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PhD THESIS

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Observables on Quantum Structures

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I declare that I carried out this PhD thesis independently, and only with the cited sources, literature and other professional sources.

In .... date .... signature of the author

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Title: Observables on quantum structures

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Abstract: In the PhD thesis, one-to-one correspondence between finite-dimensional spectral resolutions and observables is established for various classes of algebras known as Quantum structures. The main result treats the case of monotone σ-complete effect algebras with Riesz Decomposition Property. The results are achieved using a technique of lifting spectral resolutions, which is presented and which is interesting on its own. Further, the effect of the lexicographic product on the correspondence in concern is investigated. As another main result, a description of n-spectral resolutions, which extend to observables for certain types of lexicographic effect algebras is given. In addition, a classical approach to measure extension (via outer measures) is used to provide a construction of n-observables (for a given n-spectral resolution) for interval effect algebras with faithful σ-state.

Keywords: effect algebras, observables, spectral resolutions, partially ordered groups, measure extension

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Abstrakt: V disertační práci je zavedena bijektivní korespondence mezi konečnědimenzionálními observables a spektrálními rozklady pro některé třídy algeber zkoumaných v oblasti kvantových logik. Hlavní výsledek řeší situaci monotónně σ-úplných efektových algeber splňujících Rieszovu dekompoziční podmínku. Klíčovou částí důkazu je takzvané liftování spektrálních rozkladů. Metoda liftování je zajímavá sama o sobě a v práci je plně popsána. Dále je řešen efekt operace lexikografického součinu na zmíněnou korespondenci. Druhým hlavním výsledkem je popis n-spektrálních rozkladů, které lze rozšířit na observables, pro jisté typy lexikografických algeber. V závěru je předveden klasický přístup teorie míry k problému rozšiřování míry (skrze vnější míry) ke konstrukci n-observable (k danému n-spektrálnímu rozkladu) pro intervalové monotonně σ-úplné efektové algebry mající věrný σ-stav.

Klíčová slova: efektové algebry, observables, spektrální rozklady, částečně uspořádané grupy, rozšiřování míry

Jazyk: anglický

Contents

List of notation

$\mathrm{Proj}(\mathcal{H})$	classical model of sharp quantum logic, modeled on a Hilbert space $\mathcal{H}$
$\mathcal{E}(\mathcal{H})$	classical model of unsharp quantum logic, modeled on a Hilbert space $\mathcal{H}$
$\Gamma(G, u)$	interval effect algebra (as defined in the section 1.2)
$\Delta(t, s)$	Δ-operator (as defined in Definition 1.4.2)
$\partial_i(\mathcal{C}), \partial_i(\mathcal{C})$	facets of a cuboid $\mathcal{C}$ (introduction to the section 2.1)

Through the whole thesis we follow this convention in using symbols:

x	finite-dimensional observable
$\{x_t \mid t \in \mathbb{R}\}$	one-dimensional spectral resolution
F	finite-dimensional spectral resolution
$\mathbf{r}, \mathbf{s}, \mathbf{t}, \ldots$	n-tuples of reals
$r_i, s_i, t_i, \ldots$	i-th coordinates of n-tuples $\mathbf{r}, \mathbf{s}, \mathbf{t}, \ldots$
E	effect algebra
$(G, u), (H, v)$	unital po-groups
$\mathcal{C}, \mathcal{D}, \mathcal{E}, \ldots$	cuboids (as defined in the section 2.1)
$\alpha, \beta, \gamma, \ldots$	vertices of cuboids

Through the whole thesis we use the following abbreviations:

(RDP)	Riezs decomposition property
(LP)	Lifting property (Definition 2.1.3)

Introduction

The PhD thesis concerns the question for which algebras related to the logic of quantum mechanics there is a one-to-one correspondence between observables and spectral resolutions. Observables are by definition certain σ-homomorphisms from Borel σ-algebra $\mathcal{B}(\mathbb{R})$ to a quantum structure $\mathcal{L}$ (typically effect algebra) and each observable gives rise to a spectral resolution as its distributive function. The hardest part of the problem in question is to find some conditions on $\mathcal{L}$, such that the spectral resolutions (considered as an independent concept - certain mappings $\mathcal{B}(\mathbb{R}) \to \mathcal{L}$) uniquely extend to observables.

The problem is an abstraction of the well-known result in the classical probability theory, where the Borel probabilistic measures are in one-to-one correspondence with their distribution functions. The reconstruction of a measure is possible (in the classical case) by the Carathéodory's extension theorem, which states that each σ-additive measure on a ring of sets has a unique extension to a σ-additive measure on the generated σ-algebra. In our case, we take for the ring the one generated by all half-open intervals $[t, s)$, $t, s \in \mathbb{R}$ (each spectral resolution naturally defines such sub-additive measure). In the literature, looking for the extended measure is known as *the measure extension problem*, and when the answer is positive, we say that *measure extension property* (MEP) holds.

As time passes the problem of measure extension was considered for measures with values in more general algebraic structures. R. Sikorski in [Sik69] threats the case of Boolean algebras – he showed there that the so-called weak σ-distributivity condition on the Boolean algebra has a key role. Sikorski provided proof (which arises by communication with him and K. Matthes) that the weak σ-distributivity is a sufficient condition for (MEP). On the other hand, J. D. Wright in 1971 (see [Wri71]) proved, that the weak σ-distributivity is a necessary (and so equivalent) condition for the measure extension (to be precise, Sikorski distinguished strong and weak σ-MEP, where the weak one is equivalent to weak σ-distributivity). In fact, Wright dealt with the problem on a more general level of lattice ordered vector spaces. These results were later reproved by Fremlin in [Fre75] in a simpler way. Fremlin's proof is elementary in the sense that no representation theorems are used and the proof utilizes only the countable axiom of choice. The proof is proceeded inside the algebraic structure of Riesz spaces and could be interpreted as a tricky simulation of the classical ϵ, δ calculus using weak-σ-distributivity.

The Fremlin's approach was then adopted by B. Riečan, who presented a measure extension construction on MV-algebras (l-groups) in [RT97], where probability theory for MV-algebras is systematically built. Riečan also has some partial results for non-lattice ordered effect algebras in [Rie98], but a full measure extension construction is given (in the cited article) only for the σ-complete MV-algebras. Hence, it seems to be the case, that Fremlin's technique is limited by a lattice structure.

In contrast to the mentioned results, in the thesis, we will consider the measure extension problem only for the measures based on the Borel subsets of $\mathbb{R}^n$. While for the range structure we will take some quantum structure, typically an effect algebra with (RDP), hence a more general structure than an MV-algebra in the work of Riečan.

An important moment in the research of the algebraic quantum logic was an observation, that most of the important quantum structures are representable as intervals in the partially ordered Abelian groups (as is argued by Foulis and Greechie in [GF95]). This observation led to a bridge between quantum logic and the well-developed theory of po-groups. Two important representation theorems, which have prime importance in the PhD thesis, were achieved thanks to this bridge: Each effect algebra with Riesz Decomposition Property is representable as an interval in a po-group satisfying interpolation property (in fact there is a categorical equivalence between the category of effect algebras with (RDP) and the category of unital Abelian po-groups with interpolation - a result of K. Ravindran [Rav96]). The second important theorem is a kind of Loomis–Sikorski theorem: Each monotone σ-complete effect algebra with (RDP) can be represented as a σ-homomorphic image of so-called effect tribe of fuzzy-sets (proved in [BCD06]).

The second mentioned theorem is the main tool for applications of the lifting technique presented in the PhD thesis: extending spectral resolution F on an effect algebra E for which we have Loomis–Sikorski representation

$\pi : \mathcal{T} \to E$ proceeds in three steps. First, we lift the spectral resolution to a spectral resolution $\hat{F}$ on $\mathcal{T}$, the lifted spectral resolution can be using some standard results from the probability theory extended to an observable $\hat{x}$, which gives the desired observable X by composing with π. Just described technique was used in [DK14] for E being a monotone σ-complete effect algebra with (RDP). Note that this case has not been approached by Fremlin's technique.

In the PhD thesis, after introducing basic concepts in the Chapter 1, there are provided generalisations of the results of Dvurečenskij and Kuková in several directions:

(I) generalisation to finite-dimensional observables (i.e., these having as domain $\mathcal{B}(\mathbb{R}^n), n \in \mathbb{N}$),

(II) weakening the monotone completeness, by considering lexicographic interval effect algebras $\Gamma(H \overrightarrow{\times} G, (u, 0))$, where (G, u) is a Dedekind σ-complete po-group with interpolation and (H, u) is any unital (Abelian) po-group,

(III) combination of (I–II).

In the first generalization, described in Chapter 2, the lifting process become much more complicated, in contrast to the one-dimensional case, where the lifting is a rather simple part. The lifting of finite-dimensional resolutions is presented in the general situation of a σ-projection $\pi : (H, v) \to (G, u)$ of monotone σ-complete Abelian po-groups. It turns out, that so-called *lifting property* (certain strengthening of surjectivity which holds in the cases of Loomis-Sikorski theorems) is a necessary and sufficient condition for the lifting process. The lifting is achieved by iterating the inductive process and, as one might expect, it strongly utilizes (a countable version of) the axiom of choice. This part of the PhD thesis covers the results from [DL20d] and [DL20a].

In the second and the third generalisation, described in Chapter 3, only spectral resolutions satisfying certain additional properties extend to observables. The generalisations are characteristic by a need to refine the arguments of most of the proofs. The main result considering a finite-dimensional observable on a lexicographic effect algebra is reached through many technical lemmas. This part generalises the results form [DL20b],[DL19],[DL19] and [DLce].

In the last Chapter 4, a classical approach to measure extension (via outer measures) is exhibited in the case of monotone σ-complete interval effect algebras having faithful σ-state. The assumption of the existence of faithful σ-state is a strong one, for example, each monotone σ-complete effect algebra with (RDP) is a lattice, whenever obtains such a state. Nevertheless, in Chapter 4, the property of (RDP) is not assumed, hence the main result is not covered by the ones from the previous chapters achieved by the lifting procedure.

The aim of the PhD thesis

A general aim of the PhD thesis is to establish a one-to-one correspondence between (finite-dimensional) observables and spectral resolutions for as many quantum structures as possible. In more detail, the original motivation was to develop methods from [DK14] to finite-dimensional cases and to study the effect of the lexicographic product on the correspondence in question.

Methods

The approach to the problem is based on and limited by several representations theorems: the ones representing effect algebras as intervals of partially ordered groups and Loomis-Sikorski-like representations of certain effect algebras as a σ-projections of tribes of fuzzy sets. Moreover, some well-known results from the measure theory are used.

Main Results

The main results are as follows

1. A one-to-one correspondence between finite-dimensional observables and spectral resolutions is established for the monotone σ-complete effect algebras with Riesz Decomposition Property.
2. Given a σ-projection $\pi : T \to E$ of monotone σ-complete interval effect algebras satisfying so-called lifting property, a process of lifting the finite-dimensional spectral resolutions of E to the ones of T is described.

3. For interval effect algebra $\Gamma(H\vec{\times}G, (u, 0))$, where (H, u) is an unital po-group and G is a monotone σ-complete po-group with interpolation, a classification of the spectral resolutions which extend to observables is given.

4. The classical approach to the measure extension (via outer measure) is applied to establish the correspondence in question in the case of a monotone σ-complete interval effect algebras with a faithful σ-state.

Related author's publications

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[DL20a] A. Dvurečenskij and D. Lachman. "Lifting, n-dimensional spectral resolutions, and n-dimensional observables". In: *Algebra Universalis* 34 (2020), pp. 163–191. DOI: 10.1007/s00012-020-00664-8.

[DL20b] A. Dvurečenskij and D. Lachman. "Spectral resolutions and observables in n-perfect MV-algebras". In: *Soft Computing* 24 (2020), pp. 843–860. DOI: 10.1007/s00500-019-04543-w.

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[DLce] Anatolij Dvurečenskij and Dominik Lachman. "n-dimensional observables on k-perfect MV-algebras and k-perfect effect algebras. I. Characteristic points". In: *Fuzzy Sets and Systems* ((accepted) 2021). DOI: https://doi.org/10.1016/j.fss.2021.05.005.

Chapter 1

Basic concepts

In the chapter, some basic concepts are introduced: the related ordered structures (effect algebras and partially ordered groups), finite-dimensional spectral resolutions and observables, and some essential theorems used in the following chapters. However, as the content of the thesis is all about observables, the first section is devoted to motivating the definition of observables and outlining how the concept of quantum structures encodes reasoning about physical experiments.

1.1 Quantum structures from the a logico-physical perspective

In the spirit of W. Mackey, we describe an abstraction of a physical system as a triple $(\mathcal{O}, \mathcal{S}, p)$, where $\mathcal{O}$ is a set of observables (measurable physical quantities), $\mathcal{S}$ is a set of states (possible states of the physical model) and $p : \mathcal{S} \times \mathcal{O} \times \mathcal{B}(\mathbb{R}) \to [0, 1]$ is some real valued mapping. We read $p(s, A, \Delta)$ as the probability that in some state s we get a value in Δ when measuring an observable A. For such system several Mackey's axioms (I-VII) have to hold (as G. Mackey argues in [Mac63]) to ensure physically plausible properties. For the full list of Mackey's axioms see [CGG13], we will mention only some of them:

(I) $p(s, A, \cdot) : \mathcal{B}(\mathbb{R}) \to [0, 1]$ is a probability measure for each observable A and state s.

(II) Different states and observables are distinguishable by measuring:

$$[(\forall s \in \mathcal{S}, \Delta \in \mathcal{B}(\mathbb{R})), p(s, A_1, \Delta) = p(s, A_2, \Delta)] \Rightarrow A_1 = A_2,$$
$$[(\forall A \in \mathcal{O}, \Delta \in \mathcal{B}(\mathbb{R})), p(s_1, A, \Delta) = p(s_2, A, \Delta)] \Rightarrow s_1 = s_2.$$

Now we can thing of the ordered pairs (A, Δ) in $\mathcal{O} \times \mathcal{B}(\mathbb{R})$ as certain experimental propositions about the system "we observe a value in Δ, when measuring A". And we obtain a structure $\mathcal{L}$ called quantum logic after identifying those pairs (A_1, Δ_1), (A_2, Δ_2) which are indistinguishable by measuring, i.e., whenever $p(s, A_1, \Delta_1) = p(s, A_2, \Delta_2)$ holds in each state $s \in \mathcal{S}$. Equivalently, we can define $\mathcal{L}$ as a collection of [0,1]-valued mappings with definition domain $\mathcal{S}$ (those which arise from experimental propositions). On $\mathcal{L}$ we can naturally define some structure: an ordering, a compatibility relation and an ortho-complementation. If $a, b \in \mathcal{L}$ have representatives (A_1, Δ_1), (A_2, Δ_2) such that:

- $p(\cdot, A_1, \Delta_1) \leq p(\cdot, A_2, \Delta_2)$ pointwise, then we set $a \leq b$,
- $A_1 = A_2$, we call a, b compatible,
- $\Delta_1 = \Delta_2^c$, we call a the complement of b.

Mackey stated several additional axioms which guarantee richness of both states and observables and enforce the quantum logic to be a σ-orthocomplete orthomodular poset. Moreover, Mackey demanded (via axiom (VII)) the resulting logic to be isomorphic to the classical quantum logic (introduced by Neumann and Birkhoff, [BN36]) of projections on closed subspaces of a Hilbert space (we will describe this model later on in this section). This condition was considered by many authors (including Mackey) as an *ad-hoc* assumption which led to a research on how to replace the presence of axiom (VII) by additional abstract conditions.

A different approach (developed in the work of Gudder) is to take the derived logic $\mathcal{L}$ as a primitive concept, while the observables and states as derived: observables are identified with σ-homomorphisms $x : \mathcal{B}(\mathbb{R}) \to \mathcal{L}$, this follows the assignment

$$x_A \mapsto [\Delta \mapsto (\Delta, A)].$$

Similarly, states are identified with probability measures $\mathcal{L} \to [0, 1]$, this originates in the prescription

$$s \mapsto [(A, \Delta) \mapsto p(s, A, \Delta)].$$

The two described approaches are on some level equivalent: Each pair $(\mathcal{L}, \mathcal{S})$ of σ-orthocomplete orthomodular poset $\mathcal{L}$ and order-determining set of states $\mathcal{S}$ (probability measures on $\mathcal{L}$) leads to a triple $(\mathcal{S}, \mathcal{O}, p)$, where observables $\mathcal{O}$ are obtained as is described before and formula $p(s, A, \Delta) := (s \circ x_A)(\Delta)$ holds. Such triple satisfies axioms (I-VII) and the corresponding quantum logic $\mathcal{L}'$ is isomorphic to $\mathcal{L}$.

In the standard model of quantum logic (introduced by J. von Neuman and G. Birkhoff in their seminal paper [BN36]) we assume some Hilbert space $\mathcal{H}$ and we define $\mathcal{L}$ as the lattice of all projections on closed linear subspaces of $\mathcal{H}$, denoted $\mathrm{Proj}(\mathcal{H})$. In the case $\mathcal{H}$ is separable and of dimension at least 3 the states correspond to density operators (i.e., linear, bounded, trace-class operators of trace equal to 1). The one-to-one correspondence is assured by the celebrated Gleason's theorem and is performed by Born rule: given a density operator A, each projection $P \in \mathrm{Proj}(\mathcal{H})$ is sent to

$$P \mapsto \mathrm{tr}(AP).$$

For observables we take all projection valued measures $\mathcal{B}(\mathbb{R}) \to \mathrm{Proj}(\mathcal{H})$, which correspond to self-adjoint operators on $\mathcal{H}$, this is a consequence of the well-known Spectral theorem. This model satisfies the Mackey'saxioms (in fact it served as a motivating example).

Later on, various other approaches to quantum logic were introduced, which resulted in a vast class of algebras labelled as quantum structures. Even in the '70s, Fraassen described the variety of results as "Labyrinth of quantum logic" (see [Fra74]). We shall briefly describe two important classes of algebras: orthoalgebras and its generalisation: effect algebras (which is a basic structure in the thesis).

An orthoalgebra is a partial algebra $(E, \oplus, 0, 1)$, where $\oplus$ is a partially defined associative and commutative operation having 0 as a neutral element. Moreover, each element $a \in E$ asserts a unique complement a' such that $a \oplus a' = 1$ and if $a \oplus a$ exists, then $a = 0$. Orthoalgebras arise as a certain answer to the problem with tensor products of orthomodular structures. As is proved in[FR79], the category of quantum logics in Mackey sense is not closed under tensoring, which is a serious problem as the tensor product is an algebraic counterpart of compounding two physical systems. The category of orthoalgebras is argued to be the smallest category containing all the unital orthomodular lattices and yet be closed under tensor products. Moreover, orthoalgebras were advocated by a nice relation with the test spaces (an important semantic for quantum logic introduced in the 70s and developed in the 80s by D. J. Foulis and C. H. Randall).

Even more general algebraic concept for quantum logic is provided by the effect algebras. These arise from orthoalgebras by replacing the last (in previous paragraph) mentioned assumption by: $a \oplus 1$ exists implies $a = 0$. The effect algebras were introduced by Foulis and Benett in 1994 [FB94], as algebraic semantic for so-called unsharp quantum logic developed (mainly) by Ludwig ([Lud13]). The basic idea is to reflect the inaccuracy of measuring by some concrete macroscopic device. Which is sometimes called the second degree of fuzziness. We can simply demonstrate the idea on the Mackey's model following the construction in [CL94]: Mackey's axiom (III) (yet not mentioned in this section) assures for each observable A and a real Borel measurable function f existence of an observable $f(A)$ (uniquely) given by the relation

$$p(s, f(A), \Delta) = p(s, A, f^{-1}(\Delta)). \tag{1.1}$$

Now, given a question (A, Δ), we have an observable $\chi_\Delta(A)$, which is called an *event*. Hence the events are the observables, which result into yes-no answer (yield values in $\{0, 1\}$). In Gudder's approach, where an observable A corresponds to an $\mathcal{L}$-valued measure $x_A : \mathcal{B}(\mathbb{R}) \to \mathcal{L}$, the observable $f(A)$ corresponds simply to $x_A \circ f^{-1}$. We can characterize events as these observables E's satisfying

$$p(s, E, \Delta) = \begin{cases} 0 & \text{if } \{0, 1\} \subseteq \Delta^c, \\ 1 & \text{if } \{0, 1\} \subseteq \Delta, \\ p(s, E, \{1\}) & \text{if } 1 \in \Delta, 0 \notin \Delta, \\ p(s, E, \{0\}) & \text{if } 0 \in \Delta, 1 \notin \Delta. \end{cases}$$

One can prove (from Mackey's axioms), that events are in one-to-one correspondence with the elements of the logic $\mathcal{L}$. Now, if we replace χ_Δ in the definition of events, by some fuzzy set ω_Δ (i.e., $\omega_\Delta : \mathbb{R} \to [0, 1]$ is a Borel measurable function with $\omega_\Delta^{-1}(0) \subseteq \Delta^c$) we obtain kind of generalized events, which are called *effects*. And all the effects compose into so-called unsharp logic. The skip from the sharp sets coded by $\chi : \mathcal{B}(\mathbb{R}) \times \mathbb{R} \to \{0, 1\}$ to fuzzy sets coded by some $\omega : \mathcal{B}(\mathbb{R}) \times \mathbb{R} \to [0, 1]$ is interpreted in the concept of measurement as a skip from an idealized macroscopic device to a concrete (non-accurate) macroscopic device. The incorporation of fuzziness is advocated by an idea formulated by R. Gilles (in [Gil70]) as "The physical significant assertions of classical or quantum mechanics must refer to physically or concretely realizable device, for no other devices can actually be realized." A detailed investigation of the unsharp modification of Mackey's and Gudder's approach to quantum logic is in [CL94].

While $\mathrm{Proj}(\mathcal{H})$, $\mathcal{H}$ is a Hilbert space, is the prototypical example of the sharp quantum logic, in the case of the unsharp quantum logic we take $\mathcal{E}(\mathcal{H})$, whose elements are all the self-adjoin operators on $\mathcal{H}$ between $0_{\mathcal{H}}$ and $1_{\mathcal{H}}$.

1.2 Effect algebras and Abelian po-groups

As we already have outlined in the previous section the effect algebras were introduced by Foulis and Bennett in [FB94] in the nineties to capture a general concept of quantum structures algebraically.

Definition 1.2.1. *We call effect algebra a partial algebra* $(E; +, ', 0, 1)$ *of type* $(2, 1, 0, 0)$*, such that for each* $a, b, c \in E$

1. $a+b=b+a$,
2. $(a+b)+c=a+(b+c)$,
3. a' *is the unique element such that* $a + a' = 1$,
4. *if* $a + 1$ *is defined then* $a = 0$.

Where we read the first two identities as: when one of the sides is defined, then the other is defined as well and equality holds.

Moreover, we define a partial ordering on E *as* $a \leq b$ *iff there is* $c \in E$*, such that* $a + c = b$*. Then the constant* 0 *(*1*, resp.) is the lowest (the greatest, resp.) element.*

Let us list several important algebras, that could be arranged as effect algebras:

1. Boolean algebras, where we take for $+$ the union operation restricted to the disjoint elements.
2. More generally MV-algebras: That is algebras $M = (M, \oplus, ', 0, 1)$ of type (2,1,0,0), where $(M; +, 0)$ is a commutative monoid and the following identities are satisfied:

(i) $a'' = a$,

(ii) $a \oplus 1 = 1$,

(iii) $a \oplus (a \oplus b')' = b \oplus (b \oplus a')'$.

One can define on every MV-algebra M an ordering $\leq$ in analogy to the case of effect algebras. Then M turns out to be a lattice and we obtain an (lattice-ordered) effect algebra by restricting $\oplus$ to the pairs (a, b) with $a \leq b'$.

3. Even more generally, every interval $[0, u]$ in a partially ordered Abelian group $(G, +_G, \leq)$, (of course $0 < u$), where we define the complementation as $a' = u - a$ and for $+$ we take the restriction of $+_G$.

4. The classical model of sharp quantum logic $\mathrm{Proj}(\mathcal{H})$ consisting of all projection in a Hilbert space $\mathcal{H}$ on its closed sub-spaces.

5. The classical model of unsharp quantum logic $\mathcal{E}(H)$ of all Hermitian operators on a Hilbert space H between the zero operator and the identity operator.

Remark 1.2.2. *the effect algebras which arise from MV-algebras (as 2. example describes) are called effect MV-algebras. These are defined in the class of effect algebras as the lattice ordered effect algebras satisfying the identity $(a \vee b) \ominus a = b \ominus (a \wedge b)$. By a result from [CK97] there is a natural one-to-one correspondence between MV-algebras and effect MV-algebras. The totalization of the partial operation $+$ proceeds as $a \oplus b = a + (a' \wedge b)$. Having this in mind we will sometimes treat some MV-algebra as an effect algebra.*

The interval effect algebras mentioned in the third point will be the most important to us. We will denote them using the following notation:

$$\Gamma(G, u) = ([0, u]; +, ', 0, u). \tag{1.2}$$

Let us briefly introduce some basic concepts from the theory of partially ordered groups (po-groups for short). At first, note, that we will consider only Abelian groups, so we will not underline the abelian property, nevertheless, it will be indicated by using $+$ for the group binary operation. A po-group is a group $(G, +, 0)$ enriched with a partial ordering $\leq$, such that for all $a, b, c \in G$ we have $a \leq b \Leftrightarrow a + c \leq b + c$. We say that an element $u \in G$ is a strong unit, if $0 \leq u$ and each $a \in G$ is dominated by some $n \cdot u$, $n \in \mathbb{N}$. An ordered pair (G, u) of a po-group G and its strong unit is called *unital po-group*. We call a po-group G *directed* if each pair $a, b \in G$ is dominated by some $c \in G$ (note, that the unital po-groups are always directed). In particular, if each pair $a, b \in G$ has the supreme $a \vee b$, we call G a *lattice-ordered group* (l-group for short).

An important generalization of the lattice property is so called *interpolation property*, which states, that for each $g_1, g_2, h_1, h_2 \in G$, inequalities $g_1, g_1 \leq h_1, h_2$ entail an element $k \in G$, such that $g_1, g_2 \leq k \leq h_1, h_2$. If (G, u) is a unital po-group with interpolation, the resulting interval effect algebra $\Gamma(G, u)$ satisfies *Riesz decomposition property* (RDP in short), which requires that if $a_1 + a_2 = b_1 + b_2$, there are four elements $\{c_{ij} \in E \colon i, j \in \{1, 2\}\}$, such that $a_1 = c_{11} + c_{12}$, $a_2 = c_{21} + c_{22}$, $b_1 = c_{11} + c_{21}$ and $b_2 = c_{12} + c_{22}$. A basic reference to the theory of po-groups with interpolation is the monograph [Goo86].

The structures we will be working with, will always satisfy some kind of completeness, most frequently a monotone σ-completness. A poset $(P; \leq)$ is *monotone σ-complete* if each ascending (descending, resp.) countable bounded sequence $a_1 \leq a_2 \leq \cdots \leq b$ ($a_1 \geq a_2 \geq \cdots \geq b$, resp.) of elements of P, has supreme (infimum, resp.). In the case of po-groups we call this property *Dedekind σ-completeness.*

Now having a countable system $\{a_t \colon t \in T\}$ of elements of an effect algebra E, we say, that the system is *summable*, if for each finite subset P of T, the element $a_P = \sum\{a_t \colon t \in P\}$ exists in E, and in addition, if the element $a = \bigvee\{a_P \colon P \subseteq T \text{ finite }\}$ is defined in E, the element a is said to be the *sum* of $\{a_t \colon t \in T\}$ and we write $a = \sum_{t \in T} a_t$. Observe, that in the case of monotone σ-complete effect algebra, every summable system has its sum.

1.3 Representation theorems

In this section, we will state several important and well-known theorems, which play a key role. The first one (proved by Ravindran in [Rav96]) provides a representation of effect algebras with (RDP) as interval effect algebras.

Theorem 1.3.1. *Prescription Γ, given by (1.2), defines categorical equivalence between the categories of unital po-groups with interpolation and the category of effect algebras with (RDP).*

The statement of the theorem assumes a suitable concept of morphisms in the two categories. And these are naturally defined, so that they preserve the structure of effect algebras and unital po-groups, respectively.

An important addition to Ranvindran's theorem is the following Lemma (for poof see [Goo86], Prop. 16.9):

Lemma 1.3.2. *Given unital po-group (G, u), the effect algebra $\Gamma(G, u)$ is monotone σ-complete if and only if G is Dedekind σ-complete.*

The two mentioned theorems provide great comfort in calculations in a monotone σ-complete effect algebra E, as we can ignore the fact, that $+$ is only partially defined in E with (RDP). One only needs to check, that the result R of a calculation fits in the inequalities $0 \leq R \leq u$.

It is also worth noting that the categorical equivalence in Theorem 1.3.1 restricts to the one between unital l-groups and MV-algebras (see [Mun86]).

The next representation theorem is a kind of Loomis-Sikorski theorem and it represents monotone σ-complete effect algebras with (RDP) as σ-projections of effect tribes of fuzzy sets. The result is prooved in [BCD06].

Definition 1.3.3. *Let $\Omega \neq \emptyset$ be a set and $\mathcal{T}$ be some subset of $[0,1]^\Omega$. We call $\mathcal{T}$ an effect-tribe if it satisfies conditions:*

(i) $1 \in \mathcal{T}$,

(ii) if $f \in \mathcal{T}$, then $1 - f \in \mathcal{T}$,

(iii) if $f, g \in \mathcal{T}$ and $f \leq 1 - g$, then $f + g \in \mathcal{T}$,

(iv) if $(f_n)_n$ is a monotone sequence of elements of $\mathcal{T}$, then its pointwise limit f ($f_n \nearrow f$) belongs to $\mathcal{T}$.

Theorem 1.3.4. *Let E be a monotone σ-complete effect algebra with (RDP). Then there exists a convex space Ω, an effect-tribe $\mathcal{T}$ of fuzzy sets on Ω with (RDP) and a σ-homomorphism h from $\mathcal{T}$ onto E.*

We will capture some details of the proof of the just stated theorem in the section 2.2, where we will prove, that the projection π satisfies the so-called lifting property.

we will also utilize some classical theorems from probability. Assume a set Ω and a class $\mathcal{D}$ of its subsets. We call $\mathcal{D}$ a Dynkin system (or λ-system) if it contains Ω and is closed under proper differences (i.e., for $A, B \in \mathcal{D}$, $A \cap B^c$ whenever $A \supset B$) and countable monotone unions. Next, we call a class $\mathcal{C}$ of subsets of Ω a π-system if it is closed under finite intersections.

Theorem 1.3.5 (Sierpińki, [Kal02], Thm. 1.1)**.** *Let $\mathcal{C}$ be a π-system and $\mathcal{D}$ a λ-system of subsets of a set Ω, such that $\mathcal{C} \subset \mathcal{D}$. Then for the σ-algebra $\sigma(\mathcal{C})$ generated by $\mathcal{C}$ we have $\sigma(\mathcal{C}) \subseteq \mathcal{D}$.*

1.4 Basic properties of n-dimensional spectral resolutions and n-dimensional observables

In the section, we introduce the crucial notions of the thesis – n-dimensional spectral resolutions and n-dimensional observables on a monotone σ-complete effect algebra, and we present the main properties of n-dimensional spectral resolutions

We will use two kinds of orderings on the n-tuples of reals:

$$(t_1, \dots, t_n) < (s_1, \dots, s_n) \iff for\ each\ i, t_i \le s_i \text{ and for some } i, t_i < s_i,$$
$$(t_1, \dots, t_n) \ll (s_1, \dots, s_n) \iff for\ each\ i, t_i < s_i.$$

Definition 1.4.1. *Let E be a σ-complete effect algebra. Then we call n-dimensional observable any σ-homomorphism $x : \mathcal{B}(\mathbb{R}^n) \to E$, that is a mapping satisfying:*

(i) $x(\mathbb{R}^n) = 1$,

(ii) $x(A \cup B) = x(A) + x(B)$ *whenever* $A \cap B = \emptyset$,

(iii) $\{A_i\}_i \nearrow A$ *implies* $\bigvee_i x(A_i) = x(A)$.

If, given an n-dimensional observable x on $E = \Gamma(G, u)$, it gives arise to its distributive function: $F_x : \mathbb{R}^n \to \Gamma(G, u)$ by

$$F_x(s_1, \dots, s_n) = x((-\infty, s_1) \times \dots \times (-\infty, s_n)), \quad (s_1, \dots, s_n) \in \mathbb{R}^n.$$

We call F_x an n-dimensional spectral resolution of x. We will be mostly concerned with the opposite process: finding an n-observable for a given n-spectral resolution. For we will treat n-dimensional spectral resolutions as an independent concept given by Definition 1.4.3. In the definition, the most intricate condition to handle is the last stated - so-called *volume condition*. Volume conditions basically assure that an n-spectral resolution prescribes non-negative volume to certain n-dimensional cuboids in $\mathbb{R}^n$. Let us introduce the following notation, which we will use throughout the whole thesis:

Definition 1.4.2. *Given a function $F : \mathbb{R}^n \to E$ (usually pseudo n-dimensional spectral resolution), an integer i, $1 \le i \le n$ and two reals $a \le b$, by $\Delta_i(a,b)F : \mathbb{R}^n \to E$ we denote the function given by the prescription*

$$(t_1,\dots,t_{i-1},t_{i+1},\dots,t_n) \mapsto F(t_1,\dots,b,\dots,t_n) - F(t_1,\dots,a,\dots,t_n). \tag{1.3}$$

Definition 1.4.3. Let (G,u) be a Dedekind monotone σ-complete po-group and let $n \ge 1$ be an integer. An *n-dimensional spectral resolution* on $E = \Gamma(G,u)$ is any mapping $F : \mathbb{R}^n \to \Gamma(G,u)$ such that

$$F(s_1,\dots,s_n) \le F(t_1,\dots,t_n) \quad \text{if} \quad (s_1,\dots,s_n) \le (t_1,\dots,t_n), \tag{1.4}$$

$$\bigvee_{(s_1,\dots,s_n)} F(s_1,\dots,s_n) = u, \tag{1.5}$$

$$\bigvee_{(s_1,\dots,s_n)\ll(t_1,\dots,t_n)} F(s_1,\dots,s_n) = F(t_1,\dots,t_n), \tag{1.6}$$

$$\bigwedge_{t_i} F(s_1,\dots,s_{i-1},t_i,s_{i+1},\dots,s_n) = 0 \text{ for } i = 1,\dots,n, \tag{1.7}$$

$$\Delta_1(a_1,b_1)\cdots\Delta_n(a_n,b_n)F \ge 0, \ \textit{for each } a_i,b_i \in \mathbb{R}, a_i \le b_i, i = 1,\dots,n. \tag{1.8}$$

Moreover, if in the equality (1.5) *only the inequality "$\le$" holds, and the other four conditions are valid, we call F a pseudo n-dimensional resolution.*

We will often simplify the term n-dimension spectral resolution to n-spectral resolution (and similarly in the case of observables).

It is important to note that the monotonicity of F and Dedekind σ-completeness of G entail that all the suprema and infima on the left-hand sides of (1.5)–(1.7) exist in G. Indeed, to see (1.5), let $(\mathbf{s}_l)_l$ and $(\mathbf{t}_l)_l$ be two non-decreasing sequences in $\mathbb{R}^n$ going to $(+\infty,\dots,+\infty)$. The monotonicity of F entails that the following suprema exist in G and are equal

$$\bigvee_l F(\mathbf{s}_l) = \bigvee_{\mathbf{s}\in\mathbb{R}^n} F(\mathbf{s}) = \bigvee_l F(\mathbf{t}_l).$$

And we can similarly argue in the case of (1.7) and (1.8).

Next, we make several remarks on the volume conditions.

Remark 1.4.4. *When we iterate Delta operators, as we do for example in the formulation of volume conditions, there may occur a switch in coordinates. For example, an application of $\Delta_1(a_1, b_1)\Delta_2(a_2, b_2)$ has the same effect as an application of $\Delta_1(a_2, b_2)\Delta_1(a_1, b_1)$. Obviously, the Delta operators commute in certain sense, but the exact formula capturing the commutativity is rather intricate:*

$$\Delta_i(a', b')\Delta_j(a'', b'') = \begin{cases} \Delta_{j-1}(a'', b'')\Delta_i(a', b'), & \text{if } i < j, \\ \Delta_j(a'', b'')\Delta_{i+1}(a', b'), & \text{if } i \geq j. \end{cases} \tag{1.9}$$

Hence we will follow the convection that by $\prod_{i\in I} \Delta_i(a_i, b_i)$, for $I \subseteq \{1, \ldots, n\}$, we mean $\Delta_{i_1}(a_{i_1}, b_{i_1}) \cdots \Delta_{i_k}(a_{i_k}, b_{i_k})$, where $k = |I|$ and $i_1 < i_2 < \cdots < i_k$ are all the elements of I.

Remark 1.4.5. *Every n-dimensional semi-closed block $A = \langle a_1, b_1) \times \cdots \times \langle a_n, b_n)$ is given by 2^n vertices $\alpha = (\alpha_1, \ldots, \alpha_n)$, where $\alpha_i \in \{a_i, b_i\}$ for each $i = 1, \ldots, n$. For each vertex $\alpha = (\alpha_1, \ldots, \alpha_n)$, we set $|\alpha|$ to be the number of α_i's coinciding with a_i in $\alpha = (\alpha_1, \ldots, \alpha_n)$. Then the volume condition can be expressed also in the form*

$$\Delta_1(a_1, b_1) \cdots \Delta_n(a_n, b_n)F(s_1, \ldots, s_n) = \sum_{\alpha} (-1)^{|\alpha|} F(\alpha_1, \ldots, \alpha_n). \tag{1.10}$$

Remark 1.4.6. *The volume conditions and monotony are special types of more general inequalities*

$$(\prod_{i\in I} \Delta_i(a_i, b_i))F \geq 0, \tag{1.11}$$

where $I \subseteq \{1, \ldots, n\}$ and $a_i \leq b_i$ for each $i \in I$ (I is one-element set for monotony and n-element set for volume conditions). It turns out, that all inequalities of form (1.11) *are consequences of the described extreme cases, if all the conditions in Definition 1.4.3 hold. However, through the process of lifting, which is presented in the next chapter, these general volume conditions will be of equal importance, hence we will call volume condition any inequality of form* (1.11).

Proposition 1.4.7. *Let F be an n-dimensional spectral resolution. Then all the volume conditions hold, that is*

$$(\prod_{i\in I} \Delta_i(a_i, b_i))F \geq 0,$$

for each $I \subseteq \{1, \ldots, n\}$ and reals $a_i \leq b_i$, $i = 1, \ldots, n$.

Proof. We will use a downward induction along $|I|$. The case $|I| = n$ holds by assumption. Now suppose the statement holds whenever $|I| > k > 1$, and choose some I, such that $|I| = k$. The function $(\prod_{i \in I} \Delta_i(a_i, b_i))F$ is monotone in each coordinate by the induction hypothesis and we can expand it to the form $\Sigma^+(t_1, \ldots, t_{n-k}) - \Sigma^-(t_1, \ldots, t_{n-k})$, where both Σ^+ and Σ^- are sums of 2^{k-1} non-negative summands, and they are monotone, as well as the difference (by the induction hypothesis). We will prove for each $s_2, \ldots, s_{n-k} \in \mathbb{R}$ inequality

$$\bigwedge_{t_1} [\Sigma^+(t_1, s_2, \ldots, s_{n-k}) - \Sigma^-(t_1, s_2, \ldots, s_{n-k})] \geq 0, \tag{1.12}$$

which in particular implies the desired volume condition. To achieve the inequality (1.12), assume we have two non-increasing sequences $(a_i)_i, (b_i)_i$ in (G, u) (substituting Σ^+, Σ^-, respectively), such that $(a_i - b_i)_i$ is non-increasing as well (by the $(k+1)$-dimensional volume conditions) and $\wedge_i a_i = \wedge_i b_i = 0$. We have

$$\begin{aligned} &\bigwedge_i (a_i - b_i) \geq 0 \\ \Leftrightarrow\ &(\forall i)\ a_i - b_i \geq \bigwedge_j a_j - \bigwedge_j b_j \\ \Leftrightarrow\ &(\forall i)\ a_i + \bigwedge_j b_j \geq b_i + \bigwedge_j a_j. \end{aligned}$$

And the last inequality is a consequence of

$$(\forall i, j \in \mathbb{N})\ i < j \implies a_i + b_j \geq b_i + a_j,$$

which in turn follows from the monotonicity of $a_i - b_i$. □

Of course each n-spectral resolution F_x (associated to na n-observable x) satisfies the definition 1.4.3. For example the volume conditions mean that if $A = \langle a_1, b_1) \times \cdots \times \langle a_n, b_n)$, $a_i \leq b_i$ for each $i = 1, \ldots, n$, denotes an n-dimensional semi-closed interval, then

$$\Delta_1(a_1, b_1) \cdots \Delta_n(a_n, b_n) F_x(s_1, \ldots, s_n) = x(A) \geq 0.$$

We enclose the section with an example of an n-dimensional observable and consequently of an n-dimensional spectral resolution: Let $\{t_k\}_k$ be a finite or countable set of mutually different elements of $\mathbb{R}^n$ and let $\{a_k\}_k$ be a finite or countable family of summable elements of $\Gamma(G,u)$ such that $\sum_k a_k = u$, where G is a Dedekind monotone σ-complete po-group. Then

$$x(A) = \sum\{a_k \colon t_k \in A\}, \quad A \in \mathcal{B}(\mathbb{R}^n),$$

is an n-dimensional observable and F_x is an example of an n-dimensional spectral resolution.

Chapter 2

Extending n-spectral resolutions via lifting technique

In the chapter, we establish a one-to-one correspondence between n-spectral resolutions and n-observables for monotone σ-complete effect algebras with (RDP). The result for the particular case of MV-algebras is then extended to the lattice-effect algebras, using the concept of blocks. We obtain the main results as an application of the method of lifting spectral resolution through some σ-projection $\pi : T \to E$ satisfying so-called lifting property. The main result in lifting is formulated in a more general context of monotone σ-complete interval effect algebras. However, in particular applications (to monotone σ-complete effect algebras with (RDP)) such generality is unnecessary, as for π we take the Loomis-Sikorski representation, which existence is proved by construction, and provides a natural choice of lift (for more details see [DL20a], Thm. 5.2.). Nevertheless, the general result in lifting seems to be interesting on its own and may have applications beyond those mentioned in the thesis.

2.1 Lifting of n-dimensional spectral resolutions

Suppose $\pi : F \to E$ is a σ-surjection of monotone σ-complete effect algebras. If we are able to solve the spectral resolution extension problem for F, we may try to extend the result to E by lifting the spectral resolutions. In more detail: suppose F is a spectral resolution on E, find a spectral resolution $\hat{F}$

on F, such that $\pi \circ \hat{F} = F$. Then extend $\hat{F}$ to an observable $\hat{x}$ and prove, that $\pi \circ \hat{x}$ is an observable extending F.

This technique applies to the effect algebras, for which variant of Loomis-Sikorski theorem holds. In the case of ordinary (one-dimensional) spectral resolutions, the lifting part is rather easy compared to the other steps. However, assuming the general situation of an n-dimensional spectral resolution, the lifting part becomes the most difficult one.

We begin by introducing some notations that allow us to handle the process of lifting of d-cuboids and to control the volume conditions in a comfortable way.

Let $D \subseteq \{1, \dots, n\}$, $d := |D|$, and for each $i = 1, \dots, n$, let $a_i, b_i \in \mathbb{R}$ be such that $a_i < b_i$ whenever $i \in D$ and $a_i = b_i$ otherwise. Define $\mathcal{C} = \{(*_1, \dots, *_n) : *_i \in \{a_i, b_i\}\}$. We call $\mathcal{C}$ a *d-cuboid*. We will refer to the integer d by $\dim \mathcal{C} := d$, to the set D by $\mathrm{Dim}(\mathcal{C}) := D$ and to the a_i (b_i, resp.) by $a_i^{\mathcal{C}}$ ($b_i^{\mathcal{C}}$, resp.). It is easy to see that any d-cuboid $\mathcal{C}$ has 2^d elements. Next, any $\mathcal{F} \subseteq \mathcal{C}$ which itself is an e-cuboid, $e \leq d$, is called an *e-face* of $\mathcal{C}$. If $e = 0$, $\mathcal{F}$ is called a *vertex* of $\mathcal{C}$. Clearly, given a cuboid $\mathcal{C}$, the vertices of $\mathcal{C}$ correspond to the elements of $\mathcal{C}$ and by a slight abuse of notation we can identify them. It is also clear that the vertices of $\mathcal{C}$ can be partially ordered as they are elements of $\mathbb{R}^n$. We call $(b_1, \dots, b_n)$ the top one or the first one, moreover, if a vertex $\alpha = (*_1, \dots, *_n) \in \mathcal{C}$ has a_i's for m indices $i \in \mathrm{Dim}(\mathcal{C})$, we say α has an *order* $\mathrm{ord}_{\mathcal{C}}(\alpha) := m+1$ in $\mathcal{C}$ (i.e., the top vertex has an order 1 and $(a_1, \dots, a_n)$ has an order $\dim \mathcal{C} + 1$). We say that some cuboid $\mathcal{D}$ is *inside* a cuboid $\mathcal{C}$, if for each $i \leq n$, $a_i^{\mathcal{C}} \leq a_i^{\mathcal{D}} \leq b_i^{\mathcal{D}} \leq b_i^{\mathcal{C}}$. In particular, every face of a cuboid $\mathcal{C}$ is inside $\mathcal{C}$.

As almost all essential steps in the process of the lifting will be achieved by an induction over dimension, the co-dimension one faces will be very important to us. For each d-cuboid $\mathcal{C}$ and $i \in \mathrm{Dim}(\mathcal{C})$, we define a $(d-1)$-cuboid $\partial_i \mathcal{C} := \{(*_1, \dots, *_n) \in \mathcal{C} | *_i = b_i^{\mathcal{C}}\}$ and $\partial_i' \mathcal{C} := \{(*_1, \dots, *_n) \in \mathcal{C} | *_i = a_i^{\mathcal{C}}\}$. Clearly $\partial_i \mathcal{C}$ and $\partial_i' \mathcal{C}$, $i = 1, \dots, n$, are all the $(d-1)$-faces of $\mathcal{C}$ and they are called *facets* of $\mathcal{C}$. Moreover, we say a facet is an *upper* (*lower*, resp.) *facet* of $\mathcal{C}$ if it arises by ∂_i (∂_i', resp.) for some $i \in \mathrm{Dim}(\mathcal{C})$. We note that each facet is either an upper or a lower one. An easy but important observation is that whenever $i, j \in \mathrm{Dim}(\mathcal{C})$, $i \neq j$, we have

$$(\partial_i \circ \partial_j)(\mathcal{C}) = \partial_i(\mathcal{C}) \cap \partial_j(\mathcal{C}) = (\partial_j \circ \partial_i)(\mathcal{C}), \tag{2.1}$$
$$(\partial_i \circ \partial'_j)(\mathcal{C}) = \partial_i(\mathcal{C}) \cap \partial'_j(\mathcal{C}) = (\partial'_j \circ \partial_i)(\mathcal{C}), \tag{2.2}$$
$$(\partial'_i \circ \partial'_j(\mathcal{C})) = \partial'_i(\mathcal{C}) \cap \partial'_j(\mathcal{C}) = (\partial'_j \circ \partial'_i)(\mathcal{C}). \tag{2.3}$$

Take a free Abelian group $\mathcal{A}_0$ generated by all cuboids in $\mathbb{R}^n$ and factorize it by the subgroup generated by all the elements

$$\mathcal{C} - \partial_i(\mathcal{C}) + \partial'_i(\mathcal{C}),\ \mathcal{C} \text{ is a cuboid},\ i \in \mathrm{Dim}(\mathcal{C}).$$

The resulting quotient Abelian group is denoted by $\mathcal{A}$. By an abuse of notation we will still refer to elements of $\mathcal{A}$ by cuboids. So in $\mathcal{A}$

$$\mathcal{C} = \partial_i(\mathcal{C}) - \partial'_i(\mathcal{C}) \tag{2.4}$$

holds for each cuboid $\mathcal{C}$ and $i \in \mathrm{Dim}(\mathcal{C})$.

Definition 2.1.1. *Suppose we have cuboids $\mathcal{C}$, $\mathcal{C}_1$ and $\mathcal{C}_2$ of the same dimension. We say that a couple $\mathcal{C}_1$ and $\mathcal{C}_2$ is a splitting of $\mathcal{C}$, if there is $i \in \mathrm{Dim}(\mathcal{C})$, such that $\partial'_i(\mathcal{C}_1) = \partial'_i(\mathcal{C})$, $\partial_i(\mathcal{C}_1) = \mathcal{C}_1 \cap \mathcal{C}_2 = \partial'_i(\mathcal{C}_2)$ and $\partial_i(\mathcal{C}_2) = \partial_i(\mathcal{C})$. In other words, there is a real c, $a_i^{\mathcal{C}} < c < b_i^{\mathcal{C}}$, such that $\mathcal{C}_1$ shares with $\mathcal{C}$ all its coordinates unless $b_i^{\mathcal{C}_1} = c$ and $\mathcal{C}_2$ shares with $\mathcal{C}$ all its coordinates unless $a_i^{\mathcal{C}_2} = c$.*

Observe that for the three cuboids from Definition 2.1.1, we have in the group $\mathcal{A}$

$$\mathcal{C} = \partial_i(\mathcal{C}_2) - \partial'_i(\mathcal{C}_1) = (\partial_i(\mathcal{C}_2) - \partial_i(\mathcal{C}'_2)) + (\partial_i(\mathcal{C}_1) - \partial'_i(\mathcal{C}_1)) = \mathcal{C}_1 + \mathcal{C}_2. \tag{2.5}$$

Lemma 2.1.2. *Each cuboid $\mathcal{C} \in \mathcal{A}$ could be uniquely (up to order of summands) written in the form*

$$\mathcal{C} = \sum_{\alpha \text{ is a vertex in } \mathcal{C}} (-1)^{\mathrm{ord}_{\mathcal{C}}(\alpha)+1} \alpha. \tag{2.6}$$

Hence, a vertex α occurs with $+1$ sign if it is of odd order in $\mathcal{C}$ and with -1 sign if it is of even order in $\mathcal{C}$. Consequently, $\mathcal{A}$ is in fact a free Abelian group generated by all the vertices (elements of $\mathbb{R}^n$).

Proof. The formula (2.6) follows by an inductive use of the formula (2.4). $\square$

Let L be a partial mapping from $\mathbb{R}^n$ to $\Gamma(G,u)$. Using (2.6), we can extend L to a group homomorphism $|\cdot|_L : \mathcal{A}_L \to G$, where $\mathcal{A}_L$ is the free subgroup of $\mathcal{A}$ generated by all vertices in $\mathrm{Def}(L)$. Hence $|\cdot|_L$ associates to each cuboid $\mathcal{C}$ having vertices in $\mathrm{Def}(L)$ its "volume" element $|\mathcal{C}|_L \in G$.

In this section, we will suppose that we have fixed two Dedekind σ-complete unital po-groups (G,u) and (H,v) with interpolation and let $\pi : (G,u) \to (H,v)$ be a fixed homomorphism satisfying following property (note that the property implies surjectivity).

Definition 2.1.3. *Given a po-group homomorphism $\pi : (G,u) \to (H,v)$ we say that π satisfies lifting property (LP), if for each $L, U \subseteq \Gamma(G,u)$ finite and $h \in H$ such that $L \leq U$ and $\pi(L) \leq h \leq \pi(U)$, there is $g \in G$ satisfying $\pi(g) = h$ and $L \leq g \leq U$. In the same way we define (LP) for a homomorphism of effect algebras.*

Let $F : \mathbb{R}^n \to \Gamma(H,v)$ be an n-dimensional spectral resolution and let $\pi : (G,u) \to (H,v)$ be a homomorphism of unital po-groups which satisfies the lifting property. We say, that a partial mapping $L : \mathbb{R}^n \to \Gamma(G,u)$ is a *partial lift* of F, if $\pi \circ L = F|_{\mathrm{Def}(L)}$ and for each cuboid $\mathcal{C} \subseteq \mathrm{Def}(L)$ we have $|\mathcal{C}|_L \in \Gamma(G,u)$. That occurs if $0 \leq L(\alpha) \leq u$ and $|\mathcal{C}|_L \geq 0$ for all vertices α's and all cuboids $\mathcal{C}$'s in the definition domain of L. We call the inequalities of the second type *volume conditions*. We note that the volume condition of a cuboid does not imply the volume conditions of its proper faces.

Our strategy will be to find a partial lift on the countable set $D = (\mathbb{Z}[\frac{1}{2}])^n$, where $\mathbb{Z}[\frac{1}{2}] = \{k/2^l : l \geq 1, k \in \mathbb{Z}\}$, which is dense subset of $\mathbb{R}^n$. In the construction, we will be extending a partial lift in kind of point by point way. So to assure the existence of partial lift on whole D we will need the countable axiom of choice.

It is important to note, that even if the volume conditions in the definition of spectral resolutions are required only for the n-cuboids (1.4) and the 1-cuboids (1.8) (the other cases consequently hold), for partial lifts the Proposition 1.4.7 does not work. Hence we have to concern ourselves with the volume conditions of all dimensions. Following Lemma describes the case, when we extend a partial lift L to $L' = L \cup \{(\alpha, g)\}$ such that some new cuboid $\mathcal{C} \subseteq \mathrm{Def}(L')$ of any dimension occurs.

Lemma 2.1.4. *Let $\mathcal{C}$ be a d-cuboid, α be its vertex, L be a partial lift defined on $\mathcal{C} \setminus \{\alpha\}$. For an extension $L' = L \cup \{(\alpha, g)\}$, $g \in \Gamma(G,u)$, we have: If α is of odd order in $\mathcal{C}$, then $\alpha - \mathcal{C} \in \mathcal{A}_L$ and the volume condition $|\mathcal{C}|_{L'} \geq 0$*

holds iff $|\alpha - \mathcal{C}|_L \leq g$. *If* α *is of even order in* $\mathcal{C}$, *then* $\alpha + \mathcal{C} \in \mathcal{A}_L$ *and the volume condition* $|\mathcal{C}|_{L'} \geq 0$ *holds iff* $g \leq |\alpha + \mathcal{C}|_L$.

Proof. If α is of odd (even, resp.) order, it occurs in (2.6) with $+1$ (-1, resp.) sign, hence α is canceled in $\alpha - \mathcal{C}$ ($\alpha + \mathcal{C}$, resp.). Consider the odd case: $|\mathcal{C}|_{L'} \geq 0 \Leftrightarrow |\mathcal{C}-\alpha|_{L'}+|\alpha|_{L'} \geq 0 \Leftrightarrow |\alpha|_{L'} \geq -|\mathcal{C}-\alpha|_{L'} \Leftrightarrow g \geq |\alpha-\mathcal{C}|_{L'} = |\alpha - \mathcal{C}|_L$. The even case: $|\mathcal{C}|_{L'} \geq 0 \Leftrightarrow |\mathcal{C} + \alpha|_{L'} \geq |\alpha|_{L'} \Leftrightarrow |\mathcal{C} + \alpha|_L \geq g$. □

Lemma 2.1.5. *Let* $\mathcal{C}$ *be a* d*-cuboid,* α *its top vertex, and* L_1 *a partial lift defined on* $\mathcal{C} \setminus \{\alpha\}$. *Then*

$$|\alpha - \mathcal{C}|_{L_1} \geq 0. \tag{2.7}$$

If $d \geq 1$, *let* β *be some vertex which is in* $\mathcal{C}$ *of the second order and* L_2 *be a partial lift defined on* $\mathcal{C} \setminus \{\beta\}$. *Denote* $i \in \mathrm{Dim}(\mathcal{C})$ *such that* β *is the top vertex in* $\partial_i'(\mathcal{C})$. *Then*

$$u \geq |\mathcal{C} + \beta|_{L_2} \geq |\beta - \partial_i'(\mathcal{C})|_{L_2} \geq 0. \tag{2.8}$$

Proof. We first prove (2.7) by an induction on d. The case $d = 0$ is trivial. Suppose $d \geq 1$. We have $|\alpha - \mathcal{C}|_{L_1} = |\alpha - \partial_i(\mathcal{C}) + \partial_i'(\mathcal{C})|_{L_1} = |\alpha - \partial_i(\mathcal{C})|_{L_1} + |\partial_i'(\mathcal{C})|_{L_1}$. As $\partial_i'(\mathcal{C})$ misses α, $|\partial_i'(\mathcal{C})|_{L_1} \geq 0$ and by the induction hypothesis $|\alpha - \partial_i(\mathcal{C})|_{L_1} \geq 0$ as well.

Let us prove inequalities (2.8): We prove the first one by an induction on d. The case $d = 1$ is trivial. Let $j \neq i$, i.e., $\beta \notin \partial_j'(\mathcal{C})$ and β is in $\partial_j(\mathcal{C})$ of order 2. Then $|\mathcal{C} + \beta|_{L_2} = |\partial_j(\mathcal{C}) - \partial_j'(\mathcal{C}) + \beta|_{L_2} = |\partial_j(\mathcal{C}) + \beta|_{L_2} - |\partial_j'(\mathcal{C})|_{L_2} \leq |\partial_j(\mathcal{C}) + \beta|_{L_2}$, and the last one is $\leq u$ by the induction hypothesis. The next inequality follows by: $|\beta + \mathcal{C}|_{L_2} \geq |\beta - \partial_i'(\mathcal{C})|_{L_2} \Leftrightarrow |\beta + \mathcal{C}|_{L_2} - |\beta - \partial_i'(\mathcal{C})|_{L_2} \geq 0 \Leftrightarrow |\mathcal{C}|_{L_2} + |\partial_i'(\mathcal{C})|_{L_2} \geq 0$. But $|\mathcal{C} + \partial_i'(\mathcal{C})|_{L_2} = |\partial_i(\mathcal{C})|_{L_2} \geq 0$ as L_2 is defined on the cuboid $\partial_i(\mathcal{C})$.

Finally, $|\beta - \partial_i'(\mathcal{C})|_L \geq 0$ follows from the already proved inequality (2.7) (β is the top vertex in $\partial_i'(\mathcal{C})$). □

Lemma 2.1.6. *Let* $\mathcal{C}$ *be a cuboid and* $\mathcal{F}(\mathcal{C})$ *be a collection of some of its facets such that, for each* $i \in \mathrm{Dim}(\mathcal{C})$, *the facet* $\partial_i(\mathcal{C})$ *or* $\partial_i'(\mathcal{C})$ *does not belong to* $\mathcal{F}(\mathcal{C})$. *Then for each sub-cuboid* $\mathcal{D} \subseteq \mathcal{C}$, *which satisfies* $\mathcal{D} \subseteq \bigcup \mathcal{F}(\mathcal{C})$, *there is some* $\mathcal{F} \in \mathcal{F}(\mathcal{C})$ *such that* $\mathcal{D} \subseteq \mathcal{F}$. *Consequently, if* L *is a mapping* $L : \bigcup \mathcal{F} \to G$ *whose restriction to any* $\mathcal{F} \in \mathcal{F}(\mathcal{C})$ *is a partial lift, then* L *itself is a partial lift.*

Proof. Suppose a cuboid $\mathcal{D} \subseteq \bigcup \mathcal{F}(\mathcal{C})$ which is not a sub-cuboid of any $\mathcal{F} \in \mathcal{F}(\mathcal{C})$. Take any $\mathcal{F} \in \mathcal{F}(\mathcal{C})$ and denote $i_{\mathcal{F}}$ the unique integer such that

$\mathrm{Dim}(\mathcal{F})\cup\{i_{\mathcal{F}}\}=\mathrm{Dim}(\mathcal{C})$. We have either $i_{\mathcal{F}}\in\mathrm{Dim}(\mathcal{D})$ or $i_{\mathcal{F}}\notin\mathrm{Dim}(\mathcal{D})$ and $a^{\mathcal{D}}_{i_{\mathcal{F}}}=b^{\mathcal{D}}_{i_{\mathcal{F}}}\neq a^{\mathcal{F}}_{i_{\mathcal{F}}}=b^{\mathcal{F}}_{i_{\mathcal{F}}}$. Since otherwise $\mathcal{D}\subseteq\mathcal{F}$. Consequently, in $\mathcal{D}$ there is a vertex α such that $a^{\alpha}_{i_{\mathcal{F}}}\neq a^{\mathcal{F}}_{i_{\mathcal{F}}}$ for each $\mathcal{F}\in\mathcal{F}(\mathcal{C})$. So α is not a vertex of any of $\mathcal{F}\in\mathcal{F}(\mathcal{C})$, which is a contradiction. $\square$

Lemma 2.1.7. *Let $\mathcal{C}_1,\mathcal{C}_2$ be a splitting of a cuboid $\mathcal{C}$, c,i be as in Definition 2.1.1 and $\mathcal{D}\subseteq\mathcal{C}_1\cup\mathcal{C}_2$ be a cuboid such that but $\mathcal{D}\nsubseteq\mathcal{C}$. Then $\mathcal{D}\subseteq\mathcal{C}_1$ or $\mathcal{D}\subseteq\mathcal{C}_2$. Consequently, if $L:\mathcal{C}_1\cup\mathcal{C}_2\to\Gamma(G,u)$ and the restrictions of L to $\mathcal{C}_1$ and to $\mathcal{C}_2$ are both partial lifts, then L is a partial lift as well.*

Proof. The i-th coordinates of the vertices of $\mathcal{D}$ belong to $\{a^{\mathcal{C}_i},c\}$ or $\{c,b^{\mathcal{C}_i}\}$, the first case implies $\mathcal{D}\subseteq\mathcal{C}_1$, the other one implies $\mathcal{D}\subseteq\mathcal{C}_2$. $\square$

Lemma 2.1.8. *Let $\mathcal{C}$ be a cuboid in $\mathbb{R}^n$ and L be a partial lift in $\mathcal{C}$ such that the definition domain of L equals one of the following sets*

(i) $\emptyset$,

(ii) *one lower facet, that is $\partial'_i(\mathcal{C})$ for some $i\in\mathrm{Dim}(\mathcal{C})$,*

(iii) *a union of one upper facet $\partial_i(\mathcal{C})$, $i\in\mathrm{Dim}(\mathcal{C})$, and a collection of lower facets $\partial'_j(\mathcal{C})$, $j\in J$, for some $J\subseteq\mathrm{Dim}(\mathcal{C})\setminus\{i\}$.*

Then we can extend L on the whole cuboid $\mathcal{C}$.

Note that if $\mathcal{C}$ is of dimension 2, the case when all vertices up to the top one are lifted is excluded.

Proof. We will use an induction on $\dim\mathcal{C}$. The case $\dim\mathcal{C}$ equals 0 or 1 is trivial. Suppose $\dim\mathcal{C}\geq 2$ and the case (i). Take any upper facet $\mathcal{F}$ of $\mathcal{C}$ and use the case (i) of the induction hypothesis to define L on $\mathcal{F}$. We have arrived at the case (iii).

If the case (ii) holds, pick any $j\in\mathrm{Dim}(\mathcal{C})\setminus\{i\}$. If we extend L to $\mathcal{F}=\partial_j(\mathcal{C})$, we will arrive at the case (iii) again. Since $\partial'_i(\mathcal{C})\cap\mathcal{F}=\partial'_i(\mathcal{F})$ is a lower facet of $\mathcal{F}$, we can use the case (ii) of the induction hypothesis to $\mathcal{F}$. So we obtain a partial lift L' on $\mathcal{F}$ and by Lemma 2.1.6, $L\cup L'$ is a partial lift.

Hence, it remains to prove the case (iii). Take any lower facet $\mathcal{F}=\partial'_k(\mathcal{C})$. If $j\notin J$, we extend L on $\mathcal{F}$: Since the upper facet $\partial_i(\mathcal{C})$ of $\mathcal{C}$ intersects $\mathcal{F}$ in an upper facet of $\mathcal{F}$ and similarly each lower facet $\partial'_j(\mathcal{C})$, $j\in J$, intersects

$\mathcal{F}$ in a lower facet of $\mathcal{F}$, we can by the case (iii) of the induction hypothesis and Lemma 2.1.6 extend L on $\mathcal{F}$.

Hence, we can assume $J = \mathrm{Dim}(\mathcal{C}) \setminus \{i\}$. That is, the only vertex that remains to lift is $\beta := (b_1, \ldots, b_{i-1}, a_i, b_{i+1}, \ldots, b_d)$. Note that β is of order 2 in $\mathcal{C}$. We have to find an extension $L' = L \cup \{(\beta, g)\}$, $g \in \Gamma(G, u)$, such that

$$|\mathcal{C}|_{L'} \geq 0, \tag{2.9}$$

$$|\partial'_i(\mathcal{C})|_{L'} \geq 0. \tag{2.10}$$

We claim, that if these two volume conditions hold, then all the volume conditions hold in $\mathcal{C}$. At first note that $|\partial_j(\mathcal{C})|_{L'} \geq 0$ holds for each $j \in \mathrm{Dim}(\mathcal{C}) \setminus \{i\}$ (the case $j = i$ holds by the assumptions): $|\partial_j(\mathcal{C})|_{L'} = |\mathcal{C}|_{L'} + |\partial'_j(\mathcal{C})|_{L'} \geq 0$, since $|\partial'_j(\mathcal{C})|_{L'} = |\partial'_j(\mathcal{C})|_{L} \geq 0$ and (2.9). Hence the volume condition holds in each facet of $\mathcal{C}$. Next let $\mathcal{F}$ be a face of $\mathcal{C}$ which contains β and is in $\mathcal{C}$ of co-dimension $e \geq 2$. It follows $\mathcal{F} = \Theta_{i_1} \circ \cdots \circ \Theta_{i_e}(\mathcal{C})$, where each $\Theta_{i_k} \in \{\partial_{i_k}, \partial'_{i_k}\}$. Since $\mathcal{F}$ contains β which is a vertex of order 2 in $\mathcal{C}$, the number of k, so that $\Theta_{i_k} = \partial'_{i_k}$ is at most 1. In particular, there is $k \leq e$ such that $\Theta_{i_k} = \partial_{i_k}$ and so $\mathcal{F}$ is of the form $\partial_{i_k}(\mathcal{F}_0)$ (by commutativity (2.2)). Now we use the formula

$$|\mathcal{F}|_{L'} = |\partial_{i_k}(\mathcal{F}_0)|_{L'} = |\mathcal{F}_0|_{L'} + |\partial'_{i_k}(\mathcal{F}_0)|_{L'}.$$

Since $|\partial'_{i_k}(\mathcal{F}_0)|_{L'} = |\partial'_{i_k}(\mathcal{F}_0)|_{L} \geq 0$ (as it misses β) it is enough to prove the volume condition for the face with less co-dimension in $\mathcal{C}$. In other words, we can use an induction on co-dimension to finish the proof of the claim.

As β occurs in $\mathcal{C}$ as an element of order 2 and in $\partial'_i(\mathcal{C})$ as an element of order 1, by Lemma 2.1.4 we can replace inequalities (2.9), (2.10) by equivalent conditions

$$|\mathcal{C} + \beta|_L \geq g \quad \& \quad g \geq |\beta - \partial'_i(\mathcal{C})|_L.$$

We finish by applying (LP) with bounds $\{u, |\mathcal{C} + \beta|_L\} \geq \{|\beta - \partial'_i(\mathcal{C})|_L, 0\}$ which are consistent by Lemma 2.1.5. □

In the following lemma $\mathcal{S}$ figures as a section of $\mathcal{C}$ orthogonal to the i-th axis.

Lemma 2.1.9. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be a splitting of a d-cuboid $\mathcal{C}$, i, c be as in Definition 2.1.1 and denote $\mathcal{S}$ the $(d-1)$-cuboid $\partial_i(\mathcal{C}_1) = \mathcal{C}_1 \cap \mathcal{C}_2 = \partial'_i(\mathcal{C}_2)$. Next let L be a partial lift defined on all the vertices of $\mathcal{C}$ and $\bigcup \mathcal{F}(\mathcal{S})$, where*

$\mathcal{F}(\mathcal{S})$ is the collection (possibly empty) of facets of $\mathcal{S}$ such that for each $j \in \mathrm{Dim}(\mathcal{C})$, $\partial_j(\mathcal{S})$ or $\partial'_j(\mathcal{S})$ does not belong to $\mathcal{F}(\mathcal{S})$ and in $\mathcal{F}(\mathcal{S})$ is at most one upper facet of $\mathcal{S}$. Then there is a partial lift which extends L on the whole $\mathcal{S}$.

Proof. We first maximalize the collection $\mathcal{F}(\mathcal{S})$ using an induction. Let $j \in \mathrm{Dim}(\mathcal{S})$ be such that neither $\partial_j(\mathcal{S})$ nor $\partial'_j(\mathcal{S})$ belongs to F. We like to extend L on $\partial'_j(\mathcal{S})$. As each upper facet in $\mathcal{F}(\mathcal{S})$ intersects $\partial'_j(\mathcal{S})$ in an upper facet of $\partial_j(\mathcal{S})$ and each lower facet in $\mathcal{F}(\mathcal{S})$ intersects $\partial_j(\mathcal{S})$ in a lower facet of $\partial'_j(\mathcal{S})$, we can use the induction hypothesis, where we take $\partial'_j(\mathcal{C})$ for $\mathcal{C}$ and $\partial'_j(\mathcal{F}(\mathcal{S}))$ for $\mathcal{F}(\mathcal{S})$. So we obtain a partial lift L' defined on $\partial'_j(C) \cup \partial'_j(\mathcal{S})$.

We have to prove $L \cup L'$ is a partial lift. For this it is enough to realize that, whenever for some cuboid $\mathcal{D}$, we have $\mathcal{D} \subseteq (\partial'_j(\mathcal{C}) \cup \partial'_j(\mathcal{S})) \cup (\mathcal{C} \cup (\bigcup \mathcal{F}(\mathcal{S})))$ (the definition domain of $L \cup L'$), then $\mathcal{D} \subseteq \partial'_j(\mathcal{C}) \cup \partial'_j(\mathcal{S})$ (the definition domain of L) or $\mathcal{D} \subseteq \mathcal{C} \cup (\bigcup \mathcal{F}(\mathcal{S})))$ (the definition domain of L'). If $\mathcal{D} \subseteq \mathcal{C}$, we are done. Otherwise by Lemma 2.1.7 we have $\mathcal{D} \subseteq \mathcal{C}_1$ or $\mathcal{D} \subseteq \mathcal{C}_2$. We can treat the both cases in similar fashion. We present a proof of the one where $\mathcal{D} \subseteq \mathcal{C}_1$. Then $\partial_i(\mathcal{D}) \subseteq (\bigcup \mathcal{F}(\mathcal{S})) \cup \partial'_j(\mathcal{S})$. By Lemma 2.1.6 two cases may occur: $\partial_i(\mathcal{D}) \subseteq \bigcup \mathcal{F}(\mathcal{S})$, and then $\mathcal{D} \subseteq \mathcal{C} \cup (\bigcup \mathcal{F}(\mathcal{S}))$, or $\partial_i(\mathcal{D}) \subseteq \partial'_j(\mathcal{S})$, and then $\mathcal{D} \subseteq \partial'_j(\mathcal{C}) \cup \partial'_j(\mathcal{S})$.

So we can assume for each $j \in \mathrm{Dim}(\mathcal{S})$ either $\partial_j(\mathcal{S}) \in \mathcal{F}(\mathcal{S})$ or $\partial'_j(\mathcal{S}) \in \mathcal{F}(\mathcal{S})$. Two cases possibly occur: (i) all facets in $\mathcal{F}(\mathcal{S})$ are lower and the top vertex β of $\mathcal{S}$ is the only one which remains to lift, or (ii) there is k, such that $\partial_k(\mathcal{S}) \in \mathcal{F}(\mathcal{S})$ and γ the only vertex which remains to lift has order 2 in $\mathcal{S}$.

Assume the case (ii). For some $g \in \Gamma(G, u)$ define $L' = L \cup \{(\gamma, g)\}$. We note that γ has order 2 in $\mathcal{C}_1$ and in $\mathcal{S}$ and order 3 in $\mathcal{C}_2$ and in $\mathcal{C}$. To assure L is a partial lift on $\mathcal{C}_1$, the following volume condition has to hold:

$$|\mathcal{C}_1|_{L'} \geq 0, \tag{2.11}$$
$$|\partial'_k(\mathcal{C}_1)|_{L'} \geq 0. \tag{2.12}$$

The case of $\mathcal{C}_2$ requires these volume conditions:

$$|\mathcal{C}_2|_{L'} \geq 0, \tag{2.13}$$
$$|\partial'_k(\mathcal{C}_2)|_{L'} \geq 0, \tag{2.14}$$
$$|\partial'_i(\mathcal{C}_2)|_{L'} \geq 0, \tag{2.15}$$
$$|\partial'_k \circ \partial'_i(\mathcal{C}_2)|_{L'} \geq 0. \tag{2.16}$$

We claim inequations (2.13)–(2.16) are sufficient. Let $\mathcal{F}$ be any face of $\mathcal{C}_2$; it has a form

$$\mathcal{F} = \Theta_{i_1} \circ \cdots \circ \Theta_{i_e}(\mathcal{C}_2), \quad e \leq \dim \mathcal{S}, \tag{2.17}$$

where each $\Theta_{i_k} \in \{\partial_{i_k}, \partial'_{i_k}\}$. We first prove by an induction on the number of occurrences of ∂_{i_k}'s in (2.17), that it is enough to treat the cases when each $\Theta_{i_k} = \partial'_{i_k}$. Since whenever $\mathcal{F}$ has form $\mathcal{F} = \partial_j(\mathcal{F}_0)$, then $|\mathcal{F}|_{L'} = |\partial'_j(\mathcal{F}_0)|_{L'} + |\mathcal{F}_0|_{L'}$. If the two summand are ≥ 0, so is the left hand side. However, if $\mathcal{F} = \partial'_{i_1} \circ \cdots \circ \partial'_{i_e}(\mathcal{C}_2)$ contains γ, each i_k's belongs to $\{i, k\}$, as γ is the top vertex of $\partial'_k \circ \partial'_i(\mathcal{C}_2)$. So (2.13)–(2.16) are all the volume conditions that matter.

Note that (2.15) and (2.16) are volume conditions for sub-cuboids of $\mathcal{C}_1$ (since $\partial'_i(\mathcal{C}_2) = \partial_i(\mathcal{C}_1)$), and hence they follow from (2.11) and (2.12). To assure L' is a partial lift, inequalities (2.11)–(2.14) give us according to Lemma 2.1.4 the following bounds:

$$g \leq |\gamma + \mathcal{C}_1|_L, \tag{2.18}$$
$$|\gamma - \partial'_k(\mathcal{C}_1)|_L \leq g, \tag{2.19}$$
$$|\gamma - \mathcal{C}_2|_L \leq g, \tag{2.20}$$
$$g \leq |\gamma + \partial'_k(\mathcal{C}_2)|_L. \tag{2.21}$$

We already know by Lemma 2.1.5 that (2.18) and (2.19) are consistent. So are (2.18) and (2.20), since: $|\gamma - \mathcal{C}_2|_L \leq |\gamma + \mathcal{C}_1|_L \Leftrightarrow |\mathcal{C}_1 + \mathcal{C}_2|_L \geq 0 \Leftrightarrow |\mathcal{C}|_L \geq 0$. Next $|\gamma - \partial'_k(\mathcal{C}_1)|_L \leq |\gamma + \partial'_k(\mathcal{C}_2)|_L \Leftrightarrow 0 \leq |\partial'_k(\mathcal{C}_1) + \partial'_k(\mathcal{C}_2)|_L = |\partial'_k(\mathcal{C})|_L$. Finally, $|\gamma - \mathcal{C}_2|_L \leq |\gamma + \partial'_k(\mathcal{C}_2)|_L \Leftrightarrow 0 \leq |C_2 + \partial'_k(\mathcal{C}_2)|_L = |\partial_k(\mathcal{C}_2)|_L$. Hence, we can apply the lifting property to obtain the desired lift in γ. The lift necessary belongs to $\Gamma(G, u)$ due to inequalities (2.8) in Lemma 2.1.5.

Next assume the (easier) case (i). Again define $L' = L \cup \{(\beta, g)\}$ for some $g \in G$. By the analogous arguments as in the case (ii), it is enough to assure volume conditions

$$|\mathcal{C}_1|_{L'} \geq 0, \tag{2.22}$$
$$|\mathcal{C}_2|_{L'} \geq 0, \tag{2.23}$$
$$|\partial'_i(\mathcal{C}_2)|_{L'} \geq 0. \tag{2.24}$$

But $\partial'_i(\mathcal{C}_2) = \partial_i(\mathcal{C}_1)$, hence the inequality (2.22) implies the equality (2.24). By Lemma 2.1.4,

$$|\beta - \mathcal{C}_1|_L \leq g \leq |\beta + \mathcal{C}_2|_L$$

is equivalent condition to (2.22) and (2.23). These bounds are consistent as $|\beta - \mathcal{C}_1|_L \leq |\beta + \mathcal{C}_2|_L \Leftrightarrow 0 \leq |\mathcal{C}_1 + \mathcal{C}_2| = |\mathcal{C}|_L$. Moreover, Lemma 2.1.5, inequalities (2.7) and (2.8) guarantee $0 \leq |\beta - \mathcal{C}_1|_L \leq |\beta + \mathcal{C}_2|_L \leq u$. Hence, we can finish the proof by application of the lifting property. $\square$

Lemma 2.1.10. *Suppose we have real numbers $a_0^i < a_1^i < \cdots < a_{m_i}^i$, $m_i \geq 1$, for each $i \leq n$. Denote $X := \{(r_1, \ldots, r_n) \in \mathbb{R}^n : \forall i, r_i \in \{a_1^i, \ldots, a_{m_i}^i\}\}$ and $\mathcal{C}$ the cuboid given by $a_i^{\mathcal{C}} = a_0^i$, $b_i^{\mathcal{C}} = a_{m_i}^i$. Let $\mathcal{F}(\mathcal{C})$ be a collection of facets of $\mathcal{C}$, such that for each $i \leq n$ facet $\partial_i(\mathcal{C})$ or $\partial_i'(\mathcal{C})$ does not belong to $\mathcal{F}(\mathcal{C})$ and $\mathcal{F}(\mathcal{C})$ contains at most one upper facet.*

Finally, let $Y \subseteq X$ be the set of all the points in X which are inside some facet in $\mathcal{F}(\mathcal{C})$ and L be a partial lift defined on $\mathcal{C} \cup Y$. Then there is a partial lift defined on whole X, which extends L.

Proof. We proceed by an induction on $|X|$. If $m_i = 1$ for each i, $X = \mathcal{C}$, then there is nothing to prove. Otherwise some m_i is at least 2, without loss of generality assume $m_1 \geq 2$. Let $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{S}$ be as in Lemma 2.1.9 where we set $c = a_2^1$. Note that $\mathcal{S} \cap Y$ can be expressed as a union $\cup \mathcal{F}(\mathcal{S})$, where $\mathcal{F}(\mathcal{S})$ is the collection of facets of $\mathcal{S}$, which satisfies the condition in the statement of Lemma 2.1.9 (elements of $\mathcal{F}(\mathcal{S})$ correspond to those in $\mathcal{F}(\mathcal{C}) \setminus \{\partial_1(\mathcal{C}), \partial_1'(\mathcal{C})\}$). So we can apply Lemma 2.1.9 to extend L on $\mathcal{S}$; denote by L_1 the extension. We claim L_1 is a partial lift. We prove: each cuboid $\mathcal{D} \subseteq Y \cup \mathcal{C} \cup \mathcal{S}$ (the definition domain of L_1) satisfies $\mathcal{D} \subseteq \mathcal{C} \cup Y$ or $\mathcal{D} \subseteq \mathcal{C} \cup \mathcal{S}$. In the first case $\mathcal{D}$ belongs to the definition domain of L so the volume condition holds for $\mathcal{D}$, in the second case the volume condition holds as well by Lemma 2.1.9.

On the way of contradiction suppose $\mathcal{D} \subseteq Y \cup \mathcal{C} \cup \mathcal{S}$ and there are vertices $\alpha, \beta \in \mathcal{D}$, $\alpha \in \mathcal{S} \setminus Y$ and $\beta \in Y \setminus (\mathcal{S} \cup \mathcal{C})$. In coordinates $\alpha = (a_1^1, x_2, \ldots, x_n)$, where $x_j \in \{a_0^j, a_{m_j}^j\}$ and $\beta = (y_1, \ldots, y_n)$, where at least for one j, $y_j \notin \{a_0^j, a_{m_j}^j\}$ in the case $j \geq 2$ or $y_j \notin \{a_0^j, a_1^j, a_{m_j}^j\}$ in the case $j = 1$. Fix concrete such j. Since $\mathcal{D}$ is a cuboid, there is a vertex $\beta' \in \mathcal{D}$ which shares with α all the coordinates, except the j-th, where it has a value y_j (so $\beta' \in (Y \cup \mathcal{S} \cup \mathcal{C}) \setminus (\mathcal{C} \cup \mathcal{S}) = Y \setminus (\mathcal{C} \cup \mathcal{S})$). We know that β' is inside some $\mathcal{F} \in \mathcal{F}(\mathcal{C})$ (as it belongs to Y), let $\mathcal{F} = \partial_k(\mathcal{C})$ ($\mathcal{F} = \partial_k'(\mathcal{C})$, resp.). That is, β' has on k-th coordinate a value a_0^k ($a_{n_k}^k$, resp.). Hence $k \neq j$. But that entails α has on k-th coordinate the value a_0^k ($a_{n_k}^k$, resp.) as well. Therefore, α is inside $\partial_k(\mathcal{C})$ ($\partial_k'(\mathcal{C})$, resp.) and so it belongs to Y, which is a contradiction.

Now we can finally use the induction hypothesis. If $\partial_1(\mathcal{C}) \in \mathcal{F}(\mathcal{C})$ (that is, $\partial_1(\mathcal{C})$ is the only upper facet in the collection), then take for a new $\mathcal{C}$ the cuboid $\mathcal{C}_2$, for a new X take all the elements of the old X which are inside $\mathcal{C}_2$ and for the new collection of facets take $\mathcal{F}(\mathcal{C}_2)$ which contains $\partial_1(\mathcal{C}_1) = \partial_1(\mathcal{C})$ and moreover it contains $\partial'_j(\mathcal{C}_2)$ ($\partial_j(\mathcal{C}_2)$, resp.), for $j \neq 1$, whenever the original $\mathcal{F}(\mathcal{C})$ contains $\partial'_j(\mathcal{C})$ ($\partial_j(\mathcal{C})$, resp.). The induction hypothesis gives us a lift defined on all elements of X, which are inside $\mathcal{C}_2$. Then apply the induction hypothesis in a similar fashion to $\mathcal{C}_1$. In the case $\partial_1(\mathcal{C}) \notin \mathcal{F}(\mathcal{C})$ we apply the induction hypothesis to $\mathcal{C}_1$ in first place and then to $\mathcal{C}_2$. Otherwise, it could happen that the collection $\mathcal{F}(\mathcal{C}_1)$ does not meet the conditions in the statement.

We have obtained a mapping L_2, which satisfies the volume condition for each cuboid having vertices in X and which is inside $\mathcal{C}_1$ or $\mathcal{C}_2$. Each cuboid $\mathcal{D} \subseteq X$ could be in the obvious way split into $\mathcal{D}_1$ and $\mathcal{D}_2$, where $\mathcal{D}_1$ is inside $\mathcal{C}_1$, $\mathcal{D}_2$ is inside $\mathcal{C}_2$ and in $\mathcal{A}$ we have $\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2$. Consequently $|\mathcal{D}|_{L_2} = |\mathcal{D}_1|_{L_2} + |\mathcal{D}_2|_{L_2} \geq 0$. $\square$

Proposition 2.1.11. *Let (G,u) and (H,v) be unital Dedekind monotone σ-complete po-groups and let $\pi : (G,u) \to (H,v)$ be a homomorphism with (LP). Let F be an n-dimensional spectral resolution on (H,v). Then there is a countable and dense subset $D \subset \mathbb{R}^n$ and a partial lift L of F which is defined on D.*

Proof. Define for each $l \in \mathbb{N}_0$ the set $\mathcal{U}_l$ of all n-cubes with coordinates in $\frac{1}{2^l}\mathbb{Z}$ which have edges of length $\frac{1}{2^l}$ (e.g., all n-cuboids $\mathcal{C}$, for which $a_j^{\mathcal{C}}, b_j^{\mathcal{C}} \in \frac{1}{2^l}\mathbb{Z}$ and $b_j^{\mathcal{C}} - a_j^{\mathcal{C}} = \frac{1}{2^l}$ for each j, $1 \leq j \leq n$). Our strategy is as follows: We will inductively construct a sequence of partial lifts $L_0 \subset L_1 \subset \cdots$ such that each L_l is defined on $D_l := \bigcup_{i \leq l} \mathcal{U}_i$. The set $D := \bigcup_{l \geq 0} D_l$ is countable and dense in $\mathbb{R}^n$ and a partial mapping $\bigcup_{l \geq 0} L_l$ will be the desired partial lift.

Claim *We can list all the n-cubes in $\mathcal{U}_l$, $l \geq 0$, so that the following property $(*)$ holds: For each $m \in \mathbb{N}$, we have $\mathcal{E}_m \cap (\bigcup_{j<m} \mathcal{E}_j)$ is of one of the following types:* (i) *empty,* (ii) *one lower facet of $\mathcal{E}_m$,* (iii) *union of one upper facet $\partial_k(\mathcal{C}_m)$ and a collection of lower facets $\partial'_j(\mathcal{E})$, where $J \subseteq \{1,\dots,n\} \backslash \{k\}$.*

Using Claim, it is rather easy to construct the L_l's: Suppose $\mathcal{C}_1, \mathcal{C}_2, \dots$ is the list from Claim of all the elements in $\mathcal{U}_0$. We first apply Lemma 2.1.8 to $\mathcal{C}_0$, then to $\mathcal{C}_1$ and so on inductively we define a lift L_0. In next step we like to extend L_0 on each point with coordinates in $\frac{1}{2}\mathbb{Z}$, that is on $D_1 = \bigcup \mathcal{U}_1$. To achieve this, we use another induction process. At the first step, we use Lemma 2.1.10 to the first n-cube $\mathcal{C}_1 \in \mathcal{U}_0$: We leave the set $\mathcal{F}(\mathcal{C}_1)$ from the statement empty and we set a^i_j's such that the points in X are exactly the ones of D_1 we want to lift (we set for each i, $m_i = 2$, $a^i_1 = \frac{a^{\mathcal{C}_1} + b^{\mathcal{C}_1}}{2}$). In the m-th step we use Lemma 2.1.10 in a similar way, the only difference is that we cannot set $\mathcal{F}(\mathcal{C}_1)$ empty, since points inside some facets of $\mathcal{C}_m$ have already been lifted, but the collection of these facets is, by the property $(*)$, in a convenient form.

In the next step we use an analogous induction process, but with $\mathcal{U}_2$ in place of $\mathcal{U}_1$. So we find lifts in all points with coordinates in $\frac{1}{2^2}\mathbb{Z}$. Similarly we find partial lifts $L_3 \subset L_4 \subset \cdots$. The desired dense set and partial lift are $D := \bigcup_{m \in \mathbb{N}} D_m$ and $L := \bigcup_{m \in \mathbb{N}} L_m$. The construction guarantees for each L_m the volume condition $|\mathcal{C}|_{L_m} \geq 0$, only for $\mathcal{C}$ being a face of some cuboid in $\mathcal{U}_m$. However, it is clear that each cuboid with vertices in D_m has (as an element of $\mathcal{A}$) a decomposition $\mathcal{C} = \mathcal{D}_1 + \cdots + \mathcal{D}_k$, such that all $\mathcal{D}_i$'s are already faces of some cuboids in $\mathcal{U}_m$. Finally, each volume condition $|\mathcal{C}|_L \geq 0$ holds, where $\mathcal{C} \subset D$, since $\mathcal{C} \subset D_m$ for some m, as $\mathcal{C}$ has only a finite number of vertices.

Proof of Claim. It obviously suffices to consider only the case $\mathcal{U}_0$, as the lattices D_l's are all isomorphic. In the rest of the proof, we assume all mentioned cuboids have integral coordinates. Suppose we have already listed all unit cuboids inside some n-cuboid $\mathcal{C}$, such that (*) holds, denote the list l. We show, that if we enlarge $\mathcal{C}$ in any of the directions by one, then we can add all the unit n-cuboids newly occurring inside extended $\mathcal{C}$ to the list.

Suppose we enlarge $\mathcal{C}$ in some direction parallel to i-th axis, this corresponds to coordinate change: (1) $a^{\mathcal{C}}_i \rightsquigarrow a^{\mathcal{C}}_i - 1$ or (2) $b^{\mathcal{C}}_i \rightsquigarrow b^{\mathcal{C}}_i + 1$. Suppose for simplicity $i = 1$. In both cases denote $\mathcal{U}'$ the collection of $\prod_{j \geq 2}(b^{\mathcal{C}}_j - a^{\mathcal{C}}_j)$ unit cuboids, we need to add to the list l.

In the case (1) each cuboid in $\mathcal{U}'$ shares exactly one of its upper facets with a cuboid in l. As cuboid in $\mathcal{U}'$ form an $n-1$ dimensional table, we can (linearly) order $\mathcal{U}'$ using the lexicographic ordering (naturally defined). With respect to this ordering each cuboid in $\mathcal{U}'$ shares with the previous ones only lower facets. Hence

$$l \sqcup \mathcal{U}',$$

whereby $\sqcup$ we mean join of the sequences, satisfies (*).

The case (2) is more complicated. Now each cuboid in $\mathcal{U}'$ shares exactly one of its lower facets with a cuboid in l. The simple idea of using the lexicographic ordering from (1) does not work. Consider a decomposition

B	B	B
B	B	A
B	A	A
B	A	B
A	A	A
A	A	B
A	B	A
A	B	B

Figure 2.1

$\mathcal{U}' = \mathcal{U}'_1 \cup \cdots \cup \mathcal{U}'_m$, which correspond to dividing the cuboids to levels with respect to the 2nd coordinate (so $m = b_2^{\mathcal{C}} - a_2^{\mathcal{C}}$). If we manage to somehow order $\mathcal{U}'_m$ so that $l \sqcup \mathcal{U}'_m$, satisfies (*), we can add the other levels in following way. Lets order each $\mathcal{U}'_i$, $j < m$ by the lexicographic ordering and add them to the list:

$$L' := l \sqcup \mathcal{U}'_m \sqcup \mathcal{U}'_{m-1} \sqcup \cdots \sqcup \mathcal{U}'_1.$$

Hence the resulting ordering on $\mathcal{U}'_{m-1} \cup \cdots \cup \mathcal{U}'_1$ is kind of lexicographic ordering, where we turn over the direction of ordering in the coordinate with the highest priority. So each cuboid $\mathcal{D} \in \mathcal{U}'_1 \cup \cdots \mathcal{U}'_{m-1}$ would share exactly one upper facet $\partial_2 \mathcal{D}$ with the ones foregoing in l and some collection of lower facets $\partial'_j \mathcal{D}$, $j \neq 2$.

Finally, we can deal with $\mathcal{U}'_m$ by iterating the same idea. That is, we divide $\mathcal{U}'_m$ into levels with respect to the 3rd axis and so on. At the bottom of the induction process we are adding to l a single cuboid $\mathcal{D}_0 \in \mathcal{U}'$ (the one which has the highest position among cuboids form $\mathcal{U}'$, with respect to ordinary lexicographic ordering), which is allowed by the point (ii) in the condition (*). The list in figure 2.1 demonstrates the resulting "twisted lexicographic ordering" on words of length 3 over $\{A, B\}$.

Now the Claim easily follows: we can begin the list l with any cuboid in $\mathcal{U}_1$. Then we will be step by step extending the area of listed cuboids using the just described technique. Obviously, we can proceed in such a way, that we reach any unit cuboid in a finite amount of steps. □

And so we are done with the whole proof. □

Now, we present one of our main results – lifting of spectral resolutions:

Theorem 2.1.12. [*Lifting of Spectral Resolutions*] *Let $\pi : (G,u) \to (H,v)$ be a σ-homomorphism of unital Dedekind monotone σ-complete po-groups and let π satisfy (LP). Then each n-dimensional spectral resolution $F : \mathbb{R}^n \to (H,v)$ can be lifted to an n-dimensional spectral resolution $K : \mathbb{R}^n \to (G,u)$ such that $\pi \circ K = F$.*

Proof. According to Proposition 2.1.11, there is a countable subset

$$D = \{(t_1,\dots,t_n) | t_i \in \mathbb{Z}[\frac{1}{2}]\} \subset \mathbb{R}^n$$

dense in $\mathbb{R}^n$ and a partial lift $L : D \to G$. Observe that D is of the form D_π^n, where D_π is dense in $\mathbb{R}$ and L is monotone on D. Define $K_0 : \mathbb{R}^n \to G$ by prescription

$$K_0(\mathbf{t}) = \bigvee_{\mathbf{s}\in D, \mathbf{s} \ll \mathbf{t}} L(\mathbf{s}). \tag{2.25}$$

Note that the supremum exists because if we take two sequences $(\mathbf{s}_j)_j$ and $(\mathbf{u}_j)_j$ of elements of D such that $\mathbf{s}_j, \mathbf{u}_j \ll \mathbf{t} \in \mathbb{R}^n$ for each $j \geq 1$ and $\mathbf{s}_j \nearrow \mathbf{t}$ and $\mathbf{u}_j \nearrow \mathbf{t}$, then monotonicity of L implies that

$$\bigvee_j L(\mathbf{s}_j) \quad \text{and} \quad \bigvee_j L(\mathbf{u}_j)$$

exist in G and

$$\bigvee_j L(\mathbf{s}_j) = \bigvee_j L(\mathbf{u}_j).$$

Hence, $K_0(\mathbf{t})$ is correctly defined and

$$K_0(\mathbf{t}) = \bigvee_j L(\mathbf{s}_j) = \bigvee_{\mathbf{s}\in D, \mathbf{s} \ll \mathbf{t}} L(\mathbf{s}).$$

Now, K_0 is monotone in each component (directly from definition) and all volume conditions hold: Given any d-cuboid $\mathcal{C}$ in $\mathbb{R}^n$, we can write the volume condition in the form

$$K_0(\alpha_1) + \cdots + K_0(\alpha_{2^{d-1}}) \leq K_0(\beta_1) + \cdots + K_0(\beta_{2^{d-1}}), \tag{2.26}$$

where α_i's are all the vertices in $\mathcal{C}$ (hence n-tuples of reals) with even order and β_i's are all the vertices in $\mathcal{C}$ with odd order. By the construction of D, for each $\epsilon > 0$, there exists a d-cuboid $\mathcal{D} \subset D$, which is sufficiently close to $\mathcal{C}$:

For each $i = 1, \dots, n$, we find $a_i^{\mathcal{D}}, b_i^{\mathcal{D}} \in D_\pi$, such that $a_i^{\mathcal{C}} - \epsilon < a_i^{\mathcal{D}} < a_i^{\mathcal{C}}$ and $b_i^{\mathcal{C}} - \epsilon < b_i^{\mathcal{D}} < b_i^{\mathcal{C}}$ (and hence the vertices of the cuboid $\mathcal{D}$ given by $a_i^{\mathcal{D}}$'s and $a_i^{\mathcal{D}}$'s belong to D). Moreover, we can assume $a_i^{\mathcal{D}} = b_i^{\mathcal{D}}$ for each $i \notin \mathrm{Dim}(\mathcal{C})$.

Given any $\alpha'_1, \dots, \alpha'_{2^{d-1}} \in D$ such that for each $i = 1, \dots, 2^{d-1}$, $\alpha'_i \ll \alpha_i$, there is $\epsilon > 0$, for which $\alpha'_i \ll \alpha_i - \epsilon$ for each α_i, where $\epsilon = (\epsilon, \dots, \epsilon) \in \mathbb{R}^n$. Hence, as we have proved above, there is a d-cuboid $\mathcal{D}$, such that for each $i = 1, \dots, 2^{d-1}$ we have $\alpha'_i < \gamma_i < \alpha_i$, where γ_i is the corresponding vertex of $\mathcal{D}$. From the volume condition for $\mathcal{D}$ we deduce

$$\begin{aligned} L(\alpha'_1) + \cdots + L(\alpha'_{2^{d-1}}) \le L(\gamma_1) + \cdots + L(\gamma_{2^{d-1}}) \le \\ \le L(\delta_1) + \cdots + L(\delta_{2^{d-1}}) \le K_0(\beta_1) + \cdots + K_0(\beta_{2^{d-1}}), \end{aligned}$$

where δ_i's are all the vertices in $\mathcal{D}$ with even order. In the inequality, if we take the supremum over all α'_i's ($\alpha'_i \ll \alpha_i$), we get (2.26), thanks to the fact that $+$ distributes over $\vee$.

The function K_0 yet does not have to vanish when some coordinate goes to $-\infty$. We have to repair this. Define

$$K_0(-\infty, t_2, \dots, t_n) := \bigwedge_t K_0(t, t_2, \dots, t_n)$$

(it exists since K_0 is monotone) and

$$K_1(t_1, \dots, t_n) := K_0(t_1, \dots, t_n) - K_0(-\infty, t_2, \dots, t_n).$$

Obviously $\bigwedge_{t_1} K_1(t_1, \dots, t_n) = 0$. We will verify that $K_1 : \mathbb{R}^d \to G$ satisfies the volume conditions for 1 and n dimensional cuboids. The verification of the volume conditions is easy for the cuboids $\mathcal{C}$ such that $1 \in \mathrm{Dim}(\mathcal{C})$, as in that case we realize that $|\mathcal{C}|_{K_1} = |\mathcal{C}|_{K_0}$: For any edge e with $\mathrm{Dim}(e) = \{1\}$ clearly $|e|_{K_0} = |e|_{K_1}$ and $\mathcal{C}$ could be written as a sum of such edges (since $1 \in \mathrm{Dim}(\mathcal{C})$). Hence we only need to prove a monotonicity of K_1 in j-th coordinate for $j \ge 2$. Without loss of generality assume $j = 2$. Hence given some reals $s, s', t_1, t_3, \dots, t_n \in \mathbb{R}$, $s < s'$ we like to prove that the middle expression in

$$\begin{aligned} -K_1(t_1, s, t_3, \dots, t_n) \le K_1(t_1, s', t_3, \dots, t_n) - K_1(t_1, s, t_3, \dots, t_n) \le \\ \le K_1(t_1, s', t_3, \dots, t_n). \end{aligned}$$

is non-negative. From the already proved versions of volume conditions we know, that all the three parts of the later expression are monotone in the first coordinate. Hence

$$-K_1(-\infty, s, t_3, \dots, t_n) \le \bigwedge_{t_1}[K_1(t_1, s', t_3, \dots, t_n) - K_1(t_1, s, t_3, \dots, t_n)] \le$$
$$\le K_1(-\infty, s', t_3, \dots, t_n),$$

and as the left and the right hand side are equal to 0, we yield the desired monotonicity of K_1 in the second (and so in any) coordinate.

Similarly, K_1 is still continuous: assume $\alpha_i \nearrow \alpha := (t_1, \dots, t_n)$, then

$$K_1(\alpha) \ge K_1(\alpha_i) \ge$$
$$\ge K_0(\alpha_i) - K_0(-\infty, t_2, \dots, t_n) \nearrow K_0(\alpha) - K_0(-\infty, t_2, \dots, t_n) = K_1(\alpha).$$

If $K_0(t_1, \dots, t_{i-1}, -\infty, t_i, \dots, t_n) = 0$ for some i, clearly

$$K_1(t_1, \dots, t_{i-1}, -\infty, t_{i+1}, \dots, t_n) = 0$$

as well. As K_1 satisfies all volume conditions it is monotone.

Finally, the equality $\pi \circ K_1 = F$ still holds, since $\pi(K_0(-\infty, t_2, \dots, t_n)) = 0$ for each reals t_i's, because π is a σ-homomorphism.

Then we repeat the above procedure for each coordinate. That is, we inductively define for $i < n$

$$K_{i+1}(t_1, \dots, t_i, -\infty, t_{i+2}, \dots, t_n) = \bigwedge_{t_{i+1}} K_i(t_1, \dots, t_n)$$

and

$$K_{i+1}(t_1, \dots, t_n) = K_i(t_1, \dots, t_n) - K_i(t_1, \dots, t_i, -\infty, t_{i+2}, \dots, t_n).$$

The same reasoning as in the case of K_1 applies to each induction step. Hence we obtain a sequence $0 \le K_n \le K_{n-1} \le \dots \le K_1$, where the K_n satisfies all the conditions in Definition 1.4.3 except (1.5). And so K_n is pseudo n-dimensional spectral resolution. Denote $u' := \bigvee K_n(t_1, \dots, t_n)$, then $\pi(u - u') = 0$ so

$$K(t_1, \dots, t_n) = \begin{cases} K_n(t_1, \dots, t_n) + (u - u'), & \text{if } \vec{0} \ll (t_1, \dots, t_n) \\ K_n(t_1, \dots, t_n), & \text{otherwise} \end{cases} \tag{2.27}$$

is the desired lifted n-spectral resolution. $\square$

2.2 Extending of spectral resolutions

In this section we apply the results from the previous one and give various examples of effect algebras for which one-to-one correspondence between n-spectral resolutions and n-observables holds. The most general theorem is:

Theorem 2.2.1. *Let E, T be a pair of monotone σ-complete effect algebras which could be represented as intervals of po-groups $E \cong \Gamma(G,u)$, $T \cong \Gamma(H,v)$ and let $\pi : (H,v) \to (G,u)$ be a σ-homomorphism of unital po-groups satisfying lifting property. Then for E each n-spectral resolution uniquely extends to an n-observable whenever it holds for T.*

Proof. Given F an n-spectral resolution on E, we can thing about it as an n-spectral resolution on (H,v). Using Theorem 2.2.1 we can lift F to $\hat{F}$ an n-spectral resolution on (G,u). Now, $\hat{F}$ extends to an n-observable $\hat{x}$ such that $x := \pi \circ \hat{x}$ is the desired n-observable extending F.

Uniqueness of x: Let y be another n-dimensional observable extending F. The set $\mathcal{D} = \{A \subset \mathcal{B}(\mathbb{R}^n) | x(A) = y(A)\}$ is a Dynkin system (it contains $\emptyset$ and is closed under complements and countable disjoint unions) which contains all intervals $(-\infty, t_1) \times \cdots \times (-\infty, t_n)$, so that by the Sierpiński Theorem 1.3.5, $\mathcal{D} = \mathcal{B}(\mathbb{R}^n)$, which shows $x = y$. $\square$

The prime example of an effect algebra which could play the role of T in Theorem 2.2.1 is an *effect tribe* (Definition 1.3.3).

Lemma 2.2.2. *Let $\mathcal{T}$ be an effect-tribe. Then each n-spectral resolution $F : \mathbb{R}^n \to \mathcal{T}$ extends to a unique n-observable.*

Proof. For each fixed $\omega \in \Omega$, the function $F_\omega : \mathbb{R}^n \to [0,1]$ defined by

$$F_\omega(t_1, \ldots, t_n) := F(t_1, \ldots, t_n)(\omega),$$

$t_1, \ldots, t_n \in \mathbb{R}$, is left continuous, going to 0 if $t_i \to -\infty$ with non-negative increments. According to [Kal02, Thm 2.25], there is a unique σ-additive finite measure P_ω on $\mathcal{B}(\mathbb{R}^n)$ such that $P_\omega((-\infty, t_1) \times \cdots \times (-\infty, t_n)) = F_\omega(t_1, \ldots, t_n)$. Therefore, we have a mapping $x : \mathcal{B}(\mathbb{R}^n) \to [0,1]^\Omega$ such that $x(A)(\omega) = P_\omega(A)$ for all $A \in \mathcal{B}(\mathbb{R}^n)$ and all $\omega \in \Omega$. We denote by $\mathcal{K} = \{A \in \mathcal{B}(\mathbb{R}^n) \colon x(A) \in \mathcal{T}\}$. Then $\mathcal{K}$ contains $\mathbb{R}^n$, all intervals of the form $(-\infty, t_1) \times \cdots \times (-\infty, t_n)$, and is closed under complements and unions of disjoint sequences, i.e. $\mathcal{K}$ is a Dynkin system and by the Sierpiński Theorem 1.3.5, $\mathcal{K} = \mathcal{B}(\mathbb{R}^n)$. Then x is an n-dimensional observable on $\mathcal{T}$ such that $x((-\infty, t_1) \times \cdots \times (-\infty, t_n)) = F(t_1, \ldots, t_n)$, $t_1, \ldots, t_n \in \mathbb{R}$.

The uniqueness of x follows by the very same argument as in the proof of Theorem 2.2.1. □

the classical models of quantum logic turn out to be representable as tribes.

Theorem 2.2.3. *For $\mathcal{E}(\mathcal{H})$ and $\mathrm{Proj}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space over the field of reals, complex numbers or quaternions, one-to-one correspondence between n-observables and n-spectral resolutions holds.*

Proof. The statement follows from an observation, that $\mathcal{E}(\mathcal{H})$ and $\mathrm{Proj}(\mathcal{H})$ are isomorphic to effect tribes. Take for $\Omega(\mathcal{H})$ the set of unit vectors in $\mathcal{H}$. To each $A \in \mathcal{E}(\mathcal{H})$ we associate $\mu_A : \omega(\mathcal{H}) \to [0, 1]$ as $\mu_A : \omega \mapsto \langle A\omega, \omega\rangle$. Now one can proof that $\mathcal{T}(\mathcal{H}) = \{\mu_A | A \in \mathcal{E}(\mathcal{H})\}$ is an effect tribe and the described mapping $A \mapsto \mu_A$ is an isomorphism.

The case of $\mathrm{Proj}(\mathcal{H})$ could be treated in the same way. □

Theorem 2.2.4. *Let E be a monotone σ-complete effect algebra with (RDP). Then each n-spectral resolution uniquely extends to an n-observable.*

Proof. The Loomis-Sikorski theorem represents the effect algebra E as a homomorphic image of an effect tribe $\pi : \mathcal{T} \to E$. We like to apply Theorem 2.2.1 utilizing lifting through π, hence we need to verify the lifting property first. To do so, we have to capture some details of the construction of T (for the whole proof see [BCD06]).

For E (or any other monotone σ-complete effect algebra with (RDP)) the effect-tribe $\mathcal{T}$ is constructed as follows: for the set Ω we take the set $S(E)$ of all states on E. The set of states is known to be convex and the set of extremal states $\partial_e S(E)$ is endowed with certain topology (induced by pointwise convergence of the states). Following standard duality construction, we associate to each $a \in E$ an element $\hat{a} : \Omega \to [0, 1]$ by prescription $\hat{a} : s \mapsto s(a)$. For $\mathcal{T}$ we take all the fuzzy sets f's on Ω, such that there is $a \in E$ with $\{s \in \partial_e S(E) | \hat{a}(s) \neq f(s))\}$ is a meager set (i.e., countable union of nowhere dense sets); we indicate that relation as $f \sim a$. One could prove the $\mathcal{T}$ really is an effect-tribe and $\pi : \mathcal{T} \to E$ sending f to a unique a with $f \sim a$ is a σ-homomorphism.

Now we will verify the lifting property for π. Assume finite $L, U \subset \mathcal{T}$ such that $L \leq U$ and some $a \in E$ fitting in $\pi(L) \leq a \leq \pi(U)$. For any $f \in L$ and $g \in U$ we have $f(s) \leq \hat{a}(s) \leq g(s)$ up to a meager set. Hence, as the collection of meager sets is closed under finite unions (in fact it is a σ-ideal in the subsets of Ω) there is $h \sim a$, such that $\max\{f(s)|f \in L\} \leq h(s) \leq \min\{g(s)|g \in U\}$ for all $s \in S(E)$. And h is the desired lift. □

Loomis-Sikorski theorem is also known for MV-algebras and a similar proof could proceed in this case (as is done in [DL20a]). Nevertheless, one could treat MV-algebras as a special kind of effect algebras see Remark 1.2.2) with (RDP). Hence the case of MV-algebras is a consequence of Theorem 2.2.4.

Corollary 2.2.5. *Let M be a σ-complete MV-algebra, then a one-to-one correspondence between ite n-observables and n-spectral resolutions holds.*

The last corollary could be extended to the case of σ-complete effect algebras using the concept of blocks. We say that two elements a, b of an effect algebra E are compatible, if there are three elements $a_1, c, b_1 \in E$, such that $a = a_1 + c$, $b = b_1 + c$ and $a_1 + c + b_1$ exists. We call a *block* any maximal system of pairwise compatible elements. In the case of a lattice-effect algebra E, every system of pairwise compatible elements can be (using the Zorn's lemma) completed to a maximal one, and the result is a sub-effect algebra, which turns out to be (essentially) an MV-algebra (see [DP00], Thm. 1.10.20–21). Moreover, if the algebra E is σ-complete, its blocks are σ-complete MV-algebras (Thm. 4.3. in [Rie00]).

Theorem 2.2.6. *Let E be a σ-complete effect algebra, then a one-to-one correspondence between its n-observables and n-spectral resolutions holds.*

Proof. It suffices to prove, that an image of any n-spectral resolution F consists of pairwise compatible elements, then by [Rie98] there is a block containing $F(\mathbb{R}^n)$. Consequently, we can restrict ourselves to the situation of MV-algebras, which is covered by Corollary 2.2.5.

Hence, assume two n-tuples $\mathbf{t}, \mathbf{s} \in \mathbb{R}^n$, we have to prove $a := F(\mathbf{t})$ and $b := F(\mathbf{s})$ are compatible. Denote by $\mathbf{t} \wedge \mathbf{s}$ ($\mathbf{t} \vee \mathbf{s}$, resp.) infimum (supremum, resp.) in $\mathbb{R}^n$ (considered as a lattice). We set $a_1 = F(\mathbf{t}) - F(\mathbf{t} \wedge \mathbf{s})$, $b_1 = F(\mathbf{s}) - F(\mathbf{t} \wedge \mathbf{s})$ and $c = F(\mathbf{t} \wedge \mathbf{s})$. Obviously $a = a_1 + c$ and $b = b_1 + c$, so it remains to prove $a_1 + c + b_1$ exists. It is a consequence of the following formula, where only the inequality is nontrivial:

$$a_1+c+b_1 = F(\mathbf{t})+(F(\mathbf{s})-F(\mathbf{t}\wedge\mathbf{s})) \leq F(\mathbf{t})+(F(\mathbf{t}\vee\mathbf{s})-F(\mathbf{t})) = F(\mathbf{t}\vee\mathbf{s}).$$

We prove $F(\mathbf{s}) - F(\mathbf{t}\wedge\mathbf{s}) \leq F(\mathbf{t}\vee\mathbf{s}) - F(\mathbf{t})$. Denote $I := \{i|s_i < t_i\}$ and without loss of generality assume $I = \{1,\dots,k\}$, for some $k \leq n$. Consider $(k+1)\times(n-k+1)$ table $(\mathbf{r}_{i,j})_{i,j}$, $i = 0,\dots,k$, $j = 0,\dots n-k$ of n-tuples of reals, where $\mathbf{r}_{0,0} := \mathbf{t}\wedge\mathbf{s}$ and skipping in rows from the $(i-1)$-th to the i-th corresponds to rewriting (increasing) the i-th coordinate from s_i to t_i, and similarly, skipping in columns from the $(j-1)$-th to the j-th corresponds to rewriting (increasing) the j-th coordinate from t_j to s_j. Now, we have $\mathbf{r}_{k,0} = \mathbf{t}$, $\mathbf{r}_{0,n-k} = \mathbf{s}$ and $\mathbf{r}_{k,n-k} = \mathbf{t}\vee\mathbf{s}$. Moreover, the volume conditions on F guarantee

$$F(\mathbf{r}_{i,j-1}) - F(\mathbf{r}_{i-1,j-1}) \leq F(\mathbf{r}_{i,j}) - F(\mathbf{r}_{i-1,j}),$$

for each $i = 1,\dots k$ and $j = 1,\dots,n-k$. Finally, summing all these inequalities we obtain desired

$$F(\mathbf{s}) - F(\mathbf{t}\wedge\mathbf{s}) \leq F(\mathbf{t}\vee\mathbf{s}) - F(\mathbf{t}).$$

□

2.3 Join observables on MV-algebras

Finally, the Theorem 2.2.5 has an application in the construction of a joint observable on an MV-algebra. We will utilize the following lemma ([DP00], Prop. 7.1.4.):

Lemma 2.3.1. *Let $\{x_i\colon i \in I\}$ be a system of elements of an MV-algebra M.*

(1) *Let $\bigvee_{i\in I} x_i$ exist in M, and let x be any element of M. Then $\bigvee_{i\in I}(x\wedge x_i)$ exists in M and*

$$\bigvee_{i\in I}(x\wedge x_i) = x\wedge\bigvee_{i\in I} x_i. \tag{2.28}$$

(2) *If $\bigwedge_{i\in I} x_i$ exists in M, then for each $x \in M$, the element $\bigwedge_{i\in I}(x\vee x_i)$ exists in M and*

$$\bigwedge_{i\in I}(x\vee x_i) = x\vee\bigwedge_{i\in I} x_i. \tag{2.29}$$

Theorem 2.3.2. *Let $x_1, \ldots, x_n$ by one-dimensional observables on a σ-complete MV-algebra M. Then there is a unique n-observable x, such that*

$$x((-\infty, t_i) \times \cdots \times (-\infty, t_n)) = \bigwedge_{i=1}^{n} x_i((-\infty, t_i)), \tag{2.30}$$

for all $t_1, \ldots, t_n \in \mathbb{R}$.

Proof. Let $F_i(s) = x_i((-\infty, s))$, $s \in \mathbb{R}$, be a one-dimensional spectral resolution corresponding to x_i. One can prove, using Lemma 2.3.1, the mapping $F : \mathbb{R}^n \to M$ defined by $F(s_1, \ldots, s_n) = \bigwedge_{i=1}^{n} F_i(s_i)$, $s_1, \ldots, s_n \in \mathbb{R}$, satisfies conditions (1.5–1.7). We show here in detail the left continuity condition: Given $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$ we have

$$\begin{aligned}
&\bigvee_{\mathbf{s} \ll \mathbf{t}} F_1(s_1) \wedge \cdots \wedge F_n(s_n) = \bigvee_{s_1 < t_1} \cdots \bigvee_{s_n < t_n} F_1(s_1) \wedge \cdots \wedge F_n(s_n) = \\
&= \bigvee_{s_1 < t_1} \cdots \bigvee_{s_{n-1} < t_{n-1}} F_1(s_1) \wedge \cdots \wedge F_{n-1}(s_{n-1}) \wedge \bigvee_{s_n < t_n} F_n(s_n) = \\
&= \bigvee_{s_1 < t_1} \cdots \bigvee_{s_{n-1} < t_{n-1}} F_1(s_1) \wedge \cdots \wedge F_{n-1}(s_{n-1}) \wedge F_n(t_n) = \\
&= F_n(t_n) \wedge \bigvee_{s_1 < t_1} \cdots \bigvee_{s_{n-1} < t_{n-1}} F_1(s_1) \wedge \cdots \wedge F_{n-1}(s_{n-1}) = \\
&= F_n(t_n) \wedge \cdots \wedge F_1(t_1).
\end{aligned}$$

Hence it remains to verify the volume conditions on F. By the well-known Chang Subdirect Representation Theorem (see e.g. [Mun07]), each MV-algebra could be represented as a subdirect product of MV-chains (linearly ordered MV-algebras). Therefor we can without loss of generality assume M is linearly ordered and obtain the desired volume conditions as a consequence of the following Claim: *Claim: Let (L, u) be a linearly ordered unital l-group and let x_0^i, x_1^i, for $i = 1, \ldots, n$, be elements of L, such that $0 \le x_0^i \le x_1^i \le u$ for each i. Then*

$$0 \le \sum_{\phi \in \{0,1\}^{\{1,\ldots,n\}}} \operatorname{sgn}(\phi) \cdot \bigwedge_i x^i_{\phi(i)} \le u, \tag{2.31}$$

where $\operatorname{sgn}(\phi)$ *equals* $+1$ *iff* $|\phi^{-1}(0)|$ *is even and equals* -1 *otherwise.*

Proof. We use an induction on n. In the case $n = 1$ the formula (2.31) has a form $0 \leq x_1^1 - x_0^1 \leq u$, which clearly holds. Now suppose the case $n > 1$. There is some k for which x_0^k is the least element among all x_j^i's. The expression in the center of (2.31) equals

$$\sum_{\phi \in \{0,1\}^{\{1,\ldots,n\}\setminus\{k\}}} -\mathrm{sgn}(\phi) \cdot x_0^k \wedge \bigwedge_{i \neq k} x_{\phi(i)}^i + \sum_{\phi \in \{0,1\}^{\{1,\ldots,n\}\setminus\{k\}}} \mathrm{sgn}(\phi) \cdot x_1^k \wedge \bigwedge_{i \neq k} x_{\phi(i)}^i.$$

The first summand vanishes as it equals $-\sum_\phi \mathrm{sgn}(\phi) \cdot x_0^k$, where ϕ goes through the functions in $\{0,1\}^{\{1,\ldots,n\}\setminus\{k\}}$; we see that exactly a half of the functions have negative sign. The second summand satisfies (2.31) by the induction hypothesis, where we redefine x_i^j as $x_i^j \wedge x_0^k$, for each $i \neq k$, $j = 0, 1$. $\square$

Finally we can apply Corollary 2.2.5 to yield a unique n-dimensional observable x on M satisfying (2.30). $\square$

The n-dimensional observable x from the latter theorem is said to be an *n-dimensional meet joint observable* of $x_1, \ldots, x_n$. Observe, that $x \circ \pi_i^{-1}$ is an observable (where $\pi_i : \mathbb{R}^n \to \mathbb{R}$ is the i-th projection), which coincide with x_i on the intervals $(-\infty, t)$, $t \in \mathbb{R}$. Hence we can apply Sierpiński Theorem, to show

$$x(\pi_i^{-1}(A)) = x_i(A), \quad A \in \mathcal{B}(\mathbb{R}),\ i = 1, \ldots, n. \tag{2.32}$$

In addition, using (2.32) for each $i = 1, \ldots, n$, we can prove

$$x(A_1 \times \cdots \times A_n) \leq \bigwedge_{i=1}^n x_i(A_i), \quad A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R}), \tag{2.33}$$

and in general, it can happen that in (2.33) we have strict inequality.

Now, we define a second type of joint n-dimensional observables on MV-algebras with product.

Definition 2.3.3. *We say, that an algebra $M = (M, \oplus, \cdot, ', 0, 1)$ is a product MV-algebra, if $(M, \oplus, ', 0, 1)$ is an MV-algebra and $\cdot$ an additional binary operation satisfying for all $a, b, c \in M$*

(i) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,

(ii) $a \cdot b = b \cdot a$,

(iii) $(a \oplus b) \cdot c = (a \cdot b) \oplus (b \cdot c)$,

(iv) $a \cdot 1 = a$.

The MV-algebra of the real interval $M = \Gamma(\mathbb{R}, 1)$ with product of reals is a product MV-algebra. Some basic properties of product MV-algebras are

(a) $a \cdot 0 = 0 = 0 \cdot a$,

(b) if $a \leq b$, then for any $c \in M$, $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$.

(c) If M is σ-complete, then $\{a_i\}_i \nearrow a$ implies $\{b \cdot a_i\}_i \nearrow b \cdot a$.

We prove the property (c): First note that $0 \leq b \cdot a - b \cdot a_i = b \cdot (a - a_i) \leq a - a_i \searrow 0$ leads to $0 = \bigwedge_i (b \cdot a - b \cdot a_i)$. And consequently, $0 = \bigwedge_i b \cdot (a - a_i) = b \cdot a - \bigvee_i b \cdot a_i$.

Theorem 2.3.4. *Let $x_1, \ldots, x_n$ be one-dimensional observables, $n \geq 1$, on a σ-complete product MV-algebra M and let $F_1, \ldots, F_n$ be the corresponding one-dimensional spectral resolutions. If we set*

$$F(s_1, \ldots, s_n) = \prod_{i=1}^{n} F_i(s_i), \quad s_1, \ldots, s_n \in \mathbb{R},$$

then F is as n-spectral resolution on M and there is a unique n-dimensional observable x, such that

$$x(A_1 \times \cdots \times A_n) = \prod_{i=1}^{n} x_i(A_i), \quad A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R}). \tag{2.34}$$

Proof. The mapping F satisfies (1.4)–(1.7), that is rather easy to prove. To show the volume conditions, let a semi-closed rectangle $\langle a_1, b_1) \times \cdots \times \langle a_n, b_n)$ be given. Using the distributivity of the product we can transform a volume condition (inequality 1.8), to the form

$$\prod_{i=1}^{n} (F_i(b_i) - F_i(a_i)) \geq 0$$

which is obviously true. Applying Theorem 2.2.5, we see that there is a unique n-dimensional observable of $x_1, \ldots, x_n$ determined by the n-dimensional spectral resolution F.

Using mathematical induction and applying the Sierpiński Theorem, it is possible to establish (2.34). □

The n-dimensional observable x from Theorem 2.3.4 is said to be an *n-dimensional product joint observable* of $x_1, \ldots, x_n$. Clearly, we have

$$\prod_{i=1}^{n} F_i(t_i) \leq \bigwedge_{i=1}^{n} F_i(t_i), \quad t_1, \ldots, t_n \in \mathbb{R},$$

however, in general situation n-dimensional meet joint observable is different from the n-dimensional product joint observable of one-dimensional observables $x_1, \ldots, x_n$.

Chapter 3

Spectral resolutions on lexicographic effect algebras

Establishing a one-to-one correspondence for a number of effect algebras a natural question arises: is the class of effect algebras where the correspondence of our interest holds, closed under some (algebraic-categorical) operations, i.e. the product of effect algebras and so on? Denote (SRE$_n$) the property of an effect algebra that each n-spectral resolution uniquely extends to an n-observable. This chapter concerns the effect of the lexicographic product on the (SRE$_n$).

We assume unital po-group (H, u) and directed monotone σ-complete po-group G, for which (SRE$_n$) holds (e.g., interpolation group). H and G will have this meaning throughout the whole chapter. For such G, H consider the lexicographic effect algebra

$$E := \Gamma(H \overrightarrow{\times} G, (u, 0)).$$

Now E is not in general monotone σ-complete, only its radical is so. This leads to some pathological cases of spectral resolutions which do not extend to observables. However, the set of spectral resolutions which do extend to observables is easy to describe by the so-called finiteness property.

The problem in concern is treated in several papers of Dvurečenskij and various collaborators. Firstly, the simplest case of perfect MV-algebras

$$\Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0)),$$

where G is a σ-complete l-group, is in [DDL19]. Then generalization to k-perfect case and effect algebras with (RDP) is done in [Dvu19] and [DL20b].

Finally, the problem in a general setting of lexicographic effect algebras is treated in [DL19],[DL20d] and [DL20a], where the dimension is assumed to be one, two, and any finite (respectively). In this thesis, we present the one-dimensional case in the first section and then we will, in the next section, extend the result to all finite dimensions.

As in the chapter every effect algebra is of the form $E = \Gamma(H \overrightarrow{\times} G, (u, 0))$, where (H, u) is a unital Abelian po-group and G is a directed monotone σ-complete Abelian po-group (sometimes with interpolation). We will usually in the statements of the theorems take those rather lengthy assumptions for granted.

3.1 Observables on lexicographic effect algebras

For each $h \in [0, u]_H := \{h \in H : 0 \le h \le u\}$, we denote by E_h the set of elements of E, whose first coordinate is h. Clearly, $E = \bigcup\{E_h : h \in [0, u]_H\}$ is a disjoint union of all E_h's with $h \in [0, u]_H$, and $E_h \neq \emptyset$. We can also write $E = (E_h : h \in [0, u]_H)$. Clearly if $h_1, h_2 \in [0, u]_H$, $h_1 < h_2$, then $E_{h_1} \le E_{h_2}$. As posets, the set E_0 is isomorphic to G^+, E_u is isomorphic to G^- and all others E_h's are isomorphic to G. Moreover we call E_0 the *radical* of E.

Definition 3.1.1. *A g-effect tribe (g stands for group) is a system $\mathcal{T}$ of bounded real-valued functions defined on a non-empty set Ω such that (i) $1_\Omega \in \mathcal{T}$, (ii) if $f, g \in \mathcal{T}$, then $f \pm g \in \mathcal{T}$, and (iii) if $\{f_n\}_n$ is a monotone sequence of elements of $\mathcal{T}$, $f_n \le f_{n+1}$, $n \ge 1$, such that there is $f_0 \in \mathcal{T}$ with $f_n(\omega) \le f_0(\omega)$ for each $\omega \in \Omega$, then $f = \lim_n f_n \in \mathcal{T}$.*

It is evident that each g-effect tribe $(\mathcal{T}, 1_\Omega)$ is an Archimedean, monotone σ-complete unital po-group where all operations are defined pointwise.

Theorem 3.1.2. [Loomis–Sikorski Theorem for Unital po-groups] *Let (G, v) be a monotone σ-complete unital po-group with interpolation. Then there exist a g-effect tribe $\mathcal{T}$ with interpolation of bounded functions on some set $\Omega \neq \emptyset$ and a po-group σ-homomorphism π from $\mathcal{T}$ onto G and 1_Ω to v.*

Moreover, whenever $\pi(f) \le a \le \pi(g)$, where $f \le g$ are elements of $\mathcal{T}$ and $a \in G$, there is $h \in \mathcal{T}$ such that $\pi(h) = a$ and $f \le h \le g$.

Proof. The existence of $\mathcal{T}$ and of the σ-epimorphic mapping π follows directly from [BCD06, Thm 5.3]. To be more precise, given $x \in G$, let $\hat{x}$ be a mapping from $\mathcal{S}(G,v) \to \mathbb{R}$ defined by $\hat{x}(s) = s(x)$, $s \in \mathcal{S}(G,v)$. Then $\hat{x}$ is an affine and continuous mapping on the convex set $\mathcal{S}(G,v)$, and the mapping $x \mapsto \hat{x}$ is injective. Let $\mathcal{T}$ be the system of all bounded functions f on $\Omega = S(G,v)$ for which there is an element $x \in G$ such that $\{s \in \partial\mathcal{S}(G,v) \colon f(s) \neq \hat{x}(s)\}$ is a meager subset of $\partial S(G,v)$. For $f \in \mathcal{T}$ and $x \in G$ we write $f \sim x$ if $\{s \in \partial\mathcal{S}(G,u) \colon f(s) \neq \hat{x}(s)\}$ is a meager set. Then $\mathcal{T}$ is a g-effect tribe in question, and the mapping $\pi(f) = x$ iff $f \sim x$ is the desired σ-epimorphism.

The second part follows from the proof of [BCD06, Thm 4.1]. More precisely, if $h_0 \in \mathcal{T}$ is a such function that $\phi(h_0) = a$, then using technique described in the above paragraph, it is possible to show that the function $h = \max\{f, \min\{h_0, g\}\}$ belongs to $\mathcal{T}$ and $\pi(h) = a$ with $f \leq h \leq g$. □

The following lemma provides a simple criterion that enlightens which suprema and infima exist in E.

Lemma 3.1.3. *Let $(a_n)_n$ be a monotone sequence of elements from an effect algebra $E = \Gamma(H \overrightarrow{\times} G, (u,0))$, where (H,u) is a unital po-group and G is a σ-complete po-group. Then $\bigvee_n a_n$ ($\bigwedge_n a_n$), resp.) exists and belongs to E_h for some $h \in [0,u]_H$ if and only if there is some upper (lower, resp.) bound $a \in E_h$ of $(a_n)_n$ and there is some $a_n \in E_h$.*

Proof. Suppose $a = \bigvee_n a_n$ exists and belongs to E_h. Then each a_n belongs to some E_{h_n} with $h_n \leq h$, $h_n \in [0,u]_H$. If there was no a_n in E_h, then any $b \in E_h$ would dominate all a_n's, which contradicts that a is the least upper bound for $\{a_n\}_n$. So some of a_n's belongs to E_h and the supremum is the needed upper bound.

On the other hand, if $a \in E_h$ dominates $\{a_n\}_n$ and $\Lambda_h := \{n \in \mathbb{N} \colon a_n \in E_h\}$ is non-empty, then $\bigvee\{a_n \colon n \in \Lambda_h\}$ exists (since G is monotone σ-complete) and dominates all a_n's, and so it is equal to the supremum.

The dual case is analogous □

Recall that a set $\{a_n\}_{n\in\mathbb{N}}$ of elements of E is summable if all the partial finite sums of $\{a_n\}_{n\in\mathbb{N}}$ exist in E and

$$\bigvee_{I \subset \mathbb{N}, |I| < \infty} \sum_{n \in I} a_n$$

exists in E as well.

Lemma 3.1.4. *Let $E = \Gamma(H \overrightarrow{\times} G, (u, 0))$, where (H, u) is a unital po-group and G is a monotone σ-complete po-group and let $\{a_n\}_n$ be a summable finite or infinite sequence of elements of E such that $\sum_n a_n = a$. Then*

(i) all but finitely many a_n's belong to E_0,

(ii) every subsequence $\{a_{n_i}\}_i$ of $\{a_n\}_n$ is summable with sum in E,

(iii) let $\{a_n\}_n$ be infinite and let $\mathbb{N} = \bigcup_{i=1}^{\infty} N_i$, where $N_i \cap N_j = \emptyset$ for $i \neq j$, $N_i \neq \emptyset$ for each i. Then $\alpha_i = \sum\{a_n : n \in N_i\}$ exists in E for each i, and $\{\alpha_i\}_i$ is summable with $\sum_i \alpha_i = a$.

Proof. If the sequence in question is finite all points (i–ii) are trivial, so assume infinity.

(i) Define partial sums $b_n = a_1 + \cdots + a_n$, $n \in \mathbb{N}$. As $b_n \nearrow a$, all the $n \in \mathbb{N}$ with $a_n \notin E_0$ are below some n_0, by Lemma 3.1.3. Consequently, the sets $A_0 := \{n | a_n \in E_0\}$ and $A_1 := \mathbb{N} \setminus A_0$ are such that $A_0 \cup A_1 = \mathbb{N}$ and $|A_0| < \infty$. Moreover, $\sum_{n \in A_0} a_n$ exists (in E_0) as the partial sums are bounded by $a - \sum_{n \in A_1} a_n$.

(ii) For the given subsequence define $A'_0 := A_0 \cap \{n_i | i \in \mathbb{N}\}$ and denote the partial sums $c_i := a_{n_1} + \cdots + a_{n_i}$, for each $i \in \mathbb{N}$. With n_0 having the same meaning as in part (i), there is E_h, so that $c_i \in E_h$, whenever $n_i \geq n_0$. And the c_i's are bounded in E_h by $c_{n_0} + \sum_{n > n_0} a_n$, hence the sum $b = \sum_i a_{n_i}$ exists again by Lemma 3.1.3.

(iii) By point (ii) all the partial sums α_i's exist. Moreover, as is clear from the part (i), all α_i's but finite number of, belong to E_0. And $\{\alpha_i | i \in \mathbb{N}\}$ is summable as each finite partial sum is bounded by a and clearly for n great enough the sum $\alpha_1 + \cdots + \alpha_n$ belongs to the same E_h as a. Hence the sum $\alpha = \sum_n \alpha_n$ exists and equals a, as α dominates all the partial sums of a_i's. □

Theorem 3.1.5. *Let G be a directed monotone σ-complete po-group and (H, u) be a unital po-group. Let x be an observable on $E = \Gamma(H \overrightarrow{\times} G, (u, 0))$. Define for each $t \in \mathbb{R}$*

$$x_t := x((-\infty, t)). \tag{3.1}$$

We have for each $s, t \in \mathbb{R}$

$$x_t \leq x_s \text{ if } t \leq s, \tag{3.2}$$

$$\bigwedge_r x_r = 0, \quad \bigvee_r x_r = 1, \tag{3.3}$$

$$\bigvee_{r<s} x_r = x_s. \tag{3.4}$$

There is a finite sequence $0 = h_0 < h_1 < \cdots < h_n = u$ *of elements of* $[0,u]_H$ *and real numbers* $s_0 = t_1 < \cdots < t_n = t_u$ *such that*

$$x_t \in \begin{cases} E_{h_0} & \text{if } t \le t_1, \\ E_{h_i} & \text{if } t \in (t_i, t_{i+1}],\ i = 1, \ldots, n-1, \\ E_{h_n} & \text{if } t_n < t. \end{cases} \tag{3.5}$$

In addition, for each $i = 1, \ldots, n$, *the element*

$$a_i := \bigwedge_{t_i<t} x_t \tag{3.6}$$

exists in E *and it belongs to* E_{h_i}.

Moreover, for each $s \in \mathbb{R}$, *we have* $x(\{s\}) = \bigwedge_{t>s}(x_t - x_s)$.

Proof. Equation (3.2) follows from monotonicity of x. Since $\{(-\infty, n)\}_n \nearrow \mathbb{R}$ and $\{(-\infty, -n)\}_n \searrow \emptyset$ applying properties of observables (3.2), we get (3.3). In a similar way, we can deduce left continuity (3.4).

For each $h \in [0,u]_H$ such that E_h contains at least one x_t (clearly E_0 and E_u have this property), define $t_h = \inf\{t \colon x_t \in E_h\}$ and $s_h = \sup\{t \colon x_t \in E_h\}$ (obviously $t_0 = -\infty$ and $s_u = \infty$). Take any real $t \in \mathbb{R}$, then $x_t \in E_h$ for some $h \in [0,u]_H$. Define $A_i := (-\infty, t - \frac{1}{i})$, $i \ge 1$, and $A = (-\infty, t)$. We see that $\{A_i\}_i \nearrow A$, hence $x_t = x(A) = \bigvee_i x(A_i)$. According to Lemma 3.1.3, there is some $x(A_i) \in E_h$. Considering the special cases, when t equals s_h or t_h, we get $x_{s_h} \in E_h$, but $x_{t_h} \notin E_h$ (with the exception of the cases s_u,t_0, where x_t is not defined).

Hence for $h \in (0,u)_H := \{h \in H \colon 0 < h < u\}$, if $I_h := \{t \in \mathbb{R} \colon x_t \in E_h\}$ is non-void, then $I_h = (t_h, s_h]$, and $(s_0, t_u]$ is covered by these non-empty half open intervals. Now we want to prove that there are only finitely many such I_h's.

If $s_0 = t_u$, the claim is settled. Assume thus $s_0 < t_u$.

Let for $u \neq h \in [0,u]_H$, $I_h \neq \emptyset$. Define $B_i = [s_h, s_h + \frac{1}{i})$, $i > 0$, and $B = \{s_h\}$. As $\{B_i\}_i \searrow B$, we have $x(B) = \bigwedge_i x(B_i)$. So Lemma 3.1.3 states there is some i_0 such that for each $i \ge i_0$, $x(\{s_h\})$ and $x([s_h, s_h + \frac{1}{i})) = x_{s_h + \frac{1}{i}} - x_{s_h}$ both are elements of some E_{h_1}. Consequently, there is $h_2 \in [0,u]_H$ such that each x_t is an elements of E_{h_2}, whenever $s_h < t \le s_h + \frac{1}{i_0}$. We have just proved: Unless the case $u = h$, each s_h equals t_{h_1} for some $h_1 \in [0,u]_H$. Hence, for each convenient $h \in [0,u]_H$, we can define an open interval $J_h = (t_h, \hat{s}_h)$, where $\hat{s}_h$ is an appropriate real number between t_h and s_h, such that $I_h \subseteq J_h \subseteq I_h \cup I_{h_1}$. We see that each J_h covers the closed interval $[\hat{s}_0, t_h]$ (already I_h's do so). From compactness, there is a finite collection of J_{h_i}'s covering the whole interval and when we add J_0 and I_u we have a cover of the whole $\mathbb{R}$. Since each $J_{h_i} \subseteq I_{h_i} \cup I_h$ for some $h \in [0,u]_H$, there is a finite collection of intervals I_h's which covers the whole $\mathbb{R}$, and in particular it covers $[s_0, t_u]$.

Inasmuch as each x_t belongs to a unique $E_{h_t} \in [0,u]_H$, and the system $\{x_t \colon t \in \mathbb{R}\}$ is linearly ordered, then so is linearly ordered the system $\{h_t \colon t \in \mathbb{R}\}$.

Since $\{s\} = \bigcap_{t>s} \big((-\infty, t) \setminus (-\infty, s)\big)$, we have that $x(\{s\}) = \bigwedge_{t>s}(x_t - x_s)$. Hence, $\bigwedge_{t>t_i}(x_t - x_{t_i})$ exists in E, and $x(\{t_i\}) = \bigwedge_{t>t_i}(x_t - x_{t_i}) = \big(\bigwedge_{t>t_i} x_t\big) - x_{t_i}$. Hence, $x(\{t_i\}) + x_{t_i} = \bigwedge_{t>t_i} x_t = a_i$ is defined in E. Applying Lemma 3.1.3, we see that $x(t_i)$ exists in E and it belongs to E_{h_i}. $\square$

The Theorem 3.1.5 indicates that a spectral resolution $(x_t)_t$ on E satisfying conditions of Definition 1.4.3 may not corresponds to any observable. Let us see two basic examples of this phenomenon.

Example 3.1.6. *Let* $E := \Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{R}, (1,0))$ *and*

$$x_t = \begin{cases} (0,0) & \text{if } \ t \le 0, \\ (1, \frac{1}{t}) & \text{if } \ 0 < t. \end{cases}$$

Then $\{x_t \colon t \in \mathbb{R}\}$ *is a spectral resolution for which the infimum* $\wedge_{0<t} x_t$ *does not exist. Hence the hypothetical observable* x *fails to well define* $x((-\infty, 0])$.

Example 3.1.7. *Let* $E = \Gamma(\mathbb{Q} \overrightarrow{\times} \mathbb{R}, (1,0))$ *and, for every* $t \in \mathbb{R}$*, we define*

$$x_t = \begin{cases} (0,0) & \text{if } \ t \le 0, \\ (1/2^n, 0) & \text{if } \ 1/2^n < t \le 1/2^{n-1}, \ n \ge 1, \\ (1,0) & \text{if } \ 1 < t. \end{cases}$$

Then $\{x_t \colon t \in \mathbb{R}\}$ *is a spectral resolution but the finiteness property fails.*

In [DL19], there is introduced a notation of characteristic points of a spectral resolutions on lexicographic effect algebra $E = \Gamma(H \overrightarrow{\times} G, (u,0))$. Let $\{x_t \colon t \in \mathbb{R}\}$ be a system of elements of E satisfying the conditions (3.2)–(3.4). Then for each $h \in H$ denote $I_h = \{t \in \mathbb{R} \colon x_t \in E_h\} \neq \emptyset$. It is rather easy to prove that each nonempty I_h is of form $(t_h, s_h]$, $s_h, t_h \in \mathbb{R}$. And the reals t_h's are called characteristic points of $(x_t)_t$. Now $(x_t)_t$ is called a *spectral resolution (on lexicographic effect algebra)* if moreover all the infima $a_h := \bigwedge\{x_t \colon t \in I_h\}$ exist in E.

Hence the system $(x_t)_t$ from Example 3.1.6 fails to be a spectral resolution in the sense of [DL19]. While $(x_t)_t$ from Example 3.1.7 is a spectral resolution on lexicographic effect algebras but fails in the finiteness property. One can prove that both of the defects described by the two examples could be prevented by an assumption, that $\wedge_{s<t} x_t$ exists for all $s \in \mathbb{R}$.

Now we present the main result of the section which is a converse to Theorem 3.1.5. We need the following notion:

Definition 3.1.8. *Let G be a directed monotone σ-complete po-group. A mapping $x : \mathcal{B}(\mathbb{R}) \to G^+$ is said to be a G-observable if (i) $x(A \cup B) = x(A) + x(B)$ whenever A and B are disjoint Borel sets, and (ii) if $\{A_i\}_i$ is a sequence of mutually disjoint Borel sets, $A = \bigcup_i A_i$, then $\{x(A_i)\}_i$ is bounded above and $x(A) = \bigvee_i (x(A_1) + \cdots + x(A_i))$.*

Easy consequences of the previous definition are: $x(\emptyset) = 0$ and $x(B \setminus A) = x(B) - x(A)$ whenever $A \subseteq B$.

Theorem 3.1.9. *Let $E = \Gamma(H \overrightarrow{\times} G, (u, 0))$, where (H, u) is a unital po-group and G is a monotone σ-complete po-group with interpolation and with strong unit. Let $\{x_t \in E \colon t \in \mathbb{R}\}$ be a system of elements of E and $h_0 < h_1 < \cdots < h_n$ be elements of $[0, u]_H$ and real numbers $s_0 = t_1 < \cdots < t_n$, such that conditions (3.2)–(3.6) are satisfied. Then there is a unique observable x on E such that $x_t = x((-\infty, t))$ for each $t \in \mathbb{R}$.*

Proof. Let v be a strong unit of G. By the Loomis–Sikorski Theorem for monotone σ-complete unital po-groups, Theorem 3.1.2, there are a g-effect tribe of bounded functions $\mathcal{T}$ on $\Omega \neq \emptyset$ and a σ-homomorphism π from $\mathcal{T}$ onto G such that $\pi(1_\Omega) = v$. Let $\pi_H : \Gamma(H \overrightarrow{\times} \mathcal{T}, (u, 0)) \to E$ be a mapping defined by $\pi_H(h, f) := (h, \pi(f))$, $h \in [0, u]_H$, $f \in \mathcal{T}$. Then π_H is a surjective σ-homomorphism from $\Gamma(H \overrightarrow{\times} \mathcal{T}, (u, 0))$ onto $E = \Gamma(H \overrightarrow{\times} G, (u, 0))$.

We see that $h_0 = 0$ and $h_n = u$, otherwise (3.3) would not hold (see Lemma 3.1.3). In order to simplify the proof, let us define $t_0 := -\infty$, $t_{n+1} := \infty$ and $x_{-\infty} := (0, 0)$, $x_\infty := (u, 0)$. For each integer i, $0 \le i \le n$, and for each $t \in \mathbb{R}$, we define

$$\hat{x}^i_t = \begin{cases} 0 & \text{if} \quad t \le t_i, \\ x_t - \bigwedge_{t_i < s} x_s & \text{if} \quad t_i < t \le t_{i+1}, \\ x_{t_{i+1}} - \bigwedge_{t_i < s} x_s & \text{if} \quad t_{i+1} < t. \end{cases}$$

Note that the system $\{\hat{x}_t^i \colon t \in \mathbb{R}\}$ is still monotone, left continuous and all $\hat{x}_t^i$'s are in E_0. Let $r_1, r_2, \ldots$ be any enumeration of rational numbers in the interval (t_i, t_{i+1}). Set $b_0^i \in \mathcal{T}^+$ any element of $\mathcal{T}^+$ such that $(0, \pi(b_0^i)) = x_{t_{i+1}}^i$ (e.g., 1_Ω in case $i = n$). One can by an induction find further elements $b_n^i \in \mathcal{T}$, $n \geq 1$, such that $(0, \pi(b_n^i)) = \hat{x}_{r_n}^i$ and $b_n^i \leq b_m^i \leq b_0^i$ whenever $r_n \leq r_m$ (see Theorem 3.1.2).

Since rational numbers are dense in $\mathbb{R}$, we can define for each real r an element $c_r^i = \bigvee_{r_n < r} b_n^i$ (which is well defined since all b_n^i with $r_n < r$ are dominated by b_0^i). Since π is a σ-homomorphism, $(0, \pi(c_r^i)) = \hat{x}_r^i$ holds. We may replace c_t^i by $c_t^i - \bigwedge_{t_i < t} c_t^i$ (note that the infimum exists, since all c_t^i's are from $\mathcal{T}^+$ and $\pi(\bigwedge_{t_i < t} c_t^i) = 0$) to assure $\bigwedge_{s_i < t} c_t^i = 0_\Omega$. Finally for $t \leq t_i$ define $c_t^i = 0_\Omega$ and for $t_{i+1} < t$ define $c_t^i := c_{t_{i+1}}^i$. See that c_t^i, $t \in \mathbb{R}$, is non-decreasing, left continuous and $\bigwedge_t c_t^i = 0_\Omega$.

Now, for each $\omega \in \Omega$ and $t \in \mathbb{R}$ we define $F_\omega^i(t) := c_t^i(\omega)$. For each ω, F_ω^i is a non-decreasing, left continuous function such that $\lim_{t \to -\infty} F_\omega^i(t) = 0$ and $\lim_{t \to \infty} F_\omega^i(t) = c_{t_{i+1}}^i(\omega)$. That is, F_ω^i is a distribution function of some finite σ-additive measure P_ω^i on $\mathcal{B}(\mathbb{R})$, see [Hal74, Thm 43.B].

Define a mapping $\xi_i : \mathcal{B}(\mathbb{R}) \to \mathcal{T}^+$, by prescription $\xi_i(B)(\omega) = F_\omega^i(B)$, $B \in \mathcal{B}(\mathbb{R})$, specially $\xi_i((-\infty, t)) = c_t^i \in \mathcal{T}^+$. In order to prove $\xi_i(B) \in \mathcal{T}^+$ for each Borel set B, we will use following argument: The system $\mathcal{K}$ of Borel sets with this property forms a Dynkin system, i.e. a system of subsets containing its universe and which is closed under the set theoretical complements and countable unions of disjoint subsets, [Kal02], and containing all intervals $(-\infty, t)$, $t \in \mathbb{R}$. These intervals form a π-system, i.e. intersection of any two sets from the π-system is from the π-system. Hence by the Sierpiński Theorem 1.3.5, $\mathcal{K}$ is a σ-algebra, which proves $\mathcal{K} = \mathcal{B}(\mathbb{R})$.

Define $y_i(B) := (0, \pi(\xi_i(B)))$, $B \in \mathcal{B}(\mathbb{R})$. Given $a \in E$, we define $0 \cdot a := a$ and $1 \cdot a := a$. Now we finally set

$$x(B) := \sum_{i=0}^{n} y_i(B) + \sum_{i=1}^{n} \chi_{\{t_i\}}(B) \cdot \Big(\bigwedge_{t_i < t} x_t - x_{t_i}\Big). \tag{3.7}$$

The sum is well defined for every $B \in \mathcal{B}(\mathbb{R})$: For each $i \leq n$

$$\begin{aligned} y_i(B) + \chi_{\{t_i\}}(B) \cdot (\bigwedge_{t_i<t} x_t - x_{t_i}) &\leq y_i(\mathbb{R}) + \bigwedge_{t_i<t} (x_t - x_{t_i}) \\ &= [x_{t_{i+1}} - \bigwedge_{t_i<t} x_t] + [(\bigwedge_{t_i<t} x_t) - x_{t_i}] \\ &= x_{t_{i+1}} - x_{t_i} =: \delta_i. \end{aligned}$$

As $\{\delta_i \colon i = 0, \ldots, n\}$ is obviously a summable sequence with sum $x_\infty - x_{-\infty} = (u, 0)$, formula (3.7) could be summed as well. An easy computation (similar to the just computed evaluation at $\mathbb{R}$) leads to $x((-\infty, t)) = x_t$, for each real t.

It remains to prove x is an observable on E (Definition 1.4.1). We have already proved $x(\mathbb{R}) = (u, 0)$. To the second part: Let $A_j \in \mathcal{B}(\mathbb{R})$, $j \geq 1$, be pairwise disjoint sets whose union is A. At first, we prove the condition for each y_i. Using both: ξ_i is a $\mathcal{T}$-observable and π is a σ-morphism, we get

$$\begin{aligned} y_i(\bigcup_j A_j) &= (0, \pi(\xi_i(\bigcup_j A_j))) = (0, \pi(\sum_j \xi_i(A_j))) \\ &= (0, \sum_j \pi(\xi_i(A_j))) = \sum_j y_i(A_j), \end{aligned}$$

which we read as $\sum_j y_i(A_j)$ is summable and its sum is $y_i(A)$. Similarly, $\sum_j \chi_{\{t_i\}}(A_j)$ is trivially summable as it has at most one non-vanishing summand if t_i belongs to some A_j, equivalently, if t_i belongs to A. Putting these facts together, we can derive σ-additivity of x:

$$\begin{aligned} \sum_j x(A_j) &= \sum_j \left[\sum_{i=0}^n y_i(A_j) + \sum_{i=1}^n \chi_{\{t_i\}}(A_j) \cdot (\bigwedge_{t_i<t} x_t - x_{t_i}) \right] \\ &= \sum_j \left[\sum_{i=0}^n y_i(A_j) \right] + \sum_j \sum_{i=1}^n \chi_{\{t_i\}}(A_j) \cdot (\bigwedge_{t_i<t} x_t - x_{t_i}) \\ &= \left[\sum_{i=0}^n \sum_j y_i(A_j) \right] + \sum_{i=1}^n \sum_j \chi_{\{t_i\}}(A_j) \cdot (\bigwedge_{t_i<t} x_t - x_{t_i}) \\ &= \left[\sum_{i=0}^n y_i(A) \right] + \sum_{i=1}^n \chi_{\{t_i\}}(A) \cdot (\bigwedge_{t_i<t} x_t - x_{t_i}) \\ &= x(A). \end{aligned}$$

Now we establish uniqueness of x. Assume, that y is any observable on E such that $y((-\infty, t)) = x_t$, $t \in \mathbb{R}$. Let $\mathcal{F}$ be the system of Borel sets $F \in \mathcal{B}(\mathbb{R})$ such that $x(F) = y(F)$. Hence, $\mathcal{F}$ is a Dynkin system containing all intervals of the form $(-\infty, t)$, $t \in \mathbb{R}$. Similarly as above, the Sierpiński Theorem implies $\mathcal{F}$ is a σ-algebra, which proves $\mathcal{F} = \mathcal{B}(\mathbb{R})$ and $x = y$. $\square$

3.2 Finite-dimensional case

In the previous section, we have mentioned the notation of characteristic points of a spectral resolution on $\Gamma(H \overrightarrow{\times} G, (u, 0))$. Given a spectral resolution F, characteristic points of F are these real numbers t's, where the value of F skips in the first coordinate. Moreover, the number of characteristic points is finite whenever F determines an observable, in which case the characteristic points provide a decomposition of the real line into a finite number of intervals. It turns out that in the finite-dimensional case, each n-spectral resolution F which corresponds to an observable satisfies finiteness property as well (i.e., $F^{-1}(E_h)$ is non-empty only for a finite number of h's). And so provides a finite decomposition $\mathbb{R}^n = \bigcup_{h \in H} F^{-1}(E_h)$. The decompositions of $\mathbb{R}^n$ which arise in this way are rather easy to characterize.

In one dimensional case a characteristic point is an infimum of some nonempty

$$B_h := F^{-1}(E_h), \tag{3.8}$$

$h \neq 0$. However for F an n-spectral resolution, in the sense of Definition 1.4.3, $B_h := F^{-1}(E_h)$ may has more than one minimal element. Nevertheless, for each $\mathbf{s} \in B_h$ there is exactly one minimal element of B_h which is below $\mathbf{s}$.

In the section each nonempty set of form (3.8) is called a *block*.

Lemma 3.2.1. *Let F be an n-spectral resolution on $\Gamma(H \overrightarrow{\times} G, (u, 0))$, $h \in H$ and $\mathbf{s} \in F^{-1}(E_h)$. Denote*

$$\mathbf{t} := \inf\{\mathbf{r} \in B_h | \mathbf{r} \leq \mathbf{s}\}. \tag{3.9}$$

Then some $\mathbf{r} \leq \mathbf{s}$ belongs to B_h if and only if $\mathbf{t} \ll \mathbf{r} \leq \mathbf{s}$.

Proof. The if part: suppose $\mathbf{t} \ll \mathbf{r} \leq \mathbf{s}$. Then there are $\mathbf{r}_1, \ldots, \mathbf{r}_n$, elements of B_h, such that for each i, the i-th coordinate of $\mathbf{h}_i$ is smaller than the i-th coordinate of $\mathbf{r}$. Moreover, we can assume each $\mathbf{r}_i$ shares with $\mathbf{s}$ all coordinates unless the i-th one. We show that $\mathbf{r}_0 := \wedge_i \mathbf{r}_i$ belongs to B_h, which leads to $\mathbf{r} \in B_h$ as B_h is (as a subset of poset) convex. Now the $\mathbf{r}_i$'s define an n-cuboid $\mathcal{C}$, the one having $\mathbf{s}$ as the first vertex, $\mathbf{r}_i$'s as the vertices of second order and $\mathbf{r}_0$ as the bottom vertex (in the sense of Chapter 2). One can prove, using the volume conditions to faces of $\mathcal{C}$, that for each edge $(\mathbf{r}', \mathbf{r}'')$, $\mathbf{r}' < \mathbf{r}''$, of $\mathcal{C}$, we have $F(\mathbf{r}'') - F(\mathbf{r}') \leq F(\mathbf{r}_i) - F(\mathbf{s}) \in E_0$ for some i. Hence we can proof by an induction that all vertices of $\mathcal{C}$ belong to B_h, and so in particular $F(\mathbf{r}_0) \in E_h$.

Next we prove the only if part. If some $\mathbf{r} \leq \mathbf{s}$ does not satisfy $\mathbf{t} \leq \mathbf{r}$, then $\mathbf{r}$ obviously cannot belong to B_h. If $\mathbf{r}$ satisfies only $\mathbf{t} \leq \mathbf{r}$ (but not $\mathbf{t} \ll \mathbf{r}$), then there is a monotone sequence $\mathbf{r}_i \nearrow \mathbf{r}$, with $\mathbf{r}_i \ll \mathbf{r}$ for each $i \in \mathbb{N}$. The $\mathbf{r}_i$'s do not belong to E_h and so $\mathbf{r}$ neither, by the continuity of F. $\square$

We call *characteristic point* of an n-spectral resolution any n-tuple $\mathbf{t} \in \mathbb{R}^n$ given by (3.9) for some $h \in H$ and $\mathbf{s} \in F^{-1}(E_h)$.

Lemma 3.2.2. *Let $\mathbf{t_1}, \mathbf{t_2}, \mathbf{s} \in \mathbb{R}^n$, $\mathbf{t_1} < \mathbf{t_2}$, $0 \leq \mathbf{s}$ and F be an n-spectral resolution. If the set I of indices of coordinates where $\mathbf{s}$ is nonzero and the set J of indices of coordinates where $\mathbf{t_1}$ differ from $\mathbf{t_2}$ are disjoint, then*

$$F(\mathbf{t_2} + \mathbf{s}) - F(\mathbf{t_1} + \mathbf{s}) \geq F(\mathbf{t_2}) - F(\mathbf{t_1}). \tag{3.10}$$

Proof. Choose sequences $\mathbf{0} = \mathbf{s_0} < \mathbf{s_1} < \cdots < \mathbf{s_{k-1}} < \mathbf{s_k} = \mathbf{s}$ and $\mathbf{t_1} = \mathbf{r_0} < \mathbf{r_1} < \cdots < \mathbf{r_{l-1}} < \mathbf{r_l} = \mathbf{t_2}$ such that (in both sequences) any to consecutive elements differ only in one coordinate. By volume conditions we have for each $i < k, j < l$ inequality

$$F(\mathbf{r_{j+1}} + \mathbf{s_{i+1}}) - F(\mathbf{r_j} + \mathbf{s_{i+1}}) \geq F(\mathbf{r_{j+1}} + \mathbf{s_i}) - F(\mathbf{r_j} + \mathbf{s_i}).$$

Composing such inequalities for all i's we yield

$$F(\mathbf{r_{j+1}} + \mathbf{s}) - F(\mathbf{r_j} + \mathbf{s}) \geq F(\mathbf{r_{j+1}}) - F(\mathbf{r_j}), \tag{3.11}$$

for each $j < l$. An equivalent form of (3.11) is $F(\mathbf{r_{j+1}} + \mathbf{s}) - F(\mathbf{r_{j+1}}) \geq F(\mathbf{r_j} + \mathbf{s}) - F(\mathbf{r_j})$, we can again compose these inequalities to obtain the desired inequality (3.10). $\square$

Lemma 3.2.3. *Let $\mathbf{s_1}, \mathbf{s_2} \in \mathbb{R}^n$, $\mathbf{s_1} < \mathbf{s_2}$, and F an n-spectral resolution. Then $\mathbf{s_1}, \mathbf{s_2}$ belong to different blocks B_h's if and only if there is a characteristic point $\mathbf{t}$, such that $\mathbf{t} \ll \mathbf{s_2}$ but $\mathbf{t} \not\ll \mathbf{s_1}$.*

Proof. We begin with the "if" part. Suppose a characteristic point $\mathbf{t} \in B_h$, such that $\mathbf{t} \ll \mathbf{s_2}$ but $\mathbf{t} \not\ll \mathbf{s_1}$. We want to prove $F(\mathbf{s_2}) - F(\mathbf{s_1}) \notin E_0$. There has to be some $i \leq n$, such that the i-th coordinate of $\mathbf{s_1}$, denoted by y, is smaller or equal to the i-th coordinate of $\mathbf{t}$. Define by $\mathbf{s_1'}$ an element of $\mathbb{R}^n$ which arises from $\mathbf{s_2}$ by rewriting the i-th coordinate to be y. Hence we have $\mathbf{s_1} \leq \mathbf{s_1'} < \mathbf{s_2}$, $\mathbf{t} \not\ll \mathbf{s_1'}$ and by monotonicity we obtain

$$F(\mathbf{s_2}) - F(\mathbf{s_1}) \geq F(\mathbf{s_2}) - F(\mathbf{s_1'}).$$

By definition of characteristic points, $\mathbf{t}$ is given by formula (3.9) for some B_h, and there is $\mathbf{s} \in B_h$, such that $\mathbf{s} \ll \mathbf{s_2}$. As the next step define $\mathbf{s_2''}$ and $\mathbf{s_1''}$ by decreasing in $\mathbf{s_2}$ and $\mathbf{s_1'}$, respectively, all but the i-th coordinate to the corresponding coordinates of $\mathbf{s}$. By Lemma 3.2.2 and monotonicity ($\mathbf{s_2''} > \mathbf{s}$) we have

$$F(\mathbf{s_2}) - F(\mathbf{s_1'}) \geq F(\mathbf{s_2''}) - F(\mathbf{s_1''}) \geq F(\mathbf{s}) - F(\mathbf{s_1''}).$$

Finally $\mathbf{s_1''} < \mathbf{s}$ and $\mathbf{t} \not\ll \mathbf{s_1''}$, hence $\mathbf{s_1''} \notin B_h$ and so $F(\mathbf{s}) - F(\mathbf{s_1''}) \notin E_0$.

The "only if" part of the lemma is trivial, we take for $\mathbf{t}$ the characteristic point given by (3.9), where we set $\mathbf{s} = \mathbf{s_2}$ (and so B_h is the block containing $\mathbf{s_2}$). □

The following theorem gives us a picture of how the decomposition of $\mathbb{R}^n$ to the blocks B_h's for a given F, looks like.

Theorem 3.2.4. *Let F be n-spectral resolution. For each characteristic point $\mathbf{t}$ define $C_{\mathbf{t}} = \{\mathbf{s} \in \mathbb{R}^n | \mathbf{t} \leq \mathbf{s}$ and $\mathbf{t} \not\ll \mathbf{s}\}$. Each $C_{\mathbf{t}}$ cuts $\mathbb{R}^n$ into two disjoint components as*

$$\mathbf{R}^n = \{\mathbf{s} | \mathbf{t} \not\ll \mathbf{s}\} \cup \{\mathbf{s} | \mathbf{t} \ll \mathbf{s}\}. \tag{3.12}$$

The joint cutting of $\mathbb{R}^n$ along all the $C_{\mathbf{t}}$'s refines the decomposition

$$\mathbb{R}^n = \bigcup_{h \in [0,u]_H} B_h.$$

Proof. Suppose we have $\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{R}^n$ belonging to different blocks, say $\mathbf{s}_1 \in B_{h_1}$ and $\mathbf{s}_2 \in B_{h_2}$, $h_1 \neq h_2$. To prove the Theorem we need to find a characteristic point $\mathbf{t}$ which divides the two points in the sense of (3.12). If $s_1 \leq s_2$ or $s_2 \leq s_1$ then Lemma 3.2.3 assures the $\mathbf{t}$. Otherwise, we take $s := s_1 \wedge s_2$ (we thing of $\mathbb{R}^n$ as a lattice). Now $\mathbf{s}$ belongs to some B_h, where $h \leq h_1, h_2$ and for some $i = 1, 2$, $h < h_i$. Say $i = 2$. Hence Lemma 3.2.3 gives us a characteristic point $\mathbf{t}$, $\mathbf{t} \ll \mathbf{s}_2$ but $\mathbf{t} \not\ll \mathbf{s}$. The characteristic point

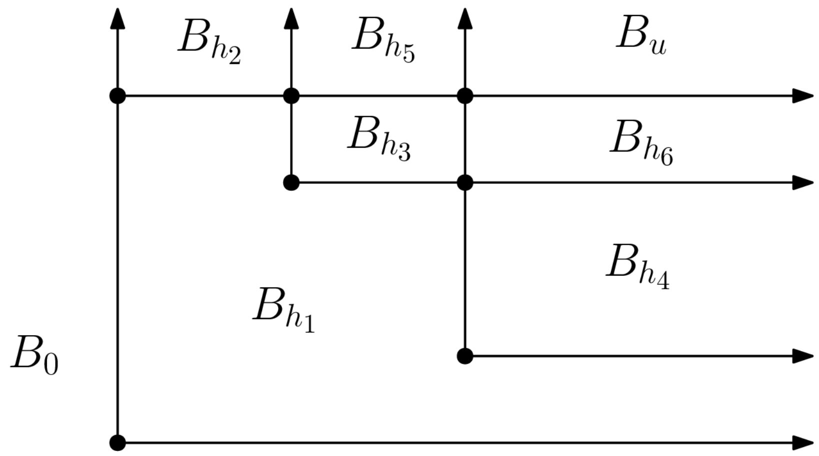

Figure 3.1: Decomposition to blocks

$\mathbf{t}$ is the desired one. Indeed, if $\mathbf{t} \ll \mathbf{s_1}$, we would have $\mathbf{s} = \mathbf{s_1} \wedge \mathbf{s_2} \gg \mathbf{t}$ as well, which is a contradiction. $\square$

In Figure 3.1 we have a schematic picture of decomposition to blocks in a two-dimensional case. In the pictured situation we have seven characteristic points and eight blocks (in the case h_i's are pairwise different). However, it may happen that $h := h_2 = h_3 = h_4$, in which case $B_h = F^{-1}(E_h)$ would consist of three disjoint components.

Theorem 3.2.5. *Each observable x on E gives arise to a spectral resolution F with only finitely many characteristic points. Moreover $\vee_{\mathbf{t} \ll \mathbf{s}} F(\mathbf{s})$ exists for each characteristic point $\mathbf{t}$.*

Proof. Take any $\mathbf{t}_u = (t_1, \ldots, t_n) \in B_u$. By Lemma 3.2.3 $\mathbf{t}_u$ is strictly over each characteristic point $\mathbf{t}$ (i.e., $\mathbf{t} \ll \mathbf{t}_u$). Next take any $i = 1, \ldots, n$ and denote by l_i the line parallel to i-th axis passing through $\mathbf{t}_u$. We can parametrize all the points on l_i by its i-th coordinate as $l_i = \{\mathbf{s}_y | y \in \mathbb{R}\}$. We claim, that two points $\mathbf{s}_y < \mathbf{s}_z$ lying on the line l_i belong to different B_h's if and only if there is some characteristic point with i-th coordinate equal to some $w \in \mathbb{R}$ such that $y \leq w < z$. This follows from Lemma 3.2.3 combined with the fact, that for some characteristic point $\mathbf{t}$ it could happen $\mathbf{t} \not\ll \mathbf{s}_y$ only in the case, when the i-th coordinate of $\mathbf{t}$ is greater or equal to the i-th coordinate of $\mathbf{s}_y$.

Now F restricted to l_i gives a (one-dimensional) t pseudo-spectral resolution F_i assigned to an observable $x_i : \mathcal{B}(l_i) \to \Gamma(H \overrightarrow{\times} G, v)$, where $v := \vee_{y \in \mathbb{R}} F_i(y)$ defined by

$$x_i : A \mapsto x((-\infty, t_1) \times \cdots \times (-\infty, t_{i-1}) \times A \times (-\infty, t_{i+1}) \times \cdots \times (-\infty, t_n)).$$

Hence, the image of F_i meets only finitely many E_h's by Theorem 3.1.5. This entails, only finitely many reals occur as the i-th coordinate of some characteristic point. Using such argument to all coordinates we observe, there are only finitely many characteristic points.

The second part of the statement is trivial. $\square$

Our next aim is to prove the opposite of Theorem 3.2.5. That is, the two mentioned conditions are sufficient for n-spectral resolution to be extendable to an n-observable.

Definition 3.2.6. *We say that a spectral resolution F satisfies the finiteness property, if F has only finitely many characteristic points and asserts infimum $\wedge_{\mathbf{t} \ll \mathbf{s}} F(\mathbf{s})$ for any characteristic point $\mathbf{t}$.*

Assume an easy case, when F is 2-spectral resolution h supreth a unique characteristic point $\mathbf{t} = (t_1, t_2)$. If we distribute the right hand side of equality

$$\mathbb{R}^2 = [(-\infty, t_1) \cup \{t_1\} \cup (t_1, \infty)] \times [(-\infty, t_2) \cup \{t_2\} \cup (t_2, \infty)], \qquad (3.13)$$

we yield a decomposition of $\mathbb{R}^2$ into nine disjoint sets, where each of them contains no characteristic point or only characteristic point. This decomposition corresponds to the decomposition of F, which arise by distributing the right hand side of (rather intuitive notation of substituting t^+ to a function is precisely defined by formula (3.15))

$$\begin{aligned} F(s_1, s_2) = \big[\Delta_1(-\infty, \min\{t_1, s_1\}) + \Delta_1(\min\{t_1, s_1\}, \min\{t_1^+, s_1\}) + \\ + \Delta_1(\min\{t_1^+, s_1\}, s_1)\big] \cdot [\Delta_2(-\infty, \min\{t_2, s_2\}) + \\ + \Delta_2(\min\{t_2, s_2\}, \min\{t_2^+, s_2\}) + \Delta_1(\min\{t_2^+, s_2\}, s_2)]F. \end{aligned}$$

The equality follows from following equation of operators

$$\begin{aligned} \Delta_1(-\infty, \min\{t_1, s_1\}) + \Delta_1(\min\{t_1, s_1\}, \min\{t_1^+, s_1\}) + \Delta_1(\min\{t_1^+, s_1\}, s_1) = \\ = \Delta_1(-\infty, s_1). \end{aligned}$$

It turns out, that the summands in the decomposition of F are in fact pseudo-spectral resolutions, which are of either kind: have no characteristic point or are two-valued. In both cases, we can extend them to pseudo n-observables, which supports are included in corresponding summands in the decomposition (3.13).

The just-described example directly generalizes to the general case of our interest. We only have to cut $\mathbb{R}^n$ in more dimensions and in more characteristic points. Our next aim is to prove that the summands in the decomposition of F are pseudo n-spectral resolutions. To achieve this, we have to prove several technical lemmas.

The following lemma states that the volume formula satisfies certain continuity properties:

Lemma 3.2.7. *Suppose we have 2^m, $m \in \mathbb{N}$, sequences $(a_\delta^i)_i$, $\delta \in \{0,1\}^m$, of elements of an effect algebra $E = \Gamma(H \overrightarrow{\times} G, (u,0))$, such that all the sequences are non-decreasing (non-increasing, resp.) and have suprema (infima, resp.) in E, for each δ. Moreover, denote by $\pi(\delta)$ the number of zero coordinates in $\delta = (\delta_1, \ldots, \delta_m)$.*

If $(\sum_\delta (-1)^{\pi(\delta)} a_\delta^i)_i$ is non-decreasing, then

$$\bigvee_i \sum_\delta (-1)^{\pi(\delta)} a_\delta^i = \sum_\delta (-1)^{\pi(\delta)} \bigvee_i a_\delta^i$$

$$\left(\bigvee_i \sum_\delta (-1)^{\pi(\delta)} a_\delta^i = \sum_\delta (-1)^{\pi(\delta)} \bigwedge_i a_\delta^i, \textit{ resp.} \right).$$

If $(\sum_\delta (-1)^{\pi(\delta)} a_\delta^i)_i$ is non-increasing, then

$$\bigwedge_i \sum_\delta (-1)^{\pi(\delta)} a_\delta^i = \sum_\delta (-1)^{\pi(\delta)} \bigvee_i a_\delta^i$$

$$\left(\bigwedge_i \sum_\delta (-1)^{\pi(\delta)} a_\delta^i = \sum_\delta (-1)^{\pi(\delta)} \bigwedge_i a_\delta^i), \textit{ resp.} \right).$$

Proof. For the sake of simplicity we demonstrate the proof in the situation where $m = 2$. In the general case only more summands are involved. Hence we assume four monotone sequences $(a_i)_i$, $(b_i)_i$, $(c_i)_i$ and $(d_i)_i$ which have sumprema (infima, resp.) a, b, c, and d. According to Lemma 3.1.3 we can without loss of generality assume the four sequences are constant in the first component (recall $E = \Gamma(H \overrightarrow{\times} G, (u,0))$), and hence so is the sequence $(a_i + d_i - b_i - c_i)_i$.

First assume the four sequences are non-decreasing (with sumprema a, b, c, d) and $(a_i - b_i - c_i + d_i)_i$ is non-decreasign as well. The supremum $S = \bigvee_i (a_i + d_i - b_i - c_i)$ exists by Lemma 3.1.3, having $a + d - b_1 - c_1$ as the upper bound. Then for each i, we have $a_i + d_i \leq S + b_i + c_i$, hence $a + d \leq S + b + c$ which gives us one inequality. To prove the second one, we have to verify that for each i we have the inequality $a_i + d_i - b_i - c_i \leq a + d - b - c$. Equivalently,

$$a_i + d_i + b + c \leq b_i + c_i + a + d. \tag{3.14}$$

But due to monotonicity of $(a_i + b_i - c_i - d_i)_i$, for each $j \geq i$, we have $a_i + d_i + b_j + c_j \leq b_i + c_i + a_j + d_j$. Which yields (3.14).

Next assume $(a_i + d_i - b_i - c_i)$ is non-increasing and denote $S = \bigwedge_i (a_i + d_i - b_i - c_i)$, which exists again by Lemma 3.1.3, having $a_1 + d_1 - b - c$ (or 0 in the case of E_0) as lower bound. We have $a_i + d_i \geq S + b_i + c_i$ for each i, which gives us $a + d - b - c \geq S$. And for each i, $a - b - c + d \leq a_i - b_i - c_i + d_i$, as this is equivalent to $a + d + b_i + c_i \leq b + c + a_i + d_i$, which follows from:

$$\forall j > i, a_j + d_j + b_i + c_i \leq b_j + c_j + a_i + d_i.$$

The other two cases (when the four sequences are non-increasing) are dual to the two just described. □

We will simplify future expressions a lot using the following notation: For any (usually a pseudo n-spectral resolution) $F : \mathbb{R} \to E$ and an integer $i \leq n$ we define

$$F(t_1, \ldots, t_i^+, \ldots, t_n) := \bigwedge_{t_i < s} F(t_1, \ldots, s, \ldots, t_n), \tag{3.15}$$

$$F(t_1, \ldots, t_i^-, \ldots, t_n) := \bigvee_{t_i < s} F(t_1, \ldots, s, \ldots, t_n). \tag{3.16}$$

For general monotone mapping F the right hand side may not be defined, however for n-spectral resolutions we can even well-define $F(t_1^{\sigma_1}, \ldots, t_n^{\sigma_n})$, where σ_i equals $+$ or $-$ or is empty (Lemma 3.2.9).

We will also naturally extend the linear order $<$: for each $r, s \in \mathbb{R}$ we set $r^- < r < r^+$ and $r^{\sigma_1} < s^{\sigma_2}$, for $s \neq s$, if and only if $r < s$, for any $\sigma_1, \sigma_2 \in \{+, -, \text{null}\}$. Moreover, $-\infty^+$,∞^- are the least and the greatest element, respectively. In following text we prefer to avoid a special treatment of $\pm\infty$, observe that computation of infima or suprema of pseudo spectral resolution

F in $\pm\infty$ behaves quite similar to the case of finite real: by definition of supremum and infimum we have

$$\bigvee_{t<-\infty} F(t) = 0 = F(-\infty), \tag{3.17}$$

$$\bigwedge_{\infty<t} F(t) = 1 \geq F(+\infty). \tag{3.18}$$

In the following part, we will be frequently proving that some mapping satisfies the condition of spectral resolutions. We have to keep an eye on the finiteness condition, as in general, some infima of an n-spectral resolution may not exist. The following equivalent formulation of the finiteness condition shows to be useful.

Lemma 3.2.8. *Let $F : \mathbb{R}^n \to E$ be an n-spectral resolution. Then F satisfies finiteness condition if and only if F has all infima of the form*

$$F(s_1^{\sigma_1}, \ldots, s_n^{\sigma_n}), \tag{3.19}$$

where for each $i = 1 \ldots, n$ we have $s_i \in \mathbb{R} \cup \{-\infty\}$ and $\sigma_i \in \{+, -, \mathrm{null}\}$ (we allow also $-\infty^+$).

Proof. The "only if" part: suppose F satisfies finiteness condition. First observe, that if in (3.19) some $s_i^{\sigma_i}$ equals $-\infty^+$, then the expression values 0 (by definition of n-spectral resolution). Hence we may assume all s_i's are finite. Now, investigating infimum (3.19) we may without loss of generality assume there is an integer $k \leq n$, such that the expression in concern is of the form $F(s_1, \ldots, s_k, s_{k+1}^+ \ldots, s_n^+)$. As F has only finitely many characteristic points, there is $\mathbf{q} \in \mathbb{R}^{n-k}$, $(s_{k+1}, \ldots, s_n) \ll \mathbf{q}$, such that for all $\mathbf{r} \in \mathbb{R}^{n-k}$, with

$$(s_{k+1}, \ldots, s_n) \ll \mathbf{r} \ll \mathbf{q}, \tag{3.20}$$

the n-tuples $(s_1, \ldots, s_k, r_1, \ldots, r_{n-k})$ belong to the same $B_h = F^{-1}(E_h)$, for some $h \in H$. Two cases could happen: either $h = 0$ in which case the $F(s_1, \ldots, s_k, r_1, \ldots, r_{n-k})$'s has 0_E as lower bound, or $h \neq 0$, in which case by Lemma 3.2.1, there is the unique characteristic point $\mathbf{t}$, such that $F(t_1^+, \ldots, t_n^+) \in E_h$ is lower bound for all $F(s_1, \ldots, s_k, r_1, \ldots, r_{n-k})$'s ($\mathbf{r}$ satisfies (3.20)). In both cases Lemma 3.1.3 assures the existence of the desired infinum.

Now the "if" part: Suppose all the infima $F(s_1^{\sigma_1}, \ldots, s_n^{\sigma_n})$, where $s_i^{\sigma_i}$'s are as in the statement, exist. We only need to verify the finiteness of the number of the characteristic points. On the way of contradiction assume F asserts infinitely many characteristic points. Hence for some integer i, the set

$$\{t \in \mathbb{R} | t \text{ occurs as } i\text{-th coordinate of some characteristic point of } F\}$$

is infinite. Suppose $i = 1$. Take any $\mathbf{t}_u \in B_u$, then $\mathbf{t}_u = (t_1^u, \ldots, t_n^u)$ is over all the characteristic points. Hence the line $L := \{(y, t_2^u, \ldots, t_n^u) | y \in \mathbb{R}\}$ meets by Lemma 3.2.3 infinitely many B_h's. Now the decomposition $\mathbb{R}^n = \bigcup_{h \in H} B_h$ restricts to infinite decomposition of L to intervals (see Theorem 3.2.4). For the sake of simplicity let us identify $L \equiv \mathbb{R}$, then the decomposition is as follows

$$L = (-\infty, s_0] \cup \left(\bigcup_{0, u \neq h \in H, B_h \neq \emptyset} (t_h, s_h] \right) \cup (t_u, \infty).$$

Where $(t_h, s_h] = L \cap B_h$. By the compactness of interval $[s_0, t_u]$ the reals s_h's have some cluster point s. However, as both supremum $F(s^-, t_2^u, \ldots, t_n^u)$ and infimum $F(s^+, t_2^u, \ldots, t_n^u)$ exist, by Lemma 3.1.3 there exists a real $\epsilon > 0$, such that there is no s_h in $(s - \epsilon, s + \epsilon) \setminus \{s\}$, which is a contradiction with s being cluster point. $\square$

Lemma 3.2.9. *Let $F : \mathbb{R}^n \to E$ be an n-spectral resolution satisfying finiteness condition. Then the value*

$$F(t_1^{\sigma_1}, \ldots, t_n^{\sigma_n}),$$

where $\sigma_i \in \{+, -\}$ is well defined. That is, whenever we have an expression of the form $\Xi_1 \cdots \Xi_n F(s_1, \ldots, s_n)$, where each Ξ_i is either $\bigvee_{t_i > s_i}$ or $\bigwedge_{t_i < s_i}$, the expression has well defined value and we may change an order of the Ξ_i's without changing the value.

Proof. We proceed by an induction over n. If $n = 1$, there is nothing to prove. Now the induction step: by fixing any coordinate in F, we get an $(n-1)$-dimensional pseudo spectral resolution, so we can freely permute the $\Xi_2, \ldots, \Xi_n$ by the induction. Using this and some re-indexing, it is enough to prove for each $\mathbf{t} \in (\mathbb{R} \cup \{\pm\infty\})^n$ the following equation is well defined and holds

$$\bigvee_{s_1 < t_1} \cdots \bigvee_{s_k < t_k} \bigwedge_{t_{k+1} < s_{k+1}} \cdots \bigwedge_{t_n < s_n} F(s_1, \ldots, s_n) =$$
$$= \bigwedge_{s_{k+1} < t_{k+1}} \cdots \bigwedge_{s_n < t_n} \bigvee_{s_1 < t_1} \cdots \bigvee_{s_k < t_k} F(s_1, \ldots, s_n).$$

Let us rewrite it in an easier form: For each $\mathbf{s} \in (\mathbb{R} \cup \{\pm\infty\})^k$ and $\mathbf{t} \in (\mathbb{R} \cup \{\pm\infty\})^{n-k}$,

$$\bigvee_{\mathbf{q} \ll \mathbf{s}} \bigwedge_{\mathbf{r} \gg \mathbf{t}} F(\mathbf{q}, \mathbf{r}) = \bigwedge_{\mathbf{r} \gg \mathbf{t}} F(\mathbf{s}, \mathbf{r}). \tag{3.21}$$

(We have used the continuity property on the right hand side.) The right hand side is well defined by Lemma 3.2.8. Hence, as for existence, we only need to look after the left hand side. For each $\mathbf{q} \ll \mathbf{s}$ and $\mathbf{r}' \gg \mathbf{t}$, we shall prove the second inequality (the first one is trivial) in

$$0 \leq \bigwedge_{\mathbf{r} \gg \mathbf{t}} F(\mathbf{s}, \mathbf{r}) - \bigwedge_{\mathbf{r} \gg \mathbf{t}} F(\mathbf{q}, \mathbf{r}) \leq F(\mathbf{s}, \mathbf{r}') - F(\mathbf{q}, \mathbf{r}'). \tag{3.22}$$

If we prove (3.22), we are done, as the last expression clearly goes to 0 as $\mathbf{q}$ goes to $\mathbf{s}$, which implies the left hand side in (3.21) exists and equals to the right one. Now inequality (3.22) obviously follows from following equality

$$\bigwedge_{\mathbf{r} \gg \mathbf{t}} F(\mathbf{s}, \mathbf{r}) - \bigwedge_{\mathbf{r} \gg \mathbf{t}} F(\mathbf{q}, \mathbf{r}) = \bigwedge_{\mathbf{r} \gg \mathbf{t}} [F(\mathbf{s}, \mathbf{r}) - F(\mathbf{q}, \mathbf{r})]. \tag{3.23}$$

In proving (3.23) we would like to apply Lemma 3.2.7 ($m = 2$), but the infima are not taken over countable non-increasing sequences. The two infima on the left hand side may be in the obvious way (using monotonicity of F) rewritten to be taken over countable monotone sequences. In the case of right hand infimum, we have to verify that whenever $\mathbf{r}_0 \leq \mathbf{r}_1$ then

$$F(\mathbf{s}, \mathbf{r}_0) - F(\mathbf{q}, \mathbf{r}_0) \leq F(\mathbf{s}, \mathbf{r}_1) - F(\mathbf{q}, \mathbf{r}_1)$$

(which in fact proves the infimum is well defined). But the last is direct consequence of Lemma 3.2.2.

Hence, we can apply the Lemma 3.2.7, to prove (3.23) and so to finish the whole proof. $\square$

There is no problem in substituting t^- (t^+, resp.) to the Δ-operators, e.g. $\Delta_i(s_i^+, -\infty^+)$ is a shortcut for $(t_1, \ldots, \hat{t}_i, \ldots, t_n) \mapsto \bigvee_{s_i < t_i} F(\mathbf{t}) - \bigwedge_{t_i} F(\mathbf{t})$. However, we have to verify several properties which were rather trivial for the ordinary Δ-operator. We shall prove the Δ-operators in this extended manner when applied to finite-dimensional pseudo spectral resolution are:

- commutative (Lemma 3.2.10),

- distributive over + (Lemma 3.2.13),
- and moreover they preserve the class of finite-dimensional pseudo spectral resolutions (Proposition 3.2.12).

In the proof of the next lemma we are using following notation which simplifies formal manipulation with substitutions to functions: given a function $F(t_1, \dots, t_n)$ of variables $t_1, \dots, t_n$ and a real s, by $F(s/t_i)$ we mean a substitution of s to the i-th coordinate in F. That is the function

$$(t_1, \dots, \hat{t_i}, \dots, t_n) \mapsto F(t_1, \dots, s, \dots, t_n).$$

Hence we can write $\Delta_i(s_i, r_i)F(\mathbf{t}) = F(r_i/t_i) - F(s_i/t_i)$.

Lemma 3.2.10. *Let F be a pseudo n-spectral resolution, $1 \leq i < j \leq n$ integers and $r_i \leq s_i$, $r_j \leq s_j$ be elements of $\mathbb{R} \cup \{\pm\infty\}$. Assume*

$$F(r_i^{\sigma_1}/t_i, r_j^{\sigma_2}/t_j), F(r_i^{\sigma_1}/t_i, s_j^{\delta_2}/t_j), F(s_i^{\delta_1}/t_i, r_j^{\sigma_2}/t_j), F(s_i^{\delta_1}/t_i, s_j^{\delta_2}/t_j)$$

exist (e.g., if F satisfies finiteness condition by Lemma 3.2.8), where σ_1, σ_2, δ_1, δ_2 freely substitute the symbols in $\{+, -, \mathrm{null}\}$ such that $+\infty^+, -\infty^-$ do not occure. Then

$$\Delta_i(r_i^{\sigma_1}, s_i^{\sigma_2})\Delta_j(r_j^{\delta_1}, s_j^{\delta_2})F = \Delta_{j-1}(r_j^{\delta_1}, s_j^{\delta_2})\Delta_i(r_i^{\sigma_1}, s_i^{\sigma_2})F.$$

Proof. We shall demonstrate the proof on one particular case

$$\Delta_i(r_i^+, s_i^-)\Delta_j(r_j^+, s_j^-)F = \Delta_{j-1}(r_j^+, s_j^-)\Delta_i(r_i^+, s_i^-)F, \tag{3.24}$$

but one can prove all the cases by the very similar way. The left hand side equals (by definition)

$$[F(s_j^-/t_j) - F(r_j^+/t_j)](s_i^-/t_i) - [F(s^+/t_j) - F(s_1^+/t_j)](r_i^+/t_i). \tag{3.25}$$

We will first evaluate the first summand of (3.25). By Lemma 3.2.7 $F(s_j^-/t_j)$ is monotone in the i-th coordinate and (by Lemma 3.2.9)

$$\bigvee_{s<s_i} F(s/t_i, s_j^+/t_j) = F(s_i^-/t_i, s_j^+/t_j).$$

Similarly, $F(r_j^+/t_j)$ is monotone in i-th coordinate and

$$\bigvee_{s<s_i} F(s/t_i, r_j^+/t_j) = F(s_i^-/t_i, r_j^+/t_j).$$

Next $F(s_j^-/t_j) - F(r_j^+/t_j)$ is monotone in i-th coordinate as well: for $s < s'$ we have

$$\begin{aligned} F(s/t_i, s_j^-/t_j) - F(s/t_i, r_j^+/t_j) &\leq F(s'/t_i, s_j^-/t_j) - F(s'/t_i, r_j^+/t_j) \\ \Leftrightarrow F(s'/t_i, r_j^+/t_j) - F(s/t_i, r_j^+/t_j) &\leq F(s'/t_i, s_j^-/t_j) - F(s'/t_i/, s_j^-/t_j) \\ \Leftrightarrow [F(s'/t_i) - F(s/t_i)](r_j^+/t_j) &\leq [F(s'/t_i) - F(s/t_i)](s_j^-/t_j), \end{aligned}$$

where the second equivalence follows by Lemma 3.2.7 (with $m = 2$). The last inequality is clearly valid if $r_j < s_j$ ($F(s'/t_i) - F(s/t_i)$ is monotone by the volume conditions). In the case $r_j = s_j$ we only need to reverse the three inequalities.

Now we can finally apply Lemma 3.2.7 (with $m = 2$) and yield

$$[F(s_j^-/t_j) - F(r_j^+/t_j)](s_i^-/t_i) = F(s_i^-/t_i, s_j^+/t_j) - F(s_i^-/t_i, s_j^-/t_j).$$

The same argumentation applies to the second summand in (3.25) and to the right hand side of (3.24) as well. Hence both, the left hand and the right hand side of (3.24) are equal to

$$F(s_i^-/t_i, s_j^+/t_j) - F(s_i^-/t_i, s_j^-/t_j) - F(r_i^+/t_i, s_j^+/t_j) + F(r_i^+/t_i, s_j^-/t_j).$$

One can realize, that the concrete choice of symbols $+, -$ we have in the desired equality affects the proof only in which variant of Lemma 3.2.7 we use. Also observe that particular cases $r_j^+ = -\infty^+$ or $s_j^- = \infty^-$ cause (in applying Lemma 3.2.7) no problems. $\square$

Observation 3.2.11. *If F is a pseudo n-spectral resolution, $i = 1, \ldots, n$ and $r < s$ are reals, then $F' := \Delta_i(r, s)F$ is a pseudo $(n-1)$-dimensional spectral resolution.*

Moreover F' satisfies finiteness condition whenever F does.

Proof. Suppose $i = 1$. monotonicity and volume conditions obviously follows from those for F. Continuity in j-th coordinate:

$$\Delta_j(s_j^-, s_j)\Delta_1(r_1, s_1)F = \Delta_1(r_1, s_1)\Delta_{j+1}(s_j^-, s_j)F = \Delta_1(r, s)0_{\mathbb{R}^{n-1}} = 0_{\mathbb{R}^{n-2}}.$$

The fact that $\bigwedge_{t_j \in \mathbb{R}} \Delta_1(r, s)F(s_2, \ldots, t_j, \ldots, s_n) = 0$, for each choice of s_i's, follows by another application of Lemma 3.2.7.

Applying Lemma 3.2.7 (in the way we have done several times) we deduce

$$\begin{aligned}&(\Delta_1(r,s)F)(s_2,\dots,s_k,s_{k+1}^+\dots,s_n^+)=\\=&F(s,s_2,\dots,s_k,s_{k+1}^+\dots,s_n^+)-F(r,s_2,\dots,s_k,s_{k+1}^+\dots,s_n^+),\end{aligned}$$

so by Lemma 3.2.8, F' satisfies finitenes condition whenever F does. $\square$

We have already proved the commutativity of two Δ-operators applied to a pseudo n-spectral resolution (Lemma 3.2.10). We like to extend this result to the commutativity of any number of Δ-operators by induction, to do so we need the following Proposition.

Proposition 3.2.12. *If F is a pseudo n-spectral resolution on effect algebra $E=\Gamma(H\overrightarrow{\times}G,(u,0))$, $i\le n$ an integer, and $r\le s$ are reals, then $F'=\Delta_i(r^{\sigma_1},s^{\sigma_2})F$ is a pseudo $(n-1)$-spectral resolution whenever $r^{\sigma_1}<s^{\sigma_2}$. Moreover, F' satisfies finiteness condition whenever F does.*

Proof. For simplicity assume $i=1$ and denote $F':=\Delta_1(r^{\sigma_1},s^{\sigma_2})F$. monotonicity in j-th coordinate is equivalent to $0\le\Delta_j(r_j,s_j)\Delta_1(r^{\sigma_1},s^{\sigma_2})F$, for each $r_j<s_j$. But the last equals (Lemma 3.2.10) $\Delta_1(r^{\sigma_1},s^{\sigma_2})\Delta_{j+1}(r_j,s_j)F$, which is non-negative (e.g., by Observable 3.2.11). Using Lemma 3.2.10 and Observation 3.2.11 we deduce

$$\begin{aligned}&\Delta_1(r_1,s_1)\cdots\Delta_{n-1}(r_{n-1},s_{n-1})\Delta_1(r^{\sigma_1},s^{\sigma_2})F=\\&\qquad=\Delta_1(r^{\sigma_1},s^{\sigma_2})\Delta_2(r_1,s_1)\cdots\Delta_n(r_{n-1},s_{n-1})F.\end{aligned}$$

and the last expression is non-negative.

Continuity in the j-th coordinate could be proved as follows

$$\begin{aligned}\Delta_j(s_j^-,s_j)\Delta_1(r^{\sigma_1},s^{\sigma_2})F=&\Delta_1(r^{\sigma_1},s^{\sigma_2})\Delta_{j+1}(s_j^-,s_j)F=\\=&\Delta_1(r^{\sigma_1},s^{\sigma_2})0_{\mathbb{R}^{n-1}}=0_{\mathbb{R}^{n-2}}.\end{aligned}$$

Infimal condition: choose any $s_j\in\mathbb{R}$

$$\begin{aligned}&F'(s_j/t_j')-F'(-\infty/t_j')=\Delta_j(-\infty,s_j)\Delta_1(r^{\sigma_1},s^{\sigma_2})F=\\=&\Delta_1(r^{\sigma_1},s^{\sigma_2})\Delta_{j+1}(-\infty,s_j)F=\Delta_1(r^{\sigma_1},s^{\sigma_2})[F(s_j/t_{j+1})-0_{\mathbb{R}^{n-1}}]=\\=&F'(s_j/t_j').\end{aligned}$$

So $F'(-\infty/t'_j) = 0_{\mathbb{R}^{n-1}}$. Suprema condition: we already know $\Delta_1(r^{\sigma_1}, s^{\sigma_2})F$ is monotone and so is $F(r^{\sigma_1}/t_1)$ and $F(s^{\sigma_1}/t_1)$. Lemma 3.2.7 (with $m = 2$) assure

$$\bigvee_{\mathbb{R}^{n-1}} \Delta_1(r^{\sigma_1}, s^{\sigma_2})F = \bigvee_{\mathbb{R}^{n-1}} F(t_1/s_2^{\sigma_2}) - \bigvee_{\mathbb{R}^{n-1}} F(t_1/r_1^{\sigma_1})$$

exists.

It remains to deduce the finiteness property. By Lemma 3.2.9, for each $k < n$ and $s_2, \ldots, s_n \in \mathbb{R} \cup \{-\infty\}$ the values $F(r^{\sigma_1}, s_2, \ldots, s_k, s_{k+1}^+, \ldots, s_n^+)$ and $F(r^{\sigma_2}, s_2, \ldots, s_k, s_{k+1}^+, \ldots, s_n^+)$ exist, and hence by Lemma 3.2.7

$$(\Delta_1(r^{\sigma_1}, s^{\sigma_2})F)(s_2, \ldots, s_k, s_{k+1}^+, \ldots, s_n^+)$$

exists as well. $\square$

Lemma 3.2.13. *Let F be a pseudo n-spectral resolution and i, j a pair of integers, such that $1 \leq i < j \leq n$. Then*

$$\begin{aligned}
&\Delta_i(r_i^{\sigma_1}, s_i^{\sigma_2})[\Delta_j(r_j^{\delta_1}, u_j^{\delta_2}) + \Delta_j(u_j^{\delta_2}, s_j^{\delta_3})]F = \\
=&\Delta_i(r_i^{\sigma_1}, s_i^{\sigma_2})\Delta_j(r_j^{\delta_1}, u_j^{\delta_2})F + \Delta_i(r_i^{\sigma_1}, s_i^{\sigma_2})\Delta_j(u_j^{\delta_2}, s_j^{\delta_3})F.
\end{aligned}$$

Proof. We can yield the identity by following deduction

$$\begin{aligned}
\text{LH'S} =&\Delta_i(r_i^{\sigma_1}, s_i^{\sigma_2})\Delta_j(r_j^{\delta_1}, s_j^{\delta_3})F \\
=&\Delta_{j-1}(r_j^{\delta_1}, s_j^{\delta_3})\Delta_i(r_i^{\sigma_1}, s_i^{\sigma_2})F \\
=&\Delta_{j-1}(r_j^{\delta_1}, u_j^{\delta_2})\Delta_i(r_i^{\sigma_1}, s_i^{\sigma_2})F + \Delta_{j-1}(u_j^{\delta_2}, s_j^{\delta_3})\Delta_i(r_i^{\sigma_1}, s_i^{\sigma_2})F \\
=&\text{RH'S}.
\end{aligned}$$

$\square$

Now let us come back to our main problem. We have an n-spectral resolution $F : \mathbb{R}^n \to E$. Suppose F has only finite amount of characteristic points and all of them are elements of the set $\prod_{i=1}^n \{t_1^i, t_2^i, \ldots, t_{l_i}^i\}$, where $t_1^i < t_2^i < \cdots < t_{l_i}^i$, $l \in \mathbb{N}$, are reals (such that all the i-th coordinate of the characteristic points of F are among them). In analogy to the motivating example in the introduction to this section we consider equality

$$F(\mathbf{t}) = \prod_{i=1}^{n} \Bigg(\Delta_i(-\infty, \min\{t_i, t_1^i\} + \sum_{j=1}^{l_i - 1} \big[\Delta_i(\min\{t_i, t_j^i\}, \min\{t_i, {t_j^i}^+\} \\ + \Delta_i(\min\{t_i, {t_j^i}^+\}, \min\{t_i, t_{j+1}^i\}) \big] + \Delta_i(\min\{t_i, t_{l_i}^i\}, \min\{t_i, {t_{l_i}^i}^+\}) \\ + \Delta_i(\min\{t_i, {t_{l_i}^i}^+\}, t_i) \Bigg). \tag{3.26}$$

Now, due to the previous lemma we can distribute the product to $\prod_i (2l_i + 1)$ summands. Each of the summands arises by applying iteration of some Δ operators on F, which (by definition) results in a real number. However, in order to acquire a decomposition of F, we need to thing of the summands as functions in variables $(t_1, \ldots, t_n)$. We need to prove the last technical lemma of this chapter, in order to verify conditions of spectral resolutions for the summands.

Lemma 3.2.14. *Let $F : \mathbb{R}^n \to E$ be an n-dimensional pseudo spectral resolution. For every $i = 1, \ldots, n$ and reals $s < r$, each of the mappings $F_i' : \mathbb{R}^n \to E$, $i = 1, 2, 3, 4$, defined as*

$$F_1'(t_1, \ldots, t_n) := (\Delta_i(-\infty, \min\{s, t_i\})F)(t_1, \ldots, \hat{t_i}, \ldots, t_n),$$
$$F_2'(t_1, \ldots, t_n) := (\Delta_i(\min\{s, t_i\}, \min\{s^+, t_i\})F)(t_1, \ldots, \hat{t_i}, \ldots, t_n),$$
$$F_3'(t_1, \ldots, t_n) := (\Delta_i(\min\{s^+, t_i\}, \min\{r, t_i\})F)(t_1, \ldots, \hat{t_i}, \ldots, t_n),$$
$$F_4'(t_1, \ldots, t_n) := (\Delta_i(\min\{r^+, t_i\}, t_i)F)(t_1, \ldots, \hat{t_i}, \ldots, t_n),$$

is an n-dimensional pseudo spectral resolution. Moreover, the characteristic points of F_i's are subsets of the characteristic points of F. In more details, a characteristic point $(s_1, \ldots, s_n)$ is characteristic point of:

- *F_1' iff $s_i < s$,*
- *F_2' iff $s_i = s$,*
- *F_3' iff $s < s_i < r$,*
- *F_4' iff $r < s_i$.*

Proof. We may without loss of generality assume $i = 1$. All the discussed cases are of the form $F'(\mathbf{t}) := (\Delta_1(\min\{s^{\sigma_1}, t_1\}, \min\{r^{\sigma_2}, t_1\})F)(t_2, \dots, t_n)$ (where $s^{\sigma_1} = -\infty^+$ in the case of F'_1 and $r^{\sigma_2} = \infty^-$ in the case of F'_4). So we can threat all the cases at once. Consider an expression $\Delta_1(s', r')F'$, for some $s' < r'$, it equals (by definition)

$$\begin{aligned} F(\min\{r^{\sigma_2}, r'\}/t_1) - &F(\min\{r^{\sigma_2}, s'\}/t_1) \\ &- F(\min\{s^{\sigma_1}, r'\}/t_1) + F(\min\{s^{\sigma_1}, s'\}/t_1). \end{aligned} \tag{3.27}$$

Two cases could occur:

$$s' = \min\{s^{\sigma_1}, r^{\sigma_2}, s', r'\} \leq s^{\sigma_1} \text{ or } s' \geq \min\{s^{\sigma_1}, r^{\sigma_2}, s', r'\} = s^{\sigma_1}.$$

In both cases two of the summands in (3.27) cancel each other. So we obtain

$$F(\min\{r^{\sigma_2}, r'\}/t_1) - F(\min\{s^{\sigma_1}, r'\}/t_1)$$

in the first case and

$$F(\min\{r^{\sigma_2}, r'\}/t_1) - F(\min\{r^{\sigma_2}, s'\}/t_1)$$

in the second case. Moreover

$$F(\min\{s^{\sigma_1}, r'\}/t_1) \geq F(s'/t_1) = F(\min\{r^{\sigma_2}, s'\}/t_1)$$

in the first case and

$$F(\min\{s^{\sigma_1}, r'\}/t_1) = F(s^{\sigma_1}/t_1) \leq F(\min\{r^{\sigma_2}, s'\}/t_1)$$

in the second case. Hence we can express (3.27) (in both cases) using only one Δ-operator as

$$\Delta_1(\max\big\{\min\{s^{\sigma_1}, r'\}, \min\{r^{\sigma_2}, s'\}\big\}, \min\{r^{\sigma_2}, r'\})F. \tag{3.28}$$

Moreover, as the four elements $s^{\sigma_1}, r^{\sigma_2}, s', r'$ are linearly ordered, the operations of min and max distribute over each other. Using distributivity it is straightforward to transform the (3.28) to the form

$$\Delta_1(\min\big\{r^{\sigma_2}, r', \max\{s^{\sigma_2}, s'\}\big\}, \min\{r^{\sigma_2}, r'\})F. \tag{3.29}$$

As consequence we can express any sequence of Δ-operators applying to F' as a sequence of Δ-operators applying to F. And so, all the volume conditions (containing monotonicity) hold for F'. One can deduce the continuity and infimal conditions in j-th coordinate, $j \geq 2$, by the similar way as is done in Lemma 3.2.12. For the case of first coordinate, $F'(\mathbf{t})$ equals 0 whenever $t_1 \leq s^{\sigma_1}$ (infimal condition) and as s' goes to r', $\Delta_1(s', r')F'$ goes to zero, this follows by discussing several cases in identity (3.29). Finally, the supremum

$$\bigvee_{\mathbf{t}\in\mathbb{R}^n} F'(\mathbf{t}) = \bigvee_{t_2,\ldots,t_n\in\mathbb{R}} \Delta_1(s^{\sigma_1}, s^{\sigma_2})F$$

exists by Lemma 3.2.12.

The identity (3.29) is also useful in concern of the characteristic points. Some $(t_1, \ldots, t_n)$ is a characteristic point of F' if and only if

$$a := \Delta_1(t_1, t_1^+) \cdots \Delta_n(t_n, t_n^+)F'$$

does not belong to the radical of E. Thanks to already proved part we have

$$a = \Delta_1(\min\left\{r^{\sigma_2}, t_1^+, \max\{s^{\sigma_2}, t_1\}\right\}, \min\{r^{\sigma_2}, t_1^+\})\Delta_2(t_2, t_2^+) \ldots \Delta_n(t_n, t_n^+)F.$$

The last expression equals 0 if $t_1 < s^{\sigma_1}$ (in which case $\Delta_1(t_1^+, t_1^+)F$ occurs) or if $r^{\sigma_2} \leq t_1$ (in which case $\Delta_1(r^{\sigma_2}, r_1^{\sigma_2})$ occurs). We are left with the case $s^{\sigma_1} \leq t_1 < r^{\sigma_2}$, when a simply equals $\Delta_1(t_1, t_1^+) \cdots \Delta_n(t_n, t_n^+)F$. Consequently $\mathbf{t}$ is a characteristic point of F' if and only if it is a characteristic point of F and $s^{\sigma_1} \leq t_1 < r^{\sigma_2}$. Hence all the cases in the statement of the lemma follow. □

Finally, we exhibit the main and the last theorem of this chapter:

Theorem 3.2.15. *Let $E = \Gamma(H \overrightarrow{\times} G, (u, 0))$, where (H, u) is a unital Abelian po-group and G is a directed monotone σ-complete Abelian po-group with interpolation. Then any n-spectral resolution F on lexicographic effect algebra E, having only finitely many characteristic points, extends to an n-observable.*

Proof. Consider decomposition of F as in (3.26). Using Lemma 3.2.14 repeatedly, we verify, that all the summands are pseudo n-spectral resolutions. Moreover each summand F' is of one of following two kinds: its image is in the radical of E and hence has no characteristic points, or it has exactly one characteristic point and its image is two-element. In any case we can extend a summand F' to the pseudo n-observable $x_{F'}$ using Theorem 2.2.4 (in the first case) or find the observable directly in the trivial second case. Now the sum $x := \sum_{F'} x_{F'}$ is the desired observable. To verify it realy extends F is rather trivial: Let $\mathbf{t} \in \mathbb{R}^n$ and $I := \{\mathbf{s} | \mathbf{s} \ll \mathbf{t}\}$. Then

$$x(I) = \sum_{F'} x_{F'}(I) = \sum_{F'} F'(\mathbf{t}) = F(\mathbf{t}).$$

$\square$

Chapter 4

Classical approach to measure extension

The problem of correspondence considered in the previous sections is related to so called measure extension problem, which Sikorski formulated as (see [Sik69], paragraph 34): Given a Boolean algebras homomorphism $h_0 : \mathcal{B}_0 \to \mathcal{S}$ such that $\bigwedge_n h(A_n) = 0_{\mathcal{S}}$, whenever $A_n \searrow 0_{\mathcal{B}_0}$. Under which circumstances it extends to a σ-homomorphism $h : \mathcal{B} \to \mathcal{S}$, where $\mathcal{B}$ is σ-generated by $\mathcal{B}_0$? When S satisfies all such extensions, we say that $\mathcal{S}$ have *weak σ-extension property.*

The problem on the level of Boolean algebras have satisfying answer: the weak σ-extension property is equivalent to *weak σ-distributivity*, which states, that given any countable matrix $(a_{i,j})_{i,j=1}^{\infty}$ of elements of $\mathcal{S}$, such that for each i we have $a_{i,j} \searrow 0$ (as $j \to \infty$), then

$$\bigwedge_{\phi \in \mathbb{N}^{\mathbb{N}}} \bigvee_i a_{\phi(i),j} = 0. \tag{4.1}$$

The sufficiency of weak σ-distributivity was given by Mathess ([Sik69], Theorem 4, paragraph 34), while the necessity is a result of Wright [Wri71], who in fact considered the problem on the level of lattice-ordered vector spaces.

Now, given an n-spectral resolution $F : \mathbb{R}^n \to E$, with values in an effect algebra E, it gives rise to a finitely additive measure $x_0 : \mathcal{B}_0 \to E$, where $\mathcal{B}_0$ is the Boolean subalgebra of $\mathcal{B}(\mathbb{R}^n)$ generated by half open intervals $[a_1, b_1) \times \cdots \times [a_n, b_n)$ (for each i, $a_i < b_i$). And the question is: For which effect algebras E the measure x_0 extends to a σ-measure x on the σ-algebra $\mathcal{B}(\mathbb{R}^n)$, which is σ-generated by $\mathcal{B}_0$. Hence, in comparison to Sikorski's question, we have restricted the situation in definition domain to one particular class of Boolean algebras ($\mathcal{B}(\mathbb{R}^n)$, $n \in \mathbb{N}$), while the Boolean algebra S is replaced by much more general structure of effect algebras.

Among the results in research of weak σ-distributivity, the ones in [RT97] seem to be the closest to our ones. In [RT97], Riečan and Neubrunn used weak σ-distributivity to provide an extension of a spectral resolution on a weak σ-distributive σ-complete MV-algebra. The proof is divided into two parts. First, it is shown, that each spectral resolution $F : \mathbb{R} \to M$ gives rise to a measure x_0 on the ring of sets generated by half-open intervals $[a, b) \subset \mathbb{R}$, and such that $x_0 : (A_i) \searrow 0$, whenever $A_i \searrow \emptyset$. This part is called (in [RT97]) Alexandrov's theorem. Then, some general results on measure extension are applied. Both parts essentially use weak σ-distributivity.

It is worth noting that the technique used by Riečan and Neubrunn was introduced by Fremlin in [Fre75], and could be interpreted as a tricky simulation of the classical ϵ, δ calculus inside an MV-algebra. Nevertheless, the Fremlin's tricks with weak σ-distributivity strongly use the lattice structure, and the attempt to move to more general effect algebras leads to only partial results, in [Rie98] Riečan proved Alexandrov's theorem for so-called weakly regular effect algebras.

In this section, we present the extension construction for the case of a monotone σ-complete interval effect algebra E, having a faithful σ-state s (i.e., $s(a) = 0 \implies a = 0$). For example, measure algebras (the case when E is a Boolean algebra) and probability MV-algebras are covered by these assumptions. The existence of the faithful σ-state is essentially a stronger assumption than weak σ-distributivity, which is a price for the absence of lattice structure. For example, any monotone σ-complete effect algebra with (RDP) having faithful σ-state is already an MV-algebra (see [Goo86], Prop. 16.5). The proof of the main result of this chapter (Theorem 4.1.4) is based on ideas in [DL20c].

4.1 Interval effect algebras with faithful σ-state

To the purpose of this chapter denote $\mathcal{R} \subset \mathcal{B}(\mathbb{R}^n)$ a ring of sets (i.e., system of sets closed under union and relative complements) generated by the semi-closed intervals of the form

$$[a_1, b_1) \times \cdots \times [a_n, b_n), \text{ where } a_i, b_i \in \mathbb{R}, a_i < b_i. \tag{4.2}$$

In the chapter by a semi-closed interval we will always mean an interval of form (4.2).

Observation 4.1.1. *Each set $A \in \mathcal{R}$ is a disjoint union of semi-closed intervals of form* (4.2)*.*

We will apply standard results concerning measure extension from a ring of sets to its generated σ-ring, which is achieved using the concept of outer measures. First, recall the concept of a measure on a ring: σ-finite measure on a ring $\mathcal{R}$ is a real-valued mapping $\mu : \mathcal{R} \to \mathbb{R}$, such that for each A, B, A_i, $i \in \mathbb{N}$, pairwise disjoint sets in $\mathcal{R}$, we have

$$\begin{aligned}
&\mu(A) \geq 0, \\
&\mu(A \cup B) = \mu(A) + \mu(B), \\
&\mu(\bigcup_i A_i) = \sum_i \mu(A_i), \text{ whenever } \sum_i A_i \in \mathcal{R}.
\end{aligned}$$

And each $B \in \mathcal{R}$ could be covered as $B \subseteq \bigcup_i B_i$, where B_i's form countable family of sets in $\mathcal{R}$ with finite measure (σ-finiteness). Following theorem is essential (for proof see [Hal74], Thm. 13.A.):

Theorem 4.1.2. *Let μ be a σ-finite measure on a ring of sets $\mathcal{R}$. Then there is a unique σ-finite measure $\bar{\mu}$, which extends μ on the σ-ring generated by $\mathcal{R}$.*

Moreover, as follows from the section 12 in [Hal74], the extended measure $\bar{\mu}$ satisfies formula

$$\bar{\mu}(A) = \inf\{\sum_{i=1}^{\infty} \mu(A_i) | A_i \in \mathcal{R}, A \subseteq \bigcup_{i=1}^{\infty} A_i, \text{ for } i \neq j, A_i \cap A_j = \emptyset\}. \tag{4.3}$$

In our case, as $\mathcal{R}$ is a ring generated by semi-closed intervals, the generated σ-ring equals $\mathcal{B}(\mathbb{R}^n)$.

Theorem 4.1.3. *Let $F : \mathbb{R}^n \to E$ be an n-spectral resolution with values in an interval effect algebra $E = \Gamma(G, u)$, where G is Dedekind σ-complete with strong unit u. Then there is a unique mapping $x_0 : \mathcal{R} \to E$ such that:*

(i) $x_0([a_1, b_1) \times \cdots \times [a_n, b_n)) = \Delta_1(a_1, b_1) \cdots \Delta_n(a_n, b_n)F$,

(ii) for each finite collection of disjoint sets $A_1, \dots, A_m \in \mathcal{R}$ *we have*

$$x_0(\cup_{i=1}^m A_i) = \sum_{i=1}^m x_0(A_i).$$

Moreover, if E *has a faithful* σ*-state* s*, we have*

(iii) if $A_i \searrow \emptyset$ *is a sequence of elements of* $\mathcal{R}$*, then* $\bigwedge_i x_0(A_i) = 0$,

(iv) Let A_i*,* $i \in \mathbb{N}$*, be a collection of disjoint Borel sets such that* $\bigcup_i A_i = A \in \mathcal{R}$*. Then* $\sum_i x_0(A_i) = x_0(A)$.

Proof. First note, that every $A \in \mathcal{R}$ is a disjoint union of semi-closed intervals (from the generating collection). Hence we can uniquely define x_0 following the conditions (i–ii).

We first prove, that x_0 is well defined mapping. Assume $A = \bigcup_i A_i = \bigcup_j A'_j$ are two disjoint unions of semi-closed intervals. We obtain a third disjoint union as

$$A = \bigcup_{i,j} A_i \cap A'_j.$$

Hence, we only need to prove

$$x_0(A_i) = \sum_j x_0(A_i \cap A'_j) \text{ and } x_0(A'_j) = \sum_i x_0(A'_j \cap A_i). \tag{4.4}$$

That is, we have to show x_0 is additive on the semi-closed intervals. If a decomposition of a semi-closed interval $B = \cup_i B_i$ arises by chopping B in a collection of hyperplanes, the additivity is obvious. Otherwise, we take a further refinement of the decomposition, which decomposes each B_i in a convenient way.

Next, we prove the condition (iii). Let us begin with a useful Claim

Claim: Given a set $A \in \mathcal{R}$ and $\epsilon > 0$, there are $C \in \mathcal{R}$ and a compact $K \in \mathcal{B}(\mathbb{R}^n)$, such that $C \subset K \subset A$ and

$$(s \circ x_0)(A \setminus C) < \epsilon.$$

Proof. It is clearly enough to verify the claim for A a semi-closed interval. So assume $A = [a_1, b_1) \times \cdots \times [a_n, b_n)$ and for each $i \in \mathbb{N}$ denote $C_i = [a_1, b_1 - \frac{1}{i}) \times \cdots \times [a_n, b_n - \frac{1}{i})$. We can apply Lemma 3.2.7 to assure $x_0(C_i) \nearrow x_0(A)$. As s is σ-state, we can take for C some C_i, with i great enough, and for K the interval $[a_1, b_1 - \frac{1}{2i}] \times \cdots \times [a_n, b_n - \frac{1}{2i}]$. □

Given a sequence $A_i \searrow \emptyset$ of elements of $\mathcal{R}$, the claim entails the elements $C_i \in \mathcal{R}$ and compact sets K_i, $i \in \mathbb{N}$, such that $C_i \subset K_i \subset A_i$ and

$$s \circ x_0(A_i \setminus C_i) < \frac{\epsilon}{2^i}.$$

Now

$$\begin{aligned}(s \circ x_0)(\bigcap_{l=1}^{i} A_l \setminus \bigcap_{l=1}^{i} C_l) &\leq (s \circ x_0)(\bigcup_{l=1}^{i}(A_l \setminus C_l)) \\ &\leq \sum_{l=1}^{i}(s \circ x_0)(A_l \setminus C_l) \leq \sum_{l=1}^{i} \frac{1}{2^i}\epsilon \leq \epsilon.\end{aligned} \tag{4.5}$$

Since A_i's have empty intersection, so do K_i's. Using compactness of K_i's, we obtain some $i_0 \in \mathbb{N}$ so that $\bigcap_{l=1}^{i} K_i = \emptyset$ for each $i \geq i_0$, and so $\bigcap_{l=1}^{i} C_i = \emptyset$, for each $i \geq i_0$, as well. Equation (4.5) for $i \geq i_0$ gives us

$$(s \circ x_0)(A_i) = (s \circ x_0)(\bigcap_{l=1}^{i} A_l) < \epsilon.$$

And as ϵ was arbitrary, there is $(s \circ x_0)(A_i) \searrow 0$. Consequently $a = \wedge_i x_0(A_i)$ satisfies $s(a) = 0$ and so $a = 0$ by faithfulness of s.

The last point (iv) is a consequence of (iii). Clearly, $\sum_i x_0(A_i) \leq x_0(A)$, and $x_0(A) - \sum_i x_0(A_i) \leq x_0(A \setminus \bigcup_{i \in I} A_i)$, for each finite $I \subset \mathbb{N}$. As I goes to $\mathbb{N}$ the $x_0(A \setminus \bigcup_{i \in I} A_i)$ goes to 0, by the point (iii). □

Now we will prove the main theorem of the chapter.

Theorem 4.1.4. *Let $E = \Gamma(G, u)$ be a monotone σ-complete interval effect algebra and s a faithful σ-state on E. Then each n-spectral resolution extends to an n-observable.*

Proof. Let us denote $\mu = s \circ x_0 : \mathcal{R} \to [0,1] \subset \mathbb{R}$, where x_0 is given by Theorem 4.1.3. We first prove, that μ is σ-additive. Assume a disjoint union $B = \bigcup_{i=1}^{\infty} B_i$, in the ring $\mathcal{R}$. Define $A_i := B \setminus \bigcup_{l=1}^{i} B_l$. The system of A_i's has empty intersection and so by Theorem 4.1.3, part (iii), $\mu(A_i) \searrow 0$, equivalently $\mu(\bigcup_{l=1}^{i} B_l) \nearrow \mu(B)$.

Now we can apply Theorem 4.1.2 to obtain an extension $\bar{\mu} : \mathcal{B}(\mathbb{R}) \to [0,1]$. Given any $A \in \mathcal{B}(\mathbb{R}^n)$, we like to define $x(A) \in E$. To do so, take for each $j \in \mathbb{N}$ a disjoint cover $A \subseteq \bigcup_{i=1}^{\infty} A_i^j$ of A by sets in $\mathcal{R}$, and such that

$$\sum_i \mu(A_i^j) \searrow \bar{\mu}(A),$$

as $j \to \infty$, which is possible by equation (4.3). Let us define

$$x(A) := \bigwedge_j \sum_i x_0(A_i^j). \tag{4.6}$$

We will verify the just defined x satisfies all conditions of observable. We begin by proving two Claims first:

Claim 1: Given two disjoint systems $(A_i | i \in \mathbb{N})$ and $(B_i | i \in \mathbb{N})$, of sets in the ring $\mathcal{R}$. If $\bigcup_i A_i \subseteq \bigcup_i B_i$, then $\sum_i x_0(A_i) \leq \sum_i x_0(B_i)$.

Proof. By definition of the infinite sum, it is enough to prove $\sum_{i \in I} x_0(A_i) \leq \sum_i x_0(B_i)$ for each finite $I \subset \mathbb{N}$. And as x_0 is additive, we can without loss of generality assume I is one-element set. Hence assume $A \subseteq \bigcup_i B_i$. By σ-additivity of x_0, we have $x_0(A) = \sum_i x_0(A \cap B_i) \leq \sum_i (B_i)$. $\square$

Claim 2: Given two systems $(A_i | i \in \mathbb{N})$ and $(B_i | i \in \mathbb{N})$, of disjoint sets in the ring $\mathcal{R}$, such that $A \subset \bigcup_i A_i$ and $A \subset \bigcup_i B_i$. Then there is a third system $(C_i | i \in \mathbb{N})$, such that

$$A \subseteq \bigcup_i C_i \subseteq (\bigcup_i A_i) \cap (\bigcup_i B_i).$$

Proof. We take C_i's such that $\{C_i | i \in \mathbb{N}\} = \{A_i \cap B_j | i, j \in \mathbb{N}\}$. $\square$

We assert the value $x(A)$ defined by (4.6) is independent on choice of the sets $(A_i^j | i \in \mathbb{N})$'s. Assume another systems $(B_i^j | i \in \mathbb{N})$, $j \in \mathbb{N}$, having the same properties as A_i^j's. Then applying Claim 2 we yield is a third collection of covering systems $(C_i^j | i \in \mathbb{N})$, $j \in \mathbb{N}$, such that for each $j \in \mathbb{N}$,

$$A \subseteq \bigcup_i C_i^j \subseteq (\bigcup_i A_i^j) \cap (\bigcup_i B_i^j).$$

Moreover, we can by inductive use of Claim 2, assume $\bigcup_i C_i^{j+1} \subseteq \bigcup_i C_i^j$. So by Claim 1: $c := \bigwedge_j \sum_i x_0(C_i^j)$ is below both $a := \bigwedge_j \sum_i x_0(A_i^j)$ and $b := \bigwedge_j \sum_i x_0(B_i^j)$, but $s(a) = s(c) = s(b)$, so $a = c = b$, by faithfulness of s.

Observe that the constructed mapping x extends x_0, as each $A \in \mathcal{R}$ obtains a trivial cover $A \subseteq A \cup \bigcup_{i=1}^{\infty} \emptyset$, which leads by Claim 1 to $x(A) = x_0(A)$. Moreover, x is monotone, that follows easily from definition of x.

Next we prove additivity. Given two disjoint sets $A, B \in \mathcal{B}(\mathbb{R}^n)$, we need to prove $x(A \cup B) = x(A) + x(B)$. Choose systems $(A_i^j | i \in \mathbb{N})$, $(B_i^j | i \in \mathbb{N})$, $j \in \mathbb{N}$, of disjoint covers of A and B, that is $A \subseteq \bigcup_i A_i^j$ and $B \subseteq \bigcup_i B_i^j$ for each $j \in \mathbb{N}$, and such that $x(A) = \bigwedge_j \sum_i x_0(A_i^j)$, $x(B) = \bigwedge_j \sum_i x_0(B_i^j)$. Moreover, denote $A^j := \bigcup_i A_i^j$, $B^j := \bigcup_i B_i^j$ and assume (using Claim 1) the two systems are monotone in j (i.e., $A^{j+1} \subseteq A^j$ and $B^{j+1} \subseteq B^j$, for each $j \in \mathbb{N}$).

We will construct inductively a disjoint covers $A \cup B \subseteq \bigcup_i C_i^j$ and $\emptyset = A \cap B \subseteq \bigcup_i D_i^j$, as follows. For each $j \in \mathbb{N}$, we set $C_1^j := A_1^j$ and $D_1^j = \emptyset$, $C_2^j := B_1^j \setminus A_1^j$ and $D_2^j := B_1^j \cap A_1^j$, $C_3^j := A_2^j \setminus B_1^j$ and $D_3^j := A_2^j \cap B_1^j$, and so on. In $(2k-1)$-th step, $k \in \mathbb{N}$, we set $C_{2k-1}^j := A_k^j \setminus (\bigcup_{i<k} B_i^j)$ and $D_k^j = A_k^j \cap \bigcup_{i<k} B_i^j$, while in $2k$-th step, $k \in \mathbb{N}$, we set $C_{2k}^j := B_k^j \setminus (\bigcup_{i\leq k} A_i^j)$ and $D_k^j = B_k^j \cap \bigcup_{i\leq k} A_i^j$. Observe that $C^j := \bigcup_i C_i^j = A^j \cup B^j$ and $D^j := \bigcup_i D_i^j = \bigcup_{i,k} A_i^j \cap B_k^j = A^j \cap B^j$, hence in particular $\bigcup_i C_i^{j+1} \subseteq \bigcup_i C_i^j$, for each $j \in \mathbb{N}$, and so do D_i^j's.

Next we define for each $j \in \mathbb{N}$ a 2×2 table $(x_{i,j})_{i,j=1,2}$ of elements of E. We set

$$x_{1,1}^j = \sum_k x_0(C_{2k-1}^j), \tag{4.7}$$

$$x_{1,2}^j = \sum_k x_0(D_{2k-1}^j), \tag{4.8}$$

$$x_{2,1}^j = \sum_k x_0(C_{2k}^j), \tag{4.9}$$

$$x_{2,2}^j = \sum_k x_0(D_{2k}^j). \tag{4.10}$$

The sums of the rows are known to converge to the values of $x(A)$ and $x(B)$:

$$\bigwedge_j (x^j_{1,1} + x^j_{1,2}) = x(A), \tag{4.11}$$

$$\bigwedge_j (x^j_{2,1} + x^j_{2,2}) = x(B). \tag{4.12}$$

While the sums of the columns satisfy

$$\bigvee_j (x^j_{1,1} + x^j_{2,1}) \geq x(A \cup B) \tag{4.13}$$

$$\bigwedge_j (x^j_{2,1} + x^j_{2,2}) = 0. \tag{4.14}$$

The second equality follows from

$$\begin{aligned}\mu(A \cup B) \leq s(x^j_{1,1}) + s(x^j_{2,1}) \leq (s(x^j_{1,1}) + s(x^j_{2,1})) + (s(x^j_{1,2}) + s(x^j_{2,2})) =\\ =(s(x^j_{1,1}) + s(x^j_{1,2})) + (s(x^j_{2,1}) + s(x^j_{2,2})) = \bar{\mu}(A^j) + \bar{\mu}(B^j) \searrow \mu(A) + \mu(B).\end{aligned}$$

Hence, by additivity of μ we have

$$0 = \bigwedge_j [s(x^j_{1,2}) + s(x^j_{2,2})] = \bigwedge_j s(x^j_{1,2} + x^j_{2,2}) = s(\bigwedge_j (x^j_{1,2} + x^j_{2,2}).$$

Where we are using the σ-additivity of s in the last step. And as s is faithful, the desired equality follows.

Finally, we calculate

$$\begin{aligned} x(A) + x(B) &= \bigwedge_j (x^j_{1,1} + x^j_{1,2}) + \bigwedge_j (x^j_{2,1} + x^j_{2,2}) \\ &= \bigwedge_j (x^j_{1,1} + x^j_{1,2} + x^j_{2,1} + x^j_{2,2}) = \\ &= \bigwedge_j (x^j_{1,1} + x^j_{2,1}) + \bigwedge_j (x^j_{1,2} + x^j_{2,2}) \geq x(A \cup B). \end{aligned}$$

When we apply the state s, we obtain $s(x(A) + x(B) - x(A \cup B)) = 0$, which leads to the desired addititvity of x by faithfulness of s.

The σ-additivity of x is now easy to prove: Given monotone sequence $A_i \nearrow A$ of Borel sets. By monotony of x, we have $\bigvee_i x(A_i) \leq x(A)$. But $s(\bigvee_i x(A_i)) = s(x(A))$, hence the σ-additivity follows by the argument with faithfulness of s again. $\square$

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Observables on Quantum Structures

Annotation on PhD Thesis

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The thesis is available at the Student Affairs Office of the Faculty of Science, Palacký University Olomouc at the address 17. listopadu 12, 771 46 Olomouc, The Czech Republic. It is also possible to download the thesis at the website stag.upol.cz. The defence will take place .... at the Department of Algebra and Geometry, Faculty of Science, Palacký University Olomouc.

Abstract: In the PhD thesis, a one-to-one correspondence between finite-dimensional spectral resolutions and observables is established for various classes of algebras known as Quantum structures. The main result treats the case of monotone σ-complete effect algebras with Riesz Decomposition Property. The results are achieved using a technique of lifting spectral resolutions, which is presented and which is interesting on its own. Further, the effect of the lexicographic product on the correspondence in concern is investigated. As another main result, a description of n-spectral resolutions, which extend to observables for certain types of lexicographic effect algebras is given. In addition, a classical approach to measure extension (via outer measures) is used to provide a construction of n-observables (for a given n-spectral resolution) for interval effect algebras with faithful σ-state.

Keywords: effect algebras, observables, spectral resolutions, partially ordered groups, measure extension

Language: English

Contents

1 Introduction

The PhD thesis concerns the question for which algebras related to the logic of quantum mechanics there is a one-to-one correspondence between observables and spectral resolutions. Observables are by definition certain σ-homomorphisms from Borel σ-algebra $\mathcal{B}(\mathbb{R})$ to a quantum structure $\mathcal{L}$ (typically an effect algebra) and each observable gives rise to a spectral resolution as its distributive function. The hardest part of the problem in question is to find some conditions on $\mathcal{L}$, such that the spectral resolutions (considered as an independent concept - certain mappings $\mathcal{B}(\mathbb{R}) \to \mathcal{L}$) uniquely extend to observables.

The problem is an abstraction of the well-known result in the classical probability theory, where the Borel probabilistic measures are in one-to-one correspondence with their distribution functions. The reconstruction of a measure is possible (in the classical case) by the Carathéodory's extension theorem, which states that each σ-additive measure on a ring of sets has a unique extension to a σ-additive measure on the generated σ-algebra. In our case, we take for the ring the one generated is by all half-open intervals $[t, s)$, $t, s \in \mathbb{R}$ (each spectral resolution naturally defines such sub-additive measure). In the literature, looking for the extended measure is known as *the measure extension problem*, and when the answer is positive, we say that *measure extension property* (MEP) holds.

As time passes the problem of measure extension was considered for measures with values in more general algebraic structures. R. Sikorski in [Sik69] threats the case of Boolean algebras – he showed there that the so-called weak

σ-distributivity condition on the Boolean algebra has a key role. Sikorski provided proof (which arises by communication with him and K. Matthes) that the weak σ-distributivity is a sufficient condition for (MEP). On the other hand, J. D. Wright in 1971 (see [Wri71]) proved, that the weak σ-distributivity is a necessary (and so equivalent) condition for the measure extension (to be precise, Sikorski distinguished strong and weak σ-MEP, where the weak one is equivalent to weak σ-distributivity). Wright dealt with the problem on a more general level of lattice ordered vector spaces. These results were later reproved by Fremlin in [Fre75] in a simpler way. Fremlin's proof is elementary in the sense that no representation theorems are used and the proof utilizes only the countable axiom of choice. The proof is proceeded inside the algebraic structure of Riesz spaces and could be interpreted as a tricky simulation of the classical ϵ, δ calculus using weak-σ-distributivity.

Fremlin's approach was then adopted by B. Riečan to measure extension construction on MV-algebras (l-groups) provided in [RT97], where probability theory for MV-algebras systematically builds. Riečan also has some partial results for non-lattice ordered effect algebras in [Rie98], but a full measure extension construction is given (in the cited article) only for the σ-complete MV-algebras. Hence, it seems to be the case, that Fremlin's technique is limited by a lattice structure.

In contrast to the mentioned results, in the thesis, we will consider the measure extension problem only for measures based on the Borel subsets of $\mathbb{R}^n$. While for the range structure we will take some quantum structure, typically an effect algebra with (RDP), hence a more general structure than an MV-algebra in the work of Riečan.

An important moment in the research of the algebraic quantum logic was an observation, that most of the important quantum structures are representable as intervals in the partially ordered Abelian groups (as is argued by Foulis and Greechie in [GF95]). This observation led to a bridge between quantum logic and the well-developed theory of po-groups. Two important representation theorems, which have prime importance in the PhD thesis, were achieved thanks to this bridge: Each effect algebra with Riesz Decomposition Property is representable as an interval in a po-group satisfying interpolation property (in fact there is a categorical equivalence between the category of effect algebras with (RDP) and the category of unital Abelian po-groups with interpolation - a result of K. Ravindran [Rav96]). The second important theorem is a kind of Loomis–Sikorski theorem: Each monotone σ-complete effect algebra with (RDP) can be represented as a σ-homomorphic image of so-called effect tribe of fuzzy-sets (proved in [BCD06]).

The second mentioned theorem is the main tool for applications of the lifting technique presented in the PhD thesis: extending spectral resolution F on an effect algebra E for which we have Loomis–Sikorski representation $\pi : \mathcal{T} \to E$ proceeds in three steps. First, we lift the spectral resolution to a spectral resolution $\hat{F}$ on $\mathcal{T}$, the lifted spectral resolution can be using some standard results from the probability theory extended to an observable $\hat{x}$, which gives the desired observable X by composing with π. Just described technique was used in [DK14] for E being a monotone σ-complete effect algebras with (RDP). Note that this case has not been approached by Fremlin's technique.

In the PhD thesis, after introducing basic concepts in the Chapter 2, there are provided generalisations of the results of Dvurečenskij and Kuková in several directions:

(I) generalisation to finite-dimensional observables (i.e., these having as domain $\mathcal{B}(\mathbb{R}^n), n \in \mathbb{N}$),

(II) weakening the monotone completeness, by considering lexicographic interval effect algebras $\Gamma(H \vec{\times} G, (u, 0))$, where (G, u) is a Dedekind σ-complete po-group with interpolation and (H, u) is any unital (Abelian) po-group,

(III) combination of (I–II).

In the first generalization, described in Chapter 2, the lifting process become much more complicated, in contrast to the one-dimensional case, where the lifting is a rather simple part. The lifting of finite-dimensional resolutions is presented in the general situation of a σ-projection $\pi : (H, v) \to (G, u)$ of monotone σ-complete Abelian po-groups. It turns out, that so-called *lifting property* (certain strengthening of surjectivity which holds in the cases of Loomis-Sikorski theorems) is a necessary and sufficient condition for the lifting process. The lifting is achieved by iterating the inductive process and, as one might expect, it strongly utilizes (a countable version of) the axiom of choice. This part of the PhD thesis covers the results from [DL20d] and [DL20a].

In the second and the third generalization, described in Chapter 4, only spectral resolutions satisfying certain additional properties extend to observables. The generalizations are characteristic by a need to refine the arguments of most of the proofs. The main result considering a finite-dimensional observable on a lexicographic effect algebra is reached through many technical lemmas. This part generalises the results form [DL20b],[DL19],[DL19] and [DL21].

In the last Chapter 5, a classical approach to measure extension (via outer measures) is exhibited in the case of monotone σ-complete interval effect algebras having faithful σ-state. The assumption of the existence of faithful σ-state is a strong one, for example, each monotone σ-complete effect algebra with (RDP) is a lattice, whenever obtains such a state. Nevertheless, in Chapter 5, the property of (RDP) is not assumed, hence the main result is not covered by the ones from the previous chapters achieved by the lifting procedure.

2 The aim of the PhD thesis

A general aim of the PhD thesis is to establish a one-to-one correspondence between (finite-dimensional) observables and spectral resolutions for as many quantum structures as possible. In more detail, the original motivation was to develop methods from [DK14] to finite-dimensional cases and to study the effect of the lexicographic product on the correspondence in question.

3 Methods

The approach to the problem is based on and limited by several representations theorems: the ones representing effect algebras as intervals of partially ordered groups and Loomis-Sikorski-like representations of certain effect algebras as a σ-projections of tribes of fuzzy sets. Moreover, some well-known results from the measure theory are used.

4 Main results

The main results are as follows

1. A one-to-one correspondence between finite-dimensional observables and spectral resolutions is established for the monotone σ-complete effect algebras with Riesz Decomposition Property.

2. Given a σ-projection $\pi : T \to E$ of monotone σ-complete interval effect algebras satisfying so-called lifting property, a process of lifting the finite-dimensional spectral resolutions of E to the ones of T is described.

3. For interval effect algebra $\Gamma(H \vec{\times} G, (u, 0))$, where (H, u) is an unital po-group and G is a monotone σ-complete po-group with interpolation, a classification of the spectral resolutions which extend to observables is given.

4. The classical approach to the measure extension (via outer measure) is applied to establish the correspondence in question in the case of a monotone σ-complete interval effect algebras with a faithful σ-state.

5 Basic definitions

The basic concept in the thesis is the effect algebras:

Definition 5.1. *We call effect algebra a partial algebra* $(E; +, ', 0, 1)$ *of type* $(0, 0, 1, 2)$*, such that for each* $a, b, c \in E$

1. *a+b=b+a,*

2. *(a+b)+c=a+(b+c),*

3. a' *is the unique element such that* $a + a' = 1$*,*

4. *if* $a + 1$ *is defined then* $a = 0$*.*

Where we read the first two identities as when one of the sides is defined then the other is defined as well and equality holds.

Moreover, we define a partial ordering on E *as* $a \leq b$ *iff there is* $c \in E$ *such that* $a + c = b$*. Then the constant* 0 *and* 1 *are the lowest and the greatest element, respectively.*

From the logico-physical perspective, we think of elements of statements about a physical system of form: during measurement x we observe the value was in Borel set B.

Let us list several important algebras, that could be arranged as effect algebras:

1. Boolean algebras, where we take for $+$ the union operation restricted to disjoint elements.

2. More generally MV-algebras $(M, \oplus, \neg)$, where we take for $+$ restriction of $\oplus$ to pairs a, b with $a \leq \neg b$.

3. Even more generally, every interval $[0, u]$ in partially ordered Abelian group $(G, +_G, \leq)$, (of course $0 < u$), where we define complement as $a' = u - a$ and for $+$ we take the restriction of $+_G$.

4. Given a set Ω, every system of functions in $\mathcal{T} \subseteq [0,1]^\Omega$ is an effect algebra if $\mathcal{T}$ (i) contains 1_Ω, (ii) is us closed under pairwise addition (whenever the result is below 1_G) and (iii) is closed under complement operation $f \mapsto 1_\Omega - f$. When $\mathcal{T}$ is also closed under pairwise monotone countable suprema, it is called an *effect-tribe*.

5. Classical model of sharp quantum logic $\Pi(\mathcal{H})$: which consists of all projections on closed subspaces of given Hilbert space $\mathcal{H}$.

6. Classical model of unsharp quantum logic: $\mathcal{E}(H)$ of all Hermitian operators on Hilbert space H between zero operators and the identity operator.

The interval effect algebra mentioned in third point is denoted $\Gamma(G, u)$ and these are the most important ones in the PhD thesis. Following notation is important in the PhD thesis: unital po-groups is an ordered pair (G, u) of (Abelian) po-group G and and element $u \in G^+$, so called stron unit, such that $\{n \cdot u | n \in \mathbb{N}\}$ dominates whole G. A po-group is said have interpolation, if given any four elements $a, b, c, d \in G$ such that $\{a, b\} \leq \{c, d\}$, there is $e \in G$, satisfying $a, b \leq e \leq c, d$. An analogue property for effect algebras is Riesz decomposition property (RDP), which requires on an effect algebra E: given $a, b_1, 2 \in E$, such that $a \leq b_1 + b_2$, there is decomposition $a = a_1 + a_2$, such that $a_i \leq b_i$, $i = 1, 2$.

Following two results on representations of effect algebras with (RDP) are of prime importance in the PhD thesis:

Theorem 5.2 (proved in [Rav96])**.** *Let E be an effect algebra satisfying (RDP). Then there is a unital po-group (G, u) with interpolation, such that $E \cong \Gamma(G, u)$.*

Theorem 5.3 (proved in [BCD06])**.** *Let E be a monotone σ-complete effect algebra. Then there is a set Ω, an effect tribe $\mathcal{T} \subset [0,1]^\Omega$ and a σ-homomorphism $\pi : \mathcal{T} \to E$, which is onto.*

Now we define the crucial notions of the PhD thesis – n-dimensional spectral resolutions and n-dimensional observables defined on monotone σ-complete effect algebras and σ-complete MV-algebras.

We will use two kinds of orderings on n-tuples of reals:

$$(t_1,\dots,t_n) < (s_1,\dots,s_n) \iff \textit{for each } i, t_i \le s_i \text{ and for some } i, t_i < s_i.$$
$$(t_1,\dots,t_n) \ll (s_1,\dots,s_n) \iff \textit{for each } i, t_i < s_i$$

Definition 5.4. *Let E be a σ-complete effect algebra. Then we call n-dimensional observable any σ-homomorphism $x : \mathbb{R}^n \to E$, that is a mapping satisfying:*

(i) $x(\mathbb{R}^n) = 1$,

(ii) $x(A \cup B) = x(A) + x(B)$ *whenever* $A \cap B = \emptyset$,

(iii) $\{A_i\}_i \nearrow A$ *implies* $\bigvee_i x(A_i) = x(A)$.

If, given an n-dimensional observable x on $E = \Gamma_a^e(G,u)$, it gives arise to its distributive function: $F_x : \mathbb{R}^n \to \Gamma_a^e(G,u)$ by

$$F_x(s_1,\dots,s_n) = x((-\infty,s_1) \times \cdots \times (-\infty,s_n)), \quad (s_1,\dots,s_n) \in \mathbb{R}^n.$$

We call F_x an n-dimensional spectral resolution of x. We are concerned with the opposite process: finding an n-observable for a given n-spectral resolutions. We will treat n-dimensional spectral resolutions as an independent concept given by Definition 5.5. In the definition, the most intricate condition to handle with, is the last stated - so called *volume condition*. Volume conditions basically assure that the n-spectral resolution prescribes non-negative volume to certain half-open intervals in $\mathbb{R}^n$. The volume conditions are handled by following notation: Let $F : \mathbb{R}^n \to E$ be any mapping (but usually pseudo m-spectral resolution), $i = 1,\dots,n$, and $a, b \in \mathbb{R}$, such that s $a \le b$. Then we define a mapping $\Delta_i(a,b)F : \mathbb{R}^n \to E$ given by prescription

$$(t_1,\dots,t_{i-1},t_{i+1},\dots,t_n) \mapsto F(t_1,\dots,b,\dots,t_n) - F(t_1,\dots,a,\dots,t_n). \quad (1)$$

Definition 5.5. Let (G,u) be a Dedekind monotone σ-complete po-group and let $n \ge 1$ be an integer. An *n-dimensional spectral resolution* on $E = \Gamma_a^e(G,u)$ is any mapping $F : \mathbb{R}^n \to \Gamma_a^e(G,u)$ such that

$$F(s_1,\dots,s_n) \le F(t_1,\dots,t_n) \quad \text{if} \quad (s_1,\dots,s_n) \le (t_1,\dots,t_n), \quad (2)$$

$$\bigvee_{(s_1,\dots,s_n)} F(s_1,\dots,s_n) = u, \quad (3)$$

$$\bigvee_{(s_1,\dots,s_n) \ll (t_1,\dots,t_n)} F(s_1,\dots,s_n) = F(t_1,\dots,t_n), \quad (4)$$

$$\bigwedge_{t_i} F(s_1,\dots,s_{i-1},t_i,s_{i+1},\dots,s_n) = 0 \text{ for } i = 1,\dots,n, \quad (5)$$

$$\Delta_1(a_1,b_1)\cdots\Delta_n(a_n,b_n)F \ge 0, \ \textit{for each } a_i, b_i \in \mathbb{R}, a_i \le b_i, i = 1,\dots,n. \quad (6)$$

6 Lifting of n-dimensional spectral resolutions

Suppose $\pi : F \to E$ is a σ-surjection of monotone σ-complete effect algebras. If we are able to solve the spectral resolution extension problem for F, we may try to extend the result to E by lifting the spectral resolutions. In more detail: suppose F is a spectral resolution on E, find a spectral resolution $\mathbf{F}$ on F, such that $\pi \circ \mathbf{F} = F$. Then extend $\mathbf{F}$ to an observable $\mathbf{x}$ and prove, that $\pi \circ \mathbf{x}$ is an observable extending F.

This technique applies to the effect algebras, for which variant of Loomis-Sikorsky theorem holds. In the case of ordinary (one-dimensional) spectral resolutions, the lifting part is rather easy compared to the other steps. However, assuming the general situation of n-dimensional spectral resolution, lifting becomes the most difficult part.

The main result in the PhD thesis concerning lifting finite-dimensional spectral is Theorem 6.1. The lifting procedure proceeds by lifting a spectral resolution $F : \mathbb{R}^n \to E$ first in the lattice of integral points $\mathbb{Z}^n \subset \mathbb{R}^n$, then the definition domain of the partial lift is inductively refined to the subsets $\frac{1}{2^n} \cdot \mathbb{Z}$, $n \in \mathbb{N}$ (the induction is over n). As a result, we obtain a partial lift defined on a dense subset of $\mathbb{R}^n$. The final desired lift is obtained by extending the previous one in a way that follows the condition of left continuity. Each step utilises some inductive process and is rather complicated. Even finding a lift only in the vertices of a unit cube in $\mathbb{R}^n$ is non-trivial () as one needs to control all volume conditions (2^n inequalities of form "$0 \le$ some expression ≤ 1") when lifting. In the construction, we also strongly use the axiom of choice, even if only a countable version.

Theorem 6.1. *[Lifting of Spectral Resolutions] Let $\pi : (G, u) \to (H, v)$ be a σ-homomorphism of unital Dedekind monotone σ-complete po-groups and let π satisfy (LP). Then each n-dimensional spectral resolution $F : \mathbb{R}^n \to H$ can be lifted to an n-dimensional spectral resolution $K : \mathbb{R}^n \to G$ such that $\pi \circ K = F$.*

7 Extending spectral resolutions

The results in lifting are then applied to several classes of effect algebras, for which a variant of Loomis-Sikorski theorem is known, to establish a one-to-one correspondence between n-spectral resolutions and n-observables. The most general theorem is:

Theorem 7.1. *Let E, T be a pair of monotone σ-complete effect algebras which could be represented as intervals of po-groups $E \cong \Gamma(H, v)$, $T \cong \Gamma(H, v)$ and let $\pi : T \to E$ be a σ-homomorphism of effect algebras satisfying lifting property. Then E has SPO_n correspondence whenever T does.*

Lemma 7.2. *Let $\mathcal{T}$ be an effect-tribe. Then each spectral resolution $F : \mathcal{R}^n \to \mathcal{T}$ extends to a unique observable.*

The previous lemma together with Loomis-Sikorski theorem and Theorem 7.1 give us:

Theorem 7.3. *Given monotonous σ-complete effect algebra E with (RDP), then one-to-one correspondence between n-spectral resolutions and n-observables holds.*

8 Spectral resolution on lexicographic effect algebras

Establishing a one-to-one correspondence for a number of effect algebras a natural question arises: under which constructions the class of algebras satisfying this correspondence (which we denote $\mathcal{SPO}$) is closed. The PhD thesis concerns the effect of the lexicographic product. Assume unital po-groups (H, u), directed monotone σ-complete po-group G with interpolation and define an effect algebra

$$E := \Gamma(H \overrightarrow{\times} G, (u, 0)).$$

Now E is not monotone σ-complete (in general), only its radical is so. This leads to some pathological cases of spectral resolutions which do not extend to observables. However, the set of spectral resolutions which do extend to observables is easy to describe by the so-called finiteness property.

The problem in concern is treated in several papers of Dvurečenskij and various collaborators. Firstly, the simplest case of perfect MV-algebras $M = \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0)$, where G is σ-complete l-group) is solved in [DDL19]. Then generalisation to k-perfect case of effect algebras with (RDP) is done in [DL20b]. Finally, the problem in the general setting of lexicographic effect algebras is treated in [DL19] and [DL21] where the dimension is assumed to be one and any finite (respectively). In the PhD thesis, the one-dimensional case is considered first, and then the general situation of any finite dimension.

Theorem 8.1. *Let G be a directed monotone σ-complete po-group and (H, u) be a unital po-group. Let x be an observable on $E = \Gamma(H \overrightarrow{\times} G, (u, 0))$. Define for each $t \in \mathbb{R}$*

$$x_t := x((-\infty, t)). \tag{7}$$

We have for each $s \in \mathbb{R}$

$$x_t \leq x_s \text{ if } t \leq s, \tag{8}$$

$$\bigwedge_t x_t = 0, \quad \bigvee_t x_t = 1, \tag{9}$$

$$\bigvee_{t<s} x_t = x_s. \tag{10}$$

There is a finite sequence $0 = h_0 < h_1 < \cdots < h_n = u$ of elements of $[0, u]_H$ and real numbers $s_0 = t_1 < \cdots < t_n = t_u$ such that

$$x_t \in \begin{cases} E_{h_0} & \text{if } t \leq t_1, \\ E_{h_i} & \text{if } t \in (t_i, t_{i+1}],\ i = 1, \ldots, n-1, \\ E_{h_n} & \text{if } t_n < t. \end{cases} \tag{11}$$

In addition, for each $i = 1, \dots, n$, the element

$$a_i := \bigwedge_{t_i < t} x_t \tag{12}$$

exists in E and it belongs to E_{h_i}.

Moreover, for each $s \in \mathbb{R}$, we have $x(\{s\}) = \bigwedge_{t>s}(x_t - x_s)$.

The opposite direction is treated in theorem:

Theorem 8.2. *Let $E = \Gamma(H \overrightarrow{\times} G, (u, 0))$, where (H, u) is a unital po-group and G is a monotone σ-complete po-group with interpolation and with strong unit. Let $\{x_t \in E \colon t \in \mathbb{R}\}$ be a system of elements of E and $h_0 < h_1 < \cdots < h_n$ be elements of $[0, u]_H$ and real numbers $s_0 = t_1 < \cdots < t_n$, such that conditions (8)–(12) are satisfied. Then there is a unique observable x on E such that $x_t = x((-\infty, t))$ for each $t \in \mathbb{R}$.*

The n-dimensional case, for $n \geq 2$, is done through several technical lemmas. The main results are:

Theorem 8.3. *Let x be an observable on $\Gamma(H \overrightarrow{\times} G, (u, 0))$. Then $\mathrm{Im}(\pi_H \circ x)$ meets only finitely many elements of H.*

A notation of characteristic points of spectral resolution is introduced. Characteristic point of $F : \mathbb{R}^n \to \Gamma(H \vec{\times} G, (u, 0))$ associated to an element $h \in H$ is an n-tuple $\mathbf{t} \in \mathbb{R}^{\mathbf{n}}$ which is roughly speaking a locally minimal over the points in $F^{-1}(\{h\} \times G)$. In the PhD thesis, a possible decomposition of $\mathbb{R}^n$ which arise as a union $\sum_{h \in H} (\pi_H \circ F)^{-1}(h)$ are investigated. The participants of the decomposition are called blocks. The following theorem provides an inside into the decomposition into blocks (see also figure 1 which illustrates possible block configuration in the two-dimensional case).

Theorem 8.4. *Let F be n-spectral resolution. For each characteristic point $\mathbf{t}$ define $C_{\mathbf{t}} = \{\mathbf{s} \in \mathbb{R}^n | \mathbf{t} \leq \mathbf{s}$ and $\mathbf{t} \not\ll \mathbf{s}\}$. Each $C_{\mathbf{t}}$ cut $\mathbb{R}^n$ into two disjoint components as*

$$\mathbf{R}^n = \{\mathbf{s} | \mathbf{t} \not\ll \mathbf{s}\} \cup \{\mathbf{s} | \mathbf{t} \ll \mathbf{s}\}. \tag{13}$$

The joint cutting of $\mathbb{R}^n$ along all the $C_{\mathbf{t}}$'s refines the decomposition $\mathbb{R}^n = \bigcup_h B_h$.

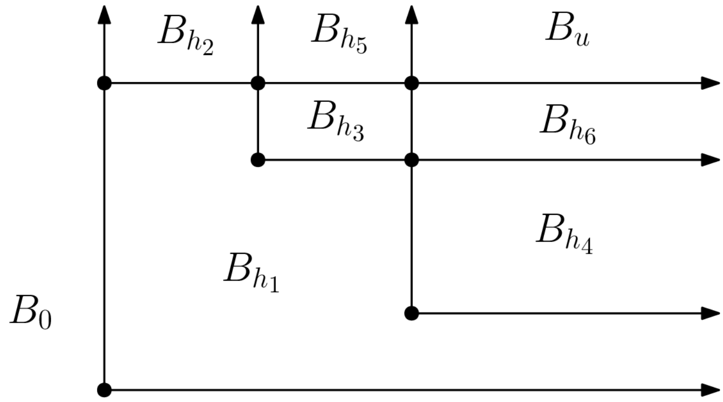

Figure 1: Decomposition to blocks

The main theorem of the section is a characterisation of n-spectral resolutions which arise from some n-observable:

Theorem 8.5. *Each observable x on E gives arise to a spectral resolution F with only finitely many characteristic points. Moreover $\vee_{\mathbf{t}\ll\mathbf{s}}F(\mathbf{s})$ exists for each characteristic point $\mathbf{t}$. These to conditions are also sufficient conditions on an n-spectral resolution have extension to an n-observable.*

9 Classical approach to measure extension

In the last chapter of the Thesis, we present the construction of extension of n-spectral resolutions for the case of a monotone σ-complete interval effect algebra E, having a faithful σ-state s (i.e., $s(a) = 0 \implies s(a) = 0$). For example the measure algebras (the case when E is boolean algebra) and the probability MV-algebras are covered by these assumptions. The existence of the faithful σ-state is a strong assumption, for example, any monotone σ-complete effect algebra with (RDP) having faithful σ-state is already an MV-algebra (see [Goo86], Prop. 16.5). The proof of the main result of this chapter (Theorem 9.3) is based on ideas in [DL20c].

Denote $\mathcal{R} \subset \mathcal{B}(\mathbb{R}^n)$ a ring of sets (i.e., system of sets closed under union and relative complements) generated by semi-closed intervals of the form

$$[a_1, b_1) \times \cdots \times [a_n, b_n), \text{ where } a_i, b_i \in \mathbb{R}, a_i < b_i. \tag{14}$$

Each n-spectral resolution on E naturally defines a measure on $\mathcal{R}$ (with values in E). The chapter aims to extend such measure to $\mathcal{B}(\mathbb{R}^n)$, which is as a field σ-generated by $\mathcal{R}$. Our strategy is to apply the following standard result concerning measure extension (using the concept of outer measures).

Theorem 9.1 ([Hal74], Thm. 13.A.)**.** *Let μ be a σ-finite measure on a ring of sets $\mathcal{R}$. Then there is a unique σ-finite measure $\bar{\mu}$, which extends μ on the σ-ring generated by $\mathcal{R}$.*

Moreover, as follows from section 12 in [Hal74], the extended measure $\bar{\mu}$ satisfies formula

$$\bar{\mu}(A) = \inf\{\sum_{i=1}^{\infty} \mu(A_i) | A_i \in \mathcal{R}, A \subseteq \bigcup_{i=1}^{\infty} A_i, \text{ for } i \neq j, A_i \cap A_j = \emptyset\}. \tag{15}$$

Theorem 9.2. *Let $F : \mathbb{R}^n \to E$ be an n-spectral resolution with values in an interval effect algebra $E = \Gamma(G, u)$, where G is Dedekind σ-complete with strong unit u. Then there is a unique mapping $x_0 : \mathcal{R} \to E$ such that:*

(i) $x_0([a_1, b_1) \times \cdots \times [a_n, b_n)) = \Delta_1(a_1, b_1) \cdots \Delta_n(a_n, b_n)F$,

(ii) for each finite collection of disjoint sets $A_1, \ldots, A_m \in \mathcal{R}$ we have

$$x_0(\cup_{i=1}^m A_i) = \sum_{i=1}^m x_0(A_i).$$

Moreover, if E has a faithful σ-state s, we have

(iii) if $A_i \searrow \emptyset$ is a sequence of elements of $\mathcal{R}$, then $\bigwedge_i x_0(A_i) = 0$,

(iv) Let A_i, $i \in \mathbb{N}$, be a collection of disjoint Borel sets such that $\bigcup_i A_i = A \in \mathcal{R}$. Then $\sum_i x_0(A_i) = x_0(A)$.

The main result of the chapter is:

Theorem 9.3. *Let $E = \Gamma(G, u)$ be a monotone σ-complete interval effect algebra and s a faithful σ-state on E. Then each n-spectral resolution extends to an n-observable.*

10 Shrnutí v českém jazyce

V disertační práci je řešen problém existence korespondence mezi observables (zaběhnutý český překlad neexistuje) a spektrálními rozklady pro různé třídy algeber zkoumaných v oblasti kvantové logiky. Motivačním příkladem je známý fakt z teorie pravděpodobnosti: Každá pravděpodobnostní míra $\nu : \mathcal{B}(\mathbb{R}) \to [0, 1]$ je jednoznačně dána svou distribuční funkcí a naopak každá monotonní zleva spojitá funkce $F : \mathcal{B}(\mathbb{R}) \to [0, 1]$ splňující jisté limitní podmínky v $\pm\infty$ je distribuční funkcí právě jedné pravděpodobnostní míry.

Přístup k problému, užitý v disertační práci, vychází ze dvou důležitých reprezentačních vět. První z nich říká, že každá efektová algebra splňující Rieszovu dekompoziční podmínku je isomorfní intervalu v částečně uspořádané (Abelovské) grupě s interpolací (dokonce existuje kategoriální ekvivalence mezi kategoriemi popsaných efektových algeber a grup; autorem výsledku je K. Ravindra [Rav96]). Druhou důležitou větou je následující zobecnění Loomis-Sikorského věty: Každá monotonně σ-úplná efektová algebra je σ-homomorfním obrazem takzvaného efektového tribu fuzzy množin ([BCD06]).

Druhá zmíněná věta je hlavním nástrojem konstrukce rozšiřování spektrálních rozkladů, která je popsána v disertační práci: Mějme efektovou algebru E, pro kterou existuje Loomis-Sikorského reprezentace $\mathcal{T} \to E$. Spektrální rozklad $F : \mathbb{R} \to E$ rozšíříme na observable ve třech krocích. Prvně zdvihneme F (vzhledem k π) na spektrální rozklad $\hat{F}$ na $\mathcal{T}$. Získaný spektrální rozklad $\hat{F}$ lze pomocí standardních vět z teorie pravděpodobnosti rozšířit na observable $\hat{x}$. Hledaný observable x dostaneme jako kompozici $\pi \circ \hat{x}$. Aplikace této metody byla předvedena v [DK14] na řadě algeber (Boolovské, MV algebry, (RDP) efektové algebry).

V disertační práci jsou popsány tři směry zobecnění výsledků z [DK14]:

(I) přejití na více dimenzionální spektrální rozklady $F : \mathbb{R}^n \to E$,

(II) oslabení podmínky monotonní-σ úplnosti (přejitím k takzvaným lexikografickým efektovým algebrám),

(III) kombinace situací (I–II).

Pro (I) je charakteristické zkomplikování procesu zdvihání spektrálních rozkladů (z nejsnadnějšího kroku se stane nejobtížnější). Tato část dizertační práce pokrývá výsledky z článků [DL20d] and [DL20a]. Pro zobecnění (II-III) (kde korespondence jedna ku jedné mezi spektrálními rozklady a observables v obecné situaci neplatí) je charakteristická potřeba zjemnit argumentaci většiny důkazů (neboť E chybí některá supréma), což je provedeno skrze množství pomocných technických lemmat. Tato část disertační práce zobecňuje výsledky publikované v článcích [DL20b], [DL19],[DL19] a [DL21].

Hlavní výsledky disertační práce jsou shrnuty v následujících větách:

Theorem 10.1. *Ať $\pi : (G, u) \to (H, v)$ je σ-homomorfismus unitární dedekindovsky monotonně σ-úplná částečně uspořádaná grupa. Dále ať π splňuje (LP). Potom každý n-dimenzionální spektrální rozklad $F : \mathbb{R}^n \to H$ lze zdvihnout na n-dimenzionální spektrální rozklad $K : \mathbb{R}^n \to G$ takový, že $\pi \circ K = F$.*

Theorem 10.2. *Pro každou monotonně σ-úplnou efektovou algebru E splňující (RDP) existuje bijektivní korespondence mezi n-spektrálními rozklady a n-observables.*

Theorem 10.3. *Ať $E = \Gamma(H \vec{\times} G, (u, 0))$ je lexikografická efektová algebra, kde (G, u) je dedekindovsky monotonně σ-úplná částečně uspořádaná grupa s interpolací a (H, u) je libovolná (Abelovská) unitární částečně uspořádaná grupa. Potom každý n-observable x zadává n-spektrální rozklad F takový, že:*

(i) $\mathrm{Im}(\pi_H \circ F)$ *je konečná množina,*

(ii) $\wedge_{\mathbf{t} \ll \mathbf{s}} F(\mathbf{s})$ *existuje pro každý charakteristický bod* $\mathbf{t}$,

kde $\mathbf{t}$ *je charakteristickým bodem, právě když existuje* $h \in H$ *a* $\mathbf{s} \in \mathbb{R}^{\mathbf{n}}$ *takové, že* $\pi_H \circ F(\mathbf{s}) = h$ *a* $\mathbf{t} := \inf\{\mathbf{r} \in \mathbb{R}^n | \pi_H \circ F(\mathbf{r}) = h, \mathbf{r} \leq \mathbf{s}\}$.

Na druhou stranu, každý n-spektrální rozklad na E splňující podmínky $(i-ii)$ lze rozšířit na observable.

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