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DISERTAČNÍ PRÁCE

Okrajové problémy s časovou singularitou

Katedra matematické analýzy a aplikací matematiky
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FACULTY OF SCIENCE

DOCTORAL THESIS

Boundary value problems with a time singularity

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Abstrakt: Tato disertační práce se zabývá problematikou obyčejných diferenciálních rovnic druhého řádu s možnou časovou singularitou v počátku, studovaných obecně na neomezeném intervalu $[0, \infty)$. Tyto vyšetřované rovnice jsou zobecněním singulárních diferenciálních rovnic, jež se vyskytují v mnoha oblastech vědy, obzvláště pak v hydrodynamice. V práci jsou vyšetřovány dva typy zobecnění těchto modelových rovnic, a to rovnice bez ϕ-Laplaciánu a s ϕ-Laplaciánem. Dané rovnice jsou vyšetřovány spolu s okrajovými podmínkami v 0 a ∞. Tyto podmínky, jakož i podmínky na datové funkce úlohy, jsou voleny s ohledem na původní hydrodynamický model a na specifický typ jeho hledaného řešení – tzv. bublinové řešení. Studium okrajové úlohy je převedeno na vyšetřování počátečních úloh. Práce se zabývá zejména otázkou existence a jednoznačnosti řešení těchto počátečních úloh a jejich asymptotickými vlastnostmi. Podstatná část práce je pak věnována vyšetřováním specifických typů řešení v závislosti na jejich supremu – tlumená, homoklinická a úniková řešení. Studuje se existence těchto jednotlivých typů řešení a jejich asymptotické vlastnosti. U rovnic bez ϕ-Laplaciánu je značná pozornost věnována tlumeným řešením a podmínkám zaručujícím jejich oscilatoričnost. U rovnic s ϕ-Laplaciánem jsou pak studována zejména úniková řešení a kritéria zaručující jejich neohraničenost.

Klíčová slova: obyčejné diferenciální rovnice druhého řádu, časová singularita, ϕ-Laplacián, asymptotické vlastnosti, existence a jednoznačnost řešení, tlumené řešení, homoklinické řešení, únikové řešení, neohraničené řešení, oscilatorické řešení, neomezený interval

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Abstract: This dissertation deals with the second order ordinary differential equations with possible time singularity at the origin, which are studied in general on the unbounded interval $[0, \infty)$. These investigated equations are the generalization of the singular differential equations, which are found in many sciencies, especially in hydrodynamics. This study investigates two types of generalizations of these model equations – equations without ϕ-Laplacian and with ϕ-Laplacian – together with the boundary conditions at 0 and ∞. These conditions as well as conditions for the data functions of our problem are chosen with respect to the original hydrodynamic model and with respect to a specific type of searched solution – so-called bubble-type solution. The study of boundary value problem is transformed into investigation of initial value problems. The thesis investigates especially the existence and uniqueness of solutions of these initial value problems and their asymptotic properties. The essential part of the thesis is dedicated to the study of specific types of solutions depending on their supremum – damped, homoclinic and escape solutions. We study the existence of these individual types of solutions and their asymptotic properties. In the case of equations without ϕ-Laplacian, considerable attention is devoted to the damped solutions and conditions guaranteeing their oscillatory behaviour. In the case of equations with ϕ-Laplacian, we study especially the escape solutions and criteria for their unboundedness.

Key words: second order ordinary differential equations, time singularity, ϕ-Laplacian, asymptotic properties, existence and uniqueness of a solution, damped solution, homoclinic solution, escape solution, unbounded solution, oscillatory solution, unbounded interval

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Prohlášení

Prohlašuji, že jsem disertační práci zpracoval samostatně pod vedením paní prof. RNDr. Ireny Rachůnkové, DrSc. a všechny použité zdroje jsem uvedl v seznamu literatury.

V Olomouci dne ....

podpis

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Poděkování

Rád bych na tomto místě poděkoval především své školitelce prof. Rachůnkové za její cenné rady, obětavou spolupráci i drahocenný čas, jenž mi věnovala během celého mého doktorského studia. Dále bych zde rád poděkoval své rodině za její podporu a trpělivost.

Notation

$\mathbb{N}$	set of all natural numbers
$\mathbb{R}$	set of all real numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
$C[a,b]$	Banach space of all continuous functions on $[a,b]$ equipped with the maximum norm $\|g\|_{C[a,b]} = \max\{\|g(t)\|\colon t \in [a,b]\}$
$C^k[a,b]$	Banach space of all functions k-times continuously differentiable on $[a,b]$ equipped with the norm $\|g\|_{C^k[a,b]} = \sum_{j=1}^{k} \|g^{(j)}\|_{C[a,b]}$
$\mathrm{Lip}(I)$	set of all Lipschitz continuous functions on the interval I
$\mathrm{Lip}_{\mathrm{loc}}(I)$	set of all locally Lipschitz continuous functions on the interval I

1 Introduction

In this thesis, we investigate the second order nonlinear ordinary differential equations (ODEs) without ϕ-Laplacian (Chapters 2–4)

$$(p(t)u'(t))' + q(t)f(u(t)) = 0 \tag{1.1}$$

and with ϕ-Laplacian (Chapters 5–7)

$$(p(t)\phi(u'(t)))' + p(t)f(\phi(u(t))) = 0. \tag{1.2}$$

The basic assumptions on functions p, q, f and ϕ are mentioned in Sections 2.1, 5.1 and 7.1. Both equations (1.1) and (1.2) are studied with the initial conditions

$$u(0) = u_0, \qquad u'(0) = 0. \tag{1.3}$$

These initial value problems (IVPs) are investigated generally on the positive half-line $[0, \infty)$.

Equations (1.1) and (1.2) can have a time singularity at the origin in the following sense. Let us consider the system of ODEs

$$x'(t) = f(t, x), \quad t \in I, \tag{1.4}$$

where $f\colon I \times \mathbb{R}^n \to \mathbb{R}^n$, $x \in \mathbb{R}^n$, $I \subset \mathbb{R}$. If the function f fulfils the Carathéodory conditions, then the system (1.4) is called *regular*, otherwise it is called *singular*. By the *time singularity at 0* we understand that

$$\int_0^{\varepsilon} |f(t, x)| \, \mathrm{d}t = \infty$$

for some $x \in \mathbb{R}$ and for each sufficiently small $\varepsilon > 0$. If we put $v = pu'$, then equation (1.1) can be expressed as a special case of system (1.4)

$$u'(t) = \frac{1}{p(t)} v(t), \qquad v'(t) = -q(t)f(u(t)).$$

Similarly, for $v = p\phi(u)$, we can assume equation (1.2) as the system

$$u'(t) = \frac{1}{p(t)} v(t), \qquad v'(t) = -p(t)f(\phi(u(t))).$$

One of our basic assumptions on the function p is that $p(0) = 0$. Hence, the integral $\int_0^1 \frac{1}{p(s)} \, \mathrm{d}s$ can be divergent, which yields the time singularity at 0. Consequently, our investigated equations (1.1) and (1.2) can have the time singularity. This contrasts with the papers that study more general equations in the regular setting, mentioned in Section 1.1. In addition, the nonlinearity f in our case does not satisfy the sign condition $xf(x) > 0$ for all $x \neq 0$. Therefore, the globally monotonous behaviour of f, which is often required in the literature, is not fulfilled here.

1.1 Recent state summary

Regular equations

A considerable amount of literature exists on the qualitative analysis of equations (1.1), (1.2) and their generalizations in the regular setting, where $p(t) > 0$ for $t \in [0, \infty)$. The monograph [31] provides a general overview of asymptotic properties of solutions of nonautonomous ODEs. Research in the last decades has focused significantly on asymptotic analysis of the second order Emden–Fowler equation

$$u''(t) + q(t)|u(t)|^{\gamma} \operatorname{sgn} u(t) = 0, \qquad \gamma > 0,\ \gamma \neq 1,$$

which is a special case of equations (1.1) and (1.2). For the historic overview, see [71]. The oscilation and nonoscilation of the second order Emden–Fowler equation is researched in [36, 41, 42, 55]. The Emden-Fowler equation of arbitrary order is analysed in [71]. Further extensions of these results have been reached for more general equations, as can be seen in, e.g. [9, 17, 18, 26, 35, 43, 72, 73]. Nonlinearities in equations in the cited papers have similar globally monotonous behaviour, characterized by the sign condition $xf(x) > 0$ for $x \in \mathbb{R} \setminus \{0\}$. We would like to emphasize that, in contrast to these papers, the nonlinearity f in our equations (1.1) and (1.2) does not have globally monotonous behaviour.

The second order Emden–Fowler equation can be generalized into the following equation with p-Laplacian

$$(p(t)\Phi_{\alpha}(u'(t)))' + q(t)\Phi_{\gamma}(u(t)) = 0, \qquad \alpha > 0,\ \gamma > 0,$$

where $\Phi_{\alpha}(u) := |u|^{\alpha} \operatorname{sgn} u$. This equation is called sub-half-linear, half-linear or super-half-linear if $\alpha > \gamma$, $\alpha = \gamma$ or $\alpha < \gamma$, respectively. The existence results of the sub-half-linear case are mentioned in [28, 37, 39], those of the half-linear case in [15, 29, 38] and those of the super-half-linear case in [16, 48].

Another approach to the asymptotic analysis is provided by the theory of regular variations [11, 47]. The asymptotic results for the related equations or systems with regularly varying functions are mentioned in [22, 27, 40, 49, 50, 67, 68]. Criteria for oscillation and nonoscillation of related two-dimensional linear and nonlinear systems can be found in [21, 46, 52].

Singular equations

The journal articles [56, 57, 60, 61, 62, 63, 64, 65, 66] are the most significant for this thesis. They contain a detailed study of the singular nonlinear equation

$$(p(t)u'(t))' + p(t)f(u(t)) = 0. \tag{1.5}$$

Equation (1.5) is a special case of equation (1.1), where $p = q$ and also a special case of equation (1.2), where $\phi(x) = x$. All types of possible solutions of IVP (1.5), (1.3) with proofs of their existence and assymptotic properties are described in [60, 61, 64]. The existence of escape and homoclinic solutions is discussed in [62, 63].

The damped oscillatory solutions of problem (1.5), (1.3) are studied in [56, 57, 66], where the conditions for their existence, convergence to zero and for another asymptotic properties are given. For the results about damped nonoscillatory solutions, we refer to [69]. The asymptotic formulas and conditions that guarantee the existence of Kneser solutions are derived there. The variational methods for $p(t) = t^k$, $k \in \mathbb{N}$ or $k \in (1, \infty)$ are used in [10] or [12], respectively, where problem (1.5), (1.3) is transformed into a problem to find positive solutions on the half-line.

Many other problems for singular equations are described in [7, 8, 53, 58, 59] and [54], where the existence theory of two-point boundary value problems on finite and semi-infinite interval is introduced. For other close existence results, see also Chapters 13 and 14 in [53], where the existence results for second order ODEs on finite, semi-finite and infinite intervals are shown. Works [58, 59] are focused on regularization and sequential techniques and contain the existence theory for a variety of singular boundary value problems, especially those with ϕ-Laplacian.

1.2 Thesis objectives

The solutions for our IVP (1.1), (1.3) without ϕ-Laplacian as well as for problem (1.2), (1.3) with ϕ-Laplacian are divided according to their supremum into damped, homoclinic and escape solutions. Chapters 2 and 5 guarantee solvability and uniqueness of our IVPs and consider all the previously mentioned types of solutions indiscriminately. On the contrary, other chapters are focused on either damped, homoclinic or escape solutions.

Equations without ϕ-Laplacian

The following objectives are concerned with the IVP (1.1), (1.3) without ϕ-Laplacian.

- The first aim of the thesis is to prove the existence and uniqueness of the damped solutions of problem (1.1), (1.3).
- Further, our effort is focussed on finding the conditions under which each damped solution is oscillatory.
- Our next goal is to prove the existence and uniqueness of escape solutions of the above-mentioned problem. Here we use the existence results of the oscillatory solutions.
- The principal objective concerning the IVP without ϕ-Laplacian is to prove the existence of homoclinic solution, which is important in applications described in Section 1.5.

Equations with ϕ-Laplacian

- Our aim is to generalize our results for damped and escape solutions of problem (1.1), (1.3) without ϕ-Laplacian to problem (1.2), (1.3) with ϕ-Laplacian.

- Moreover, we want to find conditions which guarantee that each escape solution of problem (1.2), (1.3) is unbounded and thus prove the existence of unbounded solutions.

Finally, we intend to illustrate all these main results on various examples.

Open problems and other aims of research

- The thesis contains the existence result for a homoclinic solution of problem (1.1), (1.3) without ϕ-Laplacian. The existence of homoclinic solutions for problem (1.2), (1.3) with ϕ-Laplacian stays as an open problem.

- The next open problem is finding conditions leading to the existence of the unique homoclinic solution of problem (1.1), (1.3) and problem (1.2), (1.3).

- Another interesting problem is to investigate the set of all solutions of equation (1.1) and (1.2) depending on initial values. We know that – for both of these equations – initial values in $[\bar{B}, L)$ give only damped solutions (see Theorem 3.1, Remark 3.2, Theorem 6.1). However, a structure of solutions for initial values in $(L_0, \bar{B})$ is still an open problem.

1.3 Theoretical framework and methods applied

The thesis is motivated by the research of second order singular equations initiated by I. Rachůnková, J. Tomeček et al. in [56, 57, 60, 61, 62, 63, 64, 65, 66]. These papers investigate equation (1.5) and they are based on the following basic assumptions. The function f is (locally) Lipschitz continuous on the domain, where the solution is searched for. Further, f satisfies a certain sign condition, f has either two zeros 0, $L > 0$ [56, 57, 60, 61, 65, 66] or three zeros 0, $L_0 < 0$, $L > 0$ [62, 63, 64]. The function p is continuous on $[0, \infty)$, continously differentiable and increasing on $(0, \infty)$, $p(0) = 0$ and $\lim_{t\to\infty} \frac{p'(t)}{p(t)} = 0$. For more information about contents of above cited papers, see Section 1.1.

Our effort is to generalize current results about existence and properties of three types of solutions of equation (1.5) to the more general equations (1.1) and (1.2). In this thesis, f has three zeros 0, $L_0 < 0$, $L > 0$ and the basic assumptions are mentioned in Section 2.1 for problem without ϕ-Laplacian and in Sections 5.1 and 7.1 for problem with ϕ-Laplacian.

Our results are based on the methods of differential equations and functional analysis. The fixed point theory plays an important role in the proofs of existence of solutions of our IVPs. We reduce an IVP to an operator equation and search for a fixed point of a corresponding operator. For the existence of solutions of auxiliary IVPs with and without ϕ-Laplacian, we use the Schauder fixed point theorem. Here it is necessary to prove the compactness of the operator. To prove this, we use the Arzelà–Ascoli theorem. The uniqueness of a solution is proved with the help of the Gronwall lemma. The existence and uniqueness of a solution can be proved also by the Banach fixed point theorem, which we show for the original IVP without ϕ-Laplacian with a bounded nonlinearity and some additional conditions.

Using the method of a priori estimates, we obtain estimates of solutions whose existence is not guaranteed, which is useful to prove the general existence principles. In the study of unbounded solutions of the IVP with ϕ-Laplacian, the difficulties arise in the case where the uniqueness of solution is not guaranteed. The lower and upper functions method for auxiliary mixed problem helps us to solve these difficulties in connection with the proof of existence of specific type of the solution of the IVP. The lower and upper functions satisfy the differential inequalities derived from our differential equation and fulfil the inequalities derived from the mixed boundary conditions. Our lower and upper functions are well-ordered, that is the upper function is greater or equal to the lower function and the solution is located between these functions.

1.4 Original results

This thesis contains new results in the theory of singular nonlinear ODEs of second order on the half-line $[0, \infty)$. They are based on the results published in multiple peer-reviewed journals [1, 2, 3, 4]. The author presented these results at several international conferences (see page 150).

Here we summarize the main results of the individual chapters. Chapters 2–4 are devoted to the IVP

$$(p(t)u'(t))' + q(t)f(u(t)) = 0, \qquad u(0) = u_0 \in [L_0, L], \quad u'(0) = 0 \tag{1.6}$$

with different coefficient functions p and q. This problem is established on the following basic assumptions on functions f, p and q.

- The function f has three zeros $L_0 < 0$, 0 and $L > 0$. The function f is continuous on $[L_0, L]$, negative on $(L_0, 0)$ and positive on $(0, L)$.
- The functions p and q are continuous on $[0, \infty)$, positive on $(0, \infty)$ and $p(0) = 0$. Moreover, we assume that

$$\lim_{t \to 0^+} \frac{1}{p(t)} \int_0^t q(s) \, \mathrm{d}s = 0,$$

which is the necessary condition for the existence of a solution of problem (1.6) (see Theorem 2.21).

We also study the auxiliary IVP

$$(p(t)u'(t))' + q(t)\tilde{f}(u(t)) = 0, \qquad u(0) = u_0 \in [L_0, L], \quad u'(0) = 0, \tag{1.7}$$

where

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in [L_0, L], \\ 0 & \text{for } x < L_0, \quad x > L, \end{cases}$$

which is easier due to its bounded nonlinearity $\tilde{f}$. The auxiliary nonlinearity $\tilde{f}$ is chosen in connection with our main classification of solutions into damped, homoclinic and escape solutions defined in Definitions 2.5 and 2.6.

Chapter 2 investigates the existence and uniqueness of solutions of both (1.6) and (1.7). The first main result is the existence of solutions of problem (1.7) under the basic assumptions in Theorem 2.15. If f is Lipschitz continuous on $[L_0, L]$, then the uniqueness of a solution of problem (1.7) and a continuous dependence of solutions on initial values is proved (Theorem 2.17). In all of the following results for the problem without ϕ-Laplacian, we consider the basic assumptions and, moreover, the conditions

$$\exists\, \bar{B} \in (L_0, 0) : \tilde{F}\left(\bar{B}\right) = \tilde{F}(L), \quad \text{where } \tilde{F}(x) = \int_0^x \tilde{f}(z)\,\mathrm{d}z,\ x \in \mathbb{R},$$

$$pq \text{ is nondecreasing on } [0, \infty), \quad f \in \mathrm{Lip_{loc}}\left([L_0, L] \setminus \{0\}\right).$$

Additionally, Theorem 2.15 gives that the solution u of problem (1.7) with $u_0 \in (L_0, L)$ satisfies $u > L_0$ on $[0, \infty)$. Consequently, for damped and homoclinic solutions, the function f coincides with $\tilde{f}$ and the auxiliary problem (1.7) becomes the original problem (1.6). Theorem 2.19 gives the existence and uniqueness of the original problem (1.6) provided that f is locally Lipschitz continuous on $[L_0, \infty)$ and $-C_L \le f(x) \le 0$ for $x \ge L$, for some $C_L \in (0, \infty)$.

Chapter 3 deals with damped solutions of the original problem (1.6). Theorem 3.1 gives an important result: each solution with a starting value in $\left(\bar{B}, L\right)$ is damped. Furthermore, we have guaranteed the existence of these solutions. Theorem 3.5 yields that if (3.2) and (3.4) hold, then every damped nonoscillatory solution $u(t)$ tends to 0 for $t \to \infty$. This asymptotic behaviour is valid also for $u'(t)$ provided that (3.6) holds. Significant attention is devoted to oscillatory solutions. Theorem 3.7 shows that every oscillatory solution is the damped solution and it has nonincreasing amplitudes defined in Definition 3.6. Three types of conditions which guarantee that each damped solution is oscillatory are presented in Theorems 3.11, 3.12 and 3.14. The existence of oscillatory solutions for each starting values in $\left(\bar{B}, 0\right) \cup (0, L)$ is proved under these three different criteria. The first existence result (Theorem 3.15) is proved under conditions (3.2), (3.8), (3.19) and (3.24). Theorem 3.16 yields the existence of oscillatory solutions under assumptions (3.2), (3.8)–(3.11) and (3.19), whereas Theorem 3.17 provides the third existence result reached under conditions (3.3) and (3.29).

The main aim of problem (1.6), which is the existence of a homoclinic solution, is studied in Chapter 4. According to three obtained criteria for the oscillation of solutions in Chapter 3, we get three criteria for the existence of escape and homoclinic solutions. These criteria are given either by conditions (3.2), (3.8), (3.19) and (3.24) or conditions (3.2), (3.8)–(3.11) and (3.19) or condition (3.3). Moreover, here we assume that (2.28) and (4.6)–(4.9) hold. Under these assumptions, we first present the existence of escape and homoclinic solutions of the auxiliary problem (1.7) in Theorems 4.6 and 4.7. The main results for the original problem (1.6) are contained in Theorems 4.8 and 4.9. Theorem 4.8 guarantees the existence of infinitely many escape solutions of problem (1.6) on $[0, c]$ with different starting values, where c can be different for different solutions. Using this, we are able to prove the existence of at least one homoclinic solution of problem (1.6) (Theorem 4.9). Finally, the homoclinic solution leads to the bubble-type solution (Corollary 4.10) with the physical interpretation in hydrodynamics, mentioned in Section 1.5.

Chapters 5–7 are dedicated to the IVP

$$(p(t)\phi(u'(t)))' + p(t)f(\phi(u(t))) = 0, \qquad u(0) = u_0, \quad u'(0) = 0 \tag{1.8}$$

with ϕ-Laplacian. This problem is based on the following basic assumptions on functions ϕ, f and p.

- ϕ is an increasing diffeomorphism with $\phi(\mathbb{R}) = \mathbb{R}$ and $\phi(0) = 0$.
- The function f has three zeros $\phi(L_0) < 0$, 0 and $\phi(L) > 0$. The function f is continuous on $[\phi(L_0), \phi(L)]$, negative on $(\phi(L_0), 0)$ and positive on $(0, \phi(L))$.
- The function p is continuous on $[0, \infty)$, continuously differentiable and increasing on $(0, \infty)$ and $p(0) = 0$.

We define also the auxiliary IVP

$$(p(t)\phi(u'(t)))' + p(t)\tilde{f}(\phi(u(t))) = 0, \qquad u(0) = u_0, \quad u'(0) = 0, \tag{1.9}$$

where

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in [\phi(L_0), \phi(L)], \\ 0 & \text{for } x < \phi(L_0),\ x > \phi(L). \end{cases}$$

Chapter 5 is devoted to the existence and uniqueness of a solution of problem (1.9) and the continuous dependence of solutions on initial values. The existence of solutions of auxiliary problem (1.9) under the basic assumptions is guaranteed by Theorem 5.19. The uniqueness of solution of problem (1.9) is proved in Theorem 5.21 under conditions

$$f \in \mathrm{Lip}\,[\phi(L_0), \phi(L)]\,, \tag{1.10}$$

$$\phi^{-1} \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}). \tag{1.11}$$

That is, both functions ϕ and ϕ^{-1} have to be locally Lipschitz continuous on $\mathbb{R}$, which can be problematic to satisfy. In particular, for model example $\phi(x) = |x|^\alpha \operatorname{sgn} x,\ \alpha > 1$, we have $\phi^{-1}(x) = |x|^{\frac{1}{\alpha}} \operatorname{sgn} x$, which is not locally Lipschitz continuous function on $\mathbb{R}$. Hence, we are forced to obtain crucial results also in the more difficult case, where condition (1.11) is not considered. If we asume that (1.10),

$$\exists\, \bar{B} \in (L_0, 0)\colon \tilde{F}\left(\bar{B}\right) = \tilde{F}(L), \qquad \text{where } \tilde{F}(x) = \int_0^x \tilde{f}(\phi(s))\,\mathrm{d}s, \quad x \in \mathbb{R}, \tag{1.12}$$

$$\limsup_{t\to\infty} \frac{p'(t)}{p(t)} < \infty \tag{1.13}$$

hold, then the continuous dependence of solutions on initial values in $(0, L)$ or $(L_0, 0)$ without condition (1.11) is proved in Theorem 5.24 or 5.26, respectively, under additional assumption (5.57) or (5.66), respectively.

Chapter 6 shows that conditions (1.12) and (1.13) are sufficient for the proof that each solution of the original problem (1.8) with starting value in $\left[\bar{B}, L\right)$ is damped (Theorem 6.1). Moreover, the existence of these solutions is guaranteed.

Final Chapter 7 deals with escape – especially unbounded – solutions. In this chapter, we assume

$$f \in C[\phi(L_0), \infty), \quad f \le 0 \ \text{ on } (\phi(L), \infty), \quad \limsup_{t\to\infty} \frac{p'(t)}{p(t)} = 0$$

and that (1.12) holds. Theorem 7.10 yields the existence of infinitely many escape solutions of the auxiliary problem (1.9) with different starting values in $\left(L_0, \bar{B}\right)$ under assumptions (1.10) and (1.11). If we exclude conditions (1.10) and (1.11), then, by Theorem 7.16, we get the existence of infinitely many escape solutions of problem (1.9) with not necessary different starting values in $\left[L_0, \bar{B}\right)$ with no additional condition. We provide three criteria – specified by (7.57) or (7.59) or (7.63), (7.66), (7.67) – which guarantee that each escape solution of the original problem (1.8) is unbounded (Theorem 7.19 or 7.20 or 7.22, respectively). If we combine any of these criteria with assumptions of Theorem 7.10 or Theorem 7.16, then we get the existence of unbounded solutions of problem (1.8) on $[0, b)$ (see Definition 7.1), where b can be different for different solutions. These results are contained in Theorems 7.23, 7.25, 7.27, 7.29, 7.31 and 7.33.

The thesis presents the original results reached by the author during his PhD studies of the Mathematical Analysis at Palacký University Olomouc in cooperation with:

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Chapters 2–4 are based on the papers [1] and [2]. The results of Chapters 5 and 6 were proved in [3]. Chapter 7 contains the results from the paper [4].

1.5 Motivation

We study the second order nonlinear ODEs arising in hydrodynamics. Reference [25] shows that the study of the behaviour of nonhomogeneous fluids in the Cahn–Hilliard theory can lead to the system of partial differential equations

$$\varrho_t + \operatorname{div}(\varrho\vec{v}) = 0, \qquad \frac{d\vec{v}}{dt} + \nabla(\mu(\varrho) - \gamma\Delta\varrho) = 0. \tag{1.14}$$

Here ϱ denotes the density, $\vec{v}$ is the velocity of the fluid, $\mu(\varrho)$ is its chemical potential, γ is a constant. If we suppose that a motion of the fluid is zero, then system (1.14) is reduced to the equation

$$\gamma\Delta\varrho = \mu(\varrho) - \mu_0, \tag{1.15}$$

where γ and μ_0 are suitable constants. Let us now introduce the polar system of coordinates in $\mathbb{R}^n$ and search for the spherical symmetric solution, which depends only on the radial variable r. Then, for $n = 2, 3$, partial differential equation (1.15) is reduced to the ODE

$$\gamma\left(\varrho'' + \frac{n}{r}\varrho'\right) = \mu(\varrho) - \mu(\varrho_\ell), \quad r \in (0, \infty), \tag{1.16}$$

known as the density profile equation. This equation with the boundary conditions

$$\varrho'(0) = 0, \quad \lim_{r\to\infty} \varrho(r) =: \varrho_\ell > 0 \tag{1.17}$$

describes the formation of microscopic bubbles in a liquid, in particular, vapour inside a fluid. The first condition in (1.17) follows from the spherical symmetry and is also necessary for the smoothness of solution ϱ at point $\mathrm{r} = 0$. The second condition means that the bubbles are surrounded by liquid with density ϱ_ℓ. In the simplest model for nonhomogenous fluid in $\mathbb{R}^3$, the chemical potential μ is considered as a three-degree polynomial with three distinct real roots. Then problem (1.16), (1.17) is reduced to the form

$$(t^2 u')' = \lambda^2 t^2 (u+1) u (u-\xi), \tag{1.18}$$

$$u'(0) = 0, \qquad \lim_{t\to\infty} u(t) = \xi. \tag{1.19}$$

Here $\lambda \in (0, \infty)$ and $\xi \in (0, 1)$ are parameters, which reflect different physical situations. Note that problem (1.18), (1.19) has always the constant solution $\varrho(r) \equiv \xi$, which physically corresponds to the fluid without bubbles (homogenous fluid).

Many important physical properties of the bubbles depend on the strictly increasing solution of problem (1.18), (1.19) with just one zero – so-called bubble-type solution. In particular, the gas density inside the bubble, the bubble radius and the surface tension. For more details, physical connections and numerical investigation of the problem, see [25, 33, 44, 70].

Besides hydrodynamics, equation (1.18) arises in many other areas. For instance, in the study of phase transition of Van der Waals fluids [25], in the relativistic cosmology for description of particles that can be treated as domains in the universe [45], in the homogeneous nucleation theory [5], in population genetics, where it serves as a model for spatial distribution of the genetic composition of a population [23], or in the nonlinear field theory, in particular, when describing bubbles generated by scalar fields of the Higgs type in the Minkowski spaces [20]. For other problems close to problem (1.18), (1.19), we refer to [6, 10, 13, 30, 31, 32, 51].

In the thesis, we investigate equations generalizing the density profile equation (1.18). Concretely, equations without ϕ-Laplacian

$$(p(t)u'(t))' + q(t) f(u(t)) = 0$$

and with ϕ-Laplacian

$$(p(t)\phi(u'(t)))' + p(t) f(\phi(u(t))) = 0,$$

both generally on the unbounded domain $[0, \infty)$. Especially for $L_0 < 0 < L$, we study these equations with the initial conditions

$$u(0) = u_0, \quad u'(0) = 0, \quad u_0 \in [L_0, L].$$

For these initial value problems, we derive the existence and various properties of different types of solutions and, as consequece, we obtain the existence of the bubble-type solution satisfying the boundary conditions

$$u'(0) = 0, \qquad \lim_{t\to\infty} u(t) = L$$

(see Corollary 4.10).

2 Solvability of the problem without ϕ-Laplacian

2.1 Statement of the problem

We study the equation

$$(p(t)u'(t))' + q(t)f(u(t)) = 0 \tag{2.1}$$

with the initial conditions

$$u(0) = u_0, \quad u'(0) = 0, \quad u_0 \in [L_0, L] \tag{2.2}$$

and assume the following basic assumptions:

$$L_0 < 0 < L, \quad f(L_0) = f(0) = f(L) = 0, \tag{2.3}$$
$$f \in C[L_0, L], \quad xf(x) > 0 \text{ for } x \in (L_0, L) \setminus \{0\}, \tag{2.4}$$
$$p \in C[0, \infty), \quad p(0) = 0, \quad p(t) > 0 \text{ for } t \in (0, \infty), \tag{2.5}$$
$$q \in C[0, \infty), \quad q(t) > 0 \text{ for } t \in (0, \infty). \tag{2.6}$$

A model example of (2.1), (2.2) is the following.

Example 2.1. Consider

$$p(t) = t^{\alpha}, \quad q(t) = t^{\beta}, \qquad t \in [0, \infty)\,, \ \alpha > 0, \ \beta \geq 0.$$

The functions p and q are continuous on $[0, \infty)$, positive on $(0, \infty)$ and $p(0) = 0$. Thus, (2.5) and (2.6) are fulfilled. Let us take

$$f(x) = x(x - L_0)(L - x), \quad x \in \mathbb{R}, \ L_0 < 0 < L.$$

Then the function f is continuous on $\mathbb{R}$, $f(L_0) = f(0) = f(L) = 0$, $xf(x) > 0$ for $x \in (L_0, L) \setminus \{0\}$. Hence, (2.3) and (2.4) are satisfied.

Equation (2.1) can have various types of solutions which are defined as follows.

Definition 2.2. Let $c \in (0, \infty)$. A function $u \in C^1[0, c]$ with $pu' \in C^1[0, c]$ which satisfies equation (2.1) for every $t \in [0, c]$ is called a *solution of equation* (2.1) *on* $[0, c]$. If u is solution of equation (2.1) on $[0, c]$ for every $c > 0$, then u is called a *solution of equation* (2.1).

Definition 2.3. Let $c \in (0, \infty)$. A solution u of equation (2.1) on $[0, c]$ which satisfies the initial conditions (2.2) is called a *solution of problem* (2.1), (2.2) *on* $[0, c]$. If u is solution of problem (2.1), (2.2) on $[0, c]$ for every $c > 0$, then u is called a *solution of problem* (2.1), (2.2).

Definition 2.4. A solution u of problem (2.1), (2.2) is said to be *oscillatory* if $u \not\equiv 0$ in any neighborhood of ∞ and if u has a sequence of zeros tending to ∞. Otherwise, u is called *nonoscillatory*.

Definition 2.5. Let u be a solution of problem (2.1), (2.2) with $u_0 \in (L_0, L)$. Denote

$$u_{\text{sup}} := \sup\{u(t) \colon t \in [0, \infty)\}.$$

If $u_{\text{sup}} < L$, then u is called a *damped solution* of problem (2.1), (2.2).
If $u_{\text{sup}} = L$, then u is called a *homoclinic solution* of problem (2.1), (2.2).

Definition 2.6. Assume that u is a solution of problem (2.1), (2.2) on $[0, c]$, where $c \in (0, \infty)$ and $u_0 \in (L_0, L)$. If u satisfies

$$u(c) = L, \quad u'(c) > 0,$$

then u is called an *escape solution* of problem (2.1), (2.2) *on* $[0, c]$.

Let us illustrate different types of solutions of problem (2.1), (2.2) with respect to their asymptotic behaviour in relation to Definitions 2.5 and 2.6 in Figure 2.1.

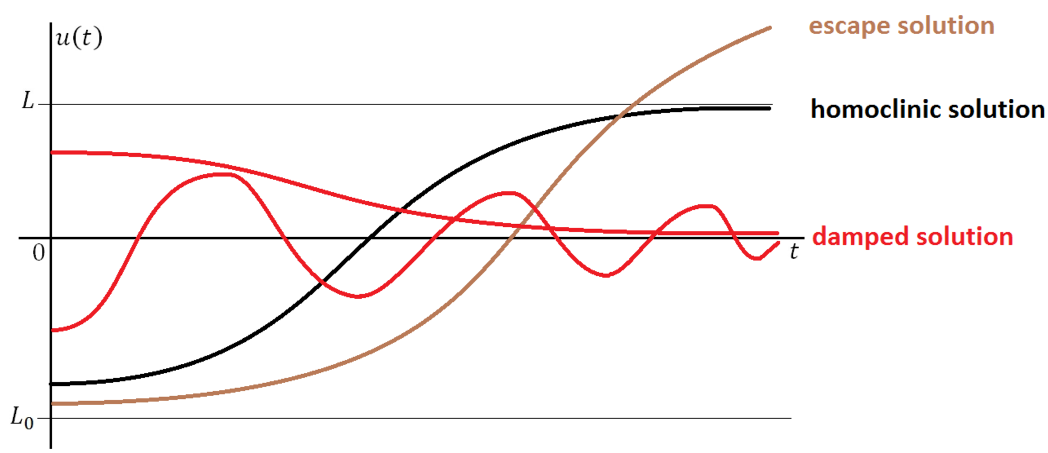

Figure 2.1: Types of solutions

One of the main goals of the thesis is to find additional conditions for functions f, p and q which guarantee that problem (2.1), (2.2) has all three types of solutions from Definitions 2.5 and 2.6, that is damped, homoclinic and escape solutions. To this aim, properties of these solutions are studied in more detail. In particular, in Chapter 4, we prove that a homoclinic solution of problem (2.1), (2.2) satisfies the boundary conditions

$$u'(0) = 0, \qquad \lim_{t\to\infty} u(t) = L$$

motivated in Section 1.5.

Definition 2.7. Let u be a solution of problem (2.1), (2.2) with $u(0) \in (L_0, 0)$ such that $u'(t) > 0$ for $t \in (0, \infty)$. If u satisfies in addition the boundary condition

$$\lim_{t\to\infty} u(t) = L, \tag{2.7}$$

then u is called a *bubble-type solution* of problem (2.1), (2.2).

Therefore, bubble-type solutions of problem (2.1), (2.2) are homoclinic solutions of (2.1), (2.2).

Note that, according to $p(0) = 0$, the integral $\int_0^1 \frac{\mathrm{d}s}{p(s)}$ may be divergent, which means that equation (2.1) can have a singularity at $t = 0$.

In order to derive the existence of all three types (damped, homoclinic, escape) of solutions of problem (2.1), (2.2), we introduce the auxiliary equation

$$(p(t)u'(t))' + q(t)\tilde{f}(u(t)) = 0, \tag{2.8}$$

where

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in [L_0, L], \\ 0 & \text{for } x < L_0, \quad x > L. \end{cases} \tag{2.9}$$

Remark 2.8. By (2.3), equations (2.1) and (2.8) have the constant solutions $u(t) \equiv L$, $u(t) \equiv 0$ and $u(t) \equiv L_0$.

2.2 Properties of solutions

Before we state the existence and uniqueness results, we provide auxiliary lemmas.

Lemma 2.9. *Assume that (2.3)–(2.6) hold and let u be a solution of equation (2.8). Assume that there exists $b \ge 0$ such that $u(b) \in (L_0, 0)$ and $u'(b) = 0$. Then $u'(t) > 0$ for $t \in (b, \theta]$, where θ is the first zero of u on (b, ∞). If such θ does not exist, then $u'(t) > 0$ for $t \in (b, \infty)$.*

Proof. Let $b \ge 0$ be such that $u(b) \in (L_0, 0)$ and $u'(b) = 0$. First, assume that there exists $\theta > b$ satisfying $u(t) < 0$ on (b, θ) and $u(\theta) = 0$. Then, according to (2.4), (2.6) and (2.9), $q(t)\tilde{f}(u(t)) < 0$ and hence,

$$(pu')'(t) > 0, \quad t \in (b, \theta).$$

Since $(pu')(b) = 0$ and since pu' is increasing on (b, θ), we get $pu' > 0$ on (b, θ) and, by (2.5), $u' > 0$ on (b, θ). Furthermore, by integrating (2.8) over $[b, \theta]$, we obtain

$$(pu')(\theta) = -\int_b^\theta q(s)\tilde{f}(u(s))\,\mathrm{d}s > 0.$$

Thus, $pu' > 0$ on $(b, \theta]$ and, due to (2.5), $u'(t) > 0$ for $t \in (b, \theta]$. If u is negative on $[b, \infty)$, we get as before $pu' > 0$ on (b, ∞) and $u'(t) > 0$ for $t \in (b, \infty)$. □

The next dual lemma can be proved analogously.

Lemma 2.10. *Let* (2.3)–(2.6) *hold and let* u *be a solution of equation* (2.8). *Assume that there exists* $a \geq 0$ *such that* $u(a) \in (0, L)$ *and* $u'(a) = 0$. *Then* $u'(t) < 0$ *for* $t \in (a, \theta]$, *where* θ *is the first zero of* u *on* (a, ∞). *If such* θ *does not exist, then* $u'(t) < 0$ *for* $t \in (a, \infty)$.

In order to obtain further important properties of solutions, we assume that

$$\exists\, \bar{B} \in (L_0, 0) : \tilde{F}\left(\bar{B}\right) = \tilde{F}(L), \quad \text{where } \tilde{F}(x) := \int_0^x \tilde{f}(z)\,\mathrm{d}z, \quad x \in \mathbb{R}. \tag{2.10}$$

By (2.4), we have $\tilde{F} \in C^1(\mathbb{R})$, $\tilde{F}(0) = 0$, $\tilde{F}$ is positive and increasing on $[0, L]$ and positive and decreasing on $[L_0, 0]$. The geometric significance of condition (2.10) is illustrated in Figure 2.2, where the both filled areas are equal.

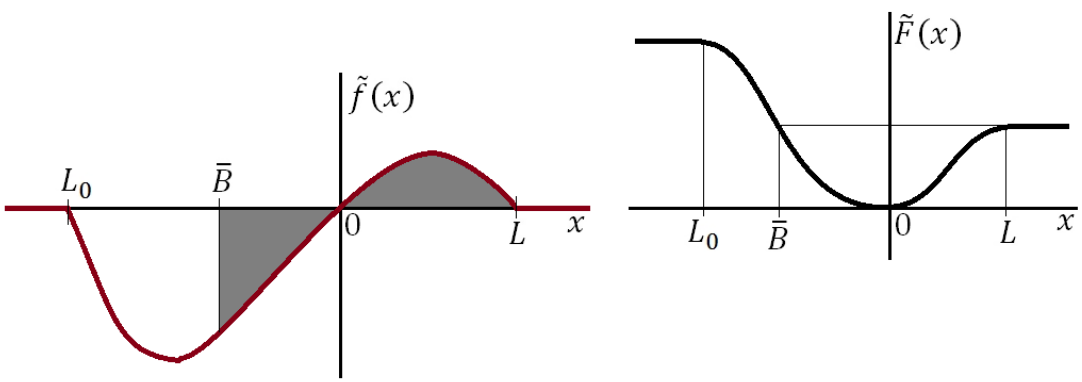

Figure 2.2: Illustration of condition (2.10)

Lemma 2.11. *Assume that conditions* (2.3)–(2.6), (2.10) *and*

$$pq \text{ is nondecreasing on } [0, \infty) \tag{2.11}$$

hold. Let u *be a solution of equation* (2.8) *such that there exist* $b \geq 0$, $\theta > b$ *satisfying*

$$u(b) \in \left(\bar{B}, 0\right), \quad u'(b) = 0, \quad u(\theta) = 0, \quad u(t) < 0 \text{ for } t \in [b, \theta).$$

Then u *fulfils either*

$$u'(t) > 0 \text{ for } t \in (b, \infty), \quad \lim_{t \to \infty} u(t) \in (0, L) \tag{2.12}$$

or

$$\exists\, c \in (\theta,\infty)\colon u(c) \in (0,L)\,, \quad u'(c)=0, \quad u'(t)>0 \text{ for } t\in(b,c)\,. \tag{2.13}$$

Furthermore, if

$$pq \text{ is increasing on } [0,\infty), \tag{2.14}$$

then the assertion holds also for $u(b)=\bar{B}$, $u'(b)=0$.

Proof. Due to Lemma 2.9, $u'(t)>0$ for $t\in(b,\theta]$. Assume that there exists $c>\theta$ such that $u'(c)=0$ and $u'(t)>0$ for $t\in(b,c)$. Let $u(c)\ge L$. Then there exists $b_1\in(\theta,c]$ such that $u(b_1)=L$, $u'>0$ on (b,b_1). By multiplying equation (2.8) by pu', integrating over $[b,b_1]$, we get

$$\begin{aligned}\int_b^{b_1} (p(t)u'(t))'\,p(t)u'(t)\,\mathrm{d}t = &-\int_b^{\theta} (pq)(t)\tilde{f}(u(t))u'(t)\,\mathrm{d}t\\ &-\int_\theta^{b_1} (pq)(t)\tilde{f}(u(t))u'(t)\,\mathrm{d}t.\end{aligned}$$

By (2.11), we obtain

$$\begin{aligned}0 \le \frac{(p(b_1)u'(b_1))^2}{2} &\le -(pq)(\theta)\int_b^{\theta}\tilde{f}(u(t))u'(t)\,\mathrm{d}t - (pq)(\theta)\int_\theta^{b_1}\tilde{f}(u(t))u'(t)\,\mathrm{d}t\\ &= (pq)(\theta)\left(\tilde{F}(u(b))-\tilde{F}(u(b_1))\right) = (pq)(\theta)\left(\tilde{F}(u(b))-\tilde{F}(L)\right).\end{aligned}$$

Therefore,

$$\tilde{F}(u(b)) \ge \tilde{F}(L). \tag{2.15}$$

On the other hand, since $\bar{B}<u(b)<0$, we obtain, by (2.10),

$$\tilde{F}(L) = \tilde{F}\left(\bar{B}\right) > \tilde{F}(u(b)). \tag{2.16}$$

This contradicts (2.15). Consequently, $u(c)\in(0,L)$ and (2.13) holds.

Let $u'(t)>0$ on (b,∞). Then u is increasing on (b,∞) and it has a limit for $t\to\infty$. Assume that $\lim_{t\to\infty}u(t)>L$. Then there exists $b_1>\theta$ such that $u(b_1)=L$, $u'(b_1)>0$, which yields a contradiction as before. Assume that $\lim_{t\to\infty}u(t)=L$. Then

$$\lim_{t\to\infty}\tilde{F}(u(t)) = \tilde{F}(L),$$

and, according to (2.16), there exists $T>b$ such that $\tilde{F}(u(T))>\tilde{F}(u(b))$. Thus, multiplying (2.8) by pu' and integrating over $[b,T]$, we get

$$0 < \frac{(p(T)u'(T))^2}{2} \le (pq)(\theta)\left(\tilde{F}(u(b))-\tilde{F}(u(T))\right) < 0.$$

This contradiction yields $\lim_{t\to\infty}u(t)\in(0,L)$ and (2.12) holds.

Let us assume that (2.14) is fulfilled and $u(b) = \bar{B}$, $u'(b) = 0$. We follow the steps in the first part of this proof. If there exists b_1 such that $u(b_1) = L, u' > 0$ on (b, b_1), then, by multiplying equation (2.8) by pu' and integrating over $[b, b_1]$, we obtain the contradiction

$$0 \le \frac{(p(b_1)u'(b_1))^2}{2} < (pq)(\theta)\left(\tilde{F}\left(\bar{B}\right) - \tilde{F}(L)\right) = 0.$$

Consequently, if there exists $c \in (0, \infty)$ such that $u'(c) = 0$, $u'(t) > 0$ for $t \in (b, c)$, then $u(c) \in (0, L)$.

Let $u'(t) > 0$ for $t \in (b, \infty)$. Due to the above arguments, $\lim_{t\to\infty} u(t) \le L$. Assume that $\lim_{t\to\infty} u(t) = L$. Then

$$\tilde{F}(u(b)) = \tilde{F}\left(\bar{B}\right) = \tilde{F}(L) = \lim_{t\to\infty} \tilde{F}(u(t)).$$

Multiplying equation (2.8) by pu', integrating over $[b, \theta]$ and over $[\theta, t]$ for $t > \theta$, we get, by Lemma 2.9,

$$0 < \frac{(p(\theta)u'(\theta))^2}{2} < (pq)(\theta)\tilde{F}\left(\bar{B}\right),$$
$$-\frac{(p(\theta)u'(\theta))^2}{2} < \frac{(p(t)u'(t))^2}{2} - \frac{(p(\theta)u'(\theta))^2}{2} < (pq)(\theta)\left(-\tilde{F}(u(t))\right).$$

Hence,

$$(pq)(\theta)\tilde{F}\left(\bar{B}\right) > \frac{(p(\theta)u'(\theta))^2}{2} > (pq)(\theta)\tilde{F}\left(u(t)\right).$$

Letting $t \to \infty$, we get

$$(pq)(\theta)\tilde{F}\left(\bar{B}\right) > \frac{(p(\theta)u'(\theta))^2}{2} \ge (pq)(\theta)\tilde{F}\left(\bar{B}\right),$$

a contradiction. □

By analogy, we get the dual lemma.

Lemma 2.12. *Assume that* (2.3)–(2.6), (2.10) *and* (2.11) *hold. Let u be a solution of equation* (2.8) *such that there exist $a \ge 0$, $\theta > a$ satisfying*

$$u(a) \in (0, L), \quad u'(a) = 0, \quad u(\theta) = 0, \quad u(t) > 0 \text{ for } t \in [a, \theta). \tag{2.17}$$

Then u fulfils either

$$u'(t) < 0 \text{ for } t \in (a, \infty), \quad \lim_{t\to\infty} u(t) \in \left(\bar{B}, 0\right) \tag{2.18}$$

or

$$\exists\, b \in (\theta, \infty)\colon u(b) \in \left(\bar{B}, 0\right), \quad u'(b) = 0, \quad u'(t) < 0 \text{ for } t \in (a, b). \tag{2.19}$$

Remark 2.13. Assume that (2.3)–(2.6) hold. If $u(0) = 0$, then u' cannot be positive on $(0, \delta)$ for some $\delta > 0$, since then u is positive on $(0, \delta)$ and integrating equation (2.1) over $[0, t]$, $t \in (0, \delta)$, we get, due to (2.4),

$$p(t)u'(t) = -\int_0^t q(s)f(u(s))\,\mathrm{d}s < 0,$$

a contradiction. Similarly, u' cannot be negative. Therefore, the solution $u(t) \equiv 0$ is the only solution of problem (2.1), (2.2) with $u_0 = 0$. Assume moreover

$$f \in \mathrm{Lip_{loc}}\left([L_0, L] \setminus \{0\}\right). \tag{2.20}$$

Then, by (2.9), (2.20), $\tilde{f} \in \mathrm{Lip_{loc}}\left(\mathbb{R} \setminus \{0\}\right)$ and the solution $u(t) \equiv L$ or $u(t) \equiv L_0$ is the only solution of problem (2.8), (2.2) satisfying for some $t_0 > 0$ conditions $u(t_0) = L$, $u'(t_0) = 0$ or $u(t_0) = L_0$, $u'(t_0) = 0$, respectively.

Lemma 2.14. *Assume that* (2.3)–(2.6) *and* (2.20) *hold. Let u be a solution of problem* (2.8), (2.2) *with $u_0 \in \left(L_0, \bar{B}\right]$. Assume that there exist $\theta > 0$, $a > \theta$ such that*

$$u(\theta) = 0, \quad u(t) < 0 \text{ for } t \in [0, \theta), \quad u'(a) = 0, \quad u'(t) > 0 \text{ for } t \in (\theta, a).$$

Then

$$u(a) \in (0, L), \qquad u'(t) > 0 \ \text{ for } t \in (0, a). \tag{2.21}$$

Proof. By virtue of Lemma 2.9, $u' > 0$ on $(0, a)$. Hence,

$$pu'(t) > 0, \quad t \in (0, a). \tag{2.22}$$

On contrary to (2.21), we assume that $u(a) \geq L$. Then, due to (2.20) and Remark 2.13, we have $u(a) > L$. Thus, there exists $a_0 \in (\theta, a)$ such that $u(t) > L$ on $(a_0, a]$. Integrating equation (2.8) over $[a_0, a]$, we obtain

$$pu'(a_0) - pu'(a) = \int_{a_0}^a q(s)\tilde{f}(u(s))\,\mathrm{d}s = 0.$$

According to (2.9), $pu'(a_0) = 0$, contrary to (2.22). □

2.3 Existence and uniqueness of a solution

In this section, we provide the existence and uniqueness results, both for the auxiliary problem (2.8), (2.2) and for the original problem (2.1), (2.2). For these results, the assumption

$$\lim_{t \to 0^+} \frac{1}{p(t)} \int_0^t q(s)\,\mathrm{d}s = 0 \tag{2.23}$$

is essential. In connection with this condition, we introduce a function φ by

$$\varphi(t) := \frac{1}{p(t)} \int_0^t q(s)\,\mathrm{d}s, \quad t \in (0,\infty), \qquad \varphi(0) = 0. \tag{2.24}$$

The function φ is continuous on $[0,\infty)$ and, by (2.23), satisfies $\lim_{t\to 0^+} \varphi(t) = 0$. Choose an arbitrary $b > 0$. Then there exists $\varphi_b > 0$ such that

$$|\varphi(t)| \le \varphi_b \quad \text{for } t \in [0,b]\,. \tag{2.25}$$

The first two theorems deal with the auxiliary problem.

Theorem 2.15 (**Existence of a solution of problem** (2.8)**,** (2.2)**).** *Assume that* (2.3)*–*(2.6) *and* (2.23) *hold. Then, for each* $u_0 \in [L_0, L]$*, problem* (2.8)*,* (2.2) *has a solution* u*.*
If moreover conditions (2.10)*,* (2.11) *and* (2.20) *hold, then the solution* u *satisfies:*

$$\text{if } u_0 \in \left[\bar{B}, L\right), \quad \text{then } u(t) > \bar{B}, \quad t \in (0,\infty)\,, \tag{2.26}$$

$$\text{if } u_0 \in \left(L_0, \bar{B}\right), \quad \text{then } u(t) > u_0, \quad t \in (0,\infty)\,. \tag{2.27}$$

Proof. *Step 1.* We prove the existence of a solution. According to Remark 2.8, for $u_0 = L_0$, $u_0 = 0$ and $u_0 = L$ there exists a solution of problem (2.8), (2.2). Assume that $u_0 \in (L_0, 0) \cup (0, L)$. By integrating equation (2.8), we get the equivalent form of problem (2.8), (2.2)

$$u(t) = u_0 - \int_0^t \frac{1}{p(s)} \int_0^s q(\tau)\tilde{f}(u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s, \quad t \in [0,\infty)\,.$$

By virtue of (2.4), (2.9), there exists $M > 0$ such that $\left|\tilde{f}(x)\right| \le M$, $x \in \mathbb{R}$. Choose an arbitrary $b > 0$. Then (2.25) is valid. Consider the Banach space $C\,[0,b]$ with the maximum norm and define an operator $\mathcal{F}\colon C\,[0,b] \to C\,[0,b]$,

$$(\mathcal{F}u)(t) := u_0 - \int_0^t \frac{1}{p(s)} \int_0^s q(\tau)\tilde{f}(u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s.$$

Denote $\Lambda := \max\{|L_0)|, L\}$ and consider the ball

$$\mathcal{B}\,(0,R) = \left\{u \in C\,[0,b] : \|u\|_{C[0,b]} \le R\right\}, \quad \text{where } R := \Lambda + M\varphi_b.$$

We estimate the norm of operator $\mathcal{F}$ as follows

$$\|\mathcal{F}u\|_{C[0,b]} = \max_{t\in[0,b]} \left| u_0 - \int_0^t \frac{1}{p(s)} \int_0^s q(\tau)\tilde{f}(u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s \right| \le \Lambda + M\varphi_b = R.$$

It means that $\mathcal{F}$ maps $\mathcal{B}\,(0,R)$ on itself. Choose a sequence $\{u_n\} \subset C\,[0,b]$ such that $\lim_{n\to\infty} \|u_n - u\|_{C[0,b]} = 0$. Since the function $\tilde{f}$ is continuous, we obtain

$$\lim_{n\to\infty} \|\mathcal{F}u_n - \mathcal{F}u\|_{C[0,b]} \le \lim_{n\to\infty} \left\|\tilde{f}(u_n) - \tilde{f}(u)\right\|_{C[0,b]} \left(\int_0^t \frac{1}{p(s)} \int_0^s q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\right) = 0,$$

that is the operator $\mathcal{F}$ is continuous. Choose an arbitrary $\varepsilon > 0$ and put $\delta := \frac{\varepsilon}{M\varphi_b}$. Then, for $t_1, t_2 \in [0, b]$ and for $u \in \mathcal{B}(0, R)$, we get

$$\begin{aligned} |t_1 - t_2| < \delta \Rightarrow |(\mathcal{F}u)(t_1) - (\mathcal{F}u)(t_2)| &= \left| \int_{t_1}^{t_2} \frac{1}{p(s)} \int_0^s q(\tau)\tilde{f}(u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s \right| \\ &\le M\varphi_b|t_2 - t_1| < M\varphi_b\delta = \varepsilon. \end{aligned}$$

Therefore, the functions in $\mathcal{F}(\mathcal{B}(0, R))$ are equicontinuous and, according to the Arzelà–Ascoli theorem, the set $\mathcal{F}(\mathcal{B}(0, R))$ is relatively compact. Hence, the operator $\mathcal{F}$ is compact on $\mathcal{B}(0, R)$.

The Schauder fixed point theorem yields a fixed point $u^\star$ of $\mathcal{F}$ in $\mathcal{B}(0, R)$. Consequently,

$$u^\star(t) = u_0^\star - \int_0^t \frac{1}{p(s)} \int_0^s q(\tau)\tilde{f}(u^\star(\tau))\,\mathrm{d}\tau\,\mathrm{d}s.$$

Thus, $u^\star(0) = u_0^\star$,

$$(p(t)(u^\star)'(t))' = -q(t)\tilde{f}\left(u^\star(t)\right), \quad t \in [0, b].$$

Since $|(u^\star)'(t)| \le M\varphi(t)$ and, due to (2.23), $\lim_{t\to 0^+}(u^\star)'(t) = 0 = (u^\star)'(0)$. By (2.9), $\tilde{f}(u^\star(t))$ is bounded on $[0, \infty)$ and hence, by Theorem 11.5 in [31], $u^\star$ can be extended to interval $[0, \infty)$ as a solution of equation (2.8).

Step 2. We prove the estimates of solutions. Assume that (2.10), (2.11) and (2.20) hold.

Let $u_0 \in (0, L)$. If $u > 0$ on $(0, \infty)$, then (2.26) holds. Assume that there exists $\theta_1 > 0$ such that $u(\theta_1) = 0$, $u(t) > 0$ for $t \in [0, \theta_1)$. Using Lemma 2.12, where $a = 0$ and $\theta = \theta_1$, we obtain that u satisfies either (2.18) or (2.19). Condition (2.18) gives (2.26). Let condition (2.19) be valid, that is

$$\exists\, b \in (\theta_1, \infty)\colon u(b) \in \left(\bar{B}, 0\right), \quad u'(b) = 0, \quad u'(t) < 0 \ \text{ for } t \in (0, b).$$

If $u < 0$ on (b, ∞), then, by Lemma 2.9, u is increasing on (b, ∞) and (2.26) is valid. Assume that there exists $\theta_2 > b$ such that $u(\theta_2) = 0$, $u(t) < 0$ for $t \in [b, \theta_2)$. Using Lemma 2.11, where $\theta = \theta_2$, we get that u satisfies either (2.12) or (2.13). Condition (2.12) gives (2.26). Let condition (2.13) be valid. Then we use previous arguments.

Let $u_0 = 0$. Due to Remark 2.13, $u(t) \equiv 0$ is the unique solution of problem (2.8), (2.2) and so, (2.26) holds.

Let $u_0 \in \left(\bar{B}, 0\right)$. If $u < 0$ on $(0, \infty)$, then, by Lemma 2.9, u is increasing on $(0, \infty)$ and (2.26) is valid. Assume that there exists $\theta_3 > 0$ such that $u(\theta_3) = 0$, $u(t) < 0$ for $t \in [0, \theta_3)$. Using Lemma 2.11, where $b = 0$ and $\theta = \theta_3$, we obtain that u satisfies either (2.12) or (2.13). Condition (2.12) gives (2.26). Let condition(2.13) be valid, that is

$$\exists\, c \in (\theta_3, \infty)\colon u(c) \in (0, L), \quad u'(c) = 0, \quad u'(t) > 0 \ \text{ for } t \in (0, c).$$

If $u > 0$ on (c, ∞), then (2.26) holds. Assume that there exists $\theta_4 > c$ such that $u(\theta_4) = 0$, $u(t) > 0$ for $t \in [c, \theta_4)$. Using Lemma 2.12, where $a = c$ and $\theta = \theta_4$, we get that u satisfies either (2.18) or (2.19). Condition (2.18) gives (2.26). Let condition (2.19) be valid. Then we use previous arguments.

Let $u_0 = \bar{B}$. If $u < 0$ on $(0, \infty)$, then, by Lemma 2.9, u is increasing on $(0, \infty)$ and (2.26) is valid. Assume that there exists $\theta_5 > 0$ such that $u(\theta_5) = 0$, $u(t) < 0$ for $t \in [0, \theta_5)$. If $u > 0$ on (θ_5, ∞), then (2.26) holds. Assume that there exists $d > \theta_5$ such that $u'(d) = 0$, $u'(t) > 0$ for $t \in (\theta_5, d)$. Using Lemma 2.14, where $\theta = \theta_5$ and $a = d$, we have that (2.21) is valid. Now, we have analogous situation as in the case $u_0 \in (0, L)$, so we argue similarly.

Let $u_0 \in \big(L_0, \bar{B}\big)$. If $u < 0$ on $(0, \infty)$, then u is increasing on $(0, \infty)$ and (2.27) is valid. Assume that there exists $\theta_6 > 0$ such that $u(\theta_6) = 0$, $u(t) < 0$ for $t \in [0, \theta_6)$. If $u > 0$ on (θ_6, ∞), then (2.27) holds. Assume that there exists $d > \theta_6$ such that $u'(d) = 0$, $u'(t) > 0$ for $t \in (\theta_6, d)$. According to Lemma 2.14, where $\theta = \theta_6$ and $a = d$, we have that (2.21) is valid. We obtain similar situation as in the case $u_0 \in (0, L)$, thus we proceed analogously. □

Remark 2.16. Under assumptions (2.3)–(2.6) and (2.23), each solution of problem (2.8), (2.2) is defined on the half-line $[0, \infty)$. Furthermore, the set of these solutions with $u_0 \in (L_0, 0) \cup (0, L)$ is composed of three disjoint classes $\mathcal{S}_d$ (damped solutions), $\mathcal{S}_h$ (homoclinic solutions), $\mathcal{S}_e$ (escape solutions). Then

1. $u \in \mathcal{S}_d$ if and only if $u_{\sup} < L$,
2. $u \in \mathcal{S}_h$ if and only if $u_{\sup} = L$,
3. $u \in \mathcal{S}_e$ if and only if $u_{\sup} > L$.

Theorem 2.17 (Uniqueness and continuous dependence on initial values). *Assume that (2.3)–(2.6) and (2.23) hold and let*

$$f \in \mathrm{Lip}[L_0, L] \tag{2.28}$$

hold. Then, for each $u_0 \in [L_0, L]$, problem (2.8), (2.2) has a unique solution. Further, for each $b > 0$, there exists $K > 0$ such that

$$\|u_1 - u_2\|_{C^1[0,b]} \le K|B_1 - B_2|. \tag{2.29}$$

Here u_i is a solution of problem (2.8), (2.2) with $u_0 = B_i$, $i = 1, 2$.

Proof. For $i \in \{1, 2\}$, choose $B_i \in [L_0, L]$. According to Theorem 2.15, there exists a solution u_i of problem (2.8), (2.2) with $u_0 = B_i$. After integrating (2.8), where $u = u_i$, we obtain, by (2.2),

$$u_i(\xi) = B_i - \int_0^{\xi} \frac{1}{p(s)} \int_0^s q(\tau) \tilde{f}(u_i(\tau))\, \mathrm{d}\tau\, \mathrm{d}s, \quad \xi \in [0, \infty). \tag{2.30}$$

Denote

$$\varrho(t) := \max\{|u_1(\xi) - u_2\,(\xi)\,| : \xi \in [0,t]\}, \quad t \in [0,\infty)\,.$$

By virtue of (2.28), there exists a Lipschitz constant $K_1 \in (0,\infty)$ for f on $[L_0, L]$. Then K_1 is the Lipschitz constant for $\tilde{f}$ on $\mathbb{R}$ and, due to (2.30), we get

$$\begin{aligned}\varrho(t) &= \max_{\xi\in[0,t]} \left| B_1 - B_2 - \int_0^\xi \frac{1}{p(s)} \int_0^s q(\tau) \left(\tilde{f}(u_1(\tau)) - \tilde{f}(u_2(\tau))\right) \mathrm{d}\tau \, \mathrm{d}s \right| \\ &\le |B_1 - B_2| + \int_0^t \frac{1}{p(s)} \int_0^s q(\tau) \left|\tilde{f}(u_1(\tau)) - \tilde{f}(u_2(\tau))\right| \mathrm{d}\tau \, \mathrm{d}s \\ &\le |B_1 - B_2| + K_1 \int_0^t \frac{1}{p(s)} \int_0^s q(\tau)\varrho(\tau)\, \mathrm{d}\tau\, \mathrm{d}s, \quad t \in [0,\infty)\,.\end{aligned}$$

Choose $b > 0$ and let φ be given by (2.24). Then (2.25) holds. Since ϱ is nondecreasing on $[0, b]$, we obtain

$$\varrho(t) \le |B_1 - B_2| + K_1 \int_0^t \varrho(s)\varphi(s)\, \mathrm{d}s \quad t \in [0,b]\,.$$

Using the Gronwall lemma, we get

$$\varrho(t) \le |B_1 - B_2| e^{K_1 \int_0^t \varphi(s)\mathrm{d}s} \le |B_1 - B_2| e^{K_1 b \varphi_b}, \quad t \in [0,b]\,. \tag{2.31}$$

Similarly, according to (2.30), we get, for $i \in \{1,2\}$,

$$u_i'(t) = -\frac{1}{p(t)} \int_0^t q(s)\tilde{f}\,(u_i(s))\;\mathrm{d}s, \quad t \in (0,\infty)\,, \quad u_i'(0) = 0.$$

Consequently,

$$\begin{aligned}|u_1'(t) - u_2'(t)| &\le \frac{1}{p(t)} \int_0^t q(s) \left|\tilde{f}\,(u_1(s)) - \tilde{f}\,(u_2(s))\right| \mathrm{d}s \le K_1 \frac{1}{p(t)} \int_0^t q(s)\varrho(s)\, \mathrm{d}s \\ &\le K_1 \varrho(t)\varphi(t), \qquad t \in [0,\infty)\,.\end{aligned}$$

By applying (2.25) and (2.31), we obtain

$$\varrho_1(b) := \max\{|u_1'(t) - u_2'(t)| : t \in [0,b]\} \le |B_1 - B_2| K_1 \varphi_b e^{K_1 b \varphi_b}.$$

Therefore, by (2.31),

$$\|u_1 - u_2\|_{C^1[0,b]} = \varrho(b) + \varrho_1(b) \le |B_1 - B_2|(1 + K_1\varphi_b) e^{K_1 b \varphi_b},$$

that is (2.29) holds for $K := (1 + K_1\varphi_b)e^{K_1 b \varphi_b}$.

If $B_1 = B_2$, then $u_1(t) = u_2(t)$ on each $[0,b] \subset \mathbb{R}$, which implies the uniqueness of solution of problem (2.8), (2.2). □

Example 2.18. Assume that $0 < L < -L_0$, $\alpha > 0$, $\beta \geq 0$, $\gamma > 0$ and $k > 0$. Consider the IVP

$$\begin{gathered}(t^\alpha u'(t))' + t^\beta \tilde{f}(u(t)) = 0, \\ u(0) = u_0 \in [L_0, L], \quad u'(0) = 0,\end{gathered} \tag{2.32}$$

where

$$\tilde{f}(x) = \begin{cases} k|x|^\gamma \operatorname{sgn} x (x - L_0)(L - x) & \text{for } x \in [L_0, L], \\ 0 & \text{for } x < L_0, \quad x > L. \end{cases}$$

We have the auxiliary equation (2.8) with

$$p(t) = t^\alpha, \quad q(t) = t^\beta, \quad t \in [0, \infty).$$

Example 2.1 shows that the functions p and q fulfil (2.5) and (2.6). In addition, pq is increasing on $[0, \infty)$, which means that (2.14) and consequently, (2.11) hold. Finally,

$$\lim_{t \to 0^+} \frac{1}{t^\alpha} \int_0^t s^\beta \, \mathrm{d}s = \lim_{t \to 0^+} \frac{1}{t^\alpha} \frac{t^{\beta+1}}{\beta + 1} = \lim_{t \to 0^+} \frac{t^{\beta+1-\alpha}}{\beta + 1} = 0 \quad \text{if } \beta > \alpha - 1.$$

Hence, if $\beta > \alpha - 1$, then (2.23) is valid.

Since, by (2.9), $f = \tilde{f}$ on $[L_0, L]$, then the function f is locally Lipschitz continuous on $[L_0, L] \setminus \{0\}$, $L_0 < 0 < L$, $f(L_0) = f(0) = f(L) = 0$, $xf(x) > 0$ for $x \in (L_0, L) \setminus \{0\}$. Thus, (2.3), (2.4) and (2.20) are satisfied.

Let us check (2.10). Define the function $g(x) := -f(-x)$ for $x \geq 0$. Then

$$g(x) = kx^\gamma(-x - L_0)(L + x), \quad f(x) = kx^\gamma(x - L_0)(L - x), \quad x \geq 0$$

and since $|L_0| > L$, we get

$$\begin{aligned} g(x) - f(x) &= kx^\gamma \left(-Lx - x^2 - L_0 L - L_0 x - Lx + x^2 + L_0 L - L_0 x\right) \\ &= 2kx^{\gamma+1}(|L_0| - L) > 0, \quad x \in (0, |L_0|]. \end{aligned}$$

Consequently, $g(x) > f(x)$ for $x \in (0, |L_0|]$ and so,

$$\tilde{F}(L_0) = \int_0^{|L_0|} g(z) \, \mathrm{d}z > \int_0^L f(z) \, \mathrm{d}z = \tilde{F}(L).$$

Therefore, there exists $\bar{B} \in (L_0, 0)$ such that $\tilde{F}\left(\bar{B}\right) = \tilde{F}(L)$, which yields (2.10).

To summarize, if

$$\beta > \alpha - 1,$$

then we have fulfilled all assumptions of Theorem 2.15. So, for each $u_0 \in [L_0, L]$, problem (2.32) has a solution u and u satisfies (2.26), (2.27). If in addition $\gamma \geq 1$, then f is Lipschitz continuous on $[L_0, L]$, which means that (2.28) holds. Then Theorem 2.17 yields the uniqueness of such solution u.

Now, let us discuss the original problem.

Theorem 2.19 (Existence and uniqueness of a solution of problem (2.1), (2.2)). *Assume that (2.3)–(2.6), (2.10), (2.11), (2.23),*

$$f \in \mathrm{Lip}_{\mathrm{loc}}[L_0, \infty), \tag{2.33}$$

$$\exists\, C_L \in (0,\infty)\colon\ -C_L \le f(x) \le 0 \quad \text{for } x \ge L \tag{2.34}$$

are satisfied. Then, for each $u_0 \in [L_0, L]$, problem (2.1), (2.2) has a unique solution u. This solution u satisfies (2.26) and (2.27).

Proof. Let $u_0 \in [L_0, L]$. Only for this proof we modify the auxiliary function $\tilde{f}$ as

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \ge L_0, \\ 0 & \text{for } x < L_0. \end{cases} \tag{2.35}$$

We see that this new function $\tilde{f}$ satisfies conditions (2.3) and (2.4) and all results from Section 2.2 can be proved the same way for this redefined function $\tilde{f}$.

Step 1. We prove the existence and uniqueness of a solution of problem (2.8), (2.2). This problem has the equivalent form

$$u(t) = u_0 - \int_0^t \frac{1}{p(s)} \int_0^s q(\tau)\tilde{f}(u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s, \quad t \in [0,\infty).$$

According to (2.33)–(2.35),

$$\exists\, M > 0\colon\ \left|\tilde{f}(x)\right| \le M, \quad x \in \mathbb{R}. \tag{2.36}$$

Put $\Lambda := \max\{|L_0|, L\}$. By (2.33), there exists $K > 0$ such that

$$|f(x) - f(y)| \le K|x - y|, \quad \forall\, x, y \in [-\Lambda - 1, \Lambda + 1]. \tag{2.37}$$

Due to (2.5), (2.6), (2.23) and (2.24), we get

$$0 < \varphi(t) < \infty \quad \text{for } t \in (0,\infty), \qquad \lim_{t\to 0^+} \varphi(t) = 0.$$

Therefore, we can find $\eta \in (0,\infty)$ such that

$$\int_0^\eta \varphi(t)\,\mathrm{d}t \le \min\left\{\frac{1}{2K}, \frac{1}{M}\right\}. \tag{2.38}$$

Consider the Banach space $C[0,\eta]$ with the maximum norm and define an operator $\mathcal{F}\colon C[0,\eta] \to C[0,\eta]$ by

$$(\mathcal{F}u)(t) := u_0 - \int_0^t \frac{1}{p(s)} \int_0^s q(\tau)\tilde{f}(u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s.$$

From (2.36) and (2.38), it follows that

$$\|\mathcal{F}u\|_{C[0,\eta]} \le \Lambda + M\int_0^\eta \varphi(s)\,\mathrm{d}s \le \Lambda + 1, \quad \forall\, u \in C[0,\eta],$$

hence $\mathcal{F}$ maps the ball $\mathcal{B}(0,\Lambda+1) = \{u \in C[0,\eta] \colon \|u\|_{C[0,\eta]} \le \Lambda + 1\}$ on itself. Choose arbitrary $u_1, u_2 \in \mathcal{B}(0,\Lambda+1)$. Then, by (2.37) and (2.38), we obtain

$$\begin{aligned}\|\mathcal{F}u_1 - \mathcal{F}u_2\|_{C[0,\eta]} &\le \int_0^\eta \frac{1}{p(s)}\int_0^s q(\tau)\Big|\tilde f(u_1(\tau)) - \tilde f(u_2(\tau))\Big|\,\mathrm{d}\tau\,\mathrm{d}s \\ &\le K\|u_1 - u_2\|_{C[0,\eta]}\int_0^\eta \varphi(s)\,\mathrm{d}s \le \frac{1}{2}\,\|u_1 - u_2\|_{C[0,\eta]},\end{aligned}$$

thus $\mathcal{F}$ is a contraction on $\mathcal{B}(0,\Lambda+1)$. The Banach fixed point theorem yields a unique fixed point u of $\mathcal{F}$ in $\mathcal{B}(0,\Lambda+1)$. Therefore

$$u(0) = u_0, \qquad u'(t) = -\frac{1}{p(t)}\int_0^t q(\tau)\tilde f(u(\tau))\,\mathrm{d}\tau, \quad t \in (0,\eta]. \tag{2.39}$$

Since $|u'(t)| \le M\varphi(t)$, it holds $\lim_{t\to 0+} u'(t) = 0$. From (2.39), it follows $(p(t)u'(t))' = -q(t)\tilde f(u(t))$, $t \in (0,\eta]$, thus the fixed point u is a solution of problem (2.8), (2.2) on $[0,\eta]$.

According to (2.36), $\tilde f(u(t))$ is bounded on $[0,\infty)$ and hence, by Theorem 11.5 in [31], u can be extended to $[0,\infty)$. Since $\tilde f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$, this extension is unique.

Step 2. We prove the estimates of solutions of problem (2.8), (2.2). Let $u_0 = L$ or $u_0 = 0$. According to Step 1 and Remark 2.13, problem (2.8), (2.2) has a unique solution $u(t) \equiv L$ or $u(t) \equiv 0$, respectively, and so, (2.26) holds.

Let $u_0 \in (L_0, 0) \cup (0, L)$. Then we argue analogously as in Step 2 in the proof of Theorem 2.15.

Step 3. We prove the existence and uniqueness of a solution of problem (2.1), (2.2). We have proved that estimates (2.26) and (2.27) are valid. By virtue of definition of $\tilde f$ (see (2.35)), the solution u of problem (2.8), (2.2) satisfies equation (2.1) on $(0,\infty)$.

Suppose that there exists another solution $\tilde u$ of problem (2.1), (2.2). We can prove as in Step 2 that $\tilde u$ fulfils (2.26) and (2.27). It means that $\tilde u$ satisfies equation (2.8) on $(0,\infty)$, too. Therefore, by Step 1, $u \equiv \tilde u$. $\square$

Example 2.20. Assume that $0 < L < -L_0$, $\alpha > 0$, $\beta \ge 0$, $\gamma > 0$ and $k > 0$. Consider the IVP

$$\begin{gathered}(t^\alpha u'(t))' + t^\beta f(u(t)) = 0, \\ u(0) = u_0 \in [L_0, L], \quad u'(0) = 0,\end{gathered} \tag{2.40}$$

where

$$f(x) = \begin{cases} k|x|^\gamma \operatorname{sgn} x (x - L_0)(L - x) & \text{for } x \in [L_0, L], \\ \frac{L-x}{x+1} & \text{for } x > L. \end{cases}$$

Here

$$p(t) = t^{\alpha}, \quad q(t) = t^{\beta}, \quad t \in [0, \infty).$$

According to Example 2.18, conditions (2.3)–(2.6), (2.10) and (2.11) are fulfilled and also that (2.23) holds provided $\beta > \alpha - 1$. If $\gamma \geq 1$, then f is locally Lipschitz continuous on $[L_0, \infty)$, which gives (2.33). Moreover, f is negative on (L, ∞) and

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{L - x}{x + 1} = -1,$$

that is (2.34) holds with $C_L = 1$.

To sum up, if

$$\beta > \alpha - 1 \ \text{ and } \ \gamma \geq 1,$$

then we have satisfied all assumptions of Theorem 2.19. Therefore, for each $u_0 \in [L_0, L]$, problem (2.40) has a unique solution u and u satisfies (2.26), (2.27).

In the next theorem, we show that condition (2.23) used in the previous results is necessary for the existence of a solution of problem (2.1), (2.2).

Theorem 2.21 (Necessity of condition (2.23)). *Assume that (2.3)–(2.6) hold and let u be a solution of problem (2.1), (2.2) with $u_0 \in (L_0, 0) \cup (0, L)$. Then (2.23) is valid.*
Vice versa, assume that (2.23) is satisfied and let u be a solution of equation (2.1) with $u(0) \in [L_0, L]$. Then $u'(0) = 0$ and u is the solution of problem (2.1), (2.2).

Proof. *Step 1.* We prove the first part of theorem. Let u be a solution of problem (2.1), (2.2) with $u_0 \in (0, L)$. Due to Lemma 2.10 there exists $t_0 > 0$ such that $u(t) > 0$, $u'(t) < 0$ for $t \in (0, t_0]$. Since $0 < u(t_0) \leq u(t) \leq u_0 < L$ for $t \in [0, t_0]$, we have

$$f(u(t)) \geq \min \{ f(x) \colon x \in [u(t_0), u_0] \} =: M_1 > 0, \quad t \in [0, t_0].$$

Integrating equation (2.1) from 0 to $t \in (0, t_0]$ and using (2.2), we obtain

$$u'(t) = -\frac{1}{p(t)} \int_0^t q(s) f(u(s)) \, \mathrm{d}s \leq -M_1 \frac{1}{p(t)} \int_0^t q(s) \, \mathrm{d}s < 0, \quad t \in (0, t_0].$$

Letting $t \to 0^+$, we get, by (2.2),

$$0 = u'(0) \leq -M_1 \lim_{t \to 0^+} \frac{1}{p(t)} \int_0^t q(s) \, \mathrm{d}s \leq 0,$$

which yields (2.23).

For $u_0 \in (L_0, 0)$, we proceed analogously.

Step 2. We prove the second part of theorem. Let u be a solution of equation (2.1) with $u_0 \in [L_0, L]$ and assume that (2.23) holds. Choose $t_1 > 0$ and put

$M_2 := \max\{|f(u(s))| : s \in [0, t_1]\} \geq 0$. Integrating equation (2.1) from 0 to $t \in (0, t_1]$ and using (2.5), we get

$$|u'(t)| = \left| \frac{1}{p(t)} \int_0^t q(s) f(u(s)) \, \mathrm{d}s \right| \leq M_2 \frac{1}{p(t)} \int_0^t q(s) \, \mathrm{d}s, \quad t \in (0, t_1].$$

We let $t \to 0^+$ and obtain, by (2.23),

$$|u'(0)| \leq M_2 \lim_{t \to 0^+} \frac{1}{p(t)} \int_0^t q(s) \, \mathrm{d}s = 0.$$

This gives $u'(0) = 0$ and so, u is the solution of problem (2.1), (2.2). □

3 Damped solutions of the problem without ϕ-Laplacian

3.1 Existence and uniqueness of damped solutions

Now, we specify an interval for starting values u_0, where the existence of damped solutions is guaranteed. Note that, by Definition 2.5 and the estimates (2.26), (2.27), each damped solution u of the auxiliary problem (2.8), (2.2) satisfies $L_0 \le u(t) < L$ for $t \in [0, \infty)$. According to (2.9), the function f coincides with $\tilde{f}$ on $[L_0, L]$ and hence, all results of Chapter 2 are valid also for damped solutions of the original problem (2.1), (2.2). In particular,

Theorem 3.1 (Existence and uniqueness of damped solutions of problem (2.1), (2.2)). *Assume that assumptions (2.3)–(2.6), (2.10), (2.11), (2.20) and (2.23) are fulfilled. Then, for each $u_0 \in \big(\bar{B}, L\big)$, problem (2.1), (2.2) has a solution u. The solution u is damped and satisfies (2.26).*
If moreover f satisfies (2.28), then the solution u is unique.

Proof. Choose $u_0 \in \big(\bar{B}, L\big)$. By Theorem 2.15, there exists a solution u of problem (2.8), (2.2) satisfying (2.26).

(i) If $u_0 = 0$, then, due to Remark 2.13, $u(t) \equiv 0$ is the only solution of problem (2.8), (2.2). It is clear that u is damped.

(ii) Let $u_0 \in (0, L)$. If $u > 0$ on $(0, \infty)$, then Lemma 2.10 yields $u' < 0$ on $(0, \infty)$ and hence, u is damped. Let $\theta > 0$ be the first zero of u. By Lemma 2.10, $u' < 0$ on $(0, \theta]$. If $u < 0$ on (θ, ∞), then u is damped. Let $\xi > \theta$ be the second zero of u. Then there is $b \in (\theta, \xi)$ such that $u'(b) = 0$. Due to (2.26), $u(b) \in \big(\bar{B}, 0\big)$. By Lemma 2.9, $u' > 0$ on $(b, \xi]$. If $u' > 0$ on (b, ∞), then, according to Lemma 2.11, $\lim_{t\to\infty} u(t) \in (0, L)$ and so, u is damped. Let there exist $c > \xi$ such that $u'(c) = 0$. Then Lemma 2.11 gives $u(c) \in (0, L)$ and we can continue as before working with $u(c)$ instead of u_0.

(iii) Let $u_0 \in \big(\bar{B}, 0\big)$. Working with u_0 in place of $u(b)$, we can use the arguments of part (ii) and prove that u is damped.

We proved that $u_{\sup} < L$ and so, u is damped. If moreover (2.28) holds, then Theorem 2.17 gives that the solution u is unique. By (2.9), $f(u(t)) = \tilde{f}(u(t))$ for $t \in [0, \infty)$ and then u is a solution of problem (2.1), (2.2).

□

Remark 3.2. In addition, if (2.14) is fulfilled, then, by Lemma 2.11, the assertion of Theorem 3.1 holds for $u_0 = \bar{B}$, too.

Example 3.3. Consider equation (2.32) from Example 2.18, that is equation (2.1) with

$$\begin{aligned} &p(t) = t^\alpha, \quad q(t) = t^\beta, \quad \alpha > 0,\ \beta \geq 0,\ \beta > \alpha - 1,\ t \in [0, \infty)\,, \\ &f(x) = k|x|^\gamma \operatorname{sgn} x(x - L_0)(L - x), \quad 0 < L < -L_0,\ \gamma > 0,\ k > 0,\ x \in [L_0, L]\,. \end{aligned}$$

By Example 2.18, conditions (2.3)–(2.6), (2.10), (2.11), (2.20) and (2.23) hold. So, Theorem 3.1 and Remark 3.2 give that, for each $u_0 \in \big[\bar{B}, L\big)$, problem (2.32), (2.2) has a solution u, u is damped and satisfies (2.26). If $\gamma \geq 1$, then f fulfils in addition (2.28) and the solution u is unique.

3.2 Properties of nonoscillatory and oscillatory solutions

In the literature, the permanent attention has been devoted to oscillatory solutions of second order nonlinear differential equations. In section 1.1, we mentioned many references on papers, where oscillatory solutions for the regular equations are studied. However, nonlinearities in equations in these cited papers have globally monotonous behaviour, in contrast to the basic asuumptions in this thesis. Moreover, we deal with solutions of (2.1) starting at possible singular point $t = 0$, and we provide an interval for starting values u_0 giving oscillatory solutions, see Theorems 3.15, 3.16 and 3.17. Therefore, theorems from these cited papers cannot be applied to the singular problem (2.1), (2.2) satisfying assumptions (2.3)–(2.6). For example, the same equation (2.1) is studied in [18] but in the regular setting, that is the function p in equation (2.1) must be a strictly positive on $[0, \infty)$. One of the basic assumptions in [18] is the convergence or divergence of integral $I_p = \int_0^\infty \frac{1}{p(t)}\,\mathrm{d}t$. In this thesis, a typical choice in equation (2.1) is $p(t) = t^\alpha,\ \alpha > 0$. Then clearly, $I_p = \infty$ and therefore, theorems in [18] which require $I_p < \infty$, cannot be applied. Other important assumption in [18] concerns the function f in equation (2.1) and has the form

$$\liminf_{|x| \to \infty} |f(x)| > 0. \tag{3.1}$$

In the thesis, the function f has three zeros $L_0 < 0 < L$ and an arbitrary behaviour for $x < L_0$ and $x > L$. Consequently, (3.1) need not be fulfilled and theorems of [18] requiring (3.1) cannot be applied here, as well.

Definition 3.4. A function u is called *eventually positive* (*eventually negative*), if there exists $t_0 > 0$ such that $u(t) > 0$ $(u(t) < 0)$ for $t \in (t_0, \infty)$.

Clearly, each nonoscillatory solution of problem (2.1), (2.2) is either eventually positive or eventually negative. Paper [69] provides an example which demonstrates that equation (2.1) can have both oscillatory damped solutions and nonoscillatory ones.

In order to obtain conditions under which every damped solution of problem (2.1), (2.2) is oscillatory, we distinguish two cases according to the convergence or divergence of the integral $\int_1^\infty \frac{1}{p(s)}\,\mathrm{d}s$.

CASE I: We assume that the function p fulfils

$$\int_1^\infty \frac{1}{p(s)}\,\mathrm{d}s < \infty. \tag{3.2}$$

CASE II: We assume that the function p fulfils

$$\int_1^\infty \frac{1}{p(s)}\,\mathrm{d}s = \infty. \tag{3.3}$$

First, we describe an asymptotic behaviour of nonoscillatory damped solutions of problem (2.1), (2.2) in Case I.

Theorem 3.5. *Assume that conditions* (2.3)–(2.6), (2.10), (2.11), (3.2) *and*

$$\lim_{t\to\infty}\int_1^t \frac{1}{p(s)}\int_1^s q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s = \infty \tag{3.4}$$

are fulfilled. If u is a damped nonoscillatory solution of problem (2.1), (2.2) *with $u_0 \in (L_0,0)\cup(0,L)$, then*

$$\lim_{t\to\infty} u(t) = 0. \tag{3.5}$$

If moreover p satisfies

$$\liminf_{t\to\infty} p(t)\int_t^\infty \frac{1}{p(s)}\,\mathrm{d}s > 0, \tag{3.6}$$

then

$$\lim_{t\to\infty} u'(t) = 0.$$

Proof. Assume that u is a damped nonoscillatory solution of problem (2.1), (2.2) with $u_0 \in (L_0,0)\cup(0,L)$. Then u is either eventually positive or eventually negative.

Step 1. We prove that $\lim_{t\to\infty} u(t) = 0$. Since u is nonoscillatory, Lemma 2.9 or Lemma 2.10 guarantees the existence of $t_0 > 1$ such that u is either increasing or decreasing on $[t_0,\infty)$. Therefore, there exists $\lim_{t\to\infty} u(t) =: c$. Since $u_{\sup} < L$, we have $c < L$. Integrating equation (2.1) from t_0 to t and dividing this by $p(t)$, we get

$$u'(t) = \frac{p(t_0)u'(t_0)}{p(t)} - \frac{1}{p(t)} \int_{t_0}^{t} q(s) f(u(s))\,\mathrm{d}s,$$
$$u(t) = u(t_0) + \int_{t_0}^{t} \frac{p(t_0)u'(t_0)}{p(s)}\,\mathrm{d}s - \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} q(\tau) f(u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s. \tag{3.7}$$

Let u be eventually positive. Then $c \in [0, L)$. Assume $c \in (0, L)$. Then there exists $M > 0$ such that $f(u(t)) \geq M$ for $t \geq t_0$. From (3.7), we obtain

$$u(t) \leq u(t_0) + p(t_0)u'(t_0) \int_{t_0}^{t} \frac{1}{p(s)}\,\mathrm{d}s - M \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s,$$
$$\lim_{t\to\infty} u(t) \leq u(t_0) + p(t_0)u'(t_0) \int_{t_0}^{\infty} \frac{1}{p(s)}\,\mathrm{d}s - M \lim_{t\to\infty} \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s$$
$$= -\infty,$$

which contradicts $c \in (0, L)$. Hence, $c = 0$.

Let u be eventually negative. If u is negative on $[0, \infty)$, then, by Lemma 2.9, we get $u'(t) > 0$ for $t \in (0, \infty)$ and thus, $c \in (L_0, 0]$. Now, assume that there exist $a \geq 0$ and $\theta > a$ satisfying (2.17) and $u(t) < 0$ for $t > \theta$. By Lemma 2.12, it occurs either (2.18) or (2.19). If (2.18) holds, then $c \in \big(\bar{B}, 0\big)$. If (2.19) holds, then, by Lemma 2.9, $c \in \big(\bar{B}, 0\big]$. Assume that $c \in (L_0, 0)$. Then there exists $M > 0$ such that $-f(u(t)) \geq M$ for $t \geq t_0$ and, similarly as in the eventually positive case, we derive a contradiction. Therefore, $c = 0$ and (3.5) is proved.

Step 2. Assume in addition that (3.6) is valid and prove that $\lim_{t\to\infty} u'(t) = 0$. Let u be eventually negative. Then, by (2.4)–(2.6) and (3.5), there exists $t_1 > 0$ such that $u'(t) > 0$ for $t \geq t_1$. Due to (3.6), there exist $c > 0$ and $t_2 \geq t_1$ such that

$$p(t) \int_{t}^{\infty} \frac{1}{p(s)}\,\mathrm{d}s \geq c > 0, \quad t \in [t_2, \infty).$$

From (2.1), (2.3), (2.4) and (2.6), it follows

$$(p(t)u'(t))' = -q(t) f(u(t)) > 0, \quad t \in [t_2, \infty).$$

So, the function pu' is increasing on $[t_2, \infty)$ and we have

$$p(\tau)u'(\tau) \leq p(s)u'(s) \leq p(t)u'(t), \quad t_2 \leq \tau \leq s \leq t.$$

Therefore,

$$u(t) - u(\tau) = \int_{\tau}^{t} u'(s)\,\mathrm{d}s = \int_{\tau}^{t} \frac{p(s)u'(s)}{p(s)}\,\mathrm{d}s \geq p(\tau)u'(\tau) \int_{\tau}^{t} \frac{1}{p(s)}\,\mathrm{d}s,$$
$$\lim_{t\to\infty} (u(t) - u(\tau)) \geq p(\tau)u'(\tau) \int_{\tau}^{\infty} \frac{1}{p(s)}\,\mathrm{d}s,$$
$$-u(\tau) \geq u'(\tau)p(\tau) \int_{\tau}^{\infty} \frac{1}{p(s)}\,\mathrm{d}s \geq u'(\tau)c > 0.$$

According to (3.5), we have $\lim_{t\to\infty} u'(t) = 0$.

For eventually positive solutions, we proceed analogously. □

In the investigation of oscillatory solutions of problem (2.1), (2.2), we use the following definition.

Definition 3.6. Let u be an oscillatory solution of problem (2.1), (2.2). Denote $\{a_n\}$ ($\{b_n\}$) sequences of local maxima (minima) of u. Assume that either $a_n < b_n < a_{n+1} < b_{n+1},\ n \in \mathbb{N}$ or $b_n < a_n < b_{n+1} < a_{n+1},\ n \in \mathbb{N}$. Then the numbers $u(a_n) - u(b_n)$, $n \in \mathbb{N}$ are called *amplitudes* of u.

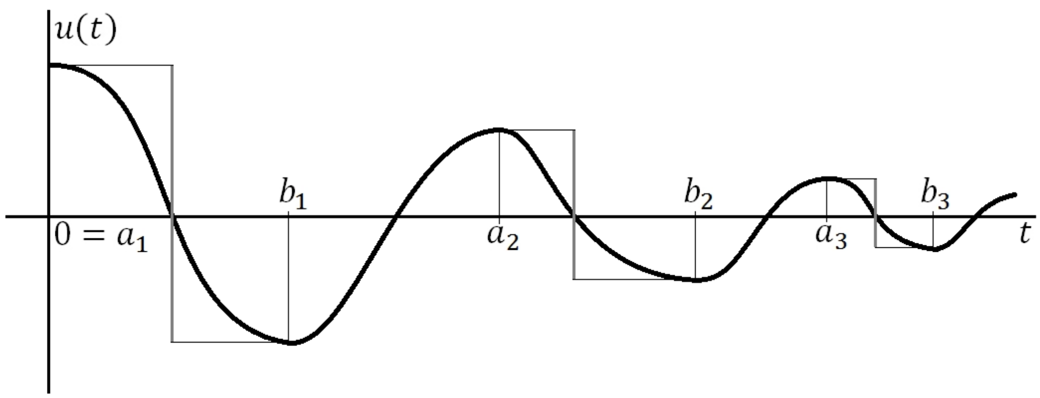

Figure 3.1: Amplitudes of oscillatory solution

Theorem 3.7. *Assume that conditions* (2.3)–(2.6), (2.10), (2.11) *and* (2.20) *hold. Let u be an oscillatory solution of problem* (2.1), (2.2) *with $u_0 \in (L_0, 0) \cup (0, L)$. Then u is a damped solution and has nonincreasing amplitudes.*
If moreover p and q satisfy (2.14), *then amplitudes of u are decreasing.*

Proof. Let u be an oscillatory solution of problem (2.8), (2.2). Then u is not a monotonous and so, by Lemmas 4.2 and 4.3, u cannot be neither an escape solution nor a homoclinic solution. Remark 2.16 yields that u is a damped solution of problem (2.8), (2.2).

(i) Let $u_0 \in (0, L)$. Since u is oscillatory, then there exists $\theta_1 > 0$ such that $u(\theta_1) = 0$, $u > 0$ on $[0, \theta_1)$. Lemma 2.12 gives that there exists $b_1 > \theta_1$ such that $u(b_1) \in \big(\bar{B}, 0\big)$, $u'(b_1) = 0$, $u'(t) < 0$ for $t \in (0, b_1)$. Multiplying equation (2.8) by pu', integrating this over $[0, b_1]$ and using the Mean value theorem, we get $\xi_1 \in [0, \theta_1]$ and $\xi_2 \in [\theta_1, b_1]$ such that

$$\int_0^{b_1} (p(s)u'(s))'p(s)u'(s)\,\mathrm{d}s = -\int_0^{\theta_1} p(s)q(s)\tilde{f}(u(s))u'(s)\,\mathrm{d}s$$
$$-\int_{\theta_1}^{b_1} p(s)q(s)\tilde{f}(u(s))u'(s)\,\mathrm{d}s = -(pq)(\xi_1)\int_0^{\theta_1} \tilde{f}(u(s))u'(s)\,\mathrm{d}s$$
$$-(pq)(\xi_2)\int_{\theta_1}^{b_1} \tilde{f}(u(s))u'(s)\,\mathrm{d}s.$$

Hence, due to (2.11),

$$0 = \frac{(p(b_1)u'(b_1))^2}{2} - \frac{(p(0)u'(0))^2}{2} = (pq)(\xi_1)\Big(\tilde{F}(u(0)) - \tilde{F}(u(\theta_1))\Big) + (pq)(\xi_2)\Big(\tilde{F}(u(\theta_1)) - \tilde{F}(u(b_1))\Big) \le (pq)(\xi_2)\left(\tilde{F}(u(0)) - \tilde{F}(u(b_1))\right).$$

Since $(pq)(\xi_2) > 0$, we have $\tilde{F}(u(0)) \ge \tilde{F}(u(b_1))$. Since u is oscillatory, then there exists $\theta_2 > b_1$ such that $u(\theta_2) = 0$, $u < 0$ on $[b_1, \theta_2)$. Due to Lemma 2.11, there exists $a_2 > \theta_2$ such that $u(a_2) \in (0, L)$, $u'(a_2) = 0$, $u'(t) > 0$ for $t \in (b_1, a_2)$. By multiplying equation (2.8) by pu', integrating this over $[b_1, a_2]$ and using the Mean value theorem and (2.11), we get $\xi_3 \in [b_1, \theta_2]$ and $\xi_4 \in [\theta_2, a_2]$ such that

$$0 = \frac{(p(a_2)u'(a_2))^2}{2} - \frac{(p(b_1)u'(b_1))^2}{2} = (pq)(\xi_3)\Big(\tilde{F}(u(b_1)) - \tilde{F}(u(\theta_2))\Big) + (pq)(\xi_4)\Big(\tilde{F}(u(\theta_2)) - \tilde{F}(u(a_2))\Big) \le (pq)(\xi_4)\left(\tilde{F}(u(b_1)) - \tilde{F}(u(a_2))\right).$$

Therefore, $\tilde{F}(u(0)) \ge \tilde{F}(u(b_1)) \ge \tilde{F}(u(a_2))$ and since $\tilde{F}$ is increasing on $[0, L]$, we get $u(a_1) := u(0) \ge u(a_2)$. Repeating this procedure, we get the sequences $\{u(a_n)\}$ and $\{u(b_n)\}$ (cf. Definition 3.6) such that $u(a_i) \ge u(a_{i+1})$, $u(b_i) \le u(b_{i+1})$ for each $i \in \mathbb{N}$. So, the sequence $\{u(a_n)\}$ is nonincreasing and $\{u(b_n)\}$ is nondecreasing, that is the sequence of amplitudes $\{u(a_n) - u(b_n)\}$ is nonincreasing.

(ii) Let $u_0 \in \big(\bar{B}, 0\big)$. Since u is oscillatory, then there exists $\theta_1 > 0$ such that $u(\theta_1) = 0$, $u < 0$ on $[0, \theta_1)$. According to Lemma 2.11, there exists $a_1 > \theta_1$ such that $u(a_1) \in (0, L)$, $u'(a_1) = 0$, $u'(t) > 0$ for $t \in (0, a_1)$. Now, we have analogous situation as in part (i), so by similar arguments we derive that the sequence of amplitudes $\{u(a_n) - u(b_n)\}$ is nonincreasing.

(iii) Let $u_0 \in \big(L_0, \bar{B}\big]$. Then there exist $\theta_1 > 0$, $a_1 > \theta_1$ such that $u(\theta_1) = 0$, $u < 0$ on $[0, \theta_1)$, $u'(a_1) = 0$, $u'(t) > 0$ for $t \in (\theta_1, a_1)$. Due to Lemma 2.14, $u(a_1) \in (0, L)$, $u' > 0$ on $(0, a_1)$. We have similar situation as in part (i), so we argue analogously.

We have proved that amplitudes of u are nonincreasing.

Let moreover (2.14) holds. Then we use the analogous arguments with difference that each mean value ξ_i, $i \in \mathbb{N}$ is located in open interval of integration. For instance, let us show that in part (i), $\xi_1 \in (0, \theta_1)$. Since pq is increasing on $[0, \infty)$ and $-\tilde{f}(u(s))u'(s) > 0$ for $s \in (0, \theta_1)$, we get

$$(pq)(\xi_1) \int_0^{\theta_1} -\tilde{f}(u(s))u'(s)\,\mathrm{d}s = \int_0^{\theta_1} -p(s)q(s)\tilde{f}(u(s))u'(s)\,\mathrm{d}s > (pq)(0) \int_0^{\theta_1} -\tilde{f}(u(s))u'(s)\,\mathrm{d}s,$$

which yields $\xi_1 > 0$. Similarly,

$$(pq)(\xi_1) \int_0^{\theta_1} -\tilde{f}(u(s))u'(s)\,\mathrm{d}s = \int_0^{\theta_1} -p(s)q(s)\tilde{f}(u(s))u'(s)\,\mathrm{d}s$$
$$< (pq)(\theta_1) \int_0^{\theta_1} -\tilde{f}(u(s))u'(s)\,\mathrm{d}s,$$

that is $\xi_1 < \theta_1$. Therefore, $\xi_1 \in (0, \theta_1)$. By virtue of (2.14), we obtain

$$0 = \frac{(p(b_1)u'(b_1))^2}{2} - \frac{(p(0)u'(0))^2}{2} = (pq)(\xi_1)\Big(\tilde{F}(u(0)) - \tilde{F}(u(\theta_1))\Big)$$
$$+ (pq)(\xi_2)\Big(\tilde{F}(u(\theta_1)) - \tilde{F}(u(b_1))\Big) < (pq)(\xi_2)\left(\tilde{F}(u(0)) - \tilde{F}(u(b_1))\right)$$

and so, $\tilde{F}(u(0)) > \tilde{F}(u(b_1))$. By analogous procedure as in part (i), we dedive that the sequence $\{u(a_n)\}$ is decreasing, $\{u(b_n)\}$ is increasing and hence, the sequence of amplitudes $\{u(a_n) - u(b_n)\}$ is decreasing.

Since $u_{\sup} < L$, according to (2.9), we have $f(u(t)) = \tilde{f}(u(t))$ for $t \in [0, \infty)$. It means that u is a solution of problem (2.1), (2.2) and u is damped. □

The next lemmas are useful for the proof of oscillatory behaviour of solution of problem (2.1), (2.2).

Lemma 3.8. *Assume that* (2.3), (2.4), (2.6), (2.10), (2.11), (3.2),

$$\liminf_{x \to 0^+} \frac{f(x)}{x} > 0, \tag{3.8}$$

$$p \in C[0, \infty) \cap C^2(0, \infty), \quad p(0) = 0, \tag{3.9}$$

$$p'(t) > 0 \;\; \textit{for } t \in (0, \infty), \qquad \lim_{t \to \infty} \frac{p'(t)}{p(t)} = 0, \quad \limsup_{t \to \infty} \left| \frac{p''(t)}{p'(t)} \right| < \infty, \tag{3.10}$$

$$\liminf_{t \to \infty} \frac{q(t)}{p(t)} > 0 \tag{3.11}$$

hold. Let u be a solution of problem (2.1), (2.2) *with $u_0 \in (0, L)$. Then there exists $\delta_1 > 0$ such that*

$$u(\delta_1) = 0, \quad u'(t) < 0 \;\; \textit{for } t \in (0, \delta_1]. \tag{3.12}$$

Proof. First, let us show that condition (3.11) implies that (3.4) holds. Condition (3.2) with $p' > 0$ on $(0, \infty)$ give

$$\lim_{t \to \infty} \frac{1}{p(t)} = 0. \tag{3.13}$$

According to (3.11), there exist $c > 0$ and $t_1 > 0$ such that $q(t) > cp(t)$ for $t > t_1$. Consequently, by (3.10), (3.13) and the l'Hôspital's rule,

$$\lim_{s \to \infty} \frac{1}{p(s)} \int_1^s q(\tau)\,\mathrm{d}\tau \geq c \lim_{s \to \infty} \frac{1}{p(s)} \int_1^s p(\tau)\,\mathrm{d}\tau = c \lim_{s \to \infty} \frac{p(s)}{p'(s)} = \infty,$$

which implies (3.4).

Now, suppose that such δ_1 satisfying (3.12) does not exist. Then u is positive on $[0,\infty)$. Due to Lemma 2.10, $u'(t)<0$ for $t\in(0,\infty)$. Therefore, u is damped and, by Theorem 3.5, u satisfies (3.5).

We define the function $v(t)=\sqrt{p(t)}\,u(t)$, $t\in[0,\infty)$. Integrating equation (2.1) from 0 to t and using $pu'\in C^1[0,\infty)$ (cf. Definition 2.2), we obtain

$$p(t)u'(t)=-\int_0^t q(s)f(u(s))\,\mathrm{d}s\in C^1[0,\infty). \tag{3.14}$$

Since $p\in C^2(0,\infty)$, we have $\frac{1}{p}\in C^2(0,\infty)$. Thus, using (3.14), we get

$$u'(t)=-\frac{1}{p(t)}\int_0^t q(s)f(u(s))\,\mathrm{d}s\in C^1(0,\infty),$$

which yields $u\in C^2(0,\infty)$ and $v\in C^2(0,\infty)$. In addition,

$$v'(t)=\frac{p'(t)u(t)}{2\sqrt{p(t)}}+\sqrt{p(t)}\,u'(t).$$

Since $p\in C[0,\infty)\cap C^2(0,\infty)$, then we can write equation (2.1) in the equivalent form

$$u''(t)+\frac{p'(t)}{p(t)}u'(t)+\frac{q(t)}{p(t)}f(u(t))=0,\qquad t\in(0,\infty). \tag{3.15}$$

Therefore,

$$\frac{p'(t)u'(t)}{p(t)u(t)}+\frac{u''(t)}{u(t)}=-\frac{q(t)}{p(t)}\frac{f(u(t))}{u(t)},\quad t\in(0,\infty),$$

which gives

$$\begin{aligned} v''(t)&=\frac{1}{2}\frac{p''(t)u(t)\sqrt{p(t)}}{p(t)}+\frac{p'(t)u'(t)}{\sqrt{p(t)}}-\frac{1}{4}\frac{p'^2(t)u(t)}{p^{\frac{3}{2}}}+\sqrt{p(t)}\,u''(t)\\ &=v(t)\left[\frac{1}{2}\frac{p''(t)}{p(t)}-\frac{1}{4}\left(\frac{p'(t)}{p(t)}\right)^2-\frac{q(t)}{p(t)}\frac{f(u(t))}{u(t)}\right],\quad t\in(0,\infty). \end{aligned} \tag{3.16}$$

Due to (3.10), we obtain

$$\lim_{t\to\infty}\left[\frac{1}{2}\frac{p''(t)}{p(t)}-\frac{1}{4}\left(\frac{p'(t)}{p(t)}\right)^2\right]=\frac{1}{2}\lim_{t\to\infty}\frac{p''(t)}{p'(t)}\frac{p'(t)}{p(t)}=0. \tag{3.17}$$

Since u is positive on $(0,\infty)$, conditions (3.5) and (3.8) give

$$\lim_{t\to\infty}\frac{f(u(t))}{u(t)}=\lim_{x\to 0^+}\frac{f(x)}{x}=:\alpha>0.$$

Denote $\lambda := \liminf_{t\to\infty} \frac{q(t)}{p(t)}$. Consequently, by (3.11) and (3.17), it follows that there exists $R > 0$ satisfying

$$-\frac{q(t)}{p(t)}\frac{f(u(t))}{u(t)} < -\frac{\alpha\lambda}{2} \quad \text{for } t \geq R,$$
$$\frac{1}{2}\frac{p''(t)}{p(t)} - \frac{1}{4}\left(\frac{p'(t)}{p(t)}\right)^2 < \frac{\alpha\lambda}{4} \quad \text{for } t \geq R.$$

Thus,

$$\frac{1}{2}\frac{p''(t)}{p(t)} - \frac{1}{4}\left(\frac{p'(t)}{p(t)}\right)^2 - \frac{q(t)}{p(t)}\frac{f(u(t))}{u(t)} < -\omega \quad \text{for } t \geq R, \quad \text{where } \omega = \frac{\alpha\lambda}{4}$$

and, according to (3.16) and $v > 0$ on $(0, \infty)$, we get

$$v''(t) < -\omega v(t) < 0 \quad \text{for } t \geq R. \tag{3.18}$$

Therefore, v' is decreasing on $[R, \infty)$ and there exists limit $\lim_{t\to\infty} v'(t) =: V$. If $V < 0$, then $\lim_{t\to\infty} v(t) = -\infty$, contrary to $v > 0$ on $(0, \infty)$. If $V \geq 0$, then $v' > 0$ on $[R, \infty)$ and $v(t) \geq v(R) > 0$ for $t \in [R, \infty)$. Then (3.18) yields

$$0 > -\omega v(R) \geq -\omega v(t) > v''(t) \quad \text{for } t \in [R, \infty).$$

We get $\lim_{t\to\infty} v'(t) = -\infty$, which contradicts $V \geq 0$. Hence, u has at least one zero on $(0, \infty)$. Let $\delta_1 > 0$ be the first zero of u. Then $u > 0$ on $[0, \delta_1)$ and Lemma 2.10 gives $u' < 0$ on $(0, \delta_1]$. □

For negative starting values u_0, we can prove a dual lemma by similar arguments.

Lemma 3.9. *Assume that (2.3), (2.4), (2.6), (2.10), (2.11), (3.2), (3.9)–(3.11) and*

$$\liminf_{x\to 0^-} \frac{f(x)}{x} > 0 \tag{3.19}$$

hold. Let u be a solution of problem (2.1), (2.2) with $u_0 \in (L_0, 0)$. Then there exists $\theta_1 > 0$ such that

$$u(\theta_1) = 0, \quad u'(t) > 0 \ \textit{ for } t \in (0, \theta_1].$$

If we argue as in the proofs of Lemma 3.8 and Lemma 3.9 working with a_1, A_1 and b_1, B_1 in place of 0, u_0, we get the next lemma.

Lemma 3.10. *Let (2.3), (2.4), (2.6), (2.10), (2.11), (3.2), (3.8)–(3.11) and (3.19) be satisfied and let u be a solution of problem (2.1), (2.2) with $u_0 \in (L_0, 0) \cup (0, L)$.*

I. Assume that there exist $b_1 > 0$ and $B_1 \in (L_0, 0)$ such that

$$u(b_1) = B_1, \quad u'(b_1) = 0. \tag{3.20}$$

Then there exists $\theta > b_1$ such that

$$u(\theta) = 0, \quad u'(t) > 0 \;\; \text{for } t \in (b_1, \theta]. \tag{3.21}$$

II. Assume that there exist $a_1 > 0$ and $A_1 \in (0, L)$ such that

$$u(a_1) = A_1, \quad u'(a_1) = 0. \tag{3.22}$$

Then there exists $\delta > a_1$ such that

$$u(\delta) = 0, \quad u'(t) < 0 \;\; \text{for } t \in (a_1, \delta]. \tag{3.23}$$

3.3 Existence of oscillatory solutions

Here we provide criteria leading to oscillatory solutions of problem (2.1), (2.2). First, we prove the results for CASE I (i.e.(3.2)) and then for CASE II (i.e. (3.3)).

Theorem 3.11 (**Damped solution is oscillatory 1, CASE I**). *Assume that (2.3)–(2.6), (2.10), (2.11), (3.2), (3.8), (3.19) and*

$$\int_1^\infty \ell^2(s) q(s)\,\mathrm{d}s = \infty, \qquad \text{where } \ell(t) = \int_t^\infty \frac{1}{p(s)}\,\mathrm{d}s \tag{3.24}$$

are fulfilled. Let u be a damped solution of problem (2.1), (2.2) with $u_0 \in (L_0, 0) \cup (0, L)$. Then u is oscillatory.

Proof. *Step 1.* We show that (3.24) implies (3.4). Let us put

$$h(t) = \int_1^t \frac{1}{p(s)} \left(\int_1^s q(\tau)\,\mathrm{d}\tau \right) \mathrm{d}s.$$

We accomplish the proof indirectly. Let

$$\lim_{t\to\infty} h(t) =: K < \infty. \tag{3.25}$$

Then integration by parts yields for every $\tau > 1$

$$\int_1^\tau \ell^2(t)q(t)\,\mathrm{d}t \left| \begin{matrix} v(t)=\ell^2(t) & v'(t)=-2\ell(t)\frac{1}{p(t)} \\ w'(t)=q(t) & w(t)=\int_1^t q(s)\,\mathrm{d}s \end{matrix} \right|$$

$$= \ell^2(\tau)\int_1^\tau q(s)\,\mathrm{d}s + 2\int_1^\tau \ell(t)\frac{1}{p(t)}\left(\int_1^t q(s)\,\mathrm{d}s\right)\mathrm{d}t$$

$$= \ell(\tau)\int_\tau^\infty \frac{1}{p(s)}\,\mathrm{d}s\int_1^\tau q(\xi)\,\mathrm{d}\xi + 2\int_1^\tau \ell(t)h'(t)\,\mathrm{d}t \left| \begin{matrix} v(t)=\ell(t) & v'(t)=-\frac{1}{p(t)} \\ w'(t)=h'(t) & w(t)=h(t) \end{matrix} \right|$$

$$= \ell(\tau)\int_\tau^\infty \frac{1}{p(s)}\left(\int_1^\tau q(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s + 2\ell(\tau)h(\tau) + 2\int_1^\tau \frac{1}{p(t)}h(t)\,\mathrm{d}t$$

$$\le \ell(\tau)\int_\tau^\infty \frac{1}{p(s)}\left(\int_1^s q(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s + 2\ell(\tau)h(\tau) + 2h(\tau)\int_1^\tau \frac{1}{p(t)}\,\mathrm{d}t.$$

Since (3.2) yields $\lim_{\tau\to\infty}\ell(\tau)=0$, we get, by (3.25) for $\tau\to\infty$,

$$\int_1^\infty \ell^2(t)q(t)\,\mathrm{d}t \le \lim_{\tau\to\infty}\ell(\tau)\int_\tau^\infty \frac{1}{p(s)}\left(\int_1^s q(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s$$
$$+ 2\lim_{\tau\to\infty}\ell(\tau)K + 2K\ell(1) = 2K\ell(1) < \infty$$

and so, (3.24) is not fulfilled. Therefore, we proved that (3.24) gives (3.4).

Step 2. Let u be a damped solution which is nonoscillatory. By Step 1, (3.4) holds and, by Lemma 3.5, we have $\lim_{t\to\infty}u(t)=0$. Thus, due to (3.8) and (3.19), we get

$$\liminf_{t\to\infty}\frac{f(u(t))}{u(t)} > 0.$$

Consequently, there exist $\alpha>0$ and $t_1>0$ such that

$$u(t)\ne 0, \quad \frac{f(u(t))}{u(t)} \ge \alpha, \quad t\in[t_1,\infty). \tag{3.26}$$

Put $\rho(t)=\frac{p(t)u'(t)}{u(t)}$ for $t\ge t_1$. By (2.1) and (3.26), we have

$$\rho'(t) = -q(t)\frac{f(u(t))}{u(t)} - \frac{1}{p(t)}\rho^2(t) \le -\alpha q(t) - \frac{1}{p(t)}\rho^2(t), \quad t\ge t_1.$$

Multiplying this inequality by ℓ^2 and integrating from t_1 to t, we obtain

$$\int_{t_1}^t \ell^2(s)\rho'(s)\,\mathrm{d}s \le -\alpha\int_{t_1}^t \ell^2(s)q(s)\,\mathrm{d}s - \int_{t_1}^t \frac{1}{p(s)}\ell^2(s)\rho^2(s)\,\mathrm{d}s, \quad t\ge t_1.$$

Integrating left side by parts, we get

$$\int_{t_1}^t \ell^2(s)\rho'(s)\,\mathrm{d}s = \ell^2(t)\rho(t) - \ell^2(t_1)\rho(t_1) + 2\int_{t_1}^t \frac{1}{p(s)}\ell(s)\rho(s)\,\mathrm{d}s$$

and hence,

$$\ell^2(t)\rho(t) - \ell^2(t_1)\rho(t_1) \le -\alpha \int_{t_1}^{t} \ell^2(s)q(s)\,\mathrm{d}s$$
$$- \int_{t_1}^{t} \frac{1}{p(s)} \left(\ell^2(s)\rho^2(s) + 2\ell(s)\rho(s) + 1\right)\,\mathrm{d}s + \int_{t_1}^{t} \frac{1}{p(s)}\,\mathrm{d}s, \quad t \in [t_1, \infty).$$

Further,

$$\ell(t)(\ell(t)\rho(t) + 1) - \ell(t) \le \ell^2(t_1)\rho(t_1) - \alpha \int_{t_1}^{t} \ell^2(s)q(s)\,\mathrm{d}s$$
$$- \int_{t_1}^{t} \frac{1}{p(s)}(\ell(s)\rho(s) + 1)^2\,\mathrm{d}s + \int_{t_1}^{\infty} \frac{1}{p(s)}\,\mathrm{d}s, \quad t \in [t_1, \infty),$$

and finally,

$$\ell(t)(\ell(t)\rho(t) + 1) \le \ell(t_1)(\ell(t_1)\rho(t_1) + 1) - \alpha \int_{t_1}^{t} \ell^2(s)q(s)\,\mathrm{d}s$$
$$- \int_{t_1}^{t} \frac{1}{p(s)}(\ell(s)\rho(s) + 1)^2\,\mathrm{d}s, \quad t \in [t_1, \infty).$$

By (3.24), there exist $t_0 \ge t_1$ such that

$$\int_{t_1}^{t} \ell^2(s)q(s)\,\mathrm{d}s \ge \frac{1}{\alpha}\ell(t_1)(\ell(t_1)\rho(t_1) + 1), \quad t \in [t_0, \infty),$$

and hence, we get

$$0 < \int_{t_1}^{t} \frac{1}{p(s)}(\ell(s)\rho(s) + 1)^2\,\mathrm{d}s \le -\ell(t)(\ell(t)\rho(t) + 1), \quad t \in [t_0, \infty). \tag{3.27}$$

Put

$$x(t) = \int_{t_1}^{t} \frac{1}{p(s)}(\ell(s)\rho(s) + 1)^2\,\mathrm{d}s, \quad t \in [t_0, \infty).$$

Then

$$x'(t) = \frac{1}{p(t)}(\ell(t)\rho(t) + 1)^2, \quad t \in [t_0, \infty)$$

and, according to (3.27),

$$x^2(t) \le \ell^2(t)(\ell(t)\rho(t) + 1)^2, \quad t \in [t_0, \infty).$$

Therefore, x fulfils the differential inequality

$$x^2(t) \le p(t)\ell^2(t)x'(t), \quad t \in [t_0, \infty). \tag{3.28}$$

Since

$$\left(-\frac{1}{x(s)}\right)' = \frac{x'(s)}{x^2(s)}, \quad s \in [t_1, \infty),$$

we obtain, by (3.28),

$$\int_{t_1}^{t} \frac{x'(s)}{x^2(s)}\,\mathrm{d}s \geq \int_{t_1}^{t} \frac{1}{p(s)\ell^2(s)}\,\mathrm{d}s, \quad t \in [t_1, \infty),$$

$$\frac{1}{x(t_1)} - \frac{1}{x(t)} \geq \frac{1}{\ell(t)} - \frac{1}{\ell(t_1)}, \quad t \in [t_1, \infty),$$

$$\frac{1}{\ell(t)} < \frac{1}{x(t_1)} + \frac{1}{\ell(t_1)}, \quad t \in [t_1, \infty).$$

Letting $t \to \infty$ and using (3.2), we get

$$\infty = \lim_{t\to\infty} \frac{1}{\ell(t)} \leq \frac{1}{x(t_1)} + \frac{1}{\ell(t_1)} < \infty.$$

This contradiction yields that u is oscillatory. □

If we replace assumptions (2.5) and (3.24) by assumptions (3.9)–(3.11), we get a modification of Theorem 3.11.

Theorem 3.12 (**Damped solution is oscillatory 2, CASE I**). *Assume that (2.3), (2.4), (2.6), (2.10), (2.11), (3.2), (3.8)–(3.11) and (3.19) hold. Let u be a damped solution of problem (2.1), (2.2) with $u_0 \in (L_0, 0) \cup (0, L)$. Then u is oscillatory.*

Proof. Let u be a damped solution of problem (2.1), (2.2) with $u_0 \in (0, L)$. By (2.26), we can find $L_1 \in (0, L)$ such that $\bar{B} < u(t) \leq L_1$ for $t \in [0, \infty)$. In the proof of Lemma 3.8, it was shown that condition (3.11) implies (3.4).

Step 1. Lemma 3.8 yields $\delta_1 > 0$ satisfying (3.12). Therefore, there exists a maximal interval (δ_1, b_1) such that $u' < 0$. If $b_1 = \infty$, then u is eventually negative and decreasing. On the other hand, due to Theorem 3.5, u satisfies (3.5), which is not possible. Hence, $b_1 < \infty$ and there exists $B_1 \in (\bar{B}, 0)$ such that (3.20) holds. Lemma 3.10 yields $\theta_1 > b_1$ satisfying (3.21) with $\theta = \theta_1$. Thus, u has just one negative local minimum $B_1 = u(b_1)$ between its first zero δ_1 and second zero θ_1.

Step 2. By virtue of 3.21, there exists a maximal interval (θ_1, a_1), where $u' > 0$. If $a_1 = \infty$, then u is eventually positive and increasing. On the other hand, by Theorem 3.5, u satisfies (3.5), a contradiction. Therefore, $a_1 < \infty$ and there exists $A_1 \in (0, L)$ such that (3.22) holds. Lemma 3.10 gives $\delta_2 > a_1$ satisfying (3.23) with $\delta = \delta_2$. Hence, u has just one positive local maximum $A_1 = u(a_1)$ between its second zero θ_1 and third zero δ_2.

Step 3. We can continue as in Step 1 and Step 2 and get the sequence

$$0 < \delta_1 < b_1 < \theta_1 < a_1 < \ldots < \delta_n < b_n < \theta_n < a_n < \ldots,$$

where $B_n = u(b_n)$ is a strict unique negative local minimum of u in (δ_n, θ_n) and $A_n = u(a_n)$ is a strict unique positive local maximum of u in (θ_n, δ_{n+1}), $n \in \mathbb{N}$. Since $\{\delta_n\}_{n=1}^{\infty}$ and $\{\theta_n\}_{n=1}^{\infty}$ are unbounded sequences of zeros of u, then u is oscillatory.

For $u_0 \in (L_0, 0)$, we proceed analogously. $\square$

Remark 3.13. Let us put $p(t) = q(t) = t^4$. Then (2.5), (3.9) and (3.10) are fulfilled. Check conditions (3.11) and (3.24).

$$\liminf_{t\to\infty} \frac{q(t)}{p(t)} = \liminf_{t\to\infty} 1 = 1 > 0,$$
$$\ell(s) = \int_s^\infty \frac{1}{\tau^4}\,\mathrm{d}\tau = \frac{1}{3s^3}, \quad \int_1^\infty \ell^2(s)q(s)\,\mathrm{d}s = \int_1^\infty \frac{1}{9s^2}\,\mathrm{d}s = \frac{1}{9} < \infty.$$

Thus, (3.11) is valid, while (3.24) is not valid.

Now, we put $p(t) = t^2$, $q(t) = t$. Then (2.5), (3.9) and (3.10) hold. Check (3.11) and (3.24) again.

$$\liminf_{t\to\infty} \frac{q(t)}{p(t)} = \liminf_{t\to\infty} \frac{1}{t} = 0,$$
$$\ell(s) = \int_s^\infty \frac{1}{\tau^2}\,\mathrm{d}\tau = \frac{1}{s}, \quad \int_1^\infty \ell^2(s)q(s)\,\mathrm{d}s = \int_1^\infty \frac{1}{s}\,\mathrm{d}s = \infty.$$

So, (3.11) is not valid, while (3.24) is valid. Therefore, conditions (3.11) and (3.24) leading to oscillatory solutions are in general different and incomparable.

Now, we provide a criterion for oscillatory solutions in CASE II (i.e. (3.3)).

Theorem 3.14 (Damped solution is oscillatory 3, CASE II). *Assume that* (2.3)–(2.6), (2.10), (2.11), (3.3) *and*

$$\int_1^\infty q(s)\,\mathrm{d}s = \infty \tag{3.29}$$

are fulfilled. Let u be a damped solution of problem (2.1), (2.2) *with $u_0 \in (L_0, 0) \cup (0, L)$. Then u is oscillatory.*

Proof. *Step 1.* Let u be damped solution of problem (2.1), (2.2) which is eventually positive. Then there exist $t_0 \ge 1$ such that $u(t) > 0$ for $t \in [t_0, \infty)$. Assume that $u' > 0$ on $[t_0, \infty)$. Then u is increasing on $[t_0, \infty)$ and there exists a limit $\lim_{t\to\infty} u(t) =: \ell_0 \in (u(t_0), L)$. Put

$$m_0 := \min\{f(x) \colon x \in [u(t_0), \ell_0]\} > 0.$$

By (2.1), we have

$$(p(t)u'(t))' = -q(t)f(u(t)) \le -q(t)m_0, \quad t \in [t_0, \infty).$$

Integrating this inequality over $[t_0, t]$ and dividing by $p(t)$, we get

$$\begin{aligned} u'(t) &\le \frac{p(t_0)u'(t_0)}{p(t)} - \frac{m_0}{p(t)} \int_{t_0}^{t} q(s)\,\mathrm{d}s, \quad t \in [t_0, \infty), \\ 0 < u(t) &\le u(t_0) + p(t_0)u'(t_0) \int_{t_0}^{t} \frac{1}{p(s)}\,\mathrm{d}s \\ &\quad - m_0 \int_{t_0}^{t} \frac{1}{p(s)} \left(\int_{t_0}^{s} q(\xi)\,\mathrm{d}\xi \right) \mathrm{d}s, \quad t \in [t_0, \infty). \end{aligned}$$

We divide this inequality by $m_0 \int_{t_0}^{t} \frac{1}{p(s)}\,\mathrm{d}s$ and get

$$\frac{\int_{t_0}^{t} \frac{1}{p(s)} \left(\int_{t_0}^{s} q(\xi)\,\mathrm{d}\xi \right) \mathrm{d}s}{\int_{t_0}^{t} \frac{1}{p(s)}\,\mathrm{d}s} < \frac{u(t_0)}{m_0 \int_{t_0}^{t} \frac{1}{p(s)}\,\mathrm{d}s} + \frac{p(t_0)u'(t_0)}{m_0}, \quad t \in [t_0, \infty),$$

$$\lim_{t\to\infty} \frac{\int_{t_0}^{t} \frac{1}{p(s)} \left(\int_{t_0}^{s} q(\xi)\,\mathrm{d}\xi \right) \mathrm{d}s}{\int_{t_0}^{t} \frac{1}{p(s)}\,\mathrm{d}s} = \lim_{t\to\infty} \frac{\frac{1}{p(t)} \int_{t_0}^{t} q(\xi)\,\mathrm{d}\xi}{\frac{1}{p(t)}} = \lim_{t\to\infty} \int_{t_0}^{t} q(\xi)\,\mathrm{d}\xi = \infty.$$

On the other hand,

$$\lim_{t\to\infty} \frac{u(t_0)}{m_0 \int_{t_0}^{t} \frac{1}{p(s)}\,\mathrm{d}s} + \frac{p(t_0)u'(t_0)}{m_0} = \frac{p(t_0)u'(t_0)}{m_0} < \infty.$$

We have $\infty \le \frac{p(t_0)u'(t_0)}{m_0} < \infty$, a contradiction. Therefore, there exists $t_1 \ge t_0$ such that $u(t_1) \in (0, L)$, $u'(t_1) \le 0$. Since u is eventually positive, equation (2.1) together with (2.4), (2.6) yields that pu' is decreasing and, by $p(t_1)u'(t_1) \le 0$, we have that pu' is negative on (t_1, ∞). Therefore, there exist $K > 0$ and $t_2 > t_1$ such that

$$\begin{aligned} pu'(t) &< -K, \quad t \in (t_2, \infty), \\ u'(t) &< -K \frac{1}{p(t)}, \quad t \in (t_2, \infty). \end{aligned}$$

By integrating this inequality from t_2 to t, we obtain

$$u(t) - u(t_2) < -K \int_{t_2}^{t} \frac{\mathrm{d}s}{p(s)}.$$

Letting $t \to \infty$ and using (3.3), we get

$$\lim_{t\to\infty} u(t) \le u(t_2) - K \int_{t_2}^{\infty} \frac{\mathrm{d}s}{p(s)} = -\infty,$$

contrary to the assumption that u is eventually positive.

Step 2. Let u be damped solution of problem (2.1), (2.2) which is eventually negative. Then there exists $t_0 \geq 1$ such that $u(t) < 0$ for $t \in [t_0, \infty)$. We show that $u(t) > L_0$ for $t \in [t_0, \infty)$. If $u(t) < 0$ for $t \in [0, \infty)$, then, by Lemma 2.9, $u(t) \in (L_0, 0)$, $u'(t) > 0$ for $t \in (0, \infty)$. Assume that there exist $a \geq 0$, $\theta \in (a, t_0)$ such that u fulfils (2.17), $u(t) < 0$ for $t \in (\theta, \infty)$. Due to Lemma 2.12, either (2.18) or (2.19) holds. If (2.18) is valid, then $u(t) \in \left(\bar{B}, 0\right)$ for $t \in (\theta, \infty)$. If (2.19) is fulfilled, then, by Lemma 2.9, $u(t) \in \left(\bar{B}, 0\right)$ for $t \in (\theta, \infty)$. We have shown that $u(t) \in (L_0, 0)$ for $t \in [t_0, \infty)$. Moreover, solution u is increasing in a neighbourhood of ∞ and that there exists $\lim_{t\to\infty} u(t) > L_0$. Analogously as in Step 1, we can derive that u cannot be eventually negative.

Consequently, u is oscillatory. □

If we combine assumptions from Theorem 3.1 and Theorem 3.7 with assumptions of Theorem 3.11 or Theorem 3.12 or Theorem 3.14, we get the main results about existence of oscillatory solutions of problem (2.1), (2.2).

Theorem 3.15 (Existence of oscillatory solutions 1, CASE I). *Assume that (2.3)–(2.6), (2.10), (2.11), (2.20), (2.23), (3.2), (3.8), (3.19) and (3.24) are fulfilled. Then, for each $u_0 \in \left(\bar{B}, 0\right) \cup (0, L)$, problem (2.1), (2.2) has a solution u. This solution u is damped, oscillatory and has nonincreasing amplitudes.*

Theorem 3.16 (Existence of oscillatory solutions 2, CASE I). *Assume that (2.3)–(2.6), (2.10), (2.11), (2.20), (2.23), (3.2), (3.8)–(3.11) and (3.19) are fulfilled. Then, for each $u_0 \in \left(\bar{B}, 0\right) \cup (0, L)$, problem (2.1), (2.2) has a solution u. This solution u is damped, oscillatory and has nonincreasing amplitudes.*

Theorem 3.17 (Existence of oscillatory solutions 3, CASE II). *Assume that (2.3)–(2.6), (2.10), (2.11), (2.20), (2.23), (3.3) and (3.29) are fulfilled. Then, for each $u_0 \in \left(\bar{B}, 0\right) \cup (0, L)$, problem (2.1), (2.2) has a solution u. This solution u is damped, oscillatory and has nonincreasing amplitudes.*

Remark 3.18. If moreover (2.14) is fulfilled, then, by Remark 3.2, the assertion of Theorems 3.15, 3.16 and 3.17 holds also for $u_0 = \bar{B}$ and, due to Theorem 3.7, the amplitudes of u are decreasing.

3.4 Examples

Here we show examples, where the functions p, q and f guarantee the existence of oscillatory solutions of problem (2.1), (2.2).

Example 3.19. Here we illustrate Theorem 3.15. Consider the IVP

$$\begin{gathered} (t^\alpha u'(t))' + t^\beta f(u(t)) = 0, \\ u(0) = u_0 \in [-2, 1], \quad u'(0) = 0, \end{gathered} \tag{3.30}$$

where

$$f(x) = \begin{cases} -|x|^a(x+2), & x \in [-2,0]\,, \\ x^b(1-x), & x \in [0,1]\,. \end{cases}$$

Here

$$p(t) = t^\alpha, \quad q(t) = t^\beta, \quad t \in [0,\infty)\,.$$

Assume that $\alpha > 0$, $\beta \geq 0$ and $0 < a \leq b$. By Example 2.18, conditions (2.5), (2.6), (2.11) and (2.14) hold and (2.23) holds for $\beta > \alpha - 1$. Since

$$\int_1^\infty \frac{1}{s^\alpha}\,\mathrm{d}s < \infty \quad \text{provided } \alpha > 1, \tag{3.31}$$

condition (3.2) is valid for $\alpha > 1$. So, we put $\alpha > 1$ and check (3.24).

$$\ell(t) = \int_t^\infty \frac{1}{s^\alpha}\,\mathrm{d}s = \frac{t^{1-\alpha}}{\alpha-1}\,,$$

$$\int_1^\infty \ell^2(s)q(s)\,\mathrm{d}s = \frac{1}{(\alpha-1)^2}\int_1^\infty s^{2-2\alpha+\beta}\,\mathrm{d}s = \infty \quad \text{if } 2-2\alpha+\beta \geq -1,$$

which yields the validity of (3.24) for $\beta \geq 2\alpha - 3$. We obtained the inequalities

$$\alpha > 1, \quad \beta \geq 0, \quad \beta > \alpha - 1, \quad \beta \geq 2\alpha - 3. \tag{3.32}$$

Since the implications

$$\alpha \in (1,2] \Rightarrow 2\alpha - 3 \leq \alpha - 1, \qquad \alpha > 2 \Rightarrow \alpha - 1 < 2\alpha - 3 \tag{3.33}$$

are valid, according to (3.32), we have satisfied all previous conditions if

$$\alpha \in (1,2]\,,\ \beta > \alpha - 1 \quad \text{or} \quad \alpha > 2,\ \beta \geq 2\alpha - 3. \tag{3.34}$$

The function f is locally Lipschitz continuous on $[L_0, L] \setminus \{0\}$, $L_0 = -2$, $L = 1$, $f(L_0) = f(0) = f(L) = 0$, $xf(x) > 0$ for $x \in (L_0, L) \setminus \{0\}$. Therefore, (2.3), (2.4) and (2.20) are satisfied. Since $0 < L < -L_0$ and $a \leq b$, we obtain, similarly as in Example 2.18, that $\tilde{F}(L) < \tilde{F}(L_0)$ and thus, there exists $\bar{B} \in (L_0, 0)$ such that $\tilde{F}\left(\bar{B}\right) = \tilde{F}(L)$, which yields (2.10). Further,

$$\liminf_{x\to 0^+} \frac{x^b(1-x)}{x} = \lim_{x\to 0^+} x^{b-1}(1-x) > 0 \quad \text{if } b \leq 1,$$

$$\liminf_{x\to 0^-} \frac{-|x|^a(x+2)}{x} > 0 \quad \text{if } a \leq 1,$$

which means that (3.8) and (3.19) are valid for $0 < a \leq b \leq 1$.

To summarize, if

$$0 < a \leq b \leq 1 \ \text{ and } \ (3.34) \text{ is valid,}$$

then we have fulfilled all assumptions of Theorem 3.15 and Remark 3.18. Therefore, for each $u_0 \in \left[\bar{B}, 0\right) \cup (0,1)$, the IVP (3.30) has a solution u, u is damped, oscillatory, and has decreasing amplitudes. If moreover $a = b = 1$, then f is Lipschitz continuous on $[L_0, L]$ and then Theorem 2.17 yields the uniqueness of such solution u.

Example 3.20. We illustrate Theorem 3.16. Let us consider the IVP

$$\begin{gathered} u''(t) + \frac{2}{t}u'(t) + tu(t)(1-u(t))(u(t)+2) = 0, \\ u(0) = u_0 \in [-2, 1], \quad u'(0) = 0. \end{gathered} \tag{3.35}$$

According to (3.15), we have

$$p(t) = t^2, \quad q(t) = t^3, \quad t \in [0, \infty), \qquad f(x) = x(1-x)(x+2), \ x \in [-2, 1].$$

By Example 2.18 (where now $\alpha = 2$ and $\beta = 3$), we know that (2.5), (2.6), (2.11), (2.14) and (2.23) are satisfied. Condition (3.2) is valid too, because

$$\int_1^\infty \frac{1}{s^2}\,\mathrm{d}s = 1 < \infty.$$

In addition,

$$\begin{gathered} p \in C[0,\infty) \cap C^2(0,\infty), \quad p(0) = 0, \quad p'(t) > 0 \text{ for } t \in (0,\infty), \\ \lim_{t\to\infty} \frac{p'(t)}{p(t)} = \lim_{t\to\infty} \frac{2t}{t^2} = \lim_{t\to\infty} \frac{2}{t} = 0, \\ \limsup_{t\to\infty} \left|\frac{p''(t)}{p'(t)}\right| = \lim_{t\to\infty} \frac{2}{2t} = 0 < \infty, \\ \liminf_{t\to\infty} \frac{q(t)}{p(t)} = \lim_{t\to\infty} \frac{t^3}{t^2} = \lim_{t\to\infty} t = \infty > 0, \end{gathered}$$

so we have fulfilled (3.9)–(3.11).

The function f is Lipschitz continuous on $[L_0, L]$, $L_0 = -2$, $L = 1$, $xf(x) > 0$ for $x \in (L_0, L) \setminus \{0\}$, $f(L_0) = f(0) = f(L) = 0$. Hence, (2.3), (2.4), (2.28) and consequently, (2.20) are satisfied. We see that

$$\liminf_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0}(1-x)(x+2) = 2 > 0,$$

that is (3.8) and (3.19) hold. Further, for $x \in [-2, 1]$, we have

$$\begin{gathered} \tilde{F}(x) = \int_0^x z(1-z)(z+2)\,\mathrm{d}z = \int_0^x (-z^3 - z^2 + 2z)\,\mathrm{d}z = -\frac{x^4}{4} - \frac{x^3}{3} + x^2, \\ \tilde{F}(-2) = \frac{8}{3}, \quad \tilde{F}(1) = \frac{5}{12}. \end{gathered}$$

Since $\tilde{F}(L_0) > \tilde{F}(L)$, there exists $\bar{B} \in (-2, 0)$ such that $\tilde{F}\left(\bar{B}\right) = \tilde{F}(L)$, that is (2.10) holds. Let us find such $\bar{B}$.

$$\frac{\tilde{F}(x) - \tilde{F}\left(\bar{B}\right)}{(x-1)^2} = \frac{-\frac{x^4}{4} - \frac{x^3}{3} + x^2 - \frac{5}{12}}{(x-1)^2} = -\frac{x^2}{4} - \frac{5x}{6} - \frac{5}{12}$$
$$= -\frac{1}{4}\left(x + \frac{\sqrt{10}+5}{3}\right)\left(x - \frac{\sqrt{10}-5}{3}\right), \quad x \in [-2, 1].$$

So, the polynomial $\tilde{F}(x) - \tilde{F}\left(\bar{B}\right)$ has the roots

$$x_1 = 1, \quad x_2 = -\frac{\sqrt{10}+5}{3} \approx -2.72, \quad x_3 = \frac{\sqrt{10}-5}{3} \approx -0.61 \in (-2, 0).$$

Therefore, $\bar{B} = \frac{\sqrt{10}-5}{3}$.

To summarize, we have satisfied all assumptions of Theorems 2.17, 3.16 and Remark 3.18. Thus, for each $u_0 \in \left[\frac{\sqrt{10}-5}{3}, 0\right) \cup (0, 1)$, problem (3.35) has a unique solution u. This solution u is damped, oscillatory and has decreasing amplitudes.

Note that we can use Theorem 3.15 here, too, since according to Example 3.19, for $\alpha = 2$ and $\beta = 3$, condition (3.24) is valid.

Example 3.21. Let us illustrate Theorems 3.15 and 3.16 once more. We show that none of these two theorems is included in the second one. Consider the IVP

$$\begin{aligned} &(t^\alpha u'(t))' + t^\beta f(u(t)) = 0, \\ &u(0) = u_0 \in [-2, 1], \quad u'(0) = 0, \end{aligned} \tag{3.36}$$

where

$$f(x) = \begin{cases} x(1-x)(x+2) & \text{for } x \le 0, \\ \frac{5}{7}x(1-x)(x+3) & \text{for } x > 0. \end{cases}$$

Here

$$p(t) = t^\alpha, \quad q(t) = t^\beta, \quad t \in [0, \infty).$$

Assume that $\alpha > 0$ and $\beta \ge 0$. By Example 2.18, conditions (2.5), (2.6), (2.11) and (2.14) hold and (2.23) is fulfilled for $\beta > \alpha - 1$. Due to (3.31), condition (3.2) is valid for $\alpha > 1$. Further,

$$p \in C[0, \infty) \cap C^2(0, \infty), \quad p(0) = 0, \quad p'(t) > 0 \text{ for } t \in (0, \infty),$$
$$\lim_{t\to\infty} \frac{p'(t)}{p(t)} = \lim_{t\to\infty} \frac{\alpha t^{\alpha-1}}{t^\alpha} = \lim_{t\to\infty} \frac{\alpha}{t} = 0,$$
$$\limsup_{t\to\infty} \left|\frac{p''(t)}{p'(t)}\right| = \lim_{t\to\infty} \frac{\alpha|\alpha-1|t^{\alpha-2}}{\alpha t^{\alpha-1}} = \lim_{t\to\infty} \frac{|\alpha-1|}{t} = 0 < \infty,$$
$$\liminf_{t\to\infty} \frac{q(t)}{p(t)} = \lim_{t\to\infty} \frac{t^\beta}{t^\alpha} = \lim_{t\to\infty} t^{\beta-\alpha} > 0 \quad \text{if } \beta \ge \alpha,$$

which means that (3.9) and (3.10) hold and (3.11) is valid for $\beta \geq \alpha$.

The function f is Lipschitz continuous on $[L_0, L]$, $L_0 = -2$, $L = 1$, $xf(x) > 0$ for $x \in (L_0, L) \setminus \{0\}$, $f(L_0) = f(0) = f(L) = 0$. Thus, (2.3), (2.4), (2.28) and consequently, (2.20) are fulfilled. Moreover,

$$\liminf_{x \to 0^-} \frac{x(1-x)(x+2)}{x} = \lim_{x \to 0^-} (1-x)(x+2) = 2 > 0,$$
$$\liminf_{x \to 0^+} \frac{\frac{5}{7}x(1-x)(x+3)}{x} = \lim_{x \to 0^+} \frac{5}{7}(1-x)(x+3) = \frac{15}{7} > 0,$$

that is (3.8) and (3.19) hold. Since L_0, L and the function f on $[L_0, 0]$ are the same as in Example 3.20, we obtain also the same $\bar{B} = \frac{\sqrt{10}-5}{3}$ satisfying (2.10) as in Example 3.20.

To sum up, provided that

$$\alpha > 1 \quad \text{and} \quad \beta \geq \alpha, \tag{3.37}$$

we have fulfilled all assumptions of Theorems 2.17, 3.16 and Remark 3.18. Therefore, for each $u_0 \in \left[\frac{\sqrt{10}-5}{3}, 0\right) \cup (0, 1)$, the IVP (3.36) has a unique solution u. The solution u is damped, oscillatory and has decreasing amplitudes. Further, by Example 3.19, if (3.34) holds, then (3.24) is satisfied and Theorem 3.15 is applicable here.

For example, if we choose $\alpha = 5$, $\beta = 6$, then (3.37) holds, whereas (3.34) falls. So, here we can use Theorem 3.16 unlike Theorem 3.15. On the other hand, by choosing $\alpha = \frac{5}{2}$, $\beta = 2$ we have fulfilled (3.34), while (3.37) does not hold. Hence, Theorem 3.15 is applicable here unlike Theorem 3.16.

Example 3.22. Now, we illustrate Theorem 3.17. Let us consider the IVP

$$\begin{gathered} (t^\alpha u'(t))' + t^\beta f(u(t)) = 0, \\ u(0) = u_0 \in \left[-2 - 2^\lambda, 2\right], \quad u'(0) = 0, \end{gathered} \tag{3.38}$$

where

$$f(x) = \begin{cases} -(x + 2^\lambda + 2) & \text{for } x \leq -2, \\ |x|^\lambda \operatorname{sgn} x & \text{for } x \in (-2, 1), \\ 2 - x & \text{for } x \geq 1. \end{cases}$$

Here

$$p(t) = t^\alpha, \quad q(t) = t^\beta, \quad t \in [0, \infty).$$

Assume $\alpha > 0$, $\beta \geq 0$ and $\lambda > 0$. Example 2.18 shows that (2.5), (2.6), (2.11) and (2.14) hold and (2.23) is valid for $\beta > \alpha - 1$. Furthermore,

$$\int_1^\infty s^\beta \, \mathrm{d}s = \infty, \qquad \int_1^\infty \frac{1}{s^\alpha} \, \mathrm{d}s = \infty \text{ if } \alpha \leq 1,$$

which means that (3.29) holds and (3.3) is valid for $\alpha \le 1$.

The function f is continuous on $\mathbb{R}$, locally Lipschitz continuous on $\mathbb{R} \setminus \{0\}$, $L_0 = -2^\lambda - 2 < -3$, $L = 2$, $f(L_0) = f(0) = f(L) = 0$, $xf(x) > 0$ for $x \in (L_0, L) \setminus \{0\}$. Thus, (2.3), (2.4) and (2.20) are satisfied. Further,

$$\tilde{F}(L_0) = \int_0^{-2^\lambda - 2} -\left(z + 2^\lambda + 2\right) \mathrm{d}z = \frac{\left(2^\lambda + 2\right)^2}{2}, \quad \tilde{F}(L) = \int_0^2 (2 - z)\, \mathrm{d}z = 2.$$

Since $\tilde{F}(L_0) > \tilde{F}(L)$, there exists $\bar{B} \in (-2^\lambda - 2\,, 0)$ such that $\tilde{F}\left(\bar{B}\right) = \tilde{F}(L)$, that is (2.10) holds.

To summarize, if

$$\alpha \in (0, 1)\,, \;\; \beta \ge 0, \;\; \beta > \alpha - 1 \;\text{ and }\; \lambda > 0,$$

then we have satisfied all assumptions of Theorem 3.17 and Remark 3.18. Therefore, for each $u_0 \in \left[\bar{B}, 0\right) \cup (0, 2)$, the IVP (3.38) has a solution u. This solution u is damped, oscillatory and has decreasing amplitudes. If in addition $\lambda \ge 1$, then f is Lipschitz continuous on $[L_0, L]$, that is (2.28) is valid. Then Theorem 2.17 gives the uniqueness of such solution u.

In the following example, we illustrate Theorem 3.17 and show that the functions p and q can be bounded.

Example 3.23. Consider the IVP

$$\begin{gathered}(\arctan t\, u'(t))' + \frac{t^2}{t^2+1}\, k|u(t)|^\gamma \operatorname{sgn}(u(t))(u(t) - L_0)(L - u(t)) = 0, \\ u(0) = u_0 \in [-4, 2]\,, \quad u'(0) = 0,\end{gathered} \tag{3.39}$$

We have equation (2.1) with

$$\begin{gathered}p(t) = \arctan t, \quad q(t) = \frac{t^2}{t^2+1}\,, \quad t \in [0, \infty), \\ f(x) = k|x|^\gamma \operatorname{sgn} x (x - L_0)(L - x), \quad x \in [L_0, L].\end{gathered}$$

Assume that $0 < L < -L_0$, $\gamma > 0$ and $k > 0$. The functions p and q are continuous on $[0, \infty)$, positive on $(0, \infty)$ and $p(0) = 0$. So, (2.5) and (2.6) are satisfied. Further, pq is increasing on $[0, \infty)$, which yields that (2.14) and consequently, (2.11) hold. In addition,

$$\begin{aligned}&\lim_{t\to 0^+} \frac{1}{\arctan t} \int_0^t \frac{s^2}{s^2+1}\, \mathrm{d}s = \lim_{t\to 0^+} \frac{1}{\arctan t} \int_0^t \left(1 - \frac{1}{s^2+1}\right) \mathrm{d}s \\ &= \lim_{t\to 0^+} \frac{1}{\arctan t}(t - \arctan t) = \lim_{t\to 0^+} \frac{t}{\arctan t} - 1 = \lim_{t\to 0^+} \frac{1}{\frac{1}{t^2+1}} - 1 = 1 - 1 = 0,\end{aligned}$$

$$\int_1^\infty \frac{s^2}{s^2+1}\,\mathrm{d}s = \int_1^\infty \left(1 - \frac{1}{s^2+1}\right)\mathrm{d}s = \lim_{t\to\infty}(t - \arctan t) - 1 + \frac{\pi}{4} = \infty,$$
$$\lim_{s\to\infty} \frac{1}{\arctan s} = \frac{1}{\frac{\pi}{2}} > 0 \quad \Longrightarrow \quad \int_1^\infty \frac{1}{\arctan s}\,\mathrm{d}s = \infty,$$

that is (2.23), (3.3) and (3.29) hold. Example 2.18 shows that the function f fulfils (2.3), (2.4), (2.10) and (2.20).

To sum up, provided that

$$0 < L < -L_0, \ \gamma > 0 \ \text{ and } \ k > 0,$$

we have satisfieded all assumptions of Theorem 3.17 and Remark 3.18. Therefore, for each $u_0 \in \left[\bar{B}, 0\right) \cup (0, L)$, problem (3.39) has a solution u. This solution u is damped, oscillatory and has decreasing amplitudes. If moreover $\gamma \geq 1$, then f is Lipschitz continuous on $[L_0, L]$ which means that (2.28) holds. Then Theorem 2.17 yields the uniqueness of such solution u.

4 Escape and homoclinic solutions of the problem without ϕ-Laplacian

4.1 Properties of escape and homoclinic solutions

In this section, we prove some important properties of escape and homoclinic solutions. In order to obtain the existence results, the monotonicity of escape and homoclinic solutions is needed, see Lemma 4.2 and Lemma 4.3. Moreover, we specify asymptotic behaviour of homoclinic solutions in Lemma 4.4.

Remark 4.1. According to Theorem 3.1, a solution of problem (2.1), (2.2) is damped if $u_0 \in \big(\bar{B}, L\big)$. Hence, if u is escape or homoclinic solution of problem (2.1), (2.2), then $u_0 \in \big(L_0, \bar{B}\big]$. If moreover (2.14) holds, then, by Remark 3.2, each escape or homoclinic solution of problem (2.1), (2.2) satisfy $u_0 \in \big(L_0, \bar{B}\big)$. Therefore, we can restrict our consideration about escape and homoclinic solutions on $u_0 \in (L_0, 0)$.

Lemma 4.2 (Escape solution is increasing). *Let assumptions* (2.3)–(2.6), (2.10) *and* (2.11) *hold. If a solution u of problem* (2.8), (2.2) *with $u_0 \in (L_0, 0)$ is an escape solution, then*

$$\exists\, c \in (0, \infty): \;\; u(c) = L, \quad u'(t) > 0 \textit{ for } t \in (0, \infty)\,. \tag{4.1}$$

Proof. Let u be an escape solution of problem (2.8), (2.2) with $u_0 \in (L_0, 0)$. By Definition 2.6, there exists a constant $c \in (0, \infty)$ such that $u(c) = L$, $u'(c) > 0$. Let $c_1 > c$ be such that $u'(c_1) = 0$, $u(t) > L$, $u'(t) > 0$ for $t \in (c, c_1)$. Integrating equation (2.8) from c to $t \in (\,c, c_1\,]$, dividing by $p(t)$ and using (2.9), we obtain

$$u'(t) = \frac{p(c)u'(c)}{p(t)} > 0 \text{ for } t \in (\,c, c_1\,]\,,$$

contrary to $u'(c_1) = 0$. Therefore, $u'(t) > 0$ for $t > c$. Now, we prove that $u'(t) > 0$ for $t \in (0, \theta_0]$. Since $u_0 \in (L_0, 0)$, Lemma 2.9 yields that there exists $\theta_0 > 0$ such that $u(\theta_0) = 0$, $u(t) < 0$ for $t \in (0, \theta_0)$, $u'(t) > 0$ for $t \in (0, \theta_0]$.

It remains to prove that $u'(t) > 0$ for $t \in (\theta_0, c)$. Assume on the contrary that there exists $a_1 \in (\theta_0, c)$ such that $u(a_1) \in (0, L)$, $u'(a_1) = 0$, $u'(t) > 0$ for $t \in (0, a_1)$. Since u is an escape solution, there exists $\theta_1 > a_1$ such that $u(\theta_1) = 0$, $u'(t) < 0$ for $t \in (a_1, \theta_1]$. Otherwise, by Lemma 2.10, u would be decreasing on

(a,∞). According to Lemma 2.12 and $u_{\sup} > L$, there exists $b_1 > \theta_1$ such that $u(b_1) \in (\bar{B},0)$, $u'(b_1) = 0$, $u'(t) < 0$ for $t \in (a_1,b_1)$. Then there exists $\theta_2 > b_1$ such that $u(\theta_2) = 0$, $u'(t) > 0$ for $t \in (b_1,\theta_2]$. By Lemma 2.11 and $u_{\sup} > L$, there exists $a_2 > \theta_2$ such that $u(a_2) \in (0,L)$, $u'(a_2) = 0$, $u'(t) > 0$ for $t \in (b_1,a_2)$. Repeating this procedure, we obtain that $u(t) \in (L_0,L)$ for $t \in [0,\infty)$, which contradicts that u is an escape solution. We have proved that $u'(t) > 0$ for $t \in (\theta_0,c)$ and, to summarize, $u'(t) > 0$ for $t > 0$. □

Lemma 4.3 (Homoclinic solution is increasing). *Let assumptions (2.3)–(2.6), (2.10), (2.11) and (2.20) hold. If a solution u of problem (2.8), (2.2) with $u_0 \in (L_0,0)$ is homoclinic, then*

$$\lim_{t\to\infty} u(t) = L, \quad u'(t) > 0 \text{ for } t \in (0,\infty). \tag{4.2}$$

Proof. Let u be a homoclinic solution of problem (2.8), (2.2) with $u_0 \in (L_0,0)$. Then, by Lemma 2.9, there exists $\theta_0 > 0$ such that $u(\theta_0) = 0$, $u(t) < 0$ for $t \in (0,\theta_0)$, $u'(t) > 0$ for $t \in (0,\theta_0]$.

Assume on the contradiction that there exists $t_1 > \theta_0$ such that $u'(t_1) = 0$, $u'(t) > 0$ for $t \in (0,t_1)$. Since u is homoclinic and (2.20) holds, $u(t_1) \in (0,L)$. By Lemma 2.10 and $u_{\sup} = L$, there exists $\theta_1 > t_1$ such that $u(\theta_1) = 0$, $u'(t) < 0$ for $t \in (t_1,\theta_1]$. According to Lemma 2.12, there exists $t_2 > \theta_1$ such that $u(t_2) \in (\bar{B},0)$, $u'(t_2) = 0$, $u'(t) < 0$ for $t \in [\theta_1,t_2)$. Repeating this procedure, we obtain a sequence of zeros $\{\theta_n\}_{n=0}^{\infty}$ of u and a sequence of local maxima $\{u(t_{2n+1})\}_{n=0}^{\infty}$ of u. Therefore, u is oscillatory.

We prove that the sequence $\{u(t_{2n+1})\}_{n=0}^{\infty}$ is nonincreasing. Choose $j = 2n+1$, $n \in \mathbb{N}_0$. Multiplying equation (2.8) by pu', integrating this from t_j to t_{j+2} and using (2.11) and the Mean value theorem, we get $\xi_1 \in [t_j,\theta_j]$, $\xi_2 \in [\theta_j,t_{j+1}]$, $\xi_3 \in [t_{j+1},\theta_{j+1}]$, $\xi_4 \in [\theta_{j+1},t_{j+2}]$ such that

$$\begin{aligned}
0 = &\int_{t_j}^{t_{j+2}} (p(t)u'(t))'p(t)u'(t)\,\mathrm{d}t = (pq)(\xi_1)\left(\tilde{F}(u(t_j)) - \tilde{F}(u(\theta_j))\right) \\
&+ (pq)(\xi_2)\left(\tilde{F}(u(\theta_j)) - \tilde{F}(u(t_{j+1}))\right) + (pq)(\xi_3)\left(\tilde{F}(u(t_{j+1})) - \tilde{F}(u(\theta_{j+1}))\right) \\
&+ (pq)(\xi_4)\left(\tilde{F}(u(\theta_{j+1})) - \tilde{F}(u(t_{j+2}))\right) \le (pq)(\xi_4)\left(\tilde{F}(u(t_j)) - \tilde{F}(u(t_{j+2}))\right).
\end{aligned}$$

Hence, $\tilde{F}(u(t_j)) \ge \tilde{F}(u(t_{j+2}))$. Since the function $\tilde{F}$ is increasing on $[0,L]$, we get $u(t_j) \ge u(t_{j+2})$. The sequence $\{u(t_{2n+1})\}_{n=0}^{\infty}$ is nonincreasing, because j is chosen arbitrarily. Thus, $u_{\sup} < L$, which cannot be fulfilled, because u is homoclinic. We have proved that $u'(t) > 0$ for $t \in (0,\infty)$. Since $u_{\sup} = L$, then $\lim_{t\to\infty} u(t) = L$. □

In order to prove further asymptotic properties of homoclinic solutions, we use the condition

$$\liminf_{t\to\infty} p(t) > 0. \tag{4.3}$$

Lemma 4.4. *Assume that* (2.3)–(2.6)*,* (2.10)*,* (2.11) *and* (2.20) *hold. Further, assume that either condition* (3.2) *is valid or conditions* (3.3) *and* (4.3) *are fulfilled. If a solution u of problem* (2.8)*,* (2.2) *with $u_0 \in (L_0, 0)$ is homoclinic, then u fulfils*

$$\lim_{t\to\infty} u'(t) = 0. \tag{4.4}$$

Proof. According to Lemma 4.3, u fulfils (4.2). Hence, there exists $t_0 > 0$ such that $u(t_0) = 0$, $u > 0$ and $\tilde{f}(u) > 0$ on (t_0, ∞). We have $(pu')' < 0$ and the function pu' is decreasing on (t_0, ∞). Since $p > 0$ and $u' > 0$ on $(0, \infty)$, there exists

$$\lim_{t\to\infty} p(t)u'(t) =: K \geq 0. \tag{4.5}$$

Assume that (3.2) holds. Then we have $\lim_{t\to\infty} \frac{1}{p(t)} = 0$ and so, $\lim_{t\to\infty} p(t) = \infty$. Therefore, using (4.5) and that pu' is decreasing, we obtain

$$0 \leq \lim_{t\to\infty} p(t)u'(t) < p(t_0)u'(t_0) < \infty,$$

which implies (4.4).

Now, assume that (3.3) and (4.3) hold. Let $K > 0$. Then $p(t)u'(t) > K$ for $t \geq t_0$ and hence,

$$u'(t) > \frac{K}{p(t)}, \quad t \geq t_0,$$

$$u(t) - u(t_0) > K \int_{t_0}^{t} \frac{\mathrm{d}s}{p(s)}, \quad t \geq t_0.$$

Letting $t \to \infty$, we get, by (3.3) and (4.2), that $L \geq K \cdot \infty$, a contradiction. Therefore, $K = 0$ and, due to (4.3), we have (4.4). □

4.2 Existence of escape and homoclinic solutions

The goal of this section is to give sufficient conditions for the existence of escape and homoclinic solutions of problem (2.1), (2.2). First, we analyse the auxiliary problem (2.8), (2.2) and we proceed by generalizing these results to the original problem (2.1), (2.2) provided that each damped solution is oscillatory.

The following lemma – which is illustrated in Figure 4.1 – is essential for the existence of escape solutions and so, we denote this lemma as basic lemma.

Lemma 4.5 (Basic lemma). *Assume that* (2.3)–(2.6)*,* (2.10)*,* (2.20) *and either assumptions* (3.2)*,* (3.8)*,* (3.19)*,* (3.24) *or assumptions* (3.2)*,* (3.8)*–*(3.11)*,* (3.19) *or assumption* (3.3) *are fulfilled. Further, we assume that*

$$(pq)' > 0 \ \ on\ (0, \infty), \tag{4.6}$$

$$\lim_{t\to\infty} \frac{(p(t)q(t))'}{q^2(t)} = 0, \tag{4.7}$$

$$\liminf_{t\to\infty} \frac{p(t)}{q(t)} > 0, \tag{4.8}$$

$$\liminf_{t\to\infty} q(t) > 0 \tag{4.9}$$

hold. Choose $C \in \left(L_0, \bar{B}\right)$ *and* $\{B_n\}_{n=1}^{\infty} \subset (L_0, C)$*. Let for each* $n \in \mathbb{N}$*,* u_n *be a solution of problem* (2.8)*,* (2.2) *with* $u_0 = B_n$ *and let* $(0, b_n)$ *be the maximal interval such that*

$$u_n(t) < L, \quad u_n'(t) > 0, \quad t \in (0, b_n). \tag{4.10}$$

Finally, assume that for $n \in \mathbb{N}$ *there exist* $\gamma_n \in (0, b_n)$ *such that*

$$u_n(\gamma_n) = C \quad \textit{and} \quad \{\gamma_n\}_{n=1}^{\infty} \textit{ is unbounded.} \tag{4.11}$$

Then the sequence $\{u_n\}_{n=1}^{\infty}$ *contains an escape solution of problem* (2.8)*,* (2.2)*.*

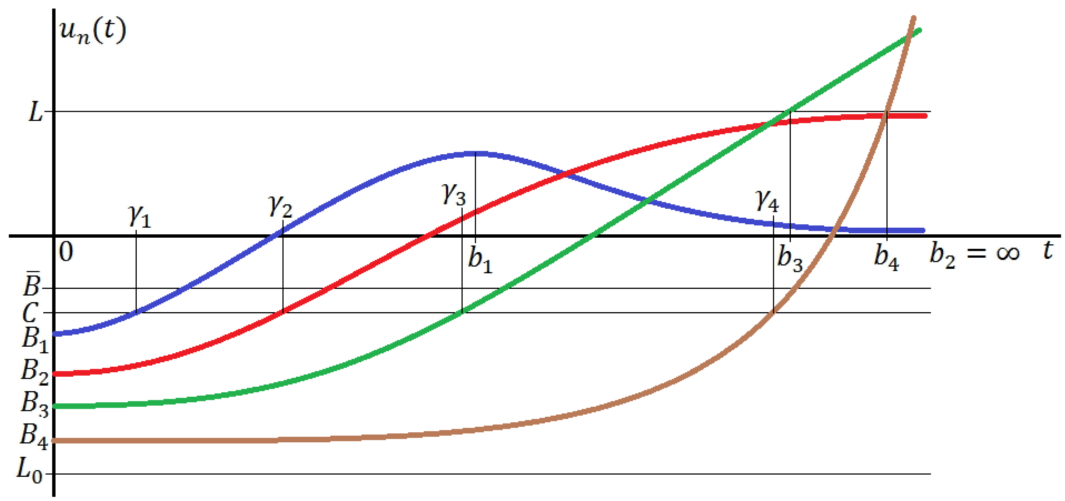

Figure 4.1: Illustration of Lemma 4.5

Proof. Since the sequence $\{\gamma_n\}_{n=1}^{\infty}$ is unbounded, there exists a subsequence going to ∞ as $n \to \infty$. For simplicity, let us denote it by $\{\gamma_n\}_{n=1}^{\infty}$. Then we have

$$\lim_{n\to\infty} \gamma_n = \infty, \quad \gamma_n < b_n, \quad n \in \mathbb{N}. \tag{4.12}$$

Assume on the contrary that for any $n \in \mathbb{N}$, u_n is not an escape solution of problem (2.8), (2.2).

Step 1. Choose $n \in \mathbb{N}$. Then we have two possibilities:

1. u_n is a damped solution. Then, if (3.2), (3.8), (3.19) and (3.24) hold, we get, by Theorem 3.11, that u_n is oscillatory. If it is satisfied (3.2), (3.8)–(3.11) and (3.19), then Theorem 3.12 yields that u_n is oscillatory. If (3.3)

and (4.3) hold, we can use Theorem 3.14, because (4.9) yields (3.29), and we get again that u_n is oscillatory.

2. u_n is a homoclinic solution, which yields $b_n = \infty$ (cf. Lemma 4.3) and we write $u_n(b_n) = \lim_{t\to\infty} u_n(t) = L$. By Lemma 4.4, u_n fulfils (4.4) and hence, $u_n'(b_n) = 0$.

Therefore, we have

$$u_n(b_n) \in (0, L], \quad u_n'(b_n) = 0, \tag{4.13}$$

for both $b_n < \infty$ and $b_n = \infty$. In addition,

$$\exists\, \overline{\gamma}_n \in [\gamma_n, b_n) \colon u_n'(\overline{\gamma}_n) = \max\{u_n'(t) \colon t \in [\gamma_n, b_n)\}. \tag{4.14}$$

Due to (2.8), u_n fulfils

$$\tilde{f}(u_n(t))u_n'(t) = -\frac{p(t)u_n'(t)(p(t)u_n'(t))'}{p(t)q(t)}, \quad t \in (0, b_n). \tag{4.15}$$

Further, we put

$$E_n(t) = \frac{(p(t)u_n'(t))^2}{2}\frac{1}{p(t)q(t)} + \tilde{F}(u_n(t)), \quad t \in (0, b_n). \tag{4.16}$$

Then, by (4.15),

$$\begin{aligned}\frac{\mathrm{d}E_n(t)}{\mathrm{d}t} &= \frac{(p(t)u_n'(t))\,(p(t)u_n'(t))'}{p(t)q(t)} + \frac{(p(t)u_n'(t))^2}{2}\left(\frac{1}{p(t)q(t)}\right)' + \tilde{f}(u_n(t))u_n'(t)\\ &= \frac{(p(t)u_n'(t))^2}{2}\left(\frac{1}{p(t)q(t)}\right)' = -\frac{(p(t)u_n'(t))^2}{2}\frac{(p(t)q(t))'}{(p(t)q(t))^2}, \quad t \in (0, b_n).\end{aligned}$$

Due to (2.6), (4.6) and (4.10), we get

$$\frac{\mathrm{d}E_n(t)}{\mathrm{d}t} = -\frac{u_n'^2(t)}{2q^2(t)}\,(p(t)q(t))' < 0, \quad t \in (0, b_n). \tag{4.17}$$

Integrating (4.17) over $[\gamma_n, b_n]$ and using (4.10), (4.14), we obtain

$$\begin{aligned}E_n(\gamma_n) - E_n(b_n) &= \int_{\gamma_n}^{b_n} \frac{u_n'^2(t)\,(p(t)q(t))'}{2q^2(t)}\,\mathrm{d}t \le u_n'(\overline{\gamma}_n)\int_{\gamma_n}^{b_n} \frac{u_n'(t)\,(p(t)q(t))'}{2q^2(t)}\,\mathrm{d}t\\ &\le u_n'(\overline{\gamma}_n)K_n\int_{\gamma_n}^{b_n} u_n'(t)\,\mathrm{d}t,\end{aligned}$$

where

$$K_n := \sup\left\{\frac{(p(t)q(t))'}{2q^2(t)} \colon t \in (\gamma_n, b_n)\right\} \in (0, \infty).$$

Consequently,

$$E_n(\gamma_n) \le E_n(b_n) + u_n'(\overline{\gamma}_n)K_n(L - C). \tag{4.18}$$

Having in mind (2.5), (2.6), (4.10) and (4.11), we get from (4.16)

$$E_n(\gamma_n) > \tilde{F}(u_n(\gamma_n)) = \tilde{F}(C). \tag{4.19}$$

Since $\tilde{F}$ is increasing on $[0, L]$, (4.13) and (4.16) give for $b_n < \infty$

$$E_n(b_n) = \tilde{F}(u_n(b_n)) \le \tilde{F}(L). \tag{4.20}$$

Let $b_n = \infty$, which means that u_n is homoclinic and $\lim_{t\to\infty} u_n(t) = L$. Then there exists $t_0 > 0$ such that $u_n(t) > 0$ and $\tilde{f}(u_n(t)) > 0$ for $t \in [t_0, \infty)$. Thus, according to (2.8), $(pu_n')' < 0$ on $[t_0, \infty)$ and so, pu_n' is decreasing on $[t_0, \infty)$. Due to (2.5) and (4.10), $p > 0, u_n' > 0$ on $(0, \infty)$ and hence,

$$0 \le \lim_{t\to\infty} p(t)u_n'(t) < p(t_0)u_n'(t_0) < \infty.$$

Therefore, using (4.9), (4.13), we get

$$0 \le \limsup_{t\to\infty} \frac{p(t)}{q(t)} u_n'(t) < \infty$$

and

$$\lim_{t\to\infty} \frac{p(t)}{q(t)} u_n'^2(t) = 0.$$

Consequently, (4.20) is valid also for $b_n = \infty$.

Using (4.18)–(4.20), we derive

$$\tilde{F}(C) < E_n(\gamma_n) \le \tilde{F}(L) + u_n'(\overline{\gamma}_n)K_n(L - C), \tag{4.21}$$

and hence,

$$\frac{\tilde{F}(C) - \tilde{F}(L)}{L - C}\frac{1}{K_n} < u_n'(\overline{\gamma}_n). \tag{4.22}$$

Step 2. Now, consider the sequence $\{u_n\}_{n=1}^{\infty}$. Assumptions (4.7) and (4.12) imply

$$\lim_{n\to\infty} K_n = 0, \tag{4.23}$$

which, by (4.22), yields

$$\lim_{n\to\infty} u_n'(\overline{\gamma}_n) = \infty. \tag{4.24}$$

Since $\tilde{F} \ge 0$ on $[L_0, L]$, we get from (4.16)

$$E_n(\overline{\gamma}_n) \ge \frac{p(\overline{\gamma}_n)u_n'^2(\overline{\gamma}_n)}{2q(\overline{\gamma}_n)}, \quad n \in \mathbb{N}.$$

Further, since E_n is decreasing on $(0, b_n)$ according to (4.17), we derive from (4.21)

$$\frac{p(\overline{\gamma}_n)u_n'^2(\overline{\gamma}_n)}{2q(\overline{\gamma}_n)} \leq E_n(\overline{\gamma}_n) \leq E_n(\gamma_n) \leq \tilde{F}(L) + u_n'(\overline{\gamma}_n)K_n(L-C), \quad n \in \mathbb{N}.$$

Consequently,

$$u_n'(\overline{\gamma}_n)\left(\frac{p(\overline{\gamma}_n)}{2q(\overline{\gamma}_n)}u_n'(\overline{\gamma}_n) - K_n(L-C)\right) \leq \tilde{F}(L) < \infty, \quad n \in \mathbb{N}. \tag{4.25}$$

Due to (4.8), (4.23) and (4.24),

$$\lim_{n\to\infty}\left(\frac{p(\overline{\gamma}_n)}{2q(\overline{\gamma}_n)}u_n'(\overline{\gamma}_n) - K_n(L-C)\right) = \infty. \tag{4.26}$$

Conditions (4.24)–(4.26) yield a contradiction. Therefore, the sequence $\{u_n\}_{n=1}^{\infty}$ contains an escape solution of problem (2.8), (2.2). □

Now, we are ready to prove our main results about the existence of escape and homoclinic solutions. All next existence theorems have the following common assumptions

$$(2.3)–(2.6),\ (2.10),\ (2.23),\ (2.28) \text{ and } (4.6)–(4.9). \tag{4.27}$$

We provide the existence results for two cases which are characterized by conditions (3.2) and (3.3). Therefore, we use in addition either assumptions

$$(3.2),\ (3.8),\ (3.19) \text{ and } (3.24) \tag{4.28}$$

or assumptions

$$(3.2),\ (3.8)–(3.11) \text{ and } (3.19) \tag{4.29}$$

or assumption (3.3). Under these assumptions, we prove that problem (2.8), (2.2) with different starting values has infinitely many escape solutions. Here let us note that condition (4.6) implies (2.11), condition (2.20) follows from (2.28), condition (4.9) gives (3.29) and condition (4.3) follows from (4.8) and (4.9). Therefore, in this section we can omit conditions (2.11), (2.20), (3.29) and (4.3).

Theorem 4.6 (Existence of escape solutions of problem (2.8), (2.2)). *Assume that (4.27) and either (4.28) or (4.29) or (3.3) hold. Then there exist a sequence $\{u_n\}_{n=1}^{\infty}$ of escape solutions of problem (2.8), (2.2) with $u_0 = B_n \in (L_0, \bar{B})$.*

Proof. Choose $n \in \mathbb{N}$, $C \in (L_0, \bar{B})$ and $B_n \in (L_0, C)$. By Theorem 2.17, there exists a unique solution u_n of problem (2.8), (2.2) with $u_0 = B_n$. Due to Lemma 2.9, there exists a maximal $a_n > 0$ such that $u_n' > 0$ on $(0, a_n)$. Since

$u_n(0) < 0$, there exists a maximal $\tilde{a}_n > 0$ such that $u_n < L$ on $[0, \tilde{a}_n)$. If we put $b_n = \min\{a_n, \tilde{a}_n\}$, then (4.10) holds.

If u_n is damped, then, by Theorem 3.11 or Theorem 3.12 or Theorem 3.14, we get that u_n is oscillatory (cf. Step 1 in the proof of Lemma 4.5). Hence, there exists $\gamma_n \in (0, b_n)$ such that $u_n(\gamma_n) = C$. If u_n is not damped, then it is either a homoclinic or an escape solution (cf. Remark 2.16) and clearly, there exists $\gamma_n \in (0, b_n)$ satisfying $u_n(\gamma_n) = C$.

Consider a sequence $\{B_n\}_{n=1}^{\infty} \subset (L_0, C)$. Then we get the sequence $\{u_n\}_{n=1}^{\infty}$ of solutions of problem (2.8), (2.2) with $u_0 = B_n$, and the corresponding sequence of $\{\gamma_n\}_{n=1}^{\infty}$. Assume that $\lim_{n\to\infty} B_n = L_0$. Then, by Theorem 2.17, the sequence $\{u_n\}_{n=1}^{\infty}$ converges locally uniformly on $[0, \infty)$ to the constant function $u \equiv L_0$. Therefore, $\lim_{n\to\infty} \gamma_n = \infty$ and (4.11) is valid. Consequently, according to Lemma 4.5, there exists $n_0 \in \mathbb{N}$ such that u_{n_0} is an escape solution of problem (2.8), (2.2). We have $u_{n_0}(0) = B_{n_0} > L_0$. Now, consider the unbounded sequence $\{\gamma_n\}_{n=n_0+1}^{\infty}$. By Lemma 4.5, there exists $n_1 \in \mathbb{N}$ such that u_{n_1} is an escape solution of problem (2.8), (2.2) such that $u_{n_1}(0) = B_{n_1} > L_0$. Repeating this procedure, we obtain the sequence $\{u_{n_k}\}_{k=0}^{\infty}$ of escape solutions of problem (2.8), (2.2). □

The following theorem provides the existence of a homoclinic solution of problem (2.8), (2.2). The proof is based on a description of sets of initial values of damped and escape solutions.

Theorem 4.7 (Existence of a homoclinic solution of problem (2.8), (2.2)). *Assume that (4.27) and either (4.28) or (4.29) or (3.3) hold. Then there exists a homoclinic solution of problem (2.8), (2.2).*

Proof. *Step 1.* Let $\mathcal{M}_d \subset (L_0, 0)$ be the set of all $u_0 \in (L_0, 0)$ such that the corresponding solutions of problem (2.8), (2.2) are damped. By Theorem 3.1, $\mathcal{M}_d$ is nonempty.

Let us choose $u_0 \in \mathcal{M}_d$ and let u be the corresponding solution of problem (2.8), (2.2). Then, according to Theorem 3.11 or Theorem 3.12 or Theorem 3.14, we have that u is oscillatory. Therefore, there exist $0 < a_1 < b_1$ such that

$$u(a_1) = A_1 > 0, \quad u(b_1) = B_1 < 0. \tag{4.30}$$

Choose $\varepsilon > 0$ satisfying

$$\varepsilon < \frac{1}{2} \min\{A_1, |B_1|\}. \tag{4.31}$$

Let v be the solution of equation (2.8) satisfying $v(0) =: v_0 \in (L_0, 0)$. By Theorem 2.17, there exists $K > 0$ such that

$$\|u - v\|_{C^1[0,b_1]} \le K|v_0 - u_0|,$$

which gives $\delta = \frac{\varepsilon}{K} > 0$ such that

$$|v_0 - u_0| < \delta \Rightarrow \|u - v\|_{C^1[0,b_1]} < \varepsilon. \tag{4.32}$$

Consequently,

$$u(t) - \varepsilon < v(t) < u(t) + \varepsilon \text{ for } t \in [0, b_1]$$

and, using (4.30) and (4.31), we get

$$v(a_1) > \frac{A_1}{2} > 0, \quad v(b_1) < \frac{B_1}{2} < 0.$$

Therefore, if $|v_0 - u_0| < \delta$, then v is not an increasing function and so, v is damped (*cf.* Lemma 4.2, Lemma 4.3 and Remark 2.16) . We have proved that if $u_0 \in \mathcal{M}_d$, then $(u_0 - \delta, u_0 + \delta) \subset \mathcal{M}_d$, that is $\mathcal{M}_d$ is open in $(L_0, 0)$.

Step 2. Let $\mathcal{M}_e \subset (L_0, 0)$ be the set of all $u_0 \in (L_0, 0)$ such that the corresponding solutions of problem (2.8), (2.2) are escape solutions. According to Theorem 4.6, $\mathcal{M}_e$ is nonempty.

Choose $u_0 \in \mathcal{M}_e$ and let u be the corresponding escape solution of problem (2.8), (2.2). Then u fulfils (4.1). Hence, there exists $c_1 > c$ such that

$$u(c_1) = L_1 > L. \tag{4.33}$$

Let us choose $\varepsilon > 0$ satisfying

$$\varepsilon < \frac{1}{2}\left(L_1 - L\right). \tag{4.34}$$

Assume that v is the solution of equation (2.8) satisfying $v(0) = v_0 \in (L_0, 0)$. Due to Theorem 2.17, there exists $\delta > 0$ such that (4.32) holds. Therefore,

$$u(t) - \varepsilon < v(t) < u(t) + \varepsilon \quad \text{for } t \in [0, c_1]$$

and, by (4.33) and (4.34),

$$v(c_1) > \frac{1}{2}(L + L_1) > L.$$

Hence, due to Remark 2.16, if $|v_0 - u_0| < \delta$, then v is an escape solution. We proved that if $u_0 \in \mathcal{M}_e$, then $(u_0 - \delta, u_0 + \delta) \subset \mathcal{M}_e$, that is $\mathcal{M}_e$ is open in $(L_0, 0)$.

Step 3. Let $\mathcal{M}_h \subset (L_0, 0)$ be defined by

$$\mathcal{M}_h = (L_0, 0) \setminus \left(\mathcal{M}_d \cup \mathcal{M}_e\right).$$

Since $\mathcal{M}_d \cup \mathcal{M}_e$ is nonempty and open set in $(L_0, 0)$, $\mathcal{M}_h$ has to be nonempty and closed in $(L_0, 0)$. In addition, if we choose $u_0 \in \mathcal{M}_h$, then the corresponding solution of problem (2.8), (2.2) fulfils $u_{\text{sup}} = L$ and, due to Remark 2.16, u is a homoclinic solution of problem (2.8), (2.2). □

Finally, we extend the existence results from Theorem 4.6 and Theorem 4.7 to the original problem (2.1), (2.2) and reach the main aim of this chapter.

Theorem 4.8 (**Existence of escape solutions of problem** (2.1)**,** (2.2)**).** *Assume that* (4.27) *and either* (4.28) *or* (4.29) *or* (3.3) *hold. Then, for each* $n \in \mathbb{N}$*, there exist constant* $c_n \in (0, \infty)$ *and function* u_n *such that* u_n *is an escape solution of problem* (2.1)*,* (2.2) *on* $[0, c_n]$ *with* $u_0 = B_n \in (L_0, \bar{B})$*.*

Proof. By Theorem 4.6, there exists a sequence $\{u_n\}_{n=1}^{\infty}$ of escape solutions of problem (2.8), (2.2) with $u_0 = B_n \in (L_0, \bar{B})$. By Lemma 4.2, for each $u \in \{u_n\}_{n=1}^{\infty}$ there exists $c \in (0, \infty)$ such that (4.1) holds. Due to (2.9), u is an escape solution of problem (2.1), (2.2) on $[0, c]$. □

Theorem 4.9 (**Existence of a homoclinic solution of problem** (2.1)**,** (2.2)**).** *Assume that* (4.27) *and either* (4.28) *or* (4.29) *or* (3.3) *hold. Then there exists a homoclinic solution of problem* (2.1)*,* (2.2)*.*

Proof. According to Theorem 4.7, there exists a homoclinic solution u of problem (2.8), (2.2). Due to (2.9), u is a homoclinic solution of problem (2.1), (2.2), as well. □

As we mentioned in Section 1.5, the significant solutions for applications are bubble-type solutions. If u is a homoclinic solution of IVP (2.1), (2.2), then, by Lemma 4.3, u satisfies (4.2) and so, u is increasing and fulfils the boundary condition (2.7). Therefore, according to Definition 2.7, the homoclinic solution of problem (2.1), (2.2) is also the bubble-type solution of (2.1), (2.2).

Corollary 4.10 (**Existence of a bubble-type solution of problem** (2.1)**,** (2.2)**).** *Assume that* (4.27) *and either* (4.28) *or* (4.29) *or* (3.3) *hold. Then there exists a bubble-type solution of problem* (2.1)*,* (2.2)*.*

4.3 Examples

We conclude this chapter with examples, where the functions p, q and f are chosen in such a way that problem (2.1), (2.2) with different starting values has infinitely many escape solutions and at least one homoclinic solution.

Example 4.11. Here we illustrate conditions (4.27) and (4.28). Consider the IVP

$$\begin{gathered}(t^{\alpha}u'(t))' + t^{\beta}f(u(t)) = 0,\\ u(0) = u_0 \in [-2, 1]\,, \quad u'(0) = 0,\end{gathered} \tag{4.35}$$

where

$$f(x) = \begin{cases} x(x+2), & x \in [-2, 0]\,, \\ x(1-x), & x \in [0, 1]\,. \end{cases}$$

Here

$$p(t) = t^{\alpha}, \quad q(t) = t^{\beta}, \quad t \in [0, \infty)\,,$$

$L_0 = -2$, $L = 1$. Assume that (3.34) holds, that is

$$\alpha \in (1, 2], \ \beta > \alpha - 1 \quad \text{or} \quad \alpha > 2, \ \beta \geq 2\alpha - 3.$$

Example 3.19 shows that conditions (2.3)–(2.6), (2.23), (2.28), (4.28) hold and that there exists $\bar{B} \in (-2, 0)$ satisfying (2.10). In addition, the function pq is continuously differentiable on $(0, \infty)$, increasing on $[0, \infty)$,

$$\lim_{t\to\infty} \frac{(p(t)q(t))'}{q^2(t)} = \lim_{t\to\infty} \frac{(\alpha+\beta)t^{\alpha+\beta+1}}{t^{2\beta}} = \lim_{t\to\infty} (\alpha+\beta)t^{\alpha-\beta-1} = 0 \quad \text{if } \beta > \alpha - 1,$$
$$\liminf_{t\to\infty} \frac{p(t)}{q(t)} = \lim_{t\to\infty} t^{\alpha-\beta} > 0 \quad \text{if } \beta \leq \alpha, \qquad \liminf_{t\to\infty} q(t) = \lim_{t\to\infty} t^{\beta} > 0.$$

It means that (4.6)–(4.9) are valid for

$$\alpha - 1 < \beta \leq \alpha.$$

To summarize, by (3.33), if

$$\alpha \in (1, 2], \ \alpha - 1 < \beta \leq \alpha \quad \text{or} \quad \alpha \in (2, 3], \ 2\alpha - 3 \leq \beta \leq \alpha, \tag{4.36}$$

then (4.27) and (4.28) hold. According to Theorem 4.8, for each $n \in \mathbb{N}$, there exist constant $c_n \in (0, \infty)$ and function u_n such that u_n is an escape solution of problem (4.35) on $[0, c_n]$ with $u_0 = B_n \in \left(-2, \bar{B}\right)$. Theorem 4.9 yields the existence of a homoclinic solution of problem (4.35).

Example 4.12. Let us show another illustration of conditions (4.27) and (4.28), where, in addition, we compute $\bar{B}$ satisfying (2.10). Consider the IVP

$$\begin{gathered} \left(t^2 u'(t)\right)' + \sqrt{t^3}\, u(t)(1-u(t))(u(t)+4) = 0, \\ u(0) = u_0 \in [-4, 1], \quad u'(0) = 0. \end{gathered} \tag{4.37}$$

Here

$$p(t) = t^2, \quad q(t) = \sqrt{t^3}, \quad t \in [0, \infty), \quad f(x) = x(1-x)(x+4), \quad x \in [-4, 1],$$

$L_0 = -4$, $L = 1$. According to Example 4.11, the functions p and q satisfy (2.5), (2.6), (2.23), (3.2), (3.24) and (4.6)–(4.9).

Example 3.23 shows that the function f fulfils (2.3), (2.4), (2.10) and (2.28). Moreover,

$$\liminf_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} (1-x)(x+4) = 4 > 0,$$

that is (3.8) and (3.19) hold. Let us find $\bar{B}$ satisfying (2.10). For $x \in [-4, 1]$, we compute

$$\begin{aligned}\tilde{F}(x) = \int_0^x z(1-z)(z+4)\,\mathrm{d}z = \int_0^x (-z^3 - 3z^2 + 4z)\,\mathrm{d}z = -\frac{x^4}{4} - x^3 + 2x^2, \\ \tilde{F}(-4) = 32, \quad \tilde{F}\left(\bar{B}\right) = \tilde{F}(1) = \frac{3}{4}, \\ \frac{\tilde{F}(x) - \tilde{F}\left(\bar{B}\right)}{(x-1)^2} = \frac{-\frac{x^4}{4} - x^3 + 2x^2 - \frac{3}{4}}{(x-1)^2} = -\frac{x^2}{4} - \frac{3x}{2} - \frac{3}{4} \\ = -\frac{1}{4}\left(x + \sqrt{6} + 3\right)\left(x - \sqrt{6} + 3\right).\end{aligned}$$

Thus, polynomial $\tilde{F}(x) - \tilde{F}\left(\bar{B}\right)$ has roots

$$x_1 = 1, \quad x_2 = -\sqrt{6} - 3 \approx -5.45, \quad x_3 = \sqrt{6} - 3 \approx -0.55 \in (-4, 0).$$

Therefore, $\bar{B} = \sqrt{6} - 3$.

To sum up, we have satisfied (4.27) and (4.28). Therefore, by Theorem 4.8, there exist infinitely many escape solutions u of problem (4.37) on $[0, c]$ with different $u_0 \in \left(-4, \sqrt{6} - 3\right)$ and generally different c for different solutions. Due to Theorem 4.9, for some $u_0 \in \left(-4, \sqrt{6} - 3\right)$, problem (4.37) has a homoclinic solution.

In the following example, we illustrate conditions (4.27) and (4.28) provided the function q is composed of power function and bounded function.

Example 4.13. Let us consider the IVP

$$\begin{gathered}\left(\sqrt[4]{t^7}\, u'(t)\right)' + \left(\sqrt[4]{t^5} + \arctan t\right) f(u(t)) = 0, \\ u(0) = u_0 \in [-2, 1], \quad u'(0) = 0,\end{gathered} \tag{4.38}$$

where

$$f(x) = \begin{cases} x(1-x)(x+3) & \text{for } x > 0, \\ \frac{7}{13}x(1-x)(x+2) & \text{for } x \le 0. \end{cases}$$

Here

$$p(t) = \sqrt[4]{t^7}, \quad q(t) = \sqrt[4]{t^5} + \arctan t, \quad t \in [0, \infty).$$

The functions p and q are continuous on $[0, \infty)$, positive on $(0, \infty)$ and $p(0) = 0$. Therefore, (2.5) and (2.6) are satisfied. Moreover, pq is continuously differentiable on $(0, \infty)$ and increasing on $[0, \infty)$, which means that (4.6) holds. Since,

$$\int_0^t \sqrt[4]{s^5}\,\mathrm{d}s = \frac{4t^{\frac{9}{4}}}{9},$$

$$\int_0^t \arctan s \,\mathrm{d}s \left| \begin{array}{ll} u = \arctan s & u' = \frac{1}{s^2+1} \\ v' = 1 & v = s \end{array} \right| = [s \arctan s]_0^t$$

$$- \int_0^t \frac{s}{s^2+1} \,\mathrm{d}s \left| \begin{array}{l} x = s^2+1 \\ \mathrm{d}x = 2s \,\mathrm{d}s \\ s = 0\colon x = 1 \\ s = t\colon x = t^2+1 \end{array} \right| = t \arctan t - \frac{1}{2} \int_1^{t^2+1} \frac{1}{x} \,\mathrm{d}x$$

$$= t \arctan t - \frac{1}{2} [\ln |x|]_1^{t^2+1} = t \arctan t - \frac{1}{2} \ln(t^2+1),$$

we get

$$\lim_{t \to 0^+} \frac{1}{\sqrt[4]{t^7}} \int_0^t \left(\sqrt[4]{s^5} + \arctan s \right) \,\mathrm{d}s = \lim_{t \to 0^+} \frac{1}{\sqrt[4]{t^7}} \left(\frac{4s^{\frac{9}{4}}}{9} + t \arctan t - \frac{1}{2} \ln(t^2+1) \right)$$

$$= \lim_{t \to 0^+} \frac{4\sqrt{t}}{9} + \lim_{t \to 0^+} \frac{\arctan t}{t^{\frac{3}{4}}} - \frac{1}{2} \lim_{t \to 0^+} \frac{\ln(t^2+1)}{t^{\frac{7}{4}}} = 0 + 0 - 0 = 0,$$

that is (2.23) holds. Further,

$$\int_1^\infty \frac{1}{\sqrt[4]{s^7}} \,\mathrm{d}s = \frac{4}{3} < \infty, \quad \ell(t) = \int_t^\infty \frac{1}{\sqrt[4]{s^7}} \,\mathrm{d}s = \frac{4}{3t^{\frac{3}{4}}},$$

$$\int_1^\infty \ell^2(s) q(s) \,\mathrm{d}s = \int_1^\infty \frac{16 \left(\sqrt[4]{s^5} + \arctan s \right)}{9s^{\frac{3}{2}}} \,\mathrm{d}s = \int_1^\infty \frac{16}{9s^{\frac{1}{4}}} \,\mathrm{d}s$$

$$+ \frac{16}{9} \int_1^\infty \frac{\arctan s}{s^{\frac{3}{2}}} \,\mathrm{d}s \geq \int_1^\infty \frac{16}{9s^{\frac{1}{4}}} \,\mathrm{d}s = \infty,$$

$$\lim_{t \to \infty} \frac{(p(t)q(t))'}{q^2(t)} = \lim_{t \to \infty} \frac{\frac{7}{4} \left(\sqrt[4]{t^5} + \arctan t \right) + \sqrt[4]{t^7} \left(\frac{5}{4} t^{\frac{1}{4}} + \frac{1}{t^2+1} \right)}{t^{\frac{5}{2}} + 2\sqrt[4]{t^5} + \arctan t + \arctan^2 t}$$

$$= \lim_{t \to \infty} \frac{\frac{7}{4\sqrt{t}} + \frac{7 \arctan t}{4t^{\frac{7}{4}}} + \frac{5}{4\sqrt{t}} + \frac{1}{t^{\frac{3}{4}}(t^2+1)}}{1 + \frac{2 \arctan t}{t^{\frac{5}{4}}} + \frac{\arctan^2 t}{t^{\frac{5}{2}}}} = 0,$$

$$\liminf_{t \to \infty} \frac{p(t)}{q(t)} = \lim_{t \to \infty} \frac{\sqrt[4]{t^7}}{\sqrt[4]{t^5} + \arctan t} = \lim_{t \to \infty} \frac{\sqrt{t}}{1 + \frac{\arctan t}{t^{\frac{5}{4}}}} = \infty > 0,$$

$$\liminf_{t \to \infty} q(t) = \lim_{t \to \infty} \left(\sqrt[4]{t^5} + \arctan t \right) = \infty > 0.$$

Thus, we checked that (3.2), (3.24) and (4.7)–(4.9) hold, respectively.

The function f is continuous on $\mathbb{R}$, Lipschitz continuous on $[L_0, L]$, $L_0 = -2$, $L = 1$, $xf(x) > 0$ for $x \in (L_0, L) \setminus \{0\}$, $f(L_0) = f(0) = f(L) = 0$. Thus, (2.3), (2.4) and (2.28) are satisfied. Furthermore,

$$\liminf_{x \to 0^-} \frac{\frac{7}{13} x(1-x)(x+2)}{x} = \lim_{x \to 0} \frac{7}{13} (1-x)(x+2) = \frac{14}{13} > 0,$$

$$\liminf_{x \to 0^+} \frac{x(1-x)(x+3)}{x} = \lim_{x \to 0} (1-x)(x+3) = 3 > 0,$$

which means that (3.8) and (3.19) hold. Since $0 < L < -L_0$, we get, similarly as in Example 2.18, that $\tilde{F}(L) < \tilde{F}(L_0)$. Therefore, there exists $\bar{B} \in (L_0, 0)$ such that $\tilde{F}\left(\bar{B}\right) = \tilde{F}(L)$, which gives (2.10). Let us find such $\bar{B}$.

$$\begin{aligned}
\tilde{F}(x) &= \int_0^x \frac{7}{13} z(1-z)(z+2)\,\mathrm{d}z = \int_0^x \frac{7}{13}(-z^3 - z^2 + 2z)\,\mathrm{d}z \\
&= \frac{7}{13}\left(-\frac{x^4}{4} - \frac{x^3}{3} + x^2\right), \quad x \in [-2, 1], \\
\tilde{F}(-2) &= \frac{56}{39}, \quad \tilde{F}\left(\bar{B}\right) = \tilde{F}(1) = \frac{35}{156}, \\
\frac{\tilde{F}(x) - \tilde{F}\left(\bar{B}\right)}{(x-1)^2} &= \frac{\frac{7}{13}\left(-\frac{x^4}{4} - \frac{x^3}{3} + x^2 - \frac{5}{12}\right)}{(x-1)^2} = -\frac{7}{13}\left(\frac{x^2}{4} + \frac{5x}{6} + \frac{5}{12}\right) \\
&= -\frac{7}{156}\left(x + \frac{\sqrt{10}+5}{3}\right)\left(x - \frac{\sqrt{10}-5}{3}\right), \quad x \in [-2, 1].
\end{aligned}$$

Hence, polynomial $\tilde{F}(x) - \tilde{F}\left(\bar{B}\right)$ has roots

$$x_1 = 1, \quad x_2 = -\frac{\sqrt{10}+5}{3} \approx -2.72, \quad x_3 = \frac{\sqrt{10}-5}{3} \approx -0.61 \in (-2, 0).$$

Therefore, $\bar{B} = \frac{\sqrt{10}-5}{3}$.

To summarize, (4.27) and (4.28) hold. Thus, by virtue of Theorem 4.8, there exist infinitely many escape solutions u of problem (4.38) on $[0, c]$ with different $u_0 \in \left(-2, \frac{\sqrt{10}-5}{3}\right)$. Here c can be different for different solutions. Theorem 4.9 yields the existence of a homoclinic solution of problem (4.38) with $u_0 \in \left(-2, \frac{\sqrt{10}-5}{3}\right)$.

Example 4.14. Now, we illustrate conditions (4.27) and (3.3). Let us consider the IVP

$$\begin{gathered}
(t^\alpha u'(t))' + kt^\beta |u(t)|^\gamma \operatorname{sgn} u(t)(u(t) - L_0)(L - u(t)) = 0, \\
u(0) = u_0 \in [L_0, L], \quad u'(0) = 0.
\end{gathered} \tag{4.39}$$

We have equation (2.1) with

$$\begin{aligned}
&p(t) = t^\alpha, \quad q(t) = t^\beta, \quad t \in [0, \infty), \\
&f(x) = k|x|^\gamma \operatorname{sgn} x(x - L_0)(L - x), \quad x \in [L_0, L].
\end{aligned}$$

According to Example 3.22, if $\alpha \in (0, 1)$, $\beta \geq 0$, $\beta > \alpha - 1$, then the functions p and q satisfy (2.5), (2.6), (2.11), (2.14), (2.23) and (3.3). Example 4.11 shows that (4.6)–(4.9) are valid for $\beta > \alpha - 1$, $\beta \leq \alpha$. Example 3.23 yields that, for $0 < L < -L_0$, $\gamma \geq 1$ and $k > 0$, the function f fulfils (2.3), (2.4), (2.10) and (2.28).

To sum up, if

$$0 < L < -L_0,\ \alpha \in (0,1],\ \beta \ge 0,\ \alpha - 1 < \beta \le \alpha,\ \gamma \ge 1 \ \text{ and } \ k > 0,$$

then (3.3) and (4.27) hold. So, the assertions of Theorems 4.8 and 4.9 for problem (4.39) are valid.

The following example illustrates the fulfilment of conditions (4.27) and (3.3) provided the function q is bounded and p is unbounded.

Example 4.15. Consider the IVP

$$\begin{gathered}\left(\sqrt{t}\,u'(t)\right)' + \tanh t\, f(u(t)) = 0,\\ u(0) = u_0 \in \left[-2 - 2^\lambda, 2\right], \quad u'(0) = 0,\end{gathered} \tag{4.40}$$

where

$$f(x) = \begin{cases} -(x + 2^\gamma + 2) & \text{for } x \le -2,\\ |x|^\gamma \operatorname{sgn} x & \text{for } x \in (-2,1),\\ 2 - x & \text{for } x \ge 1.\end{cases}$$

Here

$$p(t) = \sqrt{t}, \quad q(t) = \tanh t = \frac{e^t - e^{-t}}{e^t + e^{-t}}, \quad t \in [0,\infty),$$

$L_0 = -2 - 2^\gamma$, $L = 2$. The functions p and q are continuous on $[0,\infty)$, positive on $(0,\infty)$ and $p(0) = 0$. Therefore, (2.5) and (2.6) are satisfied. Further, pq is continuously differentiable on $(0,\infty)$ and increasing on $[0,\infty)$, which gives that (4.6) holds. Since

$$\begin{aligned}\int_0^t \tanh s \,\mathrm{d}s &= \int_0^t \frac{\sinh s}{\cosh s}\,\mathrm{d}s \left| \begin{array}{l} x = \cosh s\\ \mathrm{d}x = \sinh s \,\mathrm{d}s\\ s = 0\colon\ x = 1\\ s = t\colon\ x = \cosh t\end{array}\right| = \int_1^{\cosh t} \frac{1}{x}\,\mathrm{d}x\\ &= [\ln|x|]_1^{\cosh t} = \ln(\cosh t),\end{aligned}$$

we obtain, using the l'Hôspital's rule,

$$\lim_{t\to 0^+} \frac{1}{\sqrt{t}} \int_0^t \tanh s\,\mathrm{d}s = \lim_{t\to 0^+} \frac{\ln(\cosh t)}{\sqrt{t}} = \lim_{t\to 0^+} \frac{\frac{\sinh t}{\cosh t}}{\frac{1}{2\sqrt{t}}} = \lim_{t\to 0^+} 2\sqrt{t}\,\tanh t = 0,$$

which yields (2.23). Moreover,

$$\begin{aligned}\lim_{t\to\infty} \frac{(p(t)q(t))'}{q^2(t)} &= \lim_{t\to\infty} \frac{\frac{\tanh t}{2\sqrt{t}} + \frac{\sqrt{t}}{\cosh^2 t}}{\tanh^2 t} = \lim_{t\to\infty} \frac{1}{2\sqrt{t}\,\tanh t} + \lim_{t\to\infty} \frac{\sqrt{t}}{\cosh^2 t}\,\frac{\cosh^2 t}{\sinh^2 t}\\ &= 0 + \lim_{t\to\infty} \frac{\sqrt{t}}{\sinh^2 t} = \lim_{t\to\infty} \frac{\frac{1}{2\sqrt{t}}}{2\sinh t\cosh t} = 0,\end{aligned}$$

$$\liminf_{t\to\infty} \frac{p(t)}{q(t)} = \lim_{t\to\infty} \frac{\sqrt{t}}{\tanh t} = \infty > 0, \quad \liminf_{t\to\infty} \tanh t = 1 > 0, \quad \int_1^\infty \frac{1}{\sqrt{s}}\,\mathrm{d}s = \infty,$$

that is (3.3) and (4.7)–(4.9) hold. Example 3.22 shows that, for $\gamma \geq 1$, the function f satisfies (2.3), (2.4), (2.10) and (2.28).

To summarize, if

$$\gamma \geq 1,$$

then (3.3) and (4.27) hold. Hence, the assertions of Theorems 4.8 and 4.9 for problem (4.40) are valid.

In the next example, we illustrate conditions (4.27) and (3.3) provided that both functions p and q are bounded.

Example 4.16. Let us consider the IVP

$$\begin{gathered}(\arctan t\, u'(t))' + \frac{t^2}{t^2+1}\, u(t)(1-u(t))(u(t)+2) = 0,\\ u(0) = u_0 \in [-2,1]\,, \quad u'(0) = 0,\end{gathered} \tag{4.41}$$

Here

$$\begin{aligned} p(t) &= \arctan t, \quad q(t) = \frac{t^2}{t^2+1}\,, \quad t \in [0,\infty),\\ f(x) &= x(1-x)(x+2), \quad x \in [-2,1], \end{aligned}$$

$L_0 = -2$, $L = 1$. Example 3.23 shows that the functions p and q satisfy (2.5), (2.6), (2.23) and (3.3). In addition, pq is continuously differentiable on $(0,\infty)$ and increasing on $[0,\infty)$, which yields that (4.6) holds. Further,

$$\begin{aligned} \lim_{t\to\infty} \frac{(p(t)q(t))'}{q^2(t)} &= \lim_{t\to\infty} \frac{\left(\frac{t^2 \arctan t}{t^2+1}\right)'}{\frac{t^4}{(t^2+1)^2}}\\ &= \lim_{t\to\infty} \frac{\left(\frac{t^2}{t^2+1} + 2t\arctan t\right)(t^2+1) - 2t^3\arctan t}{t^4}\\ &= \lim_{t\to\infty} \frac{t^2+2t\arctan t}{t^4} = \lim_{t\to\infty}\left(\frac{1}{t^2} + \frac{2\arctan t}{t^3}\right) = 0, \end{aligned}$$

$$\begin{aligned} \liminf_{t\to\infty} \frac{p(t)}{q(t)} &= \lim_{t\to\infty} \frac{\arctan t}{\frac{t^2}{t^2+1}} = \lim_{t\to\infty} \arctan t \lim_{t\to\infty} \frac{t^2+1}{t^2} = \frac{\pi}{2} > 0,\\ \liminf_{t\to\infty} q(t) &= \lim_{t\to\infty} \frac{t^2}{t^2+1} = 1 > 0, \end{aligned}$$

that is (4.7)–(4.9) hold. Example 3.20 gives that the function f fulfils (2.3), (2.4), (2.10), (2.28) and that $\bar{B} = \frac{\sqrt{10}-5}{3}$.

To sum up, we have satisfied (3.3) and (4.27). According to Theorem 4.8, there exist infinitely many escape solutions u of problem (4.41) on $[0,c]$ with different

$u_0 \in \left(-2, \frac{\sqrt{10}-5}{3}\right)$, where c can be different for different solutions. Theorem 4.9 yields the existence of a homoclinic solution of problem (4.41) with $u_0 \in \left(-2, \frac{\sqrt{10}-5}{3}\right)$.

Example 4.17. We illustrate conditions (4.27) and (4.29). Consider the IVP

$$\begin{gathered}(at^\alpha u'(t))' + bt^\alpha f(u(t)) = 0,\\ u(0) = u_0 \in [-2, 1], \quad u'(0) = 0,\end{gathered} \tag{4.42}$$

where

$$f(x) = \begin{cases} x(1-x)(x+2) & \text{for } x \le 0,\\ \frac{5}{7}x(1-x)(x+3) & \text{for } x > 0.\end{cases}$$

Here

$$p(t) = at^\alpha, \quad q(t) = bt^\alpha,$$

$L_0 = -2$, $L = 1$. Assume that $\alpha > 0$ and $a, b > 0$. Then p and q are continuous on $[0, \infty)$, positive on $(0, \infty)$ and $p(0) = 0$. Hence, (2.5) and (2.6) are satisfied. Moreover, pq is continuously differentiable on $(0, \infty)$ and increasing on $[0, \infty)$, which means that (4.6) holds. Further,

$$\begin{gathered}\lim_{t\to 0^+} \frac{1}{at^\alpha}\int_0^t bs^\alpha \, \mathrm{d}s = \lim_{t\to 0^+} \frac{1}{at^\alpha}\frac{bt^{\alpha+1}}{\alpha+1} = \lim_{t\to 0^+} \frac{b}{a(\alpha+1)} t = 0,\\ p \in C[0,\infty) \cap C^2(0,\infty), \quad p'(t) > 0 \text{ for } t \in (0, \infty),\end{gathered}$$

$$\begin{aligned}
\lim_{t\to\infty} \frac{p'(t)}{p(t)} &= \lim_{t\to\infty} \frac{a\alpha t^{\alpha-1}}{at^\alpha} = \lim_{t\to\infty} \frac{\alpha}{t} = 0,\\
\limsup_{t\to\infty} \left|\frac{p''(t)}{p'(t)}\right| &= \lim_{t\to\infty} \frac{a\alpha|\alpha-1|t^{\alpha-2}}{a\alpha t^{\alpha-1}} = \lim_{t\to\infty} \frac{|\alpha-1|}{t} = 0 < \infty,\\
\liminf_{t\to\infty} \frac{q(t)}{p(t)} &= \lim_{t\to\infty} \frac{bt^\alpha}{at^\alpha} = \frac{b}{a} > 0,\\
\lim_{t\to\infty} \frac{(p(t)q(t))'}{q^2(t)} &= \lim_{t\to\infty} \frac{2ab\alpha t^{2\alpha-1}}{b^2t^{2\alpha}} = \lim_{t\to\infty} \frac{2a\alpha}{bt} = 0,\\
\liminf_{t\to\infty} \frac{p(t)}{q(t)} &= \lim_{t\to\infty} \frac{at^\alpha}{bt^\alpha} = \frac{a}{b} > 0, \quad \liminf_{t\to\infty} q(t) = \lim_{t\to\infty} bt^\alpha = \infty > 0,\\
\int_1^\infty \frac{1}{as^\alpha}\,\mathrm{d}s &< \infty \quad \text{provided } \alpha > 1.
\end{aligned}$$

Therefore, we checked that (2.23), (3.9)–(3.11), (4.7)–(4.9) hold and (3.2) holds for $\alpha > 1$. Example 3.21 shows that the function f fulfils (2.3), (2.4), (2.10), (2.28), (3.8), (3.19) and that $\bar{B} = \frac{\sqrt{10}-5}{3}$.

To summarize, if

$$\alpha > 1 \quad \text{and} \quad a, b > 0,$$

then (4.27) and (4.29) hold. Due to Theorem 4.8, there exist infinitely many escape solutions u of problem (4.42) on $[0, c]$ with different $u_0 \in \left(-2, \frac{\sqrt{10}-5}{3}\right)$, where c can be different for different solutions. Theorem 4.9 gives the existence of a homoclinic solution of problem (4.42) with $u_0 \in \left(-2, \frac{\sqrt{10}-5}{3}\right)$.

In the final example, we show that the density profile equation (1.18) from our motivation has a bubble-type solution.

Example 4.18. We illustrate conditions (4.27), (4.28) and (4.29) on the density profile equation. Let us consider the boundary value problem (1.18), (1.19), that is

$$(t^2 u'(t))' = \lambda^2 t^2 (u(t)+1) u(t) (u(t) - \xi),$$
$$u'(0) = 0, \qquad \lim_{t\to\infty} u(t) = \xi,$$

where $\lambda \in (0, \infty)$ and $\xi \in (0, 1)$. We have the special case of equation (2.1) with

$$p(t) = q(t) = t^2,\ t \in [0, \infty), \qquad f(x) = \lambda^2 x(x+1)(\xi - x),\ x \in \mathbb{R}.$$

Here $L_0 = -1$, $L = \xi$. By Example 4.11, for $\alpha = \beta = 2$, condition (4.36) holds and so, the functions p and q satisfy (2.5), (2.6), (2.23), (3.2), (3.24) and (4.6)–(4.9). According to Example 3.20, the function p fulfils (3.9) and (3.10), too. Furthermore,

$$\liminf_{t\to\infty} \frac{q(t)}{p(t)} = \lim_{t\to\infty} \frac{t^2}{t^2} = 1 > 0,$$

that is (3.11) holds.

Example 3.23 (where now $k = \lambda^2$ and $\gamma = 1$) shows that the function f satisfies (2.3), (2.4), (2.10) and (2.28). In addition,

$$\liminf_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \lambda^2 (x+1)(\xi - x) = \lambda^2 \xi > 0,$$

which yields (3.8) and (3.19).

To sum up, conditions (4.27)–(4.29) are fulfilled and, by Corollary 4.10, the IVP (1.18), (2.2) has the bubble-type solution, which is also a solution of bundary value problem (1.18), (1.19).

5 Solvability of the problem with ϕ-Laplacian

5.1 Statement of the problem

We investigate the equation

$$(p(t)\phi(u'(t)))' + p(t)f(\phi(u(t))) = 0 \tag{5.1}$$

with the initial conditions

$$u(0) = u_0, \quad u'(0) = 0, \quad u_0 \in [L_0, L] \tag{5.2}$$

and assume these basic assumptions:

$$\phi \in C^1(\mathbb{R}), \quad \phi'(x) > 0 \text{ for } x \in (\mathbb{R} \setminus \{0\}), \tag{5.3}$$

$$\phi(\mathbb{R}) = \mathbb{R}, \quad \phi(0) = 0, \tag{5.4}$$

$$L_0 < 0 < L, \quad f(\phi(L_0)) = f(0) = f(\phi(L)) = 0, \tag{5.5}$$

$$f \in C[\phi(L_0), \phi(L)], \quad xf(x) > 0 \text{ for } x \in ((\phi(L_0), \phi(L)) \setminus \{0\}), \tag{5.6}$$

$$p \in C[0,\infty) \cap C^1(0,\infty), \quad p'(t) > 0 \text{ for } t \in (0,\infty), \quad p(0) = 0. \tag{5.7}$$

A model example of (5.1), (5.2) is a problem with the α-Laplacian described below.

Example 5.1. Consider

$$\phi(x) = |x|^\alpha \operatorname{sgn} x, \quad x \in \mathbb{R},\ \alpha \ge 1.$$

Then ϕ is continuously differentiable and increasing on $\mathbb{R}$, $\phi(0) = 0$, ϕ maps $\mathbb{R}$ onto $\mathbb{R}$ and $\phi'(x) = \alpha|x|^{\alpha-1} > 0$ for $x \in (\mathbb{R} \setminus \{0\})$, that is conditions (5.3) and (5.4) are fulfilled. If we take

$$p(t) = t^\beta, \quad t \in [0,\infty),\ \beta > 0,$$

then

$$p \in C[0,\infty) \cap C^1(0,\infty), \quad p(0) = 0, \quad p'(t) = \beta t^{\beta-1} > 0 \text{ for } t \in (0,\infty),$$

which means that p fulfils (5.7). As an example of f satisfying conditions (5.5) and (5.6) we can take

$$f(x) = x\,(x - \phi(L_0))\,(\phi(L) - x), \quad x \in \mathbb{R},\ L_0 < 0 < L,$$

because f is continuous on $\mathbb{R}$, $f(\phi(L_0)) = f(0) = f(\phi(L)) = 0$ and $xf(x) > 0$ for $x \in ((\phi(L_0), \phi(L)) \setminus \{0\})$.

The both Chapters 5 and 6 are devoted to bounded solutions defined on $[0, \infty)$. Therefore, we use the next definitions.

Definition 5.2. A function $u \in C^1[0,\infty)$ with $\phi(u') \in C^1(0,\infty)$ which satisfies equation (5.1) for every $t \in (0,\infty)$ is called a *solution of equation* (5.1). If moreover u satisfies the initial conditions (5.2), then u is called a *solution of problem* (5.1), (5.2).

Definition 5.3. Consider a solution u of problem (5.1), (5.2) with $u_0 \in [L_0, L)$ and denote

$$u_{\sup} := \sup\{u(t) \colon t \in [0,\infty)\}.$$

If $u_{\sup} < L$, then u is called a *damped solution* of problem (5.1), (5.2).
If $u_{\sup} = L$, then u is called a *homoclinic solution* of problem (5.1), (5.2).
The homoclinic solution is called a *regular homoclinic solution*, if $u(t) < L$ for $t \in [0,\infty)$ and a *singular homoclinic solution*, if there exists $t_0 > 0$ such that $u(t_0) = L$.

Remark 5.4. Equation (5.1) has the constant solutions $u(t) \equiv L$, $u(t) \equiv 0$ and $u(t) \equiv L_0$. Moreover, the solution $u(t) \equiv 0$ is the only solution of problem (5.1), (5.2) with $u_0 = 0$. Really, u' cannot be positive on $(0,\delta)$ for some $\delta > 0$, since then u is positive on $(0,\delta)$ and integrating equation (5.1) from 0 to $t \in (0,\delta)$, we get, by (5.6),

$$p(t)\phi(u'(t)) = -\int_0^t p(s) f(\phi(u(s)))\,\mathrm{d}s < 0,$$

a contradiction. Similarly, u' cannot be negative.

Our goal in this chapter is to prove new existence and uniqueness results for the IVP (5.1), (5.2). The presence of ϕ-Laplacian in equation (5.1) brings difficulties in the study of the uniqueness. For example, if $\phi(x) = |x|^\alpha \operatorname{sgn} x$ and $\alpha > 1$, then ϕ fulfils the Lipschitz condition on $\mathbb{R}$. On the other hand, $\phi^{-1} = |x|^{\frac{1}{\alpha}} \operatorname{sgn} x$ and $(\phi^{-1})'(x) = \frac{1}{\alpha}|x|^{\frac{1}{\alpha}-1}$. Thus, we get $\lim_{x\to 0}(\phi^{-1})'(x) = \infty$ and the function ϕ^{-1} does not fulfil the Lipschitz condition in the neighbourhood of zero. Since both ϕ and ϕ^{-1} have to be present in the operator form of problem (5.1), (5.2), (cf. (5.45)), we cannot use the standard approach with a Lipschitz constant to prove the uniqueness near zero. Therefore, we develop a different approach near zero and show the conditions which guarantee the uniqueness of damped and regular homoclinic solutions of problem (5.1), (5.2).

For these aims, we introduce the auxiliary equation

$$(p(t)\phi(u'(t)))' + p(t)\tilde{f}(\phi(u(t))) = 0, \tag{5.8}$$

where

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in [\phi(L_0), \phi(L)], \\ 0 & \text{for } x < \phi(L_0), \ x > \phi(L). \end{cases} \tag{5.9}$$

5.2 Properties of solutions

In this section, we describe the properties of solutions of the auxiliary equation (5.8), where the nonlinearity $\tilde{f}$ is bounded and obtained from f by (5.9). By means of these results, we proceed to a priori estimates of solutions, existence and continuous dependence of solutions on initial values in next sections.

Lemma 5.5. *Assume that (5.3)–(5.7)hold and let u be a solution of equation (5.8). Assume that there exists $b \geq 0$ such that $u(b) \in (L_0, 0)$ and $u'(b) = 0$. Then $u'(t) > 0$ for $t \in (b, \theta]$, where θ is the first zero of u on (b, ∞). If such θ does not exist, then $u'(t) > 0$ for $t \in (b, \infty)$.*

Proof. Let $b \geq 0$ be such that $u(b) \in (L_0, 0)$ and $u'(b) = 0$. First, assume that there exists $\theta > b$ satisfying $u(t) < 0$ on (b, θ) and $u(\theta) = 0$. Suppose that there exists $\tau \in (b, \theta)$ such that $u'(\tau) \leq 0$, $u(t) \in (L_0, u(b)]$ for $t \in (b, \tau]$. Integrate (5.8) from b to τ and obtain

$$p(\tau)\phi(u'(\tau)) = -\int_b^\tau p(s)\tilde{f}(\phi(u(s)))\,\mathrm{d}s > 0.$$

Hence, by (5.3) and (5.7), $u'(\tau) > 0$, a contradiction. Therefore, $u' > 0$ on (b, θ). Moreover, integrating (5.8) over $[b, \theta]$, we get

$$p(\theta)\phi(u'(\theta)) = -\int_b^\theta p(s)\tilde{f}(\phi(u(s)))\,\mathrm{d}s > 0.$$

Thus, by (5.3) and (5.7), $u'(\theta) > 0$ and we have $u' > 0$ on $(b, \theta]$. If u is positive on $[b, \infty)$, we obtain as before that $u' > 0$ on (b, ∞). $\square$

By analogy, we get the dual lemma.

Lemma 5.6. *Let (5.3)–(5.7) hold and let u be a solution of equation (5.8). Assume that there exists $a \geq 0$ such that $u(a) \in (0, L)$ and $u'(a) = 0$. Then $u'(t) < 0$ for $t \in (a, \theta]$, where θ is the first zero of u on (a, ∞). If such θ does not exist, then $u'(t) < 0$ for $t \in (a, \infty)$.*

Lemma 5.7. *Let (5.3)–(5.7) hold and let u be a solution of equation (5.8). Assume that there exists $a \geq 0$ such that $u(a) = L$ and $u'(a) = 0$.*

a) Let $\theta > a$ be the first zero of u on (a, ∞). Then there exists $a_1 \in [a, \theta)$ such that

$$u(a_1) = L, \quad u'(a_1) = 0, \quad 0 \leq u(t) < L, \ u'(t) < 0, \ t \in (a_1, \theta].$$

b) *Let $u > 0$ on $[a, \infty)$ and $u \not\equiv L$ on $[a, \infty)$. Then there exists $a_1 \in [a, \infty)$ such that*

$$u(a_1) = L, \quad u'(a_1) = 0, \quad 0 < u(t) < L, \; u'(t) < 0, \; t \in (a_1, \infty).$$

In the both cases, $u(t) = L$ for $t \in [a, a_1]$.

Proof.

a) Assume that there exists $t^\star > a$ such that $u(t^\star) > L$. Then we can find $\tau \in [a, t^\star)$ fulfilling

$$u(t) > L, \; t \in (\tau, t^\star], \qquad u(\tau) = L. \tag{5.10}$$

Hence, $u'(\tau) \geq 0$. Integrating (5.8) over $[\tau, t^\star]$, we get, by (5.9),

$$p(t^\star)\phi(u'(t^\star)) = p(\tau)\phi(u'(\tau)) \geq 0.$$

Hence, by (5.3) and (5.7), $u'(t^\star) \geq 0$. Therefore, $u > L$ on $[t^\star, \infty)$ and u cannot have the zero θ, a contradiction. We have proved $0 < u \leq L$ on $[a, \theta)$ and, according to (5.8),

$$(p(t)\phi(u'(t)))' = -p(t)f(\phi(u(t))) \leq 0, \quad t \in [a, \theta].$$

Consequently, $u' \leq 0$ and u is nonincreasing on $[a, \theta]$. Hence, there exists $a_1 = [a, \theta)$ such that

$$u(a_1) = L, \quad u'(a_1) = 0, \quad 0 < u(t) < L, \; t \in (a_1, \theta).$$

Since u is monotonous on $[a, a_1]$, then $u \equiv L$ on $[a, a_1]$. Suppose that there exists $\tau_1 \in (a_1, \theta)$ such that $u'(\tau_1) = 0$. Integrate (5.8) from a_1 to τ_1 and obtain

$$p(\tau_1)\phi(u'(\tau_1)) = -\int_{a_1}^{\tau_1} p(s)\tilde{f}(\phi(u(s)))\,\mathrm{d}s < 0,$$

which yields $u'(\tau_1) < 0$, a contradiction. Therefore, $u' < 0$ on (a_1, θ). In addition, by integrating (5.8) over $[a_1, \theta]$, we get

$$p(\theta)\phi(u'(\theta)) = -\int_{a_1}^{\theta} p(s)\tilde{f}(\phi(u(s)))\,\mathrm{d}s < 0.$$

Thus, $u'(\theta) < 0$ and we have $u' < 0$ on $(a_1, \theta]$.

b) Assume as in part a) that there exists $t^\star > a$ such that $u(t^\star) > L$. Then we can find $\tau \in [a, t^\star)$ satisfying (5.10). Hence, $u'(\tau) \geq 0$. By Integrating (5.8) from τ to $t \in (\tau, t^\star]$ and using (5.9), we get

$$p(t)\phi(u'(t)) = p(\tau)\phi(u'(\tau)), \qquad t \in (\tau, t^\star].$$

If $u'(\tau) = 0$, then $u'(t) = 0$ for $t \in (\tau, t^\star]$, which contradicts $u(\tau) = L$, $u(t^\star) > L$. Therefore, $u'(\tau) > 0$. Let $\xi \in [a, \tau)$ be the minimal number fulfilling $0 < u(t) < L$, $u'(t) > 0$, $t \in (\xi, \tau)$. Since $u(\xi) < L$ and $u'(\xi) \geq 0$, we obtain $\xi > a$. Integrating (5.8) over $[a, \xi]$, we get

$$p(\xi)\phi(u'(\xi)) = -\int_a^\xi p(s)\tilde{f}(\phi(u(s)))\,\mathrm{d}s < 0.$$

Consequently, $u'(\xi) < 0$, a contradiction. We have proved that $0 < u \leq L$ on $[a, \infty)$, and that u is nonincreasing on (a, ∞). If $u \not\equiv L$ on $[a, \infty)$, we can find $a_1 \geq a$ such that the assertion b) holds using the arguments in part a). Since u is monotonous on $[a, a_1]$, we have $u \equiv L$ on $[a, a_1]$. □

In order to derive further important properties of solutions of (5.8), we assume that

$$\exists\, \bar{B} \in (L_0, 0)\colon \tilde{F}\left(\bar{B}\right) = \tilde{F}(L), \qquad \text{where } \tilde{F}(x) := \int_0^x \tilde{f}(\phi(s))\,\mathrm{d}s, \quad x \in \mathbb{R} \quad (5.11)$$

and

$$\limsup_{t\to\infty} \frac{p'(t)}{p(t)} < \infty. \quad (5.12)$$

Remark 5.8. According to (5.6), we have $\tilde{F} \in C^1(\mathbb{R})$, $\tilde{F}(0) = 0$, $\tilde{F}$ is positive and increasing on $[0, L]$ and positive and decreasing on $[L_0, 0]$.

Example 5.9. Let p, ϕ and f be from Example 5.1 and, in addition, $L < -L_0$. Since ϕ is odd and increasing on $\mathbb{R}$ and $0 < L < -L_0$, we get, similarly as in Example 2.18, that $\tilde{F}(L) < \tilde{F}(L_0)$. Therefore, there exists $\bar{B} \in (L_0, 0)$ such that $\tilde{F}\left(\bar{B}\right) = \tilde{F}(L)$, which yields (5.11). Further,

$$\limsup_{t\to\infty} \frac{p'(t)}{p(t)} = \lim_{t\to\infty} \frac{\beta t^{\beta-1}}{t^\beta} = \lim_{t\to\infty} \frac{\beta}{t} = 0 < \infty,$$

that is (5.12) holds.

Remark 5.10. From (5.3) and (5.4), we get

$$x\phi(x) > 0 \quad \text{for } x \in (\mathbb{R} \setminus \{0\}), \quad (5.13)$$

and there exists an inverse function ϕ^{-1} which is continuous and increasing on $\mathbb{R}$. By (5.7), the function p is positive and increasing on $(0, \infty)$.

Lemma 5.11. *Assume that (5.3)–(5.7), (5.11) and (5.12) hold. Let u be a solution of equation (5.8) and let there exist $b \geq 0$ and $\theta > b$ such that*

$$u(b) \in \left[\bar{B}, 0\right), \quad u'(b) = 0, \quad u(\theta) = 0, \quad u(t) < 0,\ t \in [b, \theta). \quad (5.14)$$

Then there exists $a \in (\theta, \infty)$ such that

$$u'(a) = 0, \quad u'(t) > 0, \ t \in (b, a), \quad u(a) \in (0, L).$$

Proof. Let u be a solution of equation (5.8) satisfying (5.14). Then

$$\phi'(u'(t))u''(t) + \frac{p'(t)}{p(t)}\phi(u'(t)) + \tilde{f}(\phi(u(t))) = 0, \quad t \in (0, \infty). \tag{5.15}$$

By Lemma 5.5 and (5.14), we have $u'(t) > 0$ for $t \in (b, \theta]$.

Step 1. We assume that $a > \theta$ satisfying $u'(a) = 0$ does not exist. Then we get

$$u'(t) > 0, \quad t \in (b, \infty) \tag{5.16}$$

and hence, u is increasing on (b, ∞). Since $u(\theta) = 0$, the inequality

$$u(t) > 0, \quad t \in (\theta, \infty) \tag{5.17}$$

holds. Let $(\theta, A) \subset (\theta, \infty)$ be a maximal interval with the property

$$u(t) < L, \quad t \in (\theta, A). \tag{5.18}$$

Using (5.5), (5.6) and (5.13), we obtain $\tilde{f}(\phi(u(t))) > 0$ for $t \in (\theta, A)$. Consequently, equation (5.15) yields

$$u''(t) < 0, \quad t \in (\theta, A) \tag{5.19}$$

and thus, u' is decreasing on (θ, A).

(i) Let $A < \infty$. Then (5.18) implies $u(A) = L$. Multiplying (5.15) by u' and integrating from b to A, we get

$$\int_b^A \phi'\left(u'(s)\right) u'(s)u''(s)\,\mathrm{d}s + \int_b^A \frac{p'(s)}{p(s)}\phi\left(u'(s)\right) u'(s)\,\mathrm{d}s + \int_b^A \tilde{f}(\phi(u(s)))u'(s)\,\mathrm{d}s = 0.$$

After substitutions $x = u'(s)$ in the first integral and $y = u(s)$ in the third integral, we obtain

$$\int_{u'(b)}^{u'(A)} x\phi'(x)\,\mathrm{d}x + \int_b^A \frac{p'(s)}{p(s)}\phi(u'(s))u'(s)\,\mathrm{d}s + \int_{u(b)}^{u(A)} \tilde{f}(\phi(y))\,\mathrm{d}y = 0. \tag{5.20}$$

Due to (5.14) and (5.16), we have $u'(b) = 0$ and $u'(A) > 0$. Therefore, conditions (5.7), (5.13) and (5.16) imply

$$\int_{u'(b)}^{u'(A)} x\phi'(x)\,\mathrm{d}x > 0, \qquad \int_b^A \frac{p'(s)}{p(s)}\phi(u'(s))u'(s)\,\mathrm{d}s > 0.$$

Consequently, (5.20) yields

$$\int_{u(b)}^{u(A)} \tilde{f}(\phi(y))\,\mathrm{d}y = \int_{u(b)}^{L} \tilde{f}(\phi(y))\,\mathrm{d}y < 0$$

and hence, $\tilde{F}(L) - \tilde{F}(u(b)) < 0$. By (5.11), (5.14) and Remark 5.8, we obtain

$$\tilde{F}(L) < \tilde{F}(u(b)) \le \tilde{F}\left(\bar{B}\right) = \tilde{F}(L),$$

a contradiction.

(ii) Now, we assume that $A = \infty$. Inequalities (5.17) and (5.18) give $0 < u(t) < L$ for $t \in (\theta, \infty)$. Due to (5.16), u is increasing on (θ, ∞), so there exists $\lim_{t\to\infty} u(t) =: \ell \in (0, L]$. By virtue of (5.16) and (5.19), u' is decreasing and positive on (θ, ∞) and so, $\lim_{t\to\infty} u'(t) \ge 0$. Since ℓ is finite, we have

$$\lim_{t\to\infty} u'(t) = 0. \tag{5.21}$$

Let $\ell = L$. Similarly as in part (i), we derive

$$\int_{u'(b)}^{u'(t)} x\phi'(x)\,\mathrm{d}x + \int_b^t \frac{p'(s)}{p(s)}\phi(u'(s))u'(s)\,\mathrm{d}s + \int_{u(b)}^{u(t)} \tilde{f}(\phi(y))\,\mathrm{d}y = 0, \quad t \in (b, \infty)\,.$$

Since the first integral is positive, we have

$$\int_{u(b)}^{u(t)} \tilde{f}(\phi(y))\,\mathrm{d}y < -\int_b^t \frac{p'(s)}{p(s)}\phi(u'(s))u'(s)\,\mathrm{d}s, \quad t \in (b, \infty)\,.$$

Letting $t \to \infty$ here, we get

$$\lim_{t\to\infty}\left(\tilde{F}(u(t)) - \tilde{F}(u(b))\right) = \tilde{F}(L) - \tilde{F}(u(b)) \le -\int_b^\infty \frac{p'(s)}{p(s)}\phi(u'(s))u'(s)\,\mathrm{d}s < 0.$$

Using Remark 5.8 and conditions (5.11), (5.14), we deduce

$$\tilde{F}(L) < \tilde{F}(u(b)) \le \tilde{F}\left(\bar{B}\right) = \tilde{F}(L),$$

which is a contradiction.

Let $\ell \in (0, L)$. For $t \to \infty$ in (5.15), we get, by (5.4) and (5.12),

$$\phi'(0) \lim_{t\to\infty} u''(t) = -\tilde{f}(\phi(\ell)). \tag{5.22}$$

Since $-\tilde{f}(\phi(\ell)) \in (-\infty, 0)$, we have $\lim_{t\to\infty} u''(t) < 0$, contrary to (5.21).

We have proved that there exists $a > \theta$ such that $u'(a) = 0$.

Step 2. Let $u' > 0$ on $[\theta, a)$. Then $u(a) > 0$. It remains to prove that $u(a) < L$. Multiplying (5.15) by u' and integrating from b to a, we get similarly as in part (i) of Step 1 that

$$\int_{u(b)}^{u(a)} \tilde{f}(\phi(y))\,\mathrm{d}y < 0, \quad t \in (b, a)$$

and

$$\tilde{F}(u(a)) < \tilde{F}(u(b)) \leq \tilde{F}\left(\bar{B}\right) = \tilde{F}(L).$$

According to Remark 5.8, we have $u(a) < L$. □

Lemma 5.12. *Assume that (5.3)–(5.7), (5.11) and (5.12) hold. Let u be a solution of equation (5.8) and let there exist $a \geq 0$ and $\theta > a$ such that*

$$u(a) \in (0, L], \quad u'(a) = 0, \quad u(\theta) = 0, \quad u(t) > 0, \ t \in [a, \theta). \tag{5.23}$$

Then there exists $b \in (\theta, \infty)$ such that

$$u'(b) = 0, \quad u'(t) < 0, \ t \in (a, b), \quad u(b) \in \left(\bar{B}, 0\right).$$

Proof. We argue similarly as in the proof of Lemma 5.11. Let u be a solution of equation (5.8) satisfying (5.23). By Lemma 5.6, Lemma 5.7 a) and (5.23), we have $u'(t) < 0$, for $t \in (a, \theta]$.

Step 1. We assume that $b > \theta$ satisfying $u'(b) = 0$ does not exist. Then we get

$$u(t) < 0, \ t \in (\theta, \infty), \qquad u'(t) < 0, \ t \in (a, \infty) \tag{5.24}$$

and hence, u is decreasing on (a, ∞). Let $(\theta, A) \subset (\theta, \infty)$ be the maximal interval with the property

$$u(t) > \bar{B}, \quad t \in (\theta, A). \tag{5.25}$$

Then, from (5.15), we derive

$$u''(t) > 0, \quad t \in (\theta, A) \tag{5.26}$$

and thus, u' is increasing on (θ, A).

(i) Let $A < \infty$. Then (5.24) and (5.25) imply $u(A) = \bar{B}$. Similarly as in the proof of Lemma 5.11 (Step 1, part (i)), we get the contradiction

$$\tilde{F}\left(\bar{B}\right) < \tilde{F}(u(a)) \leq \tilde{F}(L) = \tilde{F}\left(\bar{B}\right).$$

(ii) Now, we assume that $A = \infty$. By (5.24) and (5.25), u is decreasing on (θ, ∞) and $\lim_{t\to\infty} u(t) = \ell \in \left[\bar{B}, 0\right)$. Due to (5.24) and (5.26), u' is increasing and negative on (θ, ∞) and $\lim_{t\to\infty} u'(t) \leq 0$. Since ℓ is finite, we have $\lim_{t\to\infty} u'(t) = 0$. Similarly as in the proof of Lemma 5.11 (Step 1, part (ii)), we obtain a contradiction for $\ell = \bar{B}$ and for $\ell \in \left(\bar{B}, 0\right)$.

We have shown that there exists $b > \theta$ such that $u'(b) = 0$.

Step 2. Let $u' < 0$ on $[\theta, b)$. Then $u(b) < 0$ and we proceed similarly as in Step 2 of the proof of Lemma 5.11 and get $\tilde{F}(u(b)) < \tilde{F}\left(\bar{B}\right)$. Remark 5.8 yields $u(b) > \bar{B}$. ☐

Lemma 5.13. *Assume that (5.3)–(5.7) and (5.12) hold. Let u be a solution of equation (5.8) and let there exist $b \geq 0$ such that*

$$u(b) \in (L_0, 0), \quad u'(b) = 0, \quad u(t) < 0, \ t \in [b, \infty).$$

Then

$$\lim_{t\to\infty} u(t) = 0, \quad \lim_{t\to\infty} u'(t) = 0.$$

Proof. By Lemma 5.5, $u'(t) > 0$ for $t \in (b, \infty)$. Hence, u is increasing on (b, ∞),

$$L_0 < u(t) < 0, \quad t \in (b, \infty) \tag{5.27}$$

and there exists

$$\lim_{t\to\infty} u(t) =: \ell \in (u(b), 0].$$

Multiplying equation (5.15) by u', integrating it from b to t and using substitutions, we obtain

$$\psi_1(t) + \psi_2(t) + \psi_3(t) = 0, \quad t \in (b, \infty), \tag{5.28}$$

where

$$\psi_1(t) = \int_{u'(b)}^{u'(t)} x\phi'(x)\,\mathrm{d}x, \qquad \psi_2(t) = \int_b^t \frac{p'(s)}{p(s)}\phi(u'(s))u'(s)\,\mathrm{d}s,$$
$$\psi_3(t) = \int_{u(b)}^{u(t)} \tilde{f}(\phi(x))\,\mathrm{d}x.$$

We have $\psi_3(t) = \tilde{F}(u(t)) - \tilde{F}(u(b))$, where $\tilde{F}$ is defined by (5.11). Since $\tilde{F}(x)$ is decreasing for $x \in (L_0, 0)$ and u is increasing on (b, ∞), then (5.27) yields that $\tilde{F}(u(t))$ is decreasing for $t \in (b, \infty)$ and $\lim_{t\to\infty} \tilde{F}(u(t)) = \tilde{F}(\ell)$. Therefore,

$$\lim_{t\to\infty} \psi_3(t) =: Q_3 \in \left(-\tilde{F}(L_0), 0\right).$$

The positivity of ψ_1 on (b, ∞) yields the inequality $\psi_2(t) < -\psi_3(t)$ for $t \in (b, \infty)$. Since ψ_2 is continuous, increasing and positive on (b, ∞), we have

$$\lim_{t\to\infty} \psi_2(t) =: Q_2 \in (0, -Q_3].$$

Consequently, (5.28) gives

$$\lim_{t\to\infty} \psi_1(t) =: Q_1 \in \left[0, \tilde{F}(L_0)\right).$$

Hence,

$$\lim_{t\to\infty} \Phi(u'(t)) = Q_1, \quad \text{where } \ \Phi(z) := \int_0^z x\phi'(x)\,\mathrm{d}x, \ z > 0.$$

The function Φ is positive, continuous and increasing on $(0, \infty)$, so its inverse Φ^{-1} is positive, continuous and increasing, as well. Thus,

$$\lim_{t\to\infty} \Phi^{-1}(\Phi(u'(t))) = \lim_{t\to\infty} u'(t) = \Phi^{-1}(Q_1) \geq 0.$$

According to (5.27),

$$\lim_{t\to\infty} u'(t) = 0.$$

Finally, assume that $\ell \in (u(b), 0)$. Letting $t \to \infty$ in (5.15), we get, by (5.4), (5.12), that (5.22) holds. Since $-\tilde{f}(\phi(\ell)) \in (0, \infty)$, we get $\lim_{t\to\infty} u''(t) > 0$, contrary to $\lim_{t\to\infty} u'(t) = 0$. Therefore, $\ell = 0$. □

Lemma 5.14. *Assume that* (5.3)–(5.7) *and* (5.12) *hold. Let u be a solution of equation* (5.8) *and let there exist $a \geq 0$ such that*

$$u(a) \in (0, L], \quad u'(a) = 0, \quad u(t) > 0, \ t \in [a, \infty).$$

Then either

$$u(t) \equiv L, \quad t \in [a, \infty) \tag{5.29}$$

or

$$\lim_{t\to\infty} u(t) = 0, \quad \lim_{t\to\infty} u'(t) = 0. \tag{5.30}$$

Proof. *Step 1.* Let $u(a) \in (0, L)$. We continue analogously as in the proof of Lemma 5.13. According to Lemma 5.6, $u'(t) < 0$ for $t \in (a, \infty)$. Hence,

$$0 < u(t) < L, \quad t \in (a, \infty) \tag{5.31}$$

and

$$\lim_{t\to\infty} u(t) =: \ell \in [0, u(a)).$$

By multiplying equation (5.15) by u' and integrating it over $[a, t]$, we obtain (5.28) with b replaced by a. Since $\tilde{F}(x)$ is increasing for $x \in (0, L)$ and u is decreasing on (a, ∞), then (5.31) gives that $\tilde{F}(u(t))$ is decreasing for $t \in (a, \infty)$. Consequently, $\lim_{t\to\infty} \tilde{F}(u(t)) = \tilde{F}(\ell)$. Let ψ_1, ψ_2 and ψ_3 be defined as in the proof of Lemma 5.13, where b is replaced by a. Then

$$\lim_{t\to\infty} \psi_3(t) = \lim_{t\to\infty} \tilde{F}(u(t)) - \tilde{F}(u(a)) =: Q_3 \in \left(-\tilde{F}(L), 0\right).$$

The positivity of ψ_1 on (a, ∞) yields the inequality $\psi_2(t) < -\psi_3(t)$ for $t \in (a, \infty)$. Since ψ_2 is continuous, increasing and positive on (a, ∞), we get

$$\lim_{t\to\infty} \psi_2(t) =: Q_2 \in (0, -Q_3] \quad \text{and} \ \lim_{t\to\infty} \psi_1(t) =: Q_1 \in \left[0, \tilde{F}(L)\right).$$

Hence,

$$\lim_{t\to\infty}\Phi(u'(t)) = Q_1, \quad \text{where} \;\; \Phi(z) := \int_0^z x\phi'(x)\,\mathrm{d}x, \;\; z<0.$$

The function Φ is positive, continuous and decreasing on $(-\infty,0)$ and so, its inverse Φ^{-1} is positive, continuous and decreasing, as well. Therefore,

$$\lim_{t\to\infty}\Phi^{-1}(\Phi(u'(t))) = \lim_{t\to\infty}u'(t) = \Phi^{-1}(Q_1) \ge 0.$$

By virtue of (5.31), we have $\lim_{t\to\infty}u'(t)=0$.

Assume that $\ell\in(0,u(a))$. Letting $t\to\infty$ in (5.15), we obtain (5.22). Since $-\tilde f(\phi(\ell))\in(-\infty,0)$, we get $\lim_{t\to\infty}u''(t)<0$, contrary to $\lim_{t\to\infty}u'(t)=0$. Hence, $\ell=0$.

Step 2. Let $u(a)=L$. Assume that u does not fulfil (5.29). Due to Lemma 5.7 b), there exists $a_1\ge a$ such that $0<u(t)<L$, $u'(t)<0$, $t\in(a_1,\infty)$ and we can use the arguments from Step 1 to prove (5.30). □

5.3 A priori estimates of solutions

In order to prove the existence and uniqueness of solutions of the auxiliary problem (5.8), (5.2) and of the original problem (5.1), (5.2), a priori estimates derived in this section are needed.

Lemma 5.15. *Assume that* (5.3)–(5.7) *hold. Let u be a solution of problem* (5.8), (5.2) *with $u_0\in\big(L_0,\bar B\big)$. Let there exist $\theta>0$, $a>\theta$ such that*

$$u(\theta)=0, \quad u(t)<0,\; t\in[0,\theta), \quad u'(a)=0, \quad u'(t)>0,\; t\in(\theta,a). \tag{5.32}$$

Then

$$u(a)\in(0,L], \quad u'(t)>0,\; t\in(0,a). \tag{5.33}$$

Proof. According to Lemma 5.5 and (5.32), we have $u'>0$ on $(0,a)$. Therefore, $u(a)>0$. Now, assume that $u(a)>L$. Hence, there exists $a_0\in(\theta,a)$ such that $u(t)>L$ on $(a_0,a]$. Integrating equation (5.8) over (a_0,a) and using (5.9), we get

$$p(a_0)\phi(u'(a_0)) - p(a)\phi(u'(a)) = \int_{a_0}^{a} p(s)\tilde f(\phi(u(s)))\,\mathrm{d}s = 0$$

and so, $p(a_0)\phi(u'(a_0))=0$. Thus, $u'(a_0)=0$, contrary to $u'>0$ on $(0,a)$. We have proved that $u(a)\le L$. □

Lemma 5.16. *Let assumptions* (5.3)–(5.7), (5.11) *and* (5.12) *hold. Let u be a solution of problem* (5.8), (5.2) *with $u_0\in(L_0,L)$. Then*

$$u_0\in\big[\bar B,L\big) \;\Rightarrow\; \bar B<u(t)<L, \quad t\in(0,\infty), \tag{5.34}$$

$$u_0\in\big(L_0,\bar B\big) \;\Rightarrow\; u_0<u(t), \quad t\in(0,\infty). \tag{5.35}$$

Proof. Let $u(0) = u_0 = 0$. According to Remark 5.4, $u(t) \equiv 0$ is a unique solution of problem (5.8), (5.2), that is (5.34) holds.

Let $u(0) = u_0 \in (0, L)$. If $u > 0$ on $(0, \infty)$, then, by Lemma 5.6, $u' < 0$ on $(0, \infty)$ and (5.34) holds. Assume that there exists $\theta_1 > 0$ such that $u(\theta_1) = 0$, $u(t) > 0$ for $t \in [0, \theta_1)$. In view of Lemma 5.12,

$$\exists\, b \in (\theta_1, \infty)\colon u'(b) = 0, \quad u'(t) < 0,\ t \in (0, b), \quad u(b) = \left(\bar{B}, 0\right).$$

If $u < 0$ on (b, ∞), then, by Lemma 5.5, u is increasing on (b, ∞) and (5.34) is valid. Assume that there exists $\theta_2 > b$ such that $u(\theta_2) = 0$, $u(t) < 0$ for $t \in [b, \theta_2)$. Due to Lemma 5.11,

$$\exists\, a \in (\theta_2, \infty)\colon u'(a) = 0, \quad u'(t) > 0,\ t \in (b, a), \quad u(a) = (0, L).$$

Now, we use the previous arguments replacing 0 by a.

Let $u(0) = u_0 \in \left[\bar{B}, 0\right)$. We have the same situation as before, where b is replaced by 0. So, we argue similarly.

Let $u(0) = u_0 \in \left(L_0, \bar{B}\right)$. If $u < 0$ on $(0, \infty)$, then, by Lemma 5.5, $u' > 0$ on $(0, \infty)$ and (5.35) is valid. Assume that there exists $\theta_1 > 0$ such that $u(\theta_1) = 0$, $u(t) < 0$ for $t \in [0, \theta_1)$. Due to Lemma 5.5, $u' > 0$ on $(0, \theta_1]$. If $u' > 0$ on (θ_1, ∞), then (5.35) holds. Assume that there exists $a > \theta_1$ such that $u'(a) = 0$, $u'(t) > 0$ for $t \in (\theta_1, a)$. According to Lemma 5.15, (5.33) holds. If $u > 0$ on $[a, \infty)$, (5.35) is valid. Let there exist $\theta_2 > a$ such that $u(\theta_2) = 0$, $u > 0$ on $[a, \theta_2)$. We can apply Lemma 5.12 and argue as before. □

Remark 5.17. According to (5.9), (5.34), (5.35) and Definition 5.3, u is a damped or a homoclinic solution of the auxiliary problem (5.8), (5.2) if and only if u is a damped or a homoclinic solution of the original problem (5.1), (5.2).

Note that the auxiliary nonlinearity $\tilde{f}$ is bounded due to (5.9). Therefore, there exists $\tilde{M} > 0$ such that

$$\left|\tilde{f}(x)\right| \le \tilde{M}, \quad x \in \mathbb{R}. \tag{5.36}$$

For the following investigation, we introduce a function φ

$$\varphi(t) := \frac{1}{p(t)} \int_0^t p(s)\, \mathrm{d}s, \quad t \in (0, \infty), \qquad \varphi(0) = 0.$$

This function is continuous on $[0, \infty)$ and satisfies

$$0 < \varphi(t) \le t, \quad t \in (0, \infty), \qquad \lim_{t \to 0^+} \varphi(t) = 0. \tag{5.37}$$

Lemma 5.18. *Assume that (5.3)–(5.7), (5.11) and (5.12) hold. Then there exists $\tilde{c} > 0$ such that*

$$|u'(t)| \le \tilde{c}, \quad t \in [0, \infty) \tag{5.38}$$

for every solution u of problem (5.8), (5.2) with $u_0 \in (L_0, L)$.

Proof. Denote

$$\Psi_1(z) := \int_0^z x\phi'(x)\,\mathrm{d}x, \quad \Psi_2(z) := \int_0^z x\phi'(-x)\,\mathrm{d}x, \quad z \in [0,\infty).$$

Clearly, Ψ_1, Ψ_2 are positive, continuous and increasing on $(0,\infty)$. Put

$$\tilde{c} := \max\left\{\Psi_1^{-1}\left(\tilde{F}(L_0)\right), \Psi_2^{-1}\left(\tilde{F}(L)\right)\right\}, \tag{5.39}$$

where $\tilde{F}$ is defined in (5.11).

Let $u(0) = u_0 = 0$. Due to Remark 5.4, $u(t) \equiv 0$ is a unique solution of problem (5.8), (5.2). Thus, $u' = 0$ on $[0,\infty)$ and (5.38) is satisfied.

Let $u(0) = u_0 \in (L_0, 0)$, $u'(0) = 0$ and let u be a solution of equation (5.8). Then (5.15) holds.

(i) Assume that $u < 0$ on $[0,\infty)$. By Lemma 5.5, $u' > 0$ on $(0,\infty)$ and Lemma 5.13 gives $\lim_{t\to\infty} u'(t) = 0$. Thus, there exists $\xi \in (0,\infty)$ such that

$$\max_{t\in[0,\infty)} |u'(t)| = u'(\xi) > 0, \quad u(\xi) \in (u_0, 0). \tag{5.40}$$

Multiplying (5.15) by u' and integrating over $[0,\xi]$, we get

$$\int_{u'(0)}^{u'(\xi)} x\phi'(x)\,\mathrm{d}x + \int_0^\xi \frac{p'(t)}{p(t)}\phi\left(u'(t)\right)u'(t)\,\mathrm{d}t + \int_{u(0)}^{u(\xi)} \tilde{f}\left(\phi(x)\right)\,\mathrm{d}x = 0.$$

Since the second integral is positive, by using (5.40), we obtain

$$\Psi_1\left(u'(\xi)\right) < \tilde{F}(u_0) - \tilde{F}\left(u(\xi)\right) < \tilde{F}(u_0) < \tilde{F}(L_0),$$

which yields

$$0 < u'(\xi) < \Psi_1^{-1}\left(\tilde{F}(L_0)\right). \tag{5.41}$$

Due to (5.39) and (5.40), we get (5.38).

(ii) Assume that $\theta \in (0,\infty)$ is such that $u < 0$ on $[0,\theta)$, $u(\theta) = 0$. Then, by Lemma 5.5, $u' > 0$ on $(0,\theta]$. Let $a > \theta$ be such that $u' > 0$ on (θ, a), $u'(a) = 0$. We have $u > 0$, $u' > 0$ on (θ, a). Using (5.3), (5.6), (5.7) and (5.13), we get from (5.15) that $u'' < 0$ on $[\theta, a)$. Hence, u' is decreasing on $[\theta, a)$ and there exists $\xi \in (0,\theta)$ such that

$$\max_{t\in[0,a)} |u'(t)| = u'(\xi) > 0, \quad u(\xi) \in (u_0, 0). \tag{5.42}$$

Analogously as in part (i), we get (5.41) and if $a = \infty$, then estimate (5.38) is proved.

(iii) Assume that $a < \infty$ in (5.42). We have $u'(a) = 0$ and, by Lemma 5.11 and Lemma 5.15, we deduce that $u(a) \in (0, L]$. Let $u > 0$ on $[a, \infty)$. Then Lemma 5.14 gives $\lim_{t\to\infty} u'(t) = 0$ and hence, there exists $\eta \in (a, \infty)$ such that

$$\max_{t\in[a,\infty)} |u'(t)| = -u'(\eta) > 0, \quad u(\eta) \in (0, u(a)). \tag{5.43}$$

Multiplying (5.15) by u' and integrating it over $[a, \eta]$, we obtain

$$\int_{u'(a)}^{|u'(\eta)|} x\phi'(-x)\,\mathrm{d}x + \int_a^\eta \frac{p'(t)}{p(t)}\phi(u'(t))\,u'(t)\,\mathrm{d}t + \int_{u(a)}^{u(\eta)} \tilde{f}(\phi(x))\,\mathrm{d}x = 0.$$

Since the second integral is positive, by using (5.43), we get

$$\Psi_2(|u'(\eta)|) < \tilde{F}(u(a)) - \tilde{F}(u(\eta)) < \tilde{F}(L),$$

which gives

$$0 < |u'(\eta)| < \Psi_2^{-1}\left(\tilde{F}(L)\right). \tag{5.44}$$

Using (5.39) and (5.41)–(5.44), we obtain (5.38).

(iv) Assume that there exists $\chi \in (a, \infty)$, which is the next zero of u. Summarized, we have $u(a) \in (0, L]$, $u'(a) = 0$, $u(\chi) = 0$, $u > 0$ on $[a, \chi)$. In view of Lemma 5.12, there exists $b \in (\chi, \infty)$ such that $u'(b) = 0$, $u' < 0$ on (a, b), $u(b) \in (\bar{B}, 0)$. Due to (5.15), we have $u'' > 0$ on $[\chi, b)$. Consequently, there exists $\eta \in (a, \chi)$ such that

$$\max_{t\in[a,b]} |u'(t)| = -u'(\eta) > 0, \quad u(\eta) \in (0, u(a)).$$

Similarly as in part (iii), we derive (5.44).

(v) Since $u(b) < 0$, we continue repeating the argument of parts (i)–(iii) with b on place of 0 and the arguments of part (iv) writing $\tilde{b}$ instead of b. After finite or infinite number of steps, we obtain (5.38).

If $u_0 \in (0, L)$, we can argue similarly. □

5.4 Existence of a solution

This section is devoted to the existence of solutions of the auxiliary problem (5.8), (5.2), which is proved by means of the Schauder fixed point theorem.

Theorem 5.19 (Existence of solutions of problem (5.8), (5.2)). *Assume that (5.3)–(5.7) hold. Then, for each $u_0 \in [L_0, L]$, there exists a solution u of problem (5.8), (5.2).*

Proof. Clearly, for $u_0 = L_0$, $u_0 = 0$ and $u_0 = L$ there exists a solution of problem (5.8), (5.2) by Remark 5.4. Assume that $u_0 \in (L_0, 0) \cup (0, L)$. Integrating equation (5.8), we get the equivalent form of problem (5.8), (5.2)

$$u(t) = u_0 + \int_0^t \phi^{-1}\left(-\frac{1}{p(s)}\int_0^s p(\tau)\tilde{f}(\phi(u(\tau)))\,\mathrm{d}\tau\right)\mathrm{d}s, \quad t \in [0, \infty). \tag{5.45}$$

Choose a $\beta > 0$, consider the Banach space $C[0, \beta]$ with the maximum norm and define an operator $\mathcal{F}\colon C[0,\beta] \to C[0,\beta]$,

$$(\mathcal{F}u)(t) := u_0 + \int_0^t \phi^{-1}\left(-\frac{1}{p(s)}\int_0^s p(\tau)\tilde{f}(\phi(u(\tau)))\,\mathrm{d}\tau\right)\mathrm{d}s.$$

Put $\Lambda := \max\{|L_0|, L\}$, consider the ball

$$\mathcal{B}(0,R) = \left\{u \in C[0,\beta] : \|u\|_{C[0,\beta]} \le R\right\}, \quad \text{where } R := \Lambda + \beta\phi^{-1}\left(\tilde{M}\beta\right)$$

and $\tilde{M}$ is from (5.36). Since ϕ is increasing on $\mathbb{R}$, ϕ^{-1} is also increasing on $\mathbb{R}$ and, by (5.37), $\phi^{-1}\left(\tilde{M}\varphi(t)\right) \le \phi^{-1}\left(\tilde{M}\beta\right)$, $t \in [0,\beta]$. The norm of $\mathcal{F}u$ can be estimated as follows

$$\|\mathcal{F}u\|_{C[0,\beta]} = \max_{t\in[0,\beta]}\left|u_0 + \int_0^t \phi^{-1}\left(-\frac{1}{p(s)}\int_0^s p(\tau)\tilde{f}(\phi(u(\tau)))\,\mathrm{d}\tau\right)\mathrm{d}s\right| \le \Lambda$$
$$+ \int_0^t \left|\phi^{-1}\left(\tilde{M}\varphi(s)\right)\right|\mathrm{d}s \le \Lambda + \int_0^t \phi^{-1}\left(\tilde{M}\beta\right)\mathrm{d}s \le \Lambda + \beta\phi^{-1}\left(\tilde{M}\beta\right) = R,$$

which yields that $\mathcal{F}$ maps $\mathcal{B}(0,R)$ on itself.

Let us prove that $\mathcal{F}$ is compact on $\mathcal{B}(0,R)$. Choose a sequence $\{u_n\} \subset C[0,\beta]$ such that $\lim_{n\to\infty}\|u_n - u\|_{C[0,\beta]} = 0$. We have

$$(\mathcal{F}u_n)(t) - (\mathcal{F}u)(t) = \int_0^t \left(\phi^{-1}\left(-\frac{1}{p(s)}\int_0^s p(\tau)\tilde{f}(\phi(u_n(\tau)))\,\mathrm{d}\tau\right)\right.$$
$$\left. -\phi^{-1}\left(-\frac{1}{p(s)}\int_0^s p(\tau)\tilde{f}(\phi(u(\tau)))\,\mathrm{d}\tau\right)\right)\mathrm{d}s.$$

Since $\tilde{f}(\phi)$ is continuous on $\mathbb{R}$, we get

$$\lim_{n\to\infty}\left\|\tilde{f}(\phi(u_n)) - \tilde{f}(\phi(u))\right\|_{C[0,\beta]} = 0.$$

Denote

$$A_n(t) := -\frac{1}{p(t)}\int_0^t p(\tau)\tilde{f}(\phi(u_n(\tau)))\,\mathrm{d}\tau,$$
$$A(t) := -\frac{1}{p(t)}\int_0^t p(\tau)\tilde{f}(\phi(u(\tau)))\,\mathrm{d}\tau, \quad t \in (0,\beta], \quad A_n(0) = A(0) = 0, \quad n \in \mathbb{N}.$$

Then, for a fixed $n \in \mathbb{N}$,

$$|A_n(t) - A(t)| = \left| \frac{1}{p(t)} \int_0^t p(\tau) \left(\tilde{f}(\phi(u(\tau))) - \tilde{f}(\phi(u_n(\tau))) \, \mathrm{d}\tau \right) \right|, \quad t \in (0, \beta]$$

and, by (5.37) and (5.36), $\lim_{t \to 0^+} |A_n(t) - A(t)| = 0$. Therefore, $A_n - A \in C[0, \beta]$ and, using (5.37), we obtain for each $n \in \mathbb{N}$

$$\|A_n - A\|_{C[0,\beta]} \le \left\| \tilde{f}(\phi(u_n)) - \tilde{f}(\phi(u)) \right\|_{C[0,\beta]} \beta.$$

This implies that $\lim_{n \to \infty} \|A_n - A\|_{C[0,\beta]} = 0$. Using the continuity of ϕ^{-1} on $\mathbb{R}$, we have

$$\lim_{n \to \infty} \left\| \phi^{-1}(A_n) - \phi^{-1}(A) \right\|_{C[0,\beta]} = 0.$$

Consequently,

$$\begin{aligned} \lim_{n \to \infty} \|\mathcal{F}u_n - \mathcal{F}u\|_{C[0,\beta]} &= \lim_{n \to \infty} \left\| \int_0^t \left(\phi^{-1}(A_n(s)) - \phi^{-1}(A(s)) \right) \mathrm{d}s \right\|_{C[0,\beta]} \\ &\le \beta \lim_{n \to \infty} \left\| \phi^{-1}(A_n) - \phi^{-1}(A) \right\|_{C[0,\beta]} = 0, \end{aligned}$$

that is the operator $\mathcal{F}$ is continuous.

Choose an arbitrary $\varepsilon > 0$ and put $\delta := \frac{\varepsilon}{\phi^{-1}\left(\tilde{M}\beta\right)}$. Then, for $t_1, t_2 \in [0, \beta]$ and $u \in \mathcal{B}(0, R)$, we obtain

$$\begin{aligned} |t_1 - t_2| < \delta \Rightarrow &\ |(\mathcal{F}u)(t_1) - (\mathcal{F}u)(t_2)| \\ &= \left| \int_{t_2}^{t_1} \phi^{-1} \left(-\frac{1}{p(s)} \int_0^s p(\tau) \tilde{f}(\phi(u(\tau))) \, \mathrm{d}\tau \right) \mathrm{d}s \right| \le \left| \int_{t_2}^{t_1} \phi^{-1} \left(\tilde{M} \varphi(s) \right) \mathrm{d}s \right| \\ &\le \left| \int_{t_2}^{t_1} \phi^{-1} \left(\tilde{M}\beta \right) \mathrm{d}s \right| = \phi^{-1} \left(\tilde{M}\beta \right) |t_1 - t_2| < \phi^{-1} \left(\tilde{M}\beta \right) \delta = \varepsilon. \end{aligned}$$

Hence, the functions in $\mathcal{F}(\mathcal{B}(0, R))$ are equicontinuous and, by the Arzelà–Ascoli theorem, the set $\mathcal{F}(\mathcal{B}(0, R))$ is relatively compact. Consequently, the operator $\mathcal{F}$ is compact on $\mathcal{B}(0, R)$.

The Schauder fixed point theorem yields a fixed point $u^\star$ of $\mathcal{F}$ in $\mathcal{B}(0, R)$. Therefore,

$$u^\star(t) = u_0 + \int_0^t \phi^{-1} \left(-\frac{1}{p(s)} \int_0^s p(\tau) \tilde{f}(\phi(u^\star(\tau))) \, \mathrm{d}\tau \right) \mathrm{d}s.$$

Hence, $u^\star(0) = u_0$,

$$(p(t)\phi((u^\star)'(t)))' = -p(t) \tilde{f} \left(\phi(u^\star(t)) \right), \quad t \in [0, \beta].$$

Further,

$$|(u^\star)'(t)| = \left|\phi^{-1}\left(-\frac{1}{p(t)}\int_0^t p(s)\tilde{f}(\phi(u^\star(s)))\,\mathrm{d}s\right)\right| \le \phi^{-1}\left(\tilde{M}\varphi(t)\right), \quad t \in [0,\beta]\,.$$

Thus, due to (5.37),

$$\lim_{t\to 0^+} \phi^{-1}\left(\tilde{M}\varphi(t)\right) = \phi^{-1}(0) = 0$$

and therefore,

$$\lim_{t\to 0^+} (u^\star)'(t) = 0 = (u^\star)'(0).$$

According to (5.9), $\tilde{f}(\phi(u^\star(t)))$ is bounded on $[0,\infty)$ and hence, $u^\star$ can be extended to interval $[0,\infty)$ as a solution of equation (5.8). This classical extension result follows from more general Theorem 11.5 in [31]. □

Example 5.20. Consider $\phi\colon \mathbb{R}\to\mathbb{R}$ given by one of the next formulas

$$\phi(x) = |x|^\alpha \operatorname{sgn} x, \quad \alpha > 1, \tag{5.46}$$

$$\phi(x) = \left(x^4 + 2x^2\right)\operatorname{sgn} x, \tag{5.47}$$

$$\phi(x) = \sinh x = \frac{e^x - e^{-x}}{2}\,, \tag{5.48}$$

$$\phi(x) = \operatorname{arg\,sinh} x = \ln\left(x + \sqrt{x^2+1}\right), \tag{5.49}$$

$$\phi(x) = \ln(|x|+1)\operatorname{sgn} x. \tag{5.50}$$

Let us consider that

$$p(t) = t^\beta, \quad t \in [0,\infty)\,,\ \beta > 0,$$
$$f(x) = k|x|^\gamma \operatorname{sgn} x(x-\phi(L_0))(\phi(L)-x), \quad x \in [\phi(L_0),\phi(L)],\ \gamma > 0,\ k > 0,$$

where $L_0 < 0 < L$. Example 5.1 shows that the function p fulfils (5.7). The function f is continuous on $[\phi(L_0),\phi(L)]$, $f(\phi(L_0)) = f(0) = f(\phi(L)) = 0$ and $xf(x) > 0$ for $x \in ((\phi(L_0),\phi(L)) \setminus \{0\})$, that is (5.5) and (5.6) hold.

Each of ϕ given by (5.46)–(5.50) is continuously differentiable and increasing on $\mathbb{R}$, $\phi(0) = 0$ and ϕ maps $\mathbb{R}$ onto $\mathbb{R}$. Since moreover, for $x \in (\mathbb{R}\setminus\{0\})$,

$$\phi(x) = |x|^\alpha \operatorname{sgn} x,\ \alpha > 1 \implies \phi'(x) = \alpha|x|^{\alpha-1} > 0,$$
$$\phi(x) = \left(x^4+2x^2\right)\operatorname{sgn} x \implies \phi'(x) = 4(x^3+x)\operatorname{sgn} x > 0,$$
$$\phi(x) = \sinh x = \frac{e^x-e^{-x}}{2} \implies \phi'(x) = \cosh x > 0,$$
$$\phi(x) = \operatorname{arg\,sinh} x = \ln\left(x+\sqrt{x^2+1}\right) \implies \phi'(x) = \frac{1}{\sqrt{x^2+1}} > 0,$$
$$\phi(x) = \ln(|x|+1)\operatorname{sgn} x \implies \phi'(x) = \frac{1}{|x|+1} > 0,$$

we have satisfied (5.3) and (5.4).

We obtained that the functions p, ϕ and f fulfil all assumptions of Theorem 5.19. In particular, $\phi \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$ for each ϕ given by (5.46)–(5.50). Therefore, the auxiliary problem (5.8), (5.2) has a solution for every $u_0 \in [L_0, L]$.

5.5 Continuous dependence of solutions on initial values

Here we examine the uniqueness of solutions of the auxiliary problem (5.8), (5.2). Our arguments are based on a continuous dependence on initial values expressed in Theorem 5.21, Theorem 5.24 and Theorem 5.26. Assumption (5.3) implies that $\phi \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$. This need not be true for ϕ^{-1} as we have shown in Section 5.1 for $\phi(x) = |x|^\alpha \operatorname{sgn} x$, $\alpha > 1$. The special case when both ϕ and ϕ^{-1} are locally Lipschitz continuous is discussed in the next theorem.

Theorem 5.21 (**Uniqueness and continuous dependence on initial values I).** *Assume that* (5.3)–(5.7) *and*

$$f \in \mathrm{Lip}\left[\phi(L_0), \phi(L)\right], \tag{5.51}$$

$$\phi^{-1} \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}) \tag{5.52}$$

are satisfied. Let u_i be a solution of problem (5.8), (5.2) *with $u_0 = B_i \in [L_0, L]$, $i = 1, 2$. Then, for each $\beta > 0$, there exists $K > 0$ such that*

$$\|u_1 - u_2\|_{C^1[0,\beta]} \le K|B_1 - B_2|. \tag{5.53}$$

Furthermore, any solution of problem (5.8), (5.2) *with $u_0 \in [L_0, L]$ is unique on $[0, \infty)$.*

Proof. Let $i \in 1, 2$ and let u_i be a solution of problem (5.8), (5.2) with $u_0 = B_i$. By integrating (5.8) over $[0, t]$, we obtain

$$\phi(u_i'(t)) = A_i(t), \quad u_i(t) = B_i + \int_0^t \phi^{-1}\left(A_i(s)\right) \mathrm{d}s, \quad t \in [0, \infty), \tag{5.54}$$

where

$$A_i(s) := -\frac{1}{p(s)} \int_0^s p(\tau) \tilde{f}\left(\phi(u_i(\tau))\right) \mathrm{d}\tau, \quad s \in [0, \infty).$$

Choose $\beta > 0$. Since $u_i, \phi(u_i') \in C\,[0, \beta]$, there exist $m, M \in \mathbb{R}$ such that

$$m \le u_i(t) \le M, \quad m \le \phi(u_i'(t)) \le M, \quad t \in [0, \beta],\ i = 1, 2.$$

According to (5.3), (5.51) and (5.52), there exist positive constants Λ_f, Λ_ϕ, $\Lambda_{\phi^{-1}}$ satisfying

$$|f(x_1) - f(x_2)| \le \Lambda_f |x_1 - x_2|, \quad x_1, x_2 \in \left[\phi(L_0), \phi(L)\right],$$
$$|\phi(x_1) - \phi(x_2)| \le \Lambda_\phi |x_1 - x_2|, \quad x_1, x_2 \in [m, M],$$
$$\left|\phi^{-1}(x_1) - \phi^{-1}(x_2)\right| \le \Lambda_{\phi^{-1}} |x_1 - x_2|, \quad x_1, x_2 \in [m, M].$$

Denote

$$\rho(t) := \max\{|u_1(s) - u_2(s)| \colon s \in [0,t]\},\ t \in [0,\beta]\,.$$

Then, by (5.37),

$$\begin{aligned}|A_1(s) - A_2(s)| &\le \frac{1}{p(s)}\int_0^s p(\tau)\left|\tilde{f}(\phi(u_1(\tau))) - \tilde{f}(\phi(u_2(\tau)))\right| \mathrm{d}\tau \\ &\le \Lambda_f\Lambda_\phi\frac{1}{p(s)}\int_0^s p(\tau)|u_1(\tau) - u_2(\tau)|\,\mathrm{d}\tau \le \Lambda_f\Lambda_\phi\rho(s)\beta,\end{aligned}$$

and, by virtue of (5.54),

$$\begin{aligned}\rho(t) &\le \left|B_1 - B_2 + \int_0^t \phi^{-1}(A_1(s)) - \phi^{-1}(A_2(s))\,\mathrm{d}s\right| \\ &\le |B_1 - B_2| + \int_0^t \left|\phi^{-1}(A_1(s)) - \phi^{-1}(A_2(s))\right| \mathrm{d}s \\ &\le |B_1 - B_2| + \Lambda_{\phi^{-1}}\int_0^t |A_1(s) - A_2(s)|\,\mathrm{d}s \\ &\le |B_1 - B_2| + \Lambda_f\Lambda_\phi\Lambda_{\phi^{-1}}\beta\int_0^t \rho(s)\,\mathrm{d}s, \quad t \in [0,\beta]\,.\end{aligned}$$

The Gronwall lemma yields

$$\rho(t) \le |B_1 - B_2|e^{L\beta^2}, \quad t \in [0,\beta]\,, \tag{5.55}$$

where $L := \Lambda_f\Lambda_\phi\Lambda_{\phi^{-1}}$. Similarly, from (5.54), it follows

$$\begin{aligned}|u_1'(t) - u_2'(t)| = \left|\phi^{-1}(A_1(t)) - \phi^{-1}(A_2(t))\right| &\le \Lambda_{\phi^{-1}}|A_1(t) - A_2(t)| \\ &\le L\rho(t)\beta, \quad t \in [0,\beta]\,.\end{aligned}$$

Applying (5.55), we get

$$\max\{|u_1'(t) - u_2'(t)| \colon t \in [0,\beta]\} \le |B_1 - B_2|L\beta e^{L\beta^2}.$$

Consequently, using (5.55), we obtain

$$\|u_1 - u_2\|_{C^1[0,\beta]} \le |B_1 - B_2|(1 + L\beta)e^{L\beta^2}, \tag{5.56}$$

that is (5.53) holds for

$$K := (1 + L\beta)e^{L\beta^2}.$$

Clearly, if $B_1 = B_2$, we have $u_1 = u_2$ on each $[0,\beta] \subset \mathbb{R}$ and the uniqueness for problem (5.8), (5.2) on $[0,\infty)$ follows. □

If also (5.11) and (5.12) are fulfilled, we can use (5.38) and get universal estimates for $\phi(u_i')$ and u_i. This is the case that K in (5.53) does not depend on a choice of u_1, u_2. Let us show it in the next theorem.

Theorem 5.22 (**Continuous dependence on initial values II**). *Assume that* (5.3)–(5.7)*,* (5.11)*,* (5.12)*,* (5.51) *and* (5.52) *hold. Then, for each* $\beta > 0$*, there exists* $K > 0$ *such that*

$$\|u_1 - u_2\|_{C^1[0,\beta]} \le K|B_1 - B_2|,$$

where u_i *is a solution of problem* (5.8)*,* (5.2) *with* $u_0 = B_i \in [L_0, L]$*,* $i = 1, 2$*.*

Proof. Let $i \in 1, 2$ and let u_i be a solution of problem (5.8), (5.2) with $u_0 = B_i$. Then we have (5.54). Choose $\beta > 0$. By (5.38),

$$|\phi(u_i'(t))| \le \phi(\tilde{c}), \quad |u_i(t)| \le \beta\tilde{c} + \max\{|L_0|, L\} =: M, \quad t \in [0, \beta],\ i = 1, 2.$$

According to (5.3), (5.51) and (5.52), there exist positive constants Λ_f, Λ_ϕ, $\Lambda_{\phi^{-1}}$ satisfying

$$\begin{aligned} |f(x_1) - f(x_2)| &\le \Lambda_f |x_1 - x_2|, \quad x_1, x_2 \in [\phi(L_0), \phi(L)], \\ |\phi(x_1) - \phi(x_2)| &\le \Lambda_\phi |x_1 - x_2|, \quad x_1, x_2 \in [-M, M], \\ \left|\phi^{-1}(x_1) - \phi^{-1}(x_2)\right| &\le \Lambda_{\phi^{-1}} |x_1 - x_2|, \quad x_1, x_2 \in [-\phi(\tilde{c}), \phi(\tilde{c})]. \end{aligned}$$

Denote

$$\rho(t) := \max\{|u_1(s) - u_2(s)| \colon s \in [0, t]\},\ t \in [0, \beta].$$

Using the same procedure as in the proof of Theorem 5.21, we obtain the inequality (5.56) again. □

Example 5.23. In order to apply Theorem 5.21, we need both ϕ and ϕ^{-1} from $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$. Let us check the functions ϕ in Example 5.20 from this point of view. First, we find the inverse functions ϕ^{-1} for the functions ϕ given by (5.46)–(5.50).

- Let $\phi_1(x) = (x^4 + 2x^2)\operatorname{sgn} x$. Then

$$\begin{array}{ll} x \ge 0\colon & x < 0\colon \\ y = \phi_1(x) = x^4 + 2x^2 & y = \phi_1(x) = -x^4 - 2x^2 \\ y + 1 = x^4 + 2x^2 + 1 = (x^2+1)^2 & -y + 1 = x^4 + 2x^2 + 1 = (x^2+1)^2 \\ \sqrt{y+1} = x^2 + 1 & \sqrt{1-y} = x^2 + 1 \\ x^2 = \sqrt{y+1} - 1 & x^2 = \sqrt{1-y} - 1 \\ \phi_1^{-1}(y) = x = \sqrt{\sqrt{y+1} - 1},\ y \ge 0 & \phi_1^{-1}(y) = x = \sqrt{\sqrt{1-y} - 1},\ y < 0. \end{array}$$

  Hence,

$$\phi_1^{-1}(x) = \sqrt{\sqrt{|x| + 1} - 1}, \quad x \in \mathbb{R}.$$

- Let $\phi_2(x) = \ln(|x|+1)\operatorname{sgn} x$. Then

$$\begin{array}{ll} x \geq 0: & x < 0: \\ y = \phi_2(x) = \ln(x+1) & y = \phi_2(x) = -\ln(1-x) = \ln \dfrac{1}{1-x} \\ e^y = x+1 & e^y = \dfrac{1}{1-x} \\ \phi_2^{-1}(y) = x = e^y - 1,\ y \geq 0 & 1 - x = \dfrac{1}{e^y} = e^{-y} \\ & \phi_2^{-1}(y) = x = 1 - e^{-y},\ y < 0. \end{array}$$

Thus,

$$\phi_2^{-1}(x) = \left(e^{|x|} - 1\right)\operatorname{sgn} x, \quad x \in \mathbb{R}.$$

Now, we can create the following summary.

$$\begin{aligned} &\phi(x) = |x|^\alpha \operatorname{sgn} x, \quad \alpha > 1 && \Longrightarrow \phi^{-1}(x) = |x|^{\frac{1}{\alpha}} \operatorname{sgn} x && \notin \operatorname{Lip}_{\mathrm{loc}}(\mathbb{R}), \\ &\phi(x) = \left(x^4 + 2x^2\right)\operatorname{sgn} x && \Longrightarrow \phi^{-1}(x) = \sqrt{\sqrt{|x|+1} - 1} && \notin \operatorname{Lip}_{\mathrm{loc}}(\mathbb{R}), \\ &\phi(x) = \sinh x = \frac{e^x - e^{-x}}{2} && \Longrightarrow \phi^{-1}(x) = \operatorname{arg\,sinh} x && \in \operatorname{Lip}_{\mathrm{loc}}(\mathbb{R}), \\ &\phi(x) = \operatorname{arg\,sinh} x && \Longrightarrow \phi^{-1}(x) = \sinh x && \in \operatorname{Lip}_{\mathrm{loc}}(\mathbb{R}), \\ &\qquad = \ln\left(x + \sqrt{x^2+1}\right) && && \\ &\phi(x) = \ln(|x|+1)\operatorname{sgn} x && \Longrightarrow \phi^{-1}(x) = \left(e^{|x|} - 1\right)\operatorname{sgn} x && \in \operatorname{Lip}_{\mathrm{loc}}(\mathbb{R}). \end{aligned}$$

Consider

$$\begin{aligned} &p(t) = t^\beta, \quad t \in [0, \infty),\ \beta > 0, \\ &f(x) = k|x|^\gamma \operatorname{sgn} x (x - \phi(L_0))(\phi(L) - x), \quad x \in [\phi(L_0), \phi(L)],\ \gamma \geq 1,\ k > 0, \end{aligned}$$

where $L_0 < 0 < L$. Let ϕ be given by one of the formulas (5.48)–(5.50). For these ϕ, as we showed, (5.52) is valid. Example 5.20 shows that the functions ϕ and f satisfy conditions (5.3)–(5.6). In addition, f is Lipschitz continuous on $[\phi(L_0), \phi(L)]$, that is (5.51) holds. According to Example 5.1, the function p fulfils (5.7).

Therefore, all assumptions of Theorem 5.21 are fulfilled and problem (5.8), (5.2) has a unique solution for each $u_0 \in [L_0, L]$.

Note that if $\gamma \in (0,1)$, then f is not Lipschitz continuous on a neighbourhood of zero, that is (5.51) is not valid. Similarly, in the case that ϕ is given by (5.46) or (5.47), then ϕ^{-1} is not Lipschitz continuous on a neighbourhood of zero and hence, (5.52) falls.

In the next two theorems, we show the assumptions under which solutions of problem (5.8), (5.2) continuously depend on their initial values in the case that ϕ^{-1} is not locally Lipschitz continuous.

Theorem 5.24 **(Continuous dependence on initial values III).** *Assume that* (5.3)–(5.7), (5.11), (5.12), (5.51) *and*

$$\limsup_{x\to 0^-}\left(-x\left(\phi^{-1}\right)'(x)\right)<\infty,\quad \phi' \text{ is nonincreasing on } (-\infty,0) \tag{5.57}$$

are fulfilled. Let B_1, B_2 *satisfy*

$$B_1\in(2\varepsilon, L-2\varepsilon),\quad |B_1-B_2|<\varepsilon$$

for some $\varepsilon>0$. *Let* u_i *be a solution of problem* (5.8), (5.2) *with* $u_0=B_i$, $i=1,2$. *Then, for each* $\beta>0$, *where*

$$u_i'<0 \text{ on } (0,\beta],\ i=1,2,$$

there exists $K\in(0,\infty)$ *such that*

$$\|u_1-u_2\|_{C^1[0,\beta]}\le K|B_1-B_2|.$$

Proof. Let u_i be a solution of problem (5.8), (5.2) with $u_0=B_i$, $i=1,2$. Then by integrating (5.8) over $[0,t]$, we obtain

$$\phi(u_i'(t))=-\frac{1}{p(t)}\int_0^t p(s)\tilde f(\phi(u_i(s)))\,\mathrm{d}s=:A_i(t),\quad t\in[0,\infty) \tag{5.58}$$

$$u_i(t)=B_i+\int_0^t\phi^{-1}(A_i(s))\,\mathrm{d}s,\quad t\in[0,\infty),\ i=1,2.$$

Therefore,

$$|u_1(t)-u_2(t)|\le|B_1-B_2|+\int_0^t\left|\phi^{-1}(A_1(s))-\phi^{-1}(A_2(s))\right|\mathrm{d}s,\ \ t\in[0,\infty). \tag{5.59}$$

In order to reach the required estimate, we restrict our consideration on a small interval $[0,\delta]$ for a suitably chosen $\delta>0$ in Step 1. Then we prolongate the result on $[0,\beta]$ in Step 2.

Step 1. Assumptions (5.3)–(5.6), (5.51), (5.57) yield the existence of positive constants Λ_f, Λ_ϕ, K_1, K_2 such that

$$\begin{aligned}
&|f(y_1)-f(y_2)|\le\Lambda_f|y_1-y_2|,\quad y_1,y_2\in[\phi(L_0),\phi(L)],\\
&|\phi(x_1)-\phi(x_2)|\le\Lambda_\phi|x_1-x_2|,\quad x_1,x_2\in[L_0,L],\\
&K_1=\min\left\{f(\phi(x))\colon x\in[B_1-2\varepsilon,B_1+2\varepsilon]\right\}, && (5.60)\\
&0<-x\left(\phi^{-1}\right)'(x)\le K_2,\quad x\in[-1,0). && (5.61)
\end{aligned}$$

By Lemma 5.18, there exists $\tilde c>0$ such that $|u_i'|\le\tilde c$ on $[0,\infty)$, $i=1,2$. Let us choose δ such that

$$0<\delta\le\min\left\{\frac{\varepsilon}{\tilde c},\frac{1}{K_1},\frac{K_1}{2K_2\Lambda_f\Lambda_\phi}\right\}. \tag{5.62}$$

Then we get

$$|B_1 - u_1(t)| = |u_1(0) - u_1(t)| = \left|\int_0^t u_1'(s)\,\mathrm{d}s\right| \le \int_0^\delta |u_1'(s)|\,\mathrm{d}s \le \tilde{c}\delta \le \varepsilon, \quad t \in [0,\delta],$$

which yields $u_1(t) \in [B_1 - \varepsilon, B_1 + \varepsilon]$ for $t \in [0,\delta]$. Moreover,

$$\begin{aligned} |B_1 - u_2(t)| &= |B_1 - B_2 + B_2 - u_2(t)| \le |B_1 - B_2| + |u_2(0) - u_2(t)| \\ &\le \varepsilon + \left|\int_0^t u_2'(s)\,\mathrm{d}s\right| \le \varepsilon + \int_0^\delta |u_2'(s)|\,\mathrm{d}s \le \varepsilon + \tilde{c}\delta \le 2\varepsilon, \quad t \in [0,\delta], \end{aligned}$$

thus $u_2(t) \in [B_1 - 2\varepsilon, B_1 + 2\varepsilon]$ holds for $t \in [0,\delta]$. Consequently, $\tilde{f}(\phi(u_i)(t)) \ge K_1$ for $t \in [0,\delta],\ i = 1,2$. Therefore,

$$\begin{aligned} A_i(s) &:= -\int_0^s \frac{p(\tau)}{p(s)} \tilde{f}(\phi(u_i(\tau)))\,\mathrm{d}\tau \le -K_1 \int_0^s \frac{p(\tau)}{p(s)}\,\mathrm{d}\tau, \quad s \in [0,\delta], \\ |A_1(s) - A_2(s)| &\le \int_0^s \frac{p(\tau)}{p(s)} |\tilde{f}(\phi(u_1(\tau))) - \tilde{f}(\phi(u_2(\tau)))|\,\mathrm{d}\tau \\ &\le \Lambda_f \Lambda_\phi \|u_1 - u_2\|_{C[0,\delta]} \int_0^s \frac{p(\tau)}{p(s)}\,\mathrm{d}\tau, \quad s \in [0,\delta]. \end{aligned}$$

Let $s \in (0,\delta]$ be fixed. By the Mean Value Theorem, there exists $A^\star(s)$ between $A_1(s)$ and $A_2(s)$ such that

$$\left|\phi^{-1}(A_1(s)) - \phi^{-1}(A_2(s))\right| \le \left(\phi^{-1}\right)'(A^\star(s))\,|A_1(s) - A_2(s)|.$$

Since $(\phi^{-1})'$ is a nondecreasing function on $(-\infty, 0)$, we get

$$\begin{aligned} \left|\phi^{-1}(A_1(s)) - \phi^{-1}(A_2(s))\right| &\le \left(\phi^{-1}\right)'\left(-K_1 \int_0^s \frac{p(\tau)}{p(s)}\,\mathrm{d}\tau\right) |A_1(s) - A_2(s)| \\ &\le \left(\phi^{-1}\right)'\left(-K_1 \int_0^s \frac{p(\tau)}{p(s)}\,\mathrm{d}\tau\right) \frac{\Lambda_f \Lambda_\phi \|u_1 - u_2\|_{C[0,\delta]}}{K_1} K_1 \int_0^s \frac{p(\tau)}{p(s)}\,\mathrm{d}\tau. \end{aligned}$$

Using the monotonicity of p and (5.62), we have

$$0 < K_1 \int_0^s \frac{p(\tau)}{p(s)}\,\mathrm{d}\tau \le K_1 \int_0^\delta \frac{p(s)}{p(s)}\,\mathrm{d}\tau = K_1\delta \le 1$$

and hence, due to (5.61), we get

$$\begin{aligned} \left|\phi^{-1}(A_1(s)) - \phi^{-1}(A_2(s))\right| &\le \frac{K_2}{K_1} \Lambda_f \Lambda_\phi \|u_1 - u_2\|_{C[0,\delta]} K_1 \int_0^s \frac{p(\tau)}{p(s)}\,\mathrm{d}\tau \\ &\le \frac{K_2}{K_1} \Lambda_f \Lambda_\phi \|u_1 - u_2\|_{C[0,\delta]}. \end{aligned}$$

Consequently, by (5.62), we derive from (5.59) for $t \in [0, \delta]$

$$\begin{aligned}|u_1(t) - u_2(t)| &\le |B_1 - B_2| + \int_0^t \frac{K_2}{K_1} \Lambda_f \Lambda_\phi \|u_1 - u_2\|_{C[0,\delta]} \,\mathrm{d}s \\ &\le |B_1 - B_2| + \delta \frac{K_2}{K_1} \Lambda_f \Lambda_\phi \|u_1 - u_2\|_{C[0,\delta]} \le |B_1 - B_2| + \frac{1}{2} \|u_1 - u_2\|_{C[0,\delta]}.\end{aligned}$$

This yields

$$\|u_1 - u_2\|_{C[0,\delta]} \le 2|B_1 - B_2|. \tag{5.63}$$

Furthermore, by virtue of (5.58),

$$|u_1'(t) - u_2'(t)| = \left|\phi^{-1}(A_1(t)) - \phi^{-1}(A_2(t))\right| \le \frac{K_2}{K_1} \Lambda_f \Lambda_\phi \|u_1 - u_2\|_{C[0,\delta]}, \quad t \in [0, \delta].$$

Hence,

$$\|u_1' - u_2'\|_{C[0,\delta]} \le K_3 |B_1 - B_2| \tag{5.64}$$

with $K_3 := 2\frac{K_2}{K_1} \Lambda_f \Lambda_\phi$. Finally,

$$\|u_1 - u_2\|_{C^1[0,\delta]} \le K_{S1} |B_1 - B_2|,$$

where $K_{S1} := K_3 + 2$.

Step 2. In this step, we extend the continuous dependence on initial values from $[0, \delta]$ to $[0, \beta]$, where $u_i'(t) < 0$ for $t \in (0, \beta]$, $i = 1, 2$. To this aim, choose $i \in \{1, 2\}$ and denote

$$\nu_i := \max\{u_i'(t) \colon t \in [\delta, \beta]\} < 0, \quad m_1 := \max\{\nu_1, \nu_2\}, \quad m := \min\{-\tilde{c}, L_0\}.$$

Moreover, (5.3) yields the existence of positive Lipschitz constants $\Lambda_m, \Lambda_{\phi^{-1}}$ such that

$$\begin{aligned}|\phi(x_1) - \phi(x_2)| &\le \Lambda_m |x_1 - x_2|, \quad x_1, x_2 \in [m, L], \\ \left|\phi^{-1}(y_1) - \phi^{-1}(y_2)\right| &\le \Lambda_{\phi^{-1}} |y_1 - y_2|, \quad y_1, y_2 \in [\phi(-\tilde{c}), \phi(m_1)].\end{aligned}$$

By integrating (5.8) over $[\delta, t]$, $t \in [\delta, \beta]$, we get

$$\phi(u_i'(t)) = \frac{p(\delta)}{p(t)} \phi(u_i'(\delta)) - \frac{1}{p(t)} \int_\delta^t p(s) \tilde{f}(\phi(u_i(s))) \,\mathrm{d}s, \quad t \in [\delta, \beta].$$

Let us denote

$$\begin{aligned}\tilde{A}_i(t) &:= -\int_\delta^t \frac{p(s)}{p(t)} \tilde{f}(\phi(u_i(s))) \,\mathrm{d}s, \quad t \in [\delta, \beta], \\ x_i(t) &:= \frac{p(\delta)}{p(t)} \phi(u_i'(\delta)) + \tilde{A}_i(t) = \phi(u_i'(t)), \quad t \in [\delta, \beta].\end{aligned}$$

Then

$$u_i'(t) = \phi^{-1}(x_i(t)), \quad t \in [\delta, \beta]. \tag{5.65}$$

Since $-\tilde{c} \le u_i'(t) \le m_1$, then $x_i(t) \in [\phi(-\tilde{c}), \phi(m_1)]$, for $t \in [\delta, \beta]$. Integrating (5.65) from δ to $t \in [\delta, \beta]$, we get

$$u_i(t) = u_i(\delta) + \int_\delta^t \phi^{-1}(x_i(s))\,\mathrm{d}s, \quad t \in [\delta, \beta].$$

Due to (5.63), we obtain

$$\begin{aligned} |u_1(t) - u_2(t)| &\le |u_1(\delta) - u_2(\delta)| + \int_\delta^t \left|\phi^{-1}(x_1(s)) - \phi^{-1}(x_2(s))\right| \mathrm{d}s \\ &\le 2|B_1 - B_2| + \Lambda_{\phi^{-1}} \int_\delta^t |x_1(s) - x_2(s)|\,\mathrm{d}s, \quad t \in [\delta, \beta]\,. \end{aligned}$$

Further, by (5.7) and (5.64), we get

$$\begin{aligned} |x_1(s) - x_2(s)| &\le \frac{p(\delta)}{p(s)} |\phi(u_1'(\delta)) - \phi(u_2'(\delta))| + \left|\tilde{A}_1(s) - \tilde{A}_2(s)\right| \\ &\le \Lambda_m |u_1'(\delta) - u_2'(\delta)| + \int_\delta^s \frac{p(\tau)}{p(s)} \left|\tilde{f}(\phi(u_1(\tau))) - \tilde{f}(\phi(u_2(\tau)))\right| \mathrm{d}\tau \\ &\le \Lambda_m |u_1'(\delta) - u_2'(\delta)| + \int_\delta^s \left|\tilde{f}(\phi(u_1(\tau))) - \tilde{f}(\phi(u_2(\tau)))\right| \mathrm{d}\tau \\ &\le \Lambda_m K_3 |B_1 - B_2| + \Lambda_f \Lambda_m \int_\delta^s |u_1(\tau) - u_2(\tau)|\,\mathrm{d}\tau, \quad s \in [\delta, \beta]\,. \end{aligned}$$

Therefore,

$$\begin{aligned} |u_1(t) - u_2(t)| &\le 2|B_1 - B_2| + \Lambda_{\phi^{-1}} \int_\delta^t \Lambda_m K_3 |B_1 - B_2|\,\mathrm{d}s \\ &\quad + \Lambda_{\phi^{-1}} \Lambda_f \Lambda_m \int_\delta^t \int_\delta^s |u_1(\tau) - u_2(\tau)|\,\mathrm{d}\tau\,\mathrm{d}s \le K_4 |B_1 - B_2| \\ &\quad + K_5 \int_\delta^t |u_1(\tau) - u_2(\tau)|\,\mathrm{d}\tau, \quad t \in [\delta, \beta], \end{aligned}$$

where $K_4 := 2 + \Lambda_{\phi^{-1}} \Lambda_m K_3 \beta$, $K_5 := \Lambda_{\phi^{-1}} \Lambda_f \Lambda_m \beta$. Next, we set

$$\rho(t) := \max\{|u_1(s) - u_2(s)| \colon s \in [\delta, t]\}, \quad t \in [\delta, \beta].$$

Then

$$\rho(t) \le K_4 |B_1 - B_2| + K_5 \int_\delta^t \rho(\tau)\,\mathrm{d}\tau, \quad t \in [\delta, \beta].$$

The Gronwall Lemma yields that

$$\begin{gathered} \rho(t) \le K_4 |B_1 - B_2| e^{K_5 \beta}, \quad t \in [\delta, \beta] \\ \|u_2 - u_2\|_{C[\delta,\beta]} \le K_6 |B_1 - B_2| \end{gathered}$$

with $K_6 := K_4 e^{K_5\beta}$. By (5.65),

$$|u_1'(t) - u_2'(t)| = \left|\phi^{-1}(x_1(t)) - \phi^{-1}(x_2(t))\right| \le \Lambda_{\phi^{-1}}|x_1(t) - x_2(t)|$$
$$\le \Lambda_{\phi^{-1}}\Lambda_m K_3|B_1 - B_2| + \Lambda_{\phi^{-1}}\Lambda_f\Lambda_m\beta\|u_1 - u_2\|_{C[\delta,\beta]} \le K_7|B_1 - B_2|,$$

where $K_7 := \Lambda_{\phi^{-1}}\Lambda_m K_3 + \Lambda_{\phi^{-1}}\Lambda_f\Lambda_m\beta K_6$. Hence,

$$\|u_1' - u_2'\|_{C[\delta,\beta]} \le K_7|B_1 - B_2|,$$
$$\|u_1 - u_2\|_{C^1[\delta,\beta]} \le K_{S2}|B_1 - B_2|$$

with $K_{S2} := K_6 + K_7$. Finally, there exists $K := K_{S1} + K_{S2}$ such that

$$\|u_1 - u_2\|_{C^1[0,\beta]} \le K|B_1 - B_2|.$$

This completes the proof. □

Remark 5.25. The approach developed in the proof of Theorem 5.24 cannot be used for $B_1 = L$, because then the positive constant K_1 in (5.60) which is crucial in the proof, does not exist.

Theorem 5.26 (Continuous dependence on initial values IV). *Assume that (5.3)–(5.7), (5.11), (5.12), (5.51) and*

$$\limsup_{x\to 0^+}\left(x\left(\phi^{-1}\right)'(x)\right) < \infty, \quad \phi' \text{ is nondecreasing on } (0,\infty) \tag{5.66}$$

hold. Let B_1, B_2 satisfy

$$B_1 \in (L_0 + 2\varepsilon, -2\varepsilon), \quad |B_1 - B_2| < \varepsilon$$

for some $\varepsilon > 0$. Let u_i be a solution of problem (5.8), (5.2) with $u_0 = B_i$, $i = 1, 2$. Then, for each $\beta > 0$, where

$$u_i' > 0 \text{ on } (0,\beta], \; i = 1, 2,$$

there exists $K \in (0,\infty)$ such that

$$\|u_1 - u_2\|_{C^1[0,\beta]} \le K|B_1 - B_2|.$$

Proof. We proceed similarly as in the proof of Theorem 5.24. In Step 1, we replace $f(\phi(x))$ by $|f(\phi(x))|$ in (5.60), and condition (5.61) by

$$0 < x\left(\phi^{-1}\right)'(x) \le K_2, \quad x \in (0,1]\,.$$

Then we derive the inequalities

$$-\tilde f(\phi(u_i)(t)) = \left|\tilde f(\phi(u_i)(t))\right| \ge K_1, \qquad t \in [0,\delta], \; i = 1, 2,$$
$$A_i(s) := -\int_0^s \frac{p(\tau)}{p(s)}\tilde f(\phi(u_i(\tau)))\,\mathrm{d}\tau \ge K_1\int_0^s \frac{p(\tau)}{p(s)}\,\mathrm{d}\tau, \quad s \in [0,\delta].$$

Since $(\phi^{-1})'$ is nonincreasing on $(0,\infty)$, we obtain for fixed $s \in (0,\delta]$

$$\begin{aligned}\left|\phi^{-1}(A_1(s)) - \phi^{-1}(A_2(s))\right| &\le \left(\phi^{-1}\right)'\left(K_1 \int_0^s \frac{p(\tau)}{p(s)}\,\mathrm{d}\tau\right)|A_1(s) - A_2(s)| \\ &\le \left(\phi^{-1}\right)'\left(K_1 \int_0^s \frac{p(\tau)}{p(s)}\,\mathrm{d}\tau\right)\frac{\Lambda_f \Lambda_\phi \|u_1 - u_2\|_{C[0,\delta]}}{K_1} K_1 \int_0^s \frac{p(\tau)}{p(s)}\,\mathrm{d}\tau\end{aligned}$$

and follow Step 1 in the proof of Theorem 5.24. In Step 2, having $u_i'(t) > 0$ for $t \in (0,\beta]$, $i = 1,2$, we denote

$$\nu_i := \min\{u_i'(t) \colon t \in [\delta,\beta]\} > 0, \quad m_0 := \min\{\nu_1,\nu_2\}, \quad M := \max\{\tilde{c}, L\}.$$

By (5.3) there exists positive Lipschitz constants $\Lambda_m, \Lambda_{\phi^{-1}}$ such that

$$\begin{aligned}|\phi(x_1) - \phi(x_2)| \le \Lambda_m |x_1 - x_2|, \quad x_1, x_2 \in [L_0, M], \\ \left|\phi^{-1}(y_1) - \phi^{-1}(y_2)\right| \le \Lambda_{\phi^{-1}} |y_1 - y_2|, \quad y_1, y_2 \in [\phi(m_0), \phi(\tilde{c})]\,.\end{aligned}$$

We derive (5.65) and since $m_0 \le u_i'(t) \le \tilde{c}$, we get $x_i(t) \in [\phi(m_0), \phi(\tilde{c})]$ for $t \in [\delta,\beta]$, $i = 1,2$. Further, we argue as in the proof of Theorem 5.24. □

6 Damped and homoclinic solutions of the problem with ϕ-Laplacian

6.1 Existence and uniqueness of damped solutions

The existence of damped solutions of the original problem (5.1), (5.2) is proved in Theorem 6.1. Moreover, this theorem yields the uniqueness of damped solutions provided that ϕ^{-1} is Lipschitz continuous, while Theorem 6.4 gives the uniqueness of damped solutions without the Lipschitz continuity of ϕ^{-1}. Note that the results which concern damped solutions can be formulated directly for the original problem (5.1), (5.2) due to Remark 5.17.

Theorem 6.1 (**Existence and uniqueness of damped solutions of problem (5.1), (5.2)**). *Assume that (5.3)–(5.7), (5.11) and (5.12) hold. Then, for each $u_0 \in \big[\bar{B}, L\big)$, problem (5.1), (5.2) has a solution. Every solution of problem (5.1), (5.2) with $u_0 \in \big[\bar{B}, L\big)$ is damped.*
If moreover (5.51) and (5.52) hold, then the solution is unique.

Proof. By Theorem 5.19, for each $u_0 \in \big[\bar{B}, L\big)$ there exists a solution u of problem (5.8), (5.2). Lemma 5.16 gives that solution u is a damped. If conditions (5.51) and (5.52) are satisfied, then, according to Theorem 5.21, the solution u is unique. By virtue of Remark 5.17, u is solution of problem (5.1), (5.2). □

Example 6.2. Let us consider the functions p, f and ϕ from Example 5.20, where $0 < L < -L_0$. According to Example 5.20, conditions (5.3)–(5.7) are valid. Example 5.9 shows that (5.12) holds. Since these functions ϕ are odd and increasing on $\mathbb{R}$ and $0 < L < -L_0$, we get, similarly as in Example 2.18, that $\tilde{F}(L) < \tilde{F}(L_0)$. Hence, there exists $\bar{B} \in (L_0, 0)$ such that $\tilde{F}(\bar{B}) = \tilde{F}(L)$, which gives (5.11).

By Theorem 6.1, if

$$0 < L < -L_0,$$

then problem (5.1), (5.2) with p, f and ϕ from Example 5.20 has for each $u_0 \in \big[\bar{B}, L\big)$ a solution u and u is damped. If ϕ is given by one of the formulas (5.48)–(5.50) and $\gamma \geq 1$ in the formula for f, we see that also (5.51) and (5.52) hold and the solution u is unique.

Remark 6.3. By Theorem 6.1, we can get homoclinic solutions only if $u_0 \in \big[L_0, \bar{B}\big)$.

If $\phi^{-1} \notin \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$, we derive the results about uniqueness by means of Theorems 5.24 and 5.26.

Theorem 6.4 (Uniqueness of damped solutions). *Assume that (5.3)–(5.7), (5.11), (5.12), (5.51), (5.57) and (5.66) are fulfilled. Let u be a damped solution of problem (5.1), (5.2) with $u_0 \in (L_0, L)$. Then u is a unique solution of this problem.*

Proof. Assume that u is a damped solution of the auxiliary problem (5.8), (5.2) and that there exists another solution v of problem (5.8), (5.2). Definition 5.3 yields

$$u(t) < L, \qquad t \in [0, \infty). \tag{6.1}$$

By Lemma 5.16, we have

$$L_0 < u(t), \quad L_0 < v(t), \quad t \in [0, \infty). \tag{6.2}$$

Step 1. Let $u_0 \in (L_0, 0)$.

(i) According to Lemma 5.5, there exists $\beta > 0$ such that $u'(t) > 0$, $v'(t) > 0$ for $t \in (0, \beta]$. Put

$$a := \sup\{\beta > 0 \colon u'(t) > 0,\ v'(t) > 0,\ t \in (0, \beta]\},$$
$$\rho(t) := u(t) - v(t), \qquad t \in [0, \infty).$$

Since $u' > 0$, $v' > 0$ on $(0, a)$ and $B_1 := u_0 = v(0) =: B_2$, Theorem 5.26 yields

$$\rho(t) = 0, \qquad t \in [0, a). \tag{6.3}$$

If $a = \infty$, then

$$u(t) = v(t), \qquad t \in [0, \infty). \tag{6.4}$$

Consequently, by (6.1) and (6.2), u is a unique solution of problem (5.8), (5.2).
Let $a < \infty$. Since $u, v \in C^1[0, \infty)$, we get, by (6.3),

$$\begin{aligned} \lim_{t \to a^-} \rho(t) &= \rho(a) = u(a) - v(a) = 0, \\ \lim_{t \to a^-} \rho'(t) &= \rho'(a) = u'(a) - v'(a) = 0. \end{aligned} \tag{6.5}$$

Therefore, $u'(a) = v'(a)$.

(ii) According to the definition of number a, we have $u'(a) = v'(a) = 0$. By (6.1) and Lemma 5.11 or Lemma 5.15, $u(a) = v(a) \in (0, L)$. Due to Lemma 5.6, there exists $\gamma > a$ such that $u'(t) < 0$, $v'(t) < 0$ for $t \in (a, \gamma]$. Put

$$b := \sup\{\gamma > a \colon u'(t) < 0,\ v'(t) < 0,\ t \in (a, \gamma]\}.$$

Since $u' < 0$, $v' < 0$ on (a, b) and $u(a) = v(a) \in (0, L)$, by Theorem 5.24 (working with a, γ, $u(a)$ and $v(a)$ instead of 0, β, B_1 and B_2, respectively), we get

$$\rho(t) = 0, \qquad t \in [a, b). \tag{6.6}$$

If $b = \infty$, then (6.4) holds and, according to (6.1), (6.2), u is a unique solution of problem (5.8), (5.2).

Let $b < \infty$. Since $u, v \in C^1[0, \infty)$, (6.6) yields

$$\lim_{t \to b^-} \rho(t) = \rho(b) = u(b) - v(b) = 0, \quad \lim_{t \to b^-} \rho'(t) = \rho'(b) = u'(b) - v'(b) = 0.$$

Hence, $u'(b) = v'(b)$ and, due to the definition of b, $u'(b) = v'(b) = 0$. Lemma 5.12 implies $u(b) = v(b) \in \left(\bar{B}, 0\right)$. Repeating the arguments in parts (i) and (ii), we get that u is a unique solution of problem (5.8), (5.2). According to Remark 5.17, u is solution of problem (5.1), (5.2).

Step 2. Let $u_0 = 0$. Due to Remark 5.4, $u(t) \equiv 0$ is a unique solution of problem (5.1), (5.2).

Let $u_0 \in (0, L)$. We have the same situation as in part (ii) of Step 1, where a is replaced by 0 and so, we argue similarly. □

6.2 Uniqueness and properties of regular homoclinic solutions

In this section, we discuss homoclinic solutions and hence, by Remark 6.3, we take $u_0 \in \left(L_0, \bar{B}\right)$. Note that the results concerning homoclinic solutions can be formulated directly for the original problem (5.1), (5.2) due to Remark 5.17.

Theorem 6.5 (**Nonexistence of singular homoclinic solutions**). *Assume that (5.3)–(5.7), (5.51) and (5.52) hold. Then each homoclinic solution of problem (5.1), (5.2) with $u_0 \in \left(L_0, \bar{B}\right)$ is regular.*

Proof. Let u be a singular homoclinic solution of problem (5.8), (5.2) with $u_0 \in \left(L_0, \bar{B}\right)$. Then, by Definition 5.3, there exists $t_0 > 0$ such that

$$u(t_0) = L, \qquad u'(t_0) = 0 \tag{6.7}$$

and

$$u(t) < L, \qquad t \in [0, t_0). \tag{6.8}$$

Using the substitution $s = t_0 - t$, $q(s) = p(t)$, $v(s) = u(t)$ for $t \in \left[\frac{t_0}{2}, t_0\right]$, we transform the terminal value problem (5.8), (6.7) on $\left[\frac{t_0}{2}, t_0\right]$ to the IVP

$$-(q(s)\phi(-v'(s)))' + q(s)\tilde{f}(\phi(v(s))) = 0, \quad s \in \left[0, \frac{t_0}{2}\right], \quad v(0) = L, \quad v'(0) = 0.$$

By the proof of Theorem 5.21, the only possible function satisfying this problem is the constant function $v(s) = L$ for $s \in \left[0, \frac{t_0}{2}\right]$. Therefore, $u(t) = L$ for $t \in \left[\frac{t_0}{2}, t_0\right]$, which contradicts (6.8). Hence, using Remark 5.17, if u is homoclinic solution of problem (5.1), (5.2) with $u_0 \in \left(L_0, \bar{B}\right)$, then u is regular. $\square$

Theorem 6.5 discusses the case, where $\phi^{-1} \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$. Now, we study the case, where condition (5.52) falls, that is $\phi^{-1} \notin \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$. Then both regular and singular homoclinic solutions may exist and, according to Remark 5.25, we are able to prove the uniqueness just for regular ones.

Lemma 6.6 (Regular homoclinic solution is increasing). *Assume that* (5.3)–(5.7)*,* (5.11)*,* (5.12) *hold. Let u be a regular homoclinic solution of problem* (5.1)*,* (5.2) *with $u_0 \in \left(L_0, \bar{B}\right)$. Then*

$$\lim_{t\to\infty} u(t) = L, \qquad u'(t) > 0,\ t \in (0,\infty). \tag{6.9}$$

Moreover,

$$\lim_{t\to\infty} u'(t) = 0. \tag{6.10}$$

Proof. Let u be a regular homoclinic solution of problem (5.8), (5.2) with $u_0 \in (L_0, \bar{B})$. Thus, by Definition 5.3, $u_{\sup} = L$.

Step 1. By Lemma 5.5, there exists $\theta_0 > 0$ such that $u(\theta_0) = 0$, $u(t) < 0$ for $t \in (0, \theta_0)$ and $u'(t) > 0$ for $t \in (0, \theta_0]$. Assume on contrary with (6.9) that $a_1 > \theta_0$ is the first zero of u'. Since u is regular homoclinic solution, $u(a_1) \in (0, L)$. If $u > 0$ on $[a_1, \infty)$, then, by Lemma 5.6, u is decreasing, which contradicts $u_{\sup} = L$. Therefore, there exists $\theta_1 > a_1$ such that $u(\theta_1) = 0$, $u'(t) < 0$ for $t \in (a_1, \theta_1]$. Hence, we have

$$u(a_1) \in (0, L), \quad u'(a_1) = 0, \quad u'(t) > 0,\ t \in (0, a_1). \tag{6.11}$$

By Lemma 5.12, there exists $b_1 > \theta_1$ such that

$$u(b_1) \in (\bar{B}, 0), \quad u'(b_1) = 0, \quad u'(t) < 0,\ t \in [\theta_1, b_1).$$

Since, $u_{\sup} = L$, there exists $\theta_2 > b_1$ such that $u(\theta_2) = 0$, $u'(t) > 0$ for $t \in (b_1, \theta_2]$. By Lemma 5.11, there exists $a_2 > \theta_2$ such that

$$u(a_2) \in (0, L), \quad u'(a_2) = 0, \quad u'(t) > 0,\ t \in (b_1, a_2).$$

Repeating this procedure, we obtain a sequence of zeros $\{\theta_n\}_{n=0}^{\infty}$ of u and a sequence of local maxima $\{u(a_n)\}_{n=1}^{\infty}$ of u. Now, we prove that the sequence $\{u(a_n)\}_{n=1}^{\infty}$ is nonincreasing. Choose $n \in \mathbb{N}$. Equation (5.8) yields

$$p(t)\phi'(u'(t))u''(t) + p'(t)\phi(u'(t)) + p(t)\tilde{f}(\phi(u(t))) = 0.$$

Multiplying this equation by u'/p and integrating from a_n to a_{n+1}, we obtain

$$\int_{a_n}^{a_{n+1}} \phi'(u'(t))u''(t)u'(t)\,\mathrm{d}t + \int_{a_n}^{a_{n+1}} \frac{p'(t)}{p(t)}\phi(u'(t))u'(t)\,\mathrm{d}t \\ + \int_{a_n}^{a_{n+1}} \tilde{f}(\phi(u(t)))u'(t)\,\mathrm{d}t = 0.$$

Using the substitution $s = u'(t)$ in the first integral, we get

$$\int_{a_n}^{a_{n+1}} \phi'(u'(t))u''(t)u'(t)\,\mathrm{d}t = \int_{u'(a_n)}^{u'(a_{n+1})} \phi'(s)s\,\mathrm{d}s = \int_0^0 \phi'(s)s\,\mathrm{d}s = 0.$$

The second integral is nonnegative due to (5.7) and (5.13). Therefore, after the substitution $y = u(t)$, we obtain

$$0 \geq \int_{a_n}^{a_{n+1}} \tilde{f}(\phi(u(t)))u'(t)\,\mathrm{d}t = \int_{u(a_n)}^{u(a_{n+1})} \tilde{f}(\phi(y))\,\mathrm{d}y = \tilde{F}(u(a_{n+1})) - \tilde{F}(u(a_n)).$$

Since $\tilde{F}$ is increasing on $[0, L]$ (cf. Remark 5.8), we get $u(a_n) \geq u(a_{n+1})$. Since n is chosen arbitrarily, the sequence $\{u(a_n)\}_{n=1}^{\infty}$ is nonincreasing. Thus, $u_{\text{sup}} < L$, which cannot be fulfilled, because u is a homoclinic solution. This contradiction yields that

$$u'(t) > 0, \quad t \in (0, \infty).$$

Since $u_{\text{sup}} = L$, then $\lim_{t\to\infty} u(t) = L$.

Step 2. Since $u > 0$ on (θ_0, ∞), we have $f(\phi(u)) > 0$ on (θ_0, ∞). From (5.8), we obtain that

$$0 > (p(t)\phi(u'(t)))' = p'(t)\phi(u'(t)) + p(t)\left(\phi(u'(t))\right)', \quad t \in (\theta_0, \infty).$$

Since p, p', u' and $\phi(u')$ are positive on $(0, \infty)$, we get that $\phi(u')$ is decreasing on (θ_0, ∞). On the other hand, ϕ is increasing on $\mathbb{R}$. Therefore, u' is decreasing on (θ_0, ∞). Since $u' > 0$ on $(0, \infty)$, there exists a limit

$$\lim_{t\to\infty} u'(t) =: K \geq 0.$$

Assume that $K > 0$. Then

$$K(t - \theta_0) \leq \int_{\theta_0}^{t} u'(s)\,\mathrm{d}s = u(t) - u(\theta_0) = u(t), \quad t \in (\theta_0, \infty).$$

Letting $t \to \infty$, we have

$$L = \lim_{t\to\infty} u(t) \geq \lim_{t\to\infty} K(t - \theta_0) = \infty,$$

a contradiction. Therefore, (6.10) holds. Remark 5.17 yields that u is solution of problem (5.1), (5.2). □

We are ready to prove the uniqueness result for regular homoclinic solutions in the case, where ϕ^{-1} does not have to be Lipschitz continuous.

Theorem 6.7 (Uniqueness of regular homoclinic solutions). *Assume that (5.3)–(5.7), (5.11), (5.12), (5.51) and (5.66) are satisfied. Let u be a regular homoclinic solution of problem (5.1), (5.2) with $u_0 \in \big(L_0, \bar{B}\big)$. Then u is a unique solution of this problem.*

Proof. Let u be a regular homoclinic solution of problem (5.8), (5.2). According to Lemma 6.6, $u' > 0$ on $(0, \infty)$. Consider that v is another solution of problem (5.8), (5.2). Assume that there exists $t_0 \in (0, \infty)$ such that $v'(t_0) = 0$. By Lemma 5.5, there exists $\theta > 0$ such that $v(\theta) = 0$, $v'(t) > 0$ for $t \in (0, \theta]$. Therefore, $t_0 > \theta$ and there exists $a \in (\theta, t_0]$ such that $v'(a) = 0$, $v'(t) > 0$ for $t \in (0, a)$. Put

$$\rho(t) := u(t) - v(t), \quad t \in [0, \infty).$$

Since $u' > 0$ and $v' > 0$ on $(0, a)$, Using Theorem 5.26 with $u_0 = B_1 = B_2$, we obtain

$$\rho(t) = 0, \quad \rho'(t) = 0, \; t \in [0, a). \tag{6.12}$$

Since $u, v \in C^1[0, \infty)$, we get that (6.5) holds. Thus, $u'(a) = v'(a)$. According to the definition of number a, we have $u'(a) = v'(a) = 0$, which contradicts the inequality $u' > 0$ on $(0, \infty)$. Therefore, $a = \infty$ and, by (6.12), u is a unique solution of problem (5.8), (5.2). Due to Remark 5.17, u is solution of problem (5.1), (5.2). □

6.3 Examples

Here we show examples, where the functions ϕ, p, and f satisfy assumptions of Theorems 5.19, 6.4 and 6.7.

Example 6.8. Let us consider the IVP

$$\begin{gathered}\big(t^\beta \phi(u'(t))\big)' + t^\beta k |\phi(u(t))|^\gamma \operatorname{sgn} u(t) (\phi(u(t)) - \phi(L_0))(\phi(L) - \phi(u(t))) = 0,\\ u(0) = u_0, \quad u'(0) = 0, \quad u_0 \in [L_0, L],\end{gathered} \tag{6.13}$$

where

$$\phi(x) = |x|^\alpha \operatorname{sgn} x, \quad x \in \mathbb{R}.$$

We have equation (5.1) with

$$\begin{aligned}&p(t) = t^\beta, \quad t \in [0, \infty),\\ &f(x) = k|x|^\gamma \operatorname{sgn} x (x - \phi(L_0))(\phi(L) - x), \quad x \in [\phi(L_0), \phi(L)].\end{aligned}$$

Assume that $0 < L < -L_0$, $\alpha > 1$, $\beta > 0$, $\gamma \geq 1$ and $k > 0$. According to Example 5.23, conditions (5.3)–(5.7) and (5.51) are fulfilled. Example 5.9 shows that (5.12) holds. Since ϕ is odd and increasing on $\mathbb{R}$ and $0 < L < -L_0$, we obtain, similarly as in Example 2.18, that $\tilde{F}(L) < \tilde{F}(L_0)$. Hence, there exists $\bar{B} \in (L_0, 0)$ such that $\tilde{F}\left(\bar{B}\right) = \tilde{F}(L)$, which yields (5.11). Furthermore,

$$\phi^{-1}(x) = |x|^{\frac{1}{\alpha}} \operatorname{sgn} x, \qquad \left(\phi^{-1}\right)'(x) = \frac{1}{\alpha}|x|^{\frac{1}{\alpha}-1},$$
$$\limsup_{x\to 0} x\left(\phi^{-1}\right)'(x) = \lim_{x\to 0}\frac{1}{\alpha}x|x|^{\frac{1}{\alpha}-1} = 0 < \infty,$$
$$\phi'(x) = \alpha|x|^{\alpha-1}, \qquad \phi''(x) = \frac{\alpha(\alpha-1)|x|^{\alpha-1}}{x}\begin{cases}\leq 0 & \text{for } x < 0,\\ \geq 0 & \text{for } x > 0.\end{cases}$$

Hence, ϕ' is nonincreasing on $(-\infty, 0)$, nondecreasing on $(0, \infty)$ and so, conditions (5.57) and (5.66) hold.

To sum up, if

$$0 < L < -L_0,\ \alpha > 1,\ \beta > 0,\ \gamma \geq 1 \ \text{ and } \ k > 0,$$

then we have satisfied all assumptions of Theorems 5.19, 6.4 and 6.7. Hence, the auxiliary problem (6.13) with $f \equiv \tilde{f}$ has for each $u_0 \in [L_0, L]$ a solution u. If $u_0 > L_0$ and $u < L$ on $[0, \infty)$, then u is a solution of the original problem (6.13) and it is a unique solution of this problem.

Example 6.9. Now, we consider the IVP

$$\left(\arctan t\left(u'^4(t) + 2u'^2(t)\right)\operatorname{sgn} u'(t)\right)' + \arctan t\, f\left(\left(u^4(t) + 2u^2(t)\right)\operatorname{sgn} u(t)\right) = 0,$$
$$u(0) = u_0, \quad u'(0) = 0, \quad u_0 \in [L_0, L],$$

where

$$f(x) = \begin{cases} x(1-x)(x+2) & \text{for } x \leq 0,\\ \frac{5}{7}x(1-x)(x+3) & \text{for } x > 0.\end{cases}$$

We have equation (5.1) with

$$\phi(x) = \left(x^4 + 2x^2\right)\operatorname{sgn} x, \quad x \in \mathbb{R},$$
$$p(t) = \arctan t, \quad t \in [0, \infty).$$

The function p is continuously differentiable and increasing on $[0, \infty)$ and $p(0) = 0$, which yields (5.7). In addition,

$$\limsup_{t\to\infty}\frac{p'(t)}{p(t)} = \lim_{t\to\infty}\frac{\frac{1}{t^2+1}}{\arctan t} = 0,$$

that is (5.12) holds.

Here $\phi(L_0) = -L_0^4 - 2L_0^2 = -2$, $\phi(L) = L^4 + 2L^2 = 1$, which together with $L_0 < 0 < L$ yields

$$L_0 = -\sqrt{\sqrt{3} - 1} \approx -0.86, \quad L = \sqrt{\sqrt{2} - 1} \approx 0.64.$$

The function f is Lipschitz continuous on $[\phi(L_0), \phi(L)]$, $f(\phi(L_0)) = f(0) = f(\phi(L)) = 0$ and $xf(x) > 0$ for $x \in ((\phi(L_0), \phi(L)) \setminus \{0\})$, that is conditions (5.5), (5.6) and (5.51) are valid. Moreover,

$$\tilde{F}(L_0) = \int_0^{-\sqrt{\sqrt{3}-1}} \left(-s^4 - 2s^2\right)\left(1 + s^4 + 2s^2\right)\left(-s^4 - 2s^2 + 2\right) \mathrm{d}s \approx 2.56,$$
$$\tilde{F}(L) = \int_0^{\sqrt{\sqrt{2}-1}} \frac{5}{7}\left(s^4 + 2s^2\right)\left(1 - s^4 - 2s^2\right)\left(s^4 + 2s^2 + 3\right) \mathrm{d}s \approx 0.20.$$

Since $\tilde{F}(L) < \tilde{F}(L_0)$, then there exists $\bar{B} \in (L_0, 0)$ such that $\tilde{F}\left(\bar{B}\right) = \tilde{F}(L)$, which yields (5.11).

Example 5.20 shows that we have satisfied conditions (5.3) and (5.4). Further,

$$\phi^{-1}(x) = \sqrt{\sqrt{|x| + 1} - 1}\,,$$
$$\left(\phi^{-1}\right)'(x) = \frac{\frac{\operatorname{sgn} x}{2\sqrt{|x|+1}}}{2\sqrt{\sqrt{|x| + 1} - 1}} = \frac{\operatorname{sgn} x}{4\sqrt{\sqrt{|x| + 1} - 1}\,\sqrt{|x| + 1}}\,,$$
$$\limsup_{x \to 0} x\left(\phi^{-1}\right)'(x) = \lim_{x \to 0} \frac{|x|}{4\sqrt{\sqrt{|x| + 1} - 1}\,\sqrt{|x| + 1}} = 0 \in \mathbb{R},$$
$$\phi'(x) = 4\left(x^3 + x\right) \operatorname{sgn} x, \qquad \phi''(x) = 4\left(3x^2 + 1\right) \operatorname{sgn} x \begin{cases} < 0 & \text{for } x < 0, \\ > 0 & \text{for } x > 0. \end{cases}$$

Therefore, ϕ' is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$ and thus, (5.57) and (5.66) hold.

To summarize, all assumptions of Theorems 5.19, 6.4 and 6.7 are fulfilled.

Example 6.10. We consider the IVP

$$\begin{gathered} \left(\tanh t\, |u'(t)|^\alpha \operatorname{sgn} u'(t)\right)' + \tanh t\, f\left(|u(t)|^\alpha \operatorname{sgn} u(t)\right) = 0, \\ u(0) = u_0, \quad u'(0) = 0, \quad u_0 \in [L_0, L], \end{gathered} \tag{6.14}$$

where

$$f(x) = \begin{cases} -(x + 2^\lambda + 2) & \text{for } x \le -2, \\ |x|^\lambda \operatorname{sgn} x & \text{for } x \in (-2, 1), \\ 2 - x & \text{for } x \ge 1. \end{cases}$$

Here

$$\phi(x) = |x|^\alpha \operatorname{sgn} x, \quad x \in \mathbb{R},$$
$$p(t) = \tanh t = \frac{e^t - e^{-t}}{e^t + e^{-t}}, \quad t \in [0, \infty).$$

Assume that $\alpha > 1$ and $\lambda \geq 1$. The function p is continuously differentiable and increasing on $[0, \infty)$ and $p(0) = 0$, that is (5.7) holds. Furthermore,

$$\limsup_{t \to \infty} \frac{p'(t)}{p(t)} = \lim_{t \to \infty} \frac{\frac{1}{\cosh^2 t}}{\tanh t} = 0$$

yields (5.12).

Here $\phi(L_0) = -2 - 2^\lambda \leq -4$, $\phi(L) = 2$. The function f is Lipschitz continuous on $[\phi(L_0), \phi(L)]$, $f(\phi(L_0)) = f(0) = f(\phi(L)) = 0$ and $xf(x) > 0$ for $x \in ((\phi(L_0), \phi(L)) \setminus \{0\})$ and so, conditions (5.5), (5.6) and (5.51) hold. Since ϕ is odd and increasing on $\mathbb{R}$ and $0 < \phi(L) < -\phi(L_0)$, we get $0 < L < -L_0$ and, similarly as in Example 2.18, $\tilde{F}(L) < \tilde{F}(L_0)$. Thus, there exists $\bar{B} \in (L_0, 0)$ fulfilling (5.11).

According to Example 6.8, the function ϕ satisfies (5.3), (5.4), (5.57) and (5.66).

To sum up, if

$$\alpha > 1 \quad \text{and} \quad \lambda \geq 1,$$

then Theorems 5.19, 6.4 and 6.7 are applicable on problem (6.14).

7 Escape and unbounded solutions of the problem with ϕ-Laplacian

7.1 Statement of the problem

We study the same IVP with ϕ-Laplacian as before, that is

$$(p(t)\phi(u'(t)))' + p(t)f(\phi(u(t))) = 0, \tag{7.1}$$

$$u(0) = u_0, \quad u'(0) = 0, \quad u_0 \in [L_0, L]. \tag{7.2}$$

Now, we assume the following basic assumptions:

$$\phi \in C^1(\mathbb{R}), \quad \phi'(x) > 0 \text{ for } x \in (\mathbb{R} \setminus \{0\}), \tag{7.3}$$

$$\phi(\mathbb{R}) = \mathbb{R}, \quad \phi(0) = 0, \tag{7.4}$$

$$L_0 < 0 < L, \quad f(\phi(L_0)) = f(0) = f(\phi(L)) = 0, \tag{7.5}$$

$$\begin{gathered} f \in C[\phi(L_0), \infty), \qquad xf(x) > 0 \quad \text{for } x \in ((\phi(L_0), \phi(L)) \setminus \{0\}), \\ f(x) \le 0 \quad \text{for } x > \phi(L), \end{gathered} \tag{7.6}$$

$$p \in C[0,\infty) \cap C^1(0,\infty), \quad p'(t) > 0 \text{ for } t \in (0,\infty), \quad p(0) = 0. \tag{7.7}$$

We see that the only difference between these basic assumptions and those in Section 5.1 is in condition (7.6), where we consider f also on $(\phi(L), \infty)$. This is essential for the investigation of escape solutions, that is solutions whose supremum is greater than L (cf. Lemma 7.4). Since these basic assumptions contain basic assumptions from Section 5.1, all results in Chapters 5 and 6 are valid also for this chapter. A model example shows Example 5.1 again.

Definition 7.1. Let $[0, b) \subset [0, \infty)$ be a maximal interval such that a function $u \in C^1[0, b)$ with $\phi(u') \in C^1(0, b)$ satisfies equation (7.1) for every $t \in (0, b)$ and let u satisfy the initial conditions (7.2). Then u is called a *solution of problem* (7.1), (7.2) *on* $[0, b)$. If u is solution of problem (7.1), (7.2) on $[0, \infty)$, then u is called a *solution of problem* (7.1), (7.2).

Assumption (7.5) yields that the constant functions $u(t) \equiv L_0$, $u(t) \equiv 0$ and $u(t) \equiv L$ are solutions of problem (7.1), (7.2) on $[0, \infty)$ with $u_0 = L_0$, $u_0 = 0$ and $u_0 = L$, respectively.

Definition 7.2. Consider a solution of problem (7.1), (7.2) with $u_0 \in [L_0, L)$ and denote

$$u_{\sup} := \sup\{u(t) \colon t \in [0, \infty)\}.$$

If $u_{\sup} < L$, then u is called a *damped solution* of problem (7.1), (7.2).
If $u_{\sup} = L$, then u is called a *homoclinic solution* of problem (7.1), (7.2).

Definition 7.3. Let u be a solution of problem (7.1), (7.2) on $[0, b)$, where $b \in (0, \infty]$. If there exists $c \in (0, b)$ such that

$$u(c) = L, \quad u'(c) > 0, \tag{7.8}$$

then u is called an *escape solution* of problem (7.1), (7.2) *on* $[0, b)$.

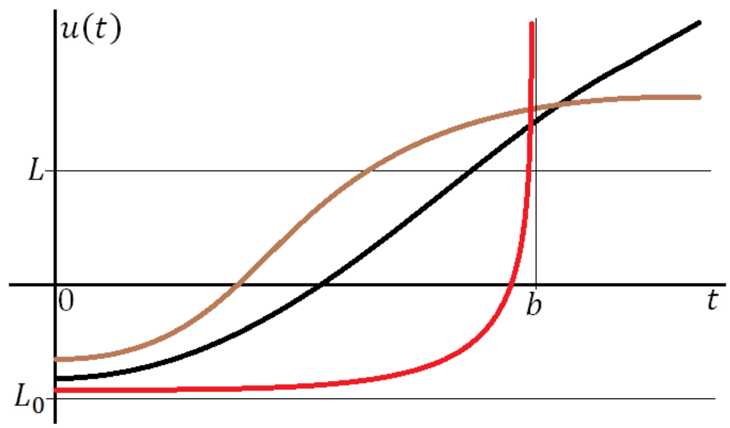

Figure 7.1: Types of escape solutions of problem (7.1), (7.2)

The goal of this chapter is to find conditions which guarantee the existence of escape solutions of problem (7.1), (7.2), which are unbounded. The analysis of problem (7.1), (7.2) with a general ϕ-Laplacian includes also $\phi(x) = |x|^\alpha \operatorname{sgn} x$ for $\alpha > 1$. Let us emphasis that in this case, $\phi^{-1}(x) = |x|^{\frac{1}{\alpha}} \operatorname{sgn} x$ is not locally Lipschitz continuous. Since ϕ^{-1} is present in the operator form of (7.1), (7.2)

$$u(t) = u_0 + \int_0^t \phi^{-1}\left(-\frac{1}{p(s)} \int_0^s p(\tau) f(\phi(u(\tau)))\, \mathrm{d}\tau\right) \mathrm{d}s, \quad t \in [0, \infty),$$

the standard technique based on the Lipschitz property is not applicable here and an another approach needs to be developed. Therefore, we distinguish two cases:

- In the first case, where the functions ϕ^{-1} and f are Lipschitz continuous, the uniqueness of solution of problem (7.1), (7.2) is guaranteed. This considerably helps to derive conditions when a sequence of solutions contains an escape solution.
- In the second case, functions ϕ^{-1} and f do not have to be Lipschitz continuous. The lack of uniqueness causes difficulties and therefore is more challenging. The problems are overcome by means of the lower and upper function method. Also here sufficient conditions for the existence of escape solutions are derived.

Since in general an escape solution need not be unbounded, criteria for an escape solution to tend to infinity are derived. In this manner, we obtain new existence results for unbounded solution of problem (7.1), (7.2).

In order to derive the main existence results about unbounded solutions of problem (7.1), (7.2), we first introduce the auxiliary equation with a bounded nonlinearity

$$(p(t)\phi(u'(t)))' + p(t)\tilde{f}(\phi(u(t))) = 0, \tag{7.9}$$

where

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in [\phi(L_0), \phi(L)], \\ 0 & \text{for } x < \phi(L_0), \quad x > \phi(L). \end{cases} \tag{7.10}$$

Since $\tilde{f}$ is bounded on $\mathbb{R}$, the maximal interval $[0, b)$ for each solution of problem (7.9), (7.2) is $[0, \infty)$.

7.2 Properties of solutions

In this section, we provide auxiliary lemmas, which are used in Section 7.3 for proofs of the existence and uniqueness of escape solutions of the auxiliary problem (7.9), (7.2). Note that all soutions of problem (7.9), (7.2) with $u_0 \in [\bar{B}, L)$ are damped solutions, see Lemma 5.16. Therefore, we consider only $u_0 \in [L_0, \bar{B})$ for investigation of escape solutions of problem (7.9), (7.2). Such solutions can be equivalently characterized as follows.

Lemma 7.4. *Assume that (7.3)–(7.7) and*

$$\limsup_{t\to\infty} \frac{p'(t)}{p(t)} < \infty \tag{7.11}$$

hold and let u be a solution of problem (7.9), (7.2). Then u is an escape solution if and only if

$$u_{\sup} = \sup\{u(t) \colon t \in [0, \infty)\} > L. \tag{7.12}$$

Proof. Let u fulfil (7.12). According to Definition 7.2, u is not a damped solution and hence, due to Lemma 5.16, $u(0) < \bar{B} < 0$. Consequently, there exists a maximal $c > 0$ such that $u(t) < L$ for $t \in [0, c)$ and

$$u(c) = L, \quad u'(c) \geq 0.$$

We exclude the case $u'(c) = 0$. Lemma 5.7 yields that if $u'(c) = 0$ then either u has a zero point $u(\theta) = 0$, $u(t) \leq L$, $t \in [c, \theta]$ or u is positive and nonincreasing on $[c, \infty)$. The later case is in contradiction with (7.12). Therefore, such zero point $\theta > c$ has to exist. We use Lemma 5.12 and repeating the arguments as in Step 1 in the proof of Lemma 6.6, we get that u has a nonincreasing sequence

$\{u(a_n)\}_{n=1}^{\infty}$ of its local maxima. Hence, $u_{\sup} = u(c) = L$, contrary to (7.12). Therefore, u fulfils (7.8). On the other hand, if u is an escape solution of problem (7.9), (7.2), then (7.12) follows immediately from Definition 7.3. □

Lemma 7.5 (Escape solution is increasing). *Assume that* (7.3)–(7.7), (7.11) *and*

$$\exists\, \bar{B} \in (L_0, 0)\colon \tilde{F}\left(\bar{B}\right) = \tilde{F}(L), \qquad \textit{where } \tilde{F}(x) := \int_0^x \tilde{f}(\phi(s))\,\mathrm{d}s, \quad x \in \mathbb{R} \tag{7.13}$$

hold. Let u be an escape solution of problem (7.9), (7.2) *with $u_0 \in \left(L_0, \bar{B}\right)$. Then*

$$u'(t) > 0, \quad t \in (0, \infty).$$

Proof. Let u be an escape solution of problem (7.9), (7.2) with $u_0 \in \left(L_0, \bar{B}\right)$. Thus, by Lemma 7.4, $u_{\sup} > L$. Then there exists $c \in (0, \infty)$ such that $u(c) = L$, $u'(c) \geq 0$ and $u(t) < L$ for $t \in [0, c)$.

We can exclude the case $u'(c) = 0$ as in the proof of Lemma 7.4. Hence, $u'(c) > 0$. Let $c_1 > c$ be such that $u'(c_1) = 0$ and $u(t) > L$, $u'(t) > 0$ for $t \in (c, c_1)$. Integrating (7.9) over $[c, c_1]$, dividing by $p(c_1)$ and using (7.3), (7.4), (7.7) and (7.10), we get

$$\phi(u'(c_1)) = \frac{p(c)\phi(u'(c))}{p(c_1)} > 0,$$

contrary to $u'(c_1) = 0$. We have proved $u'(t) > 0$ for $t > c$. Since $u_0 \in (L_0, 0)$, Lemma 5.5 yields that there exists $\theta_0 > 0$ such that $u(\theta_0) = 0$, $u(t) < 0$ for $t \in (0, \theta_0)$, $u'(t) > 0$ for $t \in (0, \theta_0]$.

It remains to prove that $u'(t) > 0$ for $t \in (\theta_0, c)$. Assume on the contrary that there exists $a_1 \in (\theta_0, c)$ such that (6.11) holds. We derive a contradiction as in Step 1 in the proof of Lemma 6.6. To summarize, $u'(t) > 0$ for $t > 0$. □

The proofs of the existence of escape solutions are based on Lemmas 7.6 and 7.9. These lemmas are denoted here as basic lemmas (cf. Lemma 4.5) because they are essential for the proof of existence of escape solutions.

Lemma 7.6 (Basic lemma 1). *Let* (7.3)–(7.7), (7.13) *and*

$$\lim_{t \to \infty} \frac{p'(t)}{p(t)} = 0 \tag{7.14}$$

hold. Choose $C \in (L_0, \bar{B})$ and a sequence $\{B_n\}_{n=1}^{\infty} \subset (L_0, C)$. Let for each $n \in \mathbb{N}$, u_n be a solution of problem (7.9), (7.2) *with $u_0 = B_n$ and let $(0, b_n)$ be the maximal interval such that*

$$u_n(t) < L, \quad u_n'(t) > 0, \quad t \in (0, b_n). \tag{7.15}$$

Finally, let $\gamma_n \in (0, b_n)$ be such that

$$u_n(\gamma_n) = C, \quad \forall\, n \in \mathbb{N}. \tag{7.16}$$

If the sequence $\{\gamma_n\}_{n=1}^\infty$ is unbounded, then the sequence $\{u_n\}_{n=1}^\infty$ contains an escape solution of problem (7.9), (7.2).

Proof. Since the sequence $\{\gamma_n\}_{n=1}^\infty$ is unbounded, there exists a subsequence going to ∞ as $n \to \infty$. For simplicity, let us denote it by $\{\gamma_n\}_{n=1}^\infty$. Then we have

$$\lim_{n\to\infty} \gamma_n = \infty, \quad \gamma_n < b_n, \quad n \in \mathbb{N}.$$

Assume on the contrary that for any $n \in \mathbb{N}$, u_n is not an escape solution of problem (7.9), (7.2). By Lemma 7.4,

$$\sup\{u_n(t)\colon t \in [0,\infty)\} \le L, \quad n \in \mathbb{N}. \tag{7.17}$$

Step 1. Choose fixed $n \in \mathbb{N}$ and consider a solution u_n of problem (7.9), (7.2) with $u_0 = B_n$.

First, assume that $u_n < 0$ on $[0,\infty)$. Then, by Lemma 5.5, we get $u_n' > 0$ on $(0,\infty)$ and for $b_n = \infty$ we obtain (7.15). In addition, Lemma 5.13 yields

$$\lim_{t\to\infty} u_n(t) = 0, \quad \lim_{t\to\infty} u_n'(t) = 0.$$

If we put

$$\lim_{t\to\infty} u_n(t) =: u_n(b_n), \quad \lim_{t\to\infty} u_n'(t) =: u_n'(b_n),$$

we have

$$u_n(b_n) = 0,\ u_n'(b_n) = 0. \tag{7.18}$$

Now, we assume that $\theta > 0$ is the first zero of u_n. By Lemma 5.5, $u_n' > 0$ on $(0,\theta]$.

(i) Let $u_n' > 0$ on (θ,∞). Then, according to (7.17), $0 < u_n < L$ on (θ,∞) and (7.15) is valid for $b_n = \infty$. We prove that

$$\lim_{t\to\infty} u_n(t) = L, \quad \lim_{t\to\infty} u_n'(t) = 0.$$

Since u_n is increasing on $(0,\infty)$, then according to (7.17), $0 < u_n < L$ on $(0,\infty)$. We denote

$$\lim_{t\to\infty} u_n(t) =: \ell \in (0, L].$$

Since u_n is a solution of equation (7.9), then

$$\phi'(u_n'(t))\, u_n''(t) + \frac{p'(t)}{p(t)}\, \phi(u_n'(t)) + \tilde{f}(\phi(u_n(t))) = 0, \quad t \in (0,\infty). \tag{7.19}$$

If we restrict the previous equation to the interval (θ, ∞) then, by (7.3)–(7.7), we have

$$\frac{p'(t)}{p(t)}\,\phi(u_n'(t)) > 0, \quad \tilde{f}(\phi(u_n(t))) > 0, \quad \phi'(u_n'(t)) > 0, \quad t \in (\theta, \infty)$$

and we deduce that $u_n''(t) < 0$, $t \in (\theta, \infty)$. Consequently, u_n' is decreasing on (θ, ∞) and so, there has to exist $\lim_{t\to\infty} u_n'(t) \ge 0$. If $\lim_{t\to\infty} u_n'(t) > 0$, then $\lim_{t\to\infty} u_n(t) = \infty$, a contradiction. Therefore,

$$\lim_{t\to\infty} u_n'(t) = 0.$$

Assume that $\ell \in (0, L)$. Letting $t \to \infty$ in (7.19), we get, by (7.4) and (7.14),

$$\phi'(0) \cdot \lim_{t\to\infty} u_n''(t) = -\tilde{f}(\phi(\ell)).$$

Since $\tilde{f}(\phi(\ell)) \in (0, \infty)$, we get

$$\lim_{t\to\infty} u_n''(t) < 0,$$

contrary to $\lim_{t\to\infty} u_n'(t) = 0$. Consequently, $\ell = L$ and so,

$$u_n(b_n) = L, \quad u_n'(b_n) = 0. \tag{7.20}$$

(ii) Let $a > \theta$ be the first zero of u_n'. According to (7.17), we have $u_n(a) \le L$. For $b_n = a$, we get (7.15) and

$$u_n(b_n) \in (0, L],\; u_n'(b_n) = 0. \tag{7.21}$$

To summarize (7.18), (7.20) and (7.21), we see that u_n fulfils

$$u_n(b_n) \in [0, L], \quad u_n'(b_n) = 0. \tag{7.22}$$

Step 2. Let n be fixed. We define

$$E_n(t) := \int_0^{u_n'(t)} x\phi'(x)\,\mathrm{d}x + \tilde{F}(u_n(t))), \quad t \in (0, b_n)$$

and

$$K_n := \sup\left\{\frac{p'(t)}{p(t)} : t \in [\gamma_n, b_n)\right\}.$$

Due to (7.14), $\lim_{n\to\infty} K_n = 0$. In addition,

$$\exists\, \overline{\gamma}_n \in [\gamma_n, b_n) \colon u_n'(\overline{\gamma}_n) = \max\{u_n'(t) \colon t \in [\gamma_n, b_n)\}. \tag{7.23}$$

Then, by (7.19), we obtain

$$\begin{aligned}\frac{\mathrm{d}E_n(t)}{\mathrm{d}t} &= u_n'(t)\,\phi'(u_n'(t))\,u_n''(t) + \tilde{f}(\phi(u_n(t)))\,u_n'(t)\\ &= -\frac{p'(t)}{p(t)}\,\phi(u_n'(t))\,u_n'(t) < 0, \quad t \in (0, b_n).\end{aligned}$$

Integrating the above equality over (γ_n, b_n) and using (7.15), (7.23), we get

$$\begin{aligned}E_n(\gamma_n) - E_n(b_n) &= \int_{\gamma_n}^{b_n} \frac{p'(t)}{p(t)}\phi(u_n'(t))u_n'(t)\,\mathrm{d}t \le \phi(u_n'(\overline{\gamma}_n)) \int_{\gamma_n}^{b_n} \frac{p'(t)}{p(t)}u_n'(t)\,\mathrm{d}t\\ &\le \phi(u_n'(\overline{\gamma}_n))K_n \int_{\gamma_n}^{b_n} u_n'(t)\,\mathrm{d}t \le \phi(u_n'(\overline{\gamma}_n))K_n(L - C).\end{aligned}$$

Hence,

$$E_n(\gamma_n) \le E_n(b_n) + \phi(u_n'(\overline{\gamma}_n))K_n(L - C). \tag{7.24}$$

Moreover, from (7.22), we have

$$E_n(\gamma_n) > F(u_n(\gamma_n)) = F(C), \quad E_n(b_n) = F(u_n(b_n)) \le F(L).$$

Therefore, using (7.24), we obtain

$$F(C) < E_n(\gamma_n) \le F(L) + \phi(u_n'(\overline{\gamma}_n)K_n(L - C), \tag{7.25}$$

which gives

$$\frac{F(C) - F(L)}{L - C}\frac{1}{K_n} < \phi(u_n'(\overline{\gamma}_n)). \tag{7.26}$$

Step 3. We consider a sequence $\{u_n\}_{n=1}^{\infty}$. Since $\lim_{n\to\infty} K_n = 0$, we derive, from (7.26),

$$\lim_{n\to\infty} \phi(u_n'(\overline{\gamma}_n)) = \infty. \tag{7.27}$$

Using (7.4), we get

$$\lim_{n\to\infty} u_n'(\overline{\gamma}_n) = \lim_{n\to\infty} \phi^{-1}(\phi(u_n'(\overline{\gamma}_n))) = \infty. \tag{7.28}$$

Since $\tilde{F} \ge 0$ on $\mathbb{R}$ and E_n is decreasing on $(0, b_n)$, we obtain, by (7.25),

$$\int_0^{u_n'(\overline{\gamma}_n)} x\phi'(x)\,\mathrm{d}x \le E_n(\overline{\gamma}_n) \le E_n(\gamma_n) \le \tilde{F}(L) + \phi(u_n'(\overline{\gamma}_n))K_n(L - C), \quad n \in \mathbb{N}$$

and so,

$$\lim_{n\to\infty}\left(\int_0^{u_n'(\overline{\gamma}_n)} x\phi'(x)\,\mathrm{d}x - \phi(u_n'(\overline{\gamma}_n))K_n(L - C)\right) \le \tilde{F}(L) < \infty. \tag{7.29}$$

According to (7.28), there exists $n_0 \in \mathbb{N}$ such that $u_n'(\overline{\gamma}_n) > 1$ for $n \geq n_0$. Therefore, we derive for all natural $n \geq n_0$ that

$$\int_0^{u_n'(\overline{\gamma}_n)} x\phi'(x)\,\mathrm{d}x > \int_1^{u_n'(\overline{\gamma}_n)} x\phi'(x)\,\mathrm{d}x > \int_1^{u_n'(\overline{\gamma}_n)} \phi'(x)\,\mathrm{d}x = \phi(u_n'(\overline{\gamma}_n)) - \phi(1)$$

and, by (7.27) and $\lim_{n\to\infty} K_n = 0$,

$$\lim_{n\to\infty} \left(\int_0^{u_n'(\overline{\gamma}_n)} x\phi'(x)\,\mathrm{d}x - \phi(u_n'(\overline{\gamma}_n))K_n(L-C) \right)$$
$$\geq \lim_{n\to\infty} \phi(u_n'(\overline{\gamma}_n))\,(1 - K_n(L-C)) - \phi(1) = \infty,$$

contrary to (7.29). Consequently, the sequence $\{u_n\}_{n=1}^{\infty}$ contains an escape solution of problem (7.9), (7.2). □

If ϕ^{-1} and f are not Lipschitz continuous, then problem (7.9), (7.2) with $u_0 \in [L_0, L] \setminus \{0\}$ can have more solutions. These solutions may belong among escape solutions. In particular, more solutions can start at L_0, not only the constant solution $u(t) \equiv L_0$. Therefore, we need to extend the assertion of Lemma 7.6 which deal with values greater than L_0 into $u_0 = L_0$. For this purpose, the next two lemmas are helpful.

Lemma 7.7. *Let (7.3)–(7.7) hold and let u be a solution of problem (7.9), (7.2) such that*

$$u_0 = L_0, \quad u(t) \not\equiv L_0, \quad u(t) \geq L_0, \quad t \in [0,\infty). \tag{7.30}$$

Then there exists $a \geq 0$ such that

$$u(t) = L_0, \quad t \in [0,a] \tag{7.31}$$

and

$$u'(t) > 0, \quad t \in (a,\theta],$$

where θ is the first zero of u on (a,∞). If such θ does not exist, then $u'(t) > 0$ for $t \in (a,\infty)$.
Let $\theta \in (a,\infty)$ and let there exist $a_1 > \theta$ such that

$$u'(a_1) = 0, \quad u'(t) > 0,\ t \in (\theta, a_1). \tag{7.32}$$

Then $u(a_1) \in (0, L]$.

Proof. By (7.30), there exists $\tau > 0$ such that

$$L_0 < u(\tau) < 0. \tag{7.33}$$

Define $a := \inf\{\tau > 0\colon (7.33) \text{ holds}\}$. Then u fulfils (7.31) and $u'(a) = 0$. Let us put $\theta := \sup\{\tau > a\colon (7.33) \text{ holds}\}$. Then

$$p(t)\,\tilde{f}(\phi(u(t))) < 0, \quad t \in (a,\theta). \tag{7.34}$$

Integrating equation (7.9) from a to $t \in (a, \theta)$, we get, by (7.34),

$$p(t)\, \phi\left(u'(t)\right) = -\int_a^t p(s)\, \tilde{f}\left(\phi(u(s))\right)\, \mathrm{d}s > 0, \quad t \in (a, \theta).$$

Since $p(t) > 0$, necessarily $u'(t) > 0$ for $t \in (a, \theta)$. If $\theta = \infty$, then the proof is finished. On the other hand, if $\theta < \infty$, then θ is the first zero of u on (a, ∞) and from

$$p(\theta)\, \phi\left(u'(\theta)\right) = -\int_a^\theta p(s)\, \tilde{f}\left(\phi(u(s))\right)\, \mathrm{d}s > 0,$$

we have $u'(\theta) > 0$.

Let $\theta \in (a, \infty)$ and (7.32) hold. Then $u(a_1) > 0$. Assume that $u(a_1) > L$. Then there exists $a_0 \in (\theta, a_1)$ such that $u > L$ on $(a_0, a_1]$. Integrating equation (7.9) over (a_0, a_1) and using (7.10), we obtain

$$p(a_0)\phi(u'(a_0)) - p(a_1)\phi(u'(a_1)) = \int_{a_0}^{a_1} p(s)\tilde{f}(\phi(u(s)))\, \mathrm{d}s = 0$$

and so, $p(a_0)\phi(u'(a_0)) = 0$. Consequently, $u'(a_0) = 0$, contrary to $u' > 0$ on (a, a_1). We have proved that $u(a_1) \le L$, which completes the proof. ☐

Lemma 7.8. *Let (7.3)–(7.7) and (7.11) hold and let u be a solution of (7.9), (7.2) satisfying (7.30). Assume that*

$$u(t) < 0, \quad t \in [0, \infty).$$

Then

$$\lim_{t\to\infty} u(t) = 0, \quad \lim_{t\to\infty} u'(t) = 0.$$

Proof. The proof is the same as the proof of Lemma 5.13, but using Lemma 7.7 instead of Lemma 5.5. ☐

Lemma 7.9 (Basic lemma 2). *Let (7.3)–(7.7), (7.13) and (7.14) hold. Choose $C \in (L_0, \bar{B})$. Let for each $n \in \mathbb{N}$, u_n be a solution of problem (7.9), (7.2) with $u_0 = L_0$ and let (a_n, b_n) be the maximal interval such that*

$$L_0 < u_n(t) < L, \quad u_n'(t) > 0, \quad t \in (a_n, b_n).$$

Finally, let $\gamma_n \in (a_n, b_n)$ be such that

$$u_n(\gamma_n) = C, \quad \forall\, n \in \mathbb{N}.$$

If the sequence $\{\gamma_n\}_{n=1}^{\infty}$ is unbounded, then the sequence $\{u_n\}_{n=1}^{\infty}$ contains an escape solution of problem (7.9), (7.2) with $u_0 = L_0$.

Proof. The proof is analogous to the proof of Lemma 7.6, but using in Step 1 Lemmas 7.7 and 7.8 instead of Lemmas 5.5 and 5.13, respectively. ☐

7.3 Existence and uniqueness of escape solutions

This section is devoted to the existence of escape solutions of problem (7.9), (7.2). First, we discuss the existence of escape solutions provided the Lipschitz continuity of ϕ^{-1} and f.

Theorem 7.10 (**Existence of escape solutions of problem** (7.9)**,** (7.2) **I).** *Let* (7.3)*–*(7.7)*,* (7.13)*,* (7.14)*,*

$$f \in \mathrm{Lip}\left[\phi(L_0), \phi(L)\right], \tag{7.35}$$

$$\phi^{-1} \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}) \tag{7.36}$$

hold. Then there exist infinitely many escape solutions of problem (7.9)*,* (7.2) *with different starting values in* $\left(L_0, \bar{B}\right)$ *converging to* L_0*.*

Proof. Choose $n \in \mathbb{N}$, $C \in \left(L_0, \bar{B}\right)$ and $B_n \in (L_0, C)$. By Theorems 5.19 and 5.21, there exists a unique solution u_n of problem (7.9), (7.2) with $u_0 = B_n$. By Lemma 5.5, there exists a maximal $a_n > 0$ such that $u_n' > 0$ on $(0, a_n)$. Since $u_n(0) < 0$, there exists a maximal $\tilde{a}_n > 0$ such that $u_n < L$ on $[0, \tilde{a}_n)$. If we put $b_n := \min\{a_n, \tilde{a}_n\}$, then (7.15) holds. Further, either $\lim_{t\to\infty} u_n(t) = 0$ or u_n has a zero $\theta_n \in (0, b_n)$ due to Lemmas 5.5 and 5.13. Consequently, there exists $\gamma_n \in (0, b_n)$ satisfying $u_n(\gamma_n) = C$ and so, (7.16) is fulfilled.

Consider a sequence $\{B_n\}_{n=1}^{\infty} \subset (L_0, C)$. Then we get the sequence $\{u_n\}_{n=1}^{\infty}$ of solutions of problem (7.9), (7.2) with $u_0 = B_n$ and the corresponding sequence of $\{\gamma_n\}_{n=1}^{\infty}$. Assume that $\lim_{n\to\infty} B_n = L_0$. Then, by Theorem 5.21, the sequence $\{u_n\}_{n=1}^{\infty}$ converges locally uniformly on $[0, \infty)$ to the constant function $u \equiv L_0$. Therefore, $\lim_{n\to\infty} \gamma_n = \infty$ and the sequence $\{\gamma_n\}_{n=1}^{\infty}$ is unbounded. By Lemma 7.6 there exists $n_0 \in \mathbb{N}$ such that u_{n_0} is an escape solution of problem (7.9), (7.2). We have $u_{n_0}(0) = B_{n_0} > L_0$. Now, consider the unbounded sequence $\{\gamma_n\}_{n=n_0+1}^{\infty}$. According to Lemma 7.6, there exists $n_1 \in \mathbb{N}$, $n_1 \geq n_0 + 1$ such that u_{n_1} is an escape solution of problem (7.9), (7.2) with $u_{n_1}(0) = B_{n_1} > L_0$. Repeating this procedure, we obtain the sequence $\{u_{n_k}\}_{k=0}^{\infty}$ of escape solutions of problem (7.9), (7.2). □

Now, we investigate the existence of escape solutions in the case, where ϕ^{-1} and f do not have to be Lipschitz continuous. In order to prove this existence result, we consider the lower and upper functions method for an auxiliary mixed problem on $[0, T]$. In particular, we use this method to find solutions of (7.9) which satisfy

$$u'(0) = 0, \quad u(T) = C, \quad C \in [L_0, L]. \tag{7.37}$$

Definition 7.11. A function $u \in \mathcal{C}^1[0, T]$ with $\phi(u') \in \mathcal{C}^1(0, T]$ is a *solution of problem* (7.9), (7.37) if u fulfils (7.9) for $t \in (0, T]$ and satisfies (7.37).

Definition 7.12. A function $\sigma_1 \in \mathcal{C}[0, T]$ is a *lower function* of problem (7.9), (7.37) if there exists a finite (possibly empty) set $\Sigma_1 \subset (0, T)$ such that $\sigma_1 \in$

$\mathcal{C}^2((0,T]\setminus\Sigma_1)$ and

$$(p(t)\,\phi(\sigma_1'(t)))' + p(t)\,\tilde f(\phi(\sigma_1(t))) \ge 0, \quad t\in(0,T]\setminus\Sigma_1, \tag{7.38}$$

$$-\infty < \sigma_1'(\tau^-) < \sigma_1'(\tau^+) < \infty, \quad \tau\in\Sigma_1, \tag{7.39}$$

$$\sigma_1'(0^+) \ge 0,\ \ \sigma_1(T) \le C. \tag{7.40}$$

Analogously,

Definition 7.13. A function $\sigma_2\in\mathcal{C}[0,T]$ is an *upper function* of problem (7.9), (7.37) if there exists a finite (possibly empty) set $\Sigma_2\subset(0,T)$ such that $\sigma_2\in\mathcal{C}^2((0,T]\setminus\Sigma_2)$ and

$$(p(t)\,\phi(\sigma_2'(t)))' + p(t)\,\tilde f(\phi(\sigma_2(t))) \le 0, \quad t\in(0,T]\setminus\Sigma_2, \tag{7.41}$$

$$-\infty < \sigma_2'(\tau^+) < \sigma_2'(\tau^-) < \infty, \quad \tau\in\Sigma_2, \tag{7.42}$$

$$\sigma_2'(0^+) \le 0,\ \ \sigma_2(T) \ge C. \tag{7.43}$$

For the following results, we define a function φ

$$\varphi(t) := \frac{1}{p(t)}\int_0^t p(s)\,\mathrm{d}s, \quad t\in(0,t], \qquad \varphi(0)=0. \tag{7.44}$$

The function φ is continuous on $[0,T]$ and fulfils

$$0<\varphi(t)\le t, \quad t\in(0,T], \qquad \lim_{t\to0^+}\varphi(t)=0. \tag{7.45}$$

Theorem 7.14 (Lower and upper functions method). *Let (7.3)–(7.7) hold and let σ_1 and σ_2 be lower and upper functions of problem (7.9), (7.37) such that*

$$\sigma_1(t)\le\sigma_2(t), \quad t\in[0,T].$$

Then problem (7.9), (7.37) has a solution u such that

$$\sigma_1(t)\le u(t)\le\sigma_2(t), \quad t\in[0,T].$$

Proof. *Step 1.* For $t\in[0,T]$ and $x\in\mathbb{R}$, we define the following auxiliary nonlinearity

$$f^*(t,x) = \begin{cases} \tilde f(\phi(\sigma_1(t))) + \frac{\sigma_1(t)-x}{\sigma_1(t)-x+1}\,, & x<\sigma_1(t),\\ \tilde f(\phi(x)), & \sigma_1(t)\le x\le\sigma_2(t),\\ \tilde f(\phi(\sigma_2(t))) - \frac{x-\sigma_2(t)}{x-\sigma_2(t)+1}\,, & x>\sigma_2(t). \end{cases} \tag{7.46}$$

Since f^* is bounded, then there exists $M^*>0$ such that

$$|f^*(t,x)|\le M^*, \quad (t,x)\in[0,T]\times\mathbb{R}. \tag{7.47}$$

Consider the auxiliary equation

$$(p(t)\,\phi(u'(t)))' + p(t)\,f^*(t,u(t)) = 0, \quad t \in (0,T]. \tag{7.48}$$

We prove that problem (7.48), (7.37) has a solution. We follow the procedure from the proof of Theorem 5.19. By integrating (7.48), we obtain the equivalent form of problem (7.48), (7.37)

$$u(t) = C - \int_t^T \phi^{-1}\left(-\frac{1}{p(s)}\int_0^s p(\tau)\,f^*(\tau,u(\tau))\,\mathrm{d}\tau\right)\,\mathrm{d}s, \quad t \in [0,T].$$

Now, consider the Banach space $C\,[0,T]$ with the maximum norm and define an operator $\mathcal{F}\colon C\,[0,T] \to C\,[0,T]$,

$$(\mathcal{F}u)(t) := C - \int_t^T \phi^{-1}\left(-\frac{1}{p(s)}\int_0^s p(\tau)\,f^*(\tau,u(\tau))\,\mathrm{d}\tau\right)\,\mathrm{d}s.$$

Let us put $\Lambda := \max\{|L_0|, L\}$ and consider the ball

$$\mathcal{B}\,(0,R) = \left\{u \in C\,[0,T] : \|u\|_{C[0,T]} \le R\right\}, \quad \text{where } R := \Lambda + T\,\phi^{-1}\,(M^*T)$$

and M^* is from (7.47). Since ϕ is increasing on $\mathbb{R}$, then ϕ^{-1} is also increasing on $\mathbb{R}$. Thus, due to (7.45), $\phi^{-1}\,(M^*\varphi(t)) \le \phi^{-1}\,(M^*T)$, $t \in [0,T]$, where φ is defined in (7.44). The norm of $\mathcal{F}u$ can be estimated as follows

$$\begin{aligned}
\|\mathcal{F}u\|_{C[0,T]} &= \max_{t\in[0,T]}\left|C - \int_t^T \phi^{-1}\left(-\frac{1}{p(s)}\int_0^s p(\tau)\,f^*(\tau,u(\tau))\,\mathrm{d}\tau\right)\,\mathrm{d}s\right| \\
&\le \Lambda + \int_t^T \left|\phi^{-1}\,(M^*\varphi(s))\right|\,\mathrm{d}s \le \Lambda + \int_t^T \phi^{-1}\,(M^*T)\,\mathrm{d}s \\
&\le \Lambda + T\,\phi^{-1}\,(M^*T) = R,
\end{aligned}$$

which gives that $\mathcal{F}$ maps $\mathcal{B}\,(0,R)$ on itself.

Now, we prove that $\mathcal{F}$ is compact on $\mathcal{B}\,(0,R)$. Choose a sequence $\{u_n\} \subset C\,[0,T]$ such that $\lim_{n\to\infty}\|u_n - u\|_{C[0,T]} = 0$. We have

$$\begin{aligned}
(\mathcal{F}u_n)(t) - (\mathcal{F}u)(t) = -\int_t^T &\left(\phi^{-1}\left(-\frac{1}{p(s)}\int_0^s p(\tau)f^*(\tau,u_n(\tau))\,\mathrm{d}\tau\right)\right. \\
&\left. + \phi^{-1}\left(-\frac{1}{p(s)}\int_0^s p(\tau)f^*(\tau,u(\tau))\,\mathrm{d}\tau\right)\right)\,\mathrm{d}s.
\end{aligned}$$

Since f^* is continuous on $[0,T]\times\mathbb{R}$, we obtain

$$\lim_{n\to\infty}\|f^*(\cdot,u_n(\cdot))) - f^*(\cdot,u(\cdot))\|_{C[0,T]} = 0.$$

Let us put

$$A_n(t) := -\frac{1}{p(t)} \int_0^t p(\tau) f^*(\tau, u_n(\tau))\, \mathrm{d}\tau,$$
$$A(t) := -\frac{1}{p(t)} \int_0^t p(\tau) f^*(\tau, u(\tau))\, \mathrm{d}\tau, \quad t \in (0,T], \quad A_n(0) = A(0) = 0, \quad n \in \mathbb{N}.$$

Then, for a fixed $n \in \mathbb{N}$, we get

$$|A_n(t) - A(t)| = \left| \frac{1}{p(t)} \int_0^t p(\tau) \left(f^*(\tau, u(\tau)) - f^*(\tau, u_n(\tau)) \,\mathrm{d}\tau \right) \right|, \quad t \in (0,T]$$

and, by virtue of (7.45) and (7.47), $\lim_{t\to 0^+} |A_n(t) - A(t)| = 0$. Hence, $A_n - A \in C[0,T]$ and

$$\|A_n - A\|_{C[0,T]} \le \|f^*(\cdot, u_n(\cdot))) - f^*(\cdot, u(\cdot)))\|_{C[0,T]}\, T, \quad n \in \mathbb{N}.$$

This yields that $\lim_{n\to\infty} \|A_n - A\|_{C[0,T]} = 0$. Using the continuity of ϕ^{-1} on $\mathbb{R}$, we have

$$\lim_{n\to\infty} \left\|\phi^{-1}(A_n) - \phi^{-1}(A)\right\|_{C[0,T]} = 0.$$

Thus,

$$\lim_{n\to\infty} \|\mathcal{F}u_n - \mathcal{F}u\|_{C[0,T]} = \lim_{n\to\infty} \left\| \int_t^T \left(\phi^{-1}(A_n(s)) - \phi^{-1}(A(s))\right) \mathrm{d}s \right\|_{C[0,T]}$$
$$\le T \lim_{n\to\infty} \left\|\phi^{-1}(A_n) - \phi^{-1}(A)\right\|_{C[0,T]} = 0,$$

that is the operator $\mathcal{F}$ is continuous.

Let us choose an arbitrary $\varepsilon > 0$ and put $\delta := \frac{\varepsilon}{\phi^{-1}(M^*T)}$. Then, for $t_1, t_2 \in [0,T]$ and $u \in \mathcal{B}(0,R)$, we get

$$|t_1 - t_2| < \delta \Rightarrow |(\mathcal{F}u)(t_1) - (\mathcal{F}u)(t_2)|$$
$$= \left| \int_{t_2}^{t_1} \phi^{-1}\left(-\frac{1}{p(s)} \int_0^s p(\tau) f^*(\tau, u(\tau))\, \mathrm{d}\tau \right) \mathrm{d}s \right| \le \left| \int_{t_2}^{t_1} \phi^{-1}\left(M^* \varphi(s)\right) \mathrm{d}s \right|$$
$$\le \left| \int_{t_2}^{t_1} \phi^{-1}\left(M^* T\right) \mathrm{d}s \right| = \phi^{-1}\left(M^*T\right) |t_1 - t_2| < \phi^{-1}\left(M^*T\right) \delta = \varepsilon.$$

Consequently, functions in $\mathcal{F}(\mathcal{B}(0,R))$ are equicontinuous, and, by the Arzelà–Ascoli theorem, the set $\mathcal{F}(\mathcal{B}(0,R))$ is relatively compact. Therefore, the operator $\mathcal{F}$ is compact on $\mathcal{B}(0,R)$.

The Schauder fixed point theorem yields a fixed point $u^\star$ of $\mathcal{F}$ in $\mathcal{B}(0,R)$. Hence,

$$u^\star(t) = C - \int_t^T \phi^{-1}\left(-\frac{1}{p(s)} \int_0^s p(\tau) f^*(\tau, u^\star(\tau))\, \mathrm{d}\tau \right) \mathrm{d}s$$

is a solution of problem (7.48), (7.37).

Step 2. We prove that any solution u of problem (7.48), (7.37) satisfies

$$\sigma_1(t) \le u(t) \le \sigma_2(t), \quad t \in [0,T]$$

and therefore, u is a solution of problem (7.1), (7.37).

Put $v(t) := u(t) - \sigma_2(t)$ for $t \in [0,T]$ and assume that

$$\max\{v(t) \colon t \in [0,T]\} =: v(t_0) > 0. \tag{7.49}$$

By (7.42), $v'(\tau^-) < v'(\tau^+)$ for each $\tau \in \Sigma_2$ and so, $t_0 \notin \Sigma_2$. Moreover, $\sigma_2(T) \ge C$ and $u(T) = C$. Thus, $v(T) \le 0$, that is $t_0 \ne T$. Therefore, $t_0 \in [0,T) \setminus \Sigma_2$.

Let $t_0 = 0$. Then (7.37) and (7.43) yield $v'(0^+) = u'(0^+) - \sigma_2'(0^+) = -\sigma_2'(0^+) \ge 0$. Furthermore, $v'(0^+) = 0$, since $v'(0^+) > 0$ give a contradiction with (7.49). If $t_0 \in (0,T) \setminus \Sigma_2$, (7.49) also implies that $v'(t_0) = 0$.

Since $t_0 \in [0,T) \setminus \Sigma_2$, there exists $\delta > 0$ such that $(t_0, t_0+\delta) \subset (0,T) \setminus \Sigma_2$ and $v(t) > 0$ for $t \in (t_0, t_0+\delta)$. Due to (7.41), (7.46) and (7.48), we obtain

$$\begin{aligned}(p(t)\,\phi(u'(t)))' - (p(t)\,\phi(\sigma_2'(t)))' &\ge p(t)\left(-f^*(t,u(t)) + \tilde f(\phi\,(\sigma_2(t)))\right)\\ &= p(t)\frac{v(t)}{v(t)+1} > 0, \qquad t \in (t_0, t_0+\delta).\end{aligned}$$

Integrating the previous expression and using $u'(t_0) - \sigma_2'(t_0) = v'(t_0) = 0$, we obtain for $t \in (t_0, t_0+\delta)$

$$\int_{t_0}^{t} \left((p(s)\,\phi(u'(s)))' - (p(s)\,\phi(\sigma_2'(s)))'\right)\,\mathrm{d}s = p(t)\,(\phi(u'(t)) - \phi(\sigma_2'(t))) > 0.$$

Therefore, since ϕ is increasing, we have that $u'(t) - \sigma_2'(t) = v'(t) > 0$ for $t \in (t_0, t_0+\delta)$, contrary to (7.49). Consequently,

$$u(t) \le \sigma_2(t), \quad t \in [0,T].$$

Analogously, it can be proved that

$$u(t) \ge \sigma_1(t), \quad t \in [0,T].$$

According to (7.46), $f^*(t,x) = f(\phi(x))$ for $t \in [0,T]$, $x \in \mathbb{R}$ and hence, the solution u of problem (7.48), (7.37) is a solution of problem (7.9), (7.37). ☐

The main result of this section is contained in Theorem 7.16. Its proof is based on Lemmas 7.6 and 7.9, where a suitable sequence $\{u_n\}$ of solutions of problem (7.9), (7.2) is used. In order to get such sequence with the starting values equal to L_0 (see part (ii) in the proof of Theorem 7.16), we need the next lemma.

Lemma 7.15. *Let (7.3)–(7.7), (7.13) and (7.14) hold. Choose $C \in (L_0, \bar{B})$ and assume that there exists at least one solution u of problem (7.9), (7.2) satisfying (7.30), that is*

$$u_0 = L_0, \quad u(t) \not\equiv L_0, \quad u(t) \geq L_0, \quad t \in [0, \infty).$$

Then there exists $\gamma > 0$ such that for each $T > \gamma$, problem (7.9), (7.2) with $u_0 = L_0$ has a solution u_T satisfying

$$u_T(T) = C, \qquad u_T(t) \geq L_0, \ t \in [0, \infty). \tag{7.50}$$

Proof. As a consequence of Lemmas 7.7 and 7.8 we know that either there exists $\theta > 0$ such that $u(\theta) = 0$ or $\lim_{t \to \infty} u(t) = 0$. Because of this, we can take

$$\gamma := \min\{t \in [0, \infty); \ u(t) = C\} > 0. \tag{7.51}$$

Now, fix $T > \gamma$.

Step 1. We construct a lower function of problem (7.9), (7.37). We prove that $\sigma_1(t) \equiv L_0$ satisfies conditions (7.38)–(7.40). First,

$$(p(t)\phi(\sigma_1'(t)))' + p(t)\tilde{f}(\phi(\sigma_1(t))) = (p(t)\phi(0))' + p(t)\tilde{f}(\phi(L_0)) = 0 \geq 0, \ t \in [0, T].$$

Moreover, in this case, $\sigma_1 \in \mathcal{C}^2[0, T]$, so $\Sigma_1 = \emptyset$. Finally, $\sigma_1'(0^+) = 0 \geq 0$ and $\sigma_1(T) = L_0 < C$. Therefore, σ_1 is a lower function of (7.9), (7.37).

Step 2. We construct an upper function of problem (7.9), (7.37). We distinguish two different cases.

(i) Let $u < 0$ on $[0, \infty)$. Then we choose $\sigma_2 = u$. We show that σ_2 fulfils conditions (7.41)–(7.43). First,

$$(p(t)\phi(\sigma_2'(t)))' + p(t)\tilde{f}(\phi(\sigma_2(t))) = 0 \leq 0, \ t \in (0, T].$$

In addition, $\sigma_2 \in \mathcal{C}^2(0, T]$, which yields $\Sigma_2 = \emptyset$. Finally, $\sigma_2'(0^+) = 0 \leq 0$ and $\sigma_2(T) > \sigma_2(\gamma) = C$. The last inequality $\sigma_2(T) > C$ is a consequence of the fact that from Lemma 7.7 we know that σ_2 is increasing on $[a, \infty)$ for some $a \in [0, \gamma)$. Hence, σ_2 satisfies conditions (7.41)–(7.43).

(ii) Let there exist $\theta > 0$ such that $u(\theta) = 0$. Then $\gamma \in (0, \theta)$ and we choose

$$\sigma_2(t) = \begin{cases} u(t), & t \in [0, \theta], \\ 0, & t \in (\theta, \infty). \end{cases}$$

We have

$$(p(t)\phi(\sigma_2'(t)))' + p(t)\tilde{f}(\phi(\sigma_2(t))) = 0 \leq 0, \ t \in (0, T] \setminus \{\theta\}.$$

In this case, $\Sigma_2 = \{\theta\}$. By Lemma 7.7, $u' > 0$ on $(a, \theta]$ for some $a \in [0, \gamma)$ and hence, $\sigma_2'(\theta^-) > 0$. It is clear that $\sigma_2'(\theta^+) = 0$ and so, $\sigma_2'(\theta^+) < \sigma_2'(\theta^-)$. Analogously to part (i), we get $\sigma_2'(0^+) = 0 \leq 0$ and $\sigma_2(T) > \sigma_2(\gamma) = u(\gamma) = C$. Therefore, σ_2 satisfies conditions (7.41)–(7.43).

Consequently, σ_2 is an upper function of (7.9), (7.37).

Step 3. We prove the existence of a solution u_T. We have found a pair of lower and upper functions which clearly satisfy for each $T > \gamma$ that $\sigma_1(t) \le \sigma_2(t)$, $t \in [0, T]$. As a consequence, Theorem 7.14 ensures the existence of a solution u_T of problem (7.9), (7.37) such that

$$L_0 \le u_T(t) \le \sigma_2(t), \quad t \in [0, T].$$

Since $\sigma_2(0) = u_T(0) = L_0$, then u fulfils (7.2) with $u_0 = L_0$. Since $\tilde{f}(\phi)$ is bounded on $\mathbb{R}$, u_T can be extended to interval $[0, \infty)$ as a solution of equation (7.9). This classical extension result follows from more general Theorem 11.5 in [31].

Step 4. We prove the estimate

$$u_T(t) \ge L_0 \quad \text{for } t \in [0, \infty). \tag{7.52}$$

We follow the proof of Lemma 5.16. If $u_T < 0$ on $[0, \infty)$, then Lemma 7.7 yields $a \ge 0$ such that $u_T = L_0$ on $[0, a]$, $u_T' > 0$ on (a, ∞) and so, (7.52) is valid. Assume that there exists $\theta_1 > 0$ such that $u_T(\theta_1) = 0$, $u_T(t) < 0$ for $t \in [0, \theta_1)$. By Lemma 7.7, there exists $a \ge 0$ such that $u_T = L_0$ on $[0, a]$ and $u_T' > 0$ on $(a, \theta_1]$. If $u_T' > 0$ on (θ_1, ∞), then (7.52) holds. Let there exist $a_1 > \theta_1$ such that $u_T'(a_1) = 0$, $u_T'(t) > 0$ for $t \in (\theta_1, a_1)$. According to Lemma 7.7, $u_T(a_1) \in (0, L]$ and $u_T' > 0$ on (a, a_1). If $u_T > 0$ on $[a_1, \infty)$, then (7.52) is valid. Assume that there exists $\theta_2 > a_1$ such that $u_T(\theta_2) = 0$, $u_T > 0$ on $[a_1, \theta_2)$. Lemma 5.12 gives $b > \theta_2$ such that $u_T'(b) = 0$, $u_T' < 0$ on (a_1, b), $u(b) \in (\bar{B}, 0)$. If $u_T < 0$ on (b, ∞), then, by Lemma 5.5, $u_T' > 0$ on (b, ∞) and (7.52) holds. Let there exist $\theta_3 > b$ such that $u_T(\theta_3) = 0$, $u_T < 0$ on $[b, \theta_3)$. Then we can apply Lemma 5.11 and argue as before.

We have proved that u_T is a solution of problem (7.9), (7.2) with $u_0 = L_0$ and satisfies (7.50). □

Theorem 7.16 (**Existence of escape solutions of problem** (7.9)**,** (7.2) **II).** *Let* (7.3)*–*(7.7)*,* (7.13) *and* (7.14) *hold. Then there exist infinitely many escape solutions of problem* (7.9)*,* (7.2) *with not necessary different starting values in* $[L_0, \bar{B})$.

Proof. Choose $n \in \mathbb{N}$, $C \in (L_0, \bar{B})$ and $B_n \in (L_0, C)$. By Theorem 5.19, there exists a solution u_n of problem (7.9), (7.2) with $u_0 = B_n$. By Lemma 5.5, there exists a maximal $a_n > 0$ such that $u_n' > 0$ on $(0, a_n)$. Since $u_n(0) < 0$, there exists a maximal $\tilde{a}_n > 0$ such that $u_n < L$ on $[0, \tilde{a}_n)$. If we put $b_n := \min\{a_n, \tilde{a}_n\}$, then (7.15) holds. Due to Lemmas 5.5 and 5.13, there exists $\gamma_n \in (0, b_n)$ such that $u_n(\gamma_n) = C$.

Consider a sequence $\{B_n\}_{n=1}^{\infty} \subset (L_0, C)$. Then we get a sequence $\{u_n\}_{n=1}^{\infty}$ of solutions of problem (7.9), (7.2) with $u_0 = B_n$ and the corresponding sequence of $\{\gamma_n\}_{n=1}^{\infty}$. Assume that $\lim_{n\to\infty} B_n = L_0$. Integrating equation (7.9), we get the equivalent form of problem (7.9), (7.2) for u_n

$$u_n(t) = B_n + \int_0^t \phi^{-1}\left(-\frac{1}{p(s)}\int_0^s p(\tau)\,\tilde{f}(\phi(u_n(\tau)))\,\mathrm{d}\tau\right)\mathrm{d}s, \quad t\in[0,\infty). \quad (7.53)$$

According to (7.10), $\tilde{f}$ is bounded. Hence, there exists $\tilde{M}>0$ such that

$$\left|\tilde{f}(x)\right| \le \tilde{M}, \quad x\in\mathbb{R}. \quad (7.54)$$

Let us choose $\beta>0$. Using (7.44), (7.45) and (7.54), we obtain from (7.53)

$$\begin{aligned}|u_n(t)| &< |L_0| + \int_0^t \left|\phi^{-1}\left(-\frac{1}{p(s)}\int_0^s p(\tau)\,\tilde{f}(\phi(u_n(\tau)))\,\mathrm{d}\tau\right)\right|\mathrm{d}s\\ &\le |L_0| + \int_0^t \left|\phi^{-1}\left(\tilde{M}\,\varphi(s)\right)\right|\mathrm{d}s \le |L_0| + \int_0^t \phi^{-1}\left(\tilde{M}\,\beta\right)\mathrm{d}s\\ &\le |L_0| + \beta\,\phi^{-1}\left(\tilde{M}\,\beta\right) =: K_\beta, \quad t\in[0,\beta].\end{aligned}$$

Since β is chosen arbitrarily, we have that the sequence $\{u_n\}_{n=1}^\infty$ is uniformly bounded on $[0,\beta]$ for all $\beta>0$. Moreover, as a consequence of Lemma 5.18, the sequence of derivatives $\{u_n'\}_{n=1}^\infty$ is uniformly bounded by number $\tilde{c}$.

Choose an arbitrary $\varepsilon>0$, put $\delta := \frac{\varepsilon}{\tilde{c}}$ and let $t_1,t_2\in[0,\beta]$. The Mean value theorem gives $\xi\in(t_1,t_2)$ such that

$$|t_1-t_2|<\delta \Rightarrow |u_n(t_1)-u_n(t_2)| = |u_n'(\xi)|\,|t_1-t_2| < \tilde{c}\delta = \varepsilon,$$

which yields that the sequence $\{u_n\}_{n=1}^\infty$ is equicontinuous on $[0,\beta]$ for all $\beta>0$. Therefore, by Arzelà–Ascoli theorem, there exists a subsequence of $\{u_n\}_{n=1}^\infty$ which converges locally uniformly on $[0,\infty)$ to a continuous function u. For the sake of simplicity, we denote this subsequence also as $\{u_n\}_{n=1}^\infty$. Since the convergence of $\{u_n\}$ is locally uniform, by letting $t\to\infty$ in (7.53), we obtain

$$u(t) = L_0 + \int_0^t \phi^{-1}\left(-\frac{1}{p(s)}\int_0^s p(\tau)\,\tilde{f}(\phi(u(\tau)))\,\mathrm{d}\tau\right)\mathrm{d}s, \quad t\in[0,\infty)$$

and hence, u is a solution of problem (7.9), (7.2) for $u_0 = L_0$.

Now, we distinguish three different cases.

(i) Let $u\equiv L_0$.

Then $\lim_{n\to\infty}\gamma_n = \infty$ and the sequence $\{\gamma_n\}_{n=1}^\infty$ is unbounded. By virtue of Lemma 7.6, there exists $n_0\in\mathbb{N}$ such that u_{n_0} is an escape solution of problem (7.9), (7.2). We have $u_{n_0}(0) = B_{n_0} > L_0$. Now consider the unbounded sequence $\{\gamma_n\}_{n=n_0+1}^\infty$. By Lemma 7.6 there exists $n_1\in\mathbb{N}$, $n_1\ge n_0+1$ such that u_{n_1} is an escape solution of problem (7.9), (7.2) with $u_{n_1}(0) = B_{n_1} > L_0$. We repeat this procedure and we obtain the sequence $\{u_{n_k}\}_{k=0}^\infty$ of escape solutions of problem (7.9), (7.2) with starting values in $\left(L_0,\bar{B}\right)$.

(ii) Let $u \not\equiv L_0$ is not the escape solution.

In this case, we take $\tilde{B}_n = L_0$ for all $n \in \mathbb{N}$ and consider γ defined in (7.51). Now, we can take an unbounded sequence $\{\tilde{\gamma}_n\}_{n=1}^{\infty}$ such that $\tilde{\gamma}_n > \gamma$ for all $n \in \mathbb{N}$. Due to Lemma 7.15, for all $n \in \mathbb{N}$ there exists a solution $\tilde{u}_n$ of problem (7.9), (7.2) with $u_0 = \tilde{B}_n$ such that

$$\tilde{u}_n(\tilde{\gamma}_n) = C, \quad \tilde{u}_n(t) \geq L_0, \quad t \in [0, \infty).$$

Therefore, we have a sequence of solutions $\{\tilde{u}_n\}_{n=1}^{\infty}$ in the conditions of Lemma 7.9 and so, this sequence contains the escape solution $\tilde{u}_{n_0}$ of (7.9), (7.2) with $u_0 = L_0$. As in the previous case, we could consider now the unbounded sequence $\{\tilde{\gamma}_n\}_{n=n_0+1}^{\infty}$ and repeat the procedure from (i). This way we obtain a sequence $\{\tilde{u}_{n_k}\}_{k=0}^{\infty}$ of escape solutions of problem (7.9), (7.2) with $u_0 = L_0$.

(iii) Let $u \not\equiv L_0$ is the escape solution.

Then we can argue as in the case (ii) and we also obtain the sequence $\{\tilde{u}_{n_k}\}_{k=0}^{\infty}$ of escape solutions of problem (7.9), (7.2) with $u_0 = L_0$.

Moreover, in this case, since the sequence $\{u_n\}_{n=0}^{\infty}$ converges locally uniformly to the escape solution of (7.9), (7.2), there exists some n_0 such that u_n is also the escape solution for all $n \geq n_0$. As a consequence, we also obtain a sequence $\{u_n\}_{n=n_0}^{\infty}$ of escape solutions of problem (7.9), (7.2) with starting values in $\big(L_0, \bar{B}\big)$. □

In the case, where ϕ^{-1} does not have to be Lipschitz continuous, the uniqueness of damped and regular homoclinic solutions is guaranteed by Theorems 6.4 and 6.7, respectively. Similarly, we can obtain also the uniqueness of escape solutions.

Theorem 7.17 (**Uniqueness of escape solutions**). *Assume that* (7.3)–(7.7), (7.11), (7.13), (7.35) *and*

$$\limsup_{x \to 0^+} \Big(x \left(\phi^{-1}\right)'(x) \Big) < \infty, \quad \phi' \textit{ is nondecreasing on } (0, \infty)$$

hold. Let u be an escape solution of problem (7.9), (7.2) *with $u_0 \in \big(L_0, \bar{B}\big)$. Then u is a unique solution of this problem.*

Proof. Let u be an escape solution of problem (7.9), (7.2) with $u_0 \in \big(L_0, \bar{B}\big)$. By Lemma 7.5, $u' > 0$ on $(0, \infty)$ and we can argue as in the proof of Theorem 6.7. □

7.4 Existence of unbounded solutions

In this section, we discuss the original problem (7.1), (7.2) and provide conditions which guarantee that an escape solution of (7.1), (7.2) is unbounded.

Note that solutions of the original problem (7.1), (7.2) and solutions of the auxiliary problem (7.9), (7.2) are related in the following way, when (7.3)–(7.7), (7.14) and (7.13) are assumed. Each solution of (7.9), (7.2) which is not an escape solution, is a bounded solution of the original problem (7.1), (7.2) in $[0,\infty)$. This result follows from Lemma 5.16 and Lemma 7.4, where such solutions of (7.9), (7.2) satisfy

$$L_0 \le u(t) \le L, \quad t \in [0,\infty)$$

and, due to (7.10),

$$\tilde{f}(\phi(u(t))) = f(\phi(u(t))), \quad t \in [0,\infty).$$

If u is an escape solution of the auxiliary problem (7.9), (7.2), i.e.

$$\exists\, c \in (0,\infty)\colon u(t) \in [L_0, L),\ t \in [0,c), \quad u(c) = L, \quad u'(c) > 0, \tag{7.55}$$

then u fulfils at once the auxiliary equation (7.9) and the original equation (7.1) on $[0,c]$. The restriction of u on $[0,c]$ can be extended as an escape solution of problem (7.1), (7.2) on some maximal interval $[0,b)$. Therefore, we search unbounded solutions of (7.1), (7.2) in the class of escape solutions of (7.1), (7.2) on $[0,b)$.

Since in general, an escape solution u of (7.1), (7.2) on $[0,b)$ need not to be unbounded, we derive criteria for u to tend to infinity. First, we show that $b < \infty$ implies the unboundedness of solution of problem (7.1), (7.2) on $[0,b)$. To do that, we prove the partial monotonicity of escape solutions under weaker assumptions compared with Lemma 7.5.

Lemma 7.18. *Assume that (7.3)–(7.7) hold. Let u be an escape solution of problem (7.1), (7.2) on $[0,b)$. Then*

$$u(t) > L, \quad u'(t) > 0, \quad t \in (c,b), \tag{7.56}$$

where c is from (7.55). If $b < \infty$, then

$$\lim_{t\to b^-} u(t) = \infty.$$

Proof. Let u be an escape solution of problem (7.1), (7.2) on $[0,b)$. Then (7.55) holds. Assume that there exists $c_1 > c$ such that $u'(c_1) = 0$, $u(t) > L$, $u'(t) > 0$ for $t \in (c,c_1)$. Integrating equation (7.1) over $[c,c_1]$, dividing by $p(c_1)$ and using (7.3), (7.4), (7.6), (7.7), we get

$$\phi(u'(c_1)) = \frac{p(c)\phi(u'(c))}{p(c_1)} - \frac{1}{p(c_1)}\int_c^{c_1} p(s) f(\phi(u(s)))\,\mathrm{d}s > 0,$$

contrary to $u'(c_1) = 0$. Hence, $u(t) > L$ and $u'(t) > 0$ for $t \in (c, b)$ which yields (7.56).

Let $b < \infty$. Since $[0, b)$ is the maximal interval, where the solution u is defined, u cannot be extended behind b. Therefore, (7.56) gives $\lim_{t\to b^-} u(t) = \infty$ and thus, the solution u is unbounded. □

Since all escape solution of (7.9), (7.2) on $[0, b)$ that cannot be extended on the half-line $[0, \infty)$ are naturally unbounded, we continue our investigation about unboundedness of escape solutions defined on $[0, \infty)$.

Theorem 7.19. *Assume that* (7.3)–(7.7) *hold and let*

$$\lim_{t\to\infty} p(t) < \infty. \tag{7.57}$$

Let u be an escape solution of problem (7.1), (7.2). *Then*

$$\lim_{t\to\infty} u(t) = \infty. \tag{7.58}$$

Proof. Let u be an escape solution of problem (7.1), (7.2). Lemma 7.18 gives (7.56) with $b = \infty$ and so, there exists $\lim_{t\to\infty} u(t) \in (L, \infty]$. Due to (7.3), (7.4), (7.7) and (7.55), $p(c)\phi(u'(c)) =: c_0 \in (0, \infty)$. Integrate equation (7.1) from c to $t > c$ and get, by (7.6), (7.7), for $t \in (c, \infty)$ that

$$u(t) = L + \int_c^t \phi^{-1}\left(\frac{c_0}{p(s)} - \frac{1}{p(s)}\int_c^s p(\tau) f(\phi(u(\tau)))\,\mathrm{d}\tau\right)\mathrm{d}s > \int_c^t \phi^{-1}\left(\frac{c_0}{p(s)}\right)\mathrm{d}s.$$

Conditions (7.7) and (7.57) give

$$\lim_{s\to\infty} \frac{c_0}{p(s)} \in (0, \infty)$$

and, by (7.3), (7.4),

$$\int_1^\infty \phi^{-1}\left(\frac{c_0}{p(s)}\right)\mathrm{d}s = \infty.$$

Therefore,

$$\lim_{t\to\infty} u(t) \ge \int_c^\infty \phi^{-1}\left(\frac{c_0}{p(s)}\right)\mathrm{d}s = \infty,$$

which gives (7.58). □

Theorem 7.20. *Assume that* (7.3)–(7.7), (7.14) *and*

$$f(x) < 0 \quad \textit{for } x > \phi(L) \tag{7.59}$$

hold. Let u be an escape solution of problem (7.1), (7.2). *Then* (7.58) *holds.*

Proof. Let u be an escape solution of problem (7.1), (7.2). According to Lemma 7.18, $u' > 0$ on (c, ∞) and hence, there exists $\lim_{t\to\infty} u(t) \in (L, \infty]$. Assume on the contrary that

$$\lim_{t\to\infty} u(t) =: A \in (L, \infty). \tag{7.60}$$

Step 1. We prove that u' is bounded. Assume that u' is unbounded. Then there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} t_n = \infty, \quad \lim_{n\to\infty} u'(t_n) = \infty.$$

The next approach is similar as in the proof of Lemma 5.13. Equation (7.1) has the equivalent form

$$\phi'(u'(t))u''(t) + \frac{p'(t)}{p(t)}\phi(u'(t)) + f(\phi(u(t))) = 0, \quad t \in (0, \infty). \tag{7.61}$$

Choose $n \in \mathbb{N}$. Multiplying this equation by u' and integrating it from c to $t > c$, we obtain for $t = t_n$ that

$$\psi_1(t_n) + \psi_2(t_n) + \psi_3(t_n) = 0, \quad t_n \in [c, \infty), \tag{7.62}$$

where

$$\psi_1(t_n) = \int_{u'(c)}^{u'(t_n)} x\phi'(x)\, \mathrm{d}x, \qquad \psi_2(t_n) = \int_c^{t_n} \frac{p'(s)}{p(s)}\phi(u'(s))u'(s)\, \mathrm{d}s,$$
$$\psi_3(t_n) = \int_L^{u(t_n)} f(\phi(x))\, \mathrm{d}x.$$

Then

$$\psi_3(t_n) = F(u(t_n)) - F(L), \qquad \text{where } F(x) := \int_0^x f(\phi(s))\, \mathrm{d}s, \quad x \in \mathbb{R}.$$

Due to (7.3), (7.4) and (7.59), $F(x)$ is decreasing for $x > \phi(L)$. Since u is increasing on (c, ∞), $F(u(t_n))$ is decreasing for $t_n \in (c, \infty)$ and $\lim_{n\to\infty} F(u(t_n)) = F(A)$. According to (7.60),

$$\lim_{n\to\infty} \psi_3(t_n) \in (-\infty, 0)\,.$$

By (7.3), (7.4) and (7.7),

$$\lim_{n\to\infty} \psi_1(t_n) = \infty, \quad \lim_{n\to\infty} \psi_2(t_n) > 0.$$

Hence, letting $n \to \infty$ in (7.62), we obtain

$$0 = \lim_{n\to\infty} (\psi_1(t_n) + \psi_2(t_n) + \psi_3(t_n)) = \infty,$$

a contradiction. So, u' is bounded.

Step 2. We prove (7.58). Since u' is bounded, letting $t \to \infty$ in (7.61) and using (7.14), (7.59), (7.60), we get

$$\lim_{t\to\infty} \phi'(u'(t))u''(t) = -f(\phi(A)) > 0.$$

Since $\phi'(u'(t)) > 0$ fot $t > c$, so there exists $\tau > c$ such that $u''(t) > 0$ for $t \geq \tau$. Therefore, u' is increasing on $[\tau, \infty)$ and there exists $\lim_{t\to\infty} u'(t) > 0$, which contradicts $\lim_{t\to\infty} u(t) = A < \infty$. Thus, (7.58) is valid. □

Remark 7.21. The proof of Theorem 7.20 yields that if a solution u of problem (7.1), (7.2) satisfies $\lim_{t\to\infty} u(t) =: A \in (L, \infty)$, then $f(\phi(A)) = 0$, which is equivalent with the fact that $u(t) \equiv A$ is a solution of equation (7.1).

For $f \equiv 0$ on $(\phi(L), \infty)$, we are able to find necessary and sufficient condition for the unboundedness of escape solutions of problem (7.1), (7.2).

Theorem 7.22. *Assume that* (7.3)–(7.7),

$$f(x) \equiv 0 \quad \text{for } x > \phi(L), \tag{7.63}$$

$$\phi(ab) = \phi(a)\phi(b), \quad a, b \in (0, \infty) \tag{7.64}$$

are satisfied. Let u be an escape solution of problem (7.1), (7.2). *Then*

$$\lim_{t\to\infty} u(t) = \infty \quad \Longleftrightarrow \quad \int_1^\infty \phi^{-1}\left(\frac{1}{p(s)}\right) \mathrm{d}s = \infty. \tag{7.65}$$

If we replace condition (7.64) *by*

$$\phi(ab) \leq \phi(a)\phi(b), \quad a, b \in (0, \infty), \tag{7.66}$$

then (7.58) *holds if*

$$\int_1^\infty \phi^{-1}\left(\frac{1}{p(s)}\right) \mathrm{d}s = \infty. \tag{7.67}$$

Proof. Let u be an escape solution of problem (7.1), (7.2). According to Lemma 7.18, $u' > 0$ on (c, ∞). Then there exists $t_0 > c$ such that $u(t_0) > L$, $u'(t) > 0$ for $t \in [t_0, \infty)$. Therefore, there exists $\lim_{t\to\infty} u(t) \in (L, \infty]$. Using (7.64), we obtain

$$\begin{aligned} \phi^{-1}(a)\phi^{-1}(b) &= \phi^{-1}(\phi(\phi^{-1}(a)\phi^{-1}(b))) = \phi^{-1}(\phi(\phi^{-1}(a))\phi(\phi^{-1}(b))) \\ &= \phi^{-1}(ab), \qquad a, b \in (0, \infty). \end{aligned} \tag{7.68}$$

Due to (7.3), (7.4), (7.7) and (7.63),

$$p(t_0)\phi(u'(t_0)) =: c_0 \in (0, \infty), \qquad f(\phi(u(t))) = 0 \;\text{ for } t \in [t_0, \infty).$$

Thus, integrating equation (7.1) from t_0 to $t > t_0$ and using (7.68), we get

$$u(t) = u(t_0) + \int_{t_0}^{t} \phi^{-1}\left(\frac{c_0}{p(s)}\right) \mathrm{d}s = u(t_0) + \phi^{-1}(c_0)\left(\int_1^t \phi^{-1}\left(\frac{1}{p(s)}\right) \mathrm{d}s - \int_1^{t_0} \phi^{-1}\left(\frac{1}{p(s)}\right) \mathrm{d}s\right), \quad t \in (t_0, \infty).$$

Letting $t \to \infty$ here, we get (7.65).

Let us consider (7.66) instead of (7.64) and assume that (7.67) holds. Then we continue analogously and obtain

$$\phi^{-1}(a)\phi^{-1}(b) = \phi^{-1}(\phi(\phi^{-1}(a)\phi^{-1}(b))) \le \phi^{-1}(\phi(\phi^{-1}(a))\phi(\phi^{-1}(b))) = \phi^{-1}(ab), \qquad a, b \in (0, \infty),$$

$$u(t) = u(t_0) + \int_{t_0}^{t} \phi^{-1}\left(\frac{c_0}{p(s)}\right) \mathrm{d}s \ge u(t_0) + \phi^{-1}(c_0)\left(\int_1^t \phi^{-1}\left(\frac{1}{p(s)}\right) \mathrm{d}s - \int_1^{t_0} \phi^{-1}\left(\frac{1}{p(s)}\right) \mathrm{d}s\right), \quad t \in (t_0, \infty).$$

We let $t \to \infty$ here and obtain, by (7.67), that (7.58) holds. □

Now, we present the existence results about unbounded solutions of the original problem (7.1), (7.2) in the case, where ϕ^{-1} and f are Lipschitz continuous, see Theorems 7.23, 7.25 and 7.27. Each of these theorems is aferwards illustrated by an example which is chosen in such a way that only this theorem is applicable, while none of the remaining two theorems can be used for this example.

Then, in Theorems 7.29, 7.31 and 7.33, we present the main existence results about unbounded solutions of the original problem (7.1), (7.2) provided ϕ^{-1} and f need not be Lipschitz continuous. The illustration by examples is done as in the previous case and shows that none of these theorems is included in any of remaining two ones.

In the rest of this section, we assume that (due to Definition 7.1) for each $n \in \mathbb{N}$, $[0, b_n) \subset [0, \infty)$ is a maximal interval such that a function u_n satisfies equation (7.1) for every $t \in (0, b_n)$.

Theorem 7.23. *Assume that (7.3)–(7.7), (7.13), (7.14), (7.35), (7.36) and (7.57) hold. Then there exist infinitely many unbounded solutions u_n of problem (7.1), (7.2) on $[0, b_n)$ with different starting values in $\big(L_0, \bar{B}\big)$, $n \in \mathbb{N}$.*

Proof. By Theorem 7.10, there exist infinitely many escape solutions u_n of problem (7.9), (7.2) with starting values in $\big(L_0, \bar{B}\big)$. Let us choose $n \in \mathbb{N}$. Then

$$\exists\, c_n \in (0, \infty)\colon u_n(t) \in (L_0, L),\ t \in [0, c_n), \quad u_n(c_n) = L, \quad u_n'(c_n) > 0.$$

Consider a restriction of u_n on $[0, c_n]$. Then there exists $b_n > c_n$ such that u_n can be extended as a solution of problem (7.1), (7.2) on $[0, b_n)$. If $b_n < \infty$, then, due to Lemma 7.18,

$$\lim_{t\to b_n^-} u_n(t) = \infty,$$

so u_n is unbounded. If $b_n = \infty$, then Theorem 7.19 yields

$$\lim_{t\to \infty} u_n(t) = \infty,$$

that is u_n is unbounded, as well. □

Example 7.24. We consider the IVP

$$\begin{aligned} (p(t)\sinh(u'(t)))' + p(t)f(\sinh(u(t))) = 0,\\ u(0) = u_0, \quad u'(0) = 0, \quad u_0 \in [L_0, L], \end{aligned} \tag{7.69}$$

where

$$f(x) = \begin{cases} x(x+\sinh 4)(\sinh 1 - x) & \text{for } x \in [-\sinh 4, \sinh 1],\\ \cos(x - \sinh 1) - 1 & \text{for } x > \sinh 1, \end{cases}$$

$$p(t) = \arctan t \qquad \text{or} \qquad p(t) = \tanh t = \frac{e^t - e^{-t}}{e^t + e^{-t}}, \qquad t \in [0, \infty).$$

We have equation (7.1) with

$$\phi(x) = \sinh x = \frac{e^x - e^{-x}}{2}, \quad x \in \mathbb{R}.$$

Here $L_0 = -4$, $L = 1$. Examples 6.9 and 6.10 shows that these functions p fulfil (7.7) and (7.14). Since

$$\lim_{t\to\infty} \arctan t = \frac{\pi}{2} < \infty, \qquad \lim_{t\to\infty} \tanh t = 1 < \infty,$$

(7.57) holds, as well.

The function f is continuous on $[\phi(L_0), \infty)$, Lipschitz continuous on $[\phi(L_0), \phi(L)]$, $f(\phi(L_0)) = f(0) = f(\phi(L)) = 0$, $xf(x) > 0$ for $x \in ((\phi(L_0), \phi(L)) \setminus \{0\})$ and f is nonpositive on $(\phi(L), \infty)$. Hence, conditions (7.5), (7.6) and (7.35) are valid. Moreover, $0 < L < -L_0$ and ϕ is odd and increasing function. Thus, we get, similarly as in Example 2.18, that $\tilde{F}(L) < \tilde{F}(L_0)$. Consequently, there exists $\bar{B} \in (L_0, 0)$ such that $\tilde{F}\left(\bar{B}\right) = \tilde{F}(L)$, which yields (7.13).

Example 5.20 gives that ϕ satisfies (7.3) and (7.4). By Example 5.23, $\phi^{-1}(x) = \arg\sinh x = \ln\left(x + \sqrt{x^2+1}\right)$ and (7.36) holds.

To summarize, we have fulfilled all assumptions of Theorem 7.23. Therefore, problem (7.69) has infinitely many unbounded solutions on $[0, b)$ with different starting values in $\left(L_0, \bar{B}\right)$ and with in general different b for different solutions. Since f has isolated zeros on $(\sinh 1, \infty)$, we cannot use Theorems 7.25 and 7.27 here.

In the same way as in the proof of Theorem 7.23, we can prove the following Theorem 7.25 or 7.27, if we use in the proof Theorem 7.20 or 7.22, respectively, instead of Theorem 7.19.

Theorem 7.25. *Let* (7.3)–(7.7)*,* (7.13)*,* (7.14)*,* (7.35)*,* (7.36) *and* (7.59) *hold. Then there exist infinitely many unbounded solutions u_n of problem* (7.1)*,* (7.2) *on $[0, b_n)$ with different starting values in $\big(L_0, \bar{B}\big)$, $n \in \mathbb{N}$.*

Example 7.26. Let us consider the IVP

$$\begin{gathered}\left(t^{\beta}\phi(u'(t))\right)' + t^{\beta}\phi(u(t))(\phi(u(t)) + \ln 4)(\ln 2 - \phi(u(t))) = 0,\\ u(0) = u_0, \quad u'(0) = 0, \quad u_0 \in [L_0, L],\end{gathered} \tag{7.70}$$

where

$$\phi(x) = \ln(|x| + 1)\operatorname{sgn} x, \quad x \in \mathbb{R}.$$

We have equation (7.1) with

$$f(x) = x(x + \ln 4)(\ln 2 - x), \quad x \in [-\ln 4, \infty), \qquad p(t) = t^{\beta}, \quad t \in [0, \infty).$$

Assume that $\beta > 0$. The function p satisfy (7.7) and (7.14) according to Examples 5.1 and 5.9.

We have $L_0 = -3$, $L = 1$. The function f is continuous on $[\phi(L_0), \infty)$, Lipschitz continuous on $[\phi(L_0), \phi(L)]$, $f(\phi(L_0)) = f(0) = f(\phi(L)) = 0$, $xf(x) > 0$ for $x \in ((\phi(L_0), \phi(L)) \setminus \{0\})$ and f is negative on $(\phi(L), \infty)$. Thus, we have fulfilled (7.5), (7.6), (7.35) and (7.59). In addition, $0 < L < -L_0$ and ϕ is odd and increasing function. Hence, we obtain, similarly as in Example 2.18, that $\tilde{F}(L) < \tilde{F}(L_0)$. Consequently, there exists $\bar{B} \in (L_0, 0)$ such that $\tilde{F}\left(\bar{B}\right) = \tilde{F}(L)$, which gives (7.13).

Example 5.20 yields that ϕ satisfies (7.3) and (7.4). Due to Example 5.23, $\phi^{-1}(x) = \left(e^{|x|} - 1\right)\operatorname{sgn} x$ and (7.36) is valid.

To sum up, if

$$\beta > 0,$$

then we can apply Theorem 7.25 on problem (7.70). Since $\lim_{t\to\infty} t^{\beta} = \infty$ and $f(x) < 0$ for $x > \ln 2$, we cannot use Theorem 7.23 as well as Theorem 7.27.

Theorem 7.27. *Assume that* (7.3)–(7.7)*,* (7.13)*,* (7.14)*,* (7.35)*,* (7.36)*,* (7.63)*,* (7.66) *and* (7.67) *hold. Then there exist infinitely many unbounded solutions u_n of problem* (7.1)*,* (7.2) *on $[0, b_n)$ with different starting values in $\big(L_0, \bar{B}\big)$, $n \in \mathbb{N}$.*

Example 7.28. We consider the IVP

$$\begin{gathered}\left(\sqrt{t}\, u'(t)\right)' + \sqrt{t}\, f(u(t)) = 0,\\ u(0) = u_0, \quad u'(0) = 0, \quad u_0 \in [L_0, L],\end{gathered}$$

where

$$f(x) = \begin{cases} x^3(x-\phi(L_0))(\phi(L)-x) & \text{for } x \in [\phi(L_0), \phi(L)], \\ 0 & \text{for } x > \phi(L), \end{cases} \qquad 0 < L < -L_0 .$$

Here

$$\phi(x) = x, \quad x \in \mathbb{R}, \qquad p(t) = \sqrt{t}\,, \quad t \in [0, \infty).$$

Due to Examples 5.1 and 5.9 $\left(\text{where now } \beta = \frac{1}{2}\right)$, the function p satisfies (7.7) and (7.14). The function f is continuous on $[\phi(L_0), \infty)$, Lipschitz continuous on $[\phi(L_0), \phi(L)]$, $f(\phi(L_0)) = f(0) = f(\phi(L)) = 0$, $xf(x) > 0$ for $x \in ((\phi(L_0), \phi(L)) \setminus \{0\})$ and $f \equiv 0$ on $(\phi(L), \infty)$. Therefore, conditions (7.5), (7.6), (7.35) and (7.63) hold. Since $f(\phi(x)) = f(x)$ and $L < -L_0$, we have $\tilde{F}(L) < \tilde{F}(L_0)$ and hence, there exists $\bar{B} \in (L_0, 0)$ such that $\tilde{F}\left(\bar{B}\right) = \tilde{F}(L)$, which yields (7.13).

Example 5.1 (where now $\alpha = 1$) shows that ϕ fulfils (7.3) and (7.4). Since $\phi(ab) = \phi(a)\phi(b)$ for each $a, b \in (0, \infty)$, (7.64) and consequently, (7.66) hold. Further, $\phi^{-1}(x) = x$ and (7.36) is valid. Since

$$\int_1^\infty \phi^{-1}\left(\frac{1}{p(s)}\right) \mathrm{d}s = \int_1^\infty \frac{1}{\sqrt{s}} \mathrm{d}s = \infty,$$

(7.67) holds.

To summarize, we have satisfied all assumptions of Theorem 7.27. Since $\lim_{t\to\infty} \sqrt{t} = \infty$ and $f(x) < 0$ for $x > \ln 2$, neither Theorem 7.23 nor Theorem 7.25 can be used.

Now, applying Theorem 7.16 instead of Theorem 7.10, we get as before the existence results about unbounded solutions in the case, where ϕ^{-1} and f do not have to be Lipschitz continuous.

Theorem 7.29. *Let (7.3)–(7.7), (7.13), (7.14) and (7.57) hold. Then there exist infinitely many unbounded solutions u_n of problem (7.1), (7.2) on $[0, b_n)$ with not necessary different starting values in $\left[L_0, \bar{B}\right)$, $n \in \mathbb{N}$.*

Example 7.30. Let us consider the IVP

$$\begin{gathered} \left(p(t)|u'(t)|^\alpha \operatorname{sgn} u'(t)\right)' + p(t) f\left(|u(t)|^\alpha \operatorname{sgn} u(t)\right) = 0, \\ u(0) = u_0, \quad u'(0) = 0, \quad u_0 \in [L_0, L], \end{gathered} \tag{7.71}$$

where

$$f(x) = \begin{cases} \sqrt{|x|}\ \operatorname{sgn} x(x-\phi(L_0))(\phi(L)-x) & \text{for } x \in [\phi(L_0), \phi(L)], \\ (\phi(L)-x)(\phi(2L)-x) & \text{for } x \in (\phi(L), \phi(2L)), \\ 0 & \text{for } x \geq \phi(2L), \end{cases} \quad 0 < L < -L_0,$$

$$p(t) = \arctan t \qquad \text{or} \qquad p(t) = \tanh t = \frac{e^t - e^{-t}}{e^t + e^{-t}}\,, \qquad t \in [0, \infty).$$

We have equation (7.1) with

$$\phi(x) = |x|^\alpha \operatorname{sgn} x, \quad x \in \mathbb{R}.$$

Assume that $\alpha > 1$. According to Example 7.24, these functions p satisfy (7.7), (7.14) and (7.57).

The function f is continuous on $[\phi(L_0), \infty)$, $f(\phi(L_0)) = f(0) = f(\phi(L)) = 0$, $xf(x) > 0$ for $x \in ((\phi(L_0), \phi(L)) \setminus \{0\})$ and f is nonpositive on $(\phi(L), \infty)$. Thus, we have fulfilled (7.5) and (7.6). Since $0 < L < -L_0$ and ϕ is odd and increasing on $\mathbb{R}$, we obtain, similarly as in Example 2.18, that $\tilde{F}(L) < \tilde{F}(L_0)$. Hence, there exists $\bar{B} \in (L_0, 0)$ such that $\tilde{F}\left(\bar{B}\right) = \tilde{F}(L)$, which gives (7.13). Example 5.20 yields that ϕ satisfies (7.3) and (7.4).

To sum up, provided that

$$\alpha > 1,$$

we have satisfied all assumptions of Theorem 7.29. Therefore, problem (7.71) has infinitely many unbounded solutions on $[0, b)$ with not necessary different starting values in $\left[L_0, \bar{B}\right)$ and with generally different b for different solutions. The form of f implies that neither Theorem 7.31 nor Theorem 7.33 can be applied.

Theorem 7.31. *Assume that (7.3)–(7.7), (7.13), (7.14) and (7.59) hold. Then there exist infinitely many unbounded solutions u_n of problem (7.1), (7.2) on $[0, b_n)$ with not necessary different starting values in $\left[L_0, \bar{B}\right)$, $n \in \mathbb{N}$.*

Example 7.32. Consider the IVP

$$\begin{gathered} \left(t^\beta u'^3(t)\right)' + t^\beta u(t)\left(u^3(t) + 8\right)\left(1 - u^3(t)\right) = 0, \\ u(0) = u_0, \quad u'(0) = 0, \quad u_0 \in [L_0, L]. \end{gathered} \tag{7.72}$$

We have equation (7.1) with

$$\begin{aligned} &\phi(x) = x^3, \quad x \in \mathbb{R}, \qquad f(x) = \sqrt[3]{x}\,(x+8)(1-x), \quad x \in [-8, \infty), \\ &p(t) = t^\beta, \quad t \in [0, \infty). \end{aligned}$$

Here $L_0 = -2$, $L = 1$, $\phi^{-1}(x) = \sqrt[3]{x}$. Assume that $\beta > 0$. By Example 7.26, the function p satisfies (7.7) and (7.14). Since f is negative on $(\phi(L), \infty)$, according to Example 5.20 $\left(\text{where now } \gamma = \frac{1}{3} \text{ and } k = 1\right)$, conditions (7.5), (7.6) and (7.59) hold. Further,

$$\begin{aligned} \tilde{F}(L_0) &= \int_0^{-2} s\left(s^3 + 8\right)\left(1 - s^3\right) \mathrm{d}s = \frac{144}{5}, \\ \tilde{F}(L) &= \int_0^{1} s\left(s^3 + 8\right)\left(1 - s^3\right) \mathrm{d}s = \frac{99}{40}. \end{aligned}$$

So, $\tilde{F}(L_0) > \tilde{F}(L)$, which yields (7.13). Example 5.20 gives that ϕ fulfils (7.3) and (7.4).

To summarize, if

$$\beta > 0,$$

then we can apply Theorem 7.31 on problem (7.72). Since $\lim_{t\to\infty} t^\beta = \infty$ and $f(x) < 0$ for $x > 1$, we cannot use Theorem 7.29 as well as Theorem 7.33.

Theorem 7.33. *Let (7.3)–(7.7), (7.13), (7.14), (7.63), (7.66) and (7.67) hold. Then there exist infinitely many unbounded solutions u_n of problem (7.1), (7.2) on $[0, b_n)$ with not necessary different starting values in $\left[L_0, \bar{B}\right)$, $n \in \mathbb{N}$.*

Example 7.34. Let us consider the IVP

$$\begin{gathered} \left(t^\beta |u'(t)|^\alpha \operatorname{sgn} u'(t)\right)' + t^\beta f\left(|u(t)|^\alpha \operatorname{sgn} u(t)\right) = 0, \\ u(0) = u_0, \quad u'(0) = 0, \quad u_0 \in [L_0, L], \end{gathered} \tag{7.73}$$

where

$$f(x) = \begin{cases} \sqrt[3]{x}\,(x - \phi(L_0))(\phi(L) - x) & \text{for } x \in [\phi(L_0), \phi(L)], \\ 0 & \text{for } x > \phi(L), \end{cases} \qquad 0 < L < -L_0\,.$$

Here

$$\phi(x) = |x|^\alpha \operatorname{sgn} x, \quad x \in \mathbb{R}, \qquad p(t) = t^\beta, \quad t \in [0, \infty).$$

Assume that $\alpha > 1$ and $\beta > 0$. According to Example 7.26, the function p fulfils (7.7) and (7.14). The function f is continuous on $[\phi(L_0), \infty)$, $f(\phi(L_0)) = f(0) = f(\phi(L)) = 0$, $xf(x) > 0$ for $x \in ((\phi(L_0), \phi(L)) \setminus \{0\})$ and $f \equiv 0$ on $(\phi(L), \infty)$. Hence, (7.5), (7.6) and (7.63) hold. Since $0 < L < -L_0$ and ϕ is odd and increasing function, we get, similarly as in Example 2.18, that $\tilde{F}(L) < \tilde{F}(L_0)$. Thus, there exists $\bar{B} \in (L_0, 0)$ such that $\tilde{F}\left(\bar{B}\right) = \tilde{F}(L)$, which yields (7.13).

Example 5.20 gives that ϕ satisfies (7.3) and (7.4). Since $\phi(ab) = \phi(a)\phi(b)$ for each $a, b \in (0, \infty)$, (7.64) and consequently, (7.66) hold. Further,

$$\begin{gathered} \phi^{-1}(x) = |x|^{\frac{1}{\alpha}} \operatorname{sgn} x, \ x \in \mathbb{R}, \qquad \phi^{-1}(x) = x^{\frac{1}{\alpha}}, \ x > 0, \\ \int_1^\infty \phi^{-1}\left(\frac{1}{p(s)}\right) \mathrm{d}s = \int_1^\infty \frac{1}{s^{\frac{\beta}{\alpha}}} \mathrm{d}s = \infty \quad \text{if } \frac{\beta}{\alpha} \le 1, \end{gathered}$$

that is (7.67) holds for $\frac{\beta}{\alpha} \le 1$.

To sum up, provided that

$$\alpha > 1, \ \beta > 0 \ \text{ and } \ \frac{\beta}{\alpha} \le 1,$$

we have satisfied all assumptions of Theorem 7.33 for problem (7.73). Since $\lim_{t\to\infty} t^\beta = \infty$ and $f(x) = 0$ for $x > \phi(L)$, neither Theorem 7.29 nor Theorem 7.31 is applicable.

It si clear that every unbounded solution of problem (7.1), (7.2) is an escape solution. According to the proofs of above theorems, we can formulate also the reverse assertion.

Corollary 7.35. *Assume all assumptions of Theorem* 7.23 *or* 7.25 *or* 7.27 *or* 7.29 *or* 7.31 *or* 7.33. *Then each escape solution of problem* (7.1), (7.2) *is unbounded.*

Conclusion

This thesis presented new contributions to the theory of singular nonlinear ordinary differential equations on an unbounded interval. We managed to generalize current results about existence and properties of three types of solutions of the singular equation

$$(p(t)u'(t))' + p(t)f(u(t)) = 0$$

to the equation with different coefficient functions p and q and to the equation with ϕ-Laplacian.

The first part of the thesis dealt with the initial value problem

$$(p(t)u'(t))' + q(t)f(u(t)) = 0, \qquad u(0) = u_0 \in [L_0, L], \quad u'(0) = 0 \tag{7.74}$$

and with an auxiliary initial value problem, where the nonlinearity f is replaced by a bounded nonlinearity $\tilde{f}$. We proved the existence (Theorem 2.15) and uniqueness of a solution of this auxiliary problem for every considered starting value as well as a continuous dependence of solutions on initial values (Theorem 2.17).

A significant attention was dedicated to the damped solutions of the original problem (7.74). Their existence was proved in Theorem 3.1 together with a starting interval giving only damped solutions. Theorem 3.7 gave that every oscillatory solution is the damped solution and has nonincreasing amplitudes. In addition, three types of conditions which guarantee that each damped solution is oscillatory were shown in Theorems 3.11, 3.12 and 3.14. The existence of oscillatory solutions was proved under these three different criteria in Theorems 3.15–3.17.

In connection with three obtained criteria for the oscillation of solutions, we reached three criteria (with additional conditions) leading to the existence of escape (Theorem 4.8) and homoclinic solutions of problem (7.74) (Theorem 4.9). The homoclinic solution is, furthermore, a bubble-type solution (Corollary 4.10), that is a solution of the boundary value problem in hydrorodynamics from our motivation.

The second part of the thesis investigated the initial value problem

$$(p(t)\phi(u'(t)))' + p(t)f(\phi(u(t))) = 0, \qquad u(0) = u_0 \in [L_0, L], \quad u'(0) = 0 \tag{7.75}$$

and an auxiliary initial value problem with a bounded nonlinearity $\tilde{f}$ instead of f. Theorem 5.19 guaranteed the existence of a solution of this auxiliary problem for each considered starting value. The uniqueness of the solution of this problem was proved in Theorem 5.21 provided that ϕ^{-1} is locally Lipschitz continuous on

$\mathbb{R}$. In the case that this condition falls, we proved the continuous dependence of solutions on positive initial values in Theorem 5.24 and for negative initial values in Theorem 5.26.

The existence of damped solutions of problem (7.75) was guaranteed by Theorem 6.1 for the same starting values as for problem (7.74). We proved the existence of escape solutions of auxiliary problem in the case, where both functions ϕ^{-1} and f are Lipschitz continuous (Theorem 7.10) and also in the more difficult opposite case (Theorem 7.16). Further, we derived three criteria guaranteeing that each escape solution of problem (7.75) is unbounded (Theorems 7.19, 7.20, 7.22). Finally, by combinations of these criteria with theorems guaranteeing the existence of escape solutions, we obtained the criteria of existence of unbounded solutions of problem (7.75) (Theorems 7.23, 7.25, 7.27, 7.29, 7.31 and 7.33).

For a better idea, we illustrated these main results on diverse examples. The existence of a homoclinic solution of problem (7.75) with ϕ-Laplacian still remains as an open problem. This is our aim for the future research as well as to find conditions leading to the existence of the unique homoclinic solution of problem (7.74) and (7.75).

List of publications

[1] M. Rohleder: *On the existence of oscillatory solutions of the second order nonlinear ODE*, Acta Univ. Palack. Olomuc. Fac. Rerum. Natur. Math. **51**, 2, (2012), 107–127.

[2] J. Burkotová, M. Rohleder, J. Stryja: *On the existence and properties of three types of solutions of singular IVPs*, Electron. J. Qual. Theory Differ. Equ. **29**, (2015), 1–25.

[3] J. Burkotová, I. Rachůnková, M. Rohleder, J. Stryja: *Existence and uniqueness of damped solutions of singular IVPs with ϕ-Laplacian*, Electron. J. Qual. Theory Differ. Equ. **121**, (2016), 1–28.

[4] M. Rohleder, J. Burkotová, L. López-Somoza, J. Stryja: *On unbounded solutions of singular IVPs with ϕ-Laplacian*, submitted.

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TU Košice	one month intership, February–March 2014 one month intership, May–June 2014

List of Publications

2017	M. Rohleder, J. Burkotová, L. López-Somoza, J. Stryja: *On unbounded solutions of singular IVPs with ϕ-Laplacian*, submitted. (corresponding author)
2016	J. Burkotová, I. Rachůnková, M. Rohleder, J. Stryja: *Existence and uniqueness of damped solutions of singular IVPs with ϕ-Laplacian*, Electron. J. Qual. Theory Differ. Equ. **121**, (2016), 1–28. (corresponding author)
2015	J. Burkotová, M. Rohleder, J. Stryja: *On the existence and properties of three types of solutions of singular IVPs*, Electron. J. Qual. Theory Differ. Equ. **29**, (2015), 1–25. (corresponding author)
2012	M. Rohleder: *On the existence of oscillatory solutions of the second order nonlinear ODE*, Acta Univ. Palack. Olomuc. Fac. Rerum. Natur. Math. **51**, 2, (2012), 107–127.

International Conferences

AIMS 2014	The 10th AIMS Conference on Dynamical Systems Differential Equations and Applications, Madrid, Spain, 7.–11. 7. 2014: Asymptotic properties of damped solutions of nonlinear singular ODE.
CDDEA 2014	Conference on Differential and Difference Equations and Applications, Jasná, Slovak Republic, 23.–27. 6. 2014: Asymptotic properties of damped solutions of nonlinear singular ODE.
EQUADIFF 2013	International conferences on differential equations, Prague, Czech Republic, 26.–30. 8. 2013: Existence of oscillatory solutions of nonlinear singular ODE.
MINI-SCHOOL 2013	Mini-School on Differential Equations, MÚ AVČR Brno, Czech Republic, 27.–31. 5. 2013: Existence of oscillatory solutions of nonlinear singular ODE.
CDDEA 2012	Conference on Differential and Difference Equations and Applications, Terchová, Slovak Republic, 25.–29. 6. 2012: On the existence of oscillatory solutions of the second order nonlinear ODE.

UNIVERZITA PALACKÉHO V OLOMOUCI
PŘÍRODOVĚDECKÁ FAKULTA

AUTOREFERÁT DISERTAČNÍ PRÁCE

Okrajové problémy s časovou singularitou

Katedra matematické analýzy a aplikací matematiky
Školitel: **prof. RNDr. Irena Rachůnková, DrSc.**
Autor: **Mgr. Martin Rohleder**
Studijní program: P1102 Matematika
Studijní obor: Matematická analýza
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Výsledky obsažené v disertační práci byly získány během doktorského studia oboru Matematická analýza na Katedře matematické analýzy a aplikací matematiky Přírodovědecké fakulty Univerzity Palackého v Olomouci.

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Autoreferát byl rozeslán dne ....

Obhajoba disertační práce se koná dne .... v .... hod. před oborovou radou doktorského studijního oboru Matematická analýza v učebně .... v budově Přf UPOL na adrese 17. listopadu 12 v Olomouci.

S disertační prací je možno se seznámit na studijním oddělení Přírodovědecké fakulty UPOL.

Contents

Notation

$\mathbb{N}$	set of all natural numbers
$\mathbb{R}$	set of all real numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
$C[a,b]$	Banach space of all continuous functions on $[a,b]$ equipped with the maximum norm $\|g\|_{C[a,b]} = \max\{|g(t)|\colon t \in [a,b]\}$
$C^k[a,b]$	Banach space of all functions k-times continuously differentiable on $[a,b]$ equipped with the norm $\|g\|_{C^k[a,b]} = \sum_{j=1}^{k} \|g^{(j)}\|_{C[a,b]}$
$\mathrm{Lip}(I)$	set of all Lipschitz continuous functions on the interval I
$\mathrm{Lip}_{\mathrm{loc}}(I)$	set of all locally Lipschitz continuous functions on the interval I

1 Abstract

This dissertation deals with the second order ordinary differential equations with possible time singularity at the origin, which are studied in general on the unbounded interval $[0, \infty)$. These investigated equations are the generalization of the singular differential equations, which are found in many sciencies, especially in hydrodynamics. This study investigates two types of generalizations of these model equations – equations without ϕ-Laplacian and with ϕ-Laplacian – together with the boundary conditions at 0 and ∞. These conditions as well as conditions for the data functions of our problem are chosen with respect to the original hydrodynamic model and with respect to a specific type of searched solution – so-called bubble-type solution. The study of boundary value problem is transformed into investigation of initial value problems.

The thesis investigates especially the existence and uniqueness of solutions of these initial value problems and their asymptotic properties. The essential part of the thesis is dedicated to the study of specific types of solutions depending on their supremum – damped, homoclinic and escape solutions. We study the existence of these individual types of solutions and their asymptotic properties. In the case of equations without ϕ-Laplacian, considerable attention is devoted to the damped solutions and conditions guaranteeing their oscillatory behaviour. In the case of equations with ϕ-Laplacian, we study especially the escape solutions and criteria for their unboundedness.

Key words: second order ordinary differential equations, time singularity, ϕ-Laplacian, asymptotic properties, existence and uniqueness of a solution, damped solution, homoclinic solution, escape solution, unbounded solution, oscillatory solution, unbounded interval

2 Abstrakt v českém jazyce

Disertační práce se zabývá problematikou obyčejných diferenciálních rovnic druhého řádu s možnou časovou singularitou v počátku, studovaných obecně na neomezeném intervalu $[0, \infty)$. Tyto vyšetřované rovnice jsou zobecněním singulárních diferenciálních rovnic, jež se vyskytují v mnoha oblastech vědy, obzvláště pak v hydrodynamice. V práci jsou vyšetřovány dva typy zobecnění těchto modelových rovnic, a to rovnice bez ϕ-Laplaciánu a s ϕ-Laplaciánem. Dané rovnice jsou vyšetřovány spolu s okrajovými podmínkami v 0 a ∞. Tyto podmínky, jakož i podmínky na datové funkce úlohy, jsou voleny s ohledem na původní hydrodynamický model a na specifický typ jeho hledaného řešení – tzv. bublinové řešení. Studium okrajové úlohy je převedeno na vyšetřování počátečních úloh.

Práce se zabývá zejména otázkou existence a jednoznačnosti řešení těchto počátečních úloh a jejich asymptotickými vlastnostmi. Podstatná část práce je pak věnována vyšetřováním specifických typů řešení v závislosti na jejich supremu – tlumená, homoklinická a úniková řešení. Studuje se existence těchto jednotlivých typů řešení a jejich asymptotické vlastnosti. U rovnic bez ϕ-Laplaciánu je značná pozornost věnována tlumeným řešením a podmínkám zaručujícím jejich oscilatoričnost. U rovnic s ϕ-Laplaciánem jsou pak studována zejména úniková řešení a kritéria zaručující jejich neohraničenost.

Klíčová slova: obyčejné diferenciální rovnice druhého řádu, časová singularita, ϕ-Laplacián, asymptotické vlastnosti, existence a jednoznačnost řešení, tlumené řešení, homoklinické řešení, únikové řešení, neohraničené řešení, oscilatorické řešení, neomezený interval

3 Introduction

In the thesis, we investigate the second order nonlinear ordinary differential equations (ODEs) without ϕ-Laplacian

$$(p(t)u'(t))' + q(t)f(u(t)) = 0 \tag{3.1}$$

and with ϕ-Laplacian

$$(p(t)\phi(u'(t)))' + p(t)f(\phi(u(t))) = 0. \tag{3.2}$$

The basic assumptions on functions p, q, f and ϕ are mentioned in Chapter 3. Both equations (3.1) and (3.2) are studied with the initial conditions

$$u(0) = u_0, \qquad u'(0) = 0. \tag{3.3}$$

These initial value problems (IVPs) are investigated generally on the positive half-line $[0, \infty)$.

Equations (3.1) and (3.2) can have a time singularity at the origin in the following sense. Let us consider the system of ODEs

$$x'(t) = f(t, x), \quad t \in I, \tag{3.4}$$

where $f\colon I \times \mathbb{R}^n \to \mathbb{R}^n$, $x \in \mathbb{R}^n$, $I \subset \mathbb{R}$. If the function f fulfils the Carathéodory conditions, then the system (3.4) is called *regular*, otherwise it is called *singular*. By the *time singularity at 0* we understand that

$$\int_0^{\varepsilon} |f(t, x)| \,\mathrm{d}t = \infty$$

for some $x \in \mathbb{R}$ and for each sufficiently small $\varepsilon > 0$. If we put $v = pu'$, then equation (3.1) can be expressed as a special case of system (3.4)

$$u'(t) = \frac{1}{p(t)} v(t), \qquad v'(t) = -q(t)f(u(t)).$$

Similarly, for $v = p\phi(u)$, we can assume equation (3.2) as the system

$$u'(t) = \frac{1}{p(t)} v(t), \qquad v'(t) = -p(t)f(\phi(u(t))).$$

One of our basic assumptions on the function p is that $p(0) = 0$. Hence, the integral $\int_0^1 \frac{1}{p(s)} \,\mathrm{d}s$ can be divergent, which yields the time singularity at 0. Consequently, our investigated equations (3.1) and (3.2) can have the time singularity.

This contrasts with the papers that study more general equations in the regular setting, mentioned in Chapter 3. In addition, the nonlinearity f in our case does not satisfy the sign condition $xf(x) > 0$ for all $x \neq 0$. Therefore, the globally monotonous behaviour of f, which is often required in the literature, is not fulfilled here.

4 Recent state summary

Regular equations

A considerable amount of literature exists on the qualitative analysis of equations (3.1), (3.2) and their generalizations in the regular setting, where $p(t) > 0$ for $t \in [0, \infty)$. The monograph [31] provides a general overview of asymptotic properties of solutions of nonautonomous ODEs. Research in the last decades has focused significantly on asymptotic analysis of the second order Emden–Fowler equation

$$u''(t) + q(t)|u(t)|^\gamma \operatorname{sgn} u(t) = 0, \qquad \gamma > 0,\ \gamma \neq 1,$$

which is a special case of equations (3.1) and (3.2). For the historic overview, see [71]. The oscilation and nonoscilation of the second order Emden–Fowler equation is researched in [36, 41, 42, 55]. The Emden-Fowler equation of arbitrary order is analysed in [71]. Further extensions of these results have been reached for more general equations, as can be seen in, e.g. [9, 17, 18, 26, 35, 43, 72, 73]. Nonlinearities in equations in the cited papers have similar globally monotonous behaviour, characterized by the sign condition $xf(x) > 0$ for $x \in \mathbb{R} \setminus \{0\}$. We would like to emphasize that, in contrast to these papers, the nonlinearity f in our equations (3.1) and (3.2) does not have globally monotonous behaviour.

The second order Emden–Fowler equation can be generalized into the following equation with p-Laplacian

$$(p(t)\Phi_\alpha(u'(t)))' + q(t)\Phi_\gamma(u(t)) = 0, \qquad \alpha > 0,\ \gamma > 0,$$

where $\Phi_\alpha(u) := |u|^\alpha \operatorname{sgn} u$. This equation is called sub-half-linear, half-linear or super-half linear if $\alpha > \gamma$, $\alpha = \gamma$ or $\alpha < \gamma$, respectively. The existence results of the sub-half-linear case are mentioned in [28, 37, 39], those of the half-linear case in [15, 29, 38] and those of the super-half-linear case in [16, 48].

Another approach to the asymptotic analysis is provided by the theory of regular variations [11, 47]. The asymptotic results for the related equations or systems with regularly varying functions are mentioned in [22, 27, 40, 49, 50, 67, 68]. Criteria for oscillation and nonoscillation of related two-dimensional linear and nonlinear systems can be found in [21, 46, 52].

Singular equations

The journal articles [56, 57, 60, 61, 62, 63, 64, 65, 66] are the most significant for the dissertation. They contain a detailed study of the singular nonlinear equation

$$(p(t)u'(t))' + p(t)f(u(t)) = 0. \tag{4.1}$$

Equation (4.1) is a special case of equation (3.1), where $p = q$ and also a special case of equation (3.2), where $\phi(x) = x$. All types of possible solutions of IVP (4.1), (3.3) with proofs of their existence and assymptotic properties are described in [60, 61, 64]. The existence of escape and homoclinic solutions is discussed in [62, 63].

The damped oscillatory solutions of problem (4.1), (3.3) are studied in [56, 57, 66], where the conditions for their existence, convergence to zero and for another asymptotic properties are given. For the results about damped nonoscillatory solutions, we refer to [69]. The asymptotic formulas and conditions that guarantee the existence of Kneser solutions are derived there. The variational methods for $p(t) = t^k$, $k \in \mathbb{N}$ or $k \in (1, \infty)$ are used in [10] or [12], respectively, where problem (4.1), (3.3) is transformed into a problem to find positive solutions on the half-line.

Many other problems for singular equations are described in [7, 8, 53, 58, 59] and [54], where the existence theory of two-point boundary value problems on finite and semi-infinite interval is introduced. For other close existence results, see also Chapters 13 and 14 in [53], where the existence results for second order ODEs on finite, semi-finite and infinite intervals are shown. Works [58, 59] are focused on regularization and sequential techniques and contain the existence theory for a variety of singular boundary value problems, especially those with ϕ-Laplacian.

5 Thesis objectives

The solutions for our IVP (3.1), (3.3) without ϕ-Laplacian as well as for problem (3.2), (3.3) with ϕ-Laplacian are divided according to their supremum into damped, homoclinic and escape solutions.

Equations without ϕ-Laplacian

The following objectives are concerned with the IVP (3.1), (3.3) without ϕ-Laplacian.

- The first aim of the thesis is to prove the existence and uniqueness of the damped solutions of problem (3.1), (3.3).
- Further, our effort is focussed on finding the conditions under which each damped solution is oscillatory.

- Our next goal is to prove the existence and uniqueness of escape solutions of the above-mentioned problem. Here we use the existence results of the oscillatory solutions.
- The principal objective concerning the IVP without ϕ-Laplacian is to prove the existence of homoclinic solution, which is important in applications.

Equations with ϕ-Laplacian

- Our aim is to generalize our results for damped and escape solutions of problem (3.1), (3.3) without ϕ-Laplacian to problem (3.2), (3.3) with ϕ-Laplacian.
- Moreover, we want to find conditions which guarantee that each escape solution of problem (3.2), (3.3) is unbounded and thus prove the existence of unbounded solutions.

Finally, we intend to illustrate all these main results on various examples.

Open problems and other aims of research

- The thesis contains the existence result for a homoclinic solution of problem (3.1), (3.3) without ϕ-Laplacian. The existence of homoclinic solutions for problem (3.2), (3.3) with ϕ-Laplacian stays as an open problem.
- The next open problem is finding conditions leading to the existence of the unique homoclinic solution of problem (3.1), (3.3) and problem (3.2), (3.3).
- Another interesting problem is to investigate the set of all solutions of equation (3.1) and (3.2) depending on initial values. We know that – for both of these equations – initial values in $\left[\bar{B}, L\right)$ give only damped solutions (see Theorem 7.10, Theorem 7.31). However, a structure of solutions for initial values in $\left(L_0, \bar{B}\right)$ is still an open problem.

6 Theoretical framework and methods applied

The thesis is motivated by the research of second order singular equations initiated by I. Rachůnková, J. Tomeček et al. in [56, 57, 60, 61, 62, 63, 64, 65, 66]. These papers investigate equation (4.1) and they are based on the following basic assumptions. The function f is (locally) Lipschitz continuous on the domain, where the solution is searched for. Further, f satisfies a certain sign condition, f has either two zeros 0, $L > 0$ [56, 57, 60, 61, 65, 66] or three zeros 0, $L_0 < 0$, $L > 0$

[62, 63, 64]. The function p is continuous on $[0,\infty)$, continously differentiable and increasing on $(0,\infty)$, $p(0)=0$ and $\lim_{t\to\infty}\frac{p'(t)}{p(t)}=0$. For more information about contents of above cited papers, see Chapter 3.

Our effort is to generalize current results about existence and properties of three types of solutions of equation (4.1) to the more general equations (3.1) and (3.2). In the dissertation, f has three zeros 0, $L_0<0$, $L>0$ and the basic assumptions are mentioned in Chapter 3.

Our results are based on the methods of differential equations and functional analysis. The fixed point theory plays an important role in the proofs of existence of solutions of our IVPs. We reduce an IVP to an operator equation and search for a fixed point of a corresponding operator. For the existence of solutions of auxiliary IVPs with and without ϕ-Laplacian, we use the Schauder fixed point theorem. Here it is necessary to prove the compactness of the operator. To prove this, we use the Arzelà–Ascoli theorem. The uniqueness of a solution is proved with the help of the Gronwall lemma. The existence and uniqueness of a solution can be proved also by the Banach fixed point theorem, which we show for the original IVP without ϕ-Laplacian with a bounded nonlinearity and some additional conditions.

Using the method of a priori estimates, we obtain estimates of solutions whose existence is not guaranteed, which is useful to prove the general existence principles. In the study of unbounded solutions of the IVP with ϕ-Laplacian, the difficulties arise in the case where the uniqueness of solution is not guaranteed. The lower and upper functions method for auxiliary mixed problem helps us to solve these difficulties in connection with the proof of existence of specific type of the solution of the IVP. The lower and upper functions satisfy the differential inequalities derived from our differential equation and fulfil the inequalities derived from the mixed boundary conditions. Our lower and upper functions are well-ordered, that is the upper function is greater or equal to the lower function and the solution is located between these functions.

7 Original results

The thesis contains new results in the theory of singular nonlinear ODEs of second order on the half-line $[0,\infty)$. They are based on the results published in multiple peer-reviewed journals [1, 2, 3, 4].

Differential equations without ϕ-Laplacian

We study the equation

$$(p(t)u'(t))' + q(t)f(u(t)) = 0 \tag{7.1}$$

with the initial conditions

$$u(0) = u_0, \quad u'(0) = 0, \quad u_0 \in [L_0, L] \tag{7.2}$$

and assume the following basic assumptions:

$$L_0 < 0 < L, \quad f(L_0) = f(0) = f(L) = 0, \tag{7.3}$$

$$f \in C[L_0, L], \quad xf(x) > 0 \text{ for } x \in (L_0, L) \setminus \{0\}, \tag{7.4}$$

$$p \in C[0, \infty), \quad p(0) = 0, \quad p(t) > 0 \text{ for } t \in (0, \infty), \tag{7.5}$$

$$q \in C[0, \infty), \quad q(t) > 0 \text{ for } t \in (0, \infty). \tag{7.6}$$

A model example of (7.1), (7.2) is the following:

$$p(t) = t^\alpha, \quad q(t) = t^\beta, \qquad t \in [0, \infty)\,, \ \alpha > 0, \ \beta \geq 0,$$
$$f(x) = x(x - L_0)(L - x), \quad x \in \mathbb{R}, \ L_0 < 0 < L.$$

Equation (7.1) can have various types of solutions which are defined as follows.

Definition 7.1. Let $c \in (0, \infty)$. A function $u \in C^1[0, c]$ with $pu' \in C^1[0, c]$ which satisfies equation (7.1) for every $t \in [0, c]$ is called a *solution of equation* (7.1) *on* $[0, c]$. If u is solution of equation (7.1) on $[0, c]$ for every $c > 0$, then u is called a *solution of equation* (7.1).

Definition 7.2. Let $c \in (0, \infty)$. A solution u of equation (7.1) on $[0, c]$ which satisfies the initial conditions (7.2) is called a *solution of problem* (7.1), (7.2) *on* $[0, c]$. If u is solution of problem (7.1), (7.2) on $[0, c]$ for every $c > 0$, then u is called a *solution of problem* (7.1), (7.2).

Definition 7.3. A solution u of problem (7.1), (7.2) is said to be *oscillatory* if $u \not\equiv 0$ in any neighborhood of ∞ and if u has a sequence of zeros tending to ∞. Otherwise, u is called *nonoscillatory*.

Definition 7.4. Let u be a solution of problem (7.1), (7.2) with $u_0 \in (L_0, L)$. Denote

$$u_{\sup} := \sup\{u(t) \colon t \in [0, \infty)\}.$$

If $u_{\sup} < L$, then u is called a *damped solution* of problem (7.1), (7.2).
If $u_{\sup} = L$, then u is called a *homoclinic solution* of problem (7.1), (7.2).

Definition 7.5. Assume that u is a solution of problem (7.1), (7.2) on $[0, c]$, where $c \in (0, \infty)$ and $u_0 \in (L_0, L)$. If u satisfies

$$u(c) = L, \quad u'(c) > 0,$$

then u is called an *escape solution* of problem (7.1), (7.2) *on* $[0, c]$.

Let us illustrate different types of solutions of problem (7.1), (7.2) with respect to their asymptotic behaviour in relation to Definitions 7.4 and 7.5 in Figure 7.1.

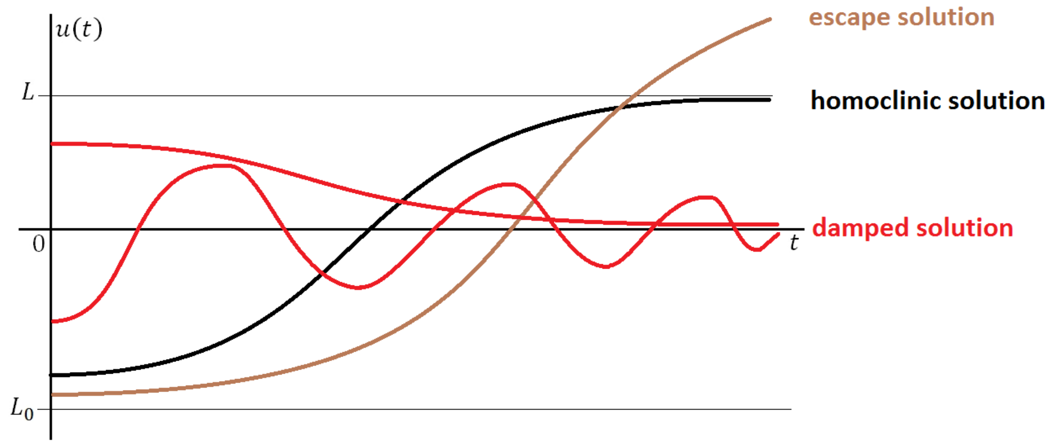

Figure 7.1: Types of solutions

Note that, according to $p(0) = 0$, the integral $\int_0^1 \frac{\mathrm{d}s}{p(s)}$ may be divergent, which means that equation (7.1) can have a singularity at $t = 0$.

In order to derive the existence of all three types of solutions of problem (7.1), (7.2), we introduce the auxiliary equation

$$(p(t)u'(t))' + q(t)\tilde{f}(u(t)) = 0, \tag{7.7}$$

where

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in [L_0, L], \\ 0 & \text{for } x < L_0, \quad x > L. \end{cases} \tag{7.8}$$

By (7.3), equations (7.1) and (7.7) have the constant solutions $u(t) \equiv L$, $u(t) \equiv 0$ and $u(t) \equiv L_0$. Moreover, the solution $u(t) \equiv 0$ is the only solution of (7.1) and (7.7) with $u_0 = 0$.

For many following results, we need, besides the basic assumptions (7.3)–(7.6), the next assumptions.

$$\exists\, \bar{B} \in (L_0, 0) : \tilde{F}\left(\bar{B}\right) = \tilde{F}(L), \quad \text{where } \tilde{F}(x) := \int_0^x \tilde{f}(z)\,\mathrm{d}z, \quad x \in \mathbb{R}, \tag{7.9}$$

$$pq \text{ is nondecreasing on } [0, \infty), \tag{7.10}$$

$$f \in \mathrm{Lip}_{\mathrm{loc}}\left([L_0, L] \setminus \{0\}\right), \tag{7.11}$$

$$\lim_{t \to 0^+} \frac{1}{p(t)} \int_0^t q(s)\,\mathrm{d}s = 0. \tag{7.12}$$

First, we provide the existence and uniqueness results, both for the auxiliary problem (7.7), (7.2) and for the original problem (7.1), (7.2). The following two theorems deal with the auxiliary problem.

Theorem 7.6 (**Existence of a solution of problem** (7.7)**,** (7.2)**).** *Assume that* (7.3)*–*(7.6) *and* (7.12) *hold. Then, for each $u_0 \in [L_0, L]$, problem* (7.7)*,* (7.2) *has a solution u.*
If moreover conditions (7.9)*–*(7.11) *hold, then the solution u satisfies:*

$$\text{if } u_0 \in \left[\bar{B}, L\right), \quad \text{then } u(t) > \bar{B}, \quad t \in (0, \infty), \tag{7.13}$$

$$\text{if } u_0 \in \left(L_0, \bar{B}\right), \quad \text{then } u(t) > u_0, \quad t \in (0, \infty). \tag{7.14}$$

Theorem 7.7 (**Uniqueness and continuous dependence on initial values).** *Assume that* (7.3)*–*(7.6) *and* (7.12) *hold and let*

$$f \in \mathrm{Lip}[L_0, L] \tag{7.15}$$

hold. Then, for each $u_0 \in [L_0, L]$, problem (7.7)*,* (7.2) *has a unique solution. Further, for each $b > 0$, there exists $K > 0$ such that*

$$\|u_1 - u_2\|_{C^1[0,b]} \le K|B_1 - B_2|.$$

Here u_i is a solution of problem (7.7)*,* (7.2) *with $u_0 = B_i$, $i = 1, 2$.*

Now, let us discuss the original problem.

Theorem 7.8 (**Existence and uniqueness of a solution of problem** (7.1)**,** (7.2)**).** *Assume that* (7.3)*–*(7.6)*,* (7.9)*,* (7.10)*,* (7.12)*,*

$$f \in \mathrm{Lip}_{\mathrm{loc}}[L_0, \infty),$$
$$\exists\, C_L \in (0, \infty)\colon\ -C_L \le f(x) \le 0 \quad \text{for } x \ge L$$

are satisfied. Then, for each $u_0 \in [L_0, L]$, problem (7.1)*,* (7.2) *has a unique solution u. This solution u satisfies* (7.13) *and* (7.14)*.*

In the next theorem, we show that condition (7.12) used in the previous results is necessary for the existence of a solution of problem (7.1), (7.2).

Theorem 7.9 (**Necessity of condition** (7.12)**).** *Assume that* (7.3)*–*(7.6) *hold and let u be a solution of problem* (7.1)*,* (7.2) *with $u_0 \in (L_0, 0) \cup (0, L)$. Then* (7.12) *is valid.*
Vice versa, assume that (7.12) *is satisfied and let u be a solution of equation* (7.1) *with $u(0) \in [L_0, L]$. Then $u'(0) = 0$ and u is the solution of problem* (7.1)*,* (7.2)*.*

Now, we specify an interval for starting values u_0, where the existence of damped solutions is guaranteed. Note that, by Definition 7.4 and the estimates

(7.13), (7.14), each damped solution u of the auxiliary problem (7.7), (7.2) satisfies $L_0 \le u(t) < L$ for $t \in [0, \infty)$. According to (7.8), the function f coincides with $\tilde{f}$ on $[L_0, L]$ and hence, all results for the damped solutions of problem (7.7), (7.2) are valid also for the original problem (7.1), (7.2). In particular,

Theorem 7.10 (**Existence and uniqueness of damped solutions of problem** (7.1)**,** (7.2)**).** *Assume that assumptions* (7.3)*–*(7.6) *and* (7.9)*–*(7.12) *are fulfilled. Then, for each* $u_0 \in \big(\bar{B}, L\big)$*, problem* (7.1)*,* (7.2) *has a solution* u*. The solution* u *is damped and satisfies* (7.13)*. If*

$$pq \text{ is increasing on } [0, \infty), \tag{7.16}$$

then this assertion holds also for $u_0 = \bar{B}$*.*
If moreover f *satisfies* (7.15)*, then the solution* u *is unique.*

Definition 7.11. A function u is called *eventually positive* (*eventually negative*), if there exists $t_0 > 0$ such that $u(t) > 0$ ($u(t) < 0$) for $t \in (t_0, \infty)$.

In order to obtain conditions under which every damped solution of problem (7.1), (7.2) is oscillatory, we distinguish two cases according to the convergence or divergence of the integral $\int_1^\infty \frac{1}{p(s)}\,\mathrm{d}s$.

CASE I: We assume that the function p fulfils

$$\int_1^\infty \frac{1}{p(s)}\,\mathrm{d}s < \infty. \tag{7.17}$$

CASE II: We assume that the function p fulfils

$$\int_1^\infty \frac{1}{p(s)}\,\mathrm{d}s = \infty. \tag{7.18}$$

First, we describe an asymptotic behaviour of nonoscillatory damped solutions of problem (7.1), (7.2) in Case I.

Theorem 7.12. *Assume that conditions* (7.3)*–*(7.6)*,* (7.9)*,* (7.10)*,* (7.17) *and*

$$\lim_{t\to\infty} \int_1^t \frac{1}{p(s)} \int_1^s q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s = \infty$$

are fulfilled. If u *is a damped nonoscillatory solution of problem* (7.1)*,* (7.2) *with* $u_0 \in (L_0, 0) \cup (0, L)$*, then*

$$\lim_{t\to\infty} u(t) = 0.$$

If moreover p *satisfies*

$$\liminf_{t\to\infty} p(t) \int_t^\infty \frac{1}{p(s)}\,\mathrm{d}s > 0,$$

then

$$\lim_{t\to\infty} u'(t) = 0.$$

In the investigation of oscillatory solutions of problem (7.1), (7.2), we use the following definition.

Definition 7.13. Let u be an oscillatory solution of problem (7.1), (7.2). Denote $\{a_n\}$ ($\{b_n\}$) sequences of local maxima (minima) of u. Assume that either $a_n < b_n < a_{n+1} < b_{n+1},\ n \in \mathbb{N}$ or $b_n < a_n < b_{n+1} < a_{n+1},\ n \in \mathbb{N}$. Then the numbers $u(a_n) - u(b_n)$, $n \in \mathbb{N}$ are called *amplitudes* of u.

Theorem 7.14. *Assume that conditions* (7.3)–(7.6) *and* (7.9)–(7.11) *hold. Let u be an oscillatory solution of problem* (7.1), (7.2) *with $u_0 \in (L_0, 0) \cup (0, L)$. Then u is a damped solution and has nonincreasing amplitudes.*
If moreover p and q satisfy (7.16), *then amplitudes of u are decreasing.*

Now, we provide criteria leading to oscillatory solutions of problem (7.1), (7.2). First, we prove the results for CASE I (i.e.(7.17)) and then for CASE II (i.e. (7.18)).

Theorem 7.15 (**Damped solution is oscillatory 1, CASE I**)**.** *Assume that* (7.3)–(7.6), (7.9), (7.10), (7.17),

$$\liminf_{x\to 0^+} \frac{f(x)}{x} > 0, \tag{7.19}$$

$$\liminf_{x\to 0^-} \frac{f(x)}{x} > 0, \tag{7.20}$$

$$\int_1^\infty \ell^2(s)q(s)\,\mathrm{d}s = \infty, \qquad \textit{where } \ell(t) = \int_t^\infty \frac{1}{p(s)}\,\mathrm{d}s \tag{7.21}$$

are fulfilled. Let u be a damped solution of problem (7.1), (7.2) *with $u_0 \in (L_0, 0) \cup (0, L)$. Then u is oscillatory.*

If we replace assumptions (7.5) and (7.21) by assumptions (7.22)–(7.24), we get a modification of Theorem 7.15.

Theorem 7.16 (**Damped solution is oscillatory 2, CASE I**)**.** *Assume that* (7.3), (7.4), (7.6), (7.9), (7.10), (7.17), (7.19), (7.20),

$$p \in C[0,\infty) \cap C^2(0,\infty), \quad p(0) = 0, \tag{7.22}$$

$$p'(t) > 0 \ \ \textit{for } t \in (0,\infty), \qquad \lim_{t\to\infty} \frac{p'(t)}{p(t)} = 0, \quad \limsup_{t\to\infty} \left|\frac{p''(t)}{p'(t)}\right| < \infty, \tag{7.23}$$

$$\liminf_{t\to\infty} \frac{q(t)}{p(t)} > 0 \tag{7.24}$$

hold. Let u be a damped solution of problem (7.1), (7.2) *with* $u_0 \in (L_0, 0) \cup (0, L)$. *Then u is oscillatory.*

In the next theorem, we provide a criterion for oscillatory solutions in CASE II (i.e. (7.18)).

Theorem 7.17 (**Damped solution is oscillatory 3, CASE II**). *Assume that* (7.3)–(7.6), (7.9), (7.10), (7.18) *and*

$$\int_1^\infty q(s)\,\mathrm{d}s = \infty \tag{7.25}$$

are fulfilled. Let u be a damped solution of problem (7.1), (7.2) *with* $u_0 \in (L_0, 0) \cup (0, L)$. *Then u is oscillatory.*

If we combine assumptions from Theorem 7.10 and Theorem 7.14 with assumptions of Theorem 7.15 or Theorem 7.16 or Theorem 7.17, we get the main results about existence of oscillatory solutions of problem (7.1), (7.2).

Theorem 7.18 (**Existence of oscillatory solutions 1, CASE I**). *Assume that* (7.3)–(7.6), (7.9)–(7.12), (7.17) *and* (7.19)–(7.21) *are fulfilled. Then, for each* $u_0 \in \big(\bar{B}, 0\big) \cup (0, L)$, *problem* (7.1), (7.2) *has a solution u. This solution u is damped, oscillatory and has nonincreasing amplitudes.*

Theorem 7.19 (**Existence of oscillatory solutions 2, CASE I**). *Assume that* (7.3)–(7.6), (7.9)–(7.12), (7.17), (7.19), (7.20) *and* (7.22)–(7.24) *are fulfilled. Then, for each* $u_0 \in \big(\bar{B}, 0\big) \cup (0, L)$, *problem* (7.1), (7.2) *has a solution u. This solution u is damped, oscillatory and has nonincreasing amplitudes.*

Theorem 7.20 (**Existence of oscillatory solutions 3, CASE II**). *Assume that* (7.3)–(7.6), (7.9)–(7.12), (7.18) *and* (7.25) *are fulfilled. Then, for each* $u_0 \in \big(\bar{B}, 0\big) \cup (0, L)$, *problem* (7.1), (7.2) *has a solution u. This solution u is damped, oscillatory and has nonincreasing amplitudes.*

Now, we study escape and homoclinic solutions. According to Theorem 7.10, provided (7.16), a solution of problem (7.1), (7.2) is damped if $u_0 \in \big[\bar{B}, L\big)$. Hence, if u is escape or homoclinic solution of problem (7.1), (7.2), then $u_0 \in \big(L_0, \bar{B}\big)$. Therefore, we can restrict our consideration about escape and homoclinic solutions on $u_0 \in (L_0, 0)$.

For the following existence theorems, we need these assumptions.

$$(pq)' > 0 \text{ on } (0, \infty), \tag{7.26}$$

$$\lim_{t\to\infty} \frac{(p(t)q(t))'}{q^2(t)} = 0, \tag{7.27}$$

$$\liminf_{t\to\infty} \frac{p(t)}{q(t)} > 0, \tag{7.28}$$

$$\liminf_{t\to\infty} q(t) > 0 \tag{7.29}$$

The next existence theorems have the following common assumptions

$$(7.3)\text{–}(7.6),\ (7.9),\ (7.12),\ (7.15) \text{ and } (7.26)\text{–}(7.29). \tag{7.30}$$

We provide the existence results for two cases which are characterized by conditions (7.17) and (7.18). Therefore, we use in addition either assumptions

$$(7.17) \text{ and } (7.19)\text{–}(7.21) \tag{7.31}$$

or assumptions

$$(7.17),\ (7.19),\ (7.20) \text{ and } (7.22)\text{–}(7.24) \tag{7.32}$$

or assumption (7.18).

Theorem 7.21 (**Existence of escape solutions of problem** (7.7)**,** (7.2)**).** *Assume that* (7.30) *and either* (7.31) *or* (7.32) *or* (7.18) *hold. Then there exist a sequence* $\{u_n\}_{n=1}^{\infty}$ *of escape solutions of problem* (7.7), (7.2) *with* $u_0 = B_n \in \big(L_0, \bar{B}\big)$.

Theorem 7.22 (**Existence of a homoclinic solution of problem** (7.7)**,** (7.2)**).** *Assume that* (7.30) *and either* (7.31) *or* (7.32) *or* (7.18) *hold. Then there exists a homoclinic solution of problem* (7.7), (7.2).

Finally, we extend the existence results from Theorem 7.21 and Theorem 7.22 to the original problem (7.1), (7.2).

Theorem 7.23 (**Existence of escape solutions of problem** (7.1)**,** (7.2)**).** *Assume that* (7.30) *and either* (7.31) *or* (7.32) *or* (7.18) *hold. Then, for each* $n \in \mathbb{N}$, *there exist constant* $c_n \in (0, \infty)$ *and function* u_n *such that* u_n *is an escape solution of problem* (7.1), (7.2) *on* $[0, c_n]$ *with* $u_0 = B_n \in \big(L_0, \bar{B}\big)$.

Theorem 7.24 (**Existence of a homoclinic solution of problem** (7.1)**,** (7.2)**).** *Assume that* (7.30) *and either* (7.31) *or* (7.32) *or* (7.18) *hold. Then there exists a homoclinic solution of problem* (7.1), (7.2).

Differential equations with ϕ-Laplacian

Now, we investigate the equation

$$(p(t)\phi(u'(t)))' + p(t)f(\phi(u(t))) = 0 \tag{7.33}$$

with the initial conditions

$$u(0) = u_0, \quad u'(0) = 0, \quad u_0 \in [L_0, L] \tag{7.34}$$

and assume these basic assumptions:

$$\phi \in C^1(\mathbb{R}), \quad \phi'(x) > 0 \text{ for } x \in (\mathbb{R} \setminus \{0\}), \tag{7.35}$$

$$\phi(\mathbb{R}) = \mathbb{R}, \quad \phi(0) = 0, \tag{7.36}$$

$$L_0 < 0 < L, \quad f(\phi(L_0)) = f(0) = f(\phi(L)) = 0, \tag{7.37}$$

$$f \in C[\phi(L_0), \phi(L)], \quad xf(x) > 0 \text{ for } x \in ((\phi(L_0), \phi(L)) \setminus \{0\}), \tag{7.38}$$

$$p \in C[0, \infty) \cap C^1(0, \infty), \quad p'(t) > 0 \text{ for } t \in (0, \infty), \quad p(0) = 0. \tag{7.39}$$

A model example of (7.33), (7.34) is a problem with the α-Laplacian:

$$\begin{aligned}
&\phi(x) = |x|^\alpha \operatorname{sgn} x, \quad x \in \mathbb{R}, \ \alpha \geq 1, \\
&p(t) = t^\beta, \quad t \in [0, \infty), \ \beta > 0, \\
&f(x) = x\,(x - \phi(L_0))\,(\phi(L) - x), \quad x \in \mathbb{R}, \ L_0 < 0 < L.
\end{aligned}$$

First, we study bounded solutions defined on $[0, \infty)$. Therefore, we use the next definitions.

Definition 7.25. A function $u \in C^1[0, \infty)$ with $\phi(u') \in C^1(0, \infty)$ which satisfies equation (7.33) for every $t \in (0, \infty)$ is called a *solution of equation* (7.33). If moreover u satisfies the initial conditions (7.34), then u is called a *solution of problem* (7.33), (7.34).

Definition 7.26. Consider a solution u of problem (7.33), (7.34) with $u_0 \in [L_0, L)$ and denote

$$u_{\sup} := \sup\{u(t) \colon t \in [0, \infty)\}.$$

If $u_{\sup} < L$, then u is called a *damped solution* of problem (7.33), (7.34).
If $u_{\sup} = L$, then u is called a *homoclinic solution* of problem (7.33), (7.34).
The homoclinic solution is called a *regular homoclinic solution*, if $u(t) < L$ for $t \in [0, \infty)$ and a *singular homoclinic solution*, if there exists $t_0 > 0$ such that $u(t_0) = L$.

Equation (7.33) has the constant solutions $u(t) \equiv L$, $u(t) \equiv 0$ and $u(t) \equiv L_0$. Moreover, the solution $u(t) \equiv 0$ is the only solution of problem (7.33), (7.34) with $u_0 = 0$.

We introduce the auxiliary equation

$$(p(t)\phi(u'(t)))' + p(t)\tilde{f}(\phi(u(t))) = 0, \tag{7.40}$$

where

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in [\phi(L_0), \phi(L)], \\ 0 & \text{for } x < \phi(L_0), \ x > \phi(L). \end{cases}$$

For many following results, we need, besides the basic assumptions (7.35)–(7.39), the next assumptions.

$$\exists\, \bar{B} \in (L_0, 0)\colon \tilde{F}\left(\bar{B}\right) = \tilde{F}(L), \qquad \text{where } \tilde{F}(x) := \int_0^x \tilde{f}(\phi(s))\,\mathrm{d}s, \quad x \in \mathbb{R} \tag{7.41}$$

and

$$\limsup_{t\to\infty} \frac{p'(t)}{p(t)} < \infty. \tag{7.42}$$

Theorem 7.27 (**Existence of solutions of problem** (7.40)**,** (7.34)**).** *Assume that* (7.35)*–*(7.39) *hold. Then, for each* $u_0 \in [L_0, L]$*, there exists a solution* u *of problem* (7.40)*,* (7.34)*.*

Now, we examine the uniqueness of solutions of the auxiliary problem (7.40), (7.34). Assumption (7.35) implies that $\phi \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$. This need not be true for ϕ^{-1}. The special case when both ϕ and ϕ^{-1} are locally Lipschitz continuous is discussed in the next theorem.

Theorem 7.28 (**Uniqueness and continuous dependence on initial values I).** *Assume that* (7.35)*–*(7.39)*,*

$$f \in \mathrm{Lip}\left[\phi(L_0), \phi(L)\right], \tag{7.43}$$

$$\phi^{-1} \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}) \tag{7.44}$$

are satisfied. Let u_i *be a solution of problem* (7.40)*,* (7.34) *with* $u_0 = B_i \in [L_0, L]$*,* $i = 1, 2$*. Then, for each* $\beta > 0$*, there exists* $K > 0$ *such that*

$$\|u_1 - u_2\|_{C^1[0,\beta]} \le K|B_1 - B_2|.$$

Furthermore, any solution of problem (7.40)*,* (7.34) *with* $u_0 \in [L_0, L]$ *is unique on* $[0, \infty)$*.*

In the next two theorems, we show the assumptions under which solutions of problem (7.40), (7.34) continuously depend on their initial values in the case that ϕ^{-1} is not locally Lipschitz continuous.

Theorem 7.29 (**Continuous dependence on initial values II).** *Assume that* (7.35)*–*(7.39)*,* (7.41)*–*(7.43) *and*

$$\limsup_{x\to 0^-} \left(-x\left(\phi^{-1}\right)'(x)\right) < \infty, \quad \phi' \text{ is nonincreasing on } (-\infty, 0) \tag{7.45}$$

are fulfilled. Let B_1*,* B_2 *satisfy*

$$B_1 \in (2\varepsilon, L - 2\varepsilon), \quad |B_1 - B_2| < \varepsilon$$

for some $\varepsilon > 0$. *Let* u_i *be a solution of problem* (7.40), (7.34) *with* $u_0 = B_i$, $i = 1, 2$. *Then, for each* $\beta > 0$, *where*

$$u_i' < 0 \text{ on } (0, \beta],\ i = 1, 2,$$

there exists $K \in (0, \infty)$ *such that*

$$\|u_1 - u_2\|_{C^1[0,\beta]} \le K|B_1 - B_2|.$$

Theorem 7.30 (Continuous dependence on initial values III). *Assume that* (7.35)–(7.39), (7.41)–(7.43) *and*

$$\limsup_{x\to 0^+} \left(x \left(\phi^{-1}\right)'(x) \right) < \infty, \quad \phi' \text{ is nondecreasing on } (0, \infty) \tag{7.46}$$

hold. Let B_1, B_2 *satisfy*

$$B_1 \in (L_0 + 2\varepsilon, -2\varepsilon), \quad |B_1 - B_2| < \varepsilon$$

for some $\varepsilon > 0$. *Let* u_i *be a solution of problem* (7.40), (7.34) *with* $u_0 = B_i$, $i = 1, 2$. *Then, for each* $\beta > 0$, *where*

$$u_i' > 0 \text{ on } (0, \beta],\ i = 1, 2,$$

there exists $K \in (0, \infty)$ *such that*

$$\|u_1 - u_2\|_{C^1[0,\beta]} \le K|B_1 - B_2|.$$

The existence of damped solutions of the original problem (7.33), (7.34) is proved in Theorem 7.31. Moreover, this theorem yields the uniqueness of damped solutions provided that ϕ^{-1} is Lipschitz continuous, while Theorem 7.32 gives the uniqueness of damped solutions without the Lipschitz continuity of ϕ^{-1}.

Theorem 7.31 (Existence and uniqueness of damped solutions of problem (7.33), (7.34)). *Assume that* (7.35)–(7.39), (7.41) *and* (7.42) *hold. Then, for each* $u_0 \in \left[\bar{B}, L\right)$, *problem* (7.33), (7.34) *has a solution. Every solution of problem* (7.33), (7.34) *with* $u_0 \in \left[\bar{B}, L\right)$ *is damped.*
If moreover (7.43) *and* (7.44) *hold, then the solution is unique.*

By Theorem 7.31, we can get homoclinic solutions only if $u_0 \in \left[L_0, \bar{B}\right)$.

Theorem 7.32 (Uniqueness of damped solutions). *Assume that* (7.35)–(7.39), (7.41)–(7.43), (7.45) *and* (7.46) *are fulfilled. Let* u *be a damped solution of problem* (7.33), (7.34) *with* $u_0 \in (L_0, L)$. *Then* u *is a unique solution of this problem.*

Further, we discuss homoclinic solutions.

Theorem 7.33 (**Nonexistence of singular homoclinic solutions).** *Assume that* (7.35)–(7.39)*,* (7.43) *and* (7.44) *hold. Then each homoclinic solution of problem* (7.33)*,* (7.34) *with* $u_0 \in \left(L_0, \bar{B}\right)$ *is regular.*

Theorem 7.33 discusses the case, where $\phi^{-1} \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$. Now, we study the case, where condition (7.44) falls, that is $\phi^{-1} \notin \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$. Then both regular and singular homoclinic solutions may exist and we are able to prove the uniqueness just for regular ones.

Theorem 7.34 (**Uniqueness of regular homoclinic solutions).** *Assume that* (7.35)–(7.39)*,* (7.41)–(7.43) *and* (7.46) *are satisfied. Let u be a regular homoclinic solution of problem* (7.33)*,* (7.34) *with* $u_0 \in \left(L_0, \bar{B}\right)$*. Then u is a unique solution of this problem.*

Now, we study escape – especially unbounded – solutions. For their investigation, the need the assumption

$$f \in C[\phi(L_0), \infty), \quad f(x) \leq 0 \text{ for } x > \phi(L) \tag{7.47}$$

and consider the following definition of the solution.

Definition 7.35. Let $[0, b) \subset [0, \infty)$ be a maximal interval such that a function $u \in C^1[0, b)$ with $\phi(u') \in C^1(0, b)$ satisfies equation (7.33) for every $t \in (0, b)$ and let u satisfy the initial conditions (7.34). Then u is called a *solution of problem* (7.33), (7.34) *on* $[0, b)$. If u is solution of problem (7.33), (7.34) on $[0, \infty)$, then u is called a *solution of problem* (7.33), (7.34).

Since $\tilde{f}$ is bounded on $\mathbb{R}$, the maximal interval $[0, b)$ for each solution of problem (7.40), (7.34) is $[0, \infty)$.

Definition 7.36. Let u be a solution of problem (7.33), (7.34) on $[0, b)$, where $b \in (0, \infty]$. If there exists $c \in (0, b)$ such that

$$u(c) = L, \quad u'(c) > 0,$$

then u is called an *escape solution* of problem (7.33), (7.34) *on* $[0, b)$.

First, we discuss the existence of escape solutions of problem (7.40), (7.34) provided the Lipschitz continuity of ϕ^{-1} and f.

Theorem 7.37 (**Existence of escape solutions of problem** (7.40)**,** (7.34) **I).** *Let* (7.35)–(7.39)*,* (7.41)*,* (7.43)*,* (7.44)*,* (7.47) *and*

$$\lim_{t \to \infty} \frac{p'(t)}{p(t)} = 0. \tag{7.48}$$

hold. Then there exist infinitely many escape solutions of problem (7.40)*,* (7.34) *with different starting values in* $\left(L_0, \bar{B}\right)$ *converging to* L_0*.*

Now, we investigate the existence of escape solutions in the case, where ϕ^{-1} and f do not have to be Lipschitz continuous. In order to prove this existence result, we consider the lower and upper functions method for an auxiliary mixed problem on $[0, T]$. In particular, we use this method to find solutions of (7.40) which satisfy

$$u'(0) = 0, \quad u(T) = C, \quad C \in [L_0, L]. \tag{7.49}$$

Definition 7.38. A function $u \in \mathcal{C}^1[0, T]$ with $\phi(u') \in \mathcal{C}^1(0, T]$ is a *solution of problem* (7.40), (7.49) if u fulfils (7.40) for $t \in (0, T]$ and satisfies (7.49).

Definition 7.39. A function $\sigma_1 \in \mathcal{C}[0, T]$ is a *lower function* of problem (7.40), (7.49) if there exists a finite (possibly empty) set $\Sigma_1 \subset (0, T)$ such that $\sigma_1 \in \mathcal{C}^2((0, T] \setminus \Sigma_1)$ and

$$\begin{gathered}(p(t)\,\phi(\sigma_1'(t)))' + p(t)\,\tilde{f}(\phi(\sigma_1(t))) \geq 0, \quad t \in (0, T] \setminus \Sigma_1,\\ -\infty < \sigma_1'(\tau^-) < \sigma_1'(\tau^+) < \infty, \quad \tau \in \Sigma_1,\\ \sigma_1'(0^+) \geq 0,\ \sigma_1(T) \leq C.\end{gathered}$$

Analogously,

Definition 7.40. A function $\sigma_2 \in \mathcal{C}[0, T]$ is an *upper function* of problem (7.40), (7.49) if there exists a finite (possibly empty) set $\Sigma_2 \subset (0, T)$ such that $\sigma_2 \in \mathcal{C}^2((0, T] \setminus \Sigma_2)$ and

$$\begin{gathered}(p(t)\,\phi(\sigma_2'(t)))' + p(t)\,\tilde{f}(\phi(\sigma_2(t))) \leq 0, \quad t \in (0, T] \setminus \Sigma_2,\\ -\infty < \sigma_2'(\tau^+) < \sigma_2'(\tau^-) < \infty, \quad \tau \in \Sigma_2,\\ \sigma_2'(0^+) \leq 0,\ \sigma_2(T) \geq C.\end{gathered}$$

Theorem 7.41 (Lower and upper functions method). *Let (7.35)–(7.39) and (7.47) hold and let σ_1 and σ_2 be lower and upper functions of problem (7.40), (7.49) such that*

$$\sigma_1(t) \leq \sigma_2(t), \quad t \in [0, T].$$

Then problem (7.40), (7.49) has a solution u such that

$$\sigma_1(t) \leq u(t) \leq \sigma_2(t), \quad t \in [0, T].$$

Theorem 7.42 (Existence of escape solutions of problem (7.40), (7.34) II). *Let (7.35)–(7.39), (7.41), (7.47) and (7.48) hold. Then there exist infinitely many escape solutions of problem (7.40), (7.34) with not necessary different starting values in $\left[L_0, \bar{B}\right)$.*

In the case, where ϕ^{-1} does not have to be Lipschitz continuous, the uniqueness of damped and regular homoclinic solutions is guaranteed by Theorems 7.32 and 7.34, respectively. Similarly, we can obtain also the uniqueness of escape solutions.

Theorem 7.43 (**Uniqueness of escape solutions).** *Assume that* (7.35)–(7.39)*,* (7.42)*,* (7.41)*,* (7.43)*,* (7.46) *and* (7.47) *hold. Let u be an escape solution of problem* (7.40)*,* (7.34) *with $u_0 \in \big(L_0, \bar{B}\big)$. Then u is a unique solution of this problem.*

Now, we discuss the original problem (7.33), (7.34) and provide conditions which guarantee that an escape solution of (7.33), (7.34) is unbounded.

Note that solutions of the original problem (7.33), (7.34) and solutions of the auxiliary problem (7.40), (7.34) are related in the following way, when (7.35)–(7.39), (7.41), (7.47) and (7.48) are assumed. Each solution of (7.40), (7.34) which is not an escape solution, is a bounded solution of the original problem (7.33), (7.34) in $[0,\infty)$. If u is an escape solution of the auxiliary problem (7.40), (7.34), i.e.

$$\exists\, c \in (0,\infty)\colon u(t) \in [L_0, L),\ t \in [0,c), \quad u(c) = L, \quad u'(c) > 0, \tag{7.50}$$

then u fulfils at once the auxiliary equation (7.40) and the original equation (7.33) on $[0,c]$. The restriction of u on $[0,c]$ can be extended as an escape solution of problem (7.33), (7.34) on some maximal interval $[0,b)$. Therefore, we search unbounded solutions of (7.33), (7.34) in the class of escape solutions of (7.33), (7.34) on $[0,b)$.

Since in general, an escape solution u of (7.33), (7.34) on $[0,b)$ need not to be unbounded, we derive criteria for u to tend to infinity.

Since all escape solution of (7.40), (7.34) on $[0,b)$ that cannot be extended on the halfline $[0,\infty)$ are naturally unbounded, we continue our investigation about unboundedness of escape solutions defined on $[0,\infty)$.

Theorem 7.44. *Assume that* (7.35)–(7.39) *and* (7.47) *hold and let*

$$\lim_{t\to\infty} p(t) < \infty. \tag{7.51}$$

Let u be an escape solution of problem (7.33)*,* (7.34)*. Then*

$$\lim_{t\to\infty} u(t) = \infty. \tag{7.52}$$

Theorem 7.45. *Assume that* (7.35)–(7.39)*,* (7.47)*,* (7.48) *and*

$$f(x) < 0 \quad \textit{for } x > \phi(L) \tag{7.53}$$

hold. Let u be an escape solution of problem (7.33)*,* (7.34)*. Then* (7.52) *holds.*

For $f \equiv 0$ on $(\phi(L),\infty)$, we are able to find necessary and sufficient condition for the unboundedness of escape solutions of problem (7.33), (7.34).

Theorem 7.46. *Assume that* (7.35)–(7.39), (7.47),

$$f(x) \equiv 0 \quad \text{for } x > \phi(L), \tag{7.54}$$

$$\phi(ab) = \phi(a)\phi(b), \quad a, b \in (0, \infty) \tag{7.55}$$

are satisfied. Let u *be an escape solution of problem* (7.33), (7.34). *Then*

$$\lim_{t\to\infty} u(t) = \infty \quad \Longleftrightarrow \quad \int_1^\infty \phi^{-1}\left(\frac{1}{p(s)}\right) \mathrm{d}s = \infty.$$

If we replace condition (7.55) *by*

$$\phi(ab) \le \phi(a)\phi(b), \quad a, b \in (0, \infty), \tag{7.56}$$

then (7.52) *holds if*

$$\int_1^\infty \phi^{-1}\left(\frac{1}{p(s)}\right) \mathrm{d}s = \infty. \tag{7.57}$$

Now, we present the existence results about unbounded solutions of the original problem (7.33), (7.34) in the case, where ϕ^{-1} and f are Lipschitz continuous, see Theorems 7.47, 7.48 and 7.49. Then, in Theorems 7.50, 7.51 and 7.52, we present the main existence results about unbounded solutions of the original problem (7.33), (7.34) provided ϕ^{-1} and f need not be Lipschitz continuous. According to Definition 7.35, we assume that for each $n \in \mathbb{N}$, $[0, b_n) \subset [0, \infty)$ is a maximal interval such that a function u_n satisfies equation (7.33) for every $t \in (0, b_n)$.

Theorem 7.47. *Assume that* (7.35)–(7.39), (7.41), (7.43), (7.44), (7.47), (7.48) *and* (7.51) *hold. Then there exist infinitely many unbounded solutions* u_n *of problem* (7.33), (7.34) *on* $[0, b_n)$ *with different starting values in* $\big(L_0, \bar{B}\big)$, $n \in \mathbb{N}$.

Theorem 7.48. *Let* (7.35)–(7.39), (7.41), (7.43), (7.44), (7.47), (7.48) *and* (7.53) *hold. Then there exist infinitely many unbounded solutions* u_n *of problem* (7.33), (7.34) *on* $[0, b_n)$ *with different starting values in* $\big(L_0, \bar{B}\big)$, $n \in \mathbb{N}$.

Theorem 7.49. *Assume that* (7.35)–(7.39), (7.41), (7.43), (7.44), (7.47), (7.48), (7.54), (7.56) *and* (7.57) *hold. Then there exist infinitely many unbounded solutions* u_n *of problem* (7.33), (7.34) *on* $[0, b_n)$ *with different starting values in* $\big(L_0, \bar{B}\big)$, $n \in \mathbb{N}$.

Theorem 7.50. *Let* (7.35)–(7.39), (7.41), (7.47), (7.48) *and* (7.51) *hold. Then there exist infinitely many unbounded solutions* u_n *of problem* (7.33), (7.34) *on* $[0, b_n)$ *with not necessary different starting values in* $\big[L_0, \bar{B}\big)$, $n \in \mathbb{N}$.

Theorem 7.51. *Assume that* (7.35)–(7.39), (7.41), (7.47), (7.48) *and* (7.53) *hold. Then there exist infinitely many unbounded solutions* u_n *of problem* (7.33), (7.34) *on* $[0, b_n)$ *with not necessary different starting values in* $\big[L_0, \bar{B}\big)$, $n \in \mathbb{N}$.

Theorem 7.52. *Let (7.35)–(7.39), (7.41), (7.47), (7.48), (7.54), (7.56) and (7.57) hold. Then there exist infinitely many unbounded solutions u_n of problem (7.33), (7.34) on $[0, b_n)$ with not necessary different starting values in $\big[L_0, \bar{B}\big)$, $n \in \mathbb{N}$.*

It si clear that every unbounded solution of problem (7.33), (7.34) is an escape solution. We can formulate also the reverse assertion.

Theorem 7.53. *Assume all assumptions of Theorem 7.47 or 7.48 or 7.49 or 7.50 or 7.51 or 7.52. Then each escape solution of problem (7.33), (7.34) is unbounded.*

8 Summary of results

The thesis presented new contributions to the theory of singular nonlinear ordinary differential equations on an unbounded interval. We managed to generalize current results about existence and properties of three types of solutions of the singular equation

$$(p(t)u'(t))' + p(t)f(u(t)) = 0$$

to the equation with different coefficient functions p and q and to the equation with ϕ-Laplacian.

The first part of the thesis dealt with the initial value problem

$$(p(t)u'(t))' + q(t)f(u(t)) = 0, \qquad u(0) = u_0 \in [L_0, L], \quad u'(0) = 0 \tag{8.1}$$

and with an auxiliary initial value problem, where the nonlinearity f is replaced by a bounded nonlinearity $\tilde{f}$. We proved the existence (Theorem 7.6) and uniqueness of a solution of this auxiliary problem for every considered starting value as well as a continuous dependence of solutions on initial values (Theorem 7.7).

A significant attention was dedicated to the damped solutions of the original problem (8.1). Their existence was proved in Theorem 7.10 together with a starting interval giving only damped solutions. Theorem 7.14 gave that every oscillatory solution is the damped solution and has nonincreasing amplitudes. In addition, three types of conditions which guarantee that each damped solution is oscillatory were shown in Theorems 7.15, 7.16 and 7.17. The existence of oscillatory solutions was proved under these three different criteria in Theorems 7.18–7.20.

In connection with three obtained criteria for the oscillation of solutions, we reached three criteria (with additional conditions) leading to the existence of escape (Theorem 7.23) and homoclinic solutions of problem (8.1) (Theorem 7.24).

The second part of the thesis investigated the initial value problem

$$(p(t)\phi(u'(t)))' + p(t)f(\phi(u(t))) = 0, \qquad u(0) = u_0 \in [L_0, L], \quad u'(0) = 0 \tag{8.2}$$

and an auxiliary initial value problem with a bounded nonlinearity $\tilde{f}$ instead of f. Theorem 7.27 guaranteed the existence of a solution of this auxiliary problem for each considered starting value. The uniqueness of the solution of this problem was proved in Theorem 7.28 provided that ϕ^{-1} is locally Lipschitz continuous on $\mathbb{R}$. In the case that this condition falls, we proved the continuous dependence of solutions on positive initial values in Theorem 7.29 and for negative initial values in Theorem 7.30.

The existence of damped solutions of problem (8.2) was guaranteed by Theorem 7.31 for the same starting values as for problem (8.1). We proved the existence of escape solutions of auxiliary problem in the case, where both functions ϕ^{-1} and f are Lipschitz continuous (Theorem 7.37) and also in the more difficult opposite case (Theorem 7.42). Further, we derived three criteria guaranteeing that each escape solution of problem (8.2) is unbounded (Theorems 7.44, 7.45, 7.46). Finally, by combinations of these criteria with theorems guaranteeing the existence of escape solutions, we obtained the criteria of existence of unbounded solutions of problem (8.2) (Theorems 7.47, 7.48, 7.49, 7.50, 7.51 and 7.52).

List of publications

[1] M. Rohleder: *On the existence of oscillatory solutions of the second order nonlinear ODE*, Acta Univ. Palack. Olomuc. Fac. Rerum. Natur. Math. **51**, 2, (2012), 107–127.

[2] J. Burkotová, M. Rohleder, J. Stryja: *On the existence and properties of three types of solutions of singular IVPs*, Electron. J. Qual. Theory Differ. Equ. **29**, (2015), 1–25.

[3] J. Burkotová, I. Rachůnková, M. Rohleder, J. Stryja: *Existence and uniqueness of damped solutions of singular IVPs with ϕ-Laplacian*, Electron. J. Qual. Theory Differ. Equ. **121**, (2016), 1–28.

[4] M. Rohleder, J. Burkotová, L. López-Somoza, J. Stryja: *On unbounded solutions of singular IVPs with ϕ-Laplacian*, submitted.

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