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MASTER'S THESIS

MV-Algebras

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I hereby affirm that I have written this master's thesis independently under the supervision of doc. Mgr. Michal Botur, Ph.D., and that I have not used any sources other than those cited.

Olomouc 28th May 2015 ....

At this point I would like to thank my supervisor doc. Mgr. Michal Botur, Ph.D. for the time, advice and patience he found for me.

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Preface

MV-algebras are algebraic models of multi-valued Łukasiewicz's logic similarly to boolean algebras being models of boolean logic. This correspondence applies to many aspects of the theory, e.g., the relation between logic tautologies and identities which can be proved from the axioms of the algebras, and it also indicates the importance of representation theorems.

The primary intent of this work is to present a new construction of the l-group corresponding to an MV-algebra. This is in fact one of three included representation theorems which together give us means for showing that the variety of MV-algebras is generated by the standard MV-algebra.

Parts of the text may require some basic knowledge of universal algebra, lattice theory and group theory. However, some of the definitions of necessary notions and several basic theorems are included in the first two sections of the text.

1 Important notions

The first section presents merely a list of notions which may appear later on.

1.1 l-groups

The algebra we intent to use consists of two different structures:

Firstly, a *group* is a set equipped with operations $+$, $^-$ and 0, i.e., an algebra of type (2,1,0), such that

(i) $(a + b) + c = a + (b + c)$,

(ii) $a + 0 = a = 0 + a$,

(iii) $a^- + a = 0 = a + a^-$.

Secondly, a *lattice* is a partially ordered set where there exist both the supremum and the infimum of every pair of elements. The supremum, resp. infimum of a and b will be denoted as $a \vee b$, resp. $a \wedge b$.

Now, we may combine these to structures to phrase one of the possible definitions of an l-group:

An *l-group* is a group with a lattice ordering $\leq$ such that the binary group operation is monotone with respect to this ordering, i.e.,

$$a \leq b \quad \Longrightarrow \quad a + c \leq b + c \text{ and } c + a \leq c + b.$$

One of the conditions which is valid in every l-group and which is also connected to the construction in the third section of this work is related to the *Riesz decomposition property* and it can be introduced as follows:

Let x_1, x_2, y_1, y_2 be elements of an l-group $\mathcal{G}$ such that $x_1 + x_2 = y_1 + y_2$. Then there exist $h_{11}, h_{12}, h_{21}, h_{22} \in \mathcal{G}$ such that

$$h_{i1} + h_{i2} = x_i \quad \text{and} \quad h_{1i} + h_{2i} = y_i$$

where $i = 1, 2$.

The proof of this property along with many other aspects of the theory of l-groups can be found for example in [4].

1.2 Operators on classes of algebras

Let $\mathcal{K}$ be a class of algebras of the same type and $\mathcal{A}$ an arbitrary algebra. Then we define operators I, H, S, P and P_U on classes of algebras as follows:

(i) $\mathcal{A} \in I(\mathcal{K})$ if and only if $\mathcal{A}$ is isomorphic to some algebra from the class $\mathcal{K}$.

(ii) $\mathcal{A} \in H(\mathcal{K})$ if and only if $\mathcal{A}$ is a homomorphic image of some algebra from the class $\mathcal{K}$.

(iii) $\mathcal{A} \in S(\mathcal{K})$ if and only if $\mathcal{A}$ is a subalgebra of some algebra from the class $\mathcal{K}$.

(iv) $\mathcal{A} \in P(\mathcal{K})$ if and only if $\mathcal{A}$ is a direct product of a set of algebras from the class $\mathcal{K}$.

(v) $\mathcal{A} \in P_U(\mathcal{K})$ if and only if $\mathcal{A}$ is an ultraproduct of a set of algebras from the class $\mathcal{K}$.

If X and Y are some of the operators above, we shall write $XY(\mathcal{K})$ instead of $X(Y(\mathcal{K}))$.

A *variety* is a class $\mathcal{K}$ of algebras of the same type such that $HSP(\mathcal{K}) = \mathcal{K}$. And since for every class $\mathcal{L}$ of algebras of the same type the condition

$$HSP(HSP(\mathcal{L})) = HSP(\mathcal{L})$$

holds, we can define a variety generated by the class of algebras $\mathcal{L}$ as follows

$$\mathcal{V}(\mathcal{L}) = HSP(\mathcal{L}).$$

A *quasi-variety* is a class $\mathcal{K}$ of algebras of the same type such that $HSP_U(\mathcal{K}) = \mathcal{K}$. And similarly to the previous, we may define a quasi-variety generated by the class of algebras $\mathcal{L}$ as follows

$$\mathcal{QV}(\mathcal{L}) = HSP_U(\mathcal{L}).$$

More detailed theory of universal algebra can be found for example in [3].

2 Properties of MV-algebras

The objective of this section is to recall some well known properties of MV-algebras which will be useful further on.

2.1 The definition

Definition 2.1.1. *An* MV-algebra $\mathcal{A} = (A, \oplus, \neg, 0)$ *is an algebra of a type (2,1,0) satisfying the following identities:*

(1) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$

(2) $a \oplus b = b \oplus a$

(3) $a \oplus 0 = a$

(4) $\neg\neg a = a$

(5) $a \oplus \neg 0 = \neg 0$

(6) $\neg(\neg a \oplus b) \oplus b = \neg(\neg b \oplus a) \oplus a$

For further use we define new derived operations by

(2.1) $1 = \neg 0,$

(2.2) $a \odot b = \neg(\neg a \oplus \neg b),$

(2.3) $a \ominus b = a \odot \neg b.$

As a direct consequence of (4) we obtain

(2.4) $a \oplus b = \neg(\neg a \odot \neg b)$

and it can be easily proved $(A, \odot, \neg, 1)$ is also an MV-algebra.
Furthermore, setting $a = 0$ in (4), resp. $b = 1$ in (6) we obtain:

(2.5) $\neg 1 = 0$

(2.6) $a \oplus \neg a = 1$

Remark. *To simplify the rest of the text let $\mathcal{A}$ always be an MV-algebra and A its underlying set.*

2.2 The natural order

In order to build an additional structure on MV-algebras we will make use of the following.

Proposition 2.2.1. *For any $a, b \in A$ the following conditions are equivalent:*

(i) $\neg a \oplus b = 1$.

(ii) There exists $c \in A$ such that $a \oplus c = b$,

(iii) $b = (b \ominus a) \oplus a$.

Proof. We will prove three necessary implications:

(i)⇒(iii) Let $\neg a \oplus b = 1$ then using (6)

$$\begin{aligned}(b \ominus a) \oplus a &= \neg(\neg b \oplus a) \oplus a \\ &= \neg(\neg a \oplus b) \oplus b \\ &= \neg 1 \oplus b \\ &= 0 \oplus b \\ &= b.\end{aligned}$$

(iii)⇒(ii) Let $b = (b \ominus a) \oplus a$ then $c = b \ominus a$ satisfies the required condition.

(ii)⇒(i) Let $c \in A$ be such that $a \oplus c = b$. Then

$$\neg a \oplus b = \neg a \oplus a \oplus c = 1 \oplus c = 1.$$ □

Henceforth, we define a binary relation $\leq$ by

(2.7) $a \leq b \Leftrightarrow \neg a \oplus b = 1$.

Proposition 2.2.2. *The relation $\leq$ is a partial order on A.*

Proof. Indeed, it is a consequence of (ii) in Proposition 2.2.1. □

This ordering is called the *natural order* of A.

Lemma 2.2.1. *In every MV-algebra the following holds:*

(i) The operation $\neg$ is antitone with respect to $\leq$, i.e.,

$$a \leq b \Longleftrightarrow \neg b \leq \neg a.$$

(ii) The operation $\oplus$ is monotone with respect to $\leq$, i.e., for every $c \in A$

$$a \leq b \Longrightarrow a \oplus c \leq b \oplus c.$$

(iii) Condition $a \ominus b \leq c$ *holds if and only if so does* $a \leq b \oplus c$*.*

Proof. If $a \leq b$ holds then according to the definition $\neg a \oplus b = 1$. Thus $\neg(\neg b) \oplus \neg a = 1$ which means $\neg b \leq \neg a$. Further, let $c \in A$ then

$$\begin{aligned}\neg(a \oplus c) \oplus (b \oplus c) &= \neg(\neg\neg a \oplus c) \oplus c \oplus b \\ &= \neg(\neg c \oplus \neg a) \oplus \neg a \oplus b \\ &= \neg(\neg c \oplus \neg a) \oplus 1 \\ &= 1.\end{aligned}$$

That is, $a \oplus c \leq b \oplus c$.

Finally, we see

$$\begin{aligned}a \ominus b \leq c &\Leftrightarrow \neg(a \ominus b) \oplus c = 1 \\ &\Leftrightarrow \neg(\neg(\neg a \oplus b)) \oplus c = 1 \\ &\Leftrightarrow \neg a \oplus (b \oplus c) = 1 \\ &\Leftrightarrow a \leq b \oplus c.\end{aligned}$$

□

Proposition 2.2.3. *The structure* $(A, \leq)$ *is a lattice where operations supremum and infimum are defined as follows*

(2.8) $a \vee b = \neg(\neg a \oplus b) \oplus b,$

(2.9) $a \wedge b = \neg(\neg a \vee \neg b).$

Proof. It is easy to see that $b \leq \neg(\neg a \oplus b) \oplus b$ and by the axiom (6) we observe the same holds for a. Thus, $\neg(\neg a \oplus b) \oplus b$ is an upper bound of a and b.

Let $a \leq c$ and $b \leq c$. Using (i) and (iii) of Lemma 2.2.1 we obtain $\neg a \oplus c = 1$ and $c = (c \ominus b) \oplus b$. Hence, we can compute

$$\begin{aligned}\neg((a \ominus b) \oplus b) \oplus c &= (\neg(a \ominus b) \ominus b) \oplus b \oplus (c \ominus b) \\ &= (b \ominus \neg(a \ominus b)) \oplus \neg(a \ominus b) \oplus (c \ominus b) \\ &= (b \ominus \neg(a \ominus b)) \oplus \neg a \oplus b \oplus (c \ominus b) \\ &= (b \ominus \neg(a \ominus b)) \oplus \neg a \oplus c \\ &= (b \ominus \neg(a \ominus b)) \oplus 1 \\ &= 1.\end{aligned}$$

Thus, $\neg(\neg a \oplus b) \oplus b \leq c$, which yields $\neg(\neg a \oplus b) \oplus b$ is indeed the supremum of a and b.

Finally, the claim (2.9) is a direct consequence of the foregoing due to Lemma 2.2.1(i). □

Proposition 2.2.4. *Every MV-algebra satisfies the following identities:*

(2.10) $a \oplus (b \wedge c) = (a \oplus b) \wedge (a \oplus c)$,

(2.11) $a \odot (b \vee c) = (a \odot b) \vee (a \odot c)$.

Proof. Firstly, it can be easily seen that $a \oplus (b \wedge c) \leq a \oplus b$, $a \oplus c$. Now, let $d \leq a \oplus b$, $a \oplus c$ then according to Lemma 2.2.1(iii) we have $d \ominus a \leq b$, c. Therefore, $d \ominus a \leq b \wedge c$ and thus $d \leq a \oplus (b \wedge c)$. Which proves the first part of the proposition.

In order to prove the second claim we compute

$$\begin{aligned} a \odot (b \vee c) &= \neg(\neg a \oplus \neg(b \vee c)) \\ &= \neg(\neg a \oplus (\neg b \wedge \neg c)) \\ &= \neg((\neg a \oplus \neg b) \wedge (\neg a \oplus \neg c)) \\ &= \neg(\neg(a \odot b) \wedge \neg(a \odot c)) \\ &= (a \odot b) \vee (a \odot c) \end{aligned}$$

using the first distributivity. □

Not every MV-algebra is totally ordered, i.e. linear. However, it always satisfies another condition called *prelinearity*:

Proposition 2.2.5. *The following identity holds in every MV-algebra*

(2.12) $(a \ominus b) \wedge (b \ominus a) = 0$.

Proof. Harnessing the basic properties of the operations $\oplus$ and $\odot$ we obtain

$$\begin{aligned} (a \ominus b) \wedge (b \ominus a) &= (a \ominus b) \odot (\neg(a \ominus b) \oplus (b \ominus a)) \\ &= a \odot \neg b \odot (b \oplus \neg a \oplus (b \ominus a)) \\ &= a \odot \neg(b \oplus \neg(b \oplus (\neg a \oplus (b \ominus a)))) \\ &= a \odot (\neg a \oplus (b \ominus a)) \odot (\neg(\neg a \oplus (b \ominus a)) \oplus \neg b) \\ &= (b \ominus a) \odot (\neg(b \ominus a) \oplus a) \odot (\neg(\neg a \oplus (b \ominus a)) \oplus \neg b) \end{aligned}$$

where we have already used (6) twice and we will use it once again

$$\begin{aligned} &= (b \ominus a) \odot (\neg(b \ominus a) \oplus a) \odot (\neg(\neg a \oplus (b \ominus a)) \oplus \neg b) \\ &= b \odot \neg a \odot (\neg(b \ominus a) \oplus a) \odot ((a \odot \neg(b \ominus a)) \oplus \neg b) \\ &= \neg a \odot (a \oplus \neg(b \ominus a)) \odot b \odot (\neg b \oplus (a \odot \neg(b \ominus a))) \\ &= \neg a \odot (a \oplus \neg(b \ominus a)) \odot (a \odot \neg(b \ominus a)) \odot (\neg(a \odot \neg(b \ominus a)) \oplus b) \\ &= 0. \end{aligned}$$

because $a \odot \neg a = 0$ and $0 \odot a = 0$. □

2.3 Ideals and congruences

At first, let us recall the definition of an ideal.

Definition 2.3.1. *A subset $I \subseteq A$ is called an* ideal *if*

(i) $0 \in I$,

(ii) I is closed under minorants, i.e., if $a \leq b$ and $b \in I$ then $a \in I$,

(iii) I is closed under the operation $\oplus$, i.e., if $a, b \in I$ then $a \oplus b \in I$.

In MV-algebras there is a close correspondence between ideals and congruences as the following propositions show.

Proposition 2.3.1. *Let I be an ideal of A. Then the binary relation $\equiv_I$ defined by*

$$a \equiv_I b \Longleftrightarrow (a \ominus b) \oplus (b \ominus a) \in I$$

is a congruence on A. Moreover, $I = \{a \in A \mid a \equiv_I 0\}$.

Proof. Let I be an ideal of A and $\equiv_I$ a relation defined above. Firstly, since $(a \ominus a) \oplus (a \ominus a) = 0$ and $(a \ominus b) \oplus (b \ominus a) = (b \ominus a) \oplus (a \ominus b)$ the relation is reflexive and symmetric.

Now, we note that

$$\begin{aligned} \neg(a \ominus c) \oplus (a \ominus b) \oplus (b \ominus c) &= (\neg a \vee \neg b) \oplus (c \vee b) \\ &\geq \neg b \oplus b \\ &= 1 \end{aligned}$$

thus, $a \ominus c \leq (a \ominus b) \oplus (b \ominus c)$. Similarly, we obtain $c \ominus a \leq (b \ominus a) \oplus (c \ominus b)$. Hence, from the monotonicity of $\oplus$ we get

$$(a \ominus c) \oplus (c \ominus a) \leq (a \ominus b) \oplus (b \ominus c) \oplus (b \ominus a) \oplus (c \ominus b)$$

and since I is closed under $\oplus$ and minorants it immediately follows $\equiv_I$ is also transitive, thus it is an equivalence.

Next, we can see that

$$\begin{aligned} \neg((a \oplus c) \ominus (b \oplus d)) \oplus (a \ominus b) \oplus (c \ominus d) &= \neg(a \oplus c) \oplus (a \vee b) \oplus (d \vee c) \\ &\geq \neg(a \oplus c) \oplus (a \oplus c) \\ &= 1. \end{aligned}$$

Thus, $(a \oplus c) \ominus (b \oplus d) \leq (a \ominus b) \oplus (c \ominus d)$ and similarly $(b \oplus d) \ominus (a \oplus c) \leq (b \ominus a) \oplus (d \ominus c)$. Which implies

$$((a \oplus c) \ominus (b \oplus d)) \oplus ((b \oplus d) \ominus (a \oplus c)) \leq (a \ominus b) \oplus (c \ominus d) \oplus (b \ominus a) \oplus (d \ominus c).$$

This immediately yields that $\equiv_I$ satisfies the substitution condition for $\oplus$. Since

$$(\neg b \ominus \neg a) \oplus (\neg a \ominus \neg b) = (a \ominus b) \oplus (b \ominus a)$$

than $\equiv_I$ also satisfies the substitution condition for $\neg$ and thus it is a congruence on A.

Moreover,

$$\begin{aligned} b \in \{a \in A \mid a \equiv_I 0\} &\Leftrightarrow b \equiv_I 0 \\ &\Leftrightarrow (b \ominus 0) \oplus (0 \ominus b) \in I \\ &\Leftrightarrow b \in I \end{aligned}$$

thus, $I = \{a \in A \mid a \equiv_I 0\}$. □

And conversely:

Proposition 2.3.2. *Let $\equiv$ be a congruence on A. Then $\{a \in A \mid a \equiv 0\}$ is an ideal of A. Moreover, the condition*

$$a \equiv b \Longleftrightarrow (a \ominus b) \oplus (b \ominus a) \equiv 0$$

holds.

Proof. Let $a, b \in \{a \in A \mid a \equiv 0\}$ then $a \equiv 0$ and $b \equiv 0$. Hence, we have $a \oplus b \equiv 0 \oplus 0 = 0$, thus $a \oplus b \in \{a \in A \mid a \equiv 0\}$.

Now, let $a \leq b$ and $b \in \{a \in A \mid a \equiv 0\}$. Hence, $a \ominus b = 0 \equiv b$ thus we can compute

$$a = a \wedge b = a \ominus (a \ominus b) \equiv a \ominus b = 0$$

i.e., $a \in \{a \in A \mid a \equiv 0\}$.

Therefore, $I = \{a \in A \mid a \equiv 0\}$ is indeed an ideal of A.

Moreover, since I is an ideal we obtain

$$\begin{aligned} (a \ominus b) \oplus (b \ominus a) \equiv 0 &\Leftrightarrow (a \ominus b) \oplus (b \ominus a) \in I \\ &\Leftrightarrow (a \ominus b), (b \ominus a) \in I \\ &\Leftrightarrow a \ominus b \equiv 0 \quad \text{and} \quad b \ominus a \equiv 0 \\ &\Leftrightarrow (a \ominus b) \oplus b \equiv b \quad \text{and} \quad (b \ominus a) \oplus a \equiv a \\ &\Leftrightarrow a \vee b \equiv b \quad \text{and} \quad b \vee a \equiv a \\ &\Leftrightarrow a \equiv b. \end{aligned}$$

□

In conclusion, the previous two propositions reveal that the correspondence $I \mapsto \equiv_I$ is in fact a bijection between the set of all ideals and the set of all congruences of $\mathcal{A}$.

Remark. *Therefore, by the factorization by an ideal we will mean the factorization by the respective congruence. More precisely, let $a \in A$ then the equivalence class with respect to $\equiv_I$ will be denoted by a/I and the quotient set $A/\equiv_I$ by A/I. The notation of the operations of the quotient algebra $\mathcal{A}/I$ will remain the same as in $\mathcal{A}$.*

2.4 Subdirect products

In this subsection we will briefly recall Chang's Subdirect Representation Theorem.

Proposition 2.4.1. *If an MV-algebra is subdirecty irreducible then it is linear.*

Proof. From the theory of universal algebra we know an algebra is subdirectly reducible if there exists a pair of congruences $\Theta, \Phi \neq \omega$ such that $\Theta \cap \Phi = \omega$ (ω stands for the trivial equivalence). Hence, an MV-algebra is subdirectly reducible if there exist two ideals $I, J \neq \{0\}$ such that $I \cap J = \{0\}$.

Claim. *Let $M \subseteq A$ then the set $M^{\perp} = \{a \in A \mid a \wedge c = 0 \quad (\forall c \in M)\}$ is an ideal of A.*

Proof. Let $M \subseteq A$. To begin with, $0 \in M^{\perp}$ since $0 \wedge c = 0$ for all $c \in M$. Then let $a \in M^{\perp}$ and $b \leq a$ so $b \wedge c \leq a \wedge c = 0$ for all $c \in M$ and thus $b \in M^{\perp}$. Finally, let $a, b \in M^{\perp}$ then

$$\begin{aligned}(a \oplus b) \wedge c &= (a \oplus b) \wedge (a \oplus c) \wedge c \\ &= (a \oplus (b \wedge c)) \wedge c \\ &= (a \oplus 0) \wedge c \\ &= a \wedge c \\ &= 0\end{aligned}$$

for all $c \in M$. Thus $a \oplus b \in M^{\perp}$. Hence, $M^{\perp}$ is an ideal of A. □

Let $\mathcal{A}$ be non-linear. Then there exists a pair of incomparable elements $a, b \in A$. Due to prelinearity the condition $(a \ominus b) \wedge (b \ominus a) = 0$ holds. Now, according to the claim both $\{a \ominus b\}^{\perp}$ and $\{a \ominus b\}^{\perp\perp}$ are ideals of A, which are nontrivial since $b \ominus a \in \{a \ominus b\}^{\perp}$ and $a \ominus b \in \{a \ominus b\}^{\perp\perp}$.

Let $c \in \{a \ominus b\}^{\perp} \cap \{a \ominus b\}^{\perp\perp}$ then $c \wedge d = 0$ for each $d \in \{a \ominus b\}^{\perp}$ thus even for $c \in \{a \ominus b\}^{\perp}$. Hence,

$$0 = c \wedge c = c,$$

i.e., $\{a \ominus b\}^{\perp} \cap \{a \ominus b\}^{\perp\perp} = \{0\}$. □

As a consequence we obtain the promised Chang's Subdirect Representation Theorem:

Theorem 2.4.1. *Every MV-algebra is a subdirect product of linear MV-algebras.*

Proof. Any MV-algebra is a subdirect product of subdirectly irreducible MV-algebras, i.e., according to the previous proposition, linear MV-algebras. □

2.5 Partial operations

As an appendix to this section we shall define partial operations $+$ and $-$ on MV-algebras and show some of their properties, which will be used further on.

(2.13) $a + b$ exists if and only if $a \leq \neg b$ and then we put $a + b = a \oplus b$.

(2.14) $b - a$ exists if and only if $a \leq b$ and then we put $b - a = b \ominus a$.

Lemma 2.5.1. *Partial operations $+$ and $-$ satisfy the following partial identities:*

(2.15) If $a + b$ and $b - c$ exist then both $(a + b) - c$ and $a + (b - c)$ exist and

$$(a + b) - c = a + (b - c).$$

(2.16) If $a + b$ exists then $(a + b) - b$ exist and

$$(a + b) - b = a.$$

(2.17) If $a + (b - c)$ and $a - c$, resp. $b - c$ and $(a - c) + b$ exist then $(a - c) + b$, resp. $a + (b - c)$ exists and

$$a + (b - c) = (a - c) + b.$$

(2.18) If $a - b$ exists then $(a - b) + b$ exists and

$$(a - b) + b = a.$$

(2.19) If $(a + b) - c$ and $c - b$, resp. $a - (c - b)$ and $a + b$ exist then $a - (c - b)$, resp. $(a + b) - c$ exists and

$$(a + b) - c = a - (c - b).$$

(2.20) If $a - b$ exists then $a - (a - b)$ exists and

$$a - (a - b) = b.$$

Proof. We will prove them in a different order:

(2.18) Let $a - b$ exist then $b \leq a$ and $b \leq \neg a \oplus b = \neg(a - b)$, i.e., $(a - b) + b$ exists. Hence,

$$(a - b) + b = (a \ominus b) \oplus b = a \vee b = a.$$

(2.16) Let $a+b$ exist then $b \leq \neg a$ and $b \leq a+b$, i.e., $(a+b)-b$ exists. Thus,

$$(a+b)-b = \neg(\neg(\neg\neg a \oplus b) \oplus b) = \neg(\neg a \vee b) = \neg\neg a = a.$$

(2.20) Let $a-b$ exist then $b \leq a$ and $a-b \leq a$, i.e., $a-(a-b)$ exists. Hence,

$$a-(a-b) = a \ominus (a \ominus b) = a \wedge b = b.$$

(2.15) Let $a+b$ and $b-c$ exist thus $c \leq b \leq a+b$ and $a \leq \neg b \leq \neg b + c = \neg(b-c)$, i.e., $(a+b)-c$ and $a+(b-c)$ exist. Then using (2.18) and (2.16) we obtain

$$(a+b)-c = (a+(b-c)+c)-c = a+(b-c).$$

(2.17) Let $a+(b-c)$ and $a-c$, resp. $b-c$ and $(a-c)+b$. From Lemma 2.2.1(iii) we obtain

$$b \leq \neg a \oplus c \Leftrightarrow b \ominus c \leq \neg a$$

which ensures the existence of $(a-c)+b$, resp. $a+(b-c)$. Then using (2.18) we get

$$a+(b-c) = (a-c)+c+(b-c) = (a-c)+b.$$

(2.19) Let $(a+b)-c$ and $c-b$, resp. $a-(c-b)$ and $a+b$ exist. Again from Lemma 2.2.1(iii) we get

$$c \leq a \oplus b \Leftrightarrow c \ominus b \leq a$$

which ensures the existence of $a-(c-b)$, resp. $(a+b)-c$. Finally, using (2.20) and (2.17) we compute

$$(a+b)-c = (a+(c-(c-b)))-c = ((a-(c-b))+c)-c = a-(c-b).$$

$\square$

3 An l-group representation of MV-algebras

3.1 The alphabet

Let $\mathcal{A}$ be an MV-algebra then the symbol A^+ will stand for the set of all non-empty finite sequences of elements of A, i.e., all words over the alphabet A. Because the partial operation $+$ is associative we are allowed to use the following notation. Let $\mathbf{x} = x_1x_2\dots x_n \in A^+$ then we define $\sum \mathbf{x} = x_1 + x_2 + \dots + x_n$ if and only if the sum on the right side exists.

Lemma 3.1.1. *Let $\mathcal{A}$ be an MV-algebra and $a_1, a_2, b_1, b_2 \in A$. If $a_1 + a_2 = b_1 + b_2$ then $b_2 - (a_1 \ominus b_1) = a_2 - (b_1 \ominus a_1)$.*

Proof. We will make use of identities proved in Lemma 2.5.1. First, we use (2.16) since $b_2 - (a_1 \ominus b_1) \leq b_2$ and (2.15) since $b_1 + b_2$ exists and then we apply the assumption of the lemma in (i).

$$\begin{aligned} b_2 - (a_1 \ominus b_1) &\overset{(2.16)}{=} (b_1 + (b_2 - (a_1 \ominus b_1)) - b_1 \\ &\overset{(2.15)}{=} ((b_1 + b_2) - (a_1 \ominus b_1)) - b_1 \\ &\overset{(i)}{=} ((a_1 + a_2) - (a_1 \ominus b_1)) - b_1 \end{aligned}$$

Further, we use the identity (2.15) since $a_1 - (a_1 \ominus b_1)$ exists and in (ii) we use the dual Łukasiewicz's axiom.

$$\begin{aligned} ((a_1 + a_2) - (a_1 \ominus b_1)) - b_1 &\overset{(2.15)}{=} (a_2 + (a_1 - (a_1 \ominus b_1)) - b_1 \\ &\overset{(ii)}{=} (a_2 + (b_1 - (b_1 \ominus a_1)) - b_1 \\ &\overset{(2.17)}{=} ((a_2 - (b_1 \ominus a_1)) + b_1) - b_1 \\ &\overset{(2.15)}{=} a_2 - (b_1 \ominus a_1) \end{aligned}$$

Finally, we used the partial identity (2.17) since $b_1 \leq a_1 + a_2$ thus $b_1 \ominus a_1 \leq a_2$. □

The set of all matrices $m \times n$ over A will be denoted as $\mathcal{M}_{m\times n}(A)$.

Definition 3.1.1. *Let us define a binary relation $\sim$ on A^+ such that for $\mathbf{x} = x_1x_2\dots x_n, \mathbf{y} = y_1y_2\dots y_m \in A^+$ the following condition holds: $\mathbf{x} \sim \mathbf{y}$ if an only if there exists a matrix $\mathrm{H}_{\mathbf{x},\mathbf{y}} \in \mathcal{M}_{\mathrm{n}\times\mathrm{m}}(\mathrm{A})$ (hereafter called relation matrix) such that*

- $\sum \mathbf{h}_i = x_i$ *for every $i = 1, 2, \dots, n$,*
- $\sum \mathbf{h}^i = y_i$ *for every $i = 1, 2, \dots, m$,*

where $\mathbf{h}_i$*, resp.* $\mathbf{h}^i$ *are rows, resp. columns of the matrix* $\mathrm{H}_{\mathbf{x},\mathbf{y}}$*.*

Remark. *The matrix introduced in the definition above can be equipped with headings*

$$\begin{array}{c|cccc} & y_1 & y_2 & \dots & y_m \\ \hline x_1 & h_{11} & h_{12} & \cdots & h_{1m} \\ x_2 & h_{21} & h_{22} & \cdots & h_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n & h_{n1} & h_{n2} & \cdots & h_{nm} \end{array}$$

to indicate the corresponding sums of rows and columns.

Proposition 3.1.1. *Relation* $\sim$ *is reflexive.*

Proof. Let $\mathbf{x} \in A^+$ then $\mathbf{x} \sim \mathbf{x}$ because there exists a matrix $\mathrm{H}_{\mathbf{x},\mathbf{x}}$:

$$\begin{array}{c|cccc} & x_1 & x_2 & \dots & x_n \\ \hline x_1 & x_1 & 0 & \cdots & 0 \\ x_2 & 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n & 0 & 0 & \cdots & x_n \end{array}$$

which satisfies both necessary conditions. □

Remark. *The diagonal matrix used in the proof above will be denoted* $\delta(\mathbf{x})$ *in the proof of Proposition 3.2.2.*

Proposition 3.1.2. *Relation* $\sim$ *is symmetric.*

Proof. Let us assume $\mathbf{x}, \mathbf{y} \in A^+$ and $\mathbf{x} \sim \mathbf{y}$, that means there exists a relation matrix $\mathrm{H}_{\mathbf{x},\mathbf{y}}$. Then the matrix $\mathrm{H}^{\mathrm{T}}_{\mathbf{x},\mathbf{y}}$, i.e. a transpose of $\mathrm{H}_{\mathbf{x},\mathbf{y}}$ is a relation matrix of the pair $(\mathbf{y}, \mathbf{x})$ concluding $\mathbf{y} \sim \mathbf{x}$. □

Proposition 3.1.3. *Relation* $\sim$ *is transitive.*

Before we resume to the proof let us prove an auxiliary lemma.

Lemma 3.1.2. *Let* $\mathbf{a}, \mathbf{b}$ *be elements of* A^+ *such that* $\sum \mathbf{a} = \sum \mathbf{b}$ *then* $\mathbf{a} \sim \mathbf{b}$*.*

Proof. Let $\mathbf{a} = a_1 a_2 \dots a_n$ and $\mathbf{b} = b_1 b_2 \dots b_m$. The situation $n = 1$ is evident because there exists a relation matrix $\mathrm{H}_{\mathbf{a},\mathbf{b}} = \mathbf{b}$. The same goes for the case $m = 1$.

The rest of the proof will be done by mathematical induction by $m \geq 2$ and then $n \geq 2$.

(i) Let $n = m = 2$. Then there exists a matrix $\mathrm{H}_{\mathbf{a},\mathbf{b}}$:

	b_1	b_2
a_1	$a_1 \wedge b_1$	$a_1 - (a_1 \wedge b_1)$
a_2	$b_1 - (a_1 \wedge b_1)$	c

where

$$c = a_2 - (b_1 - (a_1 \wedge b_1)).$$

Then we can make use of Lemma 3.1.1.

$$\begin{aligned} a_2 - (b_1 - (a_1 \wedge b_1)) &= a_2 - (b_1 \ominus a_1) \\ &= b_2 - (a_1 \ominus b_1) \\ &= b_2 - (a_1 - (a_1 \wedge b_1)) \end{aligned}$$

Thus, the matrix is properly defined.

(ii) Let $n = 2$ and $m \geq 3$ and let us suppose the claim holds for $m - 1$. Therefore, a matrix $\mathrm{H}_{\mathbf{a},\mathbf{b}^+}$ exists where $\mathbf{b}^+ = b_1 b_2 \ldots b_{m-2}(b_{m-1} + b_m) = \mathbf{b}'(b_{m-1} + b_m)$ and $\mathbf{a} = a_1 a_2$:

	$\mathbf{b}'$	$b_{m-1} + b_m$
$\mathbf{a}^T$	W	$\mathbf{c}^T$

From the definition of the relation matrix we obtain $\sum \mathbf{c} = c_1 + c_2 = b_{m-1} + b_m$ thus according to (i) there exists a relation matrix $\mathrm{M} = \mathrm{H}_{\mathbf{c},\mathrm{b_{m-1}b_m}}$. Using this matrix we can acquire a relation matrix $\mathrm{H}_{\mathbf{a},\mathbf{b}}$:

	$\mathbf{b}'$	b_{m-1}	b_m
$\mathbf{a}^T$	W	M	

(iii) Let $n, m \geq 3$ and let us suppose the claim holds for $n-1$. Hence, there exists a relation matrix $\mathrm{H}_{\mathbf{a}^+,\mathbf{b}}$ where $\mathbf{a}^+ = a_1 a_2 \ldots a_{n-2}(a_{n-1} + a_n) = \mathbf{a}'(a_{n-1} + a_n)$:

	$\mathbf{b}$
$\mathbf{a}'^T$	W
$a_{n-1} + a_n$	$\mathbf{c}$

From the last row the condition $a_{n-1} + a_n = c_1 + c_2 + \cdots + c_m = \sum \mathbf{c}$ holds. Thus according to (ii) the relation matrix $\mathrm{M} = \mathrm{H}_{\mathrm{a_{n-1}a_n},\mathbf{c}}$ exists and similarly as in (ii) it can be used to create a relation matrix $\mathrm{H}_{\mathbf{a},\mathbf{b}}$:

	$\mathbf{b}$
$\mathbf{a}'^T$	W
a_{n-1}	M
a_n	

Therefore, the claim of the lemma is proved for every lengths n and m. □

Proof of Proposition 3.1.3. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in A^+$ such that $\mathbf{x} \sim \mathbf{y}$ and $\mathbf{y} \sim \mathbf{z}$. Then there exist relation matrices $\mathrm{M} = \mathrm{H}_{\mathrm{x,y}}$ and $\mathrm{N} = \mathrm{H}_{\mathrm{y,z}}$. Moreover, let us denote $\mathbf{x} = x_1 x_2 \dots x_s$, $\mathbf{y} = y_1 y_2 \dots y_p$ and $\mathbf{z} = z_1 z_2 \dots z_r$. The condition

$$\sum \mathbf{m}^i = y_i = \sum \mathbf{n}_i$$

holds for every $i = 1, 2, \dots, p$; hence, according to Lemma 3.1.2 a relation $\mathbf{m}^i \sim \mathbf{n}_i$ is valid for every $i = 1, 2, \dots, p$ and thus there exists a p-tuple of matrices $\mathrm{H}_{\mathbf{m}^\mathrm{i},\mathbf{n}_\mathrm{i}} = \mathrm{Y}^\mathrm{i}$ representing these relations.

Summing across the p-tuple we make a matrix K such that $k_{ij} = \sum \mathbf{y}_{ij}$ where $\mathbf{y}_{ij} = y^1_{ij} y^2_{ij} \dots y^p_{ij}$. We use the fact that

$$\sum_{i=1}^{s} \sum \mathbf{y}_{ij} = z_j$$

for every $j = 1, 2, \dots, r$ and a similar condition holds for every x_i. Therefore, the obtained matrix is a relation matrix $\mathrm{K} = \mathrm{H}_{\mathbf{x},\mathbf{z}}$ representing a relation $\mathbf{x} \sim \mathbf{z}$. □

Joining the three propositions above we acquire the following result.

Theorem 3.1.1. *The relation $\sim$ is an equivalence on A^+.*

Proof. The theorem is an immediate consequence of Propositions 3.1.1, 3.1.2 and 3.1.3. □

Lemma 3.1.3. *Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in A^+$ such that $\mathbf{a} \sim \mathbf{b}$ and $\mathbf{c} \sim \mathbf{d}$, then the relation $\mathbf{ac} \sim \mathbf{bd}$ is valid (by $\mathbf{ac}$ we mean concatenation of $\mathbf{a}$ and $\mathbf{c}$).*

Proof. Let $\mathbf{a} \sim \mathbf{b}$ and $\mathbf{c} \sim \mathbf{d}$, then relation matrices $\mathrm{M} = \mathrm{H}_{\mathbf{a},\mathbf{b}}$ and $\mathrm{N} = \mathrm{H}_{\mathbf{c},\mathbf{d}}$ exist. Putting these matrices together as blocks (by O we mean a zero matrix of needed proportions) we obtain

	b	**d**
$\mathbf{a}^T$	M	O
$\mathbf{c}^T$	O	N

which is a relation matrix of a pair $(\mathbf{ac}, \mathbf{bd})$. □

In conclusion, it is easy to see the relation $\sim$ is a congruence on the semigroup $\mathcal{A}^+ = (A^+, \cdot\,)$ where $\cdot$ is the operation of concatenation.

3.2 The monoid

Provided $\sim$ is a congruence we can construct a factor semigroup $\mathcal{G}^+ = \mathcal{A}^+/_\sim$ thus the semigroup operation $+$ is defined as follows

$$[\mathbf{x}]_\sim + [\mathbf{y}]_\sim = [\mathbf{xy}]_\sim.$$

Also, it is not hard to see $[0]_\sim$ is a neutral element of $\mathcal{G}^+$, which is consequently a monoid.

Proposition 3.2.1. *In the monoid $\mathcal{G}^+ = (G^+, +)$ the canceling axiom holds, i.e.,*

$$[\mathbf{x}]_\sim + [\mathbf{y}]_\sim = [\mathbf{x}]_\sim + [\mathbf{z}]_\sim \implies [\mathbf{y}]_\sim = [\mathbf{z}]_\sim\,.$$

Proof. The claim to be proved can be rewritten to the language of $\mathcal{A}^+$ by

$$\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in A^+ (\mathbf{xy} \sim \mathbf{xz} \implies \mathbf{y} \sim \mathbf{z})$$

The proof will be done by the mathematical induction by the length of the finite sequence $\mathbf{x}$.

(i) Let $\mathbf{x} = x$, $\mathbf{y}$ and $\mathbf{z}$ be such that $\mathbf{xy} \sim \mathbf{xz}$. Hence, there exists a relation matrix $\mathrm{H}_{\mathbf{xy},\mathbf{xz}}$:

	x	$\mathbf{z}$
x	a	$\mathbf{b}$
$\mathbf{y}^T$	$\mathbf{c}^T$	A

By comparing the first row and the first column of the matrix we obtain

$$a + \sum \mathbf{b} = a + \sum \mathbf{c}$$

and using (2.16)

$$\sum \mathbf{b} = \sum \mathbf{c}\,,$$

thus according to Lemma 3.1.2 the relation $\mathbf{b} \sim \mathbf{c}$ holds and there exists a relation matrix $\mathrm{B} = \mathrm{H}_{\mathbf{b},\mathbf{c}}$ of the same type as the matrix A. Finally, the matrix of the relation $\mathbf{y} \sim \mathbf{z}$ can be made by summing matrices A and B:

$$\mathrm{H}_{\mathbf{y},\mathbf{z}} = \mathrm{A} + \mathrm{B}$$

The sum on the right side exists since we have $\sum_i (a_{ij} + b_{ij}) = z_j$ and $\sum_j (a_{ij} + b_{ij}) = y_i$.

(ii) Let us assume that the proposition holds for every $\mathbf{x}$ of length $n-1$. Now, let $\mathbf{x} = x_1 x_2 \dots x_n = \mathbf{x}' x_n$, $\mathbf{y}$ and $\mathbf{z}$ be such that $\mathbf{xy} \sim \mathbf{xz}$. Therefore, there exists a relation matrix $\mathrm{H}_{\mathbf{xy},\mathbf{xz}}$

	$\mathbf{x}'$	x_n	$\mathbf{z}$
$\mathbf{x}'^T$	X	$\mathbf{c}^T$	Z
x_n	$\mathbf{a}$	w	$\mathbf{b}$
$\mathbf{y}^T$	Y	$\mathbf{d}^T$	W

from which we obtain

$$\sum \mathbf{a} + \sum \mathbf{b} = \sum \mathbf{c} + \sum \mathbf{d}\,,$$

thus $\mathbf{ab} \sim \mathbf{cd}$ and there exists a relation matrix $\mathrm{A} = \mathrm{H}_{\mathbf{cd},\mathbf{ab}}$. Next, a convenient matrix of similar proportions is created as

$$\mathrm{B} = \begin{pmatrix} \mathrm{X} & \mathrm{Z} \\ \mathrm{Y} & \mathrm{W} \end{pmatrix}.$$

Summing matrices A and B a relation matrix

$$\mathrm{H}_{\mathbf{x}'\mathbf{y},\mathbf{x}'\mathbf{z}} = \mathrm{A} + \mathrm{B}$$

is obtained and so the relation $\mathbf{x}'\mathbf{y} \sim \mathbf{x}'\mathbf{z}$ holds. Take note of the fact that the sums of the rows, resp. columns of the matrix $\mathrm{A} + \mathrm{B}$ really correspond to the particular elements of $\mathbf{x}'$ and $\mathbf{y}$, resp. $\mathbf{z}$. And since $\mathbf{x}'$ is of length $n-1$ the induction assumption can be used, thus $\mathbf{y} \sim \mathbf{z}$.

Therefore, the claim of the proposition holds for every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in A^+$. □

One more property of this monoid will be useful later on.

Proposition 3.2.2. *The monoid $\mathcal{G}^+$ is commutative.*

Proof. Let $\overline{\mathbf{x}}, \overline{\mathbf{y}} \in G^+$ then there exists a relation matrix $\mathrm{H}_{\mathbf{xy},\mathbf{yx}}$:

	$\mathbf{y}$	$\mathbf{x}$
$\mathbf{x}^T$	O	$\delta(\mathbf{x})$
$\mathbf{y}^T$	$\delta(\mathbf{y})$	O

Hence, we obtain $[\mathbf{x}]_\sim + [\mathbf{y}]_\sim = [\mathbf{y}]_\sim + [\mathbf{x}]_\sim$. □

3.3 The l-monoid

To simplify the notation of the factor classes we will write $\overline{\mathbf{x}} = [\mathbf{x}]_\sim$. And now, we can define a natural ordering $\leq$ on $\mathcal{G}^+$ as follows

$$\overline{\mathbf{x}} \leq \overline{\mathbf{y}} \iff \exists \overline{\mathbf{z}} \in G^+(\overline{\mathbf{x}} + \overline{\mathbf{z}} = \overline{\mathbf{y}}).$$

It is easy to see that this is, indeed, an ordering on $\mathcal{G}^+$ which is monotone with respect to the operation $+$.

Definition 3.3.1. *Let us define a function $\|.\| : G^+ \longrightarrow \mathbb{N}$ such that $\|\overline{\mathbf{x}}\|$ is the lowest value among the lengths of all words in the class $\overline{\mathbf{x}}$.*

Remark. *This definition is correct due to the fact, that the set of all lengths of words of $\overline{\mathbf{x}}$ is in fact a non-empty subset of $\mathbb{N}$ which is known to always have a minimum. Furthermore, the following condition holds*

$$\overline{\mathbf{x}} \leq \overline{\mathbf{y}} \Longrightarrow \|\overline{\mathbf{x}}\| \leq \|\overline{\mathbf{y}}\|$$

since every $\overline{\mathbf{x}}$ such that $\|\overline{\mathbf{x}}\| = n$ is bounded by the class of the word of n-times 1, i.e., $\overline{\mathbf{x}} \leq \overline{1_n}$.

Proposition 3.3.1. *The mapping $F : A \longrightarrow G_1^+ = \{\overline{\mathbf{x}} \in G^+, \|\overline{\mathbf{x}}\| = 1\}$ such that*

$$F(x) = \overline{x}$$

is an isomorphism of partially ordered sets $(A, \leq)$ and $(G_1^+, \leq)$.

Proof. Let us suppose that for $\overline{\mathbf{x}} \in G_1^+$ there exist $x, y \in A$ such that $x, y \in \overline{\mathbf{x}}$, i.e., there exists a relation matrix $\mathrm{H_{x,y}}$:

	y
x	c

and thus $x = c = y$. Therefore, for every $\overline{\mathbf{x}} \in G_1^+$ there exists a unique $x \in A$ such that $x \in \overline{\mathbf{x}}$, and hence F is a bijection.

Now, let $x \leq y$ then there exists $z \in A$ such that $x + z = y$. However, that means

$$F(x) + F(z) = \overline{x} + \overline{z} = \overline{x + z} = \overline{y} = F(y)$$

which implies $F(x) \leq F(y)$.

The other way around, if $F(x) \leq F(y)$ then there exists $\overline{\mathbf{z}} \in G^+$ such that $F(x) + \overline{\mathbf{z}} = F(y)$. But that means $\overline{\mathbf{z}} \leq F(y)$ thus $\overline{\mathbf{z}} \in G_1^+$ and there exists such $z \in A$ that $F(z) = \overline{\mathbf{z}}$. Hence,

$$\overline{y} = F(y) = F(x) + F(z) = \overline{x} + \overline{z} = \overline{xz}$$

and so $x + z = y$ which implies $x \leq y$. □

For $\overline{\mathbf{x}} \leq \overline{\mathbf{y}}$ a partial operation $-$ is defined:

$$\overline{\mathbf{y}} - \overline{\mathbf{x}} = \overline{\mathbf{z}} \iff \overline{\mathbf{x}} + \overline{\mathbf{z}} = \overline{\mathbf{y}}.$$

Due to the canceling axiom the partial operation is defined properly.

Theorem 3.3.1. *The ordering $\leq$ is a lattice ordering on $\mathcal{G}^+$.*

Proof. First, let us prove the existence of $\overline{\mathbf{x}} \wedge \overline{\mathbf{y}}$ for every $\overline{\mathbf{x}}, \overline{\mathbf{y}} \in G^+$. It will be done using induction over $\|\overline{\mathbf{x}}\|$ and $\|\overline{\mathbf{y}}\|$.

(i) Let $\overline{\mathbf{x}}, \overline{\mathbf{y}} \in G_1^+$ then with respect to the isomorphism proved in Proposition 3.3.1:

$$\overline{\mathbf{x}} \wedge \overline{\mathbf{y}} = \overline{x \wedge y}.$$

And so, the infimum exists for every $\overline{\mathbf{x}}, \overline{\mathbf{y}}$ with $\|\overline{\mathbf{x}}\| = \|\overline{\mathbf{y}}\| = 1$.

(ii) Let us suppose the infimum exists for every $\overline{y} \in G_1^+$ and $\overline{\mathbf{x}} \in G^+$ such that $\|\overline{\mathbf{x}}\| \leq n$.

Now, let $\overline{\mathbf{x}} \in G^+$ be such that $\|\overline{\mathbf{x}}\| = n+1$ and $\overline{y} \in G_1^+$. Then there exist $\overline{\mathbf{x}}' \in G^+$ and $x_{n+1} \in A$ so that $\overline{\mathbf{x}}' + \overline{x}_{n+1} = \overline{\mathbf{x}}$ and $\|\overline{\mathbf{x}}'\| = n$. According to the assumption there exists $\overline{\mathbf{x}}' \wedge \overline{y}$ and since $\|\overline{y} - (\overline{\mathbf{x}}' \wedge \overline{y})\| \leq \|\overline{y}\|$ then according to (i) there also exists $(\overline{y} - (\overline{\mathbf{x}}' \wedge \overline{y})) \wedge \overline{x}_{n+1}$.

We will prove the infimum of $\overline{\mathbf{x}}$ and $\overline{y}$ exists in the following form:

$$\overline{\mathbf{x}} \wedge \overline{y} = (\overline{\mathbf{x}}' \wedge \overline{y}) + ((\overline{y} - (\overline{\mathbf{x}}' \wedge \overline{y})) \wedge \overline{x}_{n+1}).$$

First, it is a lower bound since

$$(\overline{\mathbf{x}}' \wedge \overline{y}) + ((\overline{y} - (\overline{\mathbf{x}}' \wedge \overline{y})) \wedge \overline{x}_{n+1}) \leq (\overline{y} - (\overline{\mathbf{x}}' \wedge \overline{y})) + (\overline{\mathbf{x}}' \wedge \overline{y}) = \overline{y}$$

and

$$(\overline{\mathbf{x}}' \wedge \overline{y}) + ((\overline{y} - (\overline{\mathbf{x}}' \wedge \overline{y})) \wedge \overline{x}_{n+1}) \leq \overline{\mathbf{x}}' + \overline{x}_{n+1} = \overline{\mathbf{x}}.$$

Second, suppose $\overline{\mathbf{z}} \leq \overline{\mathbf{x}}, \overline{y}$ then there exists $z \in A$ such that $z \in \overline{\mathbf{z}}$. Therefore, the inequality

$$\overline{\mathbf{z}} \wedge \overline{\mathbf{x}}' \leq \overline{y} \wedge \overline{\mathbf{x}}' \tag{1}$$

holds.

From inequalities

$$\overline{\mathbf{z}} \leq \overline{x}_{n+1} + \overline{\mathbf{x}}'$$
$$\overline{\mathbf{z}} \leq \overline{x}_{n+1} + \overline{\mathbf{z}}$$

it is obtained there exist $z_{11}, z_{12}, z_{21}, z_{22} \in A$ where

$$\overline{z}_{11} + \overline{z}_{12} = \overline{z} = \overline{z}_{21} + \overline{z}_{22}$$

and $\overline{z}_{11}, \overline{z}_{21} \leq \overline{x}_{n+1}$ and $\overline{z}_{12} \leq \overline{\mathbf{x}}'$ and $\overline{z}_{22} \leq \overline{\mathbf{z}}$ (this results from the relation matrices which belong to the inequalities).

Then $\overline{z}_{11}, \overline{z}_{21} \leq \overline{x}_{n+1} \wedge \overline{\mathbf{z}}$ and so using the canceling axiom and (2.19) we acquire

$$\overline{\mathbf{w}} = \overline{z}_{12} - ((\overline{x}_{n+1} \wedge \overline{\mathbf{z}}) - \overline{z}_{11}) = \overline{z}_{22} - ((\overline{x}_{n+1} \wedge \overline{\mathbf{z}}) - \overline{z}_{21}) \leq \overline{\mathbf{x}}', \overline{\mathbf{z}}.$$

Therefore, $\overline{\mathbf{w}} \leq \overline{\mathbf{x}}' \wedge \overline{\mathbf{z}}$ and $\overline{\mathbf{z}} = (\overline{x}_{n+1} \wedge \overline{\mathbf{z}}) + \overline{\mathbf{w}}$ and thus

$$\overline{\mathbf{z}} \leq \overline{x}_{n+1} + (\overline{\mathbf{x}}' \wedge \overline{\mathbf{z}}). \tag{2}$$

Furthermore, from inequalities

$$\overline{\mathbf{z}} \leq (\overline{y} - (\overline{\mathbf{x}}' \wedge \overline{y})) + \overline{\mathbf{z}}$$
$$\overline{\mathbf{z}} \leq (\overline{y} - (\overline{\mathbf{x}}' \wedge \overline{y})) + \overline{\mathbf{x}}'$$

we analogically obtain

$$\overline{\mathbf{z}} \leq (\overline{y} - (\overline{\mathbf{x}}' \wedge \overline{y})) + (\overline{\mathbf{x}}' \wedge \overline{\mathbf{z}}). \tag{3}$$

Consequently, from (2) and (3) we get

$$\overline{\mathbf{z}} \leq (\overline{\mathbf{x}}' \wedge \overline{\mathbf{z}}) + ((\overline{y} - (\overline{\mathbf{x}}' \wedge \overline{y})) \wedge \overline{x}_{n+1})$$

and by adding (1)

$$\overline{\mathbf{z}} \leq (\overline{\mathbf{x}}' \wedge \overline{y}) + ((\overline{y} - (\overline{\mathbf{x}}' \wedge \overline{y})) \wedge \overline{x}_{n+1}).$$

Thus, $(\overline{\mathbf{x}}' \wedge \overline{y}) + ((\overline{y} - (\overline{\mathbf{x}}' \wedge \overline{y})) \wedge \overline{x}_{n+1})$ is indeed an infimum of $\overline{\mathbf{x}}$ and $\overline{y}$.

In conclusion with (i) there exists an infimum for every $\overline{\mathbf{x}} \in G^+$ and $\overline{y} \in G_1^+$.

(C) **Claim.** *Let $\overline{\mathbf{x}} \wedge \overline{\mathbf{y}}$ exist for some $\overline{\mathbf{x}}, \overline{\mathbf{y}} \in G^+$ then for every $\overline{\mathbf{z}} \in G^+$ an infimum $(\overline{\mathbf{z}} + \overline{\mathbf{x}}) \wedge (\overline{\mathbf{z}} + \overline{\mathbf{y}})$ exists and the following condition holds:*

$$\overline{\mathbf{z}} + (\overline{\mathbf{x}} \wedge \overline{\mathbf{y}}) = (\overline{\mathbf{z}} + \overline{\mathbf{x}}) \wedge (\overline{\mathbf{z}} + \overline{\mathbf{y}}).$$

Proof. It will be done using induction over $\|\overline{\mathbf{z}}\|$.

(C.i) Let $\|\overline{\mathbf{z}}\| = 1$. It is easy to see that $\overline{\mathbf{z}} + (\overline{\mathbf{x}} \wedge \overline{\mathbf{y}}) \leq \overline{\mathbf{z}} + \overline{\mathbf{x}}, \overline{\mathbf{z}} + \overline{\mathbf{y}}$. Let $\overline{\mathbf{w}} \leq \overline{\mathbf{z}} + \overline{\mathbf{x}}, \overline{\mathbf{z}} + \overline{\mathbf{y}}$. According to (ii) there exists $\overline{\mathbf{w}} \wedge \overline{\mathbf{z}}$. So analogically to the proof of (2) in (ii) we can prove

$$\overline{\mathbf{w}} \leq \overline{\mathbf{z}} + (\overline{\mathbf{x}} \wedge \overline{\mathbf{y}}).$$

(C.ii) Let the claim hold for every $\overline{\mathbf{z}} \in G^+$, $\|\overline{\mathbf{z}}\| \leq n$. Let us have $\|\overline{\mathbf{z}}\| = n+1$, thus $\overline{\mathbf{z}} = \overline{z}_{n+1} + \overline{\mathbf{z}}'$ where $z_{n+1} \in A$ and $\|\overline{\mathbf{z}}'\| = n$. Then, while making use of the presumption twice, we finally obtain

$$\begin{aligned}
\overline{\mathbf{z}} + (\overline{\mathbf{x}} \wedge \overline{\mathbf{y}}) &= \overline{z}_{n+1} + \overline{\mathbf{z}}' + (\overline{\mathbf{x}} \wedge \overline{\mathbf{y}}) \\
&= \overline{z}_{n+1} + ((\overline{\mathbf{z}}' + \overline{\mathbf{x}}) \wedge (\overline{\mathbf{z}}' + \overline{\mathbf{y}})) \\
&= (\overline{z}_{n+1} + \overline{\mathbf{z}}' + \overline{\mathbf{x}}) \wedge (\overline{z}_{n+1} + \overline{\mathbf{z}}' + \overline{\mathbf{y}}) \\
&= (\overline{\mathbf{z}} + \overline{\mathbf{x}}) \wedge (\overline{\mathbf{z}} + \overline{\mathbf{y}}).
\end{aligned}$$

Therefore, in conclusion with (C.i) the claim holds for every $\overline{\mathbf{z}} \in G^+$. □

(iii) Let us suppose the claim of the theorem holds for every $\overline{\mathbf{x}}, \overline{\mathbf{y}} \in G^+$, $\|\overline{\mathbf{x}}\| \leq n$.

Let $\|\overline{\mathbf{x}}\| = n+1$, thus $\overline{\mathbf{x}} = \overline{\mathbf{x}}' + \overline{x}_{n+1}$ where $\|\overline{\mathbf{x}}'\| = n$. As well as in (ii) there exists $(\overline{\mathbf{x}}' \wedge \overline{\mathbf{y}}) + ((\overline{\mathbf{y}} - (\overline{\mathbf{x}}' \wedge \overline{\mathbf{y}})) \wedge \overline{x}_{n+1})$ and it is a lower bound since

$$\begin{aligned}
(\overline{\mathbf{x}}' \wedge \overline{\mathbf{y}}) + ((\overline{\mathbf{y}} - (\overline{\mathbf{x}}' \wedge \overline{\mathbf{y}})) \wedge \overline{x}_{n+1}) &\leq \overline{\mathbf{x}}, \\
(\overline{\mathbf{x}}' \wedge \overline{\mathbf{y}}) + ((\overline{\mathbf{y}} - (\overline{\mathbf{x}}' \wedge \overline{\mathbf{y}})) \wedge \overline{x}_{n+1}) &\leq \overline{\mathbf{y}}.
\end{aligned}$$

Next, let $\overline{\mathbf{z}} \leq \overline{\mathbf{x}}, \overline{\mathbf{y}}$. Analogically to (ii) the following inequalities hold

$$\overline{\mathbf{z}} \wedge \overline{\mathbf{x}}' \leq \overline{\mathbf{y}} \wedge \overline{\mathbf{x}}' \tag{1'}$$

and

$$\overline{\mathbf{z}} \leq \overline{x}_{n+1} + (\overline{\mathbf{x}}' \wedge \overline{\mathbf{z}}). \tag{2'}$$

However, since the existence of $\overline{\mathbf{z}} \wedge (\overline{\mathbf{y}} - (\overline{\mathbf{x}}' \wedge \overline{\mathbf{y}}))$ is not secured we need to prove the parallel to (3) differently. Nevertheless, according to (C) we have

$$\begin{aligned}
\overline{\mathbf{z}} + (\overline{\mathbf{x}}' \wedge \overline{\mathbf{y}}) &= (\overline{\mathbf{z}} + \overline{\mathbf{x}}') \wedge (\overline{\mathbf{z}} + \overline{\mathbf{y}}) \\
\overline{\mathbf{y}} + (\overline{\mathbf{x}}' \wedge \overline{\mathbf{z}}) &= (\overline{\mathbf{y}} + \overline{\mathbf{x}}') \wedge (\overline{\mathbf{y}} + \overline{\mathbf{z}})
\end{aligned}$$

which imply

$$\overline{\mathbf{z}} + (\overline{\mathbf{x}}' \wedge \overline{\mathbf{y}}) \leq \overline{\mathbf{y}} + (\overline{\mathbf{x}}' \wedge \overline{\mathbf{z}}).$$

Therefore, the inequality

$$\overline{\mathbf{z}} \leq (\overline{\mathbf{y}} - (\overline{\mathbf{x}}' \wedge \overline{\mathbf{y}})) + (\overline{\mathbf{x}}' \wedge \overline{\mathbf{z}}) \tag{3'}$$

holds. Using all (1'), (2') and (3') we obtain

$$\overline{\mathbf{z}} \leq (\overline{\mathbf{x}}' \wedge \overline{\mathbf{y}}) + ((\overline{\mathbf{y}} - (\overline{\mathbf{x}}' \wedge \overline{\mathbf{y}})) \wedge \overline{x}_{n+1})$$

similarly to (ii).

In conclusion, it is proved the infimum exists for every $\overline{\mathbf{x}}, \overline{\mathbf{y}} \in G^+$.

And the last part that remains to be proved is the existence of the supremum of arbitrary $\overline{\mathbf{x}}, \overline{\mathbf{y}} \in G^+$.

Let us show that $(\overline{\mathbf{x}}+\overline{\mathbf{y}}) - (\overline{\mathbf{x}} \wedge \overline{\mathbf{y}})$ is the required supremum. First of all, since $\overline{\mathbf{x}} \wedge \overline{\mathbf{y}} \leq \overline{\mathbf{x}}, \overline{\mathbf{y}}$ then

$$\begin{aligned}
\overline{\mathbf{x}} &\leq \overline{\mathbf{x}} + (\overline{\mathbf{y}} - (\overline{\mathbf{x}} \wedge \overline{\mathbf{y}})) = (\overline{\mathbf{x}} + \overline{\mathbf{y}}) - (\overline{\mathbf{x}} \wedge \overline{\mathbf{y}}) \\
\overline{\mathbf{y}} &\leq \overline{\mathbf{y}} + (\overline{\mathbf{x}} - (\overline{\mathbf{x}} \wedge \overline{\mathbf{y}})) = (\overline{\mathbf{x}} + \overline{\mathbf{y}}) - (\overline{\mathbf{x}} \wedge \overline{\mathbf{y}}).
\end{aligned}$$

Let us have $\overline{\mathbf{z}} \in G^+$, $\overline{\mathbf{x}}, \overline{\mathbf{y}} \leq \overline{\mathbf{z}}$ then according to (C)

$$\begin{aligned}
(\overline{\mathbf{x}} + \overline{\mathbf{y}}) - (\overline{\mathbf{x}} \wedge \overline{\mathbf{y}}) &= ((\overline{\mathbf{x}} + \overline{\mathbf{y}}) \wedge (\overline{\mathbf{x}} + \overline{\mathbf{y}})) - (\overline{\mathbf{x}} \wedge \overline{\mathbf{y}}) \\
&\leq ((\overline{\mathbf{z}} + \overline{\mathbf{x}}) \wedge (\overline{\mathbf{z}} + \overline{\mathbf{y}})) - (\overline{\mathbf{x}} \wedge \overline{\mathbf{y}}) \\
&= (\overline{\mathbf{z}} + (\overline{\mathbf{x}} \wedge \overline{\mathbf{y}})) - (\overline{\mathbf{x}} \wedge \overline{\mathbf{y}}) \\
&= \overline{\mathbf{z}}.
\end{aligned}$$

Thus the expression $(\overline{\mathbf{x}}+\overline{\mathbf{y}}) - (\overline{\mathbf{x}} \wedge \overline{\mathbf{y}})$ is indeed the supremum of $\overline{\mathbf{x}}$ and $\overline{\mathbf{y}}$. □

From the claim used in the proof the next theorem immediately follows.

Theorem 3.3.2. *The operation + is distributive with respect to the meet in the lattice $(G^+, \leq)$, i.e.,*

$$\overline{\mathbf{z}} + (\overline{\mathbf{x}} \wedge \overline{\mathbf{y}}) = (\overline{\mathbf{z}} + \overline{\mathbf{x}}) \wedge (\overline{\mathbf{z}} + \overline{\mathbf{y}})$$

for every $\overline{\mathbf{x}}, \overline{\mathbf{y}}, \overline{\mathbf{z}} \in G^+$.

3.4 The l-group

It is known that every abelian semigroup with canceling can be embedded into a group. Using this fact we obtain that $\mathcal{G} = ((G^+)^2/_\sim, +)$ where

$$(\overline{\mathbf{x}}_1, \overline{\mathbf{y}}_1) \sim (\overline{\mathbf{x}}_2, \overline{\mathbf{y}}_2) \iff \overline{\mathbf{x}}_1 + \overline{\mathbf{y}}_2 = \overline{\mathbf{x}}_2 + \overline{\mathbf{y}}_1$$

and

$$[(\overline{\mathbf{x}}_1, \overline{\mathbf{y}}_1)]_\sim + [(\overline{\mathbf{x}}_2, \overline{\mathbf{y}}_2)]_\sim = [(\overline{\mathbf{x}}_1 + \overline{\mathbf{x}}_2, \overline{\mathbf{y}}_1 + \overline{\mathbf{y}}_2)]_\sim$$

is an abelian group and $F : G^+ \longrightarrow G$ such that $F(\overline{\mathbf{x}}) = [(\overline{\mathbf{x}}, \overline{0})]_\sim$ is the embedding of the monoid $\mathcal{G}^+$ to the group $\mathcal{G}$. According to this construction the inverse is defined as follows

$$[(\overline{\mathbf{x}}, \overline{\mathbf{y}})]_\sim^{-1} = [(\overline{\mathbf{y}}, \overline{\mathbf{x}})]_\sim.$$

The group $\mathcal{G}$ can also be equipped with additional structure.

Proposition 3.4.1. *Binary relation $\leq$ on $\mathcal{G}$ defined as follows*

$$[(\overline{\mathbf{x}}_1, \overline{\mathbf{y}}_1)]_\sim \leq [(\overline{\mathbf{x}}_2, \overline{\mathbf{y}}_2)]_\sim \iff \overline{\mathbf{x}}_1 + \overline{\mathbf{y}}_2 \leq \overline{\mathbf{x}}_2 + \overline{\mathbf{y}}_1$$

is an ordering on $\mathcal{G}$. Furthermore, $+$ is monotone with respect to $\leq$.

In order to make the text more transparent we will simplify the notation of the factors by omitting the inner pair of brackets and the equivalence symbol.

Proof of Proposition 3.4.1. First, let us prove $\leq$ is an ordering:

(i) Condition $\overline{\mathbf{x}} + \overline{\mathbf{y}} \leq \overline{\mathbf{x}} + \overline{\mathbf{y}}$ holds for every $[\overline{\mathbf{x}}, \overline{\mathbf{y}}] \in G$ thus $[\overline{\mathbf{x}}, \overline{\mathbf{y}}] \leq [\overline{\mathbf{x}}, \overline{\mathbf{y}}]$, i.e., $\leq$ is reflexive.

(ii) Let $[\overline{\mathbf{x}}_1, \overline{\mathbf{y}}_1] \leq [\overline{\mathbf{x}}_2, \overline{\mathbf{y}}_2]$ and $[\overline{\mathbf{x}}_2, \overline{\mathbf{y}}_2] \leq [\overline{\mathbf{x}}_1, \overline{\mathbf{y}}_1]$ thus $\overline{\mathbf{x}}_1 + \overline{\mathbf{y}}_2 \leq \overline{\mathbf{x}}_2 + \overline{\mathbf{y}}_1 \leq \overline{\mathbf{x}}_1 + \overline{\mathbf{y}}_2$ which implies $[\overline{\mathbf{x}}_1, \overline{\mathbf{y}}_1] = [\overline{\mathbf{x}}_2, \overline{\mathbf{y}}_2]$, i.e., $\leq$ is antisymmetric.

(iii) Let $[\overline{\mathbf{x}}_1, \overline{\mathbf{y}}_1] \leq [\overline{\mathbf{x}}_2, \overline{\mathbf{y}}_2]$ and $[\overline{\mathbf{x}}_2, \overline{\mathbf{y}}_2] \leq [\overline{\mathbf{x}}_3, \overline{\mathbf{y}}_3]$ then by summing the corresponding inequalities $(\overline{\mathbf{x}}_1 + \overline{\mathbf{y}}_2) + (\overline{\mathbf{x}}_2 + \overline{\mathbf{y}}_3) \leq (\overline{\mathbf{x}}_2 + \overline{\mathbf{y}}_1) + (\overline{\mathbf{x}}_3 + \overline{\mathbf{y}}_2)$ and due to the canceling we get $\overline{\mathbf{x}}_1 + \overline{\mathbf{y}}_3 \leq \overline{\mathbf{x}}_3 + \overline{\mathbf{y}}_1$ thus $[\overline{\mathbf{x}}_1, \overline{\mathbf{y}}_1] \leq [\overline{\mathbf{x}}_3, \overline{\mathbf{y}}_3]$, i.e., $\leq$ is transitive.

Second, we prove $+$ is monotone. Let $[\overline{\mathbf{x}}_1, \overline{\mathbf{y}}_1] \leq [\overline{\mathbf{x}}_2, \overline{\mathbf{y}}_2]$ and $[\overline{\mathbf{x}}_3, \overline{\mathbf{y}}_3] \in G$ then due to the commutativity

$$(\overline{\mathbf{x}}_1 + \overline{\mathbf{x}}_3) + (\overline{\mathbf{y}}_2 + \overline{\mathbf{y}}_3) \leq (\overline{\mathbf{x}}_2 + \overline{\mathbf{x}}_3) + (\overline{\mathbf{y}}_1 + \overline{\mathbf{y}}_3)$$

and thus it is obtained

$$[\overline{\mathbf{x}}_1, \overline{\mathbf{y}}_1] + [\overline{\mathbf{x}}_3, \overline{\mathbf{y}}_3] = [\overline{\mathbf{x}}_1 + \overline{\mathbf{x}}_3, \overline{\mathbf{y}}_1 + \overline{\mathbf{y}}_3] \leq [\overline{\mathbf{x}}_2 + \overline{\mathbf{x}}_3, \overline{\mathbf{y}}_2 + \overline{\mathbf{y}}_3] = [\overline{\mathbf{x}}_2, \overline{\mathbf{y}}_2] + [\overline{\mathbf{x}}_3, \overline{\mathbf{y}}_3].$$

□

It is not hard to see that the embedding of $\mathcal{G}^+$ to $\mathcal{G}$ is in fact an isomorphism of partially ordered sets $(G^+, \leq)$ and $(\{[\overline{\mathbf{x}}, \overline{\mathbf{y}}] \in G \mid [\overline{0}, \overline{0}] \leq [\overline{\mathbf{x}}, \overline{\mathbf{y}}]\}, \leq)$. Therefore, it is natural to consider the following theorem.

Theorem 3.4.1. *The ordering $\leq$ is a lattice ordering on $\mathcal{G}$.*

Proof. First, we will show there exists an infimum of $[\overline{\mathbf{x}}_1, \overline{\mathbf{y}}_1], [\overline{\mathbf{x}}_2, \overline{\mathbf{y}}_2] \in G$ in the following form

$$[(\overline{\mathbf{x}}_1 + ((\overline{\mathbf{y}}_1 \vee \overline{\mathbf{y}}_2) - \overline{\mathbf{y}}_1)) \wedge (\overline{\mathbf{x}}_2 + ((\overline{\mathbf{y}}_1 \vee \overline{\mathbf{y}}_2) - \overline{\mathbf{y}}_2)), \overline{\mathbf{y}}_1 \vee \overline{\mathbf{y}}_2].$$

We will begin with an obvious inequality

$$(\overline{\mathbf{x}}_1 + (\overline{\mathbf{y}}_1 \vee \overline{\mathbf{y}}_2)) \wedge ((\overline{\mathbf{x}}_2 + ((\overline{\mathbf{y}}_1 \vee \overline{\mathbf{y}}_2) - \overline{\mathbf{y}}_2)) + \overline{\mathbf{y}}_1) \leq \overline{\mathbf{x}}_1 + (\overline{\mathbf{y}}_1 \vee \overline{\mathbf{y}}_2),$$

which leads to

$$((\overline{\mathbf{x}}_1 + ((\overline{\mathbf{y}}_1 \vee \overline{\mathbf{y}}_2) - \overline{\mathbf{y}}_1)) + \overline{\mathbf{y}}_1) \wedge ((\overline{\mathbf{x}}_2 + ((\overline{\mathbf{y}}_1 \vee \overline{\mathbf{y}}_2) - \overline{\mathbf{y}}_2)) + \overline{\mathbf{y}}_1) \leq \overline{\mathbf{x}}_1 + (\overline{\mathbf{y}}_1 \vee \overline{\mathbf{y}}_2).$$

Next, the distributivity proved in Theorem 3.3.2 is used to obtain

$$((\overline{\mathbf{x}}_1 + ((\overline{\mathbf{y}}_1 \vee \overline{\mathbf{y}}_2) - \overline{\mathbf{y}}_1)) \wedge (\overline{\mathbf{x}}_2 + ((\overline{\mathbf{y}}_1 \vee \overline{\mathbf{y}}_2) - \overline{\mathbf{y}}_2))) + \overline{\mathbf{y}}_1 \leq \overline{\mathbf{x}}_1 + (\overline{\mathbf{y}}_1 \vee \overline{\mathbf{y}}_2),$$

and thus

$$[(\overline{\mathbf{x}}_1 + ((\overline{\mathbf{y}}_1 \vee \overline{\mathbf{y}}_2) - \overline{\mathbf{y}}_1)) \wedge (\overline{\mathbf{x}}_2 + ((\overline{\mathbf{y}}_1 \vee \overline{\mathbf{y}}_2) - \overline{\mathbf{y}}_2)), \overline{\mathbf{y}}_1 \vee \overline{\mathbf{y}}_2] \leq [\overline{\mathbf{x}}_1, \overline{\mathbf{y}}_1].$$

Similarly, it can be proved for $[\overline{\mathbf{x}}_2, \overline{\mathbf{y}}_2]$.

Now, let $[\overline{\mathbf{x}}_3, \overline{\mathbf{y}}_3] \leq [\overline{\mathbf{x}}_1, \overline{\mathbf{y}}_1], [\overline{\mathbf{x}}_2, \overline{\mathbf{y}}_2]$. That means

$$\begin{aligned} \overline{\mathbf{x}}_3 + \overline{\mathbf{y}}_1 &\leq \overline{\mathbf{x}}_1 + \overline{\mathbf{y}}_3 \\ \overline{\mathbf{x}}_3 + \overline{\mathbf{y}}_2 &\leq \overline{\mathbf{x}}_2 + \overline{\mathbf{y}}_3 \end{aligned}$$

from which the following two can be obtained

$$\begin{aligned} \overline{\mathbf{x}}_3 + \overline{\mathbf{y}}_1 + \overline{\mathbf{y}}_2 &\leq \overline{\mathbf{x}}_1 + \overline{\mathbf{y}}_3 + \overline{\mathbf{y}}_2 \\ \overline{\mathbf{x}}_3 + \overline{\mathbf{y}}_2 + \overline{\mathbf{y}}_1 &\leq \overline{\mathbf{x}}_2 + \overline{\mathbf{y}}_3 + \overline{\mathbf{y}}_1. \end{aligned}$$

Putting these together we get

$$\overline{\mathbf{x}}_3 + \overline{\mathbf{y}}_1 + \overline{\mathbf{y}}_2 \leq (\overline{\mathbf{x}}_1 + \overline{\mathbf{y}}_3 + \overline{\mathbf{y}}_2) \wedge (\overline{\mathbf{x}}_2 + \overline{\mathbf{y}}_3 + \overline{\mathbf{y}}_1) = ((\overline{\mathbf{x}}_1 + \overline{\mathbf{y}}_2) \wedge (\overline{\mathbf{x}}_2 + \overline{\mathbf{y}}_1)) + \overline{\mathbf{y}}_3.$$

Therefore,

$$\overline{\mathbf{x}}_3 \leq (((\overline{\mathbf{x}}_1 + \overline{\mathbf{y}}_2) \wedge (\overline{\mathbf{x}}_2 + \overline{\mathbf{y}}_1)) + \overline{\mathbf{y}}_3) - (\overline{\mathbf{y}}_1 + \overline{\mathbf{y}}_2). \tag{1}$$

Hence, we can see that

$$\overline{\mathbf{x}}_3 + (\overline{\mathbf{y}}_1 \vee \overline{\mathbf{y}}_2) \overset{(1)}{\leq} ((\overline{\mathbf{y}}_1 \vee \overline{\mathbf{y}}_2) + ((\overline{\mathbf{x}}_1 + \overline{\mathbf{y}}_2) \wedge (\overline{\mathbf{x}}_2 + \overline{\mathbf{y}}_1)) + \overline{\mathbf{y}}_3) - (\overline{\mathbf{y}}_1 + \overline{\mathbf{y}}_2)$$

continuing to

$$= ((\overline{\mathbf{y}}_1 \vee \overline{\mathbf{y}}_2) + ((\overline{\mathbf{x}}_1 + (\overline{\mathbf{y}}_1 + \overline{\mathbf{y}}_2) - \overline{\mathbf{y}}_1) \wedge (\overline{\mathbf{x}}_2 + (\overline{\mathbf{y}}_1 + \overline{\mathbf{y}}_2) - \overline{\mathbf{y}}_2)) + \overline{\mathbf{y}}_3) - (\overline{\mathbf{y}}_1 + \overline{\mathbf{y}}_2)$$

$$= (((\overline{\mathbf{x}}_1 + (\overline{\mathbf{y}}_1 \vee \overline{\mathbf{y}}_2) - \overline{\mathbf{y}}_1) \wedge (\overline{\mathbf{x}}_2 + (\overline{\mathbf{y}}_1 \vee \overline{\mathbf{y}}_2) - \overline{\mathbf{y}}_2)) + (\overline{\mathbf{y}}_1 + \overline{\mathbf{y}}_2) + \overline{\mathbf{y}}_3) - (\overline{\mathbf{y}}_1 + \overline{\mathbf{y}}_2)$$

where we made use of the distributivity twice. Hence, we got

$$\overline{\mathbf{x}}_3 + (\overline{\mathbf{y}}_1 \vee \overline{\mathbf{y}}_2) \leq ((\overline{\mathbf{x}}_1 + ((\overline{\mathbf{y}}_1 \vee \overline{\mathbf{y}}_2) - \overline{\mathbf{y}}_1)) \wedge (\overline{\mathbf{x}}_2 + ((\overline{\mathbf{y}}_1 \vee \overline{\mathbf{y}}_2) - \overline{\mathbf{y}}_2))) + \overline{\mathbf{y}}_3.$$

Thus, we obtain

$$[\overline{\mathbf{x}}_3, \overline{\mathbf{y}}_3] \leq [(\overline{\mathbf{x}}_1 + ((\overline{\mathbf{y}}_1 \vee \overline{\mathbf{y}}_2) - \overline{\mathbf{y}}_1)) \wedge (\overline{\mathbf{x}}_2 + ((\overline{\mathbf{y}}_1 \vee \overline{\mathbf{y}}_2) - \overline{\mathbf{y}}_2)), \overline{\mathbf{y}}_1 \vee \overline{\mathbf{y}}_2],$$

which means the right-hand side is, indeed, the infimum of $[\overline{\mathbf{x}}_1, \overline{\mathbf{y}}_1]$ and $[\overline{\mathbf{x}}_2, \overline{\mathbf{y}}_2]$.

Second, there exists a supremum of $[\overline{\mathbf{x}}_1, \overline{\mathbf{y}}_1], [\overline{\mathbf{x}}_2, \overline{\mathbf{y}}_2] \in G$ in the form

$$\left([\overline{\mathbf{x}}_1, \overline{\mathbf{y}}_1]^{-1} \wedge [\overline{\mathbf{x}}_2, \overline{\mathbf{y}}_2]^{-1}\right)^{-1}$$

which follows from

$$[\overline{\mathbf{x}}_1, \overline{\mathbf{y}}_1] \leq [\overline{\mathbf{x}}_2, \overline{\mathbf{y}}_2] \iff [\overline{\mathbf{x}}_2, \overline{\mathbf{y}}_2]^{-1} \leq [\overline{\mathbf{x}}_1, \overline{\mathbf{y}}_1]^{-1}. \qquad \square$$

In conclusion of this section the following theorem holds.

Theorem 3.4.2. *If $\mathcal{A}$ is an MV-algebra then $\mathcal{G}$ constructed according to this section is an l-group and there exists an isomorphism of $\mathcal{A}$ on the interval $G_A = \left[\left[\overline{0}, \overline{0}\right], \left[\overline{1}, \overline{0}\right]\right] \subseteq G$.*

Proof. The fact, that $\mathcal{G}$ is an l-group is a direct consequence of Proposition 3.4.1 and Theorem 3.4.1.

We will prove $\mathcal{A}$ is isomorphic to $(G_A, \boxplus, \neg, \left[\overline{0}, \overline{0}\right])$ where

$$[\overline{\mathbf{x}}_1, \overline{\mathbf{y}}_1] \boxplus [\overline{\mathbf{x}}_2, \overline{\mathbf{y}}_2] = \left[\overline{1}, \overline{0}\right] \wedge ([\overline{\mathbf{x}}_1, \overline{\mathbf{y}}_1] + [\overline{\mathbf{x}}_2, \overline{\mathbf{y}}_2])$$

and

$$\neg[\overline{\mathbf{x}}, \overline{\mathbf{y}}] = \left[\overline{1}, \overline{0}\right] + [\overline{\mathbf{x}}, \overline{\mathbf{y}}]^{-1}.$$

Now, we define a mapping $E : A \longrightarrow G_A$ as an extension of F from Proposition 3.3.1 as follows

$$E(x) = \left[F(x), \overline{0}\right] = \left[\overline{\mathbf{x}}, \overline{0}\right]$$

Since F is an isomorphism it suffice to show $E' : G_1^+ \longrightarrow G_A$ defined as

$$E'(\overline{\mathbf{x}}) = \left[\overline{\mathbf{x}}, \overline{0}\right]$$

is a bijection in order to prove E is also a bijection.

Therefore, let $\overline{\mathbf{x}}, \overline{\mathbf{y}} \in G_1^+$ such that $E'(\overline{\mathbf{x}}) = E'(\overline{\mathbf{y}})$ then

$$\begin{aligned} E'(\overline{\mathbf{x}}) = E'(\overline{\mathbf{y}}) &\Rightarrow \left[\overline{\mathbf{x}}, \overline{0}\right] = \left[\overline{\mathbf{y}}, \overline{0}\right] \\ &\Rightarrow \overline{\mathbf{x}} + \overline{0} = \overline{\mathbf{y}} + \overline{0} \\ &\Rightarrow \overline{\mathbf{x}} = \overline{\mathbf{y}}, \end{aligned}$$

i.e., E' is injective. Now, let $[\overline{\mathbf{x}}, \overline{\mathbf{y}}] \in G_A$. This means

$$\left[\overline{0}, \overline{0}\right] \leq [\overline{\mathbf{x}}, \overline{\mathbf{y}}] \leq \left[\overline{1}, \overline{0}\right]$$

thus $\overline{\mathbf{y}} \leq \overline{\mathbf{x}}$ and consequently $\overline{0} \leq \overline{\mathbf{x}} - \overline{\mathbf{y}} \leq \overline{1}$ hold. Hence, $\|\overline{\mathbf{x}} - \overline{\mathbf{y}}\| = 1$ and so we obtain $\overline{\mathbf{x}} - \overline{\mathbf{y}} \in G_1^+$. Moreover,

$$E'(\overline{\mathbf{x}} - \overline{\mathbf{y}}) = \left[\overline{\mathbf{x}} - \overline{\mathbf{y}}, \overline{0}\right] = [\overline{\mathbf{x}}, \overline{\mathbf{y}}],$$

i.e. E' is surjective. In conclusion, both E' and E are bijections.

Let $x, y \in A$ then we can compute

$$\begin{aligned} E(x \oplus y) &= \left[\overline{x \oplus y}, \overline{0}\right] \\ &= \left[\overline{1} \wedge (\overline{x} + \overline{y}), \overline{0}\right] \\ &= \left[(\overline{1} + ((\overline{0} \vee \overline{0}) - \overline{0})) \wedge ((\overline{x} + \overline{y}) + ((\overline{0} \vee \overline{0}) - \overline{0})), \overline{0} \vee \overline{0}\right] \\ &= \left[\overline{1}, \overline{0}\right] \wedge \left[\overline{x} + \overline{y}, \overline{0}\right] \\ &= \left[\overline{x}, \overline{0}\right] \boxplus \left[\overline{y}, \overline{0}\right] \\ &= E(x) \boxplus E(y), \end{aligned}$$

where we used the following observation

$$\begin{aligned} \overline{x \oplus y} &= \overline{(x \oplus \neg x) \wedge (x \oplus y)} \\ &= \overline{x \oplus (\neg x \wedge y)} \\ &= \overline{x} + (\overline{\neg x} \wedge \overline{y}) \\ &= \overline{1} \wedge (\overline{x} + \overline{y}). \end{aligned}$$

And moreover, we get

$$\begin{aligned} E(\neg x) &= \left[\overline{\neg x}, \overline{0}\right] \\ &= \left[\overline{1} - \overline{x}, \overline{0}\right] \\ &= \left[\overline{1}, \overline{0}\right] + \left[\overline{0}, \overline{x}\right] \\ &= \neg \left[\overline{x}, \overline{0}\right] \\ &= \neg E(x). \end{aligned}$$

using the fact that $\overline{\neg x} = \overline{\neg x} + \overline{x} - \overline{x} = \overline{1} - \overline{x}$. Finally, since $E(0) = \left[\overline{0}, \overline{0}\right]$ we have proved E is indeed the claimed isomorphism. □

Remark. *Note that for $x \leq \neg y$, $x, y \in A$ the equality*

$$E(x) \boxplus E(y) = E(x) + E(y)$$

holds.

4 Representation theorems

This section is the last necessary part of the coveted representation theorem and it will be largely adopted from *Algebraic Methods in Quantum Logic*, see [2].

4.1 Finite embedding property

First, we need to define a natural generalization of an algebra.

Definition 4.1.1. *Let* $\mathcal{A} = (A, F)$ *be an algebra and* $X \subseteq A$. *A* partial algebra *is a pair* $\mathcal{A}|_X = (X, F)$, *where for any* $f \in F_n$ *and all* n*-tuples* $(x_1, \ldots, x_n) \in X^n$,

$$f^{\mathcal{A}|_X}(x_1, \ldots, x_n) \quad \textit{is defined if and only if} \quad f^{\mathcal{A}}(x_1, \ldots, x_n) \in X.$$

We then put

$$f^{\mathcal{A}|_X}(x_1, \ldots, x_n) := f^{\mathcal{A}}(x_1, \ldots, x_n).$$

This allows us to define the *finite embeddability property*:

Definition 4.1.2. *An algebra* $\mathcal{A} = (A, F)$ *satisfies* the finite embeddability property for the class $\mathcal{K}$ *of algebras of the same type if for any finite subset* $X \subseteq A$, *there exists a finite algebra* $\mathcal{B} \in \mathcal{K}$ *and an embedding* $\rho : \mathcal{A}|_X \hookrightarrow \mathcal{B}$, *i.e., an injective mapping* $\rho : X \to B$ *satisfying the property*

$$\rho(f^{\mathcal{A}|_X}(x_1, \ldots, x_n)) = f^{\mathcal{B}}(\rho(x_1), \ldots, \rho(x_n))$$

for every $x_1, \ldots, x_n \in X$ *and* $f \in F_n$ *such that* $f^{\mathcal{A}|_X}(x_1, \ldots, x_n)$ *is defined.*

The proof of the following theorem can be found in [2].

Theorem 4.1.1. *Let* $\mathcal{A} = (A, F)$ *be an algebra and* $\mathcal{K}$ *a class of algebras of the same finite type. If* $\mathcal{A}$ *satisfies the finite embeddability property for* $\mathcal{K}$ *then* $\mathcal{A} \in ISP_U(\mathcal{K})$.

4.2 The embedding

In the forthcoming text we will make use of this alternative formulation of the Farkas' lemma on rationals:

Theorem 4.2.1. *Let* A *be a matrix in* $\mathbb{Q}^{m\times n}$ *an let* $\mathbf{b}$ *be a column vector in* $\mathbb{Q}^m$. *The system*

$$\mathrm{A} \cdot \mathbf{x} \leq \mathbf{b}$$

has no solution if and only if there exists a row vector $\boldsymbol{\lambda} \in \mathbb{Q}^m$ *such that* $\boldsymbol{\lambda} \geq \mathbf{0}_m$, $\boldsymbol{\lambda} \cdot \mathrm{A} = \mathbf{0}_\mathrm{n}$ *and* $\boldsymbol{\lambda} \cdot \mathbf{b} < 0$.

The ideas of the proof of the following lemma are borrowed from [2].

Lemma 4.2.1. *Let $\mathcal{A}$ be a linearly ordered MV-algebra and $X \subseteq A \setminus \{0\}$ a finite subset. Then there is a rational-valued map $s : X \cup \{0,1\} \longrightarrow \mathbb{Q} \cap [0,1]$ such that the following conditions hold*

(i) $s(0) = 0$, $s(1) = 1$,

(ii) if $x, y, x \oplus y \in X \cup \{0,1\}$ *and* $x \leq \neg y$ *then* $s(x \oplus y) = s(x) + s(y)$,

(iii) $s(x) > 0$ *for every* $x \in X$.

Proof. Let us define

$$Y(X) := \{x \oplus y \mid x, y \in X \cup \{0,1\}\}.$$

Hence $Y(X) \subseteq A$ is finite and furthermore $X, X \oplus X \subseteq Y(X)$ and $0, 1 \in Y(X)$. Since $\mathcal{A}$ is linearly ordered we can assume $Y(X) = \{y_0 = 0 < y_1 < \cdots < y_n = 1\}$ and put $\mathbf{y} = (y_1, \ldots . y_n)^T \in Y(X)^n$. Now, for every $x \in X$ there is an index $1 \leq j_x \leq n$ such that $x = y_{j_x}$.

Let $x, y \in X$, $x \neq y$ and $x \leq \neg y$ then by $\mathbf{a}^1_{x,y}, \mathbf{a}^2_{x,y} \in \mathbb{Z}^n$ we denote non-negative row vectors such that

$$\mathbf{a}^1_{x,y}(j) = \begin{cases} 1 & \text{if } j = j_x \text{ or } j = j_y; \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \mathbf{a}^2_{x,y}(j) = \begin{cases} 1 & \text{if } j = j_{x \oplus y}; \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, let $x \in X$, $x \leq \neg x$ then by $\mathbf{a}^1_{x,x}, \mathbf{a}^2_{x,x} \in \mathbb{Z}^n$ we denote non-negative row vectors such that

$$\mathbf{a}^1_{x,x}(j) = \begin{cases} 2 & \text{if } j = j_x; \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \mathbf{a}^2_{x,x}(j) = \begin{cases} 1 & \text{if } j = j_{x \oplus x}; \\ 0 & \text{otherwise.} \end{cases}.$$

Let us have $i \in \{1, 2\}$ and let $\mathrm{A^i}$ be a matrix consisting of the rows $\mathbf{a}^i_{x,y}$ such that $x, y \in X$ and $x \leq \neg y$. We define $\mathrm{A} = \mathrm{A}^1 - \mathrm{A}^2$ and $m = |\{\{x, y\} \mid x, y \in X, x \leq \neg y\}|$.

According to the previous section, our linearly ordered MV-algebra $\mathcal{A}$ is isomorphic to the interval $G_A = \left[\left[\overline{0}, \overline{0}\right], \left[\overline{1}, \overline{0}\right]\right]$ in the respective l-group $\mathcal{G}$ which is evidently also linearly ordered. Thus for $x, y \in A$ such that $x \leq \neg y$ the sum $x \oplus y$ coincides with the sum $x + y$ computed in $\mathcal{G}$.

It follows that

$$\mathrm{A}^1 \cdot \mathbf{y} = \mathrm{A}^2 \cdot \mathbf{y},$$

therefore

$$\mathrm{A} \cdot \mathbf{y} = \mathbf{0}_{\mathrm{m}},$$

computed in $\mathcal{G}$.

Let us have the following system of linear inequalities with variables $z_1, \ldots, z_n$:

$$\begin{pmatrix} -\mathrm{E_n} \\ \mathrm{A} \\ -\mathrm{A} \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \leq \begin{pmatrix} -\mathbf{1}_n \\ \mathbf{0}_m \\ \mathbf{0}_m \end{pmatrix} \tag{4}$$

Due to Farkas' lemma for rationals this system does not have a solution in $\mathbb{Q}^n$ if and only if there is a row vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{n+2m}) \in \mathbb{Z}^{n+2m}$, $\boldsymbol{\lambda} \geq \mathbf{0}_{n+2m}$ such that

$$\boldsymbol{\lambda} \cdot \begin{pmatrix} -\mathrm{E_n} \\ \mathrm{A} \\ -\mathrm{A} \end{pmatrix} = 0 \quad \text{and} \quad \boldsymbol{\lambda} \cdot \begin{pmatrix} -\mathbf{1}_n \\ \mathbf{0}_m \\ \mathbf{0}_m \end{pmatrix} < 0. \tag{5}$$

Let us assume there exists a vector $\boldsymbol{\lambda} \in \mathbb{Z}^{n+2m}$ satisfying (5). Therefore, there is an index $1 \leq j_0 \leq n$ such that $\lambda_{j_0} > 0$.

Now, we can (again in $\mathcal{G}$) compute

$$0 = \boldsymbol{\lambda} \cdot \begin{pmatrix} -\mathrm{E_n} \\ \mathrm{A} \\ -\mathrm{A} \end{pmatrix} \cdot \mathbf{y} = \boldsymbol{\lambda} \cdot \begin{pmatrix} -\mathbf{y} \\ \mathbf{0}_m \\ \mathbf{0}_m \end{pmatrix} = -\sum_{j=1}^{n} \lambda_j y_j \tag{6}$$

where $\mathbf{y} = (y_1, \ldots, y_n)^T \in \mathcal{G}^n$.

Since $\lambda_{j_0} > 0$, $\lambda_1, \ldots, \lambda_{n+2m} \geq 0$ and $y_1, \ldots, y_n$ are positive non-zero elements in $\mathcal{G}$, we obtain $\sum_{j=1}^{n} \lambda_j y_j$ is a positive non-zero element of $\mathcal{G}$, which contradicts (6).

Therefore, the system (4) has a rational valued solution $(q_1, \ldots, q_n)$ and moreover it is easy to see $\mathbf{q} = (q_1, \ldots, q_n) \geq \mathbf{1}_n$.

Finally, we define a map $s : X \cup \{0,1\} \longrightarrow [0,1] \cap \mathbb{Q}$ by

$$s(x) = \begin{cases} \frac{q_{j_x}}{q_n} & \text{if } x \in X; \\ 0 & \text{if } x = 0; \\ 1 & \text{if } x = 1, \end{cases}$$

which clearly satisfies conditions (i) and (iii) of this lemma. Condition (ii) also holds since for $x, y \in X$, $x \leq \neg y$ we have

$$s(x \oplus y) = \frac{q_{j_{x \oplus y}}}{q_n} = \frac{q_{j_x} + q_{j_y}}{q_n} = s(x) + s(y)$$

where we used the fact $\mathbf{q}$ is a solution of a system $(\mathrm{A}^1 - \mathrm{A}^2) \cdot \mathbf{z} = \mathbf{0}_\mathrm{m}$ consequent upon (4) (the case of x or $y \in \{0,1\}$ is also not hard to see). □

Now, we make use of the mapping s in the construction of the embedding of any finite linear $\mathcal{A}|_X$ into a finite linear MV-algebra. The following embedding lemma including the proof with a slight correction is also adopted from [2].

Lemma 4.2.2. *Let $\mathcal{A}$ be a linearly ordered MV-algebra and $X \subseteq A$ its finite subset. Then there is an embedding $f : \mathcal{A}|_X \hookrightarrow \mathcal{L}_k$ where $\mathcal{L}_k \subseteq [0,1]$ is the linearly ordered finite MV-algebra on the set $\{0, \frac{1}{k}, \frac{2}{k}, \ldots, 1\}$.*

Proof. We may suppose X is closed under $\neg$ because adding all elements $\neg x$ where $x \in X$ would not affect the finiteness of X and the final embedding would simply be the restriction to the original set X.

We put

$$Y = \{x \ominus y \mid x, y \in X \cup \{0,1\}\} \setminus \{0\}.$$

Furthermore, let $s : Y \cup \{0,1\} \longrightarrow [0,1] \cap \mathbb{Q}$ be the respective mapping from Lemma 4.2.1 for the set Y.

Let $f = s|_X$ be the restriction of s to the set X. Now, let $x, y \in X$, $x < y$ then there exists $z \in Y$ such that $x \leq \neg z$ and $x \oplus z = y$. Thus,

$$f(y) = s(y) = s(x \oplus z) = s(x) + s(z) > s(x) = f(x)$$

since $s(z) > 0$. Hence, f is injective.

Let $f(X) \setminus \{0\} = \{\frac{p_1}{q_1}, \ldots, \frac{p_l}{q_l}\}$ for some $p_1, q_1, \ldots, p_l, q_l \in \mathbb{N}$ and let us set k as the least common multiple of $q_1, \ldots, q_l$. Then clearly $f(X) \subseteq \{0, \frac{1}{k}, \frac{2}{k}, \ldots, 1\}$.

If $0 \in X$ then by the definition of the mapping s we immediately have $f(0) = s(0) = 0$.

If both $x, \neg x \in X$ then we obtain

$$f(\neg x) = s(\neg x) = 1 - s(x) = \neg s(x) = \neg f(x)$$

since evidently $\neg x \leq \neg x$ and so $s(\neg x) + s(x) = s(\neg x \oplus x) = s(1) = 1$.

Finally, let $x, y \in X$ such that $x \oplus y \in X$. Then:

(i) If $x \leq \neg y$ then

$$f(x \oplus y) = s(x \oplus y) = s(x) + s(y) = f(x) + f(y).$$

(ii) If $\neg y < x$ then

$$f(x \oplus y) = s(x \oplus y) = s(1) = 1.$$

And conversely, $x \ominus \neg y \in Y$ and $x \ominus \neg y = y \ominus \neg x \leq y$ and thus

$$\begin{aligned} s(x) &= s((x \ominus \neg y) \oplus \neg y) \\ &= s(x \ominus \neg y) + s(\neg y) \\ &\geq s(\neg y) \\ &= 1 - s(y). \end{aligned}$$

It follows that

$$1 \geq f(x) \oplus f(y) = \min(1, \mathrm{s(x)} + \mathrm{s(y)}) \geq \min(1, (1 - \mathrm{s(y)}) + \mathrm{s(y)}) = 1,$$

i.e., the condition $f(x) + f(y) = 1 = f(x \oplus y)$ holds.

In conclusion, f is indeed the required embedding. □

4.3 Representation by the standard MV-algebra

The main objective of this work was to use the subdirect, l-group and ultraproduct representations to show there is an MV-algebra generating the whole quasi-variety of MV-algebras. To achieve that, we need to find such an MV-algebra first.

It is clear, the forthcoming properly defines an MV-algebra.

Definition 4.3.1. The standard MV-algebra $\mathcal{S} = ([0,1], \oplus, \neg, 0)$ *is an algebra of type (2,1,0) where* $[0,1]$ *is a unit interval of* $\mathbb{R}$ *and the operations are defined as follows* $x \oplus y = \min(1, \mathrm{x} + \mathrm{y})$ *and* $\neg x = 1 - x$.

Note that the MV-algebra $\mathcal{L}_k$ from Lemma 4.2.2 is in fact a finite subalgebra of $\mathcal{S}$ for any k.

Combining Theorem 4.1.1 and Lemma 4.2.2 we immediately obtain these two consequences:

Theorem 4.3.1. *Any linearly ordered MV-algebra can be embedded into an ultraproduct of finite linearly ordered MV-algebras.*

Theorem 4.3.2. *Any linearly ordered MV-algebra can be embedded into an ultrapower of the standard MV-algebra.*

Furthermore, due to Theorem 2.4.1 we obtain two more results:

Theorem 4.3.3. *Any MV-algebra can be embedded into a subdirect product of ultraproducts of finite linearly ordered MV-algebras.*

Theorem 4.3.4. *Any MV-algebra can be embedded into a subdirect product of ultrapowers of the standard MV-algebra.*

This yield one immediate consequence.

Theorem 4.3.5. *The variety, resp. the quasi-variety of MV-algebras is generated, resp. generated as a quasi-variety by the standard MV-algebra, i.e.,*

$$\mathcal{MV} = \mathcal{V}(\mathcal{S}) = \mathcal{QV}(\mathcal{S}).$$

Summary

The primary objective of this work was to formulate some of the representation theorems about MV-algebras. As a conjunction of those theorems we acquired the final representation theorem providing us with a generator of both the variety of MV-algebras and the quasi-variety of MV-algebras.

The second section presented basic properties of MV-algebras. It also set the basis for further parts of this work and it concluded by recalling Chang's Subdirect Representation Theorem.

Mundici's functor, which establishes a categorical equivalence between the categories of MV-algebras and lattice-ordered Abelian groups with a strong unit, is a necessary step towards the final theorem. The third section presented the primary outcome of this work, namely the new approach to the construction of Mundici's functor as a parallel to the original construction using good sequences. Thus it yielded the second representation theorem.

Finally, in the fourth section we recalled the definition of the finite embeddability property. In connection with the l-group representation we were able to prove the ultraproduct representation of linear MV-algebras. And in conclusion, we pointed out the fundamental position of the standard MV-algebra.

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