ISOTONE FUZZY CONCEPT-FORMING OPERATORS IN FORMAL CONCEPT ANALYSIS

BY
JAN KONECNY

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Abstract—The thesis develops mathematical foundations of isotone fuzzy Galois connections and the associated concept lattices and attribute implications. In particular, we study isotone fuzzy Galois connections and concept lattices parameterized by linguistic hedges. Isotone fuzzy Galois connections and concept lattices provide an alternative to antitone fuzzy Galois connections and concept lattices. We show that hedges enable us to control the number of formal concepts, i.e. collections of objects and attributes which represent interesting clusters in data. We present properties of isotone connections with hedges, including their axiomatization, and describe the structure of the associated concept lattices. In addition, we present a logic of if-then rules such as "if all attributes of an object are among those from A then they are among those from B." We provide basic syntactic and semantic notions, describe complete non-redundant sets of the if-then rules, and a logic for reasoning with such dependencies with its ordinary-style and graded-style completeness.

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Chapter 1

Introduction

Formal concept analysis (FCA) [15] is a method of analysis of relational data invented in Germany by Rudolf Wille. In the 1980s, solid mathematical and computational foundations of FCA have been developed. In the past decade or so, FCA enjoyed a considerable interest in various communities and many papers on applications of FCA in various domains appeared, including papers in premier journals and conferences. The method is based on formalization of a philosophical view of a conceptual knowledge. Basic notion in FCA is a formal concept which consists of two sets: extent – set of all objects sharing the same attributes, and intent – set of all the shared attributes. This definition of formal concept comes from traditional / Port-Royal logic [2, 23].

The basic input data for FCA, called a formal context, is a flat table in which rows represent objects and columns represent attributes. Entries of the table contain either a cross $\times$, which means that the corresponding object has the corresponding attribute, or a blank which means the opposite. One of the main outputs of FCA is a concept lattice – hierarchy of formal concepts present in the formal context. Extent and intent of the formal concepts are formed by particular operators, antitone Galois connections, induced by the formal context.

In the human thinking, it is natural to assume groups of objects whose attributes belong to the same set. For instance, consider a formal context containing people as objects; among attributes are: $age1, age2, \ldots, age100$ representing age in years. Each object has exactly one of the age attributes. It is natural to think, for example, about a group of people with age between 20 and 30 years. This group of people does not emerge as extent in concept lattice when antitone Galois connections are used as concept-forming operators since all people with age between 20 and 30 years do not share the same age. To obtain such groups as extents, different concept-forming operators must be used: isotone Galois connections. In a formal concept formed by isotone Galois connection, the extent is a set in which no object has other attributes than those in the intent; the intent is a set of all attributes of objects in the extent.

In everyday life we use concepts which are not sharply bounded (e.g. 'great dancer' or 'middle aged man'). In terms of FCA, objects and attributes do not need to belong to a formal concept in full degree. Similarly the relation between objects and attributes is a matter of degree. There are several approaches to generalize formal concept analysis to be able to process such indeterminancy or uncertainty [5, 6, 31, 25, 22, 1]. Many of them are based on Zadeh's theory of fuzzy sets [40]. Particularly, [5] generalizes antitone Galois connection to antitone fuzzy Galois connections which form extents and intents of formal concepts as fuzzy sets. An analogous generalization of isotone Galois connections is studied in this thesis.

The isotone Galois connections were shown to play an important role in Boolean factor analysis. Boolean factor analysis [37] concerns with reduction of space dimension of logical data. Its goal is to decompose a table describing a relation between objects and attributes (in fact, a formal context) into two tables: one describing relation between objects and factors, second describing relation between factors and objects, such that number of factors is as small as possible and composition of these two relation (with standard relational product) is the original relation. We can read the composition as: "object has an attribute iff there is a factor such that the object has the factor and the factor is a manifestation of the attribute." Belohlavek and Vychodil proved that formal concepts formed by antitone Galois connections serve as optimal and universal factors [14]. We can be interested in relational products with different meaning, for instance "object has an attribute iff for each factor we have, if the object has the factor, then the factor is a manifestation of the attribute." This kind of relational product are called triangular product and they were studied by Bandler and Kohout [26, 27]. In [7], Belohlavek proved that optimal and universal factors are formal concepts formed by isotone Galois connections.

One of the main aims of formal concept analysis is to find methods to decrease number of formal concepts [35, 11, 33]. The number of formal concepts can be too big even for input data which are not too large. A large collection of formal concepts is not directly comprehensible by a user. Development of methods to overcome the problem is thus an important task. Belohlavek and Vychodil [11] generalized antitone fuzzy Galois connections using particular unary operations, linguistic hedges. The principal aim was to control, in parametrical way, the number of formal concepts. Isotone fuzzy Galois connections generalized in similar way are studied in this thesis.

Outline of the thesis

The thesis studies isotone Galois connections in a fuzzy setting and their generalizations which use linguistic hedges. Results presented in this thesis were published in the following papers (the numbers in square brackets are the numbers of the papers in the Bibliography).

[9] R. Belohlavek and J. Konecny. A logic of attribute containment. In *KAM08*, pages 246–251, Wuhan, China, 2008.

[24] J. Konecny. Isotone fuzzy Galois connections with hedges. *Information Sciences*, 181(10):1804–1817, 2011. Special Issue on Information Engineering Applications Based on Lattices.

[3] E. Bartl, R. Belohlavek, J. Konecny, and V. Vychodil. Isotone Galois connections and concept lattices with hedges. In *IEEE IS 2008, Int. IEEE Conference on Intelligent Systems*, pages 15–24–15–28, Varna, Bulgaria, 2008.

The thesis is organized as follows. Chapter 2 describes basic notions of formal concept analysis, fuzzy sets, Galois connections, closure operators, interior operators, and linguistic hedges.

Chapter 3 describes properties and axiomatization of isotone fuzzy Galois connections generalized using particular unary operators – namely, truth-stressing hedge which corresponds to logical connective "very true" and truth-depressing hedge which corresponds to logical connective "slightly true". We describe properties and axiomatization of the generalized isotone Galois connections, structures of their fixpoints, correspondence with antitone Galois connections.

Chapter 4 concerns with influence of the linguistic hedges on number of fixpoints and provide an illustrative example.

Chapter 5 is dedicated to the second main output of formal concept analysis – particular data dependencies called attribute implications. The chapter studies the properties of new attribute dependencies which are associated to the isotone Galois connections, their non-redundant basis, and a logic for reasoning with these dependencies including an ordinal as well as a graded version of a completeness theorem.

Chapter 2

Preliminaries

We recall basic facts of residuated lattices, truth-stressing and truth-depressing hedges, and fuzzy sets.

2.1 Residuated lattices and fuzzy sets

We use complete residuated lattices as basic structures of truth-degrees. A complete residuated lattice [5, 19, 39] is a structure $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that

(i) $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice, i.e. a partially ordered set in which arbitrary infima and suprema exist;

(ii) $\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e. $\otimes$ is a binary operation which is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$;

(iii) $\otimes$ and $\rightarrow$ satisfy adjointness, i.e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$.

0 and 1 denote the least and greatest elements. The partial order of $\mathbf{L}$ is denoted by $\leq$. Throughout this thesis, $\mathbf{L}$ denotes an arbitrary complete residuated lattice.

Elements a of L are called truth degrees. $\otimes$ and $\rightarrow$ (truth functions of) "fuzzy conjunction" and "fuzzy implication".

Common examples of complete residuated lattices include those defined on a unit interval, (i.e. $L = [0, 1]$) or on a finite chain in a unit interval, e.g. $L = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$, $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding $\rightarrow$. The three most important pairs of adjoint operations on the unit interval are

$$\begin{array}{lrcl}
\text{Łukasiewicz:} & a \otimes b & = & \max(a+b-1,0) \\
 & a \to b & = & \min(1-a+b,1) \\
\text{Gödel:} & a \otimes b & = & \min(a,b) \\
 & a \to b & = & \begin{cases} 1 & a \leq b \\ b & \text{otherwise} \end{cases} \\
\text{Goguen (product):} & a \otimes b & = & a \cdot b \\
 & a \to b & = & \begin{cases} 1 & a \leq b \\ \frac{b}{a} & \text{otherwise} \end{cases}
\end{array}$$

An **L**-set (or fuzzy set) A in a universe set X is a mapping assigning to each $x \in X$ some truth degree $A(x) \in L$ where L is a support of a complete residuated lattice. The set of all **L**-sets in a universe X is denoted $\mathbf{L}^X$.

The operations with **L**-sets are defined componentwise. For instance, the intersection of **L**-sets $A, B \in \mathbf{L}^X$ is an **L**-set $A \cap B$ in X such that $(A \cap B)(x) = A(x) \wedge B(x)$ for each $x \in X$, etc. An **L**-set $A \in \mathbf{L}^X$ is also denoted $\{{}^{A(x)}/x \mid x \in X\}$. If for all $y \in X$ distinct from $x_1, x_2, \ldots, x_n$ we have $A(y) = 0$, we also write $\{{}^{A(x_1)}/x_1, {}^{A(x_2)}/x_1, \ldots, {}^{A(x_n)}/x_n\}$. If there is exactly one $x \in X$ s.t. $A(x) > 0$ (i.e. $A = \{{}^{A(x)}/x\}$) we call A a singleton.

Binary **L**-relations (binary fuzzy relations) between X and Y can be thought of as **L**-sets in the universe $X \times Y$. That is, a binary **L**-relation $I \in \mathbf{L}^{X \times Y}$ between a set X and a set Y is a mapping assigning to each $x \in X$ and each $y \in Y$ a truth degree $I(x, y) \in L$ (a degree to which x and y are related by I). An **L**-set $A \in \mathbf{L}^X$ is called crisp if $A(x) \in \{0, 1\}$ for each $x \in X$. Crisp **L**-sets can be identified with ordinary sets. For a crisp A, we also write $x \in A$ for $A(x) = 1$ and $x \notin A$ for $A(x) = 0$. An **L**-set $A \in \mathbf{L}^X$ is called empty (denoted by $\emptyset$) if $A(x) = 0$ for each $x \in X$. For $a \in L$ and $A \in \mathbf{L}^X$, $a \otimes A \in \mathbf{L}^X$ and $a \to A \in \mathbf{L}^X$ are defined by

$$(a \otimes A)(x) = a \otimes A(x) \text{ and } (a \to A)(x) = a \to A(x).$$

For universe X we define **L**-relation *graded subsethood* $L^X \times L^X \to L$ by:

$$S(A, B) = \bigwedge_{x \in X} A(x) \to B(x) \tag{2.1}$$

Graded subsethood generalizes the classical subsethood relation $\subseteq$ (note that unlike $\subseteq$, S is a binary **L**-relation on L^X. Described verbally, $S(A, B)$ represents a degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $S(A, B) = 1$. As a consequence, we have $A \subseteq B$ iff $A(x) \leq B(x)$ for each $x \in X$. In the following we use well-known properties of residuated lattices and fuzzy structures which can be found e.g. in [5, 19].

2.2 Linguistic hedges

We use unary operations called truth-stressing and truth-depressing hedges. Truth-stressing hedges were studied from the point of fuzzy logic as logical connectives "very

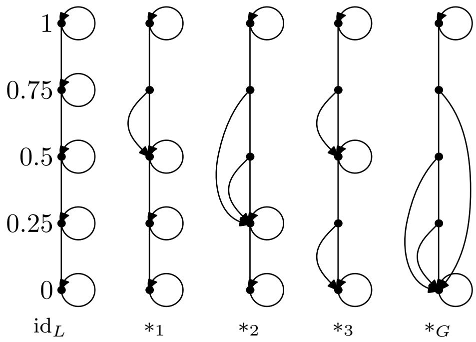


Figure 2.1: Truth-stressing hedges on 5-element chain with Łukasiewicz operations

true", see [21]. Our approach is close to that in [21]. A *truth-stressing hedge* is a mapping $^* : L \to L$ satisfying the following conditions

$$1^* = 1, \tag{2.2}$$

$$a^* \leq a, \tag{2.3}$$

$$(a \to b)^* \leq a^* \to b^*, \tag{2.4}$$

$$a^{**} = a^*, \tag{2.5}$$

for each $a, b \in L$. Truth-stressing hedges were used to parameterize antitone **L**-Galois connections e.g. in [4, 8, 13], and also to parameterize antitone **L**-Galois connections in [3].

On every complete residuated lattice **L**, there are two important truth-stressing hedges:

(i) identity, i.e. $a^* = a$ $(a \in L)$;

(ii) globalization, i.e.

$$a^* = \begin{cases} 1, & \text{if } a = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

Fig. 2.1 shows examples of truth-stressing hedges on 5-element chain with Łukasiewicz operations $\mathbf{L} = \langle\{0, 0.25, 0.5, 0.75, 1\}, \min, \max, \otimes, \to, 0, 1\rangle$. The left-most truth-stressing hedge $^{\mathrm{id}_L}$ is identity; the right-most truth-stressing hedge ^{*_G} is a globalization.

A *truth-depressing hedge* with respect to truth-stressing hedge * is a mapping $^\square$: $L \to L$ such that following conditions are satisfied

$$0^\square = 0 \tag{2.7}$$

$$a \leq a^\square \tag{2.8}$$

$$(a \to b)^* \leq a^\square \to b^\square \tag{2.9}$$

$$a^{\square\square} = a^\square \tag{2.10}$$

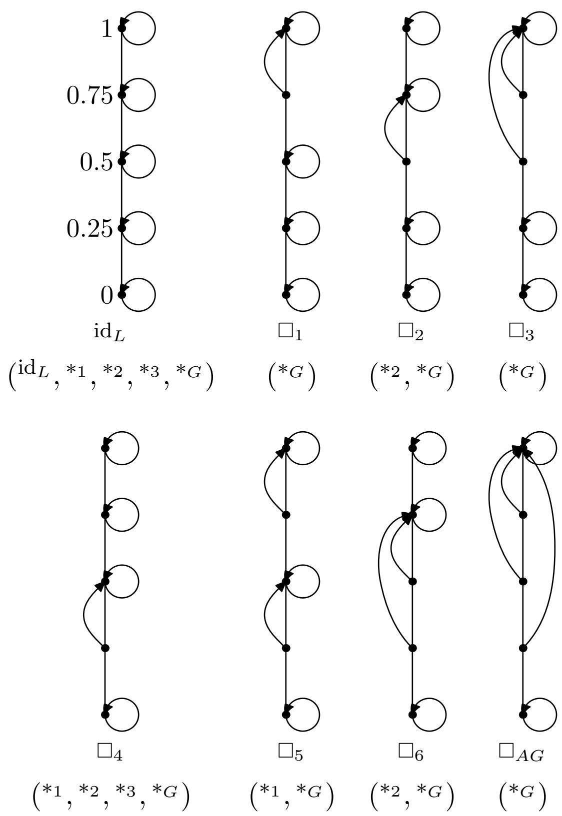


Figure 2.2: Truth-depressing hedges on 5-element chain with Łukasiewicz operations

for each $a, b \in L$. A truth-depressing hedge is a (truth function of) logical connective "slightly true", see [38].

On every complete residuated lattice $\mathbf{L}$, there are two important truth-depressing hedges:

(i) identity, i.e. $a^{\square} = a$ $(a \in L)$;

(ii) antiglobalization, i.e.

$$a^{\square} = \begin{cases} 0, & \text{if } a = 0, \\ 1, & \text{otherwise}. \end{cases} \tag{2.11}$$

Fig. 2.2 shows all truth-depressing hedges on 5-element chain with Łukasiewicz operations $\mathbf{L} = \langle \{0, 0.25, 0.5, 0.75, 1\}, \min, \max, \otimes, \rightarrow, 0, 1 \rangle$. In parentheses are listed the truth-stressing hedges for which the truth-depressing hedge satisfies (2.9). The left-most truth-depressing hedge in upper row $^{\mathrm{id}_L}$ is identity; the right-most truth-depressing hedge in lower row $^{\square_{AG}}$ is antiglobalization.

Remark 1. *(a) Note that from (2.4) follows that any truth-stressing hedge is monotone. If $a \le b$ then $(a \to b)^* = 1$. From (2.4) we have $1 \le a^* \to b^*$, i.e. $a \le b$ implies $a^* \le b^*$. Similarly, from (2.9) we have monotony of truth-depressing hedge.*

(b) The identity is a truth-depressing hedge with respect to any truth-stressing hedge.

(c) If $^\square$ is truth-depressing hedge w.r.t truth-stressing hedge * then $^\square$ is truth-depressing hedge w.r.t. globalization ^{*_G} (since $(a \to b)^{*_G} \le (a \to b)^* \le a^\square \to b^\square$). For that reason we do not declare the truth-stressing hedge for which the truth-depressing hedge satisfies (2.9), if it is not important.

We need following lemmas.

Lemma 1 ([8]). *A truth-stressing hedge * satisfies $(\bigvee_{i \in I} a_i^*)^* = \bigvee_{i \in I} a_i^*$.*

Lemma 2. *A truth-depressing hedge $^\square$ satisfies $(\bigwedge_{i \in I} a_i^\square)^\square = \bigwedge_{i \in I} a_i^\square$.*

Proof. "$\geq$" Obvious from the definition of truth-depressing hedge.

"$\leq$" Since we have $(\bigwedge_{i \in I} a_i^\square) \le a_i^\square$ and $^\square$ is monotone (see Remark 1(a)) we have $(\bigwedge_{i \in I} a_i^\square)^\square \le a_i^{\square\square} = a_i^\square$. Hence $(\bigwedge_{i \in I} a_i^\square)^\square \le \bigwedge_{i \in I} a_i^\square$.

□

2.3 Formal Concept Analysis

In this part we recall basics of formal concept analysis (FCA). The main aim in FCA is to extract interesting clusters (called formal concepts) from tabular data. A partially ordered collection of all formal concept is called a concept lattice. In the basic setting, the input data to FCA is organized in a table (formal context) such as the one in Table 2.1.

A *formal context* is a triplet $\langle X, Y, I \rangle$, where X and Y are sets of objects and attributes, respectively, and $I \subseteq X \times Y$ is a relation between X and Y. The fact that $\langle x, y \rangle \in I$ is interpreted as "object x has an attribute y".

Table 2.1: Formal context describing objects x_1, x_2, x_3 and their yes/no attributes y_1, y_2, y_3, y_4.

	y_1	y_2	y_3	y_4
x_1	1	1	0	0
x_2	0	1	1	0
x_3	0	0	1	1

A formal context $\langle X, Y, I \rangle$ *induces operators* $^{\Uparrow_I} : \mathbf{2}^X \to \mathbf{2}^Y$ and $^{\Downarrow_I} : \mathbf{2}^Y \to \mathbf{2}^X$:

$$A^{\Uparrow_I} = \{y \mid \langle x, y \rangle \in I \text{ for each } x \in A\} \tag{2.12}$$

$$B^{\Downarrow_I} = \{x \mid \langle x, y \rangle \in I \text{ for each } y \in B\}. \tag{2.13}$$

In words, we can describe the induced operators as follows: $A^{\Uparrow_I}$ is a set of all attributes shared by all objects from A. $B^{\Downarrow_I}$ is a set of all objects sharing all attributes from B.

A *formal concept* of $\langle X, Y, I\rangle$ is a pair $\langle A, B\rangle$ such that

$$A^{\Uparrow_I} = B \text{ and } B^{\Downarrow_I} = A. \tag{2.14}$$

The *set of all formal concepts* of $\langle X, Y, I\rangle$ is denoted $\mathcal{B}(X^{\Uparrow}, Y^{\Downarrow}, I)$:

$$\mathcal{B}(X^{\Uparrow}, Y^{\Downarrow}, I) = \{\langle A, B\rangle \mid A^{\Uparrow_I} = B \text{ and } B^{\Downarrow_I} = A\}. \tag{2.15}$$

A *subconcept-superconcept hierarchy* of formal concepts is a partial order $\leq$ defined a follows

$$\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle \text{ iff } A_1 \subseteq A_2 \tag{2.16}$$
$$\text{(or, equivalently, iff } B_2 \subseteq B_1\text{)} \tag{2.17}$$

for each $\langle A_1, B_1\rangle, \langle A_2, B_2\rangle \in \mathcal{B}(X^{\Uparrow}, Y^{\Downarrow}, I)$.

$\mathcal{B}(X^{\Uparrow}, Y^{\Downarrow}, I)$ with $\leq$ forms a complete lattice:

Theorem 3 (Main theorem of concept lattices, [15])**.** *Let $\langle X, Y, I\rangle$ be formal context. Then $\mathcal{B}(X^{\Uparrow}, Y^{\Downarrow}, I)$ is complete lattice whose infima and suprema are defined as follows:*

$$\bigwedge_{j\in J} \langle A_j, B_j\rangle = \langle \bigcap_{j\in J} A_j, (\bigcup_{j\in J} B_j)^{\Downarrow\Uparrow}\rangle \tag{2.18}$$
$$\bigvee_{j\in J} \langle A_j, B_j\rangle = \langle (\bigcup_{j\in J} A_j)^{\Uparrow\Downarrow}, \bigcap_{j\in J} B_j, \rangle \tag{2.19}$$

Moreover, an arbitrary complete lattice $\mathbf{K} = \langle K, \leq\rangle$ is isomorphic to $\mathcal{B}(X^{\Uparrow}, Y^{\Downarrow}, I)$ iff there are mappings $\mu : X \to K$, $\nu : Y \to K$ such that

1. *$\mu(X)$ is $\bigvee$-dense in K, $\nu(Y)$ is $\bigwedge$-dense in K;*
2. *$\mu(x) \leq \nu(y)$ iff $\langle x, y\rangle \in I$.*

[29] showed that operators $\langle {}^{\Uparrow_I}, {}^{\Downarrow_I}\rangle$ are in one-to-one correspondence with so-called *antitone Galois connections*. A pair $\langle {}^{\Uparrow}, {}^{\Downarrow}\rangle$ of mappings ${}^{\Uparrow} : 2^X \to 2^Y$, ${}^{\Downarrow} : 2^Y \to 2^X$ is said to form *antitone Galois connection* between sets X and Y if the following the conditions are satisfied:

(i) if $A_1 \subseteq A_2$ then $A_2^{\Uparrow} \subseteq A_1^{\Uparrow}$,

(ii) if $B_1 \subseteq B_2$ then $B_2^{\Downarrow} \subseteq B_1^{\Downarrow}$,

(iii) $A \subseteq A^{\Uparrow\Downarrow}$,

(iv) $B \subseteq B^{\Uparrow\Downarrow}$

for $A, A_1, A_2 \in 2^X$ and $B, B_1, B_2 \in 2^Y$.

The following theorem explains the correspondence:

Theorem 4 ([29]). *Let $\langle X, Y, I\rangle$ be formal context, $\langle {}^{\Uparrow}, {}^{\Downarrow}\rangle$ be an antitone Galois connection between X and Y. Then*

(i) $\langle {}^{\Uparrow_I}, {}^{\Downarrow_I}\rangle$ *is an antitone Galois connection.*

(ii) $I_{\langle \Uparrow, \Downarrow\rangle}$ *defined by*

$$I_{\langle \Uparrow, \Downarrow\rangle}(x, y) \text{ iff } y \in \left\{{}^1/x\right\}^{\Uparrow} \tag{2.20}$$

is a relation between X and Y and we have

(iii) $\langle {}^{\Uparrow}, {}^{\Downarrow}\rangle = \langle {}^{\Uparrow_{I_{\langle \Uparrow, \Downarrow\rangle}}}, {}^{\Downarrow_{I_{\langle \Uparrow, \Downarrow\rangle}}}\rangle$ *and* $I = I_{\langle \Uparrow_I, \Downarrow_I\rangle}$.

Chapter 3

Isotone Galois Connections with Hedges

3.1 Definition

We start by recalling the definition of and basic facts about the isotone fuzzy Galois connections [16, 32]:

Definition 1. *An isotone* **L***-Galois connection between sets* X *and* Y *is a pair* $\langle{}^{\Cap}, {}^{\Cup}\rangle$ *of mappings* ${}^{\Cap} : L^X \to L^Y$ *and* ${}^{\Cup} : L^Y \to L^X$ *satisfying* $S(A, B^{\Cup}) = S(A^{\Cap}, B)$.
(S is the graded subsethood (2.1))

Isotone **L**-Galois connections are sometimes called isotone fuzzy Galois connections. The following theorem provides an alternative definition using perhaps more comprehensible conditions [16].

Theorem 5. *A pair* $\langle{}^{\Cap}, {}^{\Cup}\rangle$ *of mappings* ${}^{\Cap} : L^X \to L^Y$ *and* ${}^{\Cup} : L^Y \to L^X$ *is an isotone Galois connection iff* ${}^{\Cap}$ *and* ${}^{\Cup}$ *satisfy*

$$S(A_1, A_2) \leq S(A_1^{\Cap}, A_2^{\Cap}), \tag{3.1}$$

$$S(B_1, B_2) \leq S(B_1^{\Cup}, B_2^{\Cup}), \tag{3.2}$$

$$A \subseteq A^{\Cap\Cup}, \tag{3.3}$$

$$B \supseteq B^{\Cup\Cap}. \tag{3.4}$$

The importance of Galois connections, both antitone and isotone, derives from the fact that they are induced in a natural way from binary relations and that the fixpoints (i.e. pairs s.t. $\langle A, B\rangle$ s.t. $A^{\Cap} = B$ and $B^{\Cup} = A$) of Galois connections have natural meaning. A canonical way an isotone Galois connection $\langle{}^{\Cap}, {}^{\Cup}\rangle$ arises from a binary fuzzy relation I between sets X and Y is described by:

$$A^{\Cap}(y) = \textstyle\bigvee_{x \in X} A(x) \otimes I(x, y), \tag{3.5}$$

$$B^{\Cup}(x) = \textstyle\bigwedge_{y \in Y} I(x, y) \to B(y). \tag{3.6}$$

If X and Y are interpreted as the set of objects and attributes and $I(x,y)$ as a degree to which object $x \in X$ has attribute $y \in Y$, then $A^{\Cap}(y)$ is just the truth degree of "there exists x in A which has y" and $B^{\Cup}(x)$ is the truth degree of "for all y: if x has y then y belongs to B". That is, $A^{\Cap}$ is the fuzzy set of attributes shared by at least one object from A and $B^{\Cup}$ is the fuzzy set of objects whose attributes are all in B.

Note that in the bivalent case, i.e. when I is an ordinary relation and A and B are ordinary sets, the operators defined by (25) and (26) are studied in [15]. The operators studied in this thesis extend those from [15] in that we assume that I is a fuzzy relation and A and B are fuzzy sets with truth degrees taken from a complete residuated lattice L. If L is the two-element Boolean algebra, operators (25) and (26) as well as their parameterized versions (27) and (28) introduced below studied coincide with those from [15]. Note also that the pairs of mappings (25) and (26) appear in [17, 23] and also in [22]. In what follows, we present and study operators which generalize (25) and (26) in that we parameterize (25) and (26). Technically, we parameterize (25) and (26) by inserting hedges at particular places in (25) and (26). Throughout the rest of the thesis, we assume that * is truth-stressing hedge on $\mathbf{L}$ and $^\square$ is truth-depressing hedge on $\mathbf{L}$ (which does not need to be a truth-depressing hedge w.r.t. *).

Let X, Y be sets of objects and attributes respectively, I be an $\mathbf{L}$-relation between X and Y, i.e. I is a mapping $I : X \times Y \to L$. $\langle X, Y, I\rangle$ is called a *formal fuzzy context*.

	$\cdots$	y	$\cdots$
$\vdots$		$\vdots$	
x	$\cdots$	$I(x,y)$	
$\vdots$			

Figure 3.1: Formal fuzzy context

For a formal fuzzy context $\langle X, Y, I\rangle$ we define a pair $\langle {}^\cap, {}^\cup\rangle$ of mappings ${}^\cap : L^X \to L^Y$ and ${}^\cup : L^Y \to L^X$ by

$$A^\cap(y) = \textstyle\bigvee_{x\in X} A(x)^* \otimes I(x,y), \tag{3.7}$$

$$B^\cup(x) = \textstyle\bigwedge_{y\in Y} I(x,y) \to B(y)^\square. \tag{3.8}$$

These mappings play a crucial role in the thesis. The meaning of $A^\cap$ and $B^\cup$ is essentially the same as that of $A^{\Cap}$ and $B^{\Cup}$. The difference is in that parts of the verbal description of $A^\cap$ and $B^\cup$ contain "very true" and "slightly true" respectively, compared to that of $A^{\Cap}$ and $B^{\Cup}$. For example, $A^\cap(y)$ is the truth degree of "there exists x for which it is very true that it belongs to A and which has y".

The fixed points of $\langle {}^\cap, {}^\cup\rangle$ (i.e. pairs $\langle A, B\rangle$ such that $A^\cap = B$ and $B^\cup = B$) are called *formal* (*fuzzy*) *concepts*. Operators induced by formal fuzzy context are usually called *concept-forming operators*. The set of all formal concepts of $\langle X, Y, I\rangle$ is denoted $\mathcal{B}(X^{*\cap}, Y^{\square\cup}, I)$.

For formal concepts $\langle A_1, B_1\rangle, \langle A_2, B_2\rangle \in \mathcal{B}(X^{*\cap}, Y^{\Box\cup}, I)$ we define

$$\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle \text{ iff } A_1 \subseteq A_2 \text{ (iff } B_1 \subseteq B_2) \tag{3.9}$$

As we show later, $\mathcal{B}(X^{*\cap}, Y^{\Box\cup}, I)$ with $\leq$ forms a complete lattice.

3.2 Basic properties

This section describes the generalization $\langle{}^\cap, {}^\cup\rangle$ of concept-forming operators $\langle{}^\Cap, {}^\Cup\rangle$ from [16] and shows basic properties of $\langle{}^\cap, {}^\cup\rangle$.

Theorem 6. *Mappings* ${}^\cap$, ${}^\cup$ *defined by (3.7) and (3.8) satisfy the following properties:*

(i) $A^\cap = A^{*\Cap}$ *and* $B^\cup = B^{\Box\Cup}$

(ii) $A^\cap = A^{*\cap}$ *and* $B^\cup = B^{\Box\cup}$

(iii) $A^\cap \subseteq A^\Cap$ *and* $B^\Cup \subseteq B^\cup$

(iv) $S(A_1, A_2)^* \leq S(A_1^*, A_2^*) \leq S(A_1^\cap, A_2^\cap)$
$S(B_1, B_2)^{*_Y} \leq S(B_1^\Box, B_2^\Box) \leq S(B_1^\cup, B_2^\cup)$
where ^{*_Y} *is a truth-stressing hedge for which (2.9) is satisfied.*

(v) $A^* \subseteq A^{\cap\cup}$ *and* $B^{\cup\cap} \subseteq B^\Box$

(vi) $A_1 \subseteq A_2$ *implies* $A_1^\cap \subseteq A_2^\cap$
$B_1 \subseteq B_2$ *implies* $B_1^\cup \subseteq B_2^\cup$

(vii) $S(A^*, B^\cup) = S(A^\cap, B^\Box)$

(viii) $\left(\bigcup_{i\in I} A_i^*\right)^\cap = \bigcup_{i\in I} A_i^\cap$ *and* $\left(\bigcap_{i\in I} B_i^\Box\right)^\cup = \bigcap_{i\in I} B_i^\cup$

(ix) $A^{\cap\cup\cap\cup} = A^{\cap\cup}$ *and* $B^{\cap\cup\cap\cup} = B^{\cap\cup}$

Proof. (i), (ii) follow immediately from definition of ${}^\cap$ and ${}^\cup$ and from properties of hedges.

(iii) follows from the fact, that $\otimes$ is monotone and $\rightarrow$ is isotone in the second argument.

(iv) $S(A_1, A_2)^* \leq S(A_1^*, A_2^*)$ and $S(B_1, B_2)^{*_Y} \leq S(B_1^\Box, B_2^\Box)$ follow from definitions of the truth-stressing and truth-depressing hedges and Lemmas 1 and 2. $S(A_1^\cap, A_2^\cap) = S(A_1^{*\Cap}, A_2^{*\Cap}) = S(A_1^*, A_2^{*\Cap\Cup}) \geq S(A_1^*, A_2^*)$. The second assertion is similar.

(v) $A^* \subseteq A^{*\Cap\Cup} = A^{\cap\Cup} \subseteq A^{\cap\cup}$; $B^{\cup\cap} \subseteq B^{\cup\Cap} = B^{\Box\Cup\Cap} \subseteq B^\Box$.

(vi) $A_1 \subseteq A_2$ implies $1 = S(A_1, A_2)^* \leq S(A_1^\cap, A_2^\cap)$. The second claim is similar.

(vii) $S(A^*, B^\cup) = S(A^*, B^{\Box\Cup}) = S(A^{*\Cap}, B^\Box) = S(A^\cap, B^\Box)$.

(viii) Using Lemma 1, we have

$$\left(\bigcup_{i\in I} A_i^*\right)^{\cap}(y) = \bigvee_{x\in X}\left(\bigvee_{i\in I}(A_i^*(x))^* \otimes I(x,y)\right) =$$
$$= \bigvee_{x\in X}\left(\left(\bigvee_{i\in I} A_i^*(x)\right)\otimes I(x,y)\right) =$$
$$= \bigvee_{i\in I}\left(\bigvee_{x\in X} A_i^*(x)\otimes I(x,y)\right) = \bigvee_{i\in I} A_i^{\cap}(y)$$

Similarly, using Lemma 2, we have

$$\left(\bigcap_{i\in I} B_i^{\Box}\right)^{\cup}(x) = \bigwedge_{y\in Y}\left(I(x,y)\to\left(\bigwedge_{i\in I} B_i^{\Box}(y)\right)^{\Box}\right) =$$
$$= \bigwedge_{y\in Y}\left(I(x,y)\to\left(\bigwedge_{i\in I} B_i^{\Box}(y)\right)\right) =$$
$$= \bigwedge_{i\in I}\left(\bigwedge_{y\in Y} I(x,y)\to B_i^{\Box}(y)\right) = \bigwedge_{i\in I} B_i^{\cup}(x)$$

(ix) Using (v) $A \subseteq A^{\cap\cup}$ and (vi) two times we get $A^{\cap\cup} \subseteq A^{\cap\cup\cap\cup}$. Using (v) with $B = A^{\cap}$ we have $A^{\cap\cup\cap} \subseteq A^{\cap\Box}$. Using (vi) we get the first claim. The second claim is similar. $\Box$

Remark 2. *Note that the induced concept-forming operators with* *, $^{\Box}$ *have very similar properties to those defined by (3.13) and (3.14) which were introduced in [3]. The following list sums up properties of these operators which are analogous to those from Theorem 6.*

(i) $A^{\cap} = A^{*_X \Cap}$ *and* $B^{\cup} = B^{*_Y \Cup}$,

(ii) $A^{\cap} = A^{*_X \cap}$ *and* $B^{\cup} = B^{*_Y \cup}$,

(iii) $A^{\cap} \subseteq A^{\Cap}$ *and* $B^{\cup} \subseteq B^{\Cup}$,

(iv) $S(A_1, A_2)^{*_X} \leq S(A_1^{*_X}, A_2^{*_X}) \leq S(A_1^{\cap}, A_2^{\cap})$,
$S(B_1, B_2)^{*_Y} \leq S(B_1^{*_Y}, B_2^{*_Y}) \leq S(B_1^{\cup}, B_2^{\cup})$,

(v) $B^{\cup\cap} \subseteq B^{*_Y}$,

(vi) $A_1 \subseteq A_2$ *implies* $A_1^{\cap} \subseteq A_2^{\cap}$,
$B_1 \subseteq B_2$ *implies* $B_1^{\cup} \subseteq B_2^{\cup}$,

(vii) $S(A^{*_X}, B^{\cup}) = S(A^{\cap}, B^{*_Y})$,

(viii) $\left(\bigcup_{i\in I} A_i^{*_X}\right)^{\cap} = \bigcup_{i\in I} A_i^{\cap}$ *and* $\left(\bigcap_{i\in I} B_i\right)^{\cup} = \left(\bigcap_{i\in I} B_i^{*_Y}\right)^{\cup}$,

(ix) $A^{\cap\cup\cap} \subseteq A^{\cap *_Y}$ *and* $B^{\cup\cap\cup} \subseteq B^{*_Y\cup}$.

We have extended (3.5) and (3.6) and made them parameterizable using truth-stressing hedge and truth-depressing hedge while we have kept most of their basic properties. In particular we have lost properties $A^{\Cap\Cup\Cap} = A^{\Cap}$ and $B^{\Cap\Cup\Cap} = B^{\Cap}$ and replaced them by the property (ix) in Theorem 6.

3.3 Axiomatization

We now turn to the problem of axiomatization of the mappings defined by (3.7) and (3.8). We present characteristic properties of these mappings.

In this part, we use subscription I to denote operations induced by context $\langle X, Y, I\rangle$ ($^{\cap_I}$ and $^{\cup_I}$) to distinguish them from operators introduced in Definition 2. At the end of this part, we show that these operations are the same, thus we do not need to distinguish them in later parts of this thesis.

Isotone Galois connections were axiomatized in [16]. We generalize the approach of [16] as follows:

Definition 2. *Let* X, Y *be two universes. A pair of mappings* $\langle {}^{\cap}, {}^{\cup}\rangle$, $^{\cap} : L^X \to L^Y$, $^{\cup} : L^Y \to L^X$ *is called* isotone **L**-Galois connection *between* X *and* Y *if*

$$S(A^*, B^{\cup}) = S(A^{\cap}, B^{\square}) \tag{3.10}$$

$$\left(\bigcup_{i\in I} A_i^*\right)^{\cap} = \bigcup_{i\in I} A_i^{\cap} \tag{3.11}$$

$$a^* \otimes \{{}^1/x\}^{\cap}(y) = \{{}^a/x\}^{\cap}(y) \tag{3.12}$$

Lemma 7. *Operators* $^{\cap_I}$ *and* $^{\cup_I}$ *defined by (3.7) and (3.8) form an isotone* **L**-*Galois connection* $\langle {}^{\cap_I}, {}^{\cup_I}\rangle$ *with hedges* * *and* $^{\square}$.

Proof. Due to Theorem 6 (vii) and (viii), it is enough to show that (3.12) is satisfied. Indeed,

$$\begin{aligned} a^* \otimes \{{}^1/x\}^{\cap_I}(y) &= a^* \otimes \bigvee_{x\in X} 1 \otimes I(x,y) = \\ &= \bigvee_{x\in X} a^* \otimes I(x,y) = \{{}^a/x\}^{\cap}(y) \end{aligned}$$

□

Lemma 8. *For every mapping* $^{\cap} : L^X \to L^Y$ *there exist at most one mapping* $^{\cup} : L^Y \to L^X$ *satisfying* $S(A^*, B^{\cup}) = S(A^{\cap}, B^{\square})$ *for every* $A \in L^X$ *and* $B \in L^Y$*.*

Proof. If $^{\cup'}$ is another such mapping, we have $S(A^*, B^{\cup'}) = S(A^{\cap}, B^{\square})$ for any A and B. Take any $x \in X$ and put $A = \{{}^1/x\}$. Then

$$B^{\cup}(x) = S(A^*, B^{\cup}) = S(A^*, B^{\cup'}) = B^{\cup'}(x)$$

Therefore, $^{\cup}$ coincides with $^{\cup'}$. $\square$

Lemma 9. *Let* $\langle^{\cap}, ^{\cup}\rangle$ *be an isotone* **L**-*Galois connection with hedges* * *and* $^{\square}$*. Then there exists an* **L**-*relation* I *between* X *and* Y *such that* $\langle^{\cap}, ^{\cup}\rangle = \langle^{\cap_I}, ^{\cup_I}\rangle$*.*

Proof. We need to find I such that $A^{\cap} = A^{\cap_I}$ and $B^{\cup} = B^{\cup_I}$ for all $A \in L^X, B \in L^Y$. Due to Lemma 8, it is sufficient to find I for which $A^{\cap} = A^{\cap_I}$. Namely, $\langle^{\cap_I}, ^{\cup_I}\rangle$ satisfy $S(A^*, B^{\cup}) = S(A^{\cap}, B^{\square})$ by Lemma 7. Hence, $^{\cup_I}$ coincides with $^{\cup}$ due to Lemma 8.

Define I by $I(x, y) = \{{}^1/x\}^{\cap}(y)$. Then we get

$$\begin{aligned} A^{\cap}(y) = A^{*\cap}(y) &= \bigvee_{x\in X, y\in Y} \{{}^{A^*(x)}/x\}^{\cap}(y) = \\ &= \bigvee_{x\in X}\bigvee_{y\in Y} A^*(x) \otimes \{{}^1/x\}^{\cap}(y) = \\ &= \bigvee_{x\in X} A^*(x) \otimes I(x, y) = A^{\cap_I}(y) \end{aligned}$$

This finishes the proof. $\square$

Theorem 10. *Let* $\langle X, Y, I\rangle$ *be formal fuzzy context,* $\langle^{\cap}, ^{\cup}\rangle$ *be an isotone* **L**-*Galois connection with hedges* * *and* $^{\square}$*. Then*

- $\langle^{\cap_I}, ^{\cup_I}\rangle$ *is isotone* **L**-*Galois connection with hedges* * *and* $^{\square}$*.*
- $I_{\langle\cap,\cup\rangle}$ *defined by* $I_{\langle\cap,\cup\rangle}(x, y) = \{{}^1/x\}^{\cap}(y)$ *is an* **L**-*relation between* X *and* Y *and we have*
- $\langle^{\cap}, ^{\cup}\rangle = \langle^{\cap_{I_{\langle\cap,\cup\rangle}}}, ^{\cup_{I_{\langle\cap,\cup\rangle}}}\rangle$ *and* $I = I_{\langle\cap,\cup\rangle}$*.*

Proof. Due to Lemma (7) and Lemma (9), it suffices to show that $I = I_{\langle\cap,\cup\rangle}$. We have

$$\begin{aligned} I_{\langle\cap,\cup\rangle}(x, y) &= \{{}^1/x\}^{\cap_I}(y) = \\ = &\bigvee_{z\in X} \{{}^{1^*}/x\}(z) \otimes I(z, y) = I(x, y) \end{aligned}$$

$\square$

Remark 3. *Note that we only need* $^{\cap}$ *to define* $I_{\langle\cap,\cup\rangle}$ *and we need* $^{\cap}$ *to satisfy only* $^{\cap}$ *and* $^{\cup}$*. Having such operation, we can use Theorem 10 to find corresponding* $^{\downarrow}$ *as* $^{\downarrow} = {}^{\cup_{I_{\langle\cap,\cup\rangle}}}$*.*

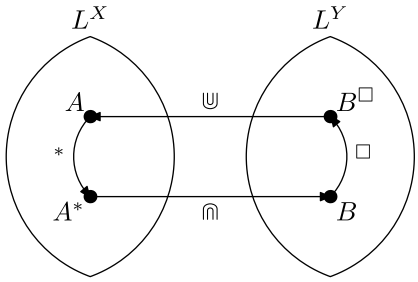


Figure 3.2: Formal concept of $\mathcal{B}(X^{*\cap}, Y^{\Box\cup}, I)$

3.4 Why we use a truth-depressing hedge?

In [3] we introduced the following concept-forming operators:

$$A^{\cap}(y) = \bigvee_{x\in X} A(x)^{*_X} \otimes I(x,y) \tag{3.13}$$

$$B^{\cup'}(x) = \bigwedge_{y\in Y} I(x,y) \rightarrow B(y)^{*_Y} \tag{3.14}$$

where ^{*_X} and ^{*_Y} are truth-stressing hedges. Note that the only difference from the concept-forming operators defined by (3.7) and (3.8) is that a truth-stressing hedge ^{*_Y} is used in (3.14) while a truth-depressing hedge $^{\Box}$ is used in (3.8). In this part we argue that the use of a truth-depressing hedge is more convenient.

Let us take a look at a geometric interpretation of a formal concept as a fixpoint of isotone **L**-Galois connection with hedges * and $^{\Box}$ (see Fig. 3.2; Arrows in Figures 3.2, 3.3, and 3.4 represent mappings. For example mapping $^* : L^X \rightarrow L^X$ is representd by arrow between A and A^* inside L^X; A^* placed under A means $A^* \subseteq A$).

If a truth-stressing hedge ^{*_Y} is used we have the situation depicted in Fig. 3.3. B and B^{*_Y} degenerate into one point, as described by the following theorem.

Theorem 11. *Let $\mathcal{B}(X^{*_X\cap}, Y^{*_Y\cup}, I)$ denote the set of all fixpoints of the operators by (3.7) and (3.8) and $\mathcal{B}(X^{*_X\cap}, Y^{\cup}, I)$ set of all fixpoints of the same operators for ^{*_Y} being identity $^{\mathrm{id}_L}$. Then we have*

$$\mathcal{B}(X^{*_X\cap}, Y^{*_Y\cup}, I) = \{\langle A, B\rangle \in \mathcal{B}(X^{*_X\cap}, Y^{\cup}, I) \mid B = B^{*_Y}\}. \tag{3.15}$$

Proof. "$\subseteq$": $B = B^{\cup'\cap} \subseteq B^{*_Y} \subseteq B$ proves, that $B = B^{*_Y}$. For intent B we have $B = B^{\cup'\cap} = B^{*_Y\Cup *_X\Cap} = B^{\Cup *_X\Cap}$ proving that $B \in \mathrm{Int}(X^{*_X\cap}, Y^{\cup}, I)$.

"$\supseteq$": $B = B^{\Cup *_X\Cap} = B^{*_Y\Cup *_X\Cap} = B^{\cup'\cap}$, thus $B \in \mathrm{Int}(X^{*_X\cap}, Y^{*_Y\cup}, I)$. $\Box$

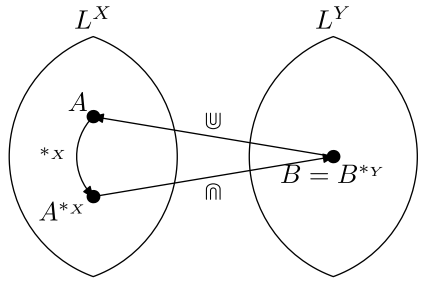


Figure 3.3: Formal concept of $\mathcal{B}(X^{*_X\cap}, Y^{*_Y\cup'}, I)$ with truth-stressing hedges

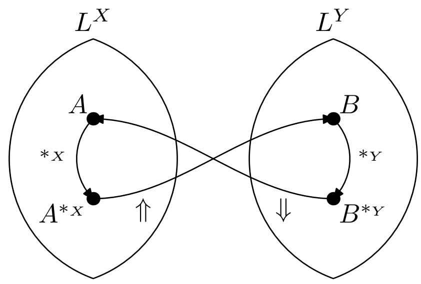


Figure 3.4: Concept of $\mathcal{B}(X^{*_X\uparrow}, Y^{*_Y\downarrow}, I)$

Note that Theorem 11 says that using ^{*_Y} brings just trivial selection of formal concepts.

The use of truth-depressing hedge brings us to analogy of the geometrical interpretation of a formal concept of $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$, which is depicted in Figure 3.4.

In the case of concept-forming operators ${}^{\uparrow}, {}^{\downarrow}$, we have both composition ${}^{\Uparrow\Downarrow}$ and ${}^{\Downarrow\Uparrow}$ being closure operators. With truth-stressing hedges ^{*_Y} and ^{*_X} the compositions ${}^{\Uparrow *_Y \Downarrow}$ and ${}^{\Downarrow *_X \Uparrow}$ keep to be closure operators. On the other hand, the truth-stressing hedges ^{*_X} and ^{*_Y} are interior operators. [1] Similarly, in the case of concept-forming operators ${}^{\cap}, {}^{\cup}$, we have the composition ${}^{\Cap\Cup}$ being a closure operator. With a truth-depressing hedge ${}^{\square}$ the composition ${}^{\Cap\square\Cup}$ keeps to be a closure operator. A truth-stressing hedge * works opposite way to the composition ${}^{\Cap\square\Cup}$. Dually, the compositions ${}^{\Cup\Cap}$ and ${}^{\Cup*\Cap}$ are interior operators, while ${}^{\square}$ is a closure operator.

The main benefits of using truth-depressing hedge in (3.8) are:

- According to Theorem 6 for any isotone **L**-Galois connection with * and ${}^{\square}$ we have convenient properties $A^{\cap\cup\cap} \subseteq A^{\cap *_Y}$ and $B^{\cup\cap\cup} \subseteq B^{*_Y\cup}$. Analogous properties do not generally hold true for isotone **L**-Galois connection with ^{*_X} and ^{*_Y}.

- By Theorem 11, using a truth-stressing hedge ^{*_Y} in (3.14) turns to be a selection of

[1] further description of $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$ is out of scope of this thesis, see f.e. [4, 8, 13] for this topic.

formal concepts from $\mathcal{B}(X^{*_X\cap}, Y^{\cup}, I)$ based on membership degrees in their intents. Particulary, all concepts from $\mathcal{B}(X^{*_X\cap}, Y^{\cup}, I)$ whose intents contain attributes in other truth-degrees than fix$(^{*_Y})$ are filtered out. This kind of selection does not seem to be reasonable.

- The reduction of the size of the associated concept lattice with two truth-stressing hedges is too drastic [3]. Especially when using $^{*_Y} = {}^{*_G}$, the resulting concept lattice commonly happens to be a trivial lattice containing no interesting information. Reduction with truth-stressing hedge and truth-depressing hedge (see Section 4) seems to be more natural in comparison with the previous one.

3.5 Main theorem on the structure of $\mathcal{B}(X^{*\cap}, X^{\square\cup}, I)$

In this part we show that concept lattice $\mathcal{B}(X^{*\cap}, Y^{\square\cup}, I)$ is isomorphic to a concept lattice of a particular ordinary formal context with $^{\Uparrow}, ^{\Downarrow}$. Moreover, we provide a variant of the main theorem of concept lattices for $\mathcal{B}(X^{*\cap}, Y^{\square\cup}, I)$. The content of this part is inspired by [4, 11].

We need the following notions:

Definition 3. *For $A \in L^X$ we define $\lfloor A \rfloor_\vee \in 2^{X\times L}$ and $\lfloor A \rfloor_\wedge \in 2^{X\times L}$ by*

$$\lfloor A \rfloor_\vee = \{\langle x, a\rangle \mid a \leq A(x)\} \tag{3.16}$$

$$\lfloor A \rfloor_\wedge = \{\langle x, a\rangle \mid A(x) \leq a\} \tag{3.17}$$

Described verbally, $\lfloor A \rfloor_\vee$ can be considered as an area in $X \times L$ under the membership function $A : X \to L$ and $\lfloor A \rfloor_\wedge$ as an area in $X \times L$ above the membership function $A : X \to L$.

For $A' \in \mathbf{2}^{X\times L}$ we define $\lceil A' \rceil_\vee \in L^X$ and $\lceil A' \rceil_\wedge \in L^X$ by

$$\lceil A' \rceil_\vee(x) = \bigvee\{a \mid \langle x, a\rangle \in A'\} \tag{3.18}$$

$$\lceil A' \rceil_\wedge(x) = \bigwedge\{a \mid \langle x, a\rangle \in A'\} \tag{3.19}$$

for each $x \in X$.

Definition 4. *For $A' \subseteq X \times L$ and (truth-stressing or truth-depressing) hedge $^\bullet : L \to L$, define $A'^\bullet = \{\langle x, a^\bullet\rangle \mid \langle x, a\rangle \in A'\}$.*

Lemma 12 ([13])**.** *For $A \subseteq \mathrm{fix}(^*) \times X$ we have $A \subseteq \lfloor \lceil A \rceil_\vee{}^* \rfloor_\vee{}^*$.*

Analogously, we have:

Lemma 13. *For $B \subseteq \mathrm{fix}(^\square) \times Y$ we have $B \subseteq \lfloor \lceil B \rceil_\wedge{}^\square \rfloor_\wedge{}^\square$.*

Proof. Let $\langle y, b\rangle \in B$. Then $b \geq \lceil B\rceil_\wedge$. Since $b \in \mathrm{fix}(^\square)$, we have $b \geq \lceil B\rceil_\wedge{}^\square$. Thus $\langle y, b\rangle \in \lfloor\lceil B\rceil_\wedge{}^\square\rfloor_\wedge$. Finally $\langle y, b\rangle \in \lfloor\lceil B\rceil_\wedge{}^\square\rfloor_\wedge{}^\square$ since $b \in \mathrm{fix}(^\square)$. $\square$

Define mappings $^{\uparrow_\times} : X \times \mathrm{fix}(^*) \to Y \times \mathrm{fix}(^\square)$ and $^{\downarrow_\times} : Y \times \mathrm{fix}(^\square) \to X \times \mathrm{fix}(^*)$ by

$$A^{\uparrow_\times} = \lfloor\lceil A\rceil_\vee{}^\cap\rfloor_\wedge{}^\square \text{ and } B^{\downarrow_\times} = \lfloor\lceil B\rceil_\wedge{}^\cup\rfloor_\vee{}^* \tag{3.20}$$

Lemma 14. *The pair $\langle^{\uparrow_\times}, ^{\downarrow_\times}\rangle$ forms an antitone Galois connection between sets $X \times \mathrm{fix}(^*)$ and $Y \times \mathrm{fix}(^\square)$.*

Proof. Antitony: $A_1 \subseteq A_2$ implies $\lceil A_1\rceil_\vee \subseteq \lceil A_2\rceil_\vee$ which implies $\lceil A_1\rceil_\vee{}^\cap \subseteq \lceil A_2\rceil_\vee{}^\cap$ which implies $\lfloor\lceil A_2\rceil_\vee{}^\cap\rfloor_\wedge \subseteq \lfloor\lceil A_1\rceil_\vee{}^\cap\rfloor_\wedge$. Similarly $B_1 \subseteq B_2$ implies $\lceil B_2\rceil_\wedge \subseteq \lceil B_1\rceil_\wedge$ which implies $\lceil B_1\rceil_\wedge{}^\cup \subseteq \lceil B_2\rceil_\wedge{}^\cup$ which implies $\lfloor\lceil B_2\rceil_\wedge{}^\cup\rfloor_\vee \subseteq \lfloor\lceil B_1\rceil_\wedge{}^\cup\rfloor_\vee$.

Extensivity: Using Lemma 12, $A^{\uparrow_\times\downarrow_\times} = \lfloor\lceil\lfloor\lceil A\rceil_\vee{}^\cap\rfloor_\vee{}^\square\rceil_\wedge{}^\cup\rfloor_\wedge{}^* = \lfloor\lceil\lfloor\lceil A\rceil_\vee{}^\cap\rfloor_\vee\rceil_\wedge{}^{\square\cup}\rfloor_\wedge{}^* = \lfloor\lceil A\rceil_\vee{}^{\cap\square\cup}\rfloor_\wedge{}^* \supseteq \lfloor\lceil A\rceil_\vee{}^*\rfloor_\wedge{}^* \supseteq A$. Similarly $B \subseteq B^{\downarrow_\times\uparrow_\times}$. $\square$

The following theorem is a direct consequence of the main theorem of concept lattices [15]. It says that concept lattice of the formal fuzzy context corresponding to isotone Galois connection $\langle^{\uparrow_\times}, ^{\downarrow_\times}\rangle$ forms a complete lattice and each complete lattice satisfying some particular technical condition is isomorphic to the concept lattice of a formal context $\langle U, V, I_{\langle\uparrow_\times,\downarrow_\times\rangle}\rangle$ which is given by the antitone the Galois connection defined by (3.20) and by (2.20).

Theorem 15. 1. *$\mathcal{B}(U^\Uparrow, V^\Downarrow, I_{\langle\uparrow_\times,\downarrow_\times\rangle})$ equipped with $\leq$, defined by $\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle$ iff $A_1 \subseteq A_2$, is a complete lattice where the infima and suprema are given by*

$$\bigwedge_{j\in J} \langle A_j, B_j\rangle = \langle\bigcap_{j\in J} A_j, (\bigcup_{j\in J} B_j)^{\downarrow_\times\uparrow_\times}\rangle, \tag{3.21}$$

$$\bigvee_{j\in J} \langle A_j, B_j\rangle = \langle(\bigcup_{j\in J} A_j)^{\uparrow_\times\downarrow_\times}, \bigcap_{j\in J} B_j\rangle \tag{3.22}$$

2. *Moreover, an arbitrary complete lattice $\mathbf{K} = \langle K, \leq\rangle$ is isomorphic to $\mathcal{B}(U, V, I_{\langle\uparrow_\times,\downarrow_\times\rangle})$ iff there are mappings $\mu : U \to K$, $\nu : V \to K$ such that*
   (a) $\mu(U)$ is $\bigvee$-dense in K, $\nu(V)$ is $\bigwedge$-dense in K;
   (b) $\mu(u) \leq \nu(v)$ iff $\langle u, v\rangle \in I_{\langle\uparrow_\times,\downarrow_\times\rangle}$.

Lemma 16. *The (crisp) relation $I^\times = I_{\langle\uparrow_\times,\downarrow_\times\rangle}$ between $X \times \mathrm{fix}(^*)$ and $Y \times \mathrm{fix}(^\square)$ corresponding to Galois connection $\langle^{\uparrow_\times}, ^{\downarrow_\times}\rangle$ defined by (3.20) is given by*

$$\langle\langle x, a\rangle, \langle y, b\rangle\rangle \in I^\times \text{ iff } I(x, y) \leq a \to b \tag{3.23}$$

Proof. We have $\langle\langle x,a\rangle,\langle y,b\rangle\rangle \in I^{\times}$ iff $\langle y,b\rangle \in \{\langle x,a\rangle\}^{\uparrow_\times}$. By definition of ${}^{\uparrow_\times}$, this is equivalent to $\langle y,b\rangle \in \lfloor\lceil\{\langle x.a\rangle\}\rceil_{\vee}{}^{\cap}\rfloor_{\vee}{}^{\Box}$. Since $\lfloor\lceil\{\langle x,a\rangle\}\rceil_{\vee}{}^{\cap}\rfloor_{\vee}{}^{\Box} = \lfloor\{{}^{a}/x\}^{\cap}\rfloor_{\vee}{}^{\Box}$ and since the smallest c such that $\langle y,c\rangle \in \lfloor\{{}^{a}/x\}^{\cap}\rfloor_{\vee}{}^{\Box}$ is $c = (\{{}^{a}/x\}^{\cap}(y))^{\Box}$, the last assertion is equivalent to $(\{{}^{a}/x\}^{\cap}(y))^{\Box} \leq b$. Since $b = b^{\Box}$, this is equivalent to $(\{{}^{a}/x\}^{\cap}(y)) \leq b$. Now, $\{{}^{a}/x\}^{\cap}(y) = a^{*} \otimes I(x,y) = a \otimes I(x,y)$, whence $\{{}^{a}/x\}^{\cap}(y) \leq b$ is equivalent to $I(x,y) \leq a \to b$ by adjointness. $\Box$

Theorem 17. *$\mathcal{B}(X^{*\cap}, Y^{\Box\cup}, I)$ (concept lattice with hedges) is isomorphic to $\mathcal{B}(X \times \mathrm{fix}(^{*})^{\uparrow_\times}, Y \times \mathrm{fix}(^{\Box})^{\downarrow_\times}, I^{\times})$ (ordinary concept lattice). The isomorphism*

$$h : \mathcal{B}(X^{*\cap}, Y^{\Box\cup}, I) \to \mathcal{B}(X \times \mathrm{fix}(^{*})^{\uparrow_\times}, Y \times \mathrm{fix}(^{\Box})^{\downarrow_\times}, I^{\times})$$

and its inverse

$$g : \mathcal{B}(X \times \mathrm{fix}(^{*})^{\uparrow_\times}, Y \times \mathrm{fix}(^{\Box})^{\downarrow_\times}, I^{\times}) \to \mathcal{B}(X^{*\cap}, Y^{\Box\cup}, I)$$

are given by

$$h(\langle A,B\rangle) = \langle \lfloor A\rfloor_{\vee}{}^{*}, \lfloor B\rfloor_{\wedge}{}^{\Box}\rangle \tag{3.24}$$

$$g(\langle A',B'\rangle) = \langle \lceil A'\rceil_{\vee}{}^{\cap\cup}, \lceil B'\rceil_{\wedge}{}^{\cup\cap}\rangle \tag{3.25}$$

Proof. We need to show, that (a) h and g are defined correctly, (b) h is order-preserving, (c) $g(h(\langle A,B\rangle)) = \langle A,B\rangle$ and $h(g(\langle A',B'\rangle)) = \langle A',B'\rangle$.

(a) For $\langle A,B\rangle$ in $\mathcal{B}(X^{*\cap}, Y^{\Box\cup}, I)$ we have $\lfloor A\rfloor_{\vee}{}^{*\uparrow_\times} = \lfloor A^{\cap}\rfloor_{\wedge}{}^{\Box}$ and $\lfloor B\rfloor_{\wedge}{}^{\Box\downarrow_\times} = \lfloor B^{\cup}\rfloor_{\vee}{}^{*}$ directly from definitions of operators ${}^{\uparrow_\times}$ and ${}^{\downarrow_\times}$ (3.20).

For $\langle A',B'\rangle$ in $\mathcal{B}(X \times \mathrm{fix}(^{*})^{\uparrow_\times}, Y \times \mathrm{fix}(^{\Box})^{\downarrow_\times}, I^{\times})$, $\lceil A'\rceil_{\vee}{}^{\cap\cup\cap} = \lceil\lfloor\lceil A'\rceil_{\vee}{}^{\cap}\rfloor_{\wedge}\rceil_{\wedge}{}^{\cup\cap} = \lceil A'^{\uparrow_\times}\rceil_{\wedge}{}^{\cup\cap} = \lceil\lfloor\lceil A'^{\uparrow_\times}\rceil_{\wedge}{}^{\cup}\rfloor_{\vee}\rceil_{\vee}{}^{\cap} = \lceil A'^{\uparrow_\times\downarrow_\times}\rceil_{\vee}{}^{\cap} = \lceil A'\rceil_{\vee}{}^{\cap}$. Similarly $(\lceil B'\rceil_{\wedge}{}^{\cup\cap})^{\cup} = \lceil B'\rceil_{\wedge}{}^{\cup}$.

(b) For $\langle A_1,B_1\rangle, \langle A_2,B_2\rangle \in \mathcal{B}(X^{*\cap}, Y^{\Box\cup}, I)$ we have $A_1 \subseteq A_2$ iff $\lfloor A_1\rfloor_{\vee} \subseteq \lfloor A_2\rfloor_{\vee}$ iff $\lfloor A_1\rfloor_{\vee}{}^{*} \subseteq \lfloor A_2\rfloor_{\vee}{}^{*}$.

(c) For $\langle A,B\rangle$ in $\mathcal{B}(X^{*\cap}, Y^{\Box\cup}, I)$ we have

$$\lceil\lfloor A\rfloor_{\vee}{}^{*}\rceil_{\vee}{}^{\cap\cup} = \lceil\lfloor A^{*}\rfloor_{\vee}\rceil_{\vee}{}^{\cap\cup} = A^{*\cap\cup} = A$$

For $\langle A',B'\rangle$ in $\mathcal{B}(X \times \mathrm{fix}(^{*})^{\uparrow_\times}, Y \times \mathrm{fix}(^{\Box})^{\downarrow_\times}, I^{\times})$ we have
$\lfloor\lceil A'\rceil_{\vee}{}^{\cap\cup}\rfloor_{\vee}{}^{*} = \lfloor\lceil\lfloor\lceil A'\rceil_{\vee}{}^{\cap}\rfloor_{\wedge}{}^{\Box}\rceil_{\wedge}{}^{\cup}\rfloor_{\vee}{}^{*} = \lfloor\lceil A^{\uparrow_\times}\rceil_{\wedge}{}^{\cup}\rfloor_{\vee}{}^{*} = A'^{\uparrow_\times\downarrow_\times} = A'$

$\Box$

Theorem 18. *1. $\mathcal{B}(X^{*\cap}, Y^{\square\cup}, I)$ equipped with $\leq$, defined by $\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle$ iff $A_1 \subseteq A_2$, is a complete lattice where the infima and suprema are given by*

$$\bigwedge_{j\in J} \langle A_j, B_j\rangle = \langle (\bigcap_{j\in J} A_j)^{\cap\cup}, (\bigcap_{j\in J} B_j^{\square})^{\cup\cap}\rangle \tag{3.26}$$

$$\bigvee_{j\in J} \langle A_j, B_j\rangle = \langle (\bigcup_{j\in J} A_j^{*})^{\cap\cup}, (\bigcup_{j\in J} B_j)^{\cup\cap}\rangle \tag{3.27}$$

2. *Moreover, an arbitrary complete lattice $\mathbf{K} = \langle K, \leq\rangle$ is isomorphic to $\mathcal{B}(X^{*\cap}, Y^{\square\cup}, I)$ iff there are mappings $\mu : \mathrm{fix}(^{*}) \times X \to K$, $\nu : \mathrm{fix}(^{\square}) \times Y \to K$ such that*

   (a) $\mu(\mathrm{fix}(^{*}) \times X)$ is $\bigvee$-dense in K, $\nu(\mathrm{fix}(^{\square}) \times Y)$ is $\bigwedge$-dense in K.

   (b) $\mu(a, x) \leq \nu(b, y)$ iff $I(x, y) \leq a \to b$.

Proof. From Theorem 15 and Theorem 17. $\square$

Chapter 4

Reducing the Size of Concept Lattices

The main idea of generalizing concept-forming operators $\langle^\cap, ^\cup\rangle$ by a truth-stressing hedge and a truth-depressing hedge is to gain control on the size of the resulting concept lattice. In the case of the original isotone concept-forming operators $\langle^\Cap, ^\Cup\rangle$, the number of formal fuzzy concepts can be inconveniently big. For instance in the example below, we obtain 207 formal fuzzy concepts from formal context with 6 objects and 4 attributes. Proper selection of the truth-stressing hedge and truth-depressing hedge decreases the number of formal fuzzy concepts in the resulting concept lattice as demonstrated in this section. We also provide a theoretical result about sizes of concept lattice.

Example 1. *We demonstrate the influence of hedges by the following example. Consider the formal fuzzy context represented by Table 4.1. The table describes six books and their graded attributes. For the five-valued Łukasiewicz chain*

$$\mathbf{L} = \langle\{0, 0.25, 0.5, 0.75, 1\}, \min, \max, \otimes, \rightarrow, 0, 1\rangle$$

as our structure of truth degrees, there are 40 combinations of truth-stressing hedge * *and truth-depressing hedge* $^\square$ *(5 possible choices of* * *and 8 possible (independent) choices of* $^\square$*, see Figures 2.1 and 2.2). For each combination of* * *and* $^\square$ *we compute the corresponding concept lattice* $\mathcal{B}(X^{*\cap}, Y^{\square\cup}, I)$*. The concept lattices are depicted in Fig. 4.2. Note that the concept lattices* $\mathcal{B}(X^{*\cap}, Y^{\square\cup}, I)$ *are displayed in a standard manner by means of their line diagrams (Hasse diagrams).*

One can notice that in Fig. 4.2 we get interesting alternating of big and small sizes of the concept lattices. For instance in the first column we have sizes 5,20,12,25,7,25. In the rest of this section we explain why this effect occurs.

Let $\mathbf{L}$ be a complete residuated lattice and $\mathrm{TD}(\mathbf{L})$ the set of all truth-depressing hedges and $\mathrm{TS}(\mathbf{L})$ set of all truth-stressing hedges.

Define partial order $\leq$ in $\mathrm{TS}(\mathbf{L})$ by

$$^{*_1} \leq {}^{*_2} \text{ iff } \mathrm{fix}(^{*_1}) \subseteq \mathrm{fix}(^{*_2}) \tag{4.1}$$

Table 4.1: Context of books and their graded properties

	High Rating	Large No. of Pages	Low Price	Top Sales Rank
1	0.75	0.00	1.00	0.00
2	0.50	1.00	0.25	0.50
3	1.00	1.00	0.25	0.50
4	0.75	0.50	0.25	1.00
5	0.75	0.25	0.75	0.00
6	1.00	0.00	0.75	0.25

And define partial order $\leq$ in $\mathrm{TD}(\mathbf{L})$ by

$$^{\square_1} \leq {}^{\square_2} \text{ iff } \mathrm{fix}(^{\square_1}) \subseteq \mathrm{fix}(^{\square_2}) \tag{4.2}$$

Note that the truth-depressing hedges from Fig. 2.2 form a partially ordered set depicted in Fig. 4.1(left), and that the truth-stressing hedges from Fig. 2.1 form partially ordered set depicted in Fig. 4.1(right).

Theorem 19. *For a formal fuzzy context $\langle X, Y, I\rangle$, truth-depressing hedges $^{\square_a}, {}^{\square_b} \in \mathrm{TD}(\mathbf{L})$ s.t. $^{\square_a} \leq {}^{\square_b}$, and truth-stressing hedges $^{*_a}, {}^{*_b} \in \mathrm{TS}(\mathbf{L})$ s.t. $^{*_a} \leq {}^{*_b}$ we have*

$$|\mathcal{B}(X^{*_a\cap}, Y^{\square_a\cup}, I)| \leq |\mathcal{B}(X^{*_b\cap}, Y^{\square_b\cup}, I)|. \tag{4.3}$$

Moreover,

$$\mathrm{Ext}(X^{\mathrm{id}_L\cap}, Y^{\square_a\cup}, I) \subseteq \mathrm{Ext}(X^{\mathrm{id}_L\cap}, Y^{\square_b\cup}, I) \tag{4.4}$$
$$\mathrm{Int}(X^{*_a\cap}, Y^{\mathrm{id}_L\cup}, I) \subseteq \mathrm{Int}(X^{*_b\cap}, Y^{\mathrm{id}_L\cup}, I) \tag{4.5}$$

Proof. Denote

$$\langle X_1', Y_1', I_1'\rangle := \langle X \times \mathrm{fix}(^{*_a}), Y \times \mathrm{fix}(^{\square_a}), I_1^{\times}\rangle$$

and

$$\langle X_2', Y_2', I_2'\rangle := \langle X \times \mathrm{fix}(^{*_b}), Y \times \mathrm{fix}(^{\square_b}), I_2^{\times}\rangle$$

. Note that $\langle X_1', Y_1', I_1'\rangle$ is a subcontext of $\langle X_2', Y_2', I_2'\rangle$; i.e. $X_1' \subseteq X_2'$, $Y_1' \subseteq Y_2'$ and I_1' is a restriction of I_2' to X_1', Y_1': $I_1' = I_2' \cap X_1' \times Y_1'$. The theorem follows from Theorem 17 and properties of subcontexts (see chapter 3 in [15]). $\square$

Remark 4. *Note that the second part of Theorem 19 does not generally hold for a truth-stresser * different from identity. For instance, for the formal fuzzy context $\langle X, Y, I\rangle$ shown in Table 4.2, truth-stressing hedges ^{*_1}, truth-depressing hedges $^{\square_5} \leq {}^{\square_1}$ (see Fig. 4.1),we have:*

$$\{^{0.75}/x_1, {}^{1.00}/x_2\} \in \mathrm{Ext}(X^{*_1\cap}, Y^{\square_5\cup}, I)$$

Table 4.2: Formal fuzzy context from Remark 4

	y_1	y_2
x_1	0.75	0.00
x_2	0.00	0.25

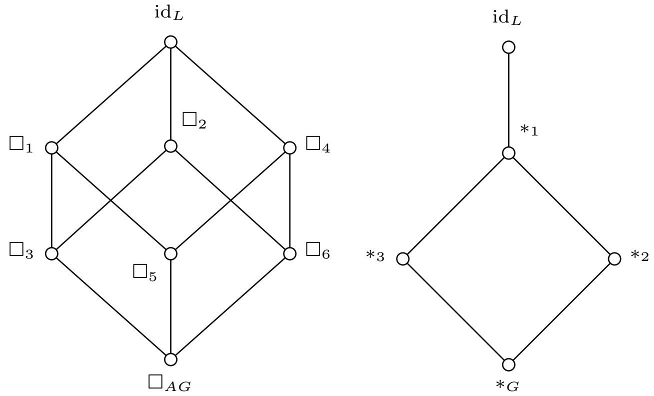


Figure 4.1: Truth-depressing hedges from Fig. 2.2 with $\leq$ (left) and truth-stressing hedges from Fig. 2.1 with $\leq$ (right)

but

$$\{{}^{0.75}/x_1, {}^{1.00}/x_2\} \notin \mathrm{Ext}(X^{*_1\cap}, Y^{\Box_1\cup}, I)$$

.

4.1 Illustrative example

In Table 4.3, linguistic terms are used instead of truth degrees. The table can be transformed into formal **L**-context with $X = \{F_1, F_2, F_3, F_4\}$ and $Y = [6-12] \times \{\text{consumption}\} \cup [140-220] \times \{\text{speed}\}$; the structure **L** of truth-degrees is three-element chain with Łukasiewicz operations. The transformed formal **L**-context is depicted in Table 4.4. This example is based on example from [31].

Note that using isotone Galois connection for this kind of data is very reasonal. For instance, we want to have "very fast car" to be a subconcept of "fast car".

Figures 4.3, 4.4, 4.5, and 4.6 show formal concepts formed with isotone Galois connection with various combinations of truth-stressing hedges and truth-depressing hedges.

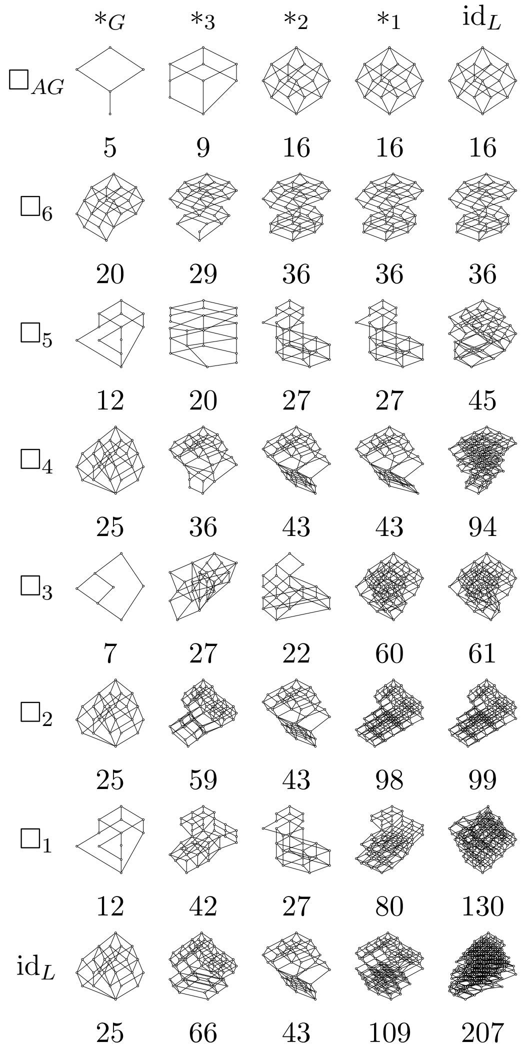
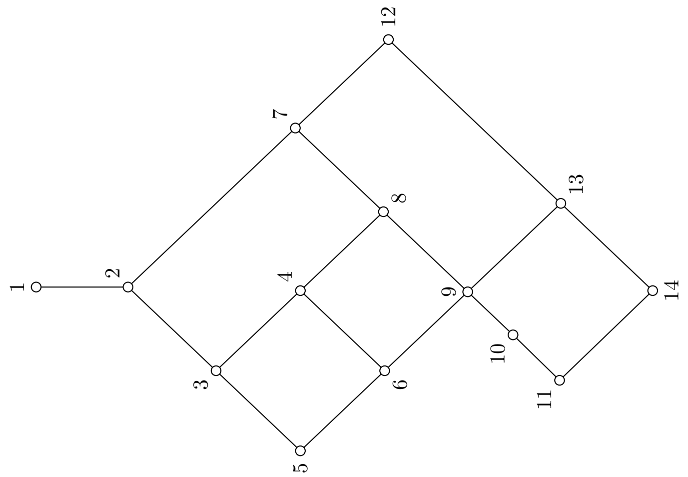
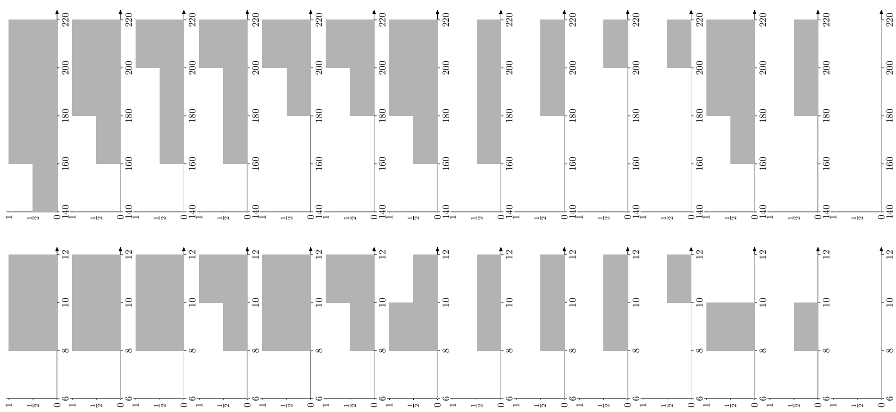


Figure 4.2: Concept lattices $\mathcal{B}(X^{*\cap}, Y^{\square\cup}, I)$ induced by the context from Table 4.1 and numbers of their formal concepts. The picture shows concept lattices resulting by all combinations of truth-stressing hedge * and truth-depressing hedge $^\square$ from Fig. 2.1 and Fig. 2.2.

	consumption	speed
F_1	very high	fast
F_2	8-10l/100km	very fast
F_3	at least 8l/100km	not so fast as F_2
F_4	at least 8l/100km	fast

Table 4.3: Table “Cars”





1	X
2	$\{{}^{1}/F_1, {}^{1}/F_2, {}^{.5}/F_3, {}^{1}/F_4\}$
3	$\{{}^{1}/F_1, {}^{.5}/F_2, {}^{.5}/F_3, {}^{1}/F_4\}$
4	$\{{}^{1}/F_1, {}^{.5}/F_2, {}^{.5}/F_3, {}^{.5}/F_4\}$
5	$\{{}^{1}/F_1, {}^{.5}/F_2, {}^{1}/F_4\}$
6	$\{{}^{1}/F_1, {}^{.5}/F_2, {}^{.5}/F_4\}$
7	$\{{}^{1}/F_1, {}^{1}/F_2, {}^{.5}/F_3, {}^{.5}/F_4\}$
8	$\{{}^{.5}/F_1, {}^{.5}/F_2, {}^{.5}/F_3, {}^{.5}/F_4\}$
9	$\{{}^{.5}/F_1, {}^{.5}/F_2, {}^{.5}/F_4\}$
10	$\{{}^{.5}/F_1, {}^{.5}/F_4\}$
11	$\{{}^{.5}/F_1\}$
12	$\{{}^{1}/F_2\}$
13	$\{{}^{.5}/F_2\}$
14	$\emptyset$

Figure 4.3: Concept lattice $\mathcal{B}(X^{\mathrm{id}_L\cap}, Y^{\mathrm{id}_L\cup}, I)$ of the formal context from Table 4.4

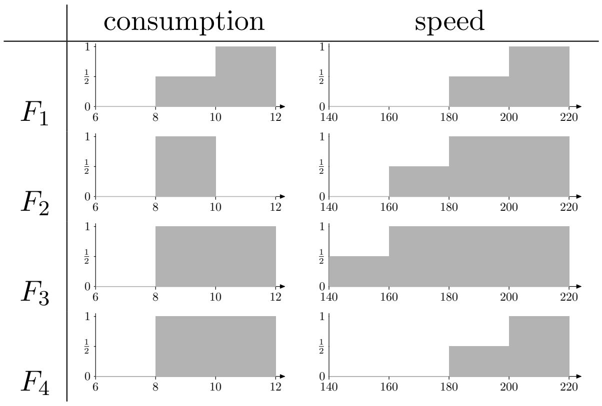
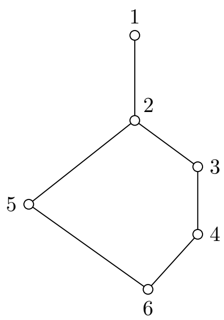


Table 4.4: Table “Cars” transformed to a formal **L**-context



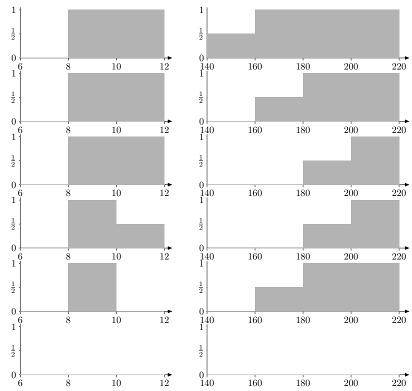
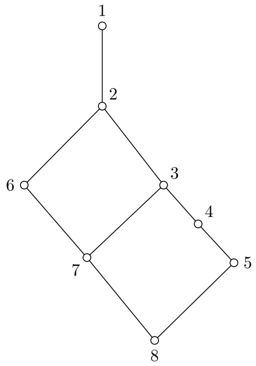
1	X	
2	$\{{}^{1}/F_1, {}^{1}/F_2, {}^{.5}/F_3, {}^{1}/F_4\}$	
3	$\{{}^{1}/F_1, {}^{.5}/F_2, {}^{1}/F_4\}$	
4	$\{{}^{1}/F_1, {}^{.5}/F_2, {}^{.5}/F_4\}$	
5	$\{{}^{1}/F_2\}$	
6	$\emptyset$	

Figure 4.4: Concept lattice $\mathcal{B}(X^{*_G\cap}, Y^{\mathrm{id}_L\cup}, I)$ of the formal context from Table 4.4



1	X
2	$\{{}^{1}\!/F_1, {}^{1}\!/F_2, {}^{.5}\!/F_3, {}^{1}\!/F_4\}$
3	$\{{}^{1}\!/F_1, {}^{.5}\!/F_2, {}^{1}\!/F_4\}$
4	$\{{}^{.5}\!/F_1, {}^{.5}\!/F_4\}$
5	$\{{}^{.5}\!/F_1\}$
6	$\{{}^{1}\!/F_2\}$
7	$\{{}^{.5}\!/F_2\}$
8	$\emptyset$

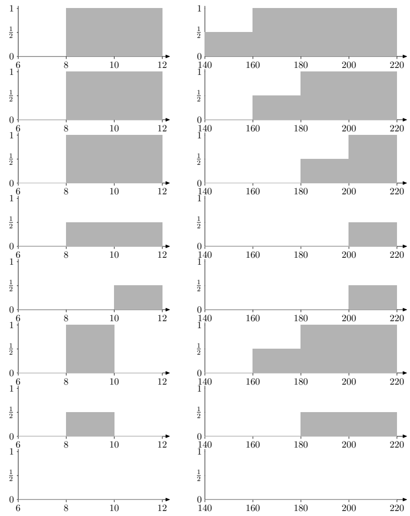
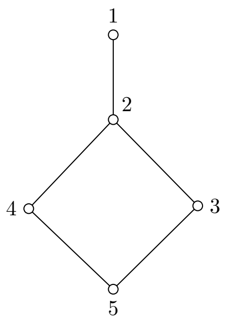


Figure 4.5: Concept lattice $\mathcal{B}(X^{\mathrm{id}_L\cap}, Y^{\square_{AG}\cup}, I)$ of the formal context from Table 4.4



1	X
2	$\{{}^{1}\!/F_1, {}^{1}\!/F_2, {}^{.5}\!/F_3, {}^{1}\!/F_4\}$
3	$\{{}^{1}\!/F_1, {}^{.5}\!/F_2, {}^{1}\!/F_4\}$
4	$\{{}^{1}\!/F_3\}$
5	$\emptyset$

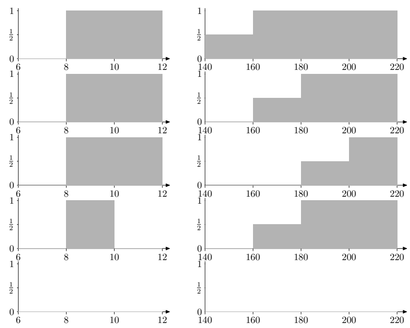


Figure 4.6: Concept lattice $\mathcal{B}(X^{*_G\cap}, Y^{\square_{AG}\cup}, I)$ of the formal context from Table 4.4

Chapter 5

Logic of Containment

Logical calculi for reasoning about binary (yes-or-no) attributes as well as computational aspects such as extraction of various types of rules from data have been intensively studied in the past. Particular attention has been paid to various types of if-then rules, see e.g. [28] for logic of binary attributes and its connection to functional dependencies; [18] and [15] for description and algorithms for extraction of a smallest complete set of if-then dependencies from binary data; [41] for an overview of association rules and [36] for related logical calculi. [36] which, however, concerns much more general dependencies in binary data, namely those in somewhat forgotten [20] which provides sophisticated logico-statistical foundations for hypotheses formation (association rules are a very particular case).

Recently, several classical as well as new aspects of reasoning about attributes have been developed in a series of papers for graded attributes, i.e. for attributes such as *red* or *good performance* which apply to objects in degrees, see e.g. [12] for an overview of results and [10] for computational aspects. In this chapter, we present a general logic of if-then rules $A \Rightarrow B$ for graded attributes which read: if all attributes of an object are contained A then they are contained in B. We introduce basic syntactic and semantic notions, covering two basic meanings of containment of graded attributes (Section 5.1), describe complete non-redundant sets of if-then rules (Section 5.2), and a logic for reasoning with such dependencies with its ordinary-style and graded-style completeness (Section 5.3). The next section provides preliminaries.

5.1 Basic concepts of syntax and semantics

Let Y be a set of (symbols of) graded attributes. A *fuzzy attribute implication* (over Y) is an expression

$$A \Rightarrow B,$$

where $A, B \in \mathbf{L}^Y$ (A and B are $\mathbf{L}$-sets of attributes). Fuzzy attribute implications (FAIs) are our basic formulas. The intended meaning of $A \Rightarrow B$ is:

> if all attributes of an object are contained in A then they are contained in B.

In a graded setting, having an attribute is a matter of degree. Hence, validity of $A \Rightarrow B$ is naturally a matter of degree as well. We need to be careful about the meaning of containment. We provide a general semantics which covers two appealing meanings of containment: bivalent containment and graded containment. Let M be an $\mathbf{L}$-set representing attributes of object x, i.e. $M(y)$ is a degree to which x has attribute y. By $||A \Rightarrow B||_M^{\cap\cup}$, we denote the truth degree of $A \Rightarrow B$ for x (we attach the superscript $^{\cap\cup}$ in order to distinguish our semantics from that of [12]). Our aim is to capture to the following intuitions: For a bivalent approach to containment, $||A \Rightarrow B||_M^{\cap\cup} = 1$ ($A \Rightarrow B$ is fully true) means:

$$\text{if } M \subseteq A \text{ then } M \subseteq B.$$

Note that $M \subseteq A$ means that $M(y) \leq A(y)$ for all $y \in Y$. For a graded approach to containment, $||A \Rightarrow B||_M^{\cap\cup} = 1$ means:

$$S(M, A) \leq S(M, B),$$

i.e. a degree to which M is contained in A is less than or equal to the degree to which M is contained in B. Both of the approaches can be obtained as particular cases of a general definition which uses a hedge:

Definition 5. *A degree $||A \Rightarrow B||_M^{\cap\cup} \in L$ to which $A \Rightarrow B$ is valid in M (M is an $\mathbf{L}$-set of attributes of some object) is defined by*

$$||A \Rightarrow B||_M^{\cap\cup} = S(M, A)^* \rightarrow S(M, B) \tag{5.1}$$

Now, one can easily check that for * being globalization and identity, this definition meets the above-described intuitive requirements regarding the bivalent and ordinary approach to containment. Note also that $||A \Rightarrow B||_M^{\cap\cup}$ is a general degree, possibly different from 0 and 1.

We are going to evaluate FAIs $A \Rightarrow B$ in data tables with graded attributes. Such tables can be regarded a triplets $\langle X, Y, I\rangle$ where X and Y are sets of objects (rows) and attributes (columns), and $I : X \times Y \rightarrow I$ is an $\mathbf{L}$-relation with $I(x, y)$ being interpreted as the degree to which attribute y applies to object x.

Definition 6. *Let $\mathcal{M}$ be a collection of $\mathbf{L}$-sets $M \in \mathbf{L}^Y$. A degree $||A \Rightarrow B||_{\mathcal{M}}^{\cap\cup}$ to which $A \Rightarrow B$ is valid in $\mathcal{M}$ is defined by*

$$||A \Rightarrow B||_{\mathcal{M}}^{\cap\cup} = \textstyle\bigwedge_{M \in \mathcal{M}} ||A \Rightarrow B||_M^{\cap\cup} \tag{5.2}$$

A degree $||A \Rightarrow B||_{\langle X,Y,I\rangle}^{\cap\cup}$ to which $A \Rightarrow B$ is valid in a data table $\langle X, Y, I\rangle$ with graded attributes is defined by

$$||A \Rightarrow B||_{\langle X,Y,I\rangle}^{\cap\cup} = ||A \Rightarrow B||_{\{I_x | x \in X\}}^{\cap\cup}, \tag{5.3}$$

where I_x denotes the "row of x", i.e. $I_x(y) = I(x, y)$.

There is a close relationship between our semantics of FAIs and isotone Galois connections and the lattices of their fixpoints which were studied in chapter 3:
${}^{\cap} : L^X \to L^Y$ and ${}^{\cup} : L^Y \to L^X$ by

$$A^{\cap}(y) = \bigvee_{x\in X}(A(x)^* \otimes I(x,y)), \tag{5.4}$$
$$B^{\cup}(x) = \bigwedge_{y\in Y}(I(x,y) \to B(y)), \tag{5.5}$$

for $A \in L^X$ and $B \in L^Y$.

Note that (5.4) and (5.5) can be considered as both isotone Galois connections with truth-stressing hedges with ${}^{*_X} = {}^{\mathrm{id}_L}$ and isotone Galois connections with truth-stressing hedge and truth-depressing hedge with ${}^{\square} = {}^{\mathrm{id}_L}$.

The following theorem shows a basic connection between our semantics and isotone Galois connection with a truth-stressing hedge. It says that validity of $A \Rightarrow B$ in a data table $\langle X, Y, I\rangle$ coincides with validity in the system of its intents and with a degree of containment of $A^{\cup\cap}$ in B.

Theorem 20. *For a FAI $A \Rightarrow B$ and a formal $\mathbf{L}$-context $\langle X, Y, I\rangle$,*

$$||A \Rightarrow B||^{\cap\cup}_{\langle X,Y,I\rangle} = ||A \Rightarrow B||^{\cap\cup}_{\mathrm{Int}(X^{\cap},Y^{\cup},I)} \tag{5.6}$$
$$= S(A^{\cup\cap}, B) \tag{5.7}$$

Proof. (5.6): $||A \Rightarrow B||^{\cap\cup}_{\langle X,Y,I\rangle} = ||A \Rightarrow B||^{\cap\cup}_{\{I_x|x\in X\}} = \bigwedge_{x\in X} S(I_x, A)^* \to S(I_x, B) = \bigwedge_{x\in X} A^{\cup}(x)^* \to B^{\cup}(x) = \bigwedge_{x\in X} A^{\cup *}(x) \to B^{\cup}(x) = S(A^{\cup *}, B^{\cup}) = S(A^{\cup *\cap}, B) = S(A^{\cup\cap}, B)$

(5.7): "$\leq$": $||A \Rightarrow B||^{\cap\cup}_{\langle X,Y,I\rangle}$
$\leq \bigwedge_{M\in\mathrm{Int}(X^{\cap},Y^{\cup},I)} S(M, A)^* \to S(M, B)$ iff $S(A^{\cup\cap}, B) \leq \bigwedge_{M\in\mathrm{Int}(X^{\cap},Y^{\cup},I)} S(M, A)^* \to S(M, B)$
iff for each $M \in \mathrm{Int}(X^{\cap}, Y^{\cup}, I)$:
$S(A^{\cup\cap}, B) \leq S(M, A)^* \to S(M, B)$
iff for each $M \in \mathrm{Int}(X^{\cap}, Y^{\cup}, I)$:
$S(M, A)^* \otimes S(A^{\cup\cap}, B) \leq S(M, B)$
iff for each $M \in \mathrm{Int}(X^{\cap}, Y^{\cup}, I)$ (note that $M = M^{\cup\cap}$):
$S(M^{\cup\cap}, A)^* \otimes S(A^{\cup\cap}, B) \leq S(M^{\cup\cap}, B)$
iff for each $M \in \mathrm{Int}(X^{\cap}, Y^{\cup}, I)$:
$S(M^{\cup *}, A^{\cup})^* \otimes S(A^{\cup *}, B^{\cup}) \leq S(M^{\cup *}, B^{\cup})$
which is true since $S(M^{\cup *}, A^{\cup})^* \leq S(M^{\cup *}, A^{\cup *})$.

"$\geq$": follows from $I_x = \{x\}^{\cap} \in \mathrm{Int}(X^{\cap}, Y^{\cup}, I)$. □

Let us now turn to theories and models.

In graded setting, a theory naturally consists of formulas to which we attach grades [17, 19]. That is, a *theory* of FAIs is an $\mathbf{L}$-set T of FAIs. The degree $T(A \Rightarrow B)$ to which $A \Rightarrow B$ belongs to T can be seen as a degree to which we assume the validity of $A \Rightarrow B$.

From another point of view, T can be seen an **L**-set of implications extracted from data such that $T(A \Rightarrow B)$ is a degree to which $A \Rightarrow B$ holds true in data. If T is an ordinary set, we call it a *crisp theory*, and write $A \Rightarrow B \in T$ if $T(A \Rightarrow B) = 1$ and $A \Rightarrow B \notin T$ if $T(A \Rightarrow B) = 0$.

For a theory T, the set $\mathrm{Mod}(T)$ of all *models* of T is defined by

$$\begin{aligned}\mathrm{Mod}(T) = \{M \in \mathbf{L}^Y \mid & \text{for each } A, B \in \mathbf{L}^Y : \\ & T(A \Rightarrow B) \leq ||A \Rightarrow B||_M^{\cap\cup}\}.\end{aligned}$$

That is, $M \in \mathrm{Mod}(T)$ means that for each attribute implication $A \Rightarrow B$, a degree to which $A \Rightarrow B$ holds in M is higher than or at least equal to a degree $T(A \Rightarrow B)$ prescribed by T. Particularly, for a crisp T, $\mathrm{Mod}(T) = \{M \in \mathbf{L}^Y \mid \text{for each } A \Rightarrow B \in T : ||A \Rightarrow B||_M^{\cap\cup} = 1\}$.

A degree $||A \Rightarrow B||_T^{\cap\cup} \in L$ to which $A \Rightarrow B$ *semantically follows* from an **L**-set T of attribute implications is defined by

$$||A \Rightarrow B||_T^{\cap\cup} = \textstyle\bigwedge_{M \in \mathrm{Mod}(T)} ||A \Rightarrow B||_M^{\cap\cup}.$$

We need the following lemma which says that validity to a degree can be reduced to validity to degree 1.

Lemma 21. *For $A, B, M \in \mathbf{L}^Y$ and $c \in L$ we have*

$$c \leq ||A \Rightarrow B||_M^{\cap\cup} \textit{ iff } ||A \Rightarrow c \to B||_M^{\cap\cup} = 1. \tag{5.8}$$

Proof. By simple derivation. □

Lemma 21 has surprising consequences. It enables us to reduce the concept of a model of an **L**-set of fuzzy attribute implications to the concept of a model of an ordinary set of fuzzy attribute implications, and to reduce the concept of semantic entailment from an **L**-set of fuzzy attribute implications to the concept of semantic entailment from an ordinary set of fuzzy attribute implications:

Lemma 22. *Let T be an **L**-set of fuzzy attribute implications and $A, B \in \mathbf{L}^Y$. Define an ordinary set $\mathrm{c}(T)$ of fuzzy attribute implications by*

$$\mathrm{c}(T) = \{A \Rightarrow T(A \Rightarrow B) \to B \mid A, B \in \mathbf{L}^Y \textit{ and } T(A \Rightarrow B) \to B \neq Y\}. \tag{5.9}$$

Then we have

$$\mathrm{Mod}(T) = \mathrm{Mod}(\mathrm{c}(T)), \tag{5.10}$$

$$||A \Rightarrow B||_T^{\cap\cup} = ||A \Rightarrow B||_{\mathrm{c}(T)}^{\cap\cup}. \tag{5.11}$$

Proof. (5.10) directly using Lemma 21. (5.11) is a consequence of (5.10). □

Furthermore, Lemma 21 enables us to reduce the concept of a degree of entailment of a fuzzy attribute implication from an **L**-set of fuzzy attribute implications to the concept of an entailment in degree 1 (full entailment) of a fuzzy attribute implication from an **L**-set of fuzzy attribute implications:

Lemma 23. *For $A, B \in \mathbf{L}^Y$ and an $\mathbf{L}$-set T of fuzzy attribute implications we have*

$$||A \Rightarrow B||_T^{\cap\cup} = \bigvee\{c \in L \mid ||A \Rightarrow c \to B||_T^{\cap\cup} = 1\}.$$

Proof. Using Lemma 21, we have

$$\begin{aligned} ||A \Rightarrow B||_T^{\cap\cup} &= \textstyle\bigwedge_{M \in \mathrm{Mod}(T)} ||A \Rightarrow B||_M^{\cap\cup} = \\ &= \textstyle\bigvee\{c \in L \mid c \leq ||A \Rightarrow B||_M^{\cap\cup} \text{ for each } M \in \mathrm{Mod}(T)\} = \\ &= \textstyle\bigvee\{c \in L \mid ||A \Rightarrow c \to B||_T^{\cap\cup} = 1\}. \end{aligned}$$

□

Therefore, we have:

Corollary 24. *For $A, B \in \mathbf{L}^Y$ and an $\mathbf{L}$-set T of fuzzy attribute implications we have*

$$||A \Rightarrow B||_T^{\cap\cup} = \bigvee\{c \in L \mid ||A \Rightarrow c \otimes B||_{\mathrm{c}(T)}^{\cap\cup} = 1\},$$

with $\mathrm{c}(T)$ defined as in Lemma 22.

Corollary 24 shows that the concept of a degree of entailment from an **L**-set of fuzzy attribute implications can be reduced to entailment in degree 1 from a set of fuzzy attribute implications. We use this fact in the subsequent development.

An ordinary set T of fuzzy attribute implications is said to be *semantically closed* if $||A \Rightarrow B||_T^{\cap\cup} = 1$ iff $A \Rightarrow B \in T$, i.e. if $T = \{A \Rightarrow B \mid ||A \Rightarrow B||_T^{\cap\cup} = 1\}$.

5.2 Non-redundant bases

In this section, we describe non-redundant bases of formal **L**-contexts $\langle X, Y, I\rangle$.

Definition 7. *A set T of FAIs is called* complete *in $\langle X, Y, I\rangle$ if $||A \Rightarrow B||_T^{\cap\cup} = ||A \Rightarrow B||_{\langle X,Y,I\rangle}^{\cap\cup}$ for each attribute implication $A \Rightarrow B$. If T is complete and no proper subset if T is complete, then T is called a* non-redundant basis *(of $\langle X, Y, I\rangle$).*

It follows that if T is complete, every $A \Rightarrow B$ from T is valid in $\langle X, Y, I\rangle$ to degree 1 and for any other $C \Rightarrow D$, the degree to which $C \Rightarrow D$ is valid in $\langle X, Y, I\rangle$ equals the degree to which T entails $C \Rightarrow D$. That is, non-redundant sets are minimal sets of FAIs with complete information about validity of FAIs in the data.

Lemma 25. *For any $A, M \in \mathbf{L}^Y$ we have*

$$||A \Rightarrow A^{\cup\cap}||_M = 1 \text{ for each } A \in \mathbf{L}^Y \text{ and } M = M^{\cup\cap} \tag{5.12}$$

Proof. Let $M = M^{\cup\cap}$. We have $S(M,A)^* \leq S(M^{\cup},A^{\cup})^* \leq S(M^{\cup *},A^{\cup *}) \leq S(M^{\cup\cap},A^{\cup\cap}) = S(M,A^{\cup\cap})$. Thus, $S(M,A)^* \to S(M,A^{\cup\cap}) = 1$, i.e. $||A \Rightarrow A^{\cup\cap}||_M = 1$ □

The following theorem characterizes complete sets.

Theorem 26. *T is complete iff* $\mathrm{Mod}(T) = \mathrm{Int}(X^{\cap}, Y^{\cup}, I)$.

Proof. Let T be complete. Suppose $M \in \mathrm{Mod}(T)$. We have $||M \Rightarrow M^{\cup\cap}||^{\cup\cap}_{\mathrm{Int}(X^{\cap},Y^{\cup},I)} = S(M^{\cup\cap}, M^{\cup\cap}) = 1$ by (5.6), i.e. $||M \Rightarrow M^{\cup\cap}||^{\cup\cap}_T = 1$ by completeness and (5.6). Since M is a model of T, we have $||M \Rightarrow M^{\cup\cap}||^{\cup\cap}_M = 1$ which immediately gives $1 = S(M,M)^* \leq S(M, M^{\cup\cap})$, i.e. $M \subseteq M^{\cup\cap}$. That is, $M \in \mathrm{Int}(X^{\cap}, Y^{\cup}, I)$ which proves that $\mathrm{Mod}(T) \subseteq \mathrm{Int}(X^{\cap}, Y^{\cup}, I)$.

Now take $M \in \mathrm{Int}(X^{\cap}, Y^{\cup}, I)$. For each implication $A \Rightarrow B \in T$ we have $||A \Rightarrow B||^{\cup\cap}_M \geq ||A \Rightarrow B||^{\cup\cap}_{\mathrm{Int}(X^{\cap},Y^{\cup},I)} = ||A \Rightarrow B||^{\cap\cup}_{\mathrm{Mod}(T)} = 1$ by (5.6), i.e. $M \in \mathrm{Mod}(T)$ showing $\mathrm{Int}(X^{\cap}, Y^{\cup}, I) \subseteq \mathrm{Mod}(T)$.

Conversely, if $\mathrm{Mod}(T) = \mathrm{Int}(X^{\cap}, Y^{\cup}, I)$ then $||A \Rightarrow B||^{\cap\cup}_T = ||A \Rightarrow B||^{\cap\cup}_{\mathrm{Int}(X^{\cap},Y^{\cup},I)} = ||A \Rightarrow B||^{\cap\cup}_{\langle X,Y,I\rangle}$. □

Definition 8. *Given $\langle X,Y,I\rangle$, $\mathcal{P} \subseteq \mathbf{L}^Y$ is called a* system of pseudo-intents *of $\langle X,Y,I\rangle$ if for each $P \in \mathbf{L}^Y$ we have:*

$P \in \mathcal{P}$ iff $P \neq P^{\cup\cap}$ and $||Q \Rightarrow Q^{\cup\cap}||_P = 1$ for each $Q \in \mathcal{P}$ with $Q \neq P$.

In what follows, $\mathcal{P}$ denotes a system of pseudo-intents.

Lemma 27. *Let $T = \{P \Rightarrow P^{\cup\cap} | P \in \mathcal{P}\}$. Then* $\mathrm{Mod}(T) \subseteq \mathrm{Int}(X^{\cap}, Y^{\cup}, I)$.

Proof. It suffices to show that each model $M \in \mathrm{Mod}(T)$ is an intent of $\mathrm{Int}(X^{\cap}, Y^{\cup}, I)$. By contradiction, let $M \in \mathrm{Mod}(T)$ and assume $M \notin \mathrm{Int}(X^{\cap}, Y^{\cup}, I)$. That is $M \neq M^{\cup\cap}$. Since $M \in \mathrm{Mod}(T)$, we have $||Q \Rightarrow Q^{\cup\cap}||^{\cup\cap}_M = 1 (Q \in \mathcal{P})$. Therefore $M \in \mathcal{P}$ by definition of $\mathcal{P}$, i.e. $M \Rightarrow M^{\cup\cap}$ belongs to T. We have

$$\begin{aligned} ||M \Rightarrow M^{\cup\cap}||^{\cup\cap}_M &= S(M,M)^* \to S(M^{\cup\cap}, M) \\ &= S(M^{\cup\cap}, M) \neq 1 \end{aligned}$$

which contradicts $M \in \mathrm{Mod}(T)$. □

Theorem 28. $T = \{P \Rightarrow P^{\cup\cap} \mid P \in \mathcal{P}\}$ *is complete.*

Proof. We show that

$$||A \Rightarrow B||^{\cap\cup}_T = ||A \Rightarrow B||^{\cap\cup}_{\mathrm{Int}(X^{\cap},Y^{\cup},I)}$$

for each FAI $A \Rightarrow B$. Completeness of T is then a consequence of (5.6). By Lemma 25, each intent $M \in \mathrm{Int}(X^{\cap}, Y^{\cup}, I)$ is a model of T, proving the $\leq$-part. The $\geq$-part follows from Lemma 27. □

The following theorem is the main result of this section. It says that in order to get a non-redundant basis of $\langle X, Y, I\rangle$, it is sufficient to compute a system of pseudo-intents of $\langle X, Y, I\rangle$.

Theorem 29. *$T = \{P \Rightarrow P^{\cup\cap} | P \in \mathcal{P}\}$ is a non-redundant basis.*

Proof. By Theorem 28, T is complete. Now we are going to show the non-redundancy. Take $T' \subset T$. Clearly, there must be $P \in \mathcal{P}$ s.t. $P \Rightarrow P^{\cup\cap}$ does not belong to T'. In addition to that, we have $||Q \Rightarrow Q^{\cup\cap}||_P^{\cup\cap} = 1$ ($Q \in \mathcal{P}, Q \neq P$) by Definition 8, i.e. $P \in \mathrm{Mod}(T')$. On the other hand, $P \notin \mathrm{Mod}(T)$ since $||P \Rightarrow P^{\cup\cap}||_P^{\cup\cap} = S(P, P^{\cup\cap} \neq 1)$. That is,

$$||P \Rightarrow P^{\cup\cap}||_{\langle X,Y,I\rangle}^{\cup\cap} = ||P \Rightarrow P^{\cup\cap}||_T^{\cup\cap} \neq ||P \Rightarrow P^{\cup\cap}||_{T'}^{\cup\cap}$$

i.e. T' is not complete, showing the non-redundancy of T. □

Lemma 30. *Let $P, Q \in \mathcal{P} \cup \mathrm{Int}(X^{\cap}, Y^{\cap}, I)$ such that*

$$S(Q, P)^* \leq S(P \cup Q, P^{\cup\cap}) \tag{5.13}$$

$$S(P, Q)^* \leq S(P \cup Q, Q^{\cup\cap}) \tag{5.14}$$

Then $P \cup Q \in \mathrm{Int}(X^{\cap}, Y^{\cap}, I)$.

Proof. Put $T' = T\ \{P \Rightarrow P^{\cup\cap}, P \Rightarrow P^{\cup\cap}\}$, where $T = \{P \Rightarrow P^{\cup\cap} | P \in \mathcal{P}\}$. Definition 8 and lemma 25 yield $P, Q \in \mathrm{Mod}(T')$. Hence, for each $A \Rightarrow B \in T'$ we have $S(P, A)^* \leq S(P, B)$ and $S(Q, A)^* \leq S(Q, B)$. Thus,

$$\begin{aligned} S(P \cup Q, A)^* &= (S(P, A) \wedge S(Q, A))^* \leq \\ &\leq S(P, A)^* \wedge S(Q, A)^* \leq \\ &\leq S(P, B) \wedge S(Q, B) = S(P \cap Q, B) \end{aligned}$$

which immediately gives that $P \cup Q$ is a model of T'. Taking into account Lemma 27, it is sufficient to show that $P \cap Q$ is a model of $\{P \Rightarrow P^{\cup\cap}, Q \Rightarrow Q^{\cup\cap}\}$. By (5.13) and (5.14) we have

$$S(P \cup Q, P)^* = S(Q, P)^* \leq S(P \cup Q, P^{\cup\cap})$$

and

$$S(P \cup Q, Q)^* = S(P, Q)^* \leq S(P \cup Q, Q^{\cup\cap})$$

i.e. $||P \Rightarrow P^{\cup\cap}||_{P \cup Q}^{\cup\cap} = 1$ and $||Q \Rightarrow Q^{\cup\cap}||_{P \cup Q}^{\cup\cap} = 1$. □

If the scale L of grades is finite and * is globalization, the non-redundant basis T given by pseudo-intents is the smallest one in terms of the number of FAIs it contains:

Theorem 31. *Let $\mathbf{L}$ be a finite residuated lattice with * being the globalization. Let T' be complete in $\langle X, Y, I\rangle$, where Y is finite. Then $|T| \leq |T'|$, where $T = \{P \Rightarrow P^{\cup\cap} | P \in \mathcal{P}\}$*

Proof. We prove the claim by showing that for each $P \in \mathcal{P}$, T' contains an implication $A \Rightarrow B$ s.t. $A^{\cup\cap} = P^{\cup\cap}$.

Take $P \in \mathcal{P}$. Since $P \neq P^{\cup\cap}$ and T' is complete, Theorem 26 yields that T' contains $A \Rightarrow B$ such that $||A \Rightarrow B||_P^{\cup\cap} \neq 1$. That is, we have $P \subseteq A$ and $P \nsubseteq B$ because * is the globalization. Completeness of T' together with (5.6) yields $S(A^{\cup\cap}, B) = 1$ i.e. $A^{\cup\cap} \subset B$. Thus, from $A^{\cup\cap} \subset B$ and $P \nsubseteq B$ we have $P \nsubseteq A^{\cup\cap}$.

As a consequence, $A^{\cup\cap} \cup P$ is not an intent, because $P \subseteq A$ and $P \nsubseteq A^{\cup\cap}$ yield $A^{\cup\cap} \subset A^{\cup\cap} \cup P \subseteq A$, i.e. the union $A^{\cup\cap} \cup P$ is not closed under $^{\cup\cap}$.

Now we claim that $A^{\cup\cap} \subseteq P$. By contradiction, $A^{\cup\cap} \nsubseteq P$ and $P \nsubseteq A^{\cup\cap}$ would give $P \cup A^{\cup\cap} \in \mathrm{Int}(X^{\cap}, Y^{\cup}, I)$ by Lemma 30 which would violate $A^{\cup\cap} \cup P \in \mathrm{Int}(X^{\cap}, Y^{\cup}, I)$ as observed lately.

Therefore, $A \subseteq P$ gives $A^{\cup\cap} \subseteq P^{\cup\cap}$ while $A^{\cup\cap} \subseteq P$ gives $A^{\cup\cap} = A^{\cup\cap\cup\cap} \subseteq P^{\cup\cap}$. □

5.3 Completeness theorems

In this section, we introduce an axiomatic system for fuzzy attribute logic (FAL) and prove completeness theorems. First, we introduce deduction rules and a notion of a proof of a fuzzy attribute implication from an ordinary set T of fuzzy attribute implications. Second, we prove that a fuzzy attribute implication $A \Rightarrow B$ is provable from an ordinary set T of fuzzy attribute implications iff $A \Rightarrow B$ semantically follows from T in degree 1. Third, we introduce a concept of a degree $|A \Rightarrow B|_T^{\cap\cup}$ of provability of a fuzzy attribute implication $A \Rightarrow B$ from an $\mathbf{L}$-set T of fuzzy attribute implications and show that $|A \Rightarrow B|_T^{\cap\cup} = ||A \Rightarrow B||_T^{\cap\cup}$.

5.3.1 Deduction rules

Our axiomatic system consists of the following *deduction rules.*

(Ax) infer $A \Rightarrow A \cup B$,

(DCut) from $A \Rightarrow B$ and $B \cap C \Rightarrow D$ infer $A \cap C \Rightarrow D$,

(Sh) from $A \Rightarrow B$ infer $c^* \rightarrow A \Rightarrow c^* \rightarrow B$

for each $A, B, C, D \in \mathbf{L}^Y$, and $c \in L$. Rules (Ax)–(Sh) are to be understood as usual deduction rules: having fuzzy attribute implications which are of the form of fuzzy attribute implication in the input part (the part preceding "infer") of a rule, a rule allows us to infer (in one step) the corresponding fuzzy attribute implication in the output part (the part following "infer") of a rule. (Ax) is a nullary rule (axiom) which says that each $A \Rightarrow A \cup B$ ($A, B \in \mathbf{L}^Y$) is inferred in one step.

Remark 5. *If * is globalization, (Sh) can be omitted. Indeed, for $c = 1$, we have $c^* = 1$ and (Sh) becomes "from $A \Rightarrow B$ infer $A \Rightarrow B$" which is a trivial rule; for $c < 1$, we have $c^* = 0$ and (Sh) becomes "from $A \Rightarrow B$ infer $Y \Rightarrow Y$" which can be omitted since $Y \Rightarrow Y$ can be inferred by (Ax).*

A fuzzy attribute implication $A \Rightarrow B$ is called *provable* from a set T of fuzzy attribute implications using a set $\mathcal{R}$ of deduction rules, written $T \vdash_{\mathcal{R}} A \Rightarrow B$, if there is a sequence $\varphi_1, \dots, \varphi_n$ of fuzzy attribute implications such that φ_n is $A \Rightarrow B$ and for each φ_i we either have $\varphi_i \in T$ or φ_i is inferred (in one step) from some of the preceding formulas (i.e., $\varphi_1, \dots, \varphi_{i-1}$) using some deduction rule from $\mathcal{R}$. If $\mathcal{R}$ consists of (Ax)–(Sh), we say just "provable ..." instead of "provable ... using $\mathcal{R}$" and write just $T \vdash A \Rightarrow B$ instead of $T \vdash_{\mathcal{R}} A \Rightarrow B$.

A deduction rule "from $\varphi_1, \dots, \varphi_n$ infer φ" (φ_i, φ are fuzzy attribute implications) is said to be derivable from a set $\mathcal{R}$ of deduction rules if $\{\varphi_1, \dots, \varphi_n\} \vdash_{\mathcal{R}} \varphi$. Again, if $\mathcal{R}$ consists of (Ax)–(Sh), we omit $\mathcal{R}$.

Lemma 32. *The following deduction rules are derivable from* (Ax) *and* (DCut):

(Ref) *infer* $A \Rightarrow A$,

(Wea) *from* $A \Rightarrow B$ *infer* $A \cap C \Rightarrow B$,

(Add) *from* $A \Rightarrow B$ *and* $A \Rightarrow C$ *infer* $A \Rightarrow B \cap C$,

(Pro) *from* $A \Rightarrow B \cap C$ *infer* $A \Rightarrow B$,

(Tra) *from* $A \Rightarrow B$ *and* $B \Rightarrow C$ *infer* $A \Rightarrow C$,

for each $A, B, C, D \in \mathbf{L}^Y$.

Proof. By simple derivation. □

5.3.2 Ordinary completeness

In this section, we show that deduction rules (Ax)–(Sh) are sound and we prove their completeness. We restrict ourselves to the case of a finite $\mathbf{L}$.

A deduction rule "from $\varphi_1, \dots, \varphi_n$ infer φ" is said to be *sound* if for each $M \in \mathrm{Mod}(\{\varphi_1, \dots, \varphi_n\})$ we have $M \in \mathrm{Mod}(\{\varphi\})$, i.e. each model of all of $\varphi_1, \dots, \varphi_n$ is also a model of φ.

Lemma 33. *Each of the deduction rules* (Ax)–(Sh) *is sound.*

Proof. For illustration, we check (Sh). Let $M \in \mathrm{Mod}(\{A \Rightarrow B\})$. We have to show that $M \in \mathrm{Mod}(\{c^* \to A \Rightarrow c^* \to B\})$.

First, $M \in \mathrm{Mod}(\{A \Rightarrow B\})$ iff $||A \Rightarrow B||_M^{\cap\cup} = 1$ iff $S(M, A)^* \le S(M, B)$ iff

$$\text{for each } y \in Y: \ M(y) \otimes S(M, A)^* \le B(y). \tag{5.15}$$

Second, $M \in \mathrm{Mod}(\{c^* \to A \Rightarrow c^* \to B\})$ iff $||c^* \to A \Rightarrow c^* \to B||_M^{\cap\cup} = 1$ iff $S(M, c^* \to A)^* \le S(M, c^* \to B)$ iff for each $y \in Y$ we have

$$M(y) \otimes S(M, c^* \to A)^* \le c^* \to B(y)$$

which is true by 5.15:

$$\begin{aligned} & M(y) \otimes S(M, c^* \to A)^* \le \\ & \le M(y) \otimes (c^{**} \to S(M, A)^*) \le \\ & \le M(y) \otimes S(M, A)^* \le B(y). \end{aligned}$$

We proved that (Sh) is sound.

Soundness of (Ax) and (DCut) can be proved analogously.

□

Remark 6. *Note that deduction rules for semantics related to antitone Galois connections truth-stressing hedge [34]* (Ax) *infer $A \cup B \Rightarrow A$,*

(Cut) *from $A \Rightarrow B$ and $B \cup C \Rightarrow D$ infer $A \cup C \Rightarrow D$,*

(Mul) *from $A \Rightarrow B$ infer $c^* \otimes A \Rightarrow c^* \otimes B$*

are not sound for our semantics.

Indeed, for (Ax) *is enough to put $M = A$, $B \subset A$. For* (Cut) *put $M \subset A \cup C; M \not\subseteq A, B, C, D, B \cup D;$. For* (Mul) *put $^* = {}^{\mathrm{id}_X}, M = B \neq \emptyset; B \subset c \otimes A$.*

A set T of fuzzy attribute implications is said to be *syntactically closed* if $T \vdash A \Rightarrow B$ iff $A \Rightarrow B \in T$, i.e. if $T = \{A \Rightarrow B \,|\, T \vdash A \Rightarrow B\}$. The following lemma is almost immediate.

Lemma 34. *A set T of fuzzy attribute implications is syntactically closed iff we have:*

(Ax)-closure $A \Rightarrow A \cup B \in T$,

(DCut)-closure *if $A \Rightarrow B \in T$ and $B \cap C \Rightarrow D \in T$ then $A \cap C \Rightarrow D \in T$,*

(Sh)-closure *if $A \Rightarrow B \in T$ then $c^* \to A \Rightarrow c^* \to B \in T$*

for each $A, B, C, D \in \mathbf{L}^Y$, and $c \in L$.

Lemma 35. *Let T be a set of fuzzy attribute implications. If T is semantically closed then T is syntactically closed.*

Proof. By Lemma 34, we have to show that for each deduction rule "from $\varphi_1, \ldots, \varphi_n$ infer φ", i.e. one of (Ax)–(Sh), we have that if $\varphi_1, \ldots \varphi_n \in T$ then $\varphi \in T$. Let thus $\varphi_1, \ldots \varphi_n \in T$. Since $\{\varphi_1, \ldots \varphi_n\} \subseteq T$, for any model $M \in \mathrm{Mod}(T)$ we have $M \in \mathrm{Mod}(\{\varphi_1, \ldots \varphi_n\})$, i.e. $M \in \mathrm{Mod}(\{\varphi_i\})$ for each $i = 1, \ldots, n$. Since each of the rules (Ax)–(Sh) is sound, we conclude $M \in \mathrm{Mod}(\{\varphi\})$. Since M is an arbitrary model of T, this shows that φ is true in each model of T. Since T is semantically closed, we get $\varphi \in T$. □

Lemma 36. *Let T be a set of fuzzy attribute implications, let both Y and L be finite. If T is syntactically closed then T is semantically closed.*

Proof. Let T be syntactically closed. In order to show that T is semantically closed, it suffices to show $\{A \Rightarrow B \,|\, ||A \Rightarrow B||_T^{\cap\cup} = 1\} \subseteq T$. We prove this by showing that if $A \Rightarrow B \notin T$ then $A \Rightarrow B \notin \{A \Rightarrow B \,|\, ||A \Rightarrow B||_T^{\cap\cup} = 1\}$. Recall that since T is syntactically closed, T is closed under all of the rules (Ref)–(Tra) of Lemma 32.

Let thus $A \Rightarrow B \notin T$. To see $A \Rightarrow B \notin \{A \Rightarrow B \,|\, ||A \Rightarrow B||_T^{\cap\cup} = 1\}$, we show that there is $M \in \mathrm{Mod}(T)$ which is not a model of $A \Rightarrow B$. For this purpose, consider $M = A^-$ where A^- is the smallest one such that $A \Rightarrow A^- \in T$. Note that A^- exists.

Namely, $S = \{C \mid A \Rightarrow C \in T\}$ is non-empty since $A \Rightarrow A \in T$ by (Ref), S is finite by finiteness of Y and L, and for $A \Rightarrow C_1, \dots, A \Rightarrow C_n \in T$, we have $A \Rightarrow \bigcap_{i=1}^{n} C_i \in T$ by a repeated use of (Add).

We now need to show that (a) $||A \Rightarrow B||_{A^-}^{\cap\cup} \neq 1$ (i.e., A^- is not a model of $A \Rightarrow B$) and (b) for each $C \Rightarrow D \in T$ we have $||C \Rightarrow D||_{A^-}^{\cap\cup} = 1$ (i.e., A^- is a model of T).

(a): By contradiction, suppose $||A \Rightarrow B||_{A^-}^{\cap\cup} = 1$. Using $A^- \subseteq A$ we then get $1 = ||A \Rightarrow B||_{A^-}^{\cap\cup} = S(A^-, A)^* \to S(A^-, B) = 1 \to S(A^-, B) = S(A^-, B)$, i.e. $A^- \subseteq B$. Since $A \Rightarrow A^- \in T$, (Pro) would give $A \Rightarrow B \in T$, a contradiction.

(b): Let $C \Rightarrow D \in T$. We need to show $||C \Rightarrow D||_{A^-}^{\cap\cup} = 1$, i.e. $S(A^-, C)^* \to S(A^-, D) = 1$ which is equivalent to $S(A^-, C)^* \otimes A^- \subseteq D$, i.e. $A^- \subseteq S(A^-, C)^* \to D$. To see this, it is sufficient to show that $A \Rightarrow S(A^-, C)^* \to D \in T$ because A^- is the smallest one for which $A \Rightarrow A^- \in T$). Note that we have (b1) $A \Rightarrow A^- \in T$ by definition of A^-, (b2) $A^- \Rightarrow S(A^-, C)^* \to C \in T$ since as $A^- \subseteq S(A^-, C)^* \to C$, $A^- \Rightarrow S(A^-, C)^* \to C$ is an instance of (Ax); and (b3) $S(A^-, C)^* \to C \Rightarrow S(A^-, C)^* \otimes D \in T$ which follows from (Sh) applied to $C \Rightarrow D \in T$. Now, $A \Rightarrow S(A^-, C)^* \to D \in T$ follows by (Tra) applied twice to (b1), (b2), and (b3). $\square$

Corollary 37. *Let T be a set of fuzzy attribute implications. T is syntactically closed iff T is semantically closed.*

Theorem 38 ((ordinary) completeness)**.** *Let $\mathbf{L}$ and Y be finite. Let T be a set of fuzzy attribute implications. Then*

$$T \vdash A \Rightarrow B \quad \textit{iff} \quad ||A \Rightarrow B||_T^{\cap\cup} = 1.$$

Proof. Denote by $syn(T)$ the least syntactically closed set of fuzzy attribute implications which contains T. It can be shown that $syn(T) = \{A \Rightarrow B \mid T \vdash A \Rightarrow B\}$. Furthermore, denote by $sem(T)$ the least semantically closed set of fuzzy attribute implications which contains T. It can be shown that $sem(T) = \{A \Rightarrow B \mid ||A \Rightarrow B||_T^{\cap\cup} = 1\}$. To prove the claim, we need to show $syn(T) = sem(T)$. As $syn(T)$ is syntactically closed, it is also semantically closed by Corollary 37 which means $sem(syn(T)) \subseteq syn(T)$. Therefore, by $T \subseteq syn(T)$ we get

$$sem(T) \subseteq sem(syn(T)) \subseteq syn(T).$$

In a similar manner we get $syn(T) \subseteq sem(T)$, showing $syn(T) = sem(T)$. The proof is complete. $\square$

5.3.3 Graded completeness

In this section, we introduce a notion of a degree $|A \Rightarrow B|_T^{\cap\cup}$ of provability of a fuzzy attribute implication $A \Rightarrow B$ from an $\mathbf{L}$-set T of attribute implications. Then, we show that $|A \Rightarrow B|_T^{\cap\cup} = ||A \Rightarrow B||_T^{\cap\cup}$, which can be understood as a graded completeness (completeness in degrees). Note that graded completeness was introduced by Pavelka [30], see also [17, 19] for further information.

For an **L**-set T of fuzzy attribute implications and for $A \Rightarrow B$ we define a *degree* $|A \Rightarrow B|_T^{\cap\cup} \in L$ *to which* $A \Rightarrow B$ *is provable from* T by

$$|A \Rightarrow B|_T = \bigvee\{c \in L \mid \mathrm{c}(T) \vdash A \Rightarrow c \otimes B\}, \tag{5.16}$$

where $\mathrm{c}(T)$ is defined by (5.9).

Theorem 39 (graded completeness)**.** *Let* **L** *and* Y *be finite. Then for every* **L**-*set* T *of fuzzy attribute implications and* $A \Rightarrow B$ *we have* $|A \Rightarrow B|_T^{\cap\cup} = ||A \Rightarrow B||_T^{\cap\cup}$.

Proof. Consequence of Corollary 24 and Theorem 38. □

5.4 Illustrative example

We close the example "Cars" from Section 4.1 by presenting non-redundant base of the formal fuzzy context from Table 4.4. The attribute implications from the base are depicted in Fig. 5.1. The base was computed using pseudointents as proposed in Theorem 29 with * being a globalization ^{*_G}. Note that the attribute implications in Fig. 5.1 have their natural meaning; for instance the last attribute implications reads: "there is no car with consumption limited to 10-12l/100km which is very fast".

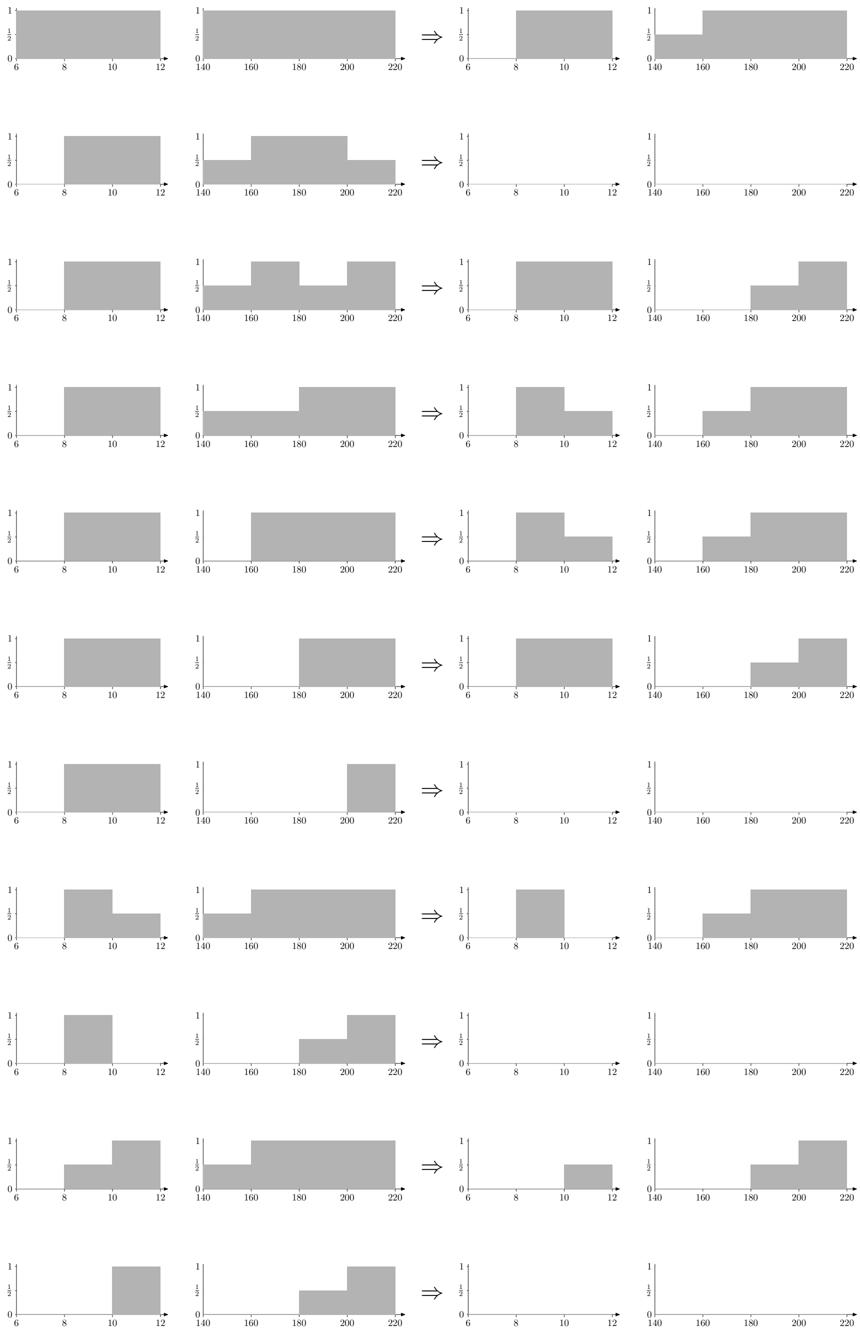


Figure 5.1: Non-redundant base of the formal **L**-context Cars from Table 4.4

Chapter 6

Conclusions

We have developed foundations of isotone Galois connections with a truth-stressing hedge and a truth-depressing hedge. We have explored basic calculus of such connections, i.e. on the properties analogous to those which are essential for the other type of Galois connections studied in the literature. We studied structure of $\mathcal{B}(X^{*\cap}, Y^{\square\cup}, I)$ and proved an analogy of the main theorem of concept lattices for our setting. Moreover, we compared our generalization with the approach studied in [3]. We showed how parameterization by hedges influences size of resulting concept lattice.

In addition, we have studied a logic of if-then rules such as "if all attributes of an object are among those from A then they are among those from B." We provided basic syntactic and semantic notions, described complete non-redundant sets of the if-then rules, and a logic for reasoning with such dependencies with its ordinary-style and graded-style completeness.

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