \triangleright J_{2\to1} \qquad \text{and} \qquad h^{-1}(D_1) = D_1 \triangleright J_{1\to2}$$

for every $D_1 \in \mathrm{Int}(X_1^{\uparrow}, Y_1^{\downarrow}, I_1)$ and $D_2 \in \mathrm{Int}(X_2^{\uparrow}, Y_2^{\downarrow}, I_2)$. Because every row of I_2 is intent of $\mathrm{Int}(X_2^{\uparrow}, Y_2^{\downarrow}, I_2)$, it follows that

$$(I_2 \triangleright J_{2\to1}) \triangleright J_{1\to2} = I_2 \triangleright (J_{2\to1} \circ J_{1\to2}) = I_2.$$

Therefore, according to Theorem 8 ($\triangleright$),

$$\mathrm{Ext}(X_2^{\uparrow}, Y_2^{\downarrow}, I_2) \subseteq \mathrm{Ext}(X_2^{\uparrow}, Y_2^{\downarrow}, I_2 \triangleright J_{2\to1}).$$

Since, according to Theorem 8 ($\triangleright$) again,

$$\mathrm{Ext}(X_2^{\uparrow}, Y_2^{\downarrow}, I_2) \supseteq \mathrm{Ext}(X_2^{\uparrow}, Y_2^{\downarrow}, I_2 \triangleright J_{2\to1}),$$

we conclude

$$\mathrm{Ext}(X_2^{\uparrow}, Y_2^{\downarrow}, I_2) = \mathrm{Ext}(X_2^{\uparrow}, Y_1^{\downarrow}, I_2 \triangleright J_{2\to1}).$$

Furthermore, if $D_1 \in \mathrm{Int}(X_1^{\uparrow}, Y_1^{\downarrow}, I_1)$, then $D_1 \triangleright J_{1\to2} = h^{-1}(D_1) \in \mathrm{Int}(Y_1^{\uparrow}, Y_2^{\downarrow}, J_{1\to2})$, hence $D_1 \triangleright J_{1\to2} = C_2 \triangleleft I_2$ for some $C_2 \in L^{X_2}$. Since $D_1 = (D_1 \triangleright J_{1\to2}) \triangleright J_{2\to1}$, we get $D_1 = (C_2 \triangleleft I_2) \triangleright J_{2\to1} = C_2 \triangleleft (I_2 \triangleright J_{2\to1})$, showing $D_1 \in \mathrm{Int}(X_2^{\uparrow}, Y_1^{\downarrow}, I_2 \triangleright J_{2\to1})$. We established $\mathrm{Int}(X_1^{\uparrow}, Y_1^{\downarrow}, I_1) \subseteq \mathrm{Int}(X_2^{\uparrow}, Y_1^{\downarrow}, I_2 \triangleright J_{2\to1})$.

If $D_1 \in \mathrm{Int}(X_2, Y_1, I_2 \triangleright J_{2\to1})$ then $D_1 = C_2 \triangleleft (I_2 \triangleright J_{2\to1}) = (C_2 \triangleleft I_2) \triangleright J_{2\to1}$ for some $C_2 \in L^{X_2}$. Since $C_2 \triangleleft I_2 \in \mathrm{Int}(X_2^{\uparrow}, Y_2^{\downarrow}, I_2)$, we get

$$D_1 = (C_2 \triangleleft I_2) \triangleright J_{2\to1} = h(C_2 \triangleleft I_2) \in \mathrm{Int}(X_1^{\uparrow}, Y_1^{\downarrow}, I_1),$$

proving $\mathrm{Int}(X_2^{\uparrow}, Y_1^{\downarrow}, I_2 \triangleright J_{2\to1}) \subseteq \mathrm{Int}(X_1^{\uparrow}, Y_1^{\downarrow}, I_1)$.

Summing up, we proved

$$\mathrm{Int}(X_2^{\uparrow}, Y_1^{\downarrow}, I_2 \triangleleft J_{2\to1}) = \mathrm{Int}(X_1^{\uparrow}, Y_1^{\downarrow}, I_1).$$

Now, $I_2 \triangleright J_{2\to1}$ yields the required **L**-relation K.

"⇐": Since we have $\mathrm{Ext}(X_2^{\uparrow}, Y_1^{\downarrow}, K) = \mathrm{Ext}(X_2^{\uparrow}, Y_2^{\downarrow}, I_2)$, an application of Theorem 10 (d) to inclusions

$$\mathrm{Ext}(X_2^{\uparrow}, Y_1^{\downarrow}, K) \subseteq \mathrm{Ext}(X_2^{\uparrow}, Y_2^{\downarrow}, I) \text{ and } \mathrm{Ext}(X_2^{\uparrow}, Y_1^{\downarrow}, K) \supseteq \mathrm{Ext}(X_2^{\uparrow}, Y_2^{\downarrow}, I),$$

respectively, yields **L**-relations $J_{1\to2} \in L^{Y_1\times Y_2}$ and $J_{2\to1} \in L^{Y_2\times Y_1}$ for which $I_2 \triangleright J_{1\to2} = K$ and $K \triangleright J_{1\to2} = I_2$.

Define mappings $f\colon \mathrm{Int}(X_2, Y_2, I_2) \to \mathrm{Int}(X_2, Y_1, K)$ and $g\colon \mathrm{Int}(X_2, Y_1, K) \to \mathrm{Int}(X_2, Y_2, I_2)$ as follows

$$f(D_2) = D_2 \triangleright J_{2\to 1} \qquad \text{and} \qquad g(D_1) = D_1 \triangleright J_{1\to 2} \tag{4.11}$$

for $D_2 \in \mathrm{Int}(X_2^{\uparrow}, Y_2^{\downarrow}, I_2)$ and $D_1 \in \mathrm{Int}(X_1^{\uparrow}, Y_1^{\downarrow}, I_1)$.
Notice that every $D_1 \in \mathrm{Int}(X_2^{\uparrow}, Y_1^{\downarrow}, K)$ is in the form $D_1 = C_2 \triangleleft K$ for some $C_2 \in L^{X_2}$ and that every $D_2 \in \mathrm{Int}(X_2^{\uparrow}, Y_2^{\downarrow}, I_2)$ is in the form $D_2 = C_2 \triangleleft I_2$ for some $C_2 \in L^{X_2}$. The mappings f and g are defined correctly. Indeed,

$$f(D_2) = D_2 \triangleright J_{2\to 1} = (C_2 \triangleleft I_2) \triangleright J_{2\to 1} = C_2 \triangleleft (I_2 \triangleright J_{2\to 1}) = C_2 \triangleleft K$$

for some C_2, and because we have $C_2 \triangleleft K \in \mathrm{Int}(X_2^{\uparrow}, Y_1^{\downarrow}, K)$, we also have $f(D_2) \in \mathrm{Int}(X_2^{\uparrow}, Y_1^{\downarrow}, K)$. In a similar way one obtains $g(D_1) \in \mathrm{Int}(X_2^{\uparrow}, Y_2^{\downarrow}, I_2)$. Next, since D_1 is in the form $D_1 = C_2 \triangleleft K$ for some $C_2 \in L^{X_2}$, we have

$$\begin{aligned} g(f(D_2)) &= ((C_2 \triangleleft K) \triangleright J_{1\to 2}) \triangleright J_{2\to 1} = (C_2 \triangleleft (K \triangleright J_{1\to 2})) \triangleright J_{2\to 1} = \\ &= (C_2 \triangleleft I_2) \triangleright J_{2\to 1} = C_2 \triangleleft (I_2 \triangleright J_{2\to 1}) = C_2 \triangleleft K = D_2 \end{aligned}$$

and, similarly, $f(g(D_1)) = D_1$, proving that f and g are mutually inverse bijections. Finally, due to (4.11), Lemma 21 implies that f and g are extendable c-morphisms. This shows that $\mathrm{Int}(X_2^{\uparrow}, Y_1^{\downarrow}, K)$ is c-isomorphic to $\mathrm{Int}(X_2^{\uparrow}, Y_2^{\downarrow}, I_2)$, and hence $\mathrm{Int}(X_1^{\uparrow}, Y_1^{\downarrow}, I_1)$ is c-isomorphic to $\mathrm{Int}(X_2^{\uparrow}, Y_2^{\downarrow}, I_2)$. □

Lemma 29. *Let $\langle X_1, Y_1, I_1\rangle$ and $\langle X_2, Y_2, I_2\rangle$ be* **L***-contexts. Let h be a lattice isomorphism $h : \mathcal{B}(X_1^{\uparrow}, Y_1^{\downarrow}, I_1) \to \mathcal{B}(X_2^{\uparrow}, Y_2^{\downarrow}, I_2)$. If its* Int*-component h_{Int} is an c-morphism then inverse h_{Ext}^{-1} its* Ext*-component is c-morphism as well.*

Proof. Due to Lemma 20, there exists an **L**-relation A_h such that

$$h_{\mathrm{Int}}(D_1) = D_1 \triangleright A_h,$$

i.e. $h_{\mathrm{Int}}(D_1) = D_1^{\wedge A_h}$ for every $D_1 \in \mathrm{Int}(X_1^{\uparrow}, Y_1^{\downarrow}, I_1)$. As a result,

$$h_{\mathrm{Ext}}(C_1) = (h_{\mathrm{Int}}(C_1^{\uparrow_{I_1}}))^{\downarrow_{I_2}} = C_1^{\uparrow_{I_1}\wedge_{A_h}\downarrow_{I_2}} \tag{4.12}$$

for every $C_1 \in \mathrm{Ext}(X_1^{\uparrow}, Y_1^{\downarrow}, I_1)$. Theorem 28 and its proof imply that the **L**-relation $K = I_1 \triangleright A_h$ satisfies $\mathrm{Int}(X_1^{\uparrow}, Y_2^{\downarrow}, K) = \mathrm{Int}(X_2^{\uparrow}, Y_2^{\downarrow}, I_2)$.

Since $\mathrm{Int}(X_1^{\uparrow}, Y_2^{\downarrow}, K) \subseteq \mathrm{Int}(X_2^{\uparrow}, Y_2^{\downarrow}, I_2)$, there is an **L**-relation J such that $K = J \triangleleft I_2$ (Theorem 10 (c)). Note that due to Theorem 7 (3.3), ${}^{\uparrow_{J\triangleleft I_2}} = {}^{\cap_J \uparrow_{I_2}}$ and ${}^{\downarrow_{J\triangleleft I_2}} = {}^{\downarrow_{I_2}\cup_J}$. As a result, (4.4) implies

$$h_{\mathrm{Ext}}(C_1) = C_1^{\uparrow_{I_1}\wedge_{A_h}\downarrow_{I_2}} = C_1^{\cap_{I_1\triangleright A_h}\downarrow_{I_2}} = C_1^{\cap_K \cup_{I_2}} = C_1^{\cap_{J\circ I_2}\downarrow_{I_2}} = C_1^{\cap_J \uparrow_{I_2}\downarrow_{I_2}}. \tag{4.13}$$

Observe now that since h_{Ext} is a bijection, we have

$$C_1^{\cap_J \uparrow_{I_2}\downarrow_{I_2}\cup_J} = C_1 \tag{4.14}$$

$$
\begin{tikzcd}
\mathrm{Int}(X_1^{\uparrow}, Y_1^{\downarrow}, I_1) \arrow[r, "h_{\mathrm{Int}}", "\text{c-morphism}"'] \arrow[d, shift right, "\downarrow_{I_1}"'] & \mathrm{Int}(X_2^{\uparrow}, Y_2^{\downarrow}, I_2) \arrow[d, shift right, "\downarrow_{I_2}"'] \\
\mathrm{Ext}(X_1^{\uparrow}, Y_1^{\downarrow}, I_1) \arrow[u, shift right, "\uparrow_{I_1}"'] & \mathrm{Ext}(X_2^{\uparrow}, Y_2^{\downarrow}, I_2) \arrow[l, "h^{-1}_{\mathrm{Ext}}", "\text{c-morphism}"'] \arrow[u, shift right, "\uparrow_{I_2}"']
\end{tikzcd}
$$

Figure 4.2: Illustration of Lemma 29.

for every $C_1 \in \mathrm{Ext}(X_1^{\uparrow}, Y_1^{\downarrow}, I_1)$. Indeed, since $C_1^{\cap_J \uparrow_{I_2} \downarrow_{I_2} \cup_J} = C_1^{\uparrow_{J \triangleleft I_2} \downarrow_{J \triangleleft I_2}}$, it follows from the general properties of isotone Galois connections that

$$C_1^{\uparrow_{J \triangleleft I_2} \downarrow_{J \triangleleft I_2}} \supseteq C_1. \tag{4.15}$$

If in (4.15), $C_1^{\uparrow_{J \triangleleft I_2} \downarrow_{J \triangleleft I_2}} \supset C_1$, i.e. $C_1^{\uparrow_{J \triangleleft I_2} \downarrow_{J \triangleleft I_2}} \neq C_1$ then applying ${}^{\uparrow_{J \triangleleft I_2} \downarrow_{I_2}}$ to both sides of the inequality and taking into account that ${}^{\uparrow_{J \triangleleft I_2} \downarrow_{I_2}} = h_{\mathrm{Ext}}$ is a bijection, we get

$$C_1^{\uparrow_{J \triangleleft I_2} \downarrow_{J \triangleleft I_2} \uparrow_{J \triangleleft I_2} \downarrow_{I_2}} \neq C_1^{\uparrow_{J \triangleleft I_2} \downarrow_{I_2}}, \tag{4.16}$$

which yields a contradiction because using ${}^{\uparrow_{J \triangleleft I_2} \downarrow_{J \triangleleft I_2} \uparrow_{J \triangleleft I_2}} = {}^{\uparrow_{J \triangleleft I_2}}$, both sides of (4.16) are equal. Thus ${}^{\cup_J}$ is equal to h^{-1}_{Ext} and due Lemma 21 it is a c-morphism. □

Figure 4.2 illustrates Lemma 29.

Theorem 30. *Let $I_1 \in L^{X_1 \times X_1}$ and $I \in L^{X_2 \times Y_2}$ be* **L***-relations. We have $\mathcal{B}(X_1^{\uparrow}, Y_2^{\downarrow}, I_2) \cong \mathcal{B}(X_2^{\uparrow}, Y_2^{\downarrow}, I_2)$ if and only if there exists an* **L***-relation $K \in L^{X_2 \times Y_1}$ such that*

$$\mathrm{Int}(X_1^{\uparrow}, Y_1^{\downarrow}, I_1) = \mathrm{Int}(X_2^{\uparrow}, Y_1^{\downarrow}, K) \text{ and } \mathrm{Ext}(X_1^{\uparrow}, Y_1^{\downarrow}, I_2) = \mathrm{Ext}(X_2^{\uparrow}, Y_1^{\downarrow}, K).$$

Proof. Follows directly from Theorem 28 and Lemma 29. □

4.3 Sensitivity Issues

In this section, we define the notion of similarity of extendable morphisms and similarity of collections of **L**-sets. We also have that the morphisms show a natural behavior in the sense that application of similar morphisms to similar spaces produces similar spaces.

Let $\mathcal{A}, \mathcal{B} \subseteq L^X$ be collections of **L**-sets, define similarity between them as

$$(\mathcal{A} \approx \mathcal{B}) = \bigwedge_{A \in \mathcal{A}} \bigvee_{B \in \mathcal{B}} (A \approx B) \wedge \bigwedge_{B \in \mathcal{B}} \bigvee_{A \in \mathcal{A}} (A \approx B).$$

That is, $\mathcal{A} \approx \mathcal{B}$ is a degree to which for **L**-set A from $\mathcal{A}$ there exists a similar **L**-set B from $\mathcal{B}$ and *vice versa*, when similarity of A to B is measured by similarity of **L**-sets (2.32).

Similarity of **L**-concept lattices $\mathcal{B}_1 = \mathcal{B}(X^{\triangle}, Y^{\triangledown}, I_1), \mathcal{B}_2 = \mathcal{B}(X^{\triangle}, Y^{\triangledown}, I_2)$ is then defined as similarity of their extents (resp. intents).

$$(\mathcal{B}_1 \approx_{\mathrm{Ext}} \mathcal{B}_2) = (\mathrm{Ext}(X^{\triangle}, Y^{\triangledown}, I_1) \approx \mathrm{Ext}(X^{\triangle}, Y^{\triangledown}, I_2))$$
$$(\mathcal{B}_1 \approx_{\mathrm{Int}} \mathcal{B}_2) = (\mathrm{Int}(X^{\triangle}, Y^{\triangledown}, I_1) \approx \mathrm{Int}(X^{\triangle}, Y^{\triangledown}, I_2))$$

This corresponds to definition of similarity between **L**-concept lattices with antitone concept-forming operators described in [3, 5].

Theorem 31. *Let $\mathcal{A}, \mathcal{B} \subseteq L^X$ and $I, J \in L^{X \times Y}$ then we have*

$$(\mathcal{A} \approx \mathcal{B}) \otimes (J \approx I) \leq (\mathcal{A}^{\uparrow_I} \approx \mathcal{B}^{\uparrow_J}) \tag{4.17}$$
$$(\mathcal{A} \approx \mathcal{B}) \otimes (J \approx I) \leq (\mathcal{A}^{\cap_I} \approx \mathcal{B}^{\cap_J}) \tag{4.18}$$
$$(\mathcal{A} \approx \mathcal{B}) \otimes (J \approx I) \leq (\mathcal{A}^{\wedge_I} \approx \mathcal{B}^{\wedge_J}) \tag{4.19}$$

where $\mathcal{A}^{\uparrow}$ means $\{A^{\uparrow} \mid A \in \mathcal{A}\}$, etc.

Proof. (4.17) We prove only

$$(I \approx J) \otimes \textstyle\bigwedge_{A \in \mathcal{A}} \bigvee_{B \in \mathcal{B}} (A \approx B) \leq \bigwedge_{A \in \mathcal{A}} \bigvee_{B \in \mathcal{B}} (A^{\uparrow_I} \approx B^{\uparrow_J}).$$

The second inequality can be proved dually.

For each $A \in \mathcal{A}, B \in \mathcal{B}$ we have $(A \approx B) \otimes (I \approx J) \leq (A \triangleleft I) \approx (B \triangleleft J)$ by properties of residuated lattices (see [5]). Thus we have

$$\begin{aligned} \bigwedge_{A \in \mathcal{A}} \bigvee_{B \in \mathcal{B}} (A^{\uparrow_I} \approx B^{\uparrow_J}) &\geq \bigwedge_{A \in \mathcal{A}} \bigvee_{B \in \mathcal{B}} (A \approx B) \otimes (I \approx J) \\ &= \bigwedge_{A \in \mathcal{A}} (I \approx J) \otimes \bigvee_{B \in \mathcal{B}} (A \approx B) \geq (I \approx J) \otimes \bigwedge_{A \in \mathcal{A}} \bigvee_{B \in \mathcal{B}} (A \approx B). \end{aligned}$$

Dually, we have $\bigwedge_{B \in \mathcal{B}} \bigvee_{A \in \mathcal{A}} (A^{\uparrow_I} \approx B^{\uparrow_J}) \geq \bigwedge_{B \in \mathcal{B}} \bigvee_{A \in \mathcal{A}} (A \approx B)$. Similarly, one can prove (4.18) and (4.19). $\square$

As a direct consequence, we get the following theorem.

Theorem 32.

$$\begin{aligned} &(\mathcal{B}(X_1^{\cap}, Y^{\cup}, A_1) \approx_{\mathrm{Int}} \mathcal{B}(X_2^{\cap}, Y^{\cup}, A_2)) \otimes (B_1 \approx B_2) \leq \\ &\quad \leq (\mathcal{B}(X_1^{\cap}, Y^{\cup}, A_1 \circ B_1) \approx_{\mathrm{Int}} \mathcal{B}(X_2^{\cap}, Y^{\cup}, A_2 \circ B_2)) \end{aligned} \tag{4.20}$$

$$\begin{aligned} &(\mathcal{B}(X_1^{\wedge}, Y^{\vee}, A_1) \approx_{\mathrm{Int}} \mathcal{B}(X_2^{\wedge}, Y^{\vee}, A_2)) \otimes (B_1 \approx B_2) \leq \\ &\quad \leq (\mathcal{B}(X_1^{\wedge}, Y^{\vee}, A_1 \circ B_1) \approx_{\mathrm{Int}} \mathcal{B}(X_2^{\wedge}, Y^{\vee}, A_2 \circ B_2)) \end{aligned} \tag{4.21}$$

$$\begin{aligned} &(\mathcal{B}(X_1^{\uparrow}, Y^{\downarrow}, A_1) \approx_{\mathrm{Int}} \mathcal{B}(X_2^{\uparrow}, Y^{\downarrow}, A_2)) \otimes (B_1 \approx B_2) \leq \\ &\quad \leq (\mathcal{B}(X_1^{\uparrow}, Y^{\downarrow}, A_1 \triangleright B_1) \approx_{\mathrm{Int}} \mathcal{B}(X_2^{\uparrow}, Y^{\downarrow}, A_2 \triangleright B_2)) \end{aligned} \tag{4.22}$$

If we define the similarity of the morphisms as the similarity of the corresponding **L**-relations, namely those which are constructed in proof of Lemma 20, Theorem 32 may be read as follows: By application of similar morphisms to similar concept lattices one obtains similar concept lattices. We use this result in Chapter 5 to obtain a result on using similar scales in conceptual scaling.

4.4 Block L-relations

The problem of the size of concept lattices is recognized as one of the most important problems of FCA. The main aim of this section is to describe approximation of formal concepts using block relations.

In [5, 3], a method of factorization of fuzzy concept lattices is presented. A similarity threshold a is supplied and the method outputs a factor lattice instead of the whole concept lattice which might be large. The elements of the factor lattice are maximal blocks of concepts from the whole concept lattice which are pairwise similar to degree at least a. We generalize the results using a special case of c-morphisms. We also show that the factor lattice can be computed from a special kind of superrelations of the incidence relation, called fuzzy block relations.

In [38], Meschke proposed a method of approximation of (crisp) concepts by tolerances and block relations. We use some of his ideas (namely the selection of important objects and attributes) in an illustrative example.

This section introduces block **L**-relations on **L**-formal contexts a their properties. We define block relations as follows:

Definition 9. Block **L**-relation *of* $I \in L^{X\times Y}$ *is an* **L**-*relation* $J \supseteq I$ *such that*

$$\mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, J) \subseteq \mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, I) \quad \textit{and} \quad \mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, J) \subseteq \mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I).$$

In the crisp setting, [41] defines block relation as a relation $J \supseteq I$ where each row is an intent of I and each column is an extent of I. Lemma 33 says that block **L**-relation is proper generalization of crisp block relation.

Lemma 33. *Let* $I \in L^{X\times Y}$*;* $J \supseteq I$ *is a block* **L**-*relation of* I *iff* $\{x\}^{\uparrow_J} \in \mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I)$ *for each* $x \in X$ *and* $\{y\}^{\downarrow_J} \in \mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, I)$ *for each* $y \in Y$*.*

Proof. Follows from properties of **L**-closure systems. □

The following theorem provides a characterization of block **L**-relations.

Theorem 34. *Let* $I \in L^{X\times Y}$ *be an* **L**-*relation. The following statements are equivalent:*

(a) J *is a block* **L**-*relation of* I*.*

(b) $J = I \triangleright S_{\mathrm{i}}$ *with* $S_{\mathrm{i}} \in L^{Y\times Y}$ *and for the induced mapping* ${}^{\wedge S_{\mathrm{i}}}$ *we have* $D^{\wedge S_{\mathrm{i}}} \in \mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I)$ *and* $D \subseteq D^{\wedge S_{\mathrm{i}}}$ *for each* $D \in \mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I)$*.*

(c) $J = S_{\mathrm{e}} \triangleleft I$ *with* $S_{\mathrm{e}} \in L^{X\times X}$ *and for the induced mapping* ${}^{\cup S_{\mathrm{e}}}$ *we have* $C^{\cup S_{\mathrm{e}}} \in \mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, I)$ *and* $C \subseteq C^{\cup S_{\mathrm{e}}}$ *for each* $C \in \mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, I)$*.*

Proof. We prove equality of (a) and (b): $\mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, J) \subseteq \mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, I)$ iff $S_{\mathrm{i}} \in L^{Y\times Y}$ s.t. $J = I \triangleright S_{\mathrm{i}}$ due Theorem 10 (d). Since $I \subseteq J$ we have $C^{\uparrow_I} \subseteq C^{\uparrow_J}$ for each $C \in L^X$, that is equal to $C^{\uparrow_I} \subseteq C^{\uparrow_I \wedge_{S_{\mathrm{i}}}}$ for each $C \in L^X$ due Due Theorem 7 (3.4). Thus $D \subseteq D^{\wedge_{S_{\mathrm{i}}}}$. Finally, $D^{\wedge_{S_{\mathrm{i}}}} \in \mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I)$ for each $D \in \mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I)$ iff $\mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I \triangleright S_{\mathrm{i}}) \subseteq \mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I)$ which holds true by Theorem 8 ($\triangleright$).

Equality of (a) and (c) can be proved analogously. □

Notice, that Theorem 34 says that block **L**-relation is given by a extensive mapping $\wedge_{S_{\mathrm{i}}}$ (or dually by $\cup_{S_{\mathrm{e}}}$) which is c-morphism due Lemma 21.

Theorem 35. *Let $I \in L^{X\times Y}$ be an **L**-relation between X and Y.*

*(a) The set of all block **L**-relations J of I is an **L**-closure system.*

*(b) The set of all S_{e} (from Theorem 34) is an **L**-interior system.*

*(c) The set of all S_{i} (from Theorem 34) is an **L**-interior system.*

Proof. (a) Let J_i be block **L**-relations of I. Let $J = \bigwedge_i J_i$ and let $D \in \mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, J)$, hence $D = C^{\uparrow_J}$ for some $C \in L^X$. By definition of ${}^{\uparrow_J}$ and properties of residuated lattices we have

$$C^{\uparrow_J}(y) = \bigwedge_{x\in X} C(x) \to J(x,y) = \bigwedge_{x\in X} C(x) \to \bigwedge_i J_i(x,y) =$$
$$= \bigwedge_i \bigwedge_{x\in X} A(x) \to J_i(x,y) = \bigwedge_i C^{\uparrow_{J_i}}.$$

Thus we have $C^{\uparrow_J} = \bigcap_i C^{\uparrow_{J_i}} \in \mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I)$. Similarly, one can show that $D^{\downarrow_J} = \bigcap_i D^{\downarrow_{J_i}} \in \mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, I)$, $C^{\uparrow_{a\to J_i}} = a \to C^{\uparrow_{J_i}} \in \mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I)$, and $D^{\downarrow_{a\to J_i}} = \bigcap_i a \to D^{\downarrow_{J_i}} \in \mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, I)$.

Finally, from $I \subseteq J_i$ we have $I \subseteq \bigwedge_i J_i$ and $I \subseteq a \to J_i$. Whence $\bigwedge_i J_i$ and $I \subseteq a \to J_i$ are block **L**-relations proving that set of all block **L**-relations J of I is an **L**-closure system.

(b) and (c) can be proved similarly. □

The induced operators of the **L**-relations S_{e} and S_{i} have following properties:

Lemma 36. *Let $I \in L^{X\times Y}$ be an **L**-relation between X and Y and let J be its block **L**-relation:*

(a) $C^{\cup_{S_{\mathrm{e}}}} = C^{\uparrow_I \downarrow_J}$ *for each* $C \in \mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, I)$,
$D^{\wedge_{S_{\mathrm{i}}}} = D^{\downarrow_I \uparrow_J}$ *for each* $D \in \mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I)$.

(b) $\langle C^{\cup_{S_{\mathrm{e}}}}, C^{\cup_{S_{\mathrm{e}}} \cap_{S_{\mathrm{e}}} \uparrow_I} \rangle \in \mathcal{B}(X^{\uparrow}, Y^{\downarrow}, J)$ *and* $\langle D^{\wedge_{S_{\mathrm{i}}} \vee_{S_{\mathrm{i}}} \downarrow_I}, D^{\wedge_{S_{\mathrm{i}}}} \rangle \in \mathcal{B}(X^{\uparrow}, Y^{\downarrow}, J)$ *for each* $\langle C, D\rangle \in \mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$

(c) $C^{\cap_{S_{\mathrm{e}}} \uparrow_I} = C^{\uparrow_I \wedge_{S_{\mathrm{i}}}} = C^{\uparrow_J}$ *for each* $C \in L^X$,
$D^{\vee_{S_{\mathrm{i}}} \downarrow_I} = D^{\downarrow_I \cup_{S_{\mathrm{e}}}} = D^{\downarrow_J}$ *for each* $D \in L^Y$.

Proof. Easy, using definition of S_{i} and S_{e} and Theorem 7. □

Remark 14. *Observe that by definition, the block* $\mathbf{L}$*-relation* J *induces smaller concept lattice than the original* $\mathbf{L}$*-relation* I*. This could suggest that* J *can be always decomposed to smaller or equal number of factors since formal concepts are the optimal and universal factors by Theorem 5 and Theorem 6. Despite of that, it is not always the case. As a counterexample consider*

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \circ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad\qquad J = \begin{pmatrix} 1 & 0.25 \\ 0.25 & 0.25 \end{pmatrix}$$

Over 5*-element chain* $\{0, 0.25, 0.5, 0.75, 1\}$ *with Łukasiewicz operations. Clearly,* J *is a block* $\mathbf{L}$*-relation of* I*, on the other hand there is no decomposition of* J *by one factor.*

Definition 10. *Let* $I \in L^{X\times Y}$ *be an* $\mathbf{L}$*-relation between* X *and* Y *and let* J *be its block relation. Denote by* ${}^{\theta_J}$ *a mapping* ${}^{\theta_J} : \mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I) \to \mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$ *defined by*

$$\langle C, D\rangle^{\theta_J} = \langle C^{\cup_{S_{\mathrm{e}}}}, C^{\cup_{S_{\mathrm{e}}}\uparrow_I}\rangle \tag{4.23}$$

and denote by ${}_{\theta_J}$ *a mapping* ${}_{\theta_J} : \mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I) \to \mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$ *defined by*

$$\langle C, D\rangle_{\theta_J} = \langle D^{\wedge_{S_{\mathrm{i}}}\downarrow_I}, D^{\wedge_{S_{\mathrm{i}}}}\rangle \tag{4.24}$$

Notice, that different block $\mathbf{L}$-relations J of an $\mathbf{L}$-relation I induce different mappings ${}^{\theta_J}$, ${}_{\theta_J}$. In what follows we omit subscript in the notation ${}^{\theta_J}$, ${}_{\theta_J}$ and write just ${}^{\theta}$, ${}_{\theta}$.

Theorem 37. *Let* $I \in L^{X\times Y}$ *be an* $\mathbf{L}$*-relation between* X *and* Y *and let* J *be its block* $\mathbf{L}$*-relation: The pairs* $\langle {}^{\theta}, {}_{\theta}\rangle$ *is isotone* $\mathbf{L}$*-Galois connection in* $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$ *and for each* $\langle C, D\rangle \in \mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$ *we have* $\langle C, D\rangle_{\theta} \leq \langle C, D\rangle \leq \langle C, D\rangle^{\theta}$*.*

Proof. Directly from Lemma 36. □

Remark 15. *According to Theorem 37 the pair* $\langle {}^{\theta}, {}_{\theta}\rangle$ *defines intervals in the concept lattice. In [34] we also study a relationship between fuzzy block relations and complete fuzzy tolerances on concept lattices. It turns out that block* $\mathbf{L}$*-relations and the induced mappings* $\langle {}^{\theta}, {}_{\theta}\rangle$ *are in one-to-one correspondence with the complete fuzzy tolerances. That way we generalize a Wille's results on (crisp) tolerances and block relations in [41] (see also [22]).*

Illustrative Example of Block L-relations

In this example, we use data, which are "inherently fuzzy". We have chosen ratings of some motion pictures obtained from reviews made by film critics. In general, movie ratings are suitable for interpreting by means of fuzzy logic. A movie rating (usually given by number of "stars" or by a percentage) can be interpreted as an answer to the question: "How do you like this movie?", given in grades. Another advantage of this example is that it is quite natural; in most cases, it is not possible to answer the above question with simple "Yes" or "No",

	c1	c2	c3	c4	c5
BV	0.4	0.4	0.2	0.4	0.6
LH	0.8	0.8	0.8	1	1
MD	0.8	0.4	1	0.8	1
TSS	0.4	0.4	0.8	0.6	0.4

Figure 4.3: Formal context of movies ("Blue Velvet" (BV), "Lost Highway" (LH), "Mulholland Drive" (MD), and "The Straight Story" (TSS)), reviewers (David Sterritt (c1), Desson Thomson (c2), Jonathan Rosenbaum (c3), Owen Gleiberman (c4) and Roger Ebert (c5)) and their ratings on the 6-element Łukasiewicz chain $L = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$. Data taken from `www.metacritic.com` on March 20, 2011.

and, in the same time, there are no doubts about the answer (especially, if given by a film critic).

For this example we use context of four selected David Lynch movies and their scores assigned by five reviewers. We rescaled the ratings the critics gave to each of the films (according to Metacritic) to the 6-element scale $L = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$ with the structure of Łukasiewicz chain. The data forms an **L**-formal context $\langle X, Y, I\rangle$, where X is the set of films and Y the set of critics. The **L**-formal context $\langle X, Y, I\rangle$ and its **L**-context lattice are displayed in Fig. 4.3 and Fig. 4.4.

Our aim is to reduce the size of the **L**-concept lattice $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$. First we take the approach from [3]. This approach is based on a choice of a threshold $a \in L$ and using the a-cut ${}^{a}\!\approx$ of the **L**-equality $\approx$ on $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$ for factorization. As noted in [15], the factor lattice is isomorphic to the concept lattice $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, a \to I)$. As it can be easily seen, these results are a special case of the results of this chapter since $a \to I$ is a block **L**-relation of I.

As an example, we present in Fig. 4.5 and Fig. 4.6 results we obtained for our formal context of movies, critics and ratings with values of the threshold a equal to 0.8 and 0.6.

We also tried a more sophisticated approach, based on [38]. The author's method is based on using a complete tolerance on a complete lattice induced by an interior and closure operator. As an example, the author uses a complete tolerance on a (crisp) concept lattice, obtained by selecting subsets $X' \subseteq X$ and $Y' \subseteq Y$ of important objects and important attributes, respectively.

Although a proper fuzzification of the results from [38] remains to be developed, it is possible to try some experiments. We provide a short outline of our method.

For a formal **L**-context $\langle X, Y, I\rangle$ we select two **L**-sets $X' \in L^X$ and $Y' \in L^Y$ and interpret them as **L**-sets of "important objects" and "important attributes", respectively. Thus, for an object $x \in X$, the value $X'(x)$ is the degree to which x is important, and similarly for attributes. Further we define two **L**-relations

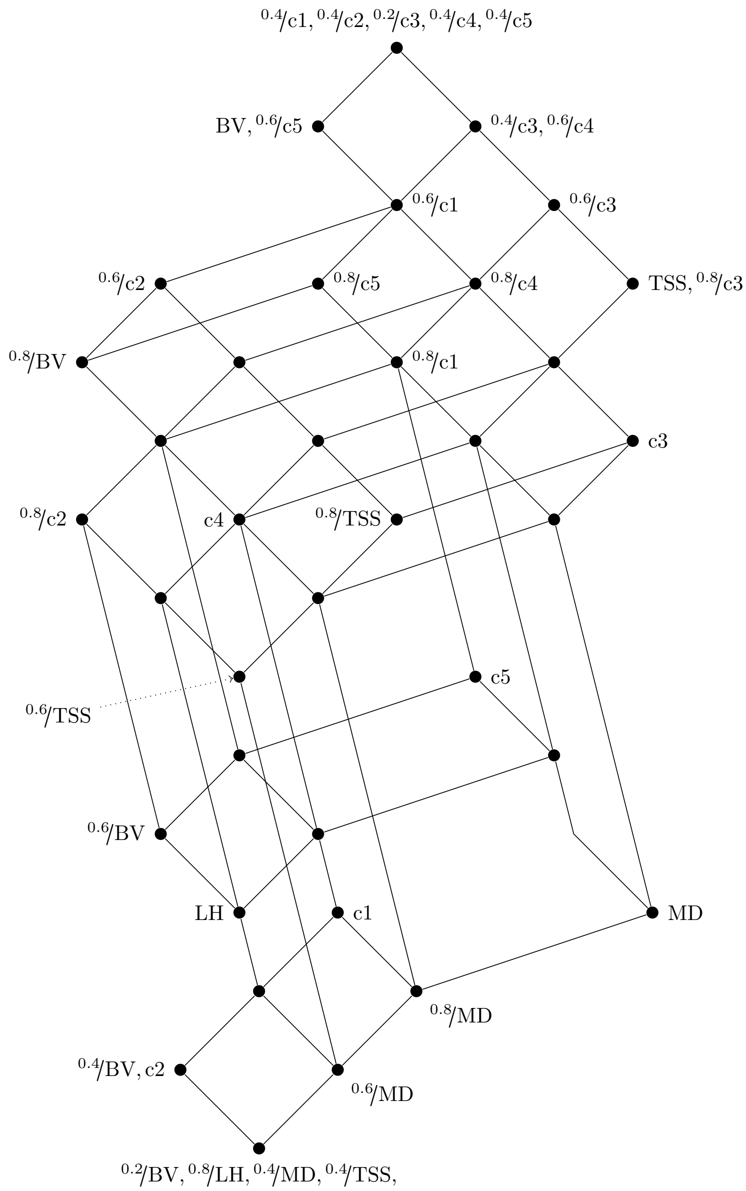
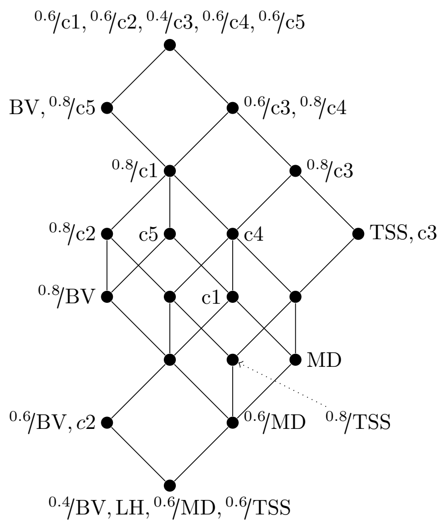


Figure 4.4: **L**-concept lattice of movies, critics and ratings (from Fig. 4.3)

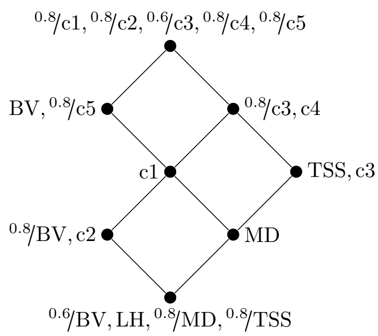


	c1	c2	c3	c4	c5
BV	0.6	0.6	0.4	0.6	0.8
LH	1.0	1.0	1.0	1.0	1.0
MD	1.0	0.6	1.0	1.0	1.0
TSS	0.6	0.6	1.0	0.8	0.6

S_{i}	c1	c2	c3	c4	c5
c1	0.8	0.8	0.4	0.6	0.6
c2	0.4	0.8	0.4	0.4	0.4
c3	0.6	0.6	0.8	0.6	0.4
c4	0.8	0.8	0.6	0.8	0.6
c5	0.8	0.8	0.4	0.6	0.8

S_{e}	BV	LH	MD	TS
BV	0.8	1.0	0.8	0.6
LH	0.2	0.8	0.4	0.4
MD	0.2	0.8	0.8	0.4
TSS	0.2	0.8	0.8	0.8

Figure 4.5: **L**-concept lattice of movies, critics and ratings (from Fig. 4.3), factorized with respect to the threshold $a = 0.8$, the corresponding block **L**-relation, and related **L**-relations S_{i} and S_{e}.



	c1	c2	c3	c4	c5
BV	0.8	0.8	0.6	0.8	1.0
LH	1.0	1.0	1.0	1.0	1.0
MD	1.0	0.8	1.0	1.0	1.0
TSS	0.8	0.8	1.0	1.0	0.8

S_{i}	c1	c2	c3	c4	c5
c1	0.6	0.6	0.4	0.4	0.4
c2	0.4	0.6	0.4	0.4	0.4
c3	0.4	0.4	0.6	0.4	0.2
c4	0.6	0.6	0.6	0.6	0.4
c5	0.6	0.6	0.4	0.4	0.6

S_{e}	BV	LH	MD	TS
BV	0.6	1.0	0.6	0.4
LH	0.2	0.8	0.4	0.4
MD	0.2	0.8	0.6	0.4
TSS	0.2	0.8	0.6	0.6

Figure 4.6: **L**-concept lattice of movies, critics and ratings (from Fig. 4.3), factorized with respect to the threshold $a = 0.6$, the corresponding block **L**-relation, and related **L**-relations S_{i} and S_{e}.

$I_{X'}, I_{Y'} \in L^{X \times Y}$ by

$$I_{X'}(x,y) = X'(x) \to I(x,y), \quad I_{Y'}(x,y) = Y'(y) \to I(x,y).$$

As it can be easily seen, $\mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I_{X'}) \subseteq \mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I)$ and $\mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, I_{Y'}) \subseteq \mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, I)$. Thus, the **L**-relations $I_{X'}$ and $I_{Y'}$ select some intents and extents of formal concepts from $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$. These intents, resp. extents, are interpreted as "important". Now, let $J_{X',Y'} \in L^{X \times Y}$ be the minimal (with respect to **L**-set inclusion) block **L**-relation of I such that intent (resp. extent) of each formal **L**-concept from $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, J_{X',Y'})$ is important (the block **L**-relation $J_{X',Y'}$ always exists because of Theorem 35 (a)). $J_{X',Y'}$ induces the concept lattice $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, J_{X',Y'})$.

We apply the above considerations to this example. Suppose we consider the film BV less important than the other films (perhaps because we have not seen BV) and the critic c2 less important than the other critics (because we do not like his opinion on MD). More precisely, set $X' = \{{}^{a}\!/\mathrm{BV}, \mathrm{LH}, \mathrm{MD}, \mathrm{TSS}\}$ and $Y' = \{\mathrm{c1}, {}^{b}\!/\mathrm{c2}, \mathrm{c3}, \mathrm{c4}, \mathrm{c5}\}$, where $a, b \in L$. In Fig. 4.8 and 4.7 we can see the resulting concept lattices in two cases: first $a = b = 0.8$ and second $a = 1$ and $b = 0.6$.

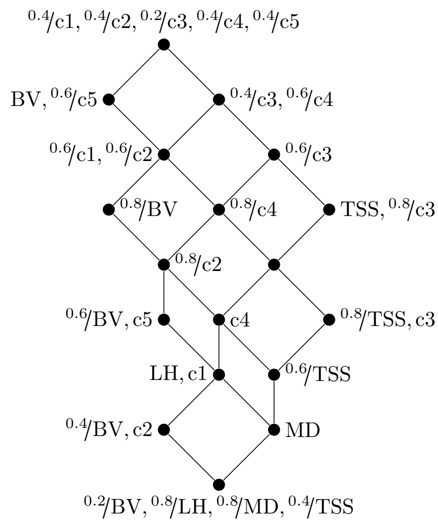
Additionally, we depict the four concept lattices from this example (from Figures 4.5,4.6,4.8,4.7), as hierarchy of intervals in the original lattice (Figs. 4.9, 4.10, 4.12, and 4.11).

4.5 Summary and Future Research

We introduced morphisms of structures associated to **L**-relations and showed the morphisms correspondence to **L**-relations. Furthermore, we described sufficient and necessary conditions under which two structures are isomorphic. Then we described special morphisms induced by block **L**-relations.

Our future research includes:

- Study of the morphisms which are not extendable.
- Block **L**-relations for isotone Galois connections and revision of the results on the one-to-one correspondence of Block **L**-relations with complete tolerances (see Remark 15).
- The new notion of a complement introduced in Section 3.4 is a bijective a-morphism. The notion of an a-isomorphisms in not yet well developed and remains for our future research.

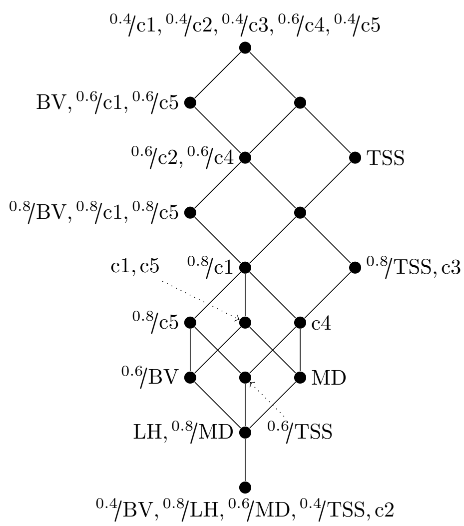


	c1	c2	c3	c4	c5
BV	0.4	0.4	0.2	0.4	0.6
LH	1.0	0.8	0.8	1.0	1.0
MD	1.0	0.6	1.0	1.0	1.0
TSS	0.4	0.4	0.8	0.6	0.4

S_{i}	c1	c2	c3	c4	c5
c1	0.8	1.0	0.6	0.8	0.8
c2	0.4	0.6	0.4	0.4	0.4
c3	0.8	0.8	1.0	0.8	0.6
c4	0.8	1.0	0.8	0.8	0.8
c5	1.0	1.0	0.6	0.8	1.0

S_{e}	BV	LH	MD	TS
BV	1.0	1.0	1.0	0.8
LH	0.4	0.8	0.6	0.4
MD	0.2	0.8	0.6	0.4
TSS	0.4	1.0	1.0	1.0

Figure 4.7: **L**-concept lattice of movies, critics and ratings (form Fig. 4.3), factorized with respect to an **L**-set of important objects $X' = \{\mathrm{BV}, \mathrm{LH}, \mathrm{MD}, \mathrm{TSS}\}$ and **L**-set of important attributes $Y' = \{\mathrm{c1}, {}^{0.6}/\mathrm{c2}, \mathrm{c3}, \mathrm{c4}, \mathrm{c5}\}$, the corresponding block **L**-relation, and related **L**-relations S_{i} and S_{e}.



	c1	c2	c3	c4	c5
BV	0.6	0.4	0.4	0.6	0.6
LH	1.0	0.8	1.0	1.0	1.0
MD	1.0	0.6	1.0	1.0	1.0
TSS	0.4	0.4	0.8	0.6	0.4

S_{i}	c1	c2	c3	c4	c5
c1	0.8	1.0	0.6	0.8	0.8
c2	0.4	0.8	0.4	0.4	0.4
c3	0.6	0.8	0.8	0.6	0.6
c4	0.8	1.0	0.8	0.8	0.8
c5	1.0	1.0	0.6	0.8	1.0

S_{e}	BV	LH	MD	TS
BV	0.8	1.0	1.0	0.8
LH	0.2	0.8	0.6	0.4
MD	0.2	0.8	0.8	0.4
TSS	0.4	1.0	1.0	1.0

Figure 4.8: **L**-concept lattice of movies, critics and ratings (form Fig. 4.3), factorized with respect to an **L**-set of important objects $X' = \{{}^{0.8}/\mathrm{BV}, \mathrm{LH}, \mathrm{MD}, \mathrm{TSS}\}$ and **L**-set of important attributes $Y' = \{\mathrm{c1}, {}^{0.8}/\mathrm{c2}, \mathrm{c3}, \mathrm{c4}, \mathrm{c5}\}$, the corresponding block **L**-relation, and related **L**-relations S_{i} and S_{e}.

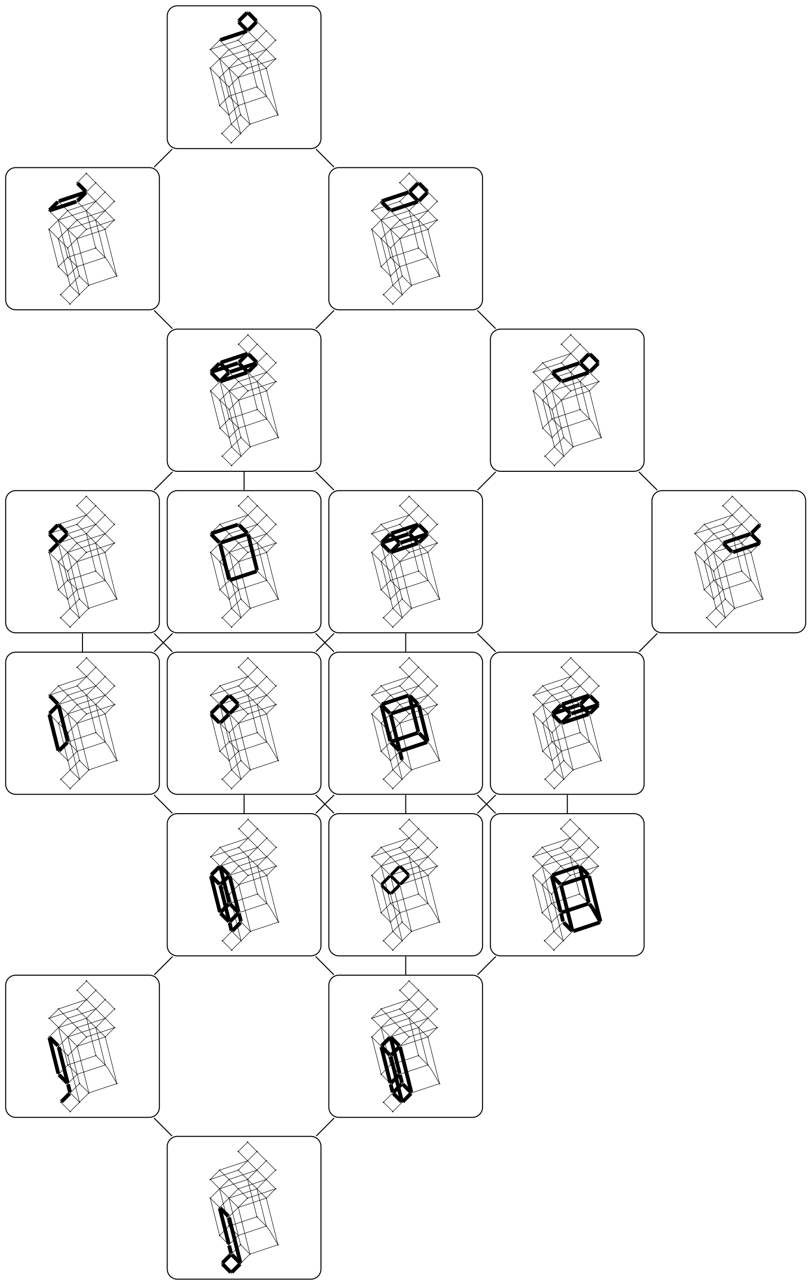


Figure 4.9: **L**-concept lattice of movies, critics and ratings (from Fig. 4.3), factorized with respect to the threshold $a = 0.8$ (Fig. 4.6) depicted as hierarchy of intervals in the original **L**-concept lattice.

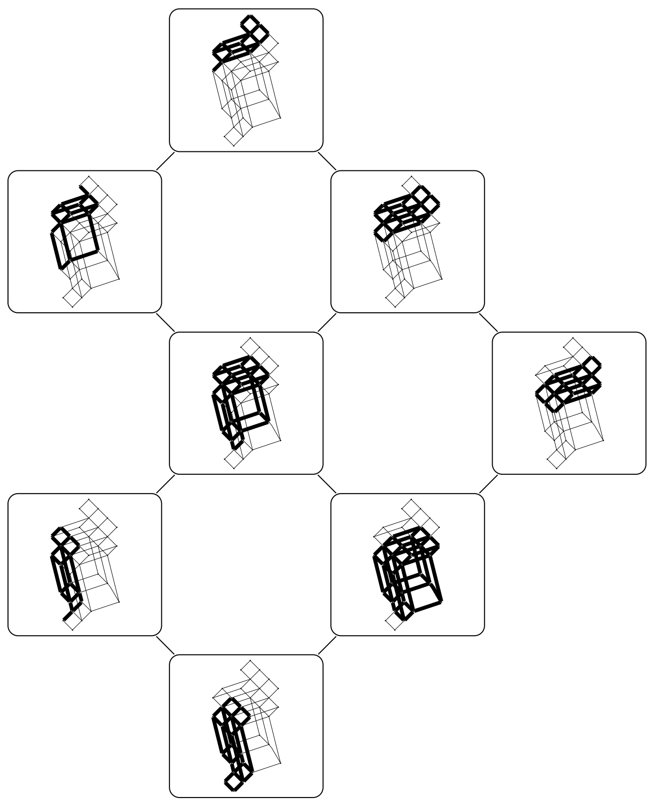


Figure 4.10: **L**-concept lattice of movies, critics and ratings (from Fig. 4.3), factorized with respect to the threshold $a = 0.6$ (Fig. 4.6) depicted as hierarchy of intervals in the original **L**-concept lattice.

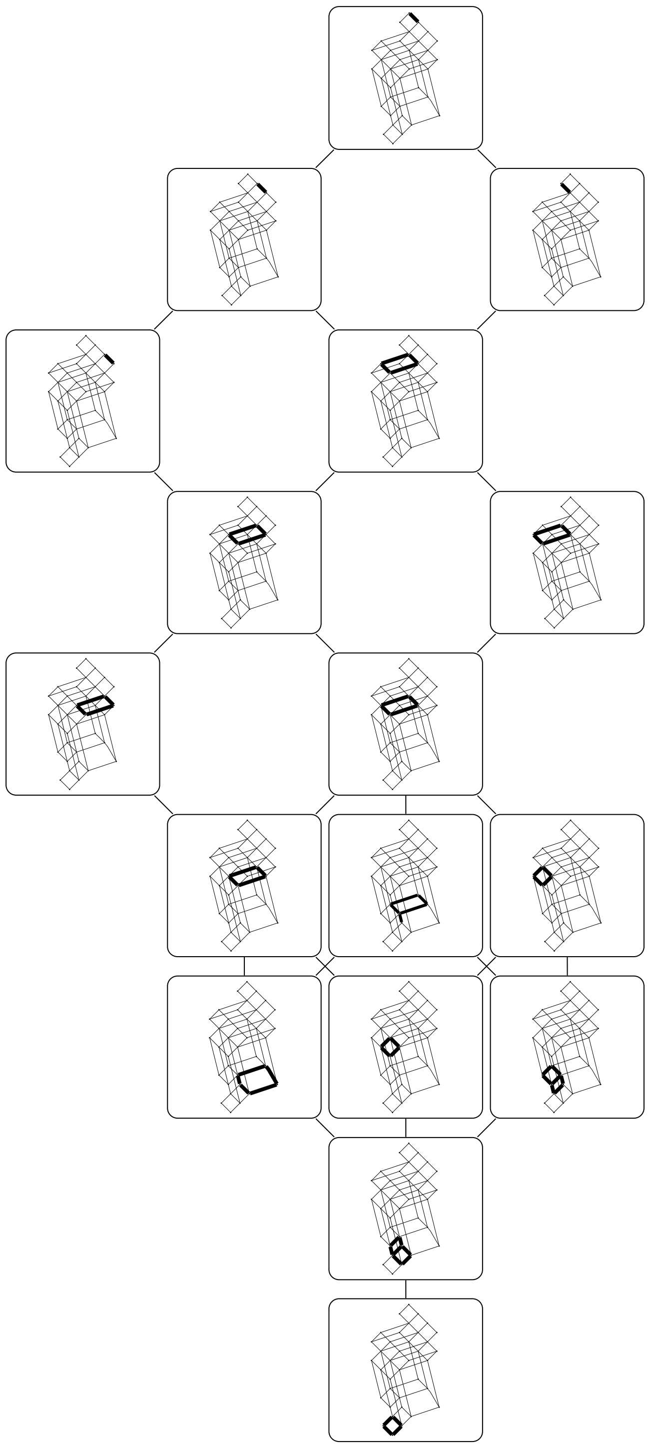


Figure 4.11: **L**-concept lattice of movies, critics and ratings (from Fig. 4.3), factorized with respect to an **L**-set of important objects $X' = \{{}^{0.8}/\mathrm{BV}, \mathrm{LH}, \mathrm{MD}, \mathrm{TSS}\}$ and **L**-set of important attributes $Y' = \{\mathrm{c1}, {}^{0.8}/\mathrm{c2}, \mathrm{c3}, \mathrm{c4}, \mathrm{c5}\}$. (Fig. 4.7) depicted as hierarchy of intervals in the original **L**-concept lattice.

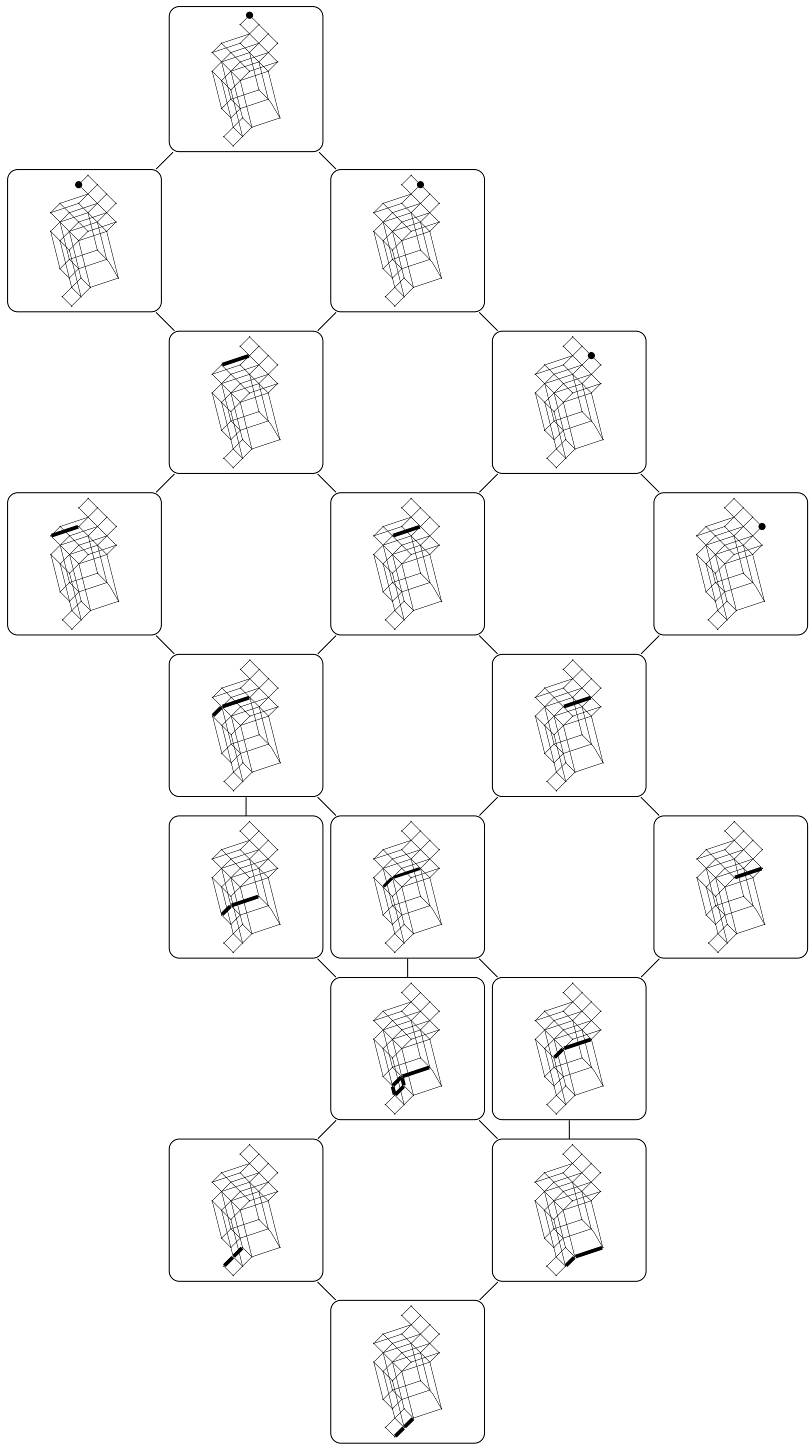


Figure 4.12: **L**-concept lattice of movies, critics and ratings (from Fig. 4.3), factorized with respect to an **L**-set of important objects $X' = \{\mathrm{BV}, \mathrm{LH}, \mathrm{MD}, \mathrm{TSS}\}$ and **L**-set of important attributes $Y' = \{\mathrm{c1}, {}^{0.6}/\mathrm{c2}, \mathrm{c3}, \mathrm{c4}, \mathrm{c5}\}$ (Fig. 4.8) depicted as hierarchy of intervals in the original **L**-concept lattice.

Chapter 5

Conceptual Scaling

In order for FCA to have the capability to analyze data with general attributes, like the one in Fig. 5.1, FCA uses so-called (conceptual) scaling. Scaling, basically, represents a transformation of a table with general attributes to a table with yes/no attributes. For instance (see below for an exact definition of conceptual scaling), attribute age could be replaced by three yes/no attributes a_y, a_m, a_o, corresponding to age intervals $[0,30]$, $[31,50]$, $[51,\infty]$, which represent "young", "middle", and "old". That is, a person with age, say, 18, has attribute a_y but has neither a_m nor a_o. In similar way, one can introduce attributes h_s, h_m, h_t, corresponding to height intervals $[0,160]$, $[161,180]$, and $[181,\infty]$. This way, data table in Fig. 5.1 can be transformed into a formal context with yes/no attributes like the one in Fig. 5.2. After the scaling, the data can be processed by means of FCA.

Scaling is a particular form of information granulation [43]. Namely, new attributes such as a_y represent granules. For instance, a_y represents a granule consisting of age values $[0,30]$. As convincingly argued by Zadeh, see e.g. [43], granules involved in human reasoning are vague rather than sharply delineated. Typical examples are granules corresponding to linguistic expressions like "young", "old", "tall", etc. Therefore, these granules should be represented

	age	height	symptom
Alice	23	165	1
Boris	30	180	0
Cyril	31	167	1
David	43	159	0
Ellen	24	155	1
Fred	64	170	0
George	30	190	0

Figure 5.1: Data table describing persons Alice, . . . , George (objects) and their attributes (age in years, height in cm, presence of symptom).

	a_y	a_m	a_o	h_s	h_m	h_t	symptom
Alice	1	0	0	0	1	0	1
Boris	1	0	0	0	1	0	0
Cyril	0	1	0	0	1	0	1
David	0	1	0	1	0	0	0
Ellen	1	0	0	1	0	0	1
Fred	0	0	1	0	1	0	0
George	1	0	0	0	0	1	0

Figure 5.2: Formal (crisp) context describing persons Alice, ..., George (objects) and their attributes as a result of scaling data table from Fig. 5.1.

	a_y	a_m	a_o	h_s	h_m	h_t	symptom
Alice	1	0.5	0	0.5	1	0	1
Boris	1	0.5	0	0	0.5	0.5	0
Cyril	0.5	1	0	0.5	1	0	1
David	0	1	0.5	1	0.5	0	0
Ellen	1	0.5	0	1	0.5	0	1
Fred	0	0	1	0.5	1	0	0
George	1	0.5	0	0	0	1	0

Figure 5.3: Formal **L**-context describing persons Alice, ..., George (objects) and their attributes as a result of scaling data table from Fig. 5.1.

by fuzzy sets rather than ordinary sets. From this point of view, scaling to crisp attributes is not appropriate. For instance if scaling is performed according to the formal context in Fig. 5.2, attribute a_y ("young") fully applies to a person of age 30, but does not apply at all to a person of age 31, which is counterintuitive.

There is an obvious way to overcome this problem. Namely, instead of crisp attributes, one can use fuzzy attributes. After a scaling to fuzzy attributes using a set $\{0, 0.5, 1\}$ of truth degrees, the resulting formal **L**-context might look like the one in Figure 5.3.

Scaling, however, is a kind of data preprocessing. After scaling, data is processed by means of FCA. That is, one can extract all formal concepts from the data, a non-redundant basis of attribute implications from the data, etc. The main aim of this chapter is to propose a general definition of scaling to fuzzy attributes, look in detail at the basic properties of this scaling, compare by means of illustrative examples to ordinary scaling, and to show that scaling may be seen as a particular case of morphisms introduced in the previous chapters. A point to emphasize is that scaling to fuzzy attributes behaves naturally w.r.t. similarity of attribute values and lends themselves to sensitivity analysis for which we present some results.

We start by introducing the notion of a many-valued context, see [22].

Definition 11. *A many-valued context (data table with general attributes) is a tuple $\mathcal{D} = \langle X, Y, W, I\rangle$ where X is a non-empty finite set of objects, Y is a finite* *set of (many-valued) attributes, W is a set of values, and I is a ternary relation between X, Y, and W, i.e., $I \subseteq X \times Y \times W$, such that*

$$\langle x, y, w\rangle \in I \text{ and } \langle x, y, v\rangle \in I \text{ imply } w = v.$$

Remark 16. (1) A many-valued context can be thought of as representing a table with rows corresponding to $x \in X$, columns corresponding to $y \in Y$, and table entries at the intersection of row x and column y containing values $w \in W$ provided by $\langle x, y, w\rangle \in I$ and containing blanks if there is no $w \in W$ with $\langle x, y, w\rangle \in I$.

(2) One can see that each $y \in Y$ can be considered a partial function from X to W. Therefore, we often write

$$y(x) = w \text{ instead of } \langle x, y, w\rangle \in I.$$

A set

$$\mathrm{dom}(y) = \{x \in X \mid \langle x, y, w\rangle \in I \text{ for some } w \in W\}$$

is called a domain of y. Attribute $y \in Y$ is called complete if $\mathrm{dom}(y) = X$, i.e. if the table contains some value in every row in the column corresponding to y. A many-valued context is called complete if each of its attributes is complete.

(3) From the point of view of theory of relational databases, a complete many-valued context is essentially a relation over a relation scheme Y, see [36]. Namely, each $y \in Y$ can be considered an attribute in the sense of relational databases and putting

$$D_y = \{w \mid \langle x, y, w\rangle \in I \text{ for some } x \in X\},$$

D_y is a domain for y.

(4) In what follows, we prefer term data table (with general attributes) over the term many-valued context.

5.1 Scales and Plain Scaling With Fuzzy Attributes

In this section, we introduce scaling using fuzzy attributes.

Definition 12. *A scale (or, **L**-scale) for attribute $y \in Y$ is a data table $\mathbb{S}_y = \langle X_y, Y_y, I_y\rangle$ with fuzzy attributes (formal fuzzy context) such that $D_y \subseteq X_y$. Objects $w \in X_y$ are called scale values, attributes of Y_y are called scale attributes.*

Remark 17. *The concept of a scale can be seen a particular case of Zadeh's concept of a linguistic variable. Zadeh's linguistic variable is defined as quintuple $\langle \chi, T, U, G, \sigma\rangle$ in which χ is a name of the linguistic variable, T denotes a set*

	a_y	a_m	a_o
0–20	1	0	0
21–30	1	0.5	0
31–40	0.5	1	0
41–50	0	1	0.5
51–60	0	0.5	1
61–150	0	0	1

Figure 5.4: Scale for attribute age for data table from Fig. 5.1.

of terms of χ (syntactic values), U is a universe, G is a syntactic rule (usually a grammar) which generates terms of χ, and σ is a semantic rule associating with each term X its meaning $\sigma(X)$, which is an **L***-set over a universe U. We can simplify the notion of a linguistic variable by removing the syntactic rule. What remains is a quadruple $\langle \chi, T, U, \sigma \rangle$.*

A scale $\mathbb{S}_y = \langle X_y, Y_y, I_y \rangle$ can then be considered as (a simplified) linguistic variable $\langle y, Y_y, X_y, \sigma \rangle$, where $(\sigma(z))(w) = I_y(w, z)$ for $z \in Y_y$ and $w \in X_y$. That is, scale attributes are considered as terms and scale values are considered as elements of the universe of the linguistic variable.

Example 3. *Consider the data table with general attributes depicted in Fig. 5.1. A set X consists of Alice, ..., George, a set Y consists of age, height, and symptom, W is a set containing all of the table entries, and we have $\langle$Alice, age, 23$\rangle \in I$, etc.*

An example of a scale for age is depicted in Fig. 5.4. Here, we use $X_{\text{age}} = \{0, 1, \ldots, 150\}$, i.e., scale objects are numbers $0, \ldots, 150$, $Y_{\text{age}} = \{\mathrm{a}_y, \mathrm{a}_m, \mathrm{a}_o\}$, i.e., scale attributes are fuzzy attributes corresponding to linguistic terms "young", "middle", "old", and we have $I_{\text{age}}(5, \mathrm{a}_y) = 1$, etc. For simplicity, all rows corresponding to $0, 1, \ldots, 20$ of the scale are represented by a single row labeled 0–20 and the same for 21–30, etc.

An example of a scale for height is depicted in Fig. 5.5. Here, $X_{\text{height}} = [120, 200]$, $Y_{\text{height}} = \{\mathrm{h}_s, \mathrm{h}_m, \mathrm{h}_t\}$, i.e., scale attributes are fuzzy attributes corresponding to linguistic terms "short", "medium", "tall", and we have then $I_{\text{height}}(165, \mathrm{h}_s) = 0.5$, etc. Again, rows corresponding to values from $[120, 150]$ are represented by a single row and the like for other groups of values.

It is often the case that domains D_y (and the scale objects X_y) for some attributes come naturally equipped with similarity relations. That is to say, we naturally consider some values $v, w \in X_y$ similar to each other, some not. This pertains in particular to numerical domains such as age or height. In fact, similarity is a matter of degree and an appropriate approach to capture this intuition is to consider sets X_y of objects equipped with similarity relations $\approx_y$. A criterion for a scale to be reasonable can then be defined as follows.

	h_s	h_m	h_t
120–150	1	0	0
151–160	1	0.5	0
161–170	0.5	1	0
171–180	0	0.5	0.5
181–200	0	0	1

Figure 5.5: Scale for attribute height for data table from Fig. 5.1.

Definition 13. *Scale $\mathbb{S}_y$ is called admissible w.r.t. $\approx_y$ if for each values $w_1, w_2 \in X_y$ and scale attribute $z \in Y_y$ we have*

$$(w_1 \approx_y w_2) \otimes I_y(w_1, z) \leq I_y(w_2, z).$$

Remark 18. *Admissibility means: if w_1 and w_2 are similar and a scale attribute z applies to value w_1 then z applies to w_2 as well. In a sense, an admissible scale respects similarity relation $\approx_y$.*

Lemma 38. *Scale $\mathbb{S}_y = \langle D_y, Y_y, I_y \rangle$ is admissible w.r.t. $\approx$ if and only if $\approx \circ I_y = I_y$.*

Proof. "$\Rightarrow$": if $\mathbb{S}_y = \langle D_y, Y_y, I_y \rangle$ is admissible w.r.t. $\approx$ then $\approx \circ I_y \subseteq I_y$ follows directly from Definition 13. The other inclusion $\approx \circ I_y \supseteq I_y$ follows from reflexivity of $\approx$ and properties of $\circ$-composition.

"$\Leftarrow$": directly from Definition 13. □

Theorem 39. *Let $\mathbb{S}_y = \langle D_y, Y_y, I_y \rangle$ be a scale and $\approx$ be a similarity relation on D_y. Then $\mathbb{S}_y = \langle D_y, Y_y, \approx \circ I_y \rangle$ is an admissible scale w.r.t. $\approx$.*

Proof. By transitivity of $\approx$ we have $\approx \circ \approx \subseteq \approx$. By monotony and associativity of $\circ$ we obtain $(\approx \circ \approx) \circ I_y = \approx \circ (\approx \circ I_y) \subseteq \approx \circ I_y$. The proposition now follows from Lemma 38. □

Example 4. *Consider fuzzy relations $\approx_y$ defined on X_y for y being both age and height by rule*

$$w_1 \approx_y w_2 = \begin{cases} 1 & \textit{if } w_1 = w_2, \\ 0.5 & \textit{if } 0 < |w_1 - w_2| < 5, \\ 0 & \textit{otherwise} \end{cases}$$

One can check that both of the scales from Fig. 5.4 and Fig. 5.5 are admissible w.r.t. $\approx_y$.

It is our contention that some ordinary crisp scales appear to be unnatural and problematic simply just because they ignore the underlying similarity on X_y (similarity is not considered in ordinary scaling). For instance, in case of the ordinary scale which is used to transform Fig. 5.1 to Fig. 5.2, Boris has scale attribute young while Cyril who is only 1 year older is considered middle aged but not young. A series of questions arises like why to separate Boris and Cyril by their ages when their ages are very similar? This kind of arbitrariness is, in our opinion, an apparent disadvantage of the ordinary concept of scale and scaling.

Example 5. *A practical consequence of what we just described is the following. For a formal concept $\langle A_1, B_1\rangle$ of the concept lattice corresponding to Fig. 5.2 which is generated by* a_y*, i.e.,* $A_1 = \{\mathrm{a}_y\}^{\downarrow}$*, we have*

$$A_1 = \{\textit{Alice, Boris, Ellen, George}\},$$

i.e. the formal concept does not apply to Cyril at all. On the other hand, for a formal concept $\langle A_2, B_2\rangle$ which is generated by a_m*, i.e.,* $A_2 = \{\mathrm{a}_m\}^{\downarrow}$*, we have*

$$A_2 = \{\textit{Cyril, David}\},$$

i.e. the formal concept does not apply to Boris at all. That is, these formal concepts completely separate Boris and Cyril.

Therefore, admissible scales seem to capture the intuitive requirement to take the underlying similarities $\approx_y$ into account.

Given scales for a data table with general attributes, we can transform the data table into a table with fuzzy attributes. The following definition says how to do it.

Definition 14 (plain scaling)**.** *For a data table $\mathcal{D} = \langle X, Y, W, I\rangle$ (as above), scales $\mathbb{S}_y$ $(y \in Y)$, the derived table with fuzzy attributes (w.r.t. plain scaling) is a table $\langle X, Z, J\rangle$ with fuzzy attributes defined by*

- $Z = \bigcup_{y \in Y}\{y\} \times Y_y$,
- $J(x, \langle y, z\rangle) = I_y(w, z)$ *for* $y(x) = w$.

Denote by $\mathcal{B}(\mathcal{D}, \mathbb{S})$, where for $\mathcal{D} = \langle X, Y, W, I\rangle$ and $\mathbb{S} = \{\mathbb{S}_y \mid y \in Y\}$, the fuzzy concept lattice corresponding to the derived table, i.e.,

$$\mathcal{B}(\mathcal{D}, \mathbb{S}) = \mathcal{B}(X^{\uparrow}, Z^{\downarrow}, J),$$

where $\langle X, Z, J\rangle$ is the table with fuzzy attributes derived from $\mathcal{D}$ and $\mathbb{S}$.

Example 6. *A derived table corresponding to table with general attributes from Fig. 5.1, scales from Fig. 5.4 and Fig. 5.5, and a trivial scale for attribute symptom, is just the* **L***-context shown in Fig. 5.3.*

Example 7. *We are going to demonstrate that the undesirable effect of separation of Boris and Cyril by both formal concepts generated by scale attributes corresponding to "young" and "middle" is not present when we use scaling to fuzzy attributes, cf. Example 5. Namely, for a formal fuzzy concept* $\langle A_1, B_1\rangle$ *of the concept lattice corresponding to Fig. 5.3 which is generated by* a_y, *i.e.,* $A_1 = \{{}^{1}\!/\mathrm{a}_y\}^{\downarrow}$, *we have*

$$A_1 = \{\mathit{Alice},\ \mathit{Boris},\ {}^{0.5}\!/\mathit{Cyril},\ \mathit{Ellen},\ \mathit{George}\},$$

i.e. the formal concept partially covers Cyril. On the other hand, for a formal concept $\langle A_2, B_2\rangle$ *which is generated by* a_m, *i.e.,* $A_2 = \{{}^{1}\!/\mathrm{a}_m\}^{\downarrow}$, *we have*

$$A_2 = \{{}^{0.5}\!/\mathit{Alice},\ {}^{0.5}\!/\mathit{Boris},\ \mathit{Cyril},\ \mathit{David},\ {}^{0.5}\!/\mathit{Ellen}, {}^{0.5}\!/\mathit{George}\},$$

i.e. the formal concept partially covers Boris (and also Alice, Ellen, and George).

The next example shows that scaling using fuzzy attributes is beneficial from the point of attribute dependencies.

Example 8. *Consider data table in Fig. 5.3 and a fuzzy attribute implication*

$$\{{}^{0.5}\!/\mathrm{a}_y, {}^{0.5}\!/\mathrm{a}_m, {}^{0.5}\!/\mathrm{h}_s, {}^{0.5}\!/\mathrm{h}_m\} \Rightarrow \{\mathrm{sym}\}. \tag{5.1}$$

We refer a reader to [16] for an overview on fuzzy attribute implications. Without going to details on semantics of fuzzy attribute implications, one can intuitively see that (5.1) is true in degree 1 *in the data from Fig. 5.3 (this is the case when we use globalization in the definition of truth degree of fuzzy attribute implications). This says that, in the data, a person who is young to degree at least* 0.5, *middle-aged to degree at least* 0.5, *short to degree at least* 0.5, *and medium-high to degree at least* 0.5, *has the symptom. That is, a person who is in between young and middle-aged and in between short and medium-high, has the symptom.*

If we use scaling with crisp attributes, i.e., one gets the data from Fig. 5.2, and use ordinary attribute implications, the situation is different. Namely, the following ordinary attribute implications which are related to (5.1) can be considered:

$$\{\mathrm{a}_m, \mathrm{h}_m\} \Rightarrow \{\mathit{symptom}\}, \tag{5.2}$$
$$\{\mathrm{a}_y, \mathrm{h}_s\} \Rightarrow \{\mathit{symptom}\}, \tag{5.3}$$
$$\{\mathrm{a}_y, \mathrm{h}_m\} \Rightarrow \{\mathit{symptom}\}, \tag{5.4}$$
$$\{\mathrm{a}_m, \mathrm{h}_s\} \Rightarrow \{\mathit{symptom}\}. \tag{5.5}$$

One can easily see that both (5.2) and (5.3) are true, but (5.4) and (5.5) are not true in Fig. 5.2. One difference w.r.t. (5.1) is that in scaling with fuzzy attributes, one has a single fuzzy attribute implication, while with ordinary attributes we have two attribute implications describing the dependency of symptom on age and height. Note also that both (5.2) and (5.3) are supported just

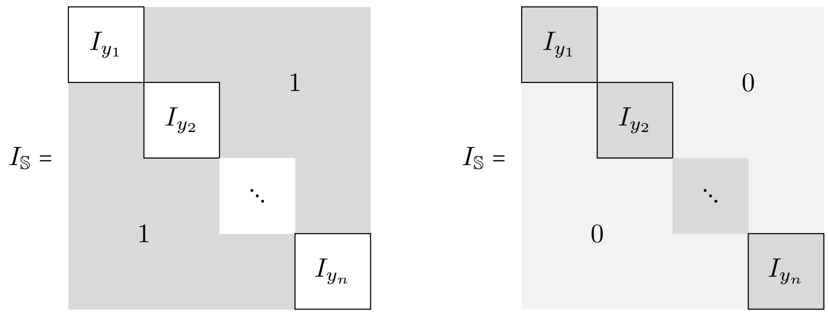


Figure 5.6: Schema of a scale **L**-context as direct sum of I_ys (left) and as direct product of I_ys (right)

by a single row in Fig. 5.2 and are, therefore, too specific. More importantly, however, if one considers the original data from Fig. 5.1, our contention is that (5.1) naturally captures the dependency of symptom on age and height.

Now, we demonstrate, that the plain scaling is a special case of c-morphism.

First, we construct a formal (crisp) context $\langle X, D, I_{\mathcal{D}} \rangle$ associated to data table $\mathcal{D} = \langle X, Y, W, I \rangle$ as follows. Let $D = \bigcup_{y \in Y} \{y\} \times D_y$ and $\langle x, \langle y, w \rangle \rangle \in I_{\mathcal{D}}$ iff $y(x) = w$.

Second, we construct **L**-context $\mathbb{S}$ as a direct sum (Fig. 5.6 (left)) of the attribute scales $\mathbb{S} = \langle D, Z, I_{\mathbb{S}} \rangle$ where $Z = \bigcup_{y \in Y} \{y\} \times Y_y$ (as in Definition 14) and

$$I_{\mathbb{S}}(\langle y_1, w \rangle, \langle y_2, z \rangle) = \begin{cases} 1 & \text{if } y_1 \neq y_2, \\ I_{y_1}(w_1, z) & \text{otherwise.} \end{cases}$$

Observe, that $I_{\mathcal{D}} \triangleleft I_{\mathbb{S}}$ is equal to J from Definition 14. Indeed, $J(x, \langle y, z \rangle) = I_y(w, z)$ for $y(x) = w$ that is equivalent to $I_y(w, z)$ for $\langle x, \langle y, w \rangle \rangle \in I_{\mathcal{D}}$; that can be rewritten as $\bigwedge_{\langle y,w \rangle} I_{\mathcal{D}}(\langle x, \langle y, w \rangle \rangle) \to I_y(w, z) = \bigwedge_{\langle y,w \rangle} I_{\mathcal{D}}(\langle x, \langle y, w \rangle \rangle) \to I_{\mathbb{S}}(\langle y, w \rangle, \langle y, z \rangle)$, hence $J = I_{\mathcal{D}} \triangleleft I_{\mathbb{S}}$

Thus plain scaling can be expressed as composition of **L**-relations and by Lemma 21 plain scaling is a c-morphism.

5.2 Sensitivity Issues in Scaling: a Theoretical Insight

One of the above-mentioned disadvantages of ordinary scaling is that it is very sensitive to user's selection of scale attributes. A very small difference in the definition of scale attributes may lead to a large difference in the resulting concept lattices. For instance, if we define attribute a_y ("young") as applying to ages from $[0, 31]$ instead of $[0, 30]$, the concepts of the resulting concept lattice will sharply change in that some objects disappear from some concepts and will appear in other concepts (e.g., Cyril disappears from a formal concept generated by "middle" and will appear in a formal concept generated by "young"). This is not desirable. Where shall a user draw a line between "young" and "middle"? This is a question we can not get rid of when using ordinary scaling.

We argued above that this effect can be mitigated when scaling with fuzzy attributes is used. However, a problem regarding the arbitrariness of boundaries, which a user defines for scale attributes, remains. The boundaries are now membership functions of scale attributes which are now fuzzy attributes. The basic question is: What happens if instead of scale $\mathbb{S}_y^1 = \langle X_y, Y_y, I_y^1 \rangle$, a user selects scale $\mathbb{S}_y^2 = \langle X_y, Y_y, I_y^2 \rangle$ which has a similar membership function, i.e., when $I_y^1(x, y)$ is close to $I_y^2(w, z)$ for any $w \in X_y$ and $z \in Y_y$? Suppose we have two sets of scales, say $\mathbb{S}^1 = \{\mathbb{S}_y^1 \,|\, y \in Y\}$ and $\mathbb{S}^2 = \{\mathbb{S}_y^2 \,|\, y \in Y\}$, such that $\mathbb{S}_y^1$ is similar to $\mathbb{S}_y^2$ for each $y \in Y$. Is it true, then, that the resulting concept lattices $\mathcal{B}(\mathcal{D}, \mathbb{S}^1)$ and $\mathcal{B}(\mathcal{D}, \mathbb{S}^2)$ are similar in some natural way of measuring similarity of concept lattices? In what follows, we are going to provide a positive answer to this question.

Let us first introduce a degree $\mathbb{S}_y^1 \approx \mathbb{S}_y^2$ to which the scales $\mathbb{S}_y^1$ and $\mathbb{S}_y^2$ are similar.

Definition 15. *For scales* $\mathbb{S}_y^1 = \langle D_y, Y_y, I_y^1 \rangle$, $\mathbb{S}_y^2 = \langle D_y, Y_y, I_y^2 \rangle$ *put*

$$(\mathbb{S}_y^1 \approx \mathbb{S}_y^2) = \bigwedge_{w \in D_y, z \in Y_y} I_y^1(w, z) \leftrightarrow I_y^2(w, z).$$

That is, $\mathbb{S}_y^1 \approx \mathbb{S}_y^2$ is a degree to which the membership functions of scale attributes z in $\mathbb{S}_y^1$ and $\mathbb{S}_y^2$ are similar if truth function $\leftrightarrow$ fuzzy equivalence is chosen to assess similarity of membership degrees.

For $\mathbb{S}_y^1 = \langle D_y, Y_y, I_y^1 \rangle$, $\mathbb{S}_y^2 = \langle D_y, Y_y, I_y^2 \rangle$ $(y \in Y)$, and $\mathbb{S}^1 = \{\mathbb{S}_y^1 \,|\, y \in Y\}$, $\mathbb{S}^2 = \{\mathbb{S}_y^2 \,|\, y \in Y\}$, put

$$(\mathbb{S}^1 \approx \mathbb{S}^2) = \bigwedge_{y \in Y} (\mathbb{S}_y^1 \approx \mathbb{S}_y^2).$$

That is, $\mathbb{S}^1 \approx \mathbb{S}^2$ is a degree to which the corresponding scales from $\mathbb{S}^1$ and $\mathbb{S}^2$ are similar.

Then we get the following theorem.

Theorem 40. *For a data table* $\mathcal{D} = \langle X, Y, W, I \rangle$, *scales* $\mathbb{S}_y^1 = \langle D_y, Y_y, I_y^1 \rangle$, $\mathbb{S}_y^2 = \langle D_y, Y_y, I_y^2 \rangle$ $(y \in Y)$, *and* $\mathbb{S}^1 = \{\mathbb{S}_y^1 \,|\, y \in Y\}$, $\mathbb{S}^2 = \{\mathbb{S}_y^2 \,|\, y \in Y\}$ *we have*

$$(\mathbb{S}^1 \approx \mathbb{S}^2) \leq (\mathcal{B}(\mathcal{D}, \mathbb{S}^1) \approx \mathcal{B}(\mathcal{D}, \mathbb{S}^2)).$$

Proof. Follows from Theorem 31 using the fact that $I_{\mathbb{S}^1} \approx I_{\mathbb{S}^2} = \mathbb{S}^1 \approx \mathbb{S}^2$. □

Theorem 40 says: "if scales $\mathbb{S}_1$ and $\mathbb{S}_2$ are similar then the resulting concept lattices $\mathcal{B}(\mathcal{D}, \mathbb{S}^1)$ and $\mathcal{B}(\mathcal{D}, \mathbb{S}^2)$ are similar too". Note that this is the exact meaning of Theorem 40 (one can see this by invoking basic rules of semantics of fuzzy logic). Therefore, small changes in definition of a scale lead to small changes in the resulting concept lattices which is a desirable property.

We can define similarity even for scales with different attributes. In this case we consider similarity as truth value of following statement: "For each combination of attributes in $\mathbb{S}^1_y$ there is an equal (similar) combination attribute in $\mathbb{S}^2_y$ and vice versa." To distinguish it from previously defined similarity we denote it by $\approx_{\mathrm{d}}$.

Definition 16. *For attribute scales* $\mathbb{S}^1_y = \langle D_y, Y^1_y, I^1_y \rangle$, $\mathbb{S}^2_y = \langle D_y, Y^2_y, I^2_y \rangle$ *put*

$$\begin{aligned}(\mathbb{S}^1_y \approx_{\mathrm{d}} \mathbb{S}^2_y) = &\textstyle\bigwedge_{Z_1 \in L^{Y^1_y}} \bigvee_{Z_2 \in L^{Y^2_y}} (Z_1{}^{\downarrow_{I^1_y}} \approx Z_2{}^{\downarrow_{I^2_y}}) \\ &\wedge \\ &\textstyle\bigwedge_{Z_2 \in L^{Y^2_y}} \bigvee_{Z_1 \in L^{Y^1_y}} (Z_1{}^{\downarrow_{I^1_y}} \approx Z_2{}^{\downarrow_{I^2_y}}).\end{aligned}$$

For $\mathbb{S}^1 = \{\mathbb{S}^1_y \mid y \in Y\}, \mathbb{S}^2 = \{\mathbb{S}^2_y \mid y \in Y\}$ *define*

$$\mathbb{S}^1 \approx_{\mathrm{d}} \mathbb{S}^2 = \bigwedge_{y \in Y} (\mathbb{S}^1_y \approx_{\mathrm{d}} \mathbb{S}^2_y)$$

Obviously, this similarity coincides with similarity of corresponding concept lattices

$$(\mathbb{S}^1_y \approx_{\mathrm{d}} \mathbb{S}^2_y) = \mathcal{B}(D_y, Y^1_y, I^1_y) \approx_{\mathrm{Ext}} \mathcal{B}(D_y, Y^2_y, I^2_y)$$

and thus

$$(\mathbb{S}^1_y \approx \mathbb{S}^2_y) \leq (\mathbb{S}^1_y \approx_{\mathrm{d}} \mathbb{S}^2_y).$$

More interestingly, we have

$$(\mathbb{S}^1 \approx_{\mathrm{d}} \mathbb{S}^2) = \mathcal{B}(X^{\uparrow}, Y^{1\downarrow}, I_{\mathbb{S}^1}) \approx_{\mathrm{Ext}} \mathcal{B}(X^{\uparrow}, Y^{2\downarrow}, I_{\mathbb{S}^2})$$

since $\mathcal{B}(X^{\uparrow}, Y^{1\downarrow}, I_{\mathbb{S}})$ is Cartesian product of $\mathcal{B}(D_y, Y_y, I_y)$s.

Using this similarity we can obtain following theorem.

Theorem 41. *Let* $\mathcal{D} = \langle X, Y, W, I \rangle$ *be a data table and let* $\mathbb{S}^1 = \{\mathbb{S}^1_y \mid y \in Y\}, \mathbb{S}^2 = \{\mathbb{S}^2_y \mid y \in Y\}$ *be scales with different attributes. Then we have*

$$\mathbb{S}^1 \approx_{\mathrm{d}} \mathbb{S}^2 \leq \mathcal{B}(\mathcal{D}, \mathbb{S}^1) \approx_{\mathrm{Ext}} \mathcal{B}(\mathcal{D}, \mathbb{S}^2).$$

Proof. By straightforward application of Theorem 31 and the discussion following Definition 16. □

Another useful result, which we are now going to present concerns the concept of admissibility of a scale w.r.t. similarity relations $\approx_y$. Suppose we have $\approx_y$ for each $y \in Y$. Note that if we do not want to consider a similarity on domain D_y, we may take the identity relation for $\approx_y$. Using $\approx_y$'s, one can define a degree $x_1 \approx_X x_2$ of similarity of objects $x_1, x_2 \in X$ for a given table $\langle X, Y, W, I \rangle$ by

$$(x_1 \approx_X x_2) = \textstyle\bigwedge_{y \in Y} y(x_1) \approx_y y(x_2).$$

That is, $x_1 \approx_X x_2$ is a degree to which for every attribute $y \in Y$, the value of x_1 on attribute y is similar to the value of x_2 on attribute y.

Then, one can prove the following theorem.

Theorem 42. *Consider a data table $\mathcal{D} = \langle X, Y, W, I\rangle$ (as above), $\mathbf{L}$-equivalences $\approx_y$ on X_y ($y \in Y$), and a collection of scales $\mathbb{S} = \{\mathbb{S}_y \,|\, y \in Y\}$ which are admissible w.r.t. $\approx_y$'s. Then for each formal concept $\langle C, D\rangle$ from $\mathcal{B}(\mathcal{D}, \mathbb{S})$ and any objects $x_1, x_2 \in X$ we have*

$$C(x_1) \otimes (x_1 \approx_X x_2) \leq C(x_2).$$

Proof. First, observe that for $H \in L^D$ we have $H^{\cup I_{\mathcal{D}}}(x) = \bigwedge_{y \in Y} H(\langle y, y(x)\rangle)$. Indeed, we have

$$\begin{aligned} H^{\cup I_{\mathcal{D}}}(x) &= \bigwedge_{\langle y,w\rangle \in D} I_{\mathcal{D}}(x, \langle y, w\rangle) \to H(\langle y, w\rangle) \\ &= \bigwedge_{\langle y,w\rangle \in D} I_y(x, \langle y, w\rangle) \to H(\langle y, w\rangle) \\ &= \bigwedge_{y \in Y} H(\langle y, y(x)\rangle) \end{aligned}$$

since $I_{\mathcal{D}}$ is a crisp relation.

From definition of admissibility we have

$$(y(x_1) \approx_y y(x_2)) \otimes I_y(y(x_1), z) \leq I_y(y(x_2), z)$$

for each $x_1, x_2 \in X, z \in Y, z \in D_y$.

Since $I_y(y(x_1), z) = I_{\mathbb{S}}(\langle y, w\rangle, \langle y, z\rangle) = \{\langle y, z\rangle\}^{\downarrow I_{\mathbb{S}}}(\langle y, w\rangle)$ we get

$$(y(x_1) \approx_y y(x_2)) \leq \{\langle y, z\rangle\}^{\downarrow I_{\mathbb{S}}}(\langle y, y(x_1)\rangle) \to \{\langle y, z\rangle\}^{\downarrow I_{\mathbb{S}}}(\langle y, y(x_2)\rangle)$$

for each $y \in Y$. Whence

$$\begin{aligned} \bigwedge_{y \in Y} (y(x_1) \approx_y y(x_2)) &\leq \bigwedge_{y \in Y} (\{\langle y, z\rangle\}^{\downarrow I_{\mathbb{S}}}(\langle y, y(x_1)\rangle) \leftrightarrow \{\langle y, z\rangle\}^{\downarrow I_{\mathbb{S}}}(\langle y, y(x_2)\rangle)) \\ &\leq \bigwedge_{y \in Y} (\{\langle y, z\rangle\}^{\downarrow I_{\mathbb{S}}}(\langle y, y(x_1)\rangle)) \leftrightarrow \bigwedge_{y \in Y} (\{\langle y, z\rangle\}^{\downarrow I_{\mathbb{S}}}(\langle y, y(x_2)\rangle)) \\ &\leq \{\langle y, z\rangle\}^{\downarrow I_{\mathbb{S}} \cup I_{\mathcal{D}}}(x_1) \leftrightarrow \{\langle y, z\rangle\}^{\downarrow I_{\mathbb{S}} \cup I_{\mathcal{D}}}(x_2) \\ &\leq B^{\downarrow I_{\mathbb{S}} \cup I_{\mathcal{D}}}(x_1) \leftrightarrow B^{\downarrow I_{\mathbb{S}} \cup I_{\mathcal{D}}}(x_2) \\ &= A(x_1) \leftrightarrow A(x_2). \end{aligned}$$

from which we immediately get the proposition. □

Note that Theorem 42 can be read as follows: If two objects are similar, they will not get separated by any formal concept of the derived table.

Using the alternative measurement of similarity $\approx_{\mathrm{d}}$ of attribute scales we can state that using similar scales on similar data tables we obtain similar concept lattices.

Theorem 43. *Le $\mathcal{D}_2 = (X, Y, W, I_1), \mathcal{D}_2 = (X, Y, W, I_2)$ be two data tables. For $i = 1, 2$ let $\mathbb{S}^i$ be a system $\{\mathbb{S}^i_y \,|\, y \in Y\}$ of scales admissible to $\approx_y$. Then we have*

$$(y_1(x) \approx_y y_2(x)) \otimes (\mathbb{S}^1 \approx_{\mathrm{d}} \mathbb{S}^2) \leq \mathcal{B}(\mathcal{D}_1, \mathbb{S}^1) \approx_{\mathrm{Ext}} \mathcal{B}(\mathcal{D}_2, \mathbb{S}^2)$$

Proof. Follows from Theorem 31 and Lemma 38. □

Remark 19. *We obtain similar results for isotone concept-forming operators if we define $\mathcal{B}(\mathcal{D}, \mathbb{S}) = \mathcal{B}(X^{\cap}, Z^{\cup}, J)$, construct $I_{\mathbb{S}}$ as direct product of attribute scales (see Fig. 5.6 (right)), and use $\circ$-composition in proofs of related theorems. The illustrative example in the next section uses isotone concept forming operators.*

5.3 Illustrative Example

We borrow the example data table "Cars" from [39]. The data table contains verbal descriptions of consumption and speed of four cars (Fig. 5.3). [39] suggests to replace each such verbal descriptions by an **L**-sets modeling their semantics and then use special concept-forming operators. In can be easily shown that this approach is equivalent to that one described in this chapter using the semantic **L**-sets as rows of the corresponding attribute scales (Fig. 5.8; three-element Łukasiewicz chain is used as the structure of truth degrees) and using isotone concept-forming operators. We use this example to show several applications of theory the from previous chapters.

Using the system of attribute scales $\mathbb{S}^1 = \{\mathbb{S}_{\text{consumption}}, \mathbb{S}_{\text{speed}}\}$ (this one corresponds to semantics used in [39]) we obtain concept lattice $\mathcal{B}(\mathcal{D}, \mathbb{S}^1)$ (Fig. 5.9). A domain expert could argue that semantics of "not as fast as F_2" is not modeled properly and that scale $\mathbb{S}'_{\text{speed}}$ should be used instead of $\mathbb{S}_{\text{speed}}$ for the attribute speed. With the altered system of scales $\mathbb{S}^2 = \{\mathbb{S}_{\text{consumption}}, \mathbb{S}'_{\text{speed}}\}$ we obtain concept lattice $\mathcal{B}(\mathcal{D}, \mathbb{S}^2)$ (Fig. 5.10).

Natural question is: how is $\mathcal{B}(\mathcal{D}, \mathbb{S}^2)$ related to $\mathcal{B}(\mathcal{D}, \mathbb{S}^1)$? As we can observe that semantics of "not as fast as F_2" from $\mathbb{S}_{\text{speed}}$ is union of "not as fast as F_2" and "quite fast" from $\mathbb{S}'_{\text{speed}}$. Thus we can easily find a binary relation A s.t. $A \circ I_{\mathbb{S}'} = I_{\mathbb{S}}$. By Theorem 8 we obtain that $\mathrm{Int}(\mathcal{D}, \mathbb{S}^1) \subseteq \mathrm{Int}(\mathcal{D}, \mathbb{S}^2)$.

Moreover, by results on similarity of concept lattices, since semantics of "not as fast as F_2" in $\mathbb{S}_{\text{speed}}$ and $\mathbb{S}'_{\text{speed}}$ are similar in degree 0.5, $\mathcal{B}(\mathcal{D}, \mathbb{S}^1)$ and $\mathcal{B}(\mathcal{D}, \mathbb{S}^2)$ are similar at least in degree 0.5.

Now, consider different attribute scale $\mathbb{S}^*_{\text{speed}}$ for speed (Fig. 5.8 (bottom right)) which uses miles per hour instead of km per hour. Notice that in $\mathbb{S}^*_{\text{speed}}$ is defined by **L**-relation between different sets than $\mathbb{S}_{\text{speed}}$ and after rescaling to the same units they do not match. Nevertheless the shapes of the **L**-relations are the same. We have $\mathrm{Ext}(\mathbb{S}_{\text{speed}}) = \mathrm{Ext}(\mathbb{S}^*_{\text{speed}})$, and thus $\mathbb{S}_{\text{speed}} \approx_{\mathrm{d}} \mathbb{S}^*_{\text{speed}} = 1$.

	consumption	speed
F_1	quite high	fast
F_2	8-10l/100 km	quite fast
F_3	at least 8l/100 km	not so fast as F_2
F_4	at least 8l/100 km	fast

Figure 5.7: Data table "Cars" from [39].

Finally, concept lattice w.r.t. this attribute scale has the same set of extents as $\mathcal{B}(\mathcal{D}, \mathbb{S}^1)$ (similarity of the two concept lattices is 1), and the two concept lattices are isomorphic.

5.4 Summary and Future Research

We presented an approach to scaling using fuzzy attributes (as an application of theory described in the previous chapters) and argued that such scaling overcomes some problematic aspects of scaling using ordinary attributes.

Our future research will focus on:

- Merge with the general framework [9]. This will bring possibility to scale not only to different attributes but also to different structures of truth-degrees.
- Study of connections of scaling to fuzzy attribute implications.

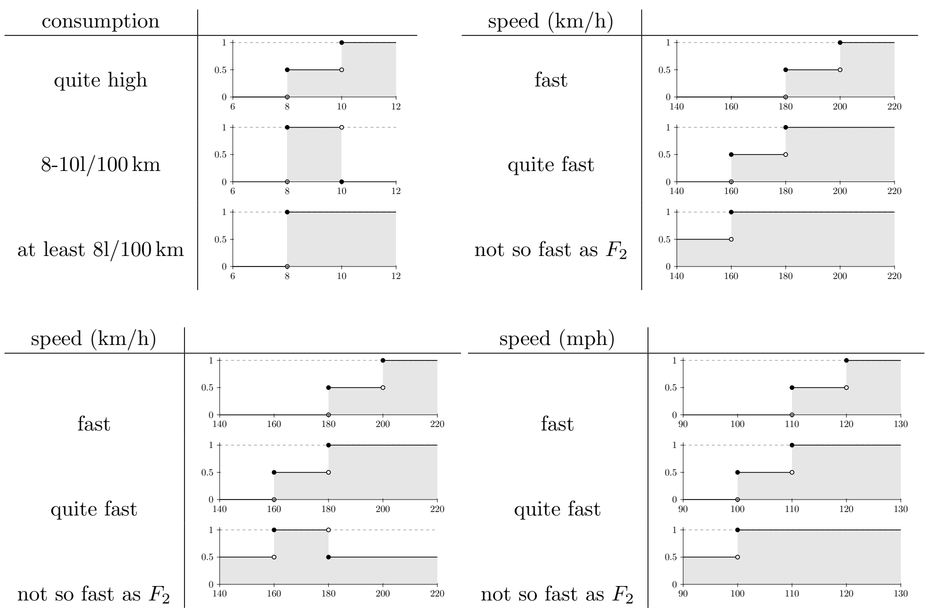


Figure 5.8: Attribute scale $\mathbb{S}_{\text{consumption}}$ for *consumption* (top left) and three different attribute scales $\mathbb{S}_{\text{speed}}$ (top right), $\mathbb{S}'_{\text{speed}}$ (bottom left), $\mathbb{S}^{*}_{\text{speed}}$ (bottom right) for attributes *consumption* and *speed* of data table "Cars".

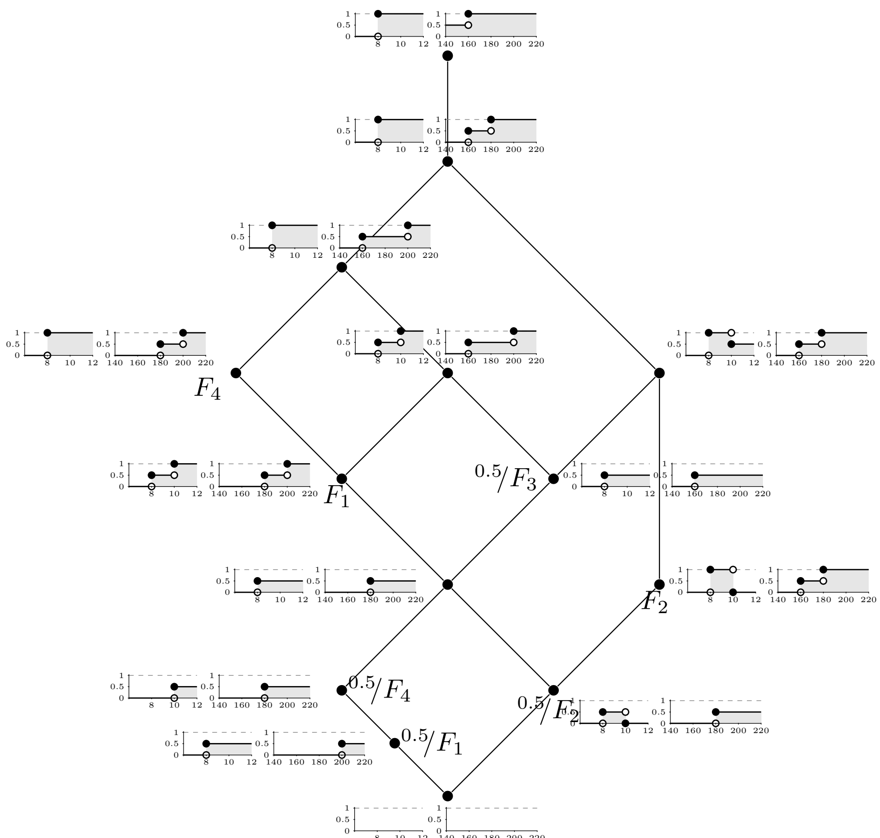


Figure 5.9: Concept lattice $\mathcal{B}(\mathbb{S}_1, \mathcal{D})$ of data table "Cars" (Fig. 5.3) with scale $\mathbb{S}_1$ (Fig. 5.8)

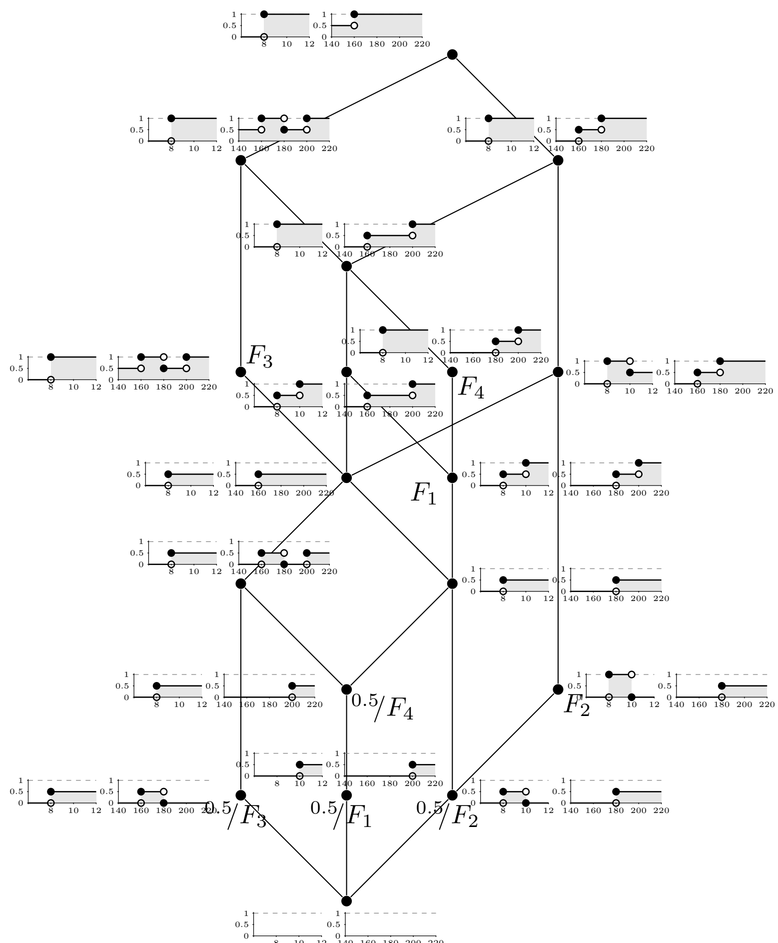


Figure 5.10: Concept lattice $\mathcal{B}(\mathbb{S}_2, \mathcal{D})$ of data table "Cars" (Fig. 5.3) with scale $\mathbb{S}_2$ (Fig. 5.8)

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