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NESTACIONÁRNÍ POHYB TUHÉHO TĚLESA V KAPALINĚ

THE NONSTATIONARY MOTION OF SOLID BODY IN A LIQUID

DIPLOMOVÁ PRÁCE
MASTER'S THESIS

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ZADÁNÍ DIPLOMOVÉ PRÁCE

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Ředitel ústavu Vám v souladu se zákonem č.111/1998 o vysokých školách a se Studijním a zkušebním řádem VUT v Brně určuje následující téma diplomové práce:

Nestacionární pohyb tuhého tělesa v kapalině

v anglickém jazyce:

The nonstationary motion of solid body in a liquid

Stručná charakteristika problematiky úkolu:

Řešení bude vycházet z nestacionárních Navierových-Stokesových rovnic. Předpokládá se laminární proudění a malé kmity tělesa. Cílem je určení rychlosti a tlaku v kapalině, stanovení přídavných silových účinků kapaliny na těleso a popis pohybu tělesa.

Cíle diplomové práce:

Formulace problému, popis algoritmu řešení, sestavení programu v MATLABu, výpočet několika praktických úloh.

Seznam odborné literatury:

[1] F. Axisa, J. Antunes: Modelling of Mechanical Systems: Fluid Structure Interaction, Vol 3, Butterworth-Heinemann, 2007.
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[4] P.G. Gresho, R.S. Sani: Incompressible Flow and the Finite Element Method. Vol 2: Isothermal Laminar Flow, John Wiley & Sons, Chichester, 2000.
[5] M.P. Paidoussis: Fluid-Structure Interactions: Slender Structures and Axial Flow, Volume 1, Academic Press, London, 1998.

Vedoucí diplomové práce: doc. RNDr. Libor Čermák, CSc.

Termín odevzdání diplomové práce je stanoven časovým plánem akademického roku 2009/2010.

V Brně, dne 20.11.2009

L.S.

prof. RNDr. Josef Šlapal, CSc.
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Děkan fakulty

Abstrakt
Obsahem této práce je numerická simulace dvoudimenzionálního proudění nestlačitelné vazké kapaliny. Uvažujeme rotující elipsu soustředně umístěnou v kružnici. Prostor mezi elipsou a kružnicí je vyplněn kapalinou. Cílem je popsat proudění kapaliny vyvolané otáčející se elipsou, tzn. stanovit rychlostní pole a rozložení tlaku. Dále pak chceme stanovit přídavné silové účinky kapaliny působící na elipsu. Tyto výsledky získáme řešením Navierových-Stokesových rovnic metodou konečných prvků. Důraz je kladen na odvození numerického schématu v maticové formě vhodné pro numerickou implementaci. Časově závislá výpočetní síť je popsána pomocí Arbitrary Lagrangian-Eulerian (ALE) formulace. Pro obdržení relevantních výsledků je nutná stabilizace metody konečných prvků. Uvedené výsledky naznačují, že odvozená metoda je dostatečně přesná.

Summary
The subject of this thesis is the numerical simulation of the two-dimensional incompressible viscous flow. We consider a rotating ellipse concentric with a circle. The space between the ellipse and the circle is filled with a fluid. Our goal is to describe the fluid flow caused by the rotating ellipse, i.e., to determine the velocity field and pressure distribution. Further, we want to determine the additional effect of the fluid acting on the ellipse. These results are obtained as a solution of the Navier-Stokes equations by the finite element method. Special emphasis has been put on the derivation of the numerical scheme in a matrix form suitable for algorithmization. The Arbitrary Lagrangian-Eulerian (ALE) method has been used to incorporate the moving domain into the algorithm. A suitable stabilization technique of the finite element method is necessary to obtain relevant outcome. Presented results indicate sufficient robustness and accuracy of the numerical algorithm.

klíčová slova
Navierovy-Stokesovy rovnice, metoda konečných prvků, ALE formulace, stabilizace

key words
Navier-Stokes equations, finite element method, ALE formulation, stabilization

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I hereby declare this master's thesis is the result of my own work and all the used sources are listed in the bibliography.

Jiří Stejskal

I would like to express my cordial thanks to doc. RNDr. Libor Čermák, CSc. for supervising my master's thesis, for his advice, valuable comments and suggestions and for his willing support.

Jiří Stejskal

Contents

Introduction

The computational fluid dynamics has experienced a huge progress in recent years, mainly due to the rapidly rising power of modern computers. Also the finite element method has emerged as one of the most used and powerful numerical methods so far. Among the main reasons of its popularity is the ease of use in modelling complex geometries, consistent treatment of various boundary conditions and the possibility to be programmed in a general and easily adaptable way. As for the applications of the finite element method, there are many for example in aircraft industry, mechanical engineering (turbines, pumps, etc.) and civil engineering.

In this thesis we focus our attention on the two-dimensional incompressible viscous flow. The rotating ellipse placed concentrically in a circle will serve us as an example. The mathematical model for this problem consists of the Navier-Stokes equations and the continuity equation. This system of equations is solved by the finite element method using the popular Taylor-Hood finite element P_2/P_1.

One encounters a lot of difficulties when solving the Navier-Stokes equations. First of all, it is the stability of a solution. In this thesis we use a stabilization using the following methods (see [2]),

- SUPG (Streamline Upwind Petrov-Galerkin),
- PSPG (Pressure Stabilizing Petrov-Galerkin),
- LSIC (Least Squares on Incompressibility Constraint).

Another possibility is a stabilization by the GLS (Galerkin Least Squares) method (see [1]). Next we face the problem of moving time-dependent computational mesh which is worked out using the Arbitrary Lagrangian-Eulerian formulation of the Navier-Stokes equations.

In section 1 we introduce the classical and weak formulation of the problem. Section 2 deals with the space discretization and the finite element approximation. One of the main parts of this thesis is the section 3. Here we derive the numerical algorithm for the solution of the Navier-Stokes equations by the finite element method. In section 4 we discuss the stabilization techniques and in section 5 we present the rotating ellipse example. The ALE form of the Navier-Stokes equations is derived here. In section 6 we present some numerical results. Throughout this thesis all the main results are presented in a consistent matrix form.

Algorithm discussed in this thesis was implemented in MATLAB by doc. RNDr. Libor Čermák, CSc. Minor changes to adjust this program to solve the rotating ellipse problem, check of the correctness of the formulas and numerical experiments were made by the author.

1 Navier-Stokes Equations

1.1 Classical formulation

Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain with the Lipschitz boundary $\partial\Omega \equiv \Gamma$ and let Γ_1, Γ_2 be parts of the boundary Γ such that $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$. The incompressible viscous flow is described by the Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + \lambda(\mathbf{u}\cdot\nabla)\mathbf{u} - 2\nu\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{u}) + \frac{1}{\varrho}\nabla p = \mathbf{f} \qquad \text{in } \Omega\times(0,T) \tag{1}$$

and the continuity equation

$$\nabla\cdot\mathbf{u} = 0 \qquad \text{in } \Omega\times(0,T)\,, \tag{2}$$

where

- $\mathbf{u} = \Big(u_1(\mathbf{x},t), u_2(\mathbf{x},t)\Big)^T = (u_1,u_2)^T$ is the velocity vector,
- $\mathbf{x} = (x_1,x_2)^T$ is a point in Ω,
- $p = p(\mathbf{x},t)$ denotes the pressure,
- ϱ is the density,
- ν denotes the kinematic viscosity,
- $\mathbf{f} = \Big(f_1(\mathbf{x},t), f_2(\mathbf{x},t)\Big)^T = (f_1,f_2)^T$ is a vector of the volume force density,
- $\boldsymbol{\varepsilon}(\mathbf{u}) = \{\varepsilon_{ij}(\mathbf{u})\}_{i,j=1}^2$,
  $$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \gamma\frac{\partial u_j}{\partial x_i}\right)\,, \quad i,j=1,2\,,$$
  is the rate-of-deformation tensor and γ is a constant which is equal to one or zero (its meaning will be explained later).
- λ is a constant which is equal to one or zero: for $\lambda = 0$ we have the linear Stokes problem and for $\lambda = 1$ we obtain the nonlinear Navier-Stokes problem.

For the sake of uniqueness of the solution we have to add the initial condition

$$\mathbf{u} = \mathbf{u}_0 \qquad \text{in } \Omega \text{ for } t = 0\,, \tag{3}$$

the Dirichlet boundary condition prescribed on Γ_1

$$\mathbf{u} = \mathbf{g} \qquad \text{on } \Gamma_1\times(0,T) \tag{4}$$

and the condition of Neumann type which gives a surface force on Γ_2

$$2\nu\boldsymbol{\varepsilon}(\mathbf{u})\mathbf{n} - \frac{p}{\varrho}\mathbf{n} = \boldsymbol{\sigma} \qquad \text{on } \Gamma_2\times(0,T)\,, \tag{5}$$

where

- $\mathbf{g} = \Big(g_1(\mathbf{x},t), g_2(\mathbf{x},t)\Big)^T = (g_1, g_2)^T$ is the given velocity vector,
- $\boldsymbol{\sigma} = \Big(\sigma_1(\mathbf{x},t), \sigma_2(\mathbf{x},t)\Big)^T = (\sigma_1, \sigma_2)^T$ is the surface force vector,
- $\mathbf{n} = \Big(n_1(\mathbf{x}), n_2(\mathbf{x})\Big)^T = (n_1, n_2)^T$ denotes the unit outer normal vector.

In the condition (5) the constant γ plays its role. For $\gamma = 1$ we get the physically meaningful boundary condition assigning a normal stress on Γ_2 , whereas $\gamma = 0$ gives an artificial boundary condition which is sometimes called the "do nothing condition" (see, e.g., [3]).

Let us now write down the equations above in a more insightful component form. We then have the Navier-Stokes equations

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} + \lambda\left(u_1\frac{\partial u_1}{\partial x_1} + u_2\frac{\partial u_1}{\partial x_2}\right) - \\
&\qquad - \frac{\partial}{\partial x_1}\left[\nu\left(\frac{\partial u_1}{\partial x_1} + \gamma\frac{\partial u_1}{\partial x_1}\right)\right] - \frac{\partial}{\partial x_2}\left[\nu\left(\frac{\partial u_1}{\partial x_2} + \gamma\frac{\partial u_2}{\partial x_1}\right)\right] + \frac{1}{\varrho}\frac{\partial p}{\partial x_1} = f_1\,, \\
&\frac{\partial u_2}{\partial t} + \lambda\left(u_1\frac{\partial u_2}{\partial x_1} + u_2\frac{\partial u_2}{\partial x_2}\right) - \\
&\qquad - \frac{\partial}{\partial x_1}\left[\nu\left(\frac{\partial u_2}{\partial x_1} + \gamma\frac{\partial u_1}{\partial x_2}\right)\right] - \frac{\partial}{\partial x_2}\left[\nu\left(\frac{\partial u_2}{\partial x_2} + \gamma\frac{\partial u_2}{\partial x_2}\right)\right] + \frac{1}{\varrho}\frac{\partial p}{\partial x_2} = f_2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \Omega\times(0,T)\,,
\end{aligned} \tag{6}$$

the continuity equation

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0 \qquad \text{in } \Omega\times(0,T) \tag{7}$$

and the boundary condition of Neumann type

$$\begin{aligned}
&\left(-\frac{p}{\varrho} + \nu\left[\frac{\partial u_1}{\partial x_1} + \gamma\frac{\partial u_1}{\partial x_1}\right]\right)n_1 + \nu\left(\frac{\partial u_1}{\partial x_2} + \gamma\frac{\partial u_2}{\partial x_1}\right)n_2 = \sigma_1\,, \\
&\nu\left(\frac{\partial u_2}{\partial x_1} + \gamma\frac{\partial u_1}{\partial x_2}\right)n_1 + \left(-\frac{p}{\varrho} + \nu\left[\frac{\partial u_2}{\partial x_2} + \gamma\frac{\partial u_2}{\partial x_2}\right]\right)n_2 = \sigma_2 \qquad \text{on } \Gamma_2\times(0,T)\,.
\end{aligned} \tag{8}$$

The component form of remaining conditions is clear. In the rest of this thesis both vector and component notation will be used. Besides the boundary conditions we have just mentioned there are also other types of conditions to be imposed, we shall not discuss them in this work, however.

The classical formulation of our problem may be stated as follows: *Find functions* $\mathbf{u} \in C^2([\Omega\times(0,T)]^2)$ *and* $p \in C^1(\Omega\times(0,T))$ *such that the equation* (1) *and the conditions* (3)-(5) *are satisfied.* Finally, let us point out that the Navier-Stokes equations are nothing but the expression of the balance of momentum and that the continuity equation is the consequence of the conservation of mass.

1.2 Weak formulation

In order to be able to introduce the weak formulation of our problem some facts from the function spaces theory are needed. These can be found in the appendix. We introduce the spaces V and V_g in the following way

$$V = \left\{ u \in H^1(\Omega);\ u = 0 \text{ on } \Gamma_1 \text{ in the sense of traces} \right\} ,$$
$$V_g = \left\{ u \in H^1(\Omega);\ u = g \text{ on } \Gamma_1 \text{ in the sense of traces} \right\} .$$

where $H^1(\Omega)$ is the Sobolev space defined in the appendix.

Let us now derive the weak formulation of the Navier-Stokes equations with the boundary conditions (4) and (5). We take the first equation in (6), multiply it by an arbitrary test function $v_1 \in V$ and integrate over Ω. After applying the divergence theorem and the fact that the functions v_1 are equal to zero on Γ_1, we get for an arbitrary $t \in (0, T)$

$$\begin{aligned} \int\limits_{\Omega} \Bigg\{ \frac{\partial u_1}{\partial t} v_1 + \lambda \left[\sum_{i=1}^{2} u_i \frac{\partial u_1}{\partial x_i} v_1 \right] + \nu \left[(1+\gamma) \frac{\partial u_1}{\partial x_1}\frac{\partial v_1}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\frac{\partial v_1}{\partial x_2} + \gamma \frac{\partial u_2}{\partial x_1}\frac{\partial v_1}{\partial x_2} \right] - \\ - \frac{p}{\varrho}\frac{\partial v_1}{\partial x_1} \Bigg\} \mathrm{d}x_1 \mathrm{d}x_2 = \int\limits_{\Omega} f_1 v_1 \, \mathrm{d}x_1 \mathrm{d}x_2 + \int\limits_{\Gamma_2} \sigma_1 v_1 \, \mathrm{d}S \, . \end{aligned} \tag{9}$$

The second equation in (6) is treated similarly. We multiply it by an arbitrary test function $v_2 \in V$ and integrate over Ω. In the same way as before we obtain

$$\begin{aligned} \int\limits_{\Omega} \Bigg\{ \frac{\partial u_2}{\partial t} v_2 + \lambda \left[\sum_{i=1}^{2} u_i \frac{\partial u_2}{\partial x_i} v_2 \right] + \nu \left[(1+\gamma) \frac{\partial u_2}{\partial x_2}\frac{\partial v_2}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\frac{\partial v_2}{\partial x_1} + \gamma \frac{\partial u_1}{\partial x_2}\frac{\partial v_2}{\partial x_1} \right] - \\ - \frac{p}{\varrho}\frac{\partial v_2}{\partial x_2} \Bigg\} \mathrm{d}x_1 \mathrm{d}x_2 = \int\limits_{\Omega} f_2 v_2 \, \mathrm{d}x_1 \mathrm{d}x_2 + \int\limits_{\Gamma_2} \sigma_2 v_2 \, \mathrm{d}S \, . \end{aligned} \tag{10}$$

The continuity equation is multiplied by the test function q/ϱ, where $q \in L^2(\Omega)$. After integration over Ω we have

$$\int\limits_{\Omega} \frac{q}{\varrho} \left[\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right] \mathrm{d}x_1 \mathrm{d}x_2 = 0 \, . \tag{11}$$

After this the weak formulation of our problem reads: *For any fixed $t \in (0, T)$ find $u_1(\cdot, t)$, $u_2(\cdot, t) \in V_g$ and $p(\cdot, t) \in L^2(\Omega)$, such that* (9)-(11) *are satisfied for arbitrary test functions $v_1, v_2 \in V$ and $q \in L^2(\Omega)$.*

If we sum the equations (9)-(11), we can write the weak formulation in somehow more elegant form: *For any fixed $t \in (0, T)$ find $\mathbf{u}(\cdot, t) \in V_g^2$ and $p(\cdot, t) \in L^2(\Omega)$, such that*

$$a(\mathbf{u}, p, \mathbf{u};\ \mathbf{v}, q) = 0 \, , \tag{12}$$

where

$$a(\mathbf{u}, p, \mathbf{w};\, \mathbf{v}, q) = \int\limits_{\Omega} \left\{ \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{v} + \lambda[(\mathbf{w} \cdot \nabla)\mathbf{u}] \cdot \mathbf{v} + \nu(\nabla \mathbf{u} :: \nabla \mathbf{v}) - \frac{p}{\varrho} \nabla \cdot \mathbf{v} + \right.$$

$$\left. + \frac{q}{\varrho} \nabla \cdot \mathbf{u} \right\} \mathrm{d}\mathbf{x} - \int\limits_{\Omega} \mathbf{f} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x} - \int\limits_{\Gamma_2} \boldsymbol{\sigma} \cdot \mathbf{v} \, \mathrm{d}S$$

for arbitrary test functions $\mathbf{v} \in V^2$ *and* $q \in L^2(\Omega)$. Here

$$\nabla \mathbf{u} :: \nabla \mathbf{v} = (1 + \gamma) \left[\frac{\partial u_1}{\partial x_1} \frac{\partial v_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \frac{\partial v_2}{\partial x_2} \right] + \frac{\partial u_1}{\partial x_2} \frac{\partial v_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \frac{\partial v_2}{\partial x_1} +$$

$$+ \gamma \left[\frac{\partial u_2}{\partial x_1} \frac{\partial v_1}{\partial x_2} + \frac{\partial u_1}{\partial x_2} \frac{\partial v_2}{\partial x_1} \right].$$

2 Space Discretization

Let us suppose that Ω is polygonal. We perform a triangulation on $\overline{\Omega}$, i. e., we cover it with a triangulation $\mathcal{T}$ consisting of triangular elements e such that

$$\overline{\Omega} = \bigcup_{e \in \mathcal{T}} \bar{e} \,.$$

Next, we shall suppose that the closures of any two distinct triangles are either disjoint, or they have a common vertex or edge. The triangles will be often called *elements* and the vertices of triangles will be often referred to as *nodes*.

2.1 Hood-Taylor finite element

The Hood-Taylor finite element P_2/P_1 will be used in this thesis for the finite element method discretization. This means that the velocity will be approximated on each element $e \in \mathcal{T}$ by a polynomial of degree 2 and the pressure will be approximated by a polynomial of degree 1. This element satisfies the *Babuška-Brezzi condition* which is substantial for the stability of given approximation.

Let $e \in \mathcal{T}$ be an element with vertices $P_1^e(x_{11}^e, x_{21}^e)$, $P_2^e(x_{12}^e, x_{22}^e)$ and $P_3^e(x_{13}^e, x_{23}^e)$ and by $\hat{e}$ denote the reference element with vertices $\hat{P}_1(0,0)$, $\hat{P}_2(1,0)$ and $\hat{P}_3(0,1)$. Now we introduce a unique mapping, see fig. (1), from the reference element $\hat{e}$ onto an element e by equations

$$\begin{aligned} x_1 &= x_1^e(\xi_1,\xi_2) = x_{11}^e + (x_{12}^e - x_{11}^e)\xi_1 + (x_{13}^e - x_{11}^e)\xi_2 \,, \\ x_2 &= x_2^e(\xi_1,\xi_2) = x_{21}^e + (x_{22}^e - x_{21}^e)\xi_1 + (x_{23}^e - x_{21}^e)\xi_2 \,, \end{aligned} \qquad (\xi_1,\xi_2) \in \hat{e} \,. \qquad (13)$$

The jacobian of this mapping is

$$J^e = \begin{vmatrix} \dfrac{\partial x_1(\xi_1,\xi_2)}{\partial \xi_1} & \dfrac{\partial x_1(\xi_1,\xi_2)}{\partial \xi_2} \\ \dfrac{\partial x_2(\xi_1,\xi_2)}{\partial \xi_1} & \dfrac{\partial x_2(\xi_1,\xi_2)}{\partial \xi_2} \end{vmatrix} = (x_{12}^e - x_{11}^e)(x_{23}^e - x_{21}^e) - (x_{13}^e - x_{11}^e)(x_{22}^e - x_{21}^e) \,.$$

For the sake of completness let us write down the inverse mapping to the mapping (13),

$$\begin{aligned} \xi_1^e = \xi_1(x_1,x_2) &= \frac{(x_1 - x_{11}^e)(x_{23}^e - x_{21}^e) - (x_2 - x_{21}^e)(x_{13}^e - x_{11}^e)}{J^e} \,, \\ \xi_2^e = \xi_2(x_1,x_2) &= \frac{(x_2 - x_{21}^e)(x_{12}^e - x_{11}^e) - (x_1 - x_{11}^e)(x_{22}^e - x_{21}^e)}{J^e} \,, \end{aligned} \qquad (x_1,x_2) \in e \,.$$

Thereinafter, we shall make use of the following notation: for a function $\varphi(x_1,x_2,t)$ defined on an element e,

$$\hat{\varphi}^e(\xi_1,\xi_2,t) = \varphi\Big(x_1^e(\xi_1,\xi_2), x_2^e(\xi_1,\xi_2), t\Big)$$

and for a function $\hat{\phi}(\xi_1, \xi_2, t)$ defined on the reference element $\hat{e}$,

$$\phi^e(x_1, x_2, t) = \hat{\phi}\Big(\xi_1^e(x_1, x_2), \xi_2^e(x_1, x_2), t\Big) .$$

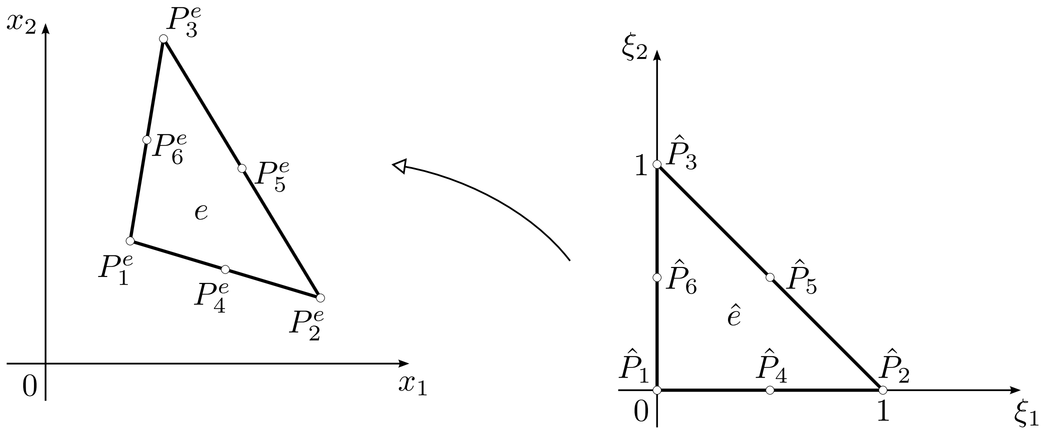


Figure 1: Mapping of the reference element $\hat{e}$ onto an element e.

So far, we have defined the mapping from the reference element $\hat{e}$ onto an element e. Using this mapping we will be able to carry all the computations onto the reference element $\hat{e}$, which will much simplify the situation. In the equations (12) there are integrals and derivatives. First, let us look at how the derivatives in the reference variables look like. According to the chain rule we have

$$\frac{\partial \varphi\Big(x_1^e(\xi_1, \xi_2), x_2^e(\xi_1, \xi_2), t\Big)}{\partial \xi_1} = \frac{\partial \varphi}{\partial x_1}\frac{\partial x_1^e}{\partial \xi_1} + \frac{\partial \varphi}{\partial x_2}\frac{\partial x_2^e}{\partial \xi_1} = \frac{\partial \varphi}{\partial x_1}(x_{12}^e - x_{11}^e) + \frac{\partial \varphi}{\partial x_2}(x_{22}^e - x_{21}^e)$$

and in the same manner we would get

$$\frac{\partial \varphi\Big(x_1^e(\xi_1, \xi_2), x_2^e(\xi_1, \xi_2), t\Big)}{\partial \xi_2} = \frac{\partial \varphi}{\partial x_1}(x_{13}^e - x_{11}^e) + \frac{\partial \varphi}{\partial x_2}(x_{23}^e - x_{21}^e) ,$$

which we may write as

$$\begin{pmatrix} \dfrac{\partial \hat{\varphi}^e(\xi_1, \xi_2, t)}{\partial \xi_1} \\ \dfrac{\partial \hat{\varphi}^e(\xi_1, \xi_2, t)}{\partial \xi_2} \end{pmatrix} = \begin{pmatrix} x_{12}^e - x_{11}^e & x_{22}^e - x_{21}^e \\ x_{13}^e - x_{11}^e & x_{23}^e - x_{21}^e \end{pmatrix} \begin{pmatrix} \dfrac{\partial \varphi^e(x_1, x_2, t)}{\partial x_1} \\ \dfrac{\partial \varphi^e(x_1, x_2, t)}{\partial x_2} \end{pmatrix} .$$

From here, by the inversion, we obtain

$$\begin{pmatrix} \dfrac{\partial \varphi^e(x_1, x_2, t)}{\partial x_1} \\ \dfrac{\partial \varphi^e(x_1, x_2, t)}{\partial x_2} \end{pmatrix} = \begin{pmatrix} h_{11}^e & h_{12}^e \\ h_{21}^e & h_{22}^e \end{pmatrix} \begin{pmatrix} \dfrac{\partial \hat{\varphi}^e(\xi_1, \xi_2, t)}{\partial \xi_1} \\ \dfrac{\partial \hat{\varphi}^e(\xi_1, \xi_2, t)}{\partial \xi_2} \end{pmatrix} , \tag{14}$$

where

$$
\begin{aligned}
h_{11}^e &= (x_{23}^e - x_{21}^e)/J^e\,, & h_{12}^e &= (x_{21}^e - x_{22}^e)/J^e\,, \\
h_{21}^e &= (x_{11}^e - x_{13}^e)/J^e\,, & h_{22}^e &= (x_{12}^e - x_{11}^e)/J^e\,.
\end{aligned}
\tag{15}
$$

Analogously, we would obtain the relation for the second derivatives

$$
\begin{pmatrix}
\dfrac{\partial^2 \varphi^e(x_1,x_2,t)}{\partial x_1^2} \\
\dfrac{\partial^2 \varphi^e(x_1,x_2,t)}{\partial x_1 \partial x_2} \\
\dfrac{\partial^2 \varphi^e(x_1,x_2,t)}{\partial x_2^2}
\end{pmatrix}
=
\begin{pmatrix}
(h_{11}^e)^2 & 2h_{11}^e h_{12}^e & (h_{12}^e)^2 \\
h_{11}^e h_{21}^e & h_{11}^e h_{22}^e + h_{12}^e h_{21}^e & h_{22}^e h_{12}^e \\
(h_{21}^e)^2 & 2h_{21}^e h_{22}^e & (h_{22}^e)^2
\end{pmatrix}
\begin{pmatrix}
\dfrac{\partial^2 \hat{\varphi}^e(\xi_1,\xi_2,t)}{\partial \xi_1^2} \\
\dfrac{\partial^2 \hat{\varphi}^e(\xi_1,\xi_2,t)}{\partial \xi_1 \partial \xi_2} \\
\dfrac{\partial^2 \hat{\varphi}^e(\xi_1,\xi_2,t)}{\partial \xi_2^2}
\end{pmatrix}
, \tag{16}
$$

which do not appear in our weak formulation but we will need them later.

The integrals appearing in (12) will be computed numerically element-wise,

$$
\int_e \varphi^e(x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2 = \int_{\hat{e}} \hat{\varphi}^e(\xi_1,\xi_2)|J^e|\,\mathrm{d}\xi_1\mathrm{d}\xi_2 \approx \sum_{k=1}^{n_{qe}} \omega_k^{qe}|J^e|\hat{\varphi}^e(\xi_{1k}^{qe},\xi_{2k}^{qe})\,, \tag{17}
$$

where ω_k^{qe} are the quadrature weights and $\boldsymbol{\xi}_k^{qe} = (\xi_{1k}^{qe}, \xi_{2k}^{qe})^T$ are the quadrature points of some quadrature rule on the reference element $\hat{e}$.

As was already stated, the Hood-Taylor finite element P_2/P_1 means approximation of velocity by a polynomial of degree two and approximation of pressure by a polynomial of degree one on each element. To this end, we will use the base functions with a property that at node P_i^e of an element e their value is 1 and at all other nodes their value is 0.

Let $P_4^e(x_{14}^e, x_{24}^e)$ be a midpoint of an edge $P_1^eP_2^e$, $P_5^e(x_{15}^e, x_{25}^e)$ be a midpoint of an edge $P_2^eP_3^e$ and $P_6^e(x_{16}^e, x_{26}^e)$ be a midpoint of an edge $P_3^eP_1^e$, see fig. 1. Similarly, $\hat{P}_4(\frac{1}{2},0)$, $\hat{P}_5(\frac{1}{2},\frac{1}{2})$ and $\hat{P}_6(0,\frac{1}{2})$ are midpoints of the edges $\hat{P}_1\hat{P}_2$, $\hat{P}_2\hat{P}_3$ and $\hat{P}_3\hat{P}_1$, respectively. Then for the velocity these functions have the following form on $\hat{e}$,

$$
\begin{aligned}
\hat{Q}_1 &= 2(1-\xi_1-\xi_2)(\tfrac{1}{2}-\xi_1-\xi_2) \\
\hat{Q}_2 &= 2\xi_1(\xi_1-\tfrac{1}{2}) \\
\hat{Q}_3 &= 2\xi_2(\xi_2-\tfrac{1}{2}) \\
\hat{Q}_4 &= 4\xi_1(1-\xi_1-\xi_2) \\
\hat{Q}_5 &= 4\xi_1\xi_2 \\
\hat{Q}_6 &= 4\xi_2(1-\xi_1-\xi_2)\,.
\end{aligned}
\tag{18}
$$

For the pressure the base functions on the reference element are

$$\begin{aligned}
\hat{L}_1 &= 1 - \xi_1 - \xi_2 \\
\hat{L}_2 &= \xi_1 \\
\hat{L}_3 &= \xi_2 \,.
\end{aligned} \tag{19}$$

2.2 Approximation by the finite element method

The spaces $H^1(\Omega)$ and $L^2(\Omega)$ where we look for a solution have an infinite dimension and consequently, they are useless for numerical computations. The principle of the finite element method is an approximation of these spaces by their finite dimensional subspaces. In our case this will be the subspace X_{hv} of continuous functions being on each element polynomials of degree 2 and the subspace X_{hp} of continuous functions being on each element linear. Then these functions are piecewise polynomials of degree 2 and piecewise polynomials of degree 1, respectively. The functions from X_{hv} are uniquely determined by their values in nodes P_i including the nodes at midpoints of the edges and the functions from X_{hp} are uniquely determined by their values at vertices P_i of the elements of triangulation $\mathcal{T}$.

The fact that a function $\varphi(x_1, x_2)$ is a polynomial of degree m on an element e will be expressed as $\varphi(x_1, x_2)|_e \in P_m(e)$. With help of this notation we will define the spaces X_{hv} and X_{hp} as follows

$$\begin{aligned}
X_{hv} &= \{u_h \in C(\Omega);\, u_h|_e \in P_2(e)\} \\
X_{hp} &= \{p_h \in C(\Omega);\, p_h|_e \in P_1(e)\} \,.
\end{aligned}$$

The functions $Q_i(x_1, x_2)$ whose values are equal to one at node P_i and zero at all other nodes are the special cases of functions from the space X_{hv}. Let PU_v be a number of all nodes including the midpoints of the edges. Then every function $u_h \in X_{hv}$ may be written in the following form,

$$u_h(x_1, x_2) = \sum_{i=1}^{PU_v} u_i Q_i(x_1, x_2) \,,$$

where $u_i = u_h(x_{1i}, x_{2i})$ is the value of a function u_h at node P_i. From this we observe that the functions Q_i form the basis of the subspace X_{hv} of the dimension PU_v. The significant property of the finite element method is the fact that the functions Q_i are nonzero only on a small portion of the domain Ω.

In an analogous way we choose special functions $L_i(x_1, x_2)$ from the space X_{hp} whose values at node P_i are equal to one and at all other nodes they are equal to zero. If PU is a number of all vertices of the triangulation, then every function $p_h \in X_{hp}$ may be expressed as

$$p_h(x_1, x_2) = \sum_{i=1}^{PU} p_i L_i(x_1, x_2) \,,$$

where $p_i = p_h(x_{1i}, x_{2i})$ is the value of a function p_h at node P_i. Hence we have chosen the basis of the subspace X_{hp} of the dimesion PU.

Let us define the spaces

$$V_h = \{u_h \in X_{hv};\, u_h(P_j) = 0 \ \forall P_j \in \overline{\Gamma}_1\}$$
$$V_{gh} = \{u_h \in X_{hv};\, u_h(P_j) = g(P_j) \ \forall P_j \in \overline{\Gamma}_1\}\,.$$

Now we can formulate the discretized weak formulation: *For any fixed* $t \in (0,T)$ *find* $\mathbf{u}_h(\cdot,t) \in V_{gh}^2$ *and* $p_h(\cdot,t) \in X_{hp}$, *such that*

$$\int\limits_\Omega \left\{\frac{\partial \mathbf{u}_h}{\partial t}\cdot \mathbf{v}_h + \lambda[(\mathbf{u}_h\cdot\nabla)\mathbf{u}_h]\cdot\mathbf{v}_h + \nu(\nabla\mathbf{u}_h :: \nabla\mathbf{v}_h) - \frac{p_h}{\varrho}\nabla\cdot\mathbf{v}_h + \frac{q_h}{\varrho}\nabla\cdot\mathbf{u}_h\right\}\mathrm{d}\mathbf{x} - \\ - \int\limits_\Omega \mathbf{f}_h\cdot\mathbf{v}_h\,\mathrm{d}\mathbf{x} - \int\limits_{\Gamma_2} \boldsymbol{\sigma}_h\cdot\mathbf{v}_h\,\mathrm{d}S = 0\,, \tag{20}$$

or

$$a(\mathbf{u}_h, p_h, \mathbf{u}_h;\, \mathbf{v}_h, q_h) = 0\,,$$

for arbitrary test functions $\mathbf{v}_h \in V_h^2$ *and* $q_h \in X_{hp}$. We approximated the function $\mathbf{f}$ by a function $\mathbf{f}_h \in X_{hv}^2$ in the same way as velocity and the function $\boldsymbol{\sigma}$ was approximated by a function $\boldsymbol{\sigma}_h \in X_{hp}^2$ in the same way as pressure.

2.2.1 Integration on elements

Because of the particular form of our basis of the space X_{hv}, every function $u_h \in X_{hv}$ has on an element e of the triangulation $\mathcal{T}$ the following form,

$$u_h(x_1,x_2)|_e = u_h^e(x_1,x_2) = \sum_{i=1}^{n_v} u_i^e Q_i^e(x_1,x_2)\,, \tag{21}$$

where $u_i^e = u_h^e(x_{1i},x_{2i})$, $i = 1,\ldots,n_v$, are values of the function u_h^e at nodes $P_i^e(x_{1i},x_{2i})$, $i = 1,\ldots,n_v$, of an element e and $Q_i^e(x_1,x_2) \in X_{hv}$, $i = 1,\ldots,n_v$, are the base functions with the values equal to one at node $P_i^e(x_{1i},x_{2i})$ and zero in all other nodes of an element e. In our case, as may be easily seen from the picture (1), $n_v = 6$. To achieve some generality we shall stick to writing n_v, however, because if we chose some other finite element the value of n_v could be different, in general.

Similarly, the form of the function $p_h \in X_{hp}$ on an element e is

$$p_h(x_1,x_2)|_e = p_h^e(x_1,x_2) = \sum_{i=1}^{n_p} p_i^e L_i^e(x_1,x_2)\,, \tag{22}$$

where $p_i^e = p_h^e(x_{1i}, x_{2i})$, $i = 1, \dots, n_p$ are values of the function p_h^e at nodes $P_i^e(x_{1i}, x_{2i})$, $i = 1, \dots, n_p$, of an element e and $L_i^e(x_1, x_2) \in X_{hp}$, $i = 1, \dots, n_p$, are the base functions with the values equal to one at vertex $P_i^e(x_{1i}, x_{2i})$ and zero at all other vertices of a triangle e. We have $n_p = 3$.

We want to transform the last two expressions onto the reference element $\hat{e}$. From (13), (21) and (22) we get

$$u_h^e\Big(x_1^e(\xi_1, \xi_2), x_2^e(\xi_1, \xi_2)\Big) = \hat{u}_h^e(\xi_1, \xi_2) = \sum_{i=1}^{n_v} u_i^e \hat{Q}_i(\xi_1, \xi_2)$$

$$p_h^e\Big(x_1^e(\xi_1, \xi_2), x_2^e(\xi_1, \xi_2)\Big) = \hat{p}_h^e(\xi_1, \xi_2) = \sum_{i=1}^{n_p} p_i^e \hat{L}_i(\xi_1, \xi_2)\,,$$

where $\hat{Q}_i$, $i = 1, \dots, n_v$, are the base functions on the reference element given by (18) and $\hat{L}_i$, $i = 1, \dots, n_p$, are the base functions given by (19). Then, on the reference element $\hat{e}$, for the velocities u_{ih} and the pressure p_h we have

$$\begin{aligned} \hat{u}_{ih}^e(\xi_1, \xi_2, t) &= \sum_{j=1}^{n_v} u_{ij}^e(t) \hat{Q}_j(\xi_1, \xi_2) = [\mathbf{u}_i^e]^T \boldsymbol{\kappa} \qquad i = 1, 2 \\ \hat{p}_h^e(\xi_1, \xi_2, t) &= \sum_{i=1}^{n_p} p_i^e(t) \hat{L}_i(\xi_1, \xi_2) = [\mathbf{p}^e]^T \mathbf{l}\,, \end{aligned} \tag{23}$$

where $\boldsymbol{\kappa} = \Big(\hat{Q}_1, \dots, \hat{Q}_{n_v}\Big)^T$, $\mathbf{l} = \Big(\hat{L}_1, \dots, \hat{L}_{n_p}\Big)^T$, $\mathbf{p}^e = \Big(p_1^e(t), \dots, p_{n_p}^e(t)\Big)^T$ and $\mathbf{u}_i^e = \Big(u_{i1}^e(t), \dots, u_{in_v}^e(t)\Big)^T$. The test functions v_{ih}, q_h and the force f_{ih} can be expressed in the same fashion

$$\begin{aligned} \hat{v}_{ih}^e(\xi_1, \xi_2) &= \sum_{j=1}^{n_v} v_{ij}^e \hat{Q}_j(\xi_1, \xi_2) = [\mathbf{v}_i^e]^T \boldsymbol{\kappa}\,, \qquad i = 1, 2\,, \\ \hat{f}_{ih}^e(\xi_1, \xi_2) &= \sum_{j=1}^{n_v} v_{ij}^e \hat{Q}_j(\xi_1, \xi_2) = [\mathbf{f}_i^e]^T \boldsymbol{\kappa}\,, \qquad i = 1, 2\,, \\ \hat{q}_h^e(\xi_1, \xi_2) &= \sum_{i=1}^{n_p} q_i^e \hat{L}_i(\xi_1, \xi_2) = [\mathbf{q}^e]^T \mathbf{l}\,, \end{aligned} \tag{24}$$

where $\mathbf{q}^e = \Big(q_1^e, \dots, q_{n_p}^e\Big)^T$, $\mathbf{f}_i^e = \Big(f_{i1}^e, \dots, f_{in_v}^e\Big)^T$ and $\mathbf{v}_i^e = \Big(v_{i1}^e, \dots, v_{in_v}^e\Big)^T$.

Let s be an edge of a triangle e with end points $P_1^s(x_{11}^s, x_{21}^s)$, $P_2^s(x_{12}^s, x_{22}^s)$. We introduce the mapping from the reference line segment $\hat{s}$ onto the edge s by

$$\begin{aligned} x_1 &= x_1^s(\xi) = x_{11}^s + (x_{12}^s - x_{11}^s)\xi \\ x_2 &= x_2^s(\xi) = x_{21}^s + (x_{22}^s - x_{21}^s)\xi\,, \end{aligned} \qquad \xi \in \langle 0, 1\rangle\,.$$

The length of the edge $P_1^s P_2^s$ will be denoted by

$$J^s = \sqrt{(x_{12}^s - x_{11}^s)^2 + (x_{22}^s - x_{21}^s)^2}\,.$$

The test functions v_{ih} on an edge s will be again expressed as linear combinations of the base functions

$$v_{ih}(x_1,x_2)|_s = v_{ih}\Big(x_1^s(\xi), x_2^s(\xi)\Big) = \hat{v}_{ih}^s(\xi) = \sum_{j=1}^{n_{sv}} v_{ij}^s \hat{R}_j(\xi) = [\mathbf{v}_i^s]^T\mathbf{r}, \quad i=1,2\,, \tag{25}$$

where $\mathbf{v}_i^s = \Big(v_{i1}^s, \dots, v_{in_{sv}}^s\Big)^T$, $\mathbf{r} = \Big(\hat{R}_1, \dots, \hat{R}_{n_{sv}}\Big)^T$ and

$$\begin{aligned}
\hat{R}_1(\xi) &= 2(1-\xi)(\tfrac{1}{2}-\xi)\\
\hat{R}_2(\xi) &= 2\xi(\xi-\tfrac{1}{2})\\
\hat{R}_3(\xi) &= 4\xi(1-\xi)\,,
\end{aligned} \tag{26}$$

from where we observe that in our case $n_{sv}=3$ and that the base functions $\hat{R}_i$ are restrictions of the corresponding base functions $\hat{Q}_k$. Thus, the function $\hat{v}_{ih}^s(\xi,t)$ is a restriction of the function $\hat{v}_{ih}^e(\xi_1,\xi_2,t)$ on an edge s. Analogously, we express the function σ_{ih} on an edge s.

$$\sigma_{ih}(x_1,x_2,t)|_s = \hat{\sigma}_{ih}^s(\xi) = \sum_{j=1}^{n_{sp}} \sigma_{ij}^s(t)\hat{S}_j(\xi) = [\boldsymbol{\sigma}_i^s]^T\mathbf{s}\,, \qquad i=1,2\,, \tag{27}$$

where $\boldsymbol{\sigma}_i^s = \Big(\sigma_{i1}^s(t), \dots, \sigma_{in_{sp}}^s(t)\Big)^T$, $\mathbf{s} = \Big(\hat{S}_1, \dots, \hat{S}_{n_{sp}}\Big)$ and

$$\begin{aligned}
\hat{S}_1 &= 1-\xi\\
\hat{S}_2 &= \xi\,,
\end{aligned} \tag{28}$$

from where we see that $n_{sp}=2$ and also that the function $\hat{\sigma}_{ih}^s(\xi,t)$ is a restriction of the function $\hat{\sigma}_{ih}^e(\xi_1,\xi_2,t)$, on an edge s.

With help of (14) we introduce the following notation

$$\begin{aligned}
\boldsymbol{\kappa}_k^e &= h_{k1}^e\frac{\partial\boldsymbol{\kappa}}{\partial\xi_1} + h_{k2}^e\frac{\partial\boldsymbol{\kappa}}{\partial\xi_2} \equiv \left\{\frac{\partial\hat{Q}_j^e}{\partial x_k}(\xi_1,\xi_2)\right\}_{j=1,\dots,n_v}, \qquad k=1,2\\
\mathbf{l}_k^e &= h_{k1}^e\frac{\partial\mathbf{l}}{\partial\xi_1} + h_{k2}^e\frac{\partial\mathbf{l}}{\partial\xi_2} \equiv \left\{\frac{\partial\hat{L}_j^e}{\partial x_k}(\xi_1,\xi_2)\right\}_{j=1,\dots,n_p}, \qquad k=1,2\,,
\end{aligned} \tag{29}$$

where

$$\frac{\partial\boldsymbol{\kappa}}{\partial\xi_k} = \left\{\frac{\partial\hat{Q}_j}{\partial\xi_k}(\xi_1,\xi_2)\right\}_{j=1,\dots,n_v}, \qquad \frac{\partial\mathbf{l}}{\partial\xi_k} = \left\{\frac{\partial\hat{L}_j}{\partial\xi_k}(\xi_1,\xi_2)\right\}_{j=1,\dots,n_p}, \quad k=1,2\,.$$

If we insert the expressions from (23), (24), (25) and (27) into the equation (20), then, regarding (17), we obtain

$$
\begin{aligned}
&\sum_e \int\limits_{\hat{e}} \Big\{ [\mathbf{v}_1^e]^T \boldsymbol{\kappa}\boldsymbol{\kappa}^T \dot{\mathbf{u}}_1^e + [\mathbf{v}_2^e]^T \boldsymbol{\kappa}\boldsymbol{\kappa}^T \dot{\mathbf{u}}_2^e + \lambda \Big[[\mathbf{v}_1^e]^T \boldsymbol{\kappa} [\boldsymbol{\kappa}_1^e]^T \mathbf{u}_1^e \boldsymbol{\kappa}^T \mathbf{u}_1^e \, + \\
&\quad + [\mathbf{v}_1^e]^T \boldsymbol{\kappa} [\boldsymbol{\kappa}_2^e]^T \mathbf{u}_1^e \boldsymbol{\kappa}^T \mathbf{u}_2^e + [\mathbf{v}_2^e]^T \boldsymbol{\kappa} [\boldsymbol{\kappa}_1^e]^T \mathbf{u}_2^e \boldsymbol{\kappa}^T \mathbf{u}_1^e \; + [\mathbf{v}_2^e]^T \boldsymbol{\kappa} [\boldsymbol{\kappa}_2^e]^T [\mathbf{u}_2^e]^T \boldsymbol{\kappa}^T \mathbf{u}_2^e \Big] + \\
&\quad + \nu \Big[(1+\gamma) \Big([\mathbf{v}_1^e]^T \boldsymbol{\kappa}_1^e [\boldsymbol{\kappa}_1^e]^T \mathbf{u}_1^e + [\mathbf{v}_2^e]^T \boldsymbol{\kappa}_2^e [\boldsymbol{\kappa}_2^e]^T [\mathbf{u}_2^e]^T \Big) + [\mathbf{v}_1^e]^T \boldsymbol{\kappa}_2^e [\boldsymbol{\kappa}_2^e]^T \mathbf{u}_1^e + \\
&\quad + [\mathbf{v}_2^e]^T \boldsymbol{\kappa}_1^e [\boldsymbol{\kappa}_1^e]^T \mathbf{u}_2^e + \gamma \Big([\mathbf{v}_1^e]^T \boldsymbol{\kappa}_1^e [\boldsymbol{\kappa}_2^e]^T [\mathbf{u}_2^e]^T + [\mathbf{v}_2^e]^T \boldsymbol{\kappa}_2^e [\boldsymbol{\kappa}_1^e]^T \mathbf{u}_1^e \Big) \Big] - \\
&\quad - \frac{1}{\varrho} \Big[[\mathbf{v}_1^e]^T \boldsymbol{\kappa}_1^e \mathbf{l}^T \mathbf{p}^e + [\mathbf{v}_2^e]^T \boldsymbol{\kappa}_2^e \mathbf{l}^T \mathbf{p}^e \Big] + \frac{1}{\varrho} \Big[[\mathbf{q}^e]^T \mathbf{l} [\boldsymbol{\kappa}_1^e]^T \mathbf{u}_1^e + [\mathbf{q}^e]^T \mathbf{l} [\boldsymbol{\kappa}_2^e]^T \mathbf{u}_2^e \Big] - \\
&\quad - [\mathbf{v}_1^e]^T \boldsymbol{\kappa}\boldsymbol{\kappa}^T \mathbf{f}_1^e - [\mathbf{v}_2^e]^T \boldsymbol{\kappa}\boldsymbol{\kappa}^T \mathbf{f}_2^e \Big\} |J^e| \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 - \\
&\quad - \sum_{s \in \overline{\Gamma}_2} \int\limits_{\hat{s}} \Big\{ [\mathbf{v}_1^s]^T \mathbf{r} \mathbf{s}^T \boldsymbol{\sigma}_1^s + [\mathbf{v}_2^s]^T \mathbf{r} \mathbf{s}^T \boldsymbol{\sigma}_2^s \Big\} |J^s| \, \mathrm{d}\xi = 0 \, ,
\end{aligned}
\tag{30}
$$

where

$$
\dot{\mathbf{u}}_i^e = \left(\frac{\mathrm{d}u_{i1}^e(t)}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}u_{in_v}^e(t)}{\mathrm{d}t} \right)^T , \qquad i = 1, 2 \, .
$$

3 Algorithm for the Finite Element Method

In the previous section, for the purpose of being able to find a weak solution, we approximated the functions from the infinite dimensional space by the functions from the space of finite dimension. We have chosen a particular basis and expressed all the functions as linear combinations of the elements of this basis. We have put these functions into the weak formulation, expressed the integral over Ω as a sum of the integrals over each element of the triangulation $\mathcal{T}$ and hence obtained the discretized weak formulation (30).

Now we have to perform the numerical integration, taking as a main task to express the results in a matrix form which is very suitable for the implementation of this algorithm. Further, we need to perform the time dicretization. After this we will have to deal with a system of nonlinear equations which have to be linearized using certain methods based on the Newton method.

3.1 Elementary matrices

In order to integrate (30) it is suitable to use the Gauss quadrature. Let us start with the first term in (30). Denote by $\boldsymbol{\xi}_k^{qe} = (\xi_{1k}^{qe}, \xi_{2k}^{qe})^T$, $k = 1, \ldots, n_{qe}$, the quadrature points on the reference element $\hat{e}$ and by ω_k^{qe}, $k = 1, \ldots, n_{qe}$, the quadrature weights. Regarding (17) we have

$$\begin{aligned} &\int_{\hat{e}} [\mathbf{v}_1^e]^T \boldsymbol{\kappa}\boldsymbol{\kappa}^T \dot{\mathbf{u}}_1^e |J^e| \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \approx [\mathbf{v}_1^e]^T \left[\sum_{k=1}^{n_{qe}} \boldsymbol{\kappa}(\boldsymbol{\xi}_k^{qe}) \omega_k^{qe} |J^e| \boldsymbol{\kappa}^T(\boldsymbol{\xi}_k^{qe}) \right] \dot{\mathbf{u}}_1^e = \\ &= [\mathbf{v}_1^e]^T \Big(\boldsymbol{\kappa}(\boldsymbol{\xi}_1^{qe}), \, \ldots \, ,\boldsymbol{\kappa}(\boldsymbol{\xi}_{n_{qe}}^{qe}) \Big) \begin{pmatrix} |J^e|\omega_1^{qe} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & |J^e|\omega_{n_{qe}}^{qe} \end{pmatrix} \begin{pmatrix} \boldsymbol{\kappa}^T(\boldsymbol{\xi}_1^{qe}) \\ \vdots \\ \boldsymbol{\kappa}^T(\boldsymbol{\xi}_{n_{qe}}^{qe}) \end{pmatrix} \dot{\mathbf{u}}_1^e = \\ &= [\mathbf{v}_1^e]^T [\mathbf{Q}^e]^T \mathbf{G}^e \mathbf{Q}^e \dot{\mathbf{u}}_1^e \,, \end{aligned} \tag{31}$$

where

$$\begin{aligned} \mathbf{Q}^e &= \Big(\boldsymbol{\kappa}(\boldsymbol{\xi}_1^{qe}), \, \ldots, \boldsymbol{\kappa}(\boldsymbol{\xi}_{n_{qe}}^{qe}) \Big)^T = \Big\{ \hat{Q}_j(\xi_{1i}^{qe}, \xi_{2i}^{qe}) \Big\}_{\substack{i=1,\ldots,n_{qe} \\ j=1,\ldots,n_v}} \,, \\ \mathbf{G}^e &= \operatorname{diag}\left\{ \omega_i^{qe} |J^e|, i = 1, \ldots, n_{qe} \right\} \,. \end{aligned}$$

Similarly, the second term in (30) yields

$$\int_{\hat{e}} [\mathbf{v}_2^e]^T \boldsymbol{\kappa}\boldsymbol{\kappa}^T \dot{\mathbf{u}}_2^e |J^e| \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \approx [\mathbf{v}_2^e]^T [\mathbf{Q}^e]^T \mathbf{G}^e \mathbf{Q}^e \dot{\mathbf{u}}_2^e \,. \tag{32}$$

Finally, after summing (31) and (32), we express the first two terms in (30) as

$$
\begin{aligned}
&[\mathbf{v}_1^e]^T[\mathbf{Q}^e]^T\mathbf{G}^e\mathbf{Q}^e\dot{\mathbf{u}}_1^e + [\mathbf{v}_2^e]^T[\mathbf{Q}^e]^T\mathbf{G}^e\mathbf{Q}^e\dot{\mathbf{u}}_2^e = \\
&= \begin{pmatrix}[\mathbf{v}_1^e]^T & [\mathbf{v}_2^e]^T\end{pmatrix} \begin{pmatrix}[\mathbf{Q}^e]^T\mathbf{G}^e\mathbf{Q}^e & \mathbf{O} \\ \mathbf{O} & [\mathbf{Q}^e]^T\mathbf{G}^e\mathbf{Q}^e\end{pmatrix} \begin{pmatrix}\dot{\mathbf{u}}_1^e \\ \dot{\mathbf{u}}_2^e\end{pmatrix} = \\
&= [\mathbf{v}^e]^T\mathbf{M}^e\dot{\mathbf{u}}^e\,,
\end{aligned} \tag{33}
$$

where

$$
\begin{aligned}
&\dot{\mathbf{u}}^e = \left([\dot{\mathbf{u}}_1^e]^T, [\dot{\mathbf{u}}_2^e]^T\right)^T\,, \quad \mathbf{v}^e = \left([\mathbf{v}_1^e]^T, [\mathbf{v}_2^e]^T\right)^T\,, \\
&\mathbf{M}^e = \begin{pmatrix}[\mathbf{Q}^e]^T\mathbf{G}^e\mathbf{Q}^e & \mathbf{O} \\ \mathbf{O} & [\mathbf{Q}^e]^T\mathbf{G}^e\mathbf{Q}^e\end{pmatrix}
\end{aligned}
$$

and $\mathbf{O}$ is the zero matrix.

By the same reasoning we treat the terms in the first square bracket in (30),

$$
\begin{aligned}
&\int_{\hat{e}} [\mathbf{v}_1^e]^T\boldsymbol{\kappa}[\boldsymbol{\kappa}_1^e]^T\mathbf{u}_1^e\boldsymbol{\kappa}^T\mathbf{u}_1^e|J^e|\,\mathrm{d}\xi_1\mathrm{d}\xi_2 \approx \\
&\approx [\mathbf{v}_1^e]^T\left[\sum_{k=1}^{n_{qe}}\boldsymbol{\kappa}(\boldsymbol{\xi}_k^{qe})\omega_k^{qe}|J^e|[\boldsymbol{\kappa}_1^e]^T(\boldsymbol{\xi}_k^{qe})\mathbf{u}_1^e\boldsymbol{\kappa}^T(\boldsymbol{\xi}_k^{qe})\right]\mathbf{u}_1^e = \\
&= [\mathbf{v}_1^e]^T[\mathbf{Q}^e]^T\mathbf{G}^e \begin{pmatrix}[\boldsymbol{\kappa}_1^e]^T(\boldsymbol{\xi}_1^{qe})\mathbf{u}_1^e & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & [\boldsymbol{\kappa}_1^e]^T(\boldsymbol{\xi}_{n_{qe}}^{qe})\mathbf{u}_1^e\end{pmatrix}\mathbf{Q}^e\mathbf{u}_1^e = \\
&= [\mathbf{v}_1^e]^T[\mathbf{Q}^e]^T\mathbf{G}^e\mathrm{diag}\left\{\mathbf{Q}_1^e\mathbf{u}_1^e\right\}\mathbf{Q}^e\mathbf{u}_1^e\,,
\end{aligned} \tag{34}
$$

where

$$
\mathbf{Q}_1^e = \left(\boldsymbol{\kappa}_1^e(\boldsymbol{\xi}_1^{qe}),\dots,\boldsymbol{\kappa}_1^e(\boldsymbol{\xi}_{n_{qe}}^{qe})\right)^T = h_{11}^e\frac{\partial\mathbf{Q}^e}{\partial\xi_1} + h_{12}^e\frac{\partial\mathbf{Q}^e}{\partial\xi_2} = \left\{\frac{\partial\hat{Q}_j^e}{\partial x_1}(\xi_{1i}^{qe},\xi_{2i}^{qe})\right\}_{\substack{i=1,\dots,n_{qe}\\ j=1,\dots,n_v}}\,.
$$

Here

$$
\frac{\partial\mathbf{Q}^e}{\partial\xi_r} = \left\{\frac{\partial\hat{Q}_j^e}{\partial\xi_r}(\xi_{1i}^{qe},\xi_{2i}^{qe})\right\}_{\substack{i=1,\dots,n_{qe}\\ j=1,\dots,n_v}}\,, \qquad r = 1,2\,.
$$

Analogously for the remaining integrals,

$$\int_{\hat{e}} [\mathbf{v}_1^e]^T \boldsymbol{\kappa} [\boldsymbol{\kappa}_2^e]^T \mathbf{u}_1^e \boldsymbol{\kappa}^T \mathbf{u}_2^e |J^e| \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \approx [\mathbf{v}_1^e]^T [\mathbf{Q}^e]^T \mathbf{G}^e \mathrm{diag}\left\{\mathbf{Q}_2^e \mathbf{u}_1^e\right\} \mathbf{Q}^e \mathbf{u}_2^e \,, \tag{35}$$

$$\int_{\hat{e}} [\mathbf{v}_2^e]^T \boldsymbol{\kappa} [\boldsymbol{\kappa}_1^e]^T \mathbf{u}_2^e \boldsymbol{\kappa}^T \mathbf{u}_1^e |J^e| \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \approx [\mathbf{v}_2^e]^T [\mathbf{Q}^e]^T \mathbf{G}^e \mathrm{diag}\left\{\mathbf{Q}_1^e \mathbf{u}_2^e\right\} \mathbf{Q}^e \mathbf{u}_1^e \,, \tag{36}$$

$$\int_{\hat{e}} [\mathbf{v}_2^e]^T \boldsymbol{\kappa} [\boldsymbol{\kappa}_2^e]^T \mathbf{u}_2^e \boldsymbol{\kappa}^T \mathbf{u}_2^e |J^e| \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \approx [\mathbf{v}_2^e]^T [\mathbf{Q}^e]^T \mathbf{G}^e \mathrm{diag}\left\{\mathbf{Q}_2^e \mathbf{u}_2^e\right\} \mathbf{Q}^e \mathbf{u}_2^e \,, \tag{37}$$

where

$$\mathbf{Q}_2^e = \left(\boldsymbol{\kappa}_2^e(\boldsymbol{\xi}_1^{qe}), \ldots, \boldsymbol{\kappa}_2^e(\boldsymbol{\xi}_{n_{qe}}^{qe})\right)^T = h_{21}^e \frac{\partial \mathbf{Q}^e}{\partial \xi_1} + h_{22}^e \frac{\partial \mathbf{Q}^e}{\partial \xi_2} = \left\{ \frac{\partial \hat{Q}_j^e}{\partial x_2}(\xi_{1i}^{qe}, \xi_{2i}^{qe}) \right\}_{\substack{i=1,\ldots,n_{qe} \\ j=1,\ldots,n_v}} .$$

We add (34)-(37) and write the convective term in (30),

$$\begin{aligned}
&\lambda\Big[[\mathbf{v}_1^e]^T [\mathbf{Q}^e]^T \mathbf{G}^e \mathrm{diag}\left\{\mathbf{Q}_1^e \mathbf{u}_1^e\right\} \mathbf{Q}^e \mathbf{u}_1^e + [\mathbf{v}_1^e]^T [\mathbf{Q}^e]^T \mathbf{G}^e \mathrm{diag}\left\{\mathbf{Q}_2^e \mathbf{u}_1^e\right\} \mathbf{Q}^e \mathbf{u}_2^e + \\
&+ [\mathbf{v}_2^e]^T [\mathbf{Q}^e]^T \mathbf{G}^e \mathrm{diag}\left\{\mathbf{Q}_1^e \mathbf{u}_2^e\right\} \mathbf{Q}^e \mathbf{u}_1^e + [\mathbf{v}_2^e]^T [\mathbf{Q}^e]^T \mathbf{G}^e \mathrm{diag}\left\{\mathbf{Q}_2^e \mathbf{u}_2^e\right\} \mathbf{Q}^e \mathbf{u}_2^e\Big] = \\
&= \lambda \left([\mathbf{v}_1^e]^T \quad [\mathbf{v}_2^e]^T\right) \begin{pmatrix} [\mathbf{Q}^e]^T \mathbf{G}^e \mathrm{diag}\left\{\mathbf{Q}_1^e \mathbf{u}_1^e\right\} \mathbf{Q}^e & [\mathbf{Q}^e]^T \mathbf{G}^e \mathrm{diag}\left\{\mathbf{Q}_2^e \mathbf{u}_1^e\right\} \mathbf{Q}^e \\ [\mathbf{Q}^e]^T \mathbf{G}^e \mathrm{diag}\left\{\mathbf{Q}_1^e \mathbf{u}_2^e\right\} \mathbf{Q}^e & [\mathbf{Q}^e]^T \mathbf{G}^e \mathrm{diag}\left\{\mathbf{Q}_2^e \mathbf{u}_2^e\right\} \mathbf{Q}^e \end{pmatrix} \begin{pmatrix} \mathbf{u}_1^e \\ \mathbf{u}_2^e \end{pmatrix} = \\
&= [\mathbf{v}^e]^T \lambda \mathbf{C}^{e1}(\mathbf{u}^e) \mathbf{u}^e \,,
\end{aligned} \tag{38}$$

where

$$\mathbf{u}^e = \left([\mathbf{u}_1^e]^T, [\mathbf{u}_2^e]^T\right)^T \,,$$

$$\mathbf{C}^{e1}(\mathbf{u}^e) = \begin{pmatrix} [\mathbf{Q}^e]^T \mathbf{G}^e \mathrm{diag}\left\{\mathbf{Q}_1^e \mathbf{u}_1^e\right\} \mathbf{Q}^e & [\mathbf{Q}^e]^T \mathbf{G}^e \mathrm{diag}\left\{\mathbf{Q}_2^e \mathbf{u}_1^e\right\} \mathbf{Q}^e \\ [\mathbf{Q}^e]^T \mathbf{G}^e \mathrm{diag}\left\{\mathbf{Q}_1^e \mathbf{u}_2^e\right\} \mathbf{Q}^e & [\mathbf{Q}^e]^T \mathbf{G}^e \mathrm{diag}\left\{\mathbf{Q}_2^e \mathbf{u}_2^e\right\} \mathbf{Q}^e \end{pmatrix} .$$

Let us introduce an auxiliary matrix

$$\boldsymbol{\Phi}^e = \begin{pmatrix} [\mathbf{Q}^e]^T \mathbf{G}^e & \mathbf{O} \\ \mathbf{O} & [\mathbf{Q}^e]^T \mathbf{G}^e \end{pmatrix}$$

and matrices

$$\mathbf{H}_{ij}^e(\mathbf{u}_i^e) = \mathrm{diag}\{\mathbf{Q}_j^e \mathbf{u}_i^e\} \mathbf{Q}^e \,, \qquad i, j = 1, 2 \,,$$

$$\mathbf{H}^e(\mathbf{u}^e) = \mathrm{diag}\{\mathbf{Q}^e \mathbf{u}_1^e\} \mathbf{Q}_1^e + \mathrm{diag}\{\mathbf{Q}^e \mathbf{u}_2^e\} \mathbf{Q}_2^e \,.$$

Using these matrices we may write down the matrix $\mathbf{C}^{e1}(\mathbf{u}^e)$ as follows

$$\mathbf{C}^{e1}(\mathbf{u}^e) = \mathbf{\Phi}^e \begin{pmatrix} \mathbf{H}^e_{11}(\mathbf{u}^e_1) & \mathbf{H}^e_{12}(\mathbf{u}^e_1) \\ \mathbf{H}^e_{21}(\mathbf{u}^e_2) & \mathbf{H}^e_{22}(\mathbf{u}^e_2) \end{pmatrix} .$$

If we interchanged the order of functions in the integrands of the convective term, we could alternatively express these integrals as

$$[\mathbf{v}^e]^T \lambda \mathbf{C}^{e2}(\mathbf{u}^e)\mathbf{u}^e , \tag{39}$$

where

$$\mathbf{C}^{e2}(\mathbf{u}^e) = \mathbf{\Phi}^e \begin{pmatrix} \mathbf{H}^e(\mathbf{u}^e) & \mathbf{O} \\ \mathbf{O} & \mathbf{H}^e(\mathbf{u}^e) \end{pmatrix} .$$

Now we move on to the second square bracket in (30). We proceed under the same scenario as above, and thus we may immediately write

$$\begin{aligned} &(1+\gamma)\Big([\mathbf{v}^e_1]^T[\mathbf{Q}^e_1]^T\mathbf{G}^e\mathbf{Q}^e_1\mathbf{u}^e_1 + [\mathbf{v}^e_2]^T[\mathbf{Q}^e_2]^T\mathbf{G}^e\mathbf{Q}^e_2\mathbf{u}^e_2\Big) + [\mathbf{v}^e_1]^T[\mathbf{Q}^e_2]^T\mathbf{G}^e\mathbf{Q}^e_2\mathbf{u}^e_1 + \\ &+ [\mathbf{v}^e_2]^T[\mathbf{Q}^e_1]^T\mathbf{G}^e\mathbf{Q}^e_1\mathbf{u}^e_2 + \gamma[\mathbf{v}^e_1]^T[\mathbf{Q}^e_2]^T\mathbf{G}^e\mathbf{Q}^e_1\mathbf{u}^e_2 + \gamma[\mathbf{v}^e_2]^T[\mathbf{Q}^e_1]^T\mathbf{G}^e\mathbf{Q}^e_2\mathbf{u}^e_1 = \\ &= [\mathbf{v}^e]^T\mathbf{K}^e\mathbf{u}^e , \end{aligned} \tag{40}$$

where the matrix $\mathbf{K}^e$ has the following form,

$$\mathbf{K}^e = \nu \begin{pmatrix} (1+\gamma)[\mathbf{Q}^e_1]^T\mathbf{G}^e\mathbf{Q}^e_1 + \gamma[\mathbf{Q}^e_2]^T\mathbf{G}^e\mathbf{Q}^e_2 & \gamma[\mathbf{Q}^e_2]^T\mathbf{G}^e\mathbf{Q}^e_1 \\ \gamma[\mathbf{Q}^e_1]^T\mathbf{G}^e\mathbf{Q}^e_2 & (1+\gamma)[\mathbf{Q}^e_2]^T\mathbf{G}^e\mathbf{Q}^e_2 + \gamma[\mathbf{Q}^e_1]^T\mathbf{G}^e\mathbf{Q}^e_1 \end{pmatrix} .$$

For subsequent integrals in (30), we have

$$\begin{aligned} &-\int_{\hat{e}} \frac{1}{\varrho}\Big[[\mathbf{v}^e_1]^T\boldsymbol{\kappa}^e_1\mathbf{l}^T\mathbf{p}^e + [\mathbf{v}^e_2]^T\boldsymbol{\kappa}^e_2\mathbf{l}^T\mathbf{p}^e\Big]|J^e|\,\mathrm{d}\xi_1\mathrm{d}\xi_2 \approx \\ &\approx -\frac{1}{\varrho}\Big[[\mathbf{v}^e_1]^T[\mathbf{Q}^e_1]^T\mathbf{G}^e\mathbf{L}^e\mathbf{p}^e + [\mathbf{v}^e_2]^T[\mathbf{Q}^e_2]^T\mathbf{G}^e\mathbf{L}^e\mathbf{p}^e\Big] = \\ &= \Big([\mathbf{v}^e_1]^T \quad [\mathbf{v}^e_2]^T\Big)\left(-\frac{1}{\varrho}\right)\begin{pmatrix} [\mathbf{Q}^e_1]^T\mathbf{G}^e\mathbf{L}^e \\ [\mathbf{Q}^e_2]^T\mathbf{G}^e\mathbf{L}^e \end{pmatrix}\mathbf{p}^e = \\ &= [\mathbf{v}^e]^T\mathbf{D}^e\mathbf{p}^e , \end{aligned} \tag{41}$$

where

$$\mathbf{D}^e = -\frac{1}{\varrho}\begin{pmatrix} [\mathbf{Q}^e_1]^T\mathbf{G}^e\mathbf{L}^e \\ [\mathbf{Q}^e_2]^T\mathbf{G}^e\mathbf{L}^e \end{pmatrix} , \qquad \mathbf{L}^e = \Big\{\hat{L}_j(\xi^{qe}_{1i}, \xi^{qe}_{2i})\Big\}_{\substack{i=1,\ldots,n_{qe} \\ j=1,\ldots,n_p}} .$$

Similarly,

$$\begin{aligned}
&\int\limits_{\hat{e}} \frac{1}{\varrho}\Big[[\mathbf{q}^e]^T\mathbf{l}[\boldsymbol{\kappa}_1^e]^T\mathbf{u}_1^e + [\mathbf{q}^e]^T\mathbf{l}[\boldsymbol{\kappa}_2^e]^T\mathbf{u}_2^e\Big]|J^e|\,\mathrm{d}\xi_1\mathrm{d}\xi_2 \approx \\
&\quad\approx \frac{1}{\varrho}\Big[[\mathbf{q}^e]^T[\mathbf{L}^e]^T\mathbf{G}^e\mathbf{Q}_1^e\mathbf{u}_1^e + [\mathbf{q}^e]^T[\mathbf{L}^e]^T\mathbf{G}^e\mathbf{Q}_2^e\mathbf{u}_2^e\Big] = \\
&\quad= \frac{1}{\varrho}[\mathbf{q}^e]^T\begin{pmatrix}[\mathbf{L}^e]^T\mathbf{G}^e\mathbf{Q}_1^e & [\mathbf{L}^e]^T\mathbf{G}^e\mathbf{Q}_2^e\end{pmatrix}\begin{pmatrix}\mathbf{u}_1^e\\ \mathbf{u}_2^e\end{pmatrix} = \\
&\quad= [\mathbf{q}^e]^T\left(-[\mathbf{D}^e]^T\right)\mathbf{u}^e\,.
\end{aligned} \tag{42}$$

Finally, the integral where the volume force occurs yields

$$\begin{aligned}
&-\int\limits_{\hat{e}} \Big[[\mathbf{v}_1^e]^T\boldsymbol{\kappa}\boldsymbol{\kappa}^T\mathbf{f}_1^e + [\mathbf{v}_2^e]^T\boldsymbol{\kappa}\boldsymbol{\kappa}^T\mathbf{f}_2^e\Big]|J^e|\,\mathrm{d}\xi_1\mathrm{d}\xi_2 \approx \\
&\quad\approx -\Big[[\mathbf{v}_1^e]^T[\mathbf{Q}^e]^T\mathbf{G}^e\mathbf{Q}^e\mathbf{f}_1^e + [\mathbf{v}_2^e]^T[\mathbf{Q}^e]^T\mathbf{G}^e\mathbf{Q}^e\mathbf{f}_2^e\Big] = \\
&\quad= -\begin{pmatrix}[\mathbf{v}_1^e]^T & [\mathbf{v}_2^e]^T\end{pmatrix}\begin{pmatrix}[\mathbf{Q}^e]^T\mathbf{G}^e\mathbf{Q}^e & \mathbf{O}\\ \mathbf{O} & [\mathbf{Q}^e]^T\mathbf{G}^e\mathbf{Q}^e\end{pmatrix}\begin{pmatrix}\mathbf{f}_1^e\\ \mathbf{f}_2^e\end{pmatrix} = \\
&\quad= [\mathbf{v}^e]^T\mathbf{M}^e\mathbf{f}^e\,,
\end{aligned} \tag{43}$$

where $\mathbf{f}^e = \left([\mathbf{f}_1^e]^T, [\mathbf{f}_2^e]^T\right)^T$. This completes the integration over elements.

There are still the integrals over the edges $s \in \overline{\Gamma}_2$ left, though. Denote by ξ_k^{qs}, $k = 1,\dots,n_{qs}$, the quadrature points and by ω_k^{qs}, $k = 1,\dots,n_{qs}$, the quadrature weights of a quadrature rule over an edge $\hat{s}$. Then, following the same recipe as when integrating over the elements, we get

$$\begin{aligned}
&-\int\limits_{\hat{s}} \Big\{[\mathbf{v}_1^s]^T\mathbf{r}\mathbf{s}^T\boldsymbol{\sigma}_1^s + [\mathbf{v}_2^s]^T\mathbf{r}\mathbf{s}^T\boldsymbol{\sigma}_2^s\Big\}|J^s|\,\mathrm{d}\xi \approx \\
&\quad\approx -\Big[[\mathbf{v}_1^s]^T[\mathbf{R}^s]^T\mathbf{G}^s\mathbf{S}^s\boldsymbol{\sigma}_1 + [\mathbf{v}_2^s]^T[\mathbf{R}^s]^T\mathbf{G}^s\mathbf{S}^s\boldsymbol{\sigma}_2\Big] = \\
&\quad= -\begin{pmatrix}[\mathbf{v}_1^s]^T & [\mathbf{v}_2^s]^T\end{pmatrix}\begin{pmatrix}[\mathbf{R}^s]^T\mathbf{G}^s\mathbf{S}^s & \mathbf{O}\\ \mathbf{O} & [\mathbf{R}^s]^T\mathbf{G}^s\mathbf{S}^s\end{pmatrix}\begin{pmatrix}\boldsymbol{\sigma}_1^s\\ \boldsymbol{\sigma}_2^s\end{pmatrix} = \\
&\quad= -[\mathbf{v}^s]^T\mathbf{N}^s\boldsymbol{\sigma}^s\,,
\end{aligned} \tag{44}$$

where

$$\boldsymbol{\sigma}^s = \left([\boldsymbol{\sigma}_1^s]^T, [\boldsymbol{\sigma}_2^s]^T\right)^T , \quad \mathbf{v}^s = \left([\mathbf{v}_1^s]^T, [\mathbf{v}_2^s]^T\right)^T \mathbf{R}^s = \left\{\hat{R}_j(\xi_i^{qs})\right\}_{\substack{i=1,\dots,n_{qs} \\ j=1,\dots,n_{sv}}} ,$$

$$\mathbf{S}^s = \left\{\hat{S}_j(\xi_i^{qs})\right\}_{\substack{i=1,\dots,n_{qs} \\ j=1,\dots,n_{sp}}} , \quad \mathbf{G}^s = \mathrm{diag}\{\omega_i^{qs}|J^s|, i = 1, \dots, n_{qs}\} .$$

After inserting the expessions (33), (38), (40), (41), (42), (43) and (44) into the equation (30), we finally obtain

$$\sum_e \left\{ [\mathbf{v}^e]^T \mathbf{M}^e \dot{\mathbf{u}}^e + \begin{pmatrix} \mathbf{v}^e \\ \mathbf{q}^e \end{pmatrix}^T \left[\begin{pmatrix} \mathbf{K}^e + \lambda \mathbf{C}^e(\mathbf{u}^e) & \mathbf{D}^e \\ -[\mathbf{D}^e]^T & \mathbf{O} \end{pmatrix} \begin{pmatrix} \mathbf{u}^e \\ \mathbf{p}^e \end{pmatrix} - \begin{pmatrix} \mathbf{M}^e \mathbf{f}^e \\ \mathbf{o} \end{pmatrix} \right] \right\} - \\ - \sum_s [\mathbf{v}^s]^T \mathbf{N}^s \boldsymbol{\sigma}^s = 0 . \tag{45}$$

For better convenience we repeat the meaning of individual matrices in (45) here. On an element e the vectors of parameters are

$$\mathbf{u}^e = \left([\mathbf{u}_1^e]^T, [\mathbf{u}_2^e]^T\right)^T , \quad \dot{\mathbf{u}}^e = \frac{\mathrm{d}\mathbf{u}^e}{\mathrm{d}t} , \quad \mathbf{v}^e = \left([\mathbf{v}_1^e]^T, [\mathbf{v}_2^e]^T\right)^T , \quad \mathbf{f}^e = \left([\mathbf{f}_1^e]^T, [\mathbf{f}_2^e]^T\right)^T , \tag{46}$$

and on an edge s we have

$$\mathbf{v}^s = \left([\mathbf{v}_1^s]^T, [\mathbf{v}_2^s]^T\right)^T , \qquad \boldsymbol{\sigma}^s = \left([\boldsymbol{\sigma}_1^s]^T, [\boldsymbol{\sigma}_2^s]^T\right)^T . \tag{47}$$

In (45), $\mathbf{O}$ is a zero matrix and $\mathbf{o}$ is a zero vector. We have defined the auxiliary matrix $\boldsymbol{\Phi}^e$ by

$$\boldsymbol{\Phi}^e = \begin{pmatrix} [\mathbf{Q}^e]^T \mathbf{G}^e & \mathbf{O} \\ \mathbf{O} & [\mathbf{Q}^e]^T \mathbf{G}^e \end{pmatrix} . \tag{48}$$

The elementary matrix $\mathbf{M}^e$ is

$$\mathbf{M}^e = \begin{pmatrix} [\mathbf{Q}^e]^T \mathbf{G}^e \mathbf{Q}^e & \mathbf{O} \\ \mathbf{O} & [\mathbf{Q}^e]^T \mathbf{G}^e \mathbf{Q}^e \end{pmatrix} \tag{49}$$

and the elementary matrix $\mathbf{K}^e$ was defined as

$$\mathbf{K}^e = \nu \begin{pmatrix} (1+\gamma)[\mathbf{Q}_1^e]^T \mathbf{G}^e \mathbf{Q}_1^e + \gamma [\mathbf{Q}_2^e]^T \mathbf{G}^e \mathbf{Q}_2^e & \gamma [\mathbf{Q}_2^e]^T \mathbf{G}^e \mathbf{Q}_1^e \\ \gamma [\mathbf{Q}_1^e]^T \mathbf{G}^e \mathbf{Q}_2^e & (1+\gamma)[\mathbf{Q}_2^e]^T \mathbf{G}^e \mathbf{Q}_2^e + \gamma [\mathbf{Q}_1^e]^T \mathbf{G}^e \mathbf{Q}_1^e \end{pmatrix} . \tag{50}$$

To express $\mathbf{C}^e(\mathbf{u}^e)$ we will make use of the matrices

$$\begin{aligned} &\mathbf{H}^e_{ij}(\mathbf{u}^e_i) = \mathrm{diag}\{\mathbf{Q}^e_j \mathbf{u}^e_i\}\mathbf{Q}^e\,, \qquad i,j=1,2\,, \\ &\mathbf{H}^e(\mathbf{u}^e) = \mathrm{diag}\{\mathbf{Q}^e\mathbf{u}^e_1\}\mathbf{Q}^e_1 + \mathrm{diag}\{\mathbf{Q}^e\mathbf{u}^e_2\}\mathbf{Q}^e_2\,. \end{aligned} \tag{51}$$

The matrix $\mathbf{C}^e(\mathbf{u}^e)$ may be written in two ways. Either as $\mathbf{C}^e(\mathbf{u}^e) = \mathbf{C}^{e1}(\mathbf{u}^e)$, where

$$\mathbf{C}^{e1}(\mathbf{u}^e) = \mathbf{\Phi}^e \begin{pmatrix} \mathbf{H}^e_{11}(\mathbf{u}^e_1) & \mathbf{H}^e_{12}(\mathbf{u}^e_1) \\ \mathbf{H}^e_{21}(\mathbf{u}^e_2) & \mathbf{H}^e_{22}(\mathbf{u}^e_2) \end{pmatrix}\,, \tag{52}$$

or as $\mathbf{C}^e(\mathbf{u}^e) = \mathbf{C}^{e2}(\mathbf{u}^e)$, where

$$\mathbf{C}^{e2}(\mathbf{u}^e) = \mathbf{\Phi}^e \begin{pmatrix} \mathbf{H}^e(\mathbf{u}^e) & \mathbf{O} \\ \mathbf{O} & \mathbf{H}^e(\mathbf{u}^e) \end{pmatrix}\,. \tag{53}$$

The matrix $\mathbf{D}^e$ is defined as

$$\mathbf{D}^e = -\frac{1}{\varrho} \begin{pmatrix} [\mathbf{Q}^e_1]^T\mathbf{G}^e\mathbf{L}^e \\ [\mathbf{Q}^e_2]^T\mathbf{G}^e\mathbf{L}^e \end{pmatrix}\,, \tag{54}$$

and finally,

$$\mathbf{N}^s = \begin{pmatrix} [\mathbf{R}^s]^T\mathbf{G}^s\mathbf{S}^s & \mathbf{O} \\ \mathbf{O} & [\mathbf{R}^s]^T\mathbf{G}^s\mathbf{S}^s \end{pmatrix}\,. \tag{55}$$

Further, in these matrices the following occur,

$$\begin{aligned} &\mathbf{Q}^e = \left\{\hat{Q}_j(\xi^{qe}_{1i},\xi^{qe}_{2i})\right\}_{\substack{i=1,\dots,n_{qe}\\ j=1,\dots,n_v}}\,, \qquad \mathbf{L}^e = \left\{\hat{L}_j(\xi^{qe}_{1i},\xi^{qe}_{2i})\right\}_{\substack{i=1,\dots,n_{qe}\\ j=1,\dots,n_p}}\,, \\ &\mathbf{R}^s = \left\{\hat{R}_j(\xi^{qs}_i)\right\}_{\substack{i=1,\dots,n_{qs}\\ j=1,\dots,n_{sv}}}\,, \qquad \mathbf{S}^s = \left\{\hat{S}_j(\xi^{qs}_i)\right\}_{\substack{i=1,\dots,n_{qs}\\ j=1,\dots,n_{sp}}}\,, \\ &\mathbf{G}^e = \mathrm{diag}\{\omega^{qe}_i|J^e|, i=1,\dots,n_{qe}\}\,, \quad \mathbf{G}^s = \mathrm{diag}\{\omega^{qs}_i|J^s|, i=1,\dots,n_{qs}\} \end{aligned} \tag{56}$$

and

$$\mathbf{Q}^e_k = h^e_{k1}\frac{\partial\mathbf{Q}^e}{\partial\xi_1} + h^e_{k2}\frac{\partial\mathbf{Q}^e}{\partial\xi_2} \equiv \left\{\frac{\partial\hat{Q}^e_j}{\partial x_k}(\xi^{qe}_{1i},\xi^{qe}_{2i})\right\}_{\substack{i=1,\dots,n_{qe}\\ j=1,\dots,n_v}}\,, \quad k=1,2\,. \tag{57}$$

3.2 Time discretization

We consider a partition

$$0 = t_0 < t_1 < \ldots < t_{n-1} < t_n < \ldots < t_N = T$$

of the interval $\langle 0, T \rangle$ into N intervals and $N+1$ time layers. The timestep between individual time layers will be assumed constant and will be denoted by Δt. Functions evaluated at time t_n will be denoted by a superscript n. This means that, for instance, $\mathbf{u}^{e,n}$ will stand for the vector of velocity parameters at time t_n. Similarly, $\mathbf{p}^{e,n}$ will be the vector of pressure parameters at time t_n. Further, $\mathbf{u}^{e,n-1}$ is a vector of velocity parameters at time t_{n-1} and $\mathbf{f}^{e,n}$ or $\boldsymbol{\sigma}^{s,n}$ is a vector of parameters of $\mathbf{f}^e$ or $\boldsymbol{\sigma}^s$, respectively, at time t_n.

The time derivative of the vector of velocity parameters $\dot{\mathbf{u}}^e$ will be approximated by the backward difference

$$\dot{\mathbf{u}}^e(t_n) \approx \frac{\mathbf{u}^{e,n} - \mathbf{u}^{e,n-1}}{\Delta t} \,.$$

Then, the implicit Euler method in every time step leads to the following: *find* $\mathbf{u}^{e,n}$ *and* $\mathbf{p}^{e,n}$, *such that*

$$\sum_e \Big\{ [\mathbf{v}^e]^T \Big[\mathbf{M}^e \frac{\mathbf{u}^{e,n} - \mathbf{u}^{e,n-1}}{\Delta t} + [\mathbf{K}^e + \lambda \mathbf{C}^e(\mathbf{u}^{e,n})] \mathbf{u}^{e,n} + \mathbf{D}^e \mathbf{p}^{e,n} - \mathbf{M}^e \mathbf{f}^{e,n} \Big] - \\ - [\mathbf{q}^e]^T [\mathbf{D}^e]^T \mathbf{u}^{e,n} \Big\} - \sum_s [\mathbf{v}^s]^T \mathbf{N}^s \boldsymbol{\sigma}^{s,n} = 0 \tag{58}$$

holds. The values of $\mathbf{u}^{e,0}$ and $\mathbf{p}^{e,0}$ are determined from the initial condition.

3.3 Linearization

The equation (58) is nonlinear because of the convective term $\lambda \mathbf{C}^e(\mathbf{u}^{e,n})\mathbf{u}^{e,n}$. Therefore, we have to iterate to solve it for $\mathbf{u}^{e,n}$ and $\mathbf{p}^{e,n}$. Using $\mathbf{u}^{e,n,k-1}$ and $\mathbf{p}^{e,n,k-1}$ from the previous iteration we will compute the new approximations $\mathbf{u}^{e,n,k}$ and $\mathbf{p}^{e,n,k}$.

Let us look back at the discretized weak formulation

$$a(\mathbf{u}_h, p_h, \mathbf{u}_h;\, \mathbf{v}_h, q_h) = 0 \,.$$

First of all, we will approximate the time derivative in the form a by the difference quotient and then split it into two forms b and c as follows,

$$b(\mathbf{u}^n, p^n;\, \mathbf{v}, q) = \int_\Omega \Big\{ \frac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\Delta t} \cdot \mathbf{v} + \nu (\nabla \mathbf{u}^n :: \nabla \mathbf{v}) - \frac{p^n}{\varrho} \nabla \cdot \mathbf{v}^n + \frac{q}{\varrho} \nabla \cdot \mathbf{u}^n - \\ - \mathbf{f}^n \cdot \mathbf{v} \Big\} \, \mathrm{d}\mathbf{x} - \int_{\Gamma_2} \boldsymbol{\sigma}^n \cdot \mathbf{v} \, \mathrm{d}S \,, \tag{59}$$

$$c(\mathbf{u}^n, \mathbf{w}^n;\, \mathbf{v}) = \int_\Omega \lambda \Big[(\mathbf{w}^n \cdot \nabla) \mathbf{u}^n \Big] \cdot \mathbf{v} \, \mathrm{d}\mathbf{x} \,.$$

Hence we have separated the convective term using the form c. Thus in the n-th time step we are seeking $\mathbf{u}_h^n$ and p_h^n, such that

$$b(\mathbf{u}_h^n, p_h^n;\, \mathbf{v}_h, q_h) + c(\mathbf{u}_h^n, \mathbf{u}_h^n;\, \mathbf{v}_h) = 0 \tag{60}$$

for each $\mathbf{v}_h \in V_h^2$ and $q_h \in X_{hp}$.

Let us now introduce the functionals $F_1(\mathbf{u}, p)$ and $F_2(\mathbf{u})$ through relations

$$F_1(\mathbf{u}^n, p^n) \equiv F_1(\mathbf{u}^n, p^n)(\mathbf{v}, q) = b(\mathbf{u}^n, p^n;\, \mathbf{v}, q)\,,$$

$$F_2(\mathbf{u}^n) \equiv F_2(\mathbf{u}^n)(\mathbf{v}) = c(\mathbf{u}^n, \mathbf{u}^n;\, \mathbf{v})\,.$$

If we define $\mathbf{d} = \mathbf{u}_h^{n,k} - \mathbf{u}_h^{n,k-1}$ and $\delta = p_h^{n,k} - p_h^{n,k-1}$, we may write down the scheme of the Newton method as

$$\begin{aligned} &\mathrm{d}F_1(\mathbf{u}_h^{n,k-1}, p_h^{n,k-1})(\mathbf{d}, \delta) + \mathrm{d}F_2(\mathbf{u}_h^{n,k-1})(\mathbf{d}) = \\ &\qquad = -F_1(\mathbf{u}_h^{n,k-1}, p_h^{n,k-1}) - F_2(\mathbf{u}_h^{n,k-1})\,, \end{aligned} \tag{61}$$

where $\mathrm{d}F_1(\mathbf{u}_h^{n,k-1}, p_h^{n,k-1})$ and $\mathrm{d}F_2(\mathbf{u}_h^{n,k-1})$ denote the Gateaux derivative, (see [5]). Because of the fact that F_1 is linear, we have

$$\mathrm{d}F_1(\mathbf{u}_h^{n,k-1}, p_h^{n,k-1})(\mathbf{d}, \delta) = F_1(\mathbf{d}, \delta) = F_1(\mathbf{u}_h^{n,k}, p_h^{n,k}) - F_1(\mathbf{u}_h^{n,k-1}, p_h^{n,k-1})\,,$$

which, after substitution in (61), yields

$$F_1(\mathbf{u}_h^{n,k}, p_h^{n,k}) + \mathrm{d}F_2(\mathbf{u}_h^{n,k-1})(\mathbf{d}) = -F_2(\mathbf{u}_h^{n,k-1})\,. \tag{62}$$

Let us now compute the Gateaux derivative of F_2. First, we shall rewrite the convective term as

$$\lambda\Big[(\mathbf{u}_h^n \cdot \nabla)\mathbf{u}_h^n\Big] \cdot \mathbf{v}_h = \lambda u_{hj}^n \frac{\partial u_{hi}^n}{\partial x_j} v_{hi}\,,$$

where we used the summation convention. It means we sum up over the index occuring twice in a single term. In this case we sum up over i and j, $i, j = 1, 2$. The Gateaux derivative of F_2 is given by

$$\begin{aligned} \mathrm{d}F_2(\mathbf{u}_h^{n,k-1})(\mathbf{d}) &= \lambda\Big[\frac{\mathrm{d}}{\mathrm{d}\tau} F_2(\mathbf{u}_h^{n,k-1} + \tau\mathbf{d})\Big]_{\tau=0} = \\ &= \lambda\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\int_\Omega (u_{hj}^{n,k-1} + \tau d_j)\frac{\partial(u_{hi}^{n,k-1} + \tau d_i)}{\partial x_j} v_{hi}\,\mathrm{d}\Omega\right]_{\tau=0} = \\ &= \lambda\left[\int_\Omega d_j \frac{\partial(u_{hi}^{n,k-1} + \tau d_i)}{\partial x_j} v_{hi} + (u_{hj}^{n,k-1} + \tau d_j)\frac{\partial d_i}{\partial x_j} v_{hi}\,\mathrm{d}\Omega\right]_{\tau=0} = \\ &= \lambda\int_\Omega \left[d_j \frac{\partial u_{hi}^{n,k-1}}{\partial x_j} v_{hi} + u_{hj}^{n,k-1}\frac{\partial d_i}{\partial x_j} v_{hi}\right]\mathrm{d}\Omega = \\ &= c(\mathbf{u}_h^{n,k-1}, \mathbf{d};\, \mathbf{v}_h) + c(\mathbf{d}, \mathbf{u}_h^{n,k-1};\, \mathbf{v}_h)\,, \end{aligned}$$

from where, after substitution in (62), we obtain the Newton method scheme

$$\begin{aligned} b(\mathbf{u}_h^{n,k}, p_h^{n,k}; \mathbf{v}_h, q_h) + c(\mathbf{u}_h^{n,k}, \mathbf{u}_h^{n,k-1}; \mathbf{v}_h) + c(\mathbf{u}_h^{n,k-1}, \mathbf{u}_h^{n,k}; \mathbf{v}_h) - \\ - c(\mathbf{u}_h^{n,k-1}, \mathbf{u}_h^{n,k-1}; \mathbf{v}_h) = 0\,. \end{aligned} \tag{63}$$

From here one easily sees how the equation (58) will be affected by the Newton method. The convective term will be approximated by

$$\begin{aligned} \mathbf{C}^e(\mathbf{u}^{e,n,k})\mathbf{u}^{e,n,k} \approx \mathbf{C}^{e2}(\mathbf{u}^{e,n,k-1})\mathbf{u}^{e,n,k} + \\ + \beta\Big[\mathbf{C}^{e1}(\mathbf{u}^{e,n,k-1})\mathbf{u}^{e,n,k} - \mathbf{C}^{e1}(\mathbf{u}^{e,n,k-1})\mathbf{u}^{e,n,k-1}\Big]\,, \end{aligned}$$

where, for $\beta = 1$ we have a linearization by the Newton method, and for $\beta = 0$ we obtain a simplified linearization of Oseen type. The unknown parameters $\mathbf{u}^{e,n,k}$ and $\mathbf{p}^{e,n,k}$ are then computed from the equation

$$\begin{aligned} \sum_e \Big\{ [\mathbf{v}^e]^T \Big[\Big(\frac{1}{\Delta t}\mathbf{M}^e + \mathbf{K}^e + \lambda\beta\mathbf{C}^{e1}(\mathbf{u}^{e,n,k-1}) + \lambda\mathbf{C}^{e2}(\mathbf{u}^{e,n,k-1}) \Big) \mathbf{u}^{e,n,k} + \\ + \mathbf{D}^e\mathbf{p}^{e,n,k} - \lambda\beta\mathbf{C}^{e1}(\mathbf{u}^{e,n,k-1})\mathbf{u}^{e,n,k-1} - \mathbf{M}^e\left(\frac{1}{\Delta t}\mathbf{u}^{e,n-1} + \mathbf{f}^{e,n}\right)\Big] - \\ - [\mathbf{q}^e]^T[\mathbf{D}^e]^T\mathbf{u}^{e,n,k}\Big\} - \sum_s [\mathbf{v}^s]^T\mathbf{N}^s\boldsymbol{\sigma}^{s,n} = 0\,. \end{aligned} \tag{64}$$

We iterate according to this scheme in every time step. Iterations are stopped if the difference of the two successive iterations is sufficiently small or if the number of iterations overruns some preassigned value. If this scheme converges, we put $\mathbf{u}^{e,n} = \mathbf{u}^{e,n,k}$, $\mathbf{p}^{e,n} = \mathbf{p}^{e,n,k}$. As the initial approximation we take a solution from the previous time, i. e. $\mathbf{u}^{e,n,0} = \mathbf{u}^{e,n-1}$, $\mathbf{p}^{e,n,0} = \mathbf{p}^{e,n-1}$.

When doing the computations, from (64) we form

$$\mathbf{v}^T(\mathbf{A}\mathbf{u} - \mathbf{f}) = 0$$

using a standard algorithm. Because the vector $\mathbf{v}$ may be arbitrary, it must be true that

$$\mathbf{A}\mathbf{u} = \mathbf{f}\,.$$

This system of linear equations is then solved by some suitable method.

4 Stabilization

Stability of (64) is restricted by the Reynolds number. When the Reynolds number is too high, the convective term dominates the Navier-Stokes equations and the scheme (64) becomes unstable. Thus it is necessary to stabilize it. Stabilized finite element method is formed by adding to (12) a stabilizing term causing a small perturbation. One of the main questions arising quite naturally is “how much of the perturbation term one has to add to obtain satisfactory results”. This problem is addressed by the stability parameters whose suitable design may achieve the stability of given scheme. In general, the design of these parameters depends on the particular method.

4.1 Stability parameters

In this section we assume $\lambda = 1$ since otherwise the stabilization is not needed. Let us define the stabilizing term

$$a_s(\mathbf{u}, p, \mathbf{w}; \mathbf{v}, q) = \sum_e \int_e \left[\frac{\partial \mathbf{u}}{\partial t} + \lambda(\mathbf{w} \cdot \nabla)\mathbf{u} - 2\nu \nabla \cdot \boldsymbol{\varepsilon}(\mathbf{u}) + \frac{1}{\varrho}\nabla p - \mathbf{f} \right] \psi^e(\mathbf{w}; \mathbf{v}, q) \, \mathrm{d}\mathbf{x} + \\ + \sum_e \int_e \delta^e [\nabla \cdot \mathbf{u}][\nabla \cdot \mathbf{v}] \, \mathrm{d}\mathbf{x} \,. \tag{65}$$

Here, as the test function we take

$$\psi^e(\mathbf{w}; \mathbf{v}, q) = \tau_u^e (\mathbf{w} \cdot \nabla)\mathbf{v} - \tau_s^e 2\nu \nabla \cdot \boldsymbol{\varepsilon}(\mathbf{v}) + \tau_p^e \frac{1}{\varrho} \nabla q \,. \tag{66}$$

We may point out here that if $\mathbf{u}$ and p is the classical solution of (1) and (2), then

$$a_s(\mathbf{u}, p, \mathbf{u}; \mathbf{v}, q) = 0 \,.$$

The stability parameters τ_u^e, τ_p^e, τ_s^e and δ^e are adjusted using one, or some combination, of the following methods

- SUPG (Streamline Upwind Petrov-Galerkin)
- PSPG (Pressure Stabilizing Petrov-Galerkin)
- LSIC (Least Squares on Incompressibility Constraint)
- GLS (Galerkin Least Squares)

We perform the space discretization in the same way as in the previous sections. Now we seek $\mathbf{u}_h$, p_h, such that

$$a(\mathbf{u}_h, p_h, \mathbf{u}_h; \mathbf{v}_h, q_h) + a_s(\mathbf{u}_h, p_h, \mathbf{u}_h; \mathbf{v}_h, q_h) = 0\,. \tag{67}$$

If we approximate the time derivative in the form a_s by the difference quotient, we obtain a form b_s given by

$$\begin{aligned} & b_s(\mathbf{u}^n, p^n, \mathbf{w}^n; \mathbf{v}, q) = \\ & = \sum_e \int_e \left[\frac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\Delta t} + \lambda(\mathbf{w}^n \cdot \nabla)\mathbf{u}^n - 2\nu \nabla \cdot \boldsymbol{\varepsilon}(\mathbf{u}^n) + \frac{1}{\varrho}\nabla p^n - \mathbf{f}^n \right] \psi^e(\mathbf{w}^n; \mathbf{v}, q)\,\mathrm{d}\mathbf{x} + \\ & + \sum_e \int_e \delta^e [\nabla \cdot \mathbf{u}^n][\nabla \cdot \mathbf{v}]\,\mathrm{d}\mathbf{x}\,. \end{aligned} \tag{68}$$

Define a form $B(\mathbf{u}^n, p^n, \mathbf{w}^n; \mathbf{v}, q)$ as

$$B(\mathbf{u}^n, p^n, \mathbf{w}^n; \mathbf{v}, q) = b(\mathbf{u}^n, p^n; \mathbf{v}, q) + c(\mathbf{u}^n, \mathbf{w}^n; \mathbf{v}) + b_s(\mathbf{u}^n, p^n, \mathbf{w}^n; \mathbf{v}, q)\,. \tag{69}$$

Thus in the n-th time step we seek the approximate solution $\mathbf{u}_h^n \in V_{gh}^2$ and $p_h^n \in X_{hp}$ satisfying

$$B(\mathbf{u}_h^n, p_h^n, \mathbf{w}_h^n; \mathbf{v}_h, q_h) = 0 \tag{70}$$

for each $\mathbf{v}_h \in V_h^2$, $q_h \in X_{hp}$.

Now the description of the stability parameters in particular cases follows,

1. SUPG+PSPG+LSIC for a case where the velocities are approximated by a polynomial of degree higher than that of a polynomial used for the pressure approximation. We choose (see [2])

$$\tau_u^e = \tau_p^e = \frac{1}{4}[h_{\max}^e]^2\,, \qquad \delta^e = 1\,, \tag{71}$$

   where $h_{\max}^e$ is the largest edge of a triangle e.

2. In the case of the GLS method we choose (see [1])

$$\tau_u^e = \tau_p^e = \tau_s^e = \begin{cases} \dfrac{1}{4\nu\lambda_{\max}^e}\,, & 0 \le \mathrm{Re}^e < 1\,, \\ \dfrac{1}{\sqrt{\lambda_{\max}^e}|\mathbf{u}^e|}\,, & \mathrm{Re}^e \ge 1\,, \end{cases} \tag{72}$$

$$\delta^e = \tau_s^e |\mathbf{u}^e|^2\,,$$

where

$$\mathrm{Re}^e = \frac{|\mathbf{u}^e|}{4\nu\sqrt{\lambda^e_{\max}}}\,.$$

Further,

$$|\mathbf{u}^e| = \max_{i,j} |u^e_{ij}|\,, \qquad i = 1, 2\,, \quad j = 1, \ldots, n_v$$

and $\lambda^e_{\max}$ is the greatest eigenvalue of the problem

$$\int_e \frac{\partial \varepsilon_{ij}(\mathbf{u}_h)}{\partial x_j}\frac{\partial \varepsilon_{ik}(\mathbf{v}_h)}{\partial x_k}\,\mathrm{d}\mathbf{x} = \lambda^e \int_e \varepsilon_{ij}(\mathbf{u}_h)\varepsilon_{ij}(\mathbf{v}_h)\,\mathrm{d}\mathbf{x} \qquad \forall \mathbf{v}_h \in P^2_2(e)/\mathrm{Z} \tag{73}$$

for $\gamma = 1$, where $\mathrm{Z} = \{\mathbf{v} \in P^2_2(e); \boldsymbol{\varepsilon}(\mathbf{v}) = \mathbf{o}\}$, $\dim \mathrm{Z} = 3$, or

$$\int_e \Delta \mathbf{u}_h \cdot \Delta \mathbf{v}_h\,\mathrm{d}\mathbf{x} = \lambda^e \int_e \nabla \mathbf{u}_h : \nabla \mathbf{v}_h\,\mathrm{d}\mathbf{x} \qquad \forall \mathbf{v}_h \in (P_2(e)/\mathbb{R})^2 \tag{74}$$

for $\gamma = 0$. In (73) we used the summation convention again, and

$$\nabla \mathbf{u} : \nabla \mathbf{v} = \frac{\partial u_1}{\partial x_1}\frac{\partial v_1}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\frac{\partial v_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\frac{\partial v_2}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\frac{\partial v_2}{\partial x_2}\,.$$

4.2 Matrix form of the stabilizing term

Obviously, the stabilizing term is nonlinear. Several methods may be used to linearize it. Probably the most simple one, known as Oseen method, computes $\mathbf{u}^{n,k}$ and $p^{n,k}$ from the equation

$$B(\mathbf{u}^{n,k}_h, p^{n,k}_h, \mathbf{u}^{n,k-1}_h; \mathbf{v}_h, q_h) = 0\,.$$

A little better linearization, based on the Newton method applied to the convective term c, yields the scheme

$$B(\mathbf{u}^{n,k}_h, p^{n,k}_h, \mathbf{u}^{n,k-1}_h; \mathbf{v}_h, q_h) + c(\mathbf{u}^{n,k-1}_h, \mathbf{u}^{n,k}_h; \mathbf{v}_h) - c(\mathbf{u}^{n,k-1}_h, \mathbf{u}^{n,k-1}_h; \mathbf{v}_h) = 0\,.$$

In this thesis we will use yet more sophisticated scheme, although still not the full Newton method. It is obtained by applying the Newton method also to the convective term occuring in the stabilizing term. To be more precise, we add to the left hand side of the equation (64) an approximation of the expression

$$\begin{aligned}
\sum_e \int_e \Bigg[&\frac{\mathbf{u}^{n,k}_h - \mathbf{u}^{n-1}_h}{\Delta t} + \lambda(\mathbf{u}^{n,k-1}_h \cdot \nabla)\mathbf{u}^{n,k}_h + \lambda\beta(\mathbf{u}^{n,k}_h \cdot \nabla)\mathbf{u}^{n,k-1}_h - \\
&- \lambda\beta(\mathbf{u}^{n,k-1}_h \cdot \nabla)\mathbf{u}^{n,k-1}_h + \mathbf{e}(\mathbf{u}^{n,k}_h) + \frac{1}{\varrho}\nabla p^{n,k}_h - \mathbf{f}^n_h\Bigg] \cdot \Bigg[\tau^e_u(\mathbf{u}^{n,k-1}_h \cdot \nabla)\mathbf{v}_h + \\
&+ \tau^e_s \mathbf{e}(\mathbf{v}_h) + \tau^e_p \frac{1}{\varrho}\nabla q_h\Bigg]\,\mathrm{d}\mathbf{x} + \sum_e \int_e \delta^e [\nabla \cdot \mathbf{u}^{n,k}_h][\nabla \cdot \mathbf{v}_h]\,\mathrm{d}\mathbf{x}\,,
\end{aligned} \tag{75}$$

where $\mathbf{e}(\mathbf{w}) = \Big(e_1(\mathbf{w}), e_2(\mathbf{w})\Big)^T$ with

$$e_1(\mathbf{w}) = -\nu \left[(1+\gamma)\frac{\partial^2 w_1}{\partial x_1^2} + \frac{\partial^2 w_1}{\partial x_2^2} + \gamma \frac{\partial^2 w_2}{\partial x_1 \partial x_2}\right],$$

$$e_2(\mathbf{w}) = -\nu \left[(1+\gamma)\frac{\partial^2 w_2}{\partial x_2^2} + \frac{\partial^2 w_2}{\partial x_1^2} + \gamma \frac{\partial^2 w_1}{\partial x_1 \partial x_2}\right].$$

To integrate (75), we proceed in exactly the same way as we did when deriving the equation (45). Using (16) we employ the following notation

$$\mathbf{Q}^e_{kl} = h^e_{k1}h^e_{l1}\frac{\partial^2 \mathbf{Q}^e}{\partial \xi_1^2} + (h^e_{k1}h^e_{l2} + h^e_{k2}h^e_{l1})\frac{\partial^2 \mathbf{Q}^e}{\partial \xi_1 \partial \xi_2} + h^e_{k2}h^e_{l2}\frac{\partial^2 \mathbf{Q}^e}{\partial \xi_2^2} \equiv$$

$$\equiv \left\{ \frac{\partial^2 \hat{Q}^e_j}{\partial x_k \partial x_l}(\xi^{qe}_{1i}, \xi^{qe}_{2i}) \right\}_{\substack{i=1,\dots,n_{qe}\\ j=1,\dots,n_v}}, \tag{76}$$

$$\mathbf{L}^e_k = h^e_{k1}\frac{\partial \mathbf{L}^e}{\partial \xi_1} + h^e_{k2}\frac{\partial \mathbf{L}^e}{\partial \xi_2} \equiv \left\{ \frac{\partial \hat{L}^e_j}{\partial x_k}(\xi^{qe}_{1i}, \xi^{qe}_{2i}) \right\}_{\substack{i=1,\dots,n_{qe}\\ j=1,\dots,n_p}}.$$

Let us start with the second term in (75) multiplied by the first term in the second square bracket. Then we get

$$\lambda \tau^e_u \int_e \sum_{i=1}^{2} \left\{ u^{n,k-1}_{h1}\frac{\partial u^{n,k}_{hi}}{\partial x_1} u^{n,k-1}_{h1}\frac{\partial v_{hi}}{\partial x_1} + u^{n,k-1}_{h1}\frac{\partial u^{n,k}_{hi}}{\partial x_1} u^{n,k-1}_{h2}\frac{\partial v_{hi}}{\partial x_2} + \right.$$

$$\left. + u^{n,k-1}_{h2}\frac{\partial u^{n,k}_{hi}}{\partial x_2} u^{n,k-1}_{h1}\frac{\partial v_{hi}}{\partial x_1} + u^{n,k-1}_{h2}\frac{\partial u^{n,k}_{hi}}{\partial x_2} u^{n,k-1}_{h2}\frac{\partial v_{hi}}{\partial x_2} \right\} \mathrm{d}\mathbf{x}$$

and, after integration, this is approximately equal to

$$\lambda \tau^e_u \sum_{i=1}^{2} \Big[[\mathbf{v}^e_i]^T [\mathbf{Q}^e_1]^T \mathrm{diag}\{\mathbf{Q}^e \mathbf{u}^{e,n,k-1}_1\} \mathbf{G}^e \mathrm{diag}\{\mathbf{Q}^e \mathbf{u}^{e,n,k-1}_1\} \mathbf{Q}^e_1 \mathbf{u}^{e,n,k}_i +$$

$$+ [\mathbf{v}^e_i]^T [\mathbf{Q}^e_2]^T \mathrm{diag}\{\mathbf{Q}^e \mathbf{u}^{e,n,k-1}_2\} \mathbf{G}^e \mathrm{diag}\{\mathbf{Q}^e \mathbf{u}^{e,n,k-1}_1\} \mathbf{Q}^e_1 \mathbf{u}^{e,n,k}_i +$$

$$+ [\mathbf{v}^e_i]^T [\mathbf{Q}^e_1]^T \mathrm{diag}\{\mathbf{Q}^e \mathbf{u}^{e,n,k-1}_1\} \mathbf{G}^e \mathrm{diag}\{\mathbf{Q}^e \mathbf{u}^{e,n,k-1}_2\} \mathbf{Q}^e_2 \mathbf{u}^{e,n,k}_i +$$

$$+ [\mathbf{v}^e_i]^T [\mathbf{Q}^e_2]^T \mathrm{diag}\{\mathbf{Q}^e \mathbf{u}^{e,n,k-1}_2\} \mathbf{G}^e \mathrm{diag}\{\mathbf{Q}^e \mathbf{u}^{e,n,k-1}_2\} \mathbf{Q}^e_2 \mathbf{u}^{e,n,k}_i \Big],$$

which can be written as

$$\lambda \tau^e_u \Big([\mathbf{v}^e_1]^T \quad [\mathbf{v}^e_2]^T \Big) \begin{pmatrix} \mathbf{H}^e(\mathbf{u}^{e,n,k-1}) & \mathbf{O} \\ \mathbf{O} & \mathbf{H}^e(\mathbf{u}^{e,n,k-1}) \end{pmatrix}^T \begin{pmatrix} \mathbf{G}^e & \mathbf{O} \\ \mathbf{O} & \mathbf{G}^e \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{H}^e(\mathbf{u}^{e,n,k-1}) & \mathbf{O} \\ \mathbf{O} & \mathbf{H}^e(\mathbf{u}^{e,n,k-1}) \end{pmatrix} \begin{pmatrix} \mathbf{u}^{e,n,k}_1 \\ \mathbf{u}^{e,n,k}_2 \end{pmatrix}. \tag{77}$$

Writing down the integration of the second term in the first square bracket multiplied by the second term in the second square bracket would be quite cumbersome so we confine ourselves to stating just the result

$$\int_e \Big[\lambda(\mathbf{u}_h^{n,k-1}\cdot\nabla)\mathbf{u}_h^{n,k}\Big]\cdot\Big[\tau_s^e\mathbf{e}(\mathbf{v}_h)\Big]\,\mathrm{d}\mathbf{x}\approx$$

$$\approx -\lambda\tau_s^e\nu\left([\mathbf{v}_1^e]^T\quad[\mathbf{v}_2^e]^T\right)\begin{pmatrix}(1+\gamma)\mathbf{Q}_{11}^e+\mathbf{Q}_{22}^e & \gamma\mathbf{Q}_{12}^e\\ \gamma\mathbf{Q}_{12}^e & (1+\gamma)\mathbf{Q}_{22}^e+\mathbf{Q}_{11}^e\end{pmatrix}^T \tag{78}$$

$$\begin{pmatrix}\mathbf{G}^e & \mathbf{O}\\ \mathbf{O} & \mathbf{G}^e\end{pmatrix}\begin{pmatrix}\mathbf{H}^e(\mathbf{u}^{e,n,k-1}) & \mathbf{O}\\ \mathbf{O} & \mathbf{H}^e(\mathbf{u}^{e,n,k-1})\end{pmatrix}\begin{pmatrix}\mathbf{u}_1^{e,n,k}\\ \mathbf{u}_2^{e,n,k}\end{pmatrix}.$$

Finally, the second term in the first square bracket in (75) multiplied by the last term in the second square bracket after integration yields

$$\frac{\lambda}{\varrho}\tau_p^e\Big[[\mathbf{q}^e]^T[\mathbf{L}_1^e]^T\mathbf{G}\,\mathrm{diag}\{\mathbf{Q}^e\mathbf{u}_1^{e,n,k-1}\}\mathbf{Q}_1^e\mathbf{u}_1^{n,k}+[\mathbf{q}^e]^T[\mathbf{L}_1^e]^T\mathbf{G}\,\mathrm{diag}\{\mathbf{Q}^e\mathbf{u}_2^{e,n,k-1}\}\mathbf{Q}_2^e\mathbf{u}_1^{n,k}+$$

$$+[\mathbf{q}^e]^T[\mathbf{L}_2^e]^T\mathbf{G}\,\mathrm{diag}\{\mathbf{Q}^e\mathbf{u}_1^{e,n,k-1}\}\mathbf{Q}_1^e\mathbf{u}_2^{n,k}+[\mathbf{q}^e]^T[\mathbf{L}_2^e]^T\mathbf{G}\,\mathrm{diag}\{\mathbf{Q}^e\mathbf{u}_2^{e,n,k-1}\}\mathbf{Q}_2^e\mathbf{u}_2^{n,k}\Big],$$

which we write in a matrix form as

$$\lambda\frac{1}{\varrho}\tau_{pe}^e[\mathbf{q}^e]^T\left([\mathbf{L}_1^e]^T\quad[\mathbf{L}_2^e]^T\right)\begin{pmatrix}\mathbf{G}^e & \mathbf{O}\\ \mathbf{O} & \mathbf{G}^e\end{pmatrix}\begin{pmatrix}\mathbf{H}^e(\mathbf{u}^{e,n,k-1}) & \mathbf{O}\\ \mathbf{O} & \mathbf{H}^e(\mathbf{u}^{e,n,k-1})\end{pmatrix}\begin{pmatrix}\mathbf{u}_1^{e,n,k}\\ \mathbf{u}_2^{e,n,k}\end{pmatrix}. \tag{79}$$

Let us define matrices

$$\mathbf{V}^e=\begin{pmatrix}(1+\gamma)\mathbf{Q}_{11}^e+\mathbf{Q}_{22}^e & \gamma\mathbf{Q}_{12}^e\\ \gamma\mathbf{Q}_{12}^e & (1+\gamma)\mathbf{Q}_{22}^e+\mathbf{Q}_{11}^e\end{pmatrix}$$

and

$$\mathbf{\Phi}^{se}(\mathbf{u}^e)=\left(\tau_u^e\begin{pmatrix}\mathbf{H}^e(\mathbf{u}^e) & \mathbf{O}\\ \mathbf{O} & \mathbf{H}^e(\mathbf{u}^e)\end{pmatrix}-\tau_s^e\nu\mathbf{V}^e\qquad\tau_p^e\frac{1}{\varrho}\begin{pmatrix}\mathbf{L}_1^e\\ \mathbf{L}_2^e\end{pmatrix}\right)^T\begin{pmatrix}\mathbf{G}^e & \mathbf{O}\\ \mathbf{O} & \mathbf{G}^e\end{pmatrix}.$$

Now, if we sum up (77), (78) and (79), we may write the second term multiplied by the second square bracket in (75) as

$$\lambda\left([\mathbf{v}_1^e]^T\quad[\mathbf{v}_2^e]^T\quad[\mathbf{q}^e]^T\right)\mathbf{\Phi}^{se}(\mathbf{u}^{e,n,k-1})\mathbf{C}^{se2}(\mathbf{u}^{e,n,k-1})\begin{pmatrix}\mathbf{u}_1^{e,n,k}\\ \mathbf{u}_2^{e,n,k}\end{pmatrix}, \tag{80}$$

where

$$\mathbf{C}^{se2}(\mathbf{u}^e)=\begin{pmatrix}\mathbf{H}^e(\mathbf{u}^e) & \mathbf{O}\\ \mathbf{O} & \mathbf{H}^e(\mathbf{u}^e)\end{pmatrix}.$$

Our experience now suggests that the third term multiplied by the second square bracket in (75) will have the following form,

$$\lambda\beta \begin{pmatrix}[\mathbf{v}_1^e]^T & [\mathbf{v}_2^e]^T & [\mathbf{q}^e]^T\end{pmatrix} \mathbf{\Phi}^{se}(\mathbf{u}^{e,n,k-1})\mathbf{C}^{se1}(\mathbf{u}^{e,n,k-1}) \begin{pmatrix}\mathbf{u}_1^{e,n,k} \\ \mathbf{u}_2^{e,n,k}\end{pmatrix} , \tag{81}$$

where

$$\mathbf{C}^{se1}(\mathbf{u}^e) = \begin{pmatrix}\mathbf{H}_{11}^e(\mathbf{u}_1^e) & \mathbf{H}_{12}^e(\mathbf{u}_1^e) \\ \mathbf{H}_{21}^e(\mathbf{u}_2^e) & \mathbf{H}_{22}^e(\mathbf{u}_2^e)\end{pmatrix} .$$

Further, for the fourth term multiplied by the second square bracket in (75) we have

$$-\lambda\beta \begin{pmatrix}[\mathbf{v}_1^e]^T & [\mathbf{v}_2^e]^T & [\mathbf{q}^e]^T\end{pmatrix} \mathbf{\Phi}^{se}(\mathbf{u}^{e,n,k-1})\mathbf{C}^{se1}(\mathbf{u}^{e,n,k-1}) \begin{pmatrix}\mathbf{u}_1^{e,n,k-1} \\ \mathbf{u}_2^{e,n,k-1}\end{pmatrix} . \tag{82}$$

Integrating in the same way again and again, for the fifth, sixth and seventh term in (75), all multiplied by the second square bracket, we arrive at

$$\begin{pmatrix}[\mathbf{v}_1^e]^T & [\mathbf{v}_2^e]^T & [\mathbf{q}^e]^T\end{pmatrix} (-\nu)\mathbf{\Phi}^{se}(\mathbf{u}^{e,n,k-1})\mathbf{V}^e \begin{pmatrix}\mathbf{u}_1^{e,n,k} \\ \mathbf{u}_2^{e,n,k}\end{pmatrix} , \tag{83}$$

$$\begin{pmatrix}[\mathbf{v}_1^e]^T & [\mathbf{v}_2^e]^T & [\mathbf{q}^e]^T\end{pmatrix} \frac{1}{\varrho}\mathbf{\Phi}^{se}(\mathbf{u}^{e,n,k-1}) \begin{pmatrix}\mathbf{L}_1^e \\ \mathbf{L}_2^e\end{pmatrix} \mathbf{p}^{e,n,k} , \tag{84}$$

$$\begin{pmatrix}[\mathbf{v}_1^e]^T & [\mathbf{v}_2^e]^T & [\mathbf{q}^e]^T\end{pmatrix} \mathbf{\Phi}^{se}(\mathbf{u}^{e,n,k-1}) \begin{pmatrix}\mathbf{Q}^e & \mathbf{O} \\ \mathbf{O} & \mathbf{Q}^e\end{pmatrix} \begin{pmatrix}\mathbf{f}_1^{e,n} \\ \mathbf{f}_2^{e,n}\end{pmatrix} , \tag{85}$$

respectively. Analogously, for the difference quotient we have

$$\frac{1}{\Delta t} \begin{pmatrix}[\mathbf{v}_1^e]^T & [\mathbf{v}_2^e]^T & [\mathbf{q}^e]^T\end{pmatrix} \mathbf{\Phi}^{se}(\mathbf{u}^{e,n,k-1}) \begin{pmatrix}\mathbf{Q}^e & \mathbf{O} \\ \mathbf{O} & \mathbf{Q}^e\end{pmatrix} \left[\begin{pmatrix}\mathbf{u}_1^{e,n,k} \\ \mathbf{u}_2^{e,n,k}\end{pmatrix} - \begin{pmatrix}\mathbf{u}_1^{e,n-1} \\ \mathbf{u}_2^{e,n-1}\end{pmatrix}\right] , \tag{86}$$

and finally, integration of the last term in (75) yields

$$\delta^e \begin{pmatrix}[\mathbf{v}_1^e]^T & [\mathbf{v}_2^e]^T\end{pmatrix} \begin{pmatrix}[\mathbf{Q}_1^e]^T\mathbf{G}^e\mathbf{Q}_1^e & [\mathbf{Q}_1^e]^T\mathbf{G}^e\mathbf{Q}_2^e \\ [\mathbf{Q}_2^e]^T\mathbf{G}^e\mathbf{Q}_1^e & [\mathbf{Q}_2^e]^T\mathbf{G}^e\mathbf{Q}_2^e\end{pmatrix} \begin{pmatrix}\mathbf{u}_1^{e,n,k} \\ \mathbf{u}_2^{e,n,k}\end{pmatrix} . \tag{87}$$

Now, if we insert the expressions (80)-(87) into (75), we obtain the stabilizing term in the following matrix form,

$$\sum_e \left([\mathbf{v}^e]^T \quad [\mathbf{q}^e]^T\right) \mathbf{\Phi}^{se}(\mathbf{u}^{e,n,k-1})\Bigg\{\Bigg[\frac{1}{\Delta t}\mathbf{M}^{se} + \mathbf{K}^{se} + \\ + \lambda\beta\mathbf{C}^{se1}(\mathbf{u}^{e,n,k-1}) + \lambda\mathbf{C}^{se2}(\mathbf{u}^{e,n,k-1})\Bigg]\mathbf{u}^{e,n,k} + \mathbf{D}^{se}\mathbf{p}^{e,n,k} - \\ - \lambda\beta\mathbf{C}^{se1}(\mathbf{u}^{e,n,k-1}) - \mathbf{M}^{se}\left[\frac{1}{\Delta t}\mathbf{u}^{e,n-1} + \mathbf{f}^{e,n}\right]\Bigg\} + \\ + \sum_e \delta^e[\mathbf{v}^e]^T\mathbf{K}^{ge}\mathbf{u}^{e,n,k}\,. \tag{88}$$

The description of matrices occuring in (88) now follows. We have already defined matrices

$$\mathbf{\Phi}^{se}(\mathbf{u}^e) = \begin{pmatrix} \tau_u^e \begin{pmatrix} \mathbf{H}^e(\mathbf{u}^e) & \mathbf{O} \\ \mathbf{O} & \mathbf{H}^e(\mathbf{u}^e) \end{pmatrix} - \tau_s^e \nu \mathbf{V}^e & \tau_p^e \frac{1}{\varrho} \begin{pmatrix} \mathbf{L}_1^e \\ \mathbf{L}_2^e \end{pmatrix} \end{pmatrix}^T \begin{pmatrix} \mathbf{G}^e & \mathbf{O} \\ \mathbf{O} & \mathbf{G}^e \end{pmatrix} \tag{89}$$

and

$$\mathbf{V}^e = \begin{pmatrix} (1+\gamma)\mathbf{Q}_{11}^e + \mathbf{Q}_{22}^e & \gamma\mathbf{Q}_{12}^e \\ \gamma\mathbf{Q}_{12}^e & (1+\gamma)\mathbf{Q}_{22}^e + \mathbf{Q}_{11}^e \end{pmatrix}. \tag{90}$$

Then we have,

$$\mathbf{M}^{se} = \begin{pmatrix} \mathbf{Q}^e & \mathbf{O} \\ \mathbf{O} & \mathbf{Q}^e \end{pmatrix}, \tag{91}$$

$$\mathbf{K}^{se} = -\nu\mathbf{V}^e, \tag{92}$$

$$\mathbf{C}^{se1}(\mathbf{u}^e) = \begin{pmatrix} \mathbf{H}_{11}^e(\mathbf{u}_1^e) & \mathbf{H}_{12}^e(\mathbf{u}_1^e) \\ \mathbf{H}_{21}^e(\mathbf{u}_2^e) & \mathbf{H}_{22}^e(\mathbf{u}_2^e) \end{pmatrix}, \tag{93}$$

$$\mathbf{C}^{se2}(\mathbf{u}^e) = \begin{pmatrix} \mathbf{H}^e(\mathbf{u}^e) & \mathbf{O}) \\ \mathbf{O} & \mathbf{H}^e(\mathbf{u}^e) \end{pmatrix}, \tag{94}$$

$$\mathbf{D}^{se} = \frac{1}{\varrho}\begin{pmatrix} \mathbf{L}_1^e \\ \mathbf{L}_2^e \end{pmatrix}, \tag{95}$$

$$\mathbf{K}^{ge} = \begin{pmatrix} [\mathbf{Q}_1^e]^T \mathbf{G}^e \mathbf{Q}_1^e & [\mathbf{Q}_1^e]^T \mathbf{G}^e \mathbf{Q}_2^e \\ [\mathbf{Q}_2^e]^T \mathbf{G}^e \mathbf{Q}_1^e & [\mathbf{Q}_2^e]^T \mathbf{G}^e \mathbf{Q}_2^e \end{pmatrix} . \tag{96}$$

In practice we proceed as follows. We add the stabilizing term (88) to the left hand side of the equation (64) and from this equation we form

$$\mathbf{v}^T(\mathbf{A}\mathbf{u} - \mathbf{f}) = 0 .$$

Because the vector $\mathbf{v}$ may be arbitrary,

$$\mathbf{A}\mathbf{u} = \mathbf{f}$$

must hold. We solve this system of linear equations in every iteration.

We have still not shown how to solve the eigenvalue problem (73). Using the same integration process as always in this thesis, we arrive at the generalized eigenvalue problem

$$\mathbf{A}^e \mathbf{u}^e = \lambda \mathbf{B}^e \mathbf{u}^e , \tag{97}$$

where

$$\mathbf{A}^e = \frac{1}{4}[\mathbf{V}^e]^T \begin{pmatrix} \mathbf{G}^e & \mathbf{O} \\ \mathbf{O} & \mathbf{G}^e \end{pmatrix} \mathbf{V}^e , \qquad \mathbf{B}^e = \frac{1}{2\nu}\mathbf{K}^e$$

for $\gamma = 1$, and

$$\mathbf{A}^e = [\mathbf{V}^e]^T \begin{pmatrix} \mathbf{G}^e & \mathbf{O} \\ \mathbf{O} & \mathbf{G}^e \end{pmatrix} \mathbf{V}^e ,$$

$$\mathbf{B}^e = \begin{pmatrix} [\mathbf{Q}_1^e]^T \mathbf{G}^e \mathbf{Q}_1^e + [\mathbf{Q}_2^e]^T \mathbf{G}^e \mathbf{Q}_2^e & \mathbf{O} \\ \mathbf{O} & [\mathbf{Q}_1^e]^T \mathbf{G}^e \mathbf{Q}_1^e + [\mathbf{Q}_2^e]^T \mathbf{G}^e \mathbf{Q}_2^e \end{pmatrix}$$

for $\gamma = 0$.

We may note here that this eigenvalue problem does not depend neither on n nor on k. Therefore, for each element we compute the greatest eigenvalue only once and for all.

5 Example: Rotating Ellipse

In this section we finally arrive at an example where we can test our algorithm. Let us consider an ellipse inside a circle (see fig. 2), where

- r is a radius of the circle,
- a is a semi-major axis of the ellipse,
- b denotes a semi-minor axis of the ellipse,
- $\omega(t)$ stands for angular velocity of the ellipse with ω_m the maximal angular velocity,
- Ω is the domain,
- Γ_1 and Γ_2 are two parts of the boundary $\partial\Omega$.

The hatched area between the circle and ellipse is occupied by a fluid.

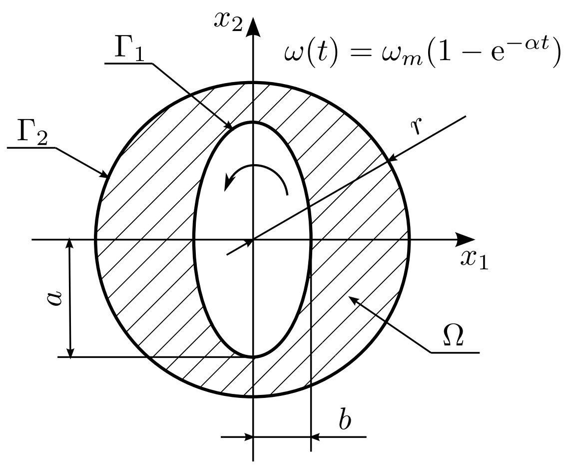


Figure 2: Ellipse rotating in a circle

So we are given the geometry of the problem and the expression for angular velocity to assure smooth start of the rotation. Our aim is to describe the velocity field and the pressure of a fluid as the ellipse rotates. Moreover, from these data we shall compute a force and momentum acting on the ellipse. The boundary condition on Γ_1 is defined by the angular velocity $\omega(t)$. On Γ_2 we impose $\mathbf{u} = \mathbf{0}$. Note that on both Γ_1 and Γ_2 we assign the Dirichlet boundary condition, which means there will be no surface integral in the weak formulation. There are also no sources, which implies $\mathbf{f} = \mathbf{0}$.

5.1 ALE formulation

In the example presented above we have to deal with a moving domain Ω, a problem we could not solve using the algorithm presented in the previous sections. Thus we need to use the ALE (Arbitrary Lagrangian Eulerian) formulation. Very good description of the ALE formulation may be found in [4]. To describe the motion of a fluid one usually works with material domain $R_{\mathbf{X}}$ and/or spatial domain $R_{\mathbf{x}}$. To pass from one domain to another one introduces a mapping

$$\boldsymbol{\varphi}: \quad (\mathbf{X}, t) \longmapsto \boldsymbol{\varphi}(\mathbf{X}, t) = (\mathbf{x}, t)\,, \tag{98}$$

which assigns to every material point $\mathbf{X}$ a spatial position $\mathbf{x}$ at time t. Obviously, for the material velocity $\mathbf{u}$ we have

$$\mathbf{u}\Big(\boldsymbol{\varphi}(\mathbf{X}, t), t\Big) = \frac{\partial \boldsymbol{\varphi}(\mathbf{X}, t)}{\partial t}\,. \tag{99}$$

To describe the moving domain we introduce yet another domain, $R_{\boldsymbol{\chi}}$, and a mapping $\boldsymbol{\Phi}$ (see fig. 3),

$$\boldsymbol{\Phi}: \quad (\boldsymbol{\chi}, t) \longmapsto \boldsymbol{\Phi}(\boldsymbol{\chi}, t) = (\mathbf{x}, t), \tag{100}$$

which describes the motion of the domain in spatial coordinates $\mathbf{x}$. Then the domain point velocity is

$$\mathbf{c}\Big(\boldsymbol{\Phi}(\boldsymbol{\chi}, t), t\Big) = \frac{\partial \boldsymbol{\Phi}(\boldsymbol{\chi}, t)}{\partial t}\,. \tag{101}$$

Usually, the domains $R_{\mathbf{X}}$ and $R_{\boldsymbol{\chi}}$ are fixed and correspond to some initial configuration at time t_0.

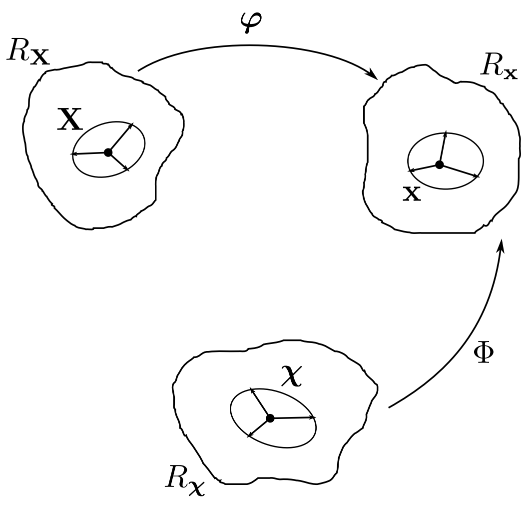


Figure 3: ALE description

In what follows, for a function $f(\mathbf{x}, t)$ defined in spatial domain, the material time derivative will be denoted as

$$\frac{\mathrm{d}}{\mathrm{d}t} \equiv \frac{\partial}{\partial t}\bigg|_{\mathbf{X}}\,,$$

the spatial time derivative as

$$\frac{\partial}{\partial t} \equiv \frac{\partial}{\partial t}\bigg|_{\mathbf{x}} ,$$

and the ALE time derivative as

$$\frac{\partial^a}{\partial t} \equiv \frac{\partial}{\partial t}\bigg|_{\mathcal{X}} ,$$

where, for example, $\big|_{\mathbf{X}}$ means "holding $\mathbf{X}$ fixed".

To derive the Navier-Stokes equations in the ALE formulation, we need to consider the derivatives of the integrals over a moving volume occupied by a fluid. Let V_t be an arbitrary volume at time t with ∂V_t its boundary and let $f(\mathbf{x}, t)$ be a scalar function defined in the spatial domain. Then, (see, e. g., [7])

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{V_t} f(\mathbf{x},t)\,\mathrm{d}V &= \int\limits_{V_t}\left(\frac{\partial f(\mathbf{x},t)}{\partial t} + \nabla\cdot[f(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)]\right)\mathrm{d}\mathbf{x} = \\ &= \int\limits_{V_t}\frac{\partial f(\mathbf{x},t)}{\partial t}\,\mathrm{d}V + \int\limits_{\partial V_t} f(\mathbf{x},t)\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S\,,\end{aligned} \tag{102}$$

where $\mathbf{n}$ denotes the unit outward normal to the surface ∂V_t at time t. The last identity is known as the Reynolds transport theorem.

Accordingly, if we interchange the material time derivative with the ALE time derivative, we obtain

$$\begin{aligned}\frac{\partial^a}{\partial t}\int\limits_{V_t} f(\mathbf{x},t)\,\mathrm{d}V &= \int\limits_{V_t}\left(\frac{\partial f(\mathbf{x},t)}{\partial t} + \nabla\cdot[f(\mathbf{x},t)\mathbf{c}(\mathbf{x},t)]\right)\mathrm{d}\mathbf{x} = \\ &= \int\limits_{V_t}\frac{\partial f(\mathbf{x},t)}{\partial t}\,\mathrm{d}V + \int\limits_{\partial V_t} f(\mathbf{x},t)\mathbf{c}\cdot\mathbf{n}\,\mathrm{d}S\,.\end{aligned} \tag{103}$$

Now, if we subtract (103) from (102), we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{V_t} f(\mathbf{x},t)\,\mathrm{d}V = \frac{\partial^a}{\partial t}\int\limits_{V_t} f(\mathbf{x},t)\,\mathrm{d}V + \int\limits_{\partial V_t} f(\mathbf{x},t)[\mathbf{u}-\mathbf{c}]\cdot\mathbf{n}\,\mathrm{d}S\,. \tag{104}$$

The equation (104) allows us to express the material time derivative in terms of the ALE time derivative.

Let us note that the continuity equation remains unchanged in the ALE formulation. Hence $\nabla\cdot\mathbf{u} = 0$ holds. Using (104) we may write down the second Newton law for an arbitrary control volume V_t as

$$\frac{\partial^a}{\partial t}\int\limits_{V_t} \varrho\mathbf{u}(\mathbf{x},t)\,\mathrm{d}V + \int\limits_{\partial V_t} \varrho\mathbf{u}[\mathbf{u}-\mathbf{c}]\cdot\mathbf{n}\,\mathrm{d}S = \int\limits_{V_t} \varrho\mathbf{f}(\mathbf{x},t)\,\mathrm{d}\mathbf{x} + \int\limits_{\partial V_t} \boldsymbol{\tau}\mathbf{n}\,\mathrm{d}S\,,$$

where $\mathbf{f}$ is the density of the body force and $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1}^{n}$ is the stress tensor. Applying (103) to the first integral and the divergence theorem to the last integral in the above identity yields

$$\int_{V_t} \left(\frac{\partial^a}{\partial t} \varrho \mathbf{u}(\mathbf{x}, t) + \varrho \mathbf{u}(\mathbf{x}, t) \nabla \cdot \mathbf{c}(\mathbf{x}, t) \right) \mathrm{d}\mathbf{x} + \int_{\partial V_t} \varrho \mathbf{u}[\mathbf{u} - \mathbf{c}] \cdot \mathbf{n} \, \mathrm{d}S = \\ = \int_{V_t} \varrho \mathbf{f}(\mathbf{x}, t) \, \mathrm{d}\mathbf{x} + \int_{V_t} \nabla \cdot \boldsymbol{\tau} \, \mathrm{d}\mathbf{x} \,. \tag{105}$$

Next, we apply the divergence theorem to the second integral in (105). Thus

$$\int_{\partial V_t} \varrho \mathbf{u}(\mathbf{x}, t)[\mathbf{u}(\mathbf{x}, t) - \mathbf{c}(\mathbf{x}, t)] \cdot \mathbf{n} \, \mathrm{d}S = \int_{V_t} \varrho \Big[\Big(\mathbf{u}(\mathbf{x}, t) - \mathbf{c}(\mathbf{x}, t) \Big) \cdot \nabla \Big] \mathbf{u}(\mathbf{x}, t) \, \mathrm{d}\mathbf{x} + \\ + \int_{V_t} \varrho \mathbf{u}(\mathbf{x}, t) [\nabla \cdot \mathbf{u}(\mathbf{x}, t) - \nabla \cdot \mathbf{c}(\mathbf{x}, t)] \, \mathrm{d}\mathbf{x} \,.$$

Substituting this identity in (105) and using the continuity equation, we obtain

$$\int_{V_t} \frac{\partial^a}{\partial t} \varrho \mathbf{u} \, \mathrm{d}\mathbf{x} + \int_{V_t} \varrho [(\mathbf{u} - \mathbf{c}) \cdot \nabla] \mathbf{u} \, \mathrm{d}\mathbf{x} = \int_{V_t} \varrho \mathbf{f} \, \mathrm{d}\mathbf{x} + \int_{V_t} \nabla \cdot \boldsymbol{\tau} \, \mathrm{d}\mathbf{x} \,,$$

and hence

$$\frac{\partial^a}{\partial t} \varrho \mathbf{u} + \varrho [(\mathbf{u} - \mathbf{c}) \cdot \nabla] \mathbf{u} = \varrho \mathbf{f} + \nabla \cdot \boldsymbol{\tau} \,. \tag{106}$$

For the Newtonian fluids one may derive the following form of $\boldsymbol{\tau}$, (see, e. g., [6])

$$\boldsymbol{\tau} = -p \mathbf{I} + 2 \mu \boldsymbol{\varepsilon}(\mathbf{u}) \,,$$

where $\mathbf{I}$ is the unit tensor and μ is the dynamic viscosity assumed constant. Using this expression we finally obtain the Navier-Stokes equations in the ALE formulation,

$$\frac{\partial^a}{\partial t} \mathbf{u} + [(\mathbf{u} - \mathbf{c}) \cdot \nabla] \mathbf{u} - 2 \nu \nabla \cdot \boldsymbol{\varepsilon}(\mathbf{u}) + \frac{1}{\varrho} \nabla p = \mathbf{f} \qquad \text{in } \Omega_t \,, \tag{107}$$

where $\nu = \mu / \varrho$ and Ω_t is the domain at time t. Obviously, we have to add the continuity equation

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega_t \,. \tag{108}$$

5.1.1 ALE discretization

Let us fix some point $\boldsymbol{\chi}$ in the reference domain $R_{\boldsymbol{\chi}}$. We may imagine this point as a mesh node, for example. For given time steps t_n and t_{n-1}, we denote by

$$\mathbf{x}^n \equiv \boldsymbol{\Phi}(\boldsymbol{\chi}, t_n)$$

and

$$\mathbf{x}^{n-1} \equiv \mathbf{\Phi}(\boldsymbol{\chi}, t_{n-1})$$

the images of the point $\boldsymbol{\chi}$ at times t_n and t_{n-1}, respectively. We shall also use the following notation,

$$\mathbf{u}^*(\boldsymbol{\chi}, t) \equiv \mathbf{u}\Big(\mathbf{\Phi}(\boldsymbol{\chi}, t), t\Big) .$$

Then,

$$\frac{\partial^a}{\partial t}\mathbf{u}(\mathbf{x}, t_n) = \frac{\partial \mathbf{u}^*(\boldsymbol{\chi}, t_n)}{\partial t} \approx \frac{\mathbf{u}^*(\boldsymbol{\chi}, t_n) - \mathbf{u}^*(\boldsymbol{\chi}, t_{n-1})}{\Delta t} = \frac{\mathbf{u}^n(\mathbf{x}^n) - \mathbf{u}^{n-1}(\mathbf{x}^{n-1})}{\Delta t} .$$

This is only a little variance with regard to the algorithm derived in the previous sections. We only have to note that the new value $\mathbf{u}^n$, and also p^n, computed at time t_n belongs to the node $\mathbf{x}^n$, translated with respect to the node $\mathbf{x}^{n-1}$ due to the ALE mapping $\mathbf{\Phi}$.

If we take a look at our algorithm, we can easily see that the moving domain will affect it as follows

- We compute the new position of the points $\mathbf{x}^n$ of the computational mesh and then we find the velocity

$$\mathbf{c}^{e,n} = \frac{\mathbf{x}^{e,n} - \mathbf{x}^{e,n-1}}{\Delta t} .$$

- In matrices $\mathbf{C}^{e2}$, $\mathbf{C}^{se2}$ and $\mathbf{\Phi}^{se}$, we substitute the velocity $\mathbf{u}^{e,n,k-1}$ by the difference of velocities $\mathbf{u}^{e,n,k-1} - \mathbf{c}^{e,n,k-1}$. Also in (72) and subsequent expression for Re^e we consider $|\mathbf{u}^{e,n,k-1} - \mathbf{c}^{e,n,k-1}|$ instead of $|\mathbf{u}^{e,n,k-1}|$.

In computations presented below we proceeded differently. To avoid computation of the new position of points $\mathbf{x}^n$ in every time step and to assure better convergence of the Newton method we use the difference $\mathbf{u}^{e,n,k-1} - \mathbf{c}^{e,n-1}$ instead. This is possible since the mesh velocity is relatively small.

5.2 Resulting force and moment acting on the ellipse

In every time step t_n we want to compute the force and moment acting on the rotating ellipse. For the force we have

$$F_i = -\int_{\Gamma_1} \sum_{j=1}^{2} \tau_{ij} n_j \,\mathrm{d}S\,, \qquad i = 1, 2\,, \tag{109}$$

and the moment is obtained from

$$M = \int\limits_{\Gamma_1} \sum_{i,j=1}^{2} \tau_{ij} n_j \tilde{r}_i \, \mathrm{d}S \,, \tag{110}$$

where

$$\tilde{r}_1 = -(x_2 - x_{02}) \,, \qquad \tilde{r}_2 = x_1 - x_{01} \,,$$

$\mathbf{x}_0 = (x_{01}, x_{02})^T$ is a point we compute the moment with respect to. In our case this will be the origin. We shall proceed according to [9].

5.2.1 Force

We start by writing the time-discretized Navier-Stokes equations in the ALE form component-wise,

$$\varrho \frac{u_i^n - u_i^{n-1}}{\Delta t} + \varrho\lambda \sum_{j=1}^{2} (u_j^n - c_j^n) \frac{\partial u_i^n}{\partial x_j} = \sum_{j=1}^{2} \frac{\partial \tau_{ij}}{\partial x_j} + \varrho f_i \qquad \text{in } \Omega_n \,, \quad i = 1, 2 \,, \tag{111}$$

where $\Omega_n \equiv \Omega_{t_n}$. Let us define $\Omega_{\Gamma_1} = \cup\{e \in \mathcal{T}_n; \bar{e} \cap \Omega_{\Gamma_1} \neq \emptyset\}$, which represents the union of finite elements having nonempty intersection with Γ_1. $\mathcal{T}_n$ denotes the triangulation at time t_n. Next, we choose a test function $\varphi \in X_{hv}$ such that $\varphi(\mathbf{x}) = 1$ for $\mathbf{x} \in \Gamma_1$ and $\varphi(\mathbf{x}) = 0$ outside Ω_{Γ_1}. Multiplying the equation (111) by φ, integrating over Ω_{Γ_1} and using the divergence theorem, we arrive at

$$\int\limits_{\Omega_{\Gamma_1}} \varrho \frac{u_i^n - u_i^{n-1}}{\Delta t} \varphi \, \mathrm{d}x_1 \mathrm{d}x_2 + \int\limits_{\Omega_{\Gamma_1}} \varrho\lambda \sum_{j=1}^{2} (u_j^n - c_j^n) \frac{\partial u_i^n}{\partial x_j} \varphi \, \mathrm{d}x_1 \mathrm{d}x_2 = \int\limits_{\Gamma_1} \sum_{j=1}^{2} \tau_{ij} n_j \, \mathrm{d}S -$$
$$- \int\limits_{\Omega_{\Gamma_1}} \tau_{ij} \frac{\partial \varphi}{\partial x_j} \, \mathrm{d}x_1 \mathrm{d}x_2 + \int\limits_{\Omega_{\Gamma_1}} \varrho f_i \varphi \, \mathrm{d}x_1 \mathrm{d}x_2 \,, \qquad i = 1, 2 \,.$$

From here we see that

$$F_i = -\Bigg\{ \int\limits_{\Omega_{\Gamma_1}} \varrho \left[\frac{u_i^n - u_i^{n-1}}{\Delta t} + \lambda \sum_{j=1}^{2} (u_j^n - c_j^n) \frac{\partial u_i^n}{\partial x_j} \right] \varphi \, \mathrm{d}x_1 \mathrm{d}x_2 - \int\limits_{\Omega_{\Gamma_1}} p^n \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}x_1 \mathrm{d}x_2 +$$
$$+ 2\nu \int\limits_{\Omega_{\Gamma_1}} \varrho \sum_{j=1}^{2} \varepsilon_{ij}(\mathbf{u}^n) \frac{\partial \varphi}{\partial x_j} \, \mathrm{d}x_1 \mathrm{d}x_2 - \int\limits_{\Omega_{\Gamma_1}} \varrho f_i \varphi \, \mathrm{d}x_1 \mathrm{d}x_2 \Bigg\} \,, \qquad i = 1, 2 \,. \tag{112}$$

Now we perform the finite element approximation as in previous sections and the function φ will be approximated as velocities, i. e.,

$$\varphi(\mathbf{x})|_e \approx \sum_{i=1}^{n_v} \varphi_i^e \hat{Q}_i(\xi_1, \xi_2) = [\boldsymbol{\varphi}^e]^T \boldsymbol{\kappa} \,,$$

where $\boldsymbol{\varphi} = (\varphi_1^e, \dots, \varphi_{n_v}^e)^T$ is the vector of parameters of φ. Applying the usual integration process, we may express the resulting force as

$$\mathbf{F} \approx -\varrho \sum_e [\tilde{\boldsymbol{\varphi}}^e]^T \left\{ \mathbf{M}^e \frac{\mathbf{u}^{e,n} - \mathbf{u}^{e,n-1}}{\Delta t} + [\lambda \mathbf{C}^{e1}(\mathbf{u}^{e,n} - \mathbf{c}^{e,n}) + \mathbf{K}^e]\mathbf{u}^{e,n} + \mathbf{D}^e \mathbf{p}^{e,n} - \mathbf{M}^e \mathbf{f}^{e,n} \right\}, \tag{113}$$

where $\tilde{\boldsymbol{\varphi}}^e = \left([\boldsymbol{\varphi}^e]^T, [\boldsymbol{\varphi}^e]^T\right)^T$,

$$\varphi_j^e = \begin{cases} 1 & \text{for } P_j^e \in \overline{\Gamma}_1 \\ 0 & \text{for } P_j^e \notin \overline{\Gamma}_1 \end{cases}, \qquad j = 1, \dots, n_v,$$

and $\mathbf{F} = (F_1, F_2)^T$.

5.2.2 Moment

Again, we start from the equation (111). This time we choose a function $\tilde{\mathbf{v}} = (\tilde{v}_1, \tilde{v}_2)^T = (\varphi \tilde{r}_1, \varphi \tilde{r}_2)^T$. We approximate $\tilde{r}_i$, $i = 1, 2$, same as φ, i. e.,

$$\varphi(\mathbf{x})|_e \approx [\boldsymbol{\varphi}^e]^T \boldsymbol{\kappa}, \qquad \tilde{r}_i|_e \approx [\tilde{\mathbf{r}}_i^e]^T \boldsymbol{\kappa}, \quad i = 1, 2.$$

Let us multiply (111) by $\tilde{v}_i$, sum over $i = 1, 2$, and integrate over Ω_{Γ_1}. Similarly as for the force, we obtain

$$M = \sum_{i=1}^{2} \left\{ \int_{\Omega_{\Gamma_1}} \varrho \left[\frac{u_i^n - u_i^{n-1}}{\Delta t} + \lambda \sum_{j=1}^{2} (u_j^n - c_j^n) \frac{\partial u_i^n}{\partial x_j} \right] \tilde{v}_i \, \mathrm{d}x_1 \mathrm{d}x_2 - \int_{\Omega_{\Gamma_1}} p^n \frac{\partial \tilde{v}_i}{\partial x_i} \, \mathrm{d}x_1 \mathrm{d}x_2 + \right.$$

$$\left. + 2\nu \int_{\Omega_{\Gamma_1}} \varrho \sum_{j=1}^{2} \varepsilon_{ij}(\mathbf{u}^n) \frac{\partial \tilde{v}_i}{\partial x_j} \, \mathrm{d}x_1 \mathrm{d}x_2 - \int_{\Omega_{\Gamma_1}} \varrho f_i \tilde{v}_i \, \mathrm{d}x_1 \mathrm{d}x_2 \right\}.$$

After numerical integration we get

$$\begin{aligned} M = \varrho \sum_e [\tilde{\boldsymbol{\varphi}}^e]^T \Bigg\{ \left[\frac{1}{\Delta t} \tilde{\mathbf{M}}^e + \lambda \tilde{\mathbf{C}}^{e1}(\mathbf{u}^{e,n} - \mathbf{c}^{e,n}) + \tilde{\mathbf{K}}^e \right] \mathbf{u}^{e,n} + \tilde{\mathbf{D}}^e \mathbf{p}^{e,n} - \\ - \tilde{\mathbf{M}}^e \left[\mathbf{f}^{e,n} + \frac{1}{\Delta t} \mathbf{u}^{e,n-1} \right] \Bigg\}. \end{aligned} \tag{114}$$

The description of the matrices occuring in (114) now follows. Let us define the auxiliary matrices

$$\begin{aligned} \mathbf{R}_i^e &= \mathrm{diag}\{\mathbf{Q}^e \tilde{\mathbf{r}}_i\} \mathbf{Q}^e, \qquad i = 1, 2, \\ \mathbf{R}_{ij}^e &= \mathrm{diag}\{\mathbf{Q}^e \tilde{\mathbf{r}}_i\} \mathbf{Q}_j^e + \mathrm{diag}\{\mathbf{Q}_j^e \tilde{\mathbf{r}}_i\} \mathbf{Q}^e, \qquad i, j = 1, 2 \end{aligned} \tag{115}$$

and

$$\tilde{\boldsymbol{\Phi}}^e = \begin{pmatrix} [\mathbf{R}_1^e]^T\mathbf{G}^e & \mathbf{O} \\ \mathbf{O} & [\mathbf{R}_2^e]^T\mathbf{G}^e \end{pmatrix} . \tag{116}$$

Then

$$\tilde{\mathbf{K}}^e = \nu \begin{pmatrix} (1+\gamma)[\mathbf{R}_{11}^e]^T\mathbf{G}\mathbf{Q}_1^e + [\mathbf{R}_{12}^e]^T\mathbf{G}\mathbf{Q}_2^e & \gamma[\mathbf{R}_{12}^e]^T\mathbf{G}\mathbf{Q}_1^e \\ \gamma[\mathbf{R}_{21}^e]^T\mathbf{G}\mathbf{Q}_2^e & (1+\gamma)[\mathbf{R}_{22}^e]^T\mathbf{G}\mathbf{Q}_2^e + [\mathbf{R}_{21}^e]^T\mathbf{G}\mathbf{Q}_1^e \end{pmatrix} , \tag{117}$$

and

$$\tilde{\mathbf{M}}^e = \tilde{\boldsymbol{\Phi}}^e \begin{pmatrix} \mathbf{Q}^e & \mathbf{O} \\ \mathbf{O} & \mathbf{Q}^e \end{pmatrix} , \tag{118}$$

$$\tilde{\mathbf{C}}^{e1}(\mathbf{u}) = \tilde{\boldsymbol{\Phi}}^e \begin{pmatrix} \mathbf{H}_{11}^e(\mathbf{u}_1^e) & \mathbf{H}_{12}^e(\mathbf{u}_1^e) \\ \mathbf{H}_{21}^e(\mathbf{u}_2^e) & \mathbf{H}_{22}^e(\mathbf{u}_2^e) \end{pmatrix} , \tag{119}$$

$$\tilde{\mathbf{D}}^e = -\frac{1}{\varrho} \begin{pmatrix} [\mathbf{R}_{11}^e]^T\mathbf{G}\mathbf{L}^e \\ [\mathbf{R}_{22}^e]^T\mathbf{G}\mathbf{L}^e \end{pmatrix} . \tag{120}$$

6 Numerical Results

In this section we present the results of the numerical computations using the algorithm described in this thesis. The algorithm will be tested on the rotating ellipse introduced in the previous section, see fig. (2). First of all, we define all the necessary data.

Geometry of the problem

- The radius $r = 0{,}2\,\mathrm{m}$
- The semi-major axis $a = 0{,}15\,\mathrm{m}$
- The semi-minor axis $b = 0{,}1\,\mathrm{m}$

Properties of the fluid

- The density $\varrho = 1000\,\mathrm{kg/m^3}$
- The dynamic viscosity $\mu = 0{,}05\,\mathrm{Ns/m^2}$
- The kinematic viscosity $\nu = \mu/\varrho = 5 \cdot 10^{-5}\,\mathrm{m^2/s}$

The angular velocity is defined by

$$\omega(t) = \omega_m(1 - \mathrm{e}^{-\alpha t}),$$

where $\omega_m = 50\,\mathrm{rad/s}$ and $\alpha = 2$.

6.1 Triangulation

Here we show how the triangulation of the computational domain is implemented and how it changes its shape as the ellipse rotates. The domain is discretized by n_c nodes around the circumference and n_r nodes in the radial direction. The nodes of the triangulation at the initial time can be seen in figure (4).

There is a refinement near the boundary of the ellipse since it is a critical place where good approximation must be assured. As the ellipse rotates, the nodes of the computational mesh are moving only in the radial direction. This may be observed in figure (5).

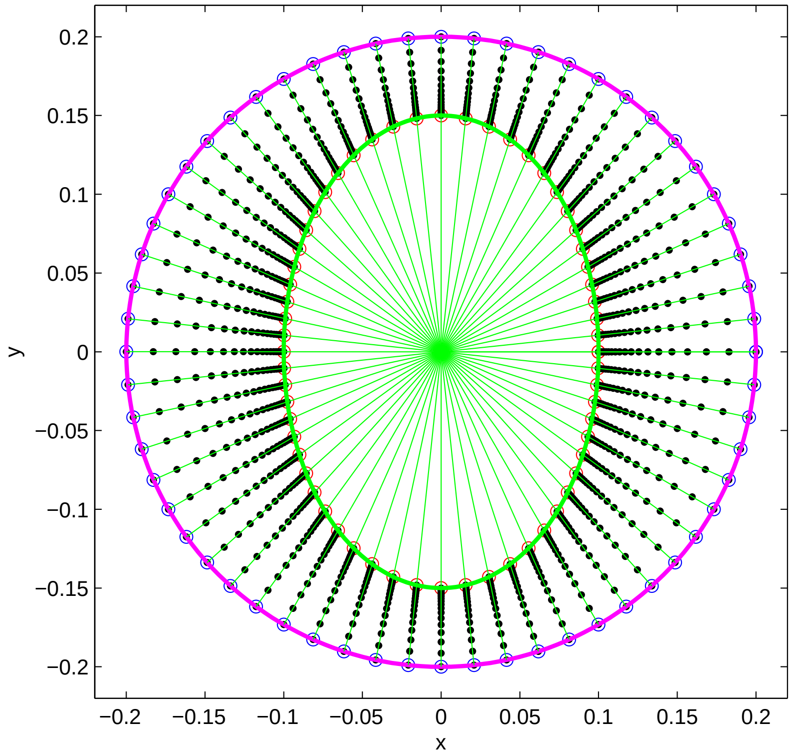


Figure 4: Nodes of the triangulation at time $t = 0$, $n_c = 60$, $n_r = 20$.

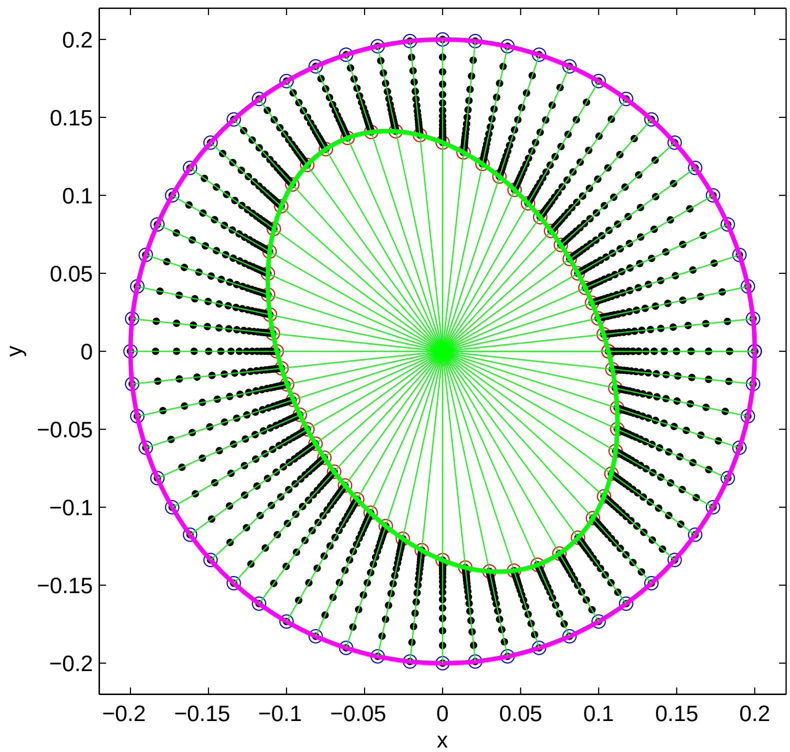


Figure 5: Nodes of the rotated triangulation, $n_c = 60$, $n_r = 20$.

These nodes are the vertices of the triangles as shown in figure (6).

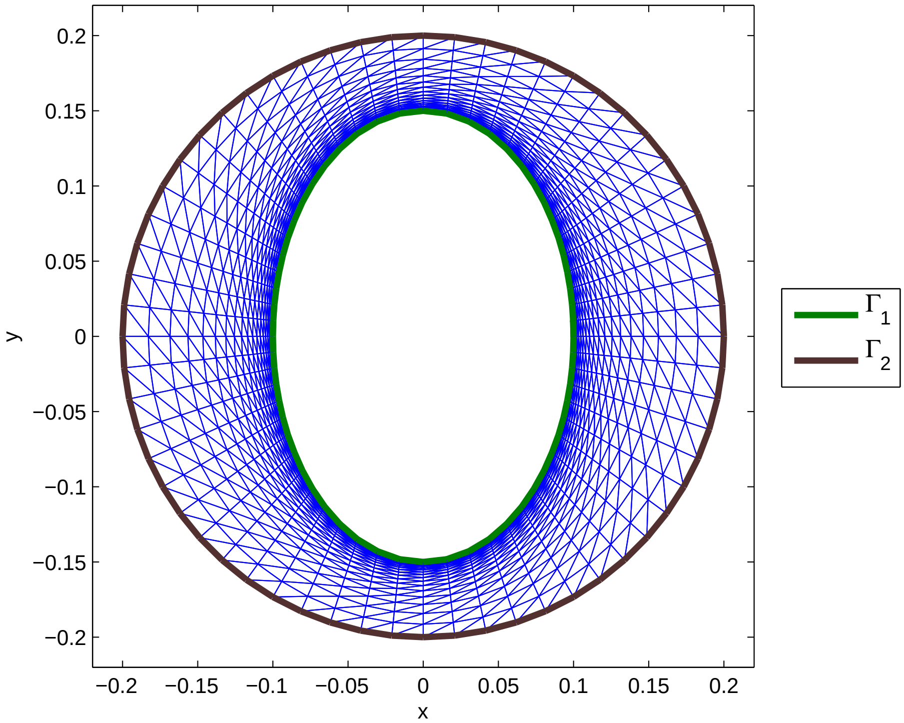
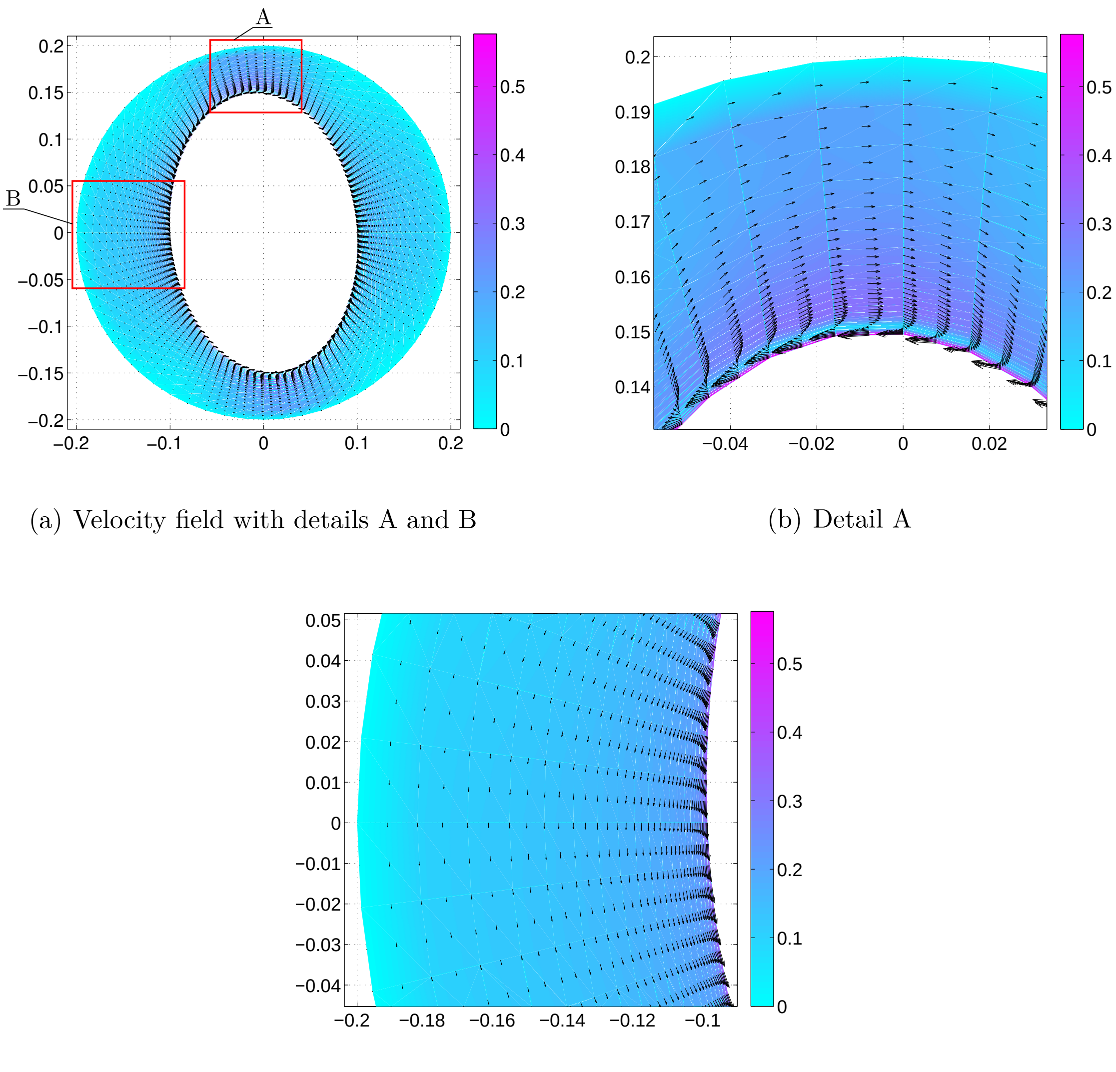


Figure 6: Triangulation at time $t = 0$, $n_c = 60$, $n_r = 20$.

6.2 Unsteady solution

The computation was performed for a time period of 5 seconds with a time step $\Delta t = 0{,}01\,\mathrm{s}$, which means 500 time steps. With $n_c = 60$ and $n_r = 20$ we have 2280 elements, 4680 nodes and 1200 vertices.

We start with a velocity field at some early time, say, 0,04 s. The velocity field is depicted in figure (7). Let us note that the length of arrows does not correspond to the real length of the velocity vectors. These arrows are proportional, however. One thing we may point out here is that the fluid starts to swirl at the narrow part



(a) Velocity field with details A and B

(b) Detail A

(c) Detail B

Figure 7: Velocity field at time $t = 0{,}04\,\mathrm{s}$.

of the domain while at the wider part it moves in the direction of rotation, see fig. 7(b) and 7(c). As we will see this swirling will pass away as time goes ahead.

Next we examine the pressure at time $t = 0{,}04\,\mathrm{s}$. Its filled contour plot is in figure (8). We see from this figure that the pressure is symmetric. With regard to the geometry of the problem it is something we could expect and hence we may convince ourselves it is right. Pressure is uniquely determined except for a constant.

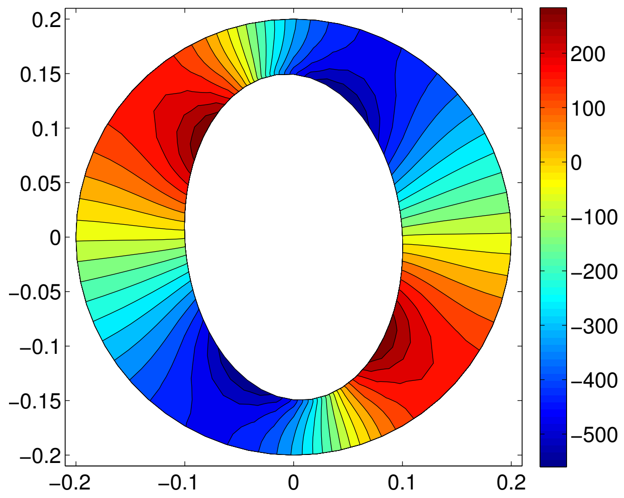


Figure 8: Filled contour plot of the pressure measured in Pa. Time $t = 0{,}04\,\mathrm{s}$.

The last thing we present at this time step is the magnitude of the velocity, see fig. (9). We see very steep decline near the boundary of the ellipse.

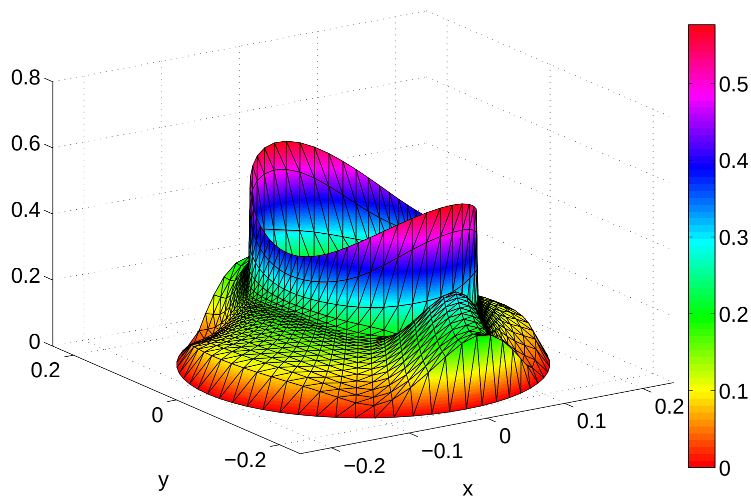
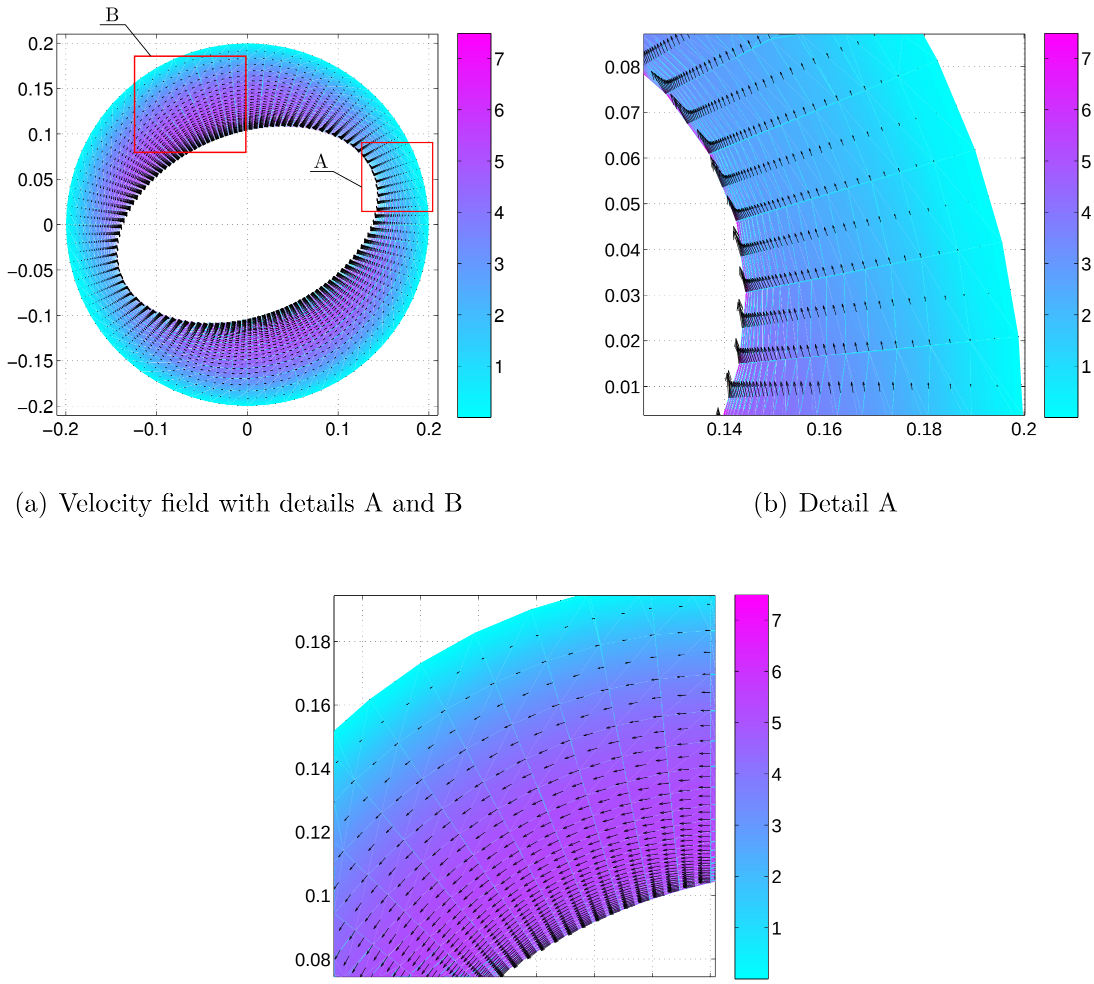


Figure 9: Magnitude of the velocity in m/s. Time $t = 0{,}04\,\mathrm{s}$.

Now we check the results at time $t = 5\,\mathrm{s}$. Again, we start with a velocity field, see fig. (10). Compared to the previous case we observe that the fluid now moves in direction of rotation at the narrow part of the domain and there is no swirling, see fig. 10(b).



(a) Velocity field with details A and B

(b) Detail A

(c) Detail B

Figure 10: Velocity field at time $t = 5\,\mathrm{s}$.

As for the pressure, it still remains symmetric. This may be observed in figure (11).

Finally, we show the magnitude of the velocity, see fig. (12). Note that the steep decline now remains only at the narrow part of the domain. The decrease of the magnitude of the velocity is now smooth elsewhere. It is expected to be smooth also at this narrow part as time goes on.

We have shown how the velocity and pressure evolve with time. The last thing to do is to compute the resulting force and moment acting on the ellipse. Using the method described above we arrived at the resulting force equal identically to

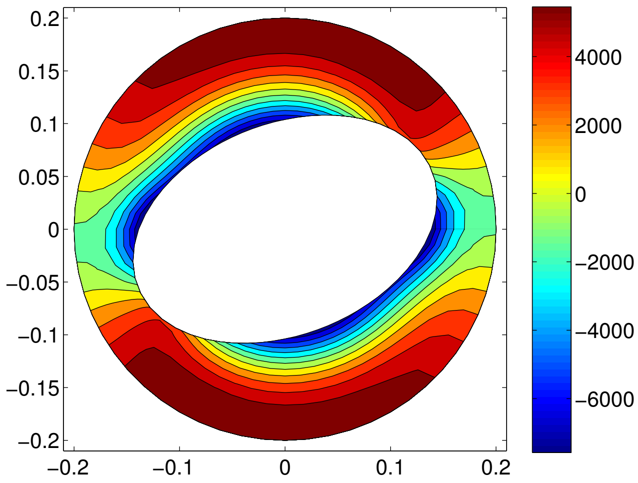


Figure 11: Filled contour plot of the pressure measured in Pa. Time $t = 5\,\mathrm{s}$.

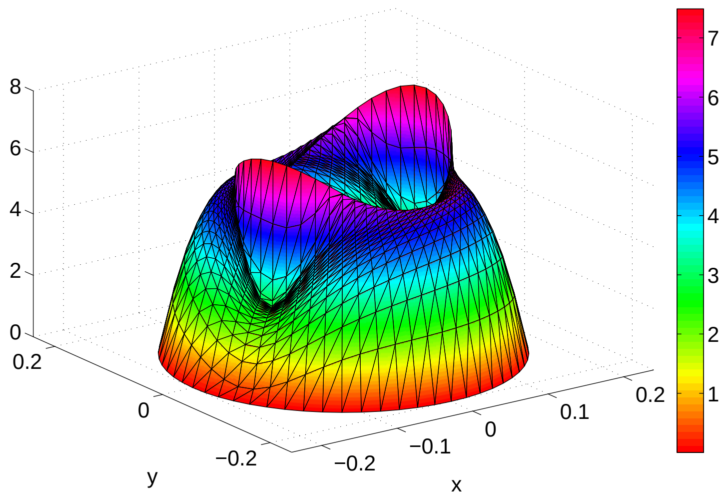


Figure 12: Magnitude of the velocity in m/s. Time $t = 5\,\mathrm{s}$.

zero. This is the consequence of the symmetry of the geometry and pressure. The evolution of the moment for $t \in (0, 5)$ is in figure (13).

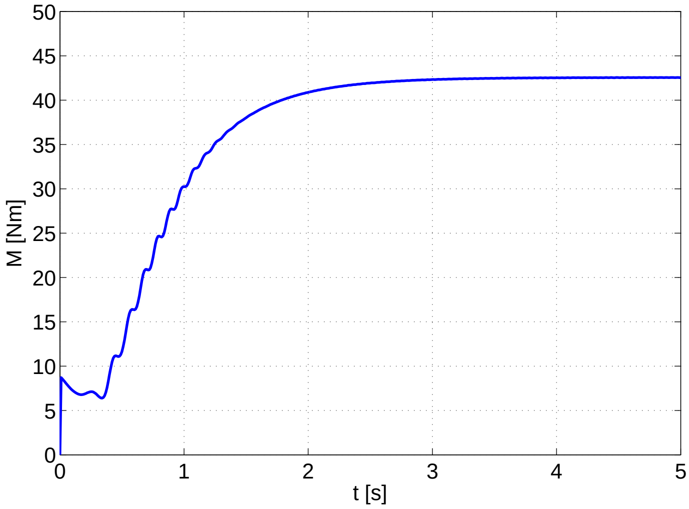


Figure 13: Evolution of the moment, $t \in (0, 5)$.

7 Appendix

We shall denote the Lebesgue measurable functions in Ω by $\mathfrak{M}(\Omega)$. Let $u : \Omega \to \mathbb{R}$ be a function. We define the functional $\|\cdot\| \equiv \|\cdot\|_\Omega$ as

$$\|u\| = \left[\int_\Omega u^2(\mathbf{x})\,\mathrm{d}\mathbf{x}\right]^{\frac{1}{2}} ,$$

where the integral is meant in the Lebesgue sense. Next, we introduce a subset of the set of measurable functions $\mathfrak{M}(\Omega)$ as follows,

$$\mathcal{L}^2(\Omega) = \{u \in \mathfrak{M}(\Omega);\ \|u\| < \infty\} .$$

The set $\mathcal{L}^2(\Omega)$ forms a linear space. However, the functional $\|\cdot\|_2$ does not satisfy the third axiom of norm, for it gives the same value for the functions that are distinct on a set of measure zero. Therefore, we identify such functions in the space $\mathcal{L}^2(\Omega)$ using the equality almost everywhere. We then obtain the Lebesgue space

$$L^2(\Omega) = \mathcal{L}^2(\Omega)\Big|_{a.e.} .$$

The elements of $L^2(\Omega)$ are the classes of functions that are distinct at most on a set of measure zero. Thus, $L^2(\Omega)$ together with the norm $\|\cdot\|_2$ forms the normed linear space. It is possible to define the scalar product in this space by

$$(u,v) = \int_\Omega uv\,\mathrm{d}x .$$

We are now ready to define the Sobolev space $H^1(\Omega)$, where we shall seek the solution of our problem,

$$H^1(\Omega) = \left\{u \in L^2(\Omega);\ \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2} \in L^2(\Omega)\right\} ,$$

where the derivatives are understood in the sense of distributions. One defines the scalar product in this space by

$$(u,v)_{1,\Omega} \equiv (u,v)_1 = \int_\Omega \left[uv + \frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}\right] \mathrm{d}\mathbf{x} .$$

This scalar product defines the norm

$$\|u\|_{1,\Omega} \equiv \|u\|_1 = \sqrt{(u,u)_1} .$$

The norm $\|\cdot\|_1$ may be easilly generalized for the vector function $\mathbf{u}$ from the space $\left[H^1(\Omega)\right]^2$ by

$$\|\mathbf{u}\|_1 = \|u_1\|_1 + \|u_2\|_1 ,$$

and in the same fashion we generalize the scalar product of two functions $\mathbf{u}$, $\mathbf{v} \in \left[H^1(\Omega)\right]^2$,

$$(\mathbf{u}, \mathbf{v})_1 = (u_1, v_1)_1 + (u_2, v_2)_1 \,.$$

It may be shown that the space $H^1(\Omega)$ and also $\left[H^1(\Omega)\right]^2$ is a separable and reflexive Banach space. Moreover, together with the scalar products $(u, v)_1$ and $(\mathbf{u}, \mathbf{v})_1$, respectively, they form the Hilbert spaces with scalar products.

For the reasons of the weak formulation we define the spaces

$$V = \left\{u \in H^1(\Omega);\, u = 0 \text{ on } \Gamma_1 \text{ in the sense of traces}\right\} ,$$

$$V_g = \left\{u \in H^1(\Omega);\, u = g \text{ on } \Gamma_1 \text{ in the sense of traces}\right\} .$$

8 Conclusion

This work is mainly focused on the solution of two-dimensional incompressible viscous flow by the finite element method. Such a problem may be addressed in various ways and finds applications in many engineering problems.

We have derived the comprehensive and directly applicable algorithm for the solution of two-dimensional Navier-Stokes equations. Further, we have seen how it can be readily modified for the case of moving computational domain using the ALE formulation. A stabilization of the finite element method was necessary to achieve convergence of the Newton method. The algorithm was tested on the rotating ellipse problem (cf. section 5) and its results were presented in section 6. We may conclude these results are satisfactory.

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