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GEOMETRIC CONTROL OF NONHOLONOMIC SYSTEMS

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DIPLOMOVÁ PRÁCE

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Assignment Master's Thesis

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As provided for by the Act No. 111/98 Coll. on higher education institutions and the BUT Study and Examination Regulations, the director of the Institute hereby assigns the following topic of Master's Thesis:

Geometric control of nonholonomic systems

Brief Description:

Nonholonomic control systems are defined by nonholonomic constrains, i.e. linear constrains on (generalized) velocities that are not integrable. They are control systems linearly dependent on control functions, and their underlying geometry is the so–called sub–Riemannian geometry. This geometry then determines the controllability and stabilizability of the system and its motion planning. In particular, the usual first–order approximation necessary for control purposes must be defined in terms of this geometry.

Master's Thesis goals:

– The student will learn the basics of the geometry of non–holonomic systems and sub–Riemannian geometry.
– Furthermore, the student will understand the method of local approximation of these systems by the Carnot group and the use of this approximation to control them.
– The student will then design his own method of (sub–)optimal control for a specific non–holonomic mechanism and perform the relevant simulations in the selected software.

Recommended bibliography:

JEAN, F. Control of Nonholonomic Systems: from Sub-Riemannian Geometry to Motion Planning. Springer, 2004.

BELLAICHE, A., JEAN, F., RISLER, J.J. Geometry of nonholonomic systems. In: Laumond, J.P. (eds) Robot Motion Planning and Control. Lecture Notes in Control and Information Sciences, vol 229. Springer, 1998.

AGRACHEV, A., BARILARI, D., BOSCAIN, U. A Comprehensive Introduction to Sub-Riemannian Geometry (Cambridge Studies in Advanced Mathematics, pp. I-Iv). Cambridge: Cambridge University Press, 2019.

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In Brno,

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Abstract

This thesis focuses on a mathematical model for a three-body space robot with the objective of reconfiguring its structure using only internal joint torques. The aim is to minimize fuel consumption and achieve efficient reconfiguration without relying on external actuators. The system exhibits one holonomic and non-holonomic constraint, making the analysis and control design challenging. To address the complexity of the non-holonomic system, the local behavior is studied through the nilpotent approximation. The thesis emphasizes understanding the nilpotent approximation and constructing nilpotent system of the space robot using algebraic coordinates, along with transforming them into exponential coordinates within the Maple environment.

Keywords

Keywords: Three-body Space Robot, Non-Holonomic System, Nilpotent System, Nilpotent Approximation, Algebraic Coordinates, Exponential Coordinates

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I declare that I wrote the diploma thesis *Geometric Control of Non-Holonomic Systems* independently under the guidance of *Mgr. et Mgr. Aleš Návrat, Ph.D.* using the literature included in the list of references.

Sri Ram Prasath Ramasubramaniyan

I am deeply grateful to my Appa, Amma, and Anna for their unwavering support, continuous encouragement, love, and belief in me have been my driving force. I extend my heartfelt thanks to my supervisor for his patience, expertise, and guidance, which have been instrumental in my academic growth. His dedication and the invaluable feedback have shaped the outcome of this thesis.

Sri Ram Prasath Ramasubramaniyan

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1 Introduction

The control of nonholonomic systems, such as car parking, robot-trailer systems, poses significant challenges due to their non-integrable constraints. Nonholonomic systems are characterized by kinematic constraints that limit their feasible motions, making traditional control techniques insufficient. To address these challenges, various approaches have been proposed, including differential-geometric methods and the use of nilpotent approximation.

The differential-geometric approach has proven to be one of the most effective techniques for studying and controlling nonholonomic systems [12, 19]. By leveraging the geometric properties of the systems, this approach provides valuable insights into their control behavior. Geometric control theory, pioneered by Godbillon [15], Abraham and Marsden [2], and Arnol'd [9], has laid the foundation for understanding the geometrization of mechanical and control systems. Subsequent works by Nijmeijer and van der Schaft [23], Jurdjevic [18], and Agrachev and Sachkov [4] have further extended the geometric principles to control theory, offering valuable insights into the controllability and stabilization of nonlinear systems.

Murray and Sastry [22] investigated the use of trigonometric controls for nonholonomic systems, specifically those transformable into a chain form. Their work demonstrated the benefits of trigonometric controls in achieving desired system behaviors. Building upon this, Tilbury et al. [27] proposed trajectory generation methods for the n-trailer problem using Goursat normal form. They showed how trigonometric controls can be employed to move a system along all coordinates simultaneously, facilitating precise control.

Nilpotent systems represent another class of control systems for which exact solutions can be found. A nilpotent system is defined by the property that the Lie brackets of its control vector fields become zero after a certain bracket length [20]. The nilpotent approximation method, introduced by Bellaïche et al. [10], enables the transformation of a general nonholonomic system into a nilpotent approximating system. By applying the Baker-Campbell-Hausdorff formula, admissible piecewise constant controls can be calculated to precisely steer the nonholonomic system to the desired final state [7]. The nilpotent approximation retains the essential properties of the original system while simplifying its control. This thesis aims to investigate the approximation of 3D space robots with kinematic constraints

In recent years, several significant contributions have advanced the field of control for nonholonomic systems. Ardentov and Sachkov [7] proposed a method for controlling mobile robots with trailers based on the construction of a nilpotent approximation. Their work demonstrated the effectiveness of the nilpotent approximation approach in simplifying the control of nonholonomic systems. By preserving the important properties of the original system, the nilpotent approximation provides an efficient means to solve the control problem.

The differential-geometric approach has been widely adopted in the control of nonholonomic systems. Chitour et al. [12] introduced a global steering method for nonholonomic systems based on geometric techniques. By exploiting the underlying geometric structure, their approach allows for effective control design and trajectory planning. Kushner et al. [19] developed contact geometry and nonlinear differential equations, providing a comprehensive understanding of the geometric principles applicable to control systems.

In the context of optimal control, Fernandes et al. [14] presented a variational approach to optimal nonholonomic motion planning. Their method involved expanding the control function in a Fourier series and truncating the series to a certain order. The resulting solution approximated the optimal control as the truncation order increased. Agrachev and Sachkov [5] developed invariant geometric methods for solving optimal control problems on Lie groups. These techniques leverage the geometric structure of the control system to derive optimal control laws.

Thesis Overview:

This thesis provides an in-depth exploration of non-holonomic systems and their analysis using differential geometry, Lie algebra, and approximation techniques. The main objective is to understand the behavior of constrained mechanical systems and develop effective approximation methods.

The thesis begins with an introduction that sets the stage for the research, emphasizing the importance and motivation behind the study. It outlines the research objectives and highlights the contributions of the thesis.

Chapter 2 focuses on the basics of differential geometry and Lie algebra. It covers essential concepts such as manifolds, with a particular emphasis on sub-Riemannian manifolds. The chapter provides an overview of Lie algebra and discusses the flow of vector fields.

Chapter 3 introduces non-holonomic systems and their connection to control theory. It starts with an introduction to control theory and then delves into the characteristics of non-holonomic systems, including their constraints and examples. The chapter also explores the application of sub-Riemannian geometry in the context of non-holonomic systems, discussing the sub-Riemannian distance, distribution, growth vector, and adapted frames. Additionally, reachability in non-holonomic systems is examined, encompassing Chow's Condition and Chow-Rashevsky's theorem.

In Chapter 4, the focus shifts to nilpotent systems and exponential coordinates. The chapter introduces exponential coordinates, both of the first and second kinds, along with their significance in the analysis of these systems.

Chapter 5 delves into approximation theory specifically tailored for non-holonomic systems. The concept of non-holonomic order for functions and vector fields is discussed, laying the foundation for studying approximation techniques. Privileged coordinates, such as algebraic coordinates, are introduced as valuable tools. The chapter also covers first-order approximation techniques and introduces the concept of nilpotent approximation, a powerful tool for approximating non-holonomic systems.

In the concluding chapter, Chapter 6, we apply the concepts and techniques established in the preceding chapters to examine a specific case study involving a 3-body space robot. We begin by setting up the model and presenting the mathematical representation of the space robot, taking into account the essential assumptions. Then we focus on the transformation of vector fields from their original coordinate system to a privileged coordinate system. This crucial conversion enables us to advance our analysis by employing the powerful tool of nilpotent approximation. Within this framework, we extensively explore the process of transforming the nilpotent system into exponential coordinate transformations, encompassing both the first and second kinds. Through this meticulous investigation, we gain insights into the local behavior of the space robot.

In summary, this thesis provides a comprehensive examination of differential geometry, Lie algebra, and control theory in the context of non-holonomic systems. It investigates the application of approximation techniques and their relevance in understanding the behavior of constrained mechanical systems. The thesis contributes to the field by offering insights into the analysis and approximation of complex mechanical systems, illustrated through the case study of a 3-body space robot.

2 Basics of Differential Geometry and Lie Algebra

Differential geometry is a field of mathematics that employs calculus and analysis to study the properties of curves, surfaces, and other geometric objects under small changes. Given its wide-ranging applications, this topic encompasses numerous definitions and theorems.

This thesis aims to contribute to the understanding of differential geometry by providing definitions and theorems relevant to our work. Furthermore, [24], [13], [17], [3] provide detailed descriptions of differential geometry and Non-holonomic systems to offer a comprehensive overview of the subject matter. We will start with basic definitions in calculus and analysis.

2.1 Differential Geometry

Given that the readers of this thesis are assumed to have a basic understanding of sets and functions, we can proceed with introducing some relevant definitions and concepts.

Given that all functions utilized in this thesis are *real-valued function*, henceforth they shall be referred to as *function*.

Tangent vectors and tangent space:

Definition 2.1. Let f be a differentiable function on $\mathbb{R}^n$, and let v be any vector (direction) on $\mathbb{R}^n$. Then the derivative of f with respect to v at p [24] is defined as

$$v_p[f] = \frac{d}{dt}(f(p + t.v))|_{t=0} \tag{2.1}$$

Basically, it defined as the rate of change of f in the direction of v at p. In the context of Euclidean setting, this is referred to as the *directional derivative*. In a more general setting, such as on a manifold, we call $v_p[f]$ a *tangent vector* at p. If we consider a trajectory of an object, we refer to it as a *velocity vector*.

Definition 2.2. Let p be a point on $\mathbb{R}^n$. The set $T_p\mathbb{R}^n$ consisting of all tangent vectors that have p as a point of application is called the tangent space of $\mathbb{R}^n$ at p [24].

The tangent space captures the notion of the tangent vectors associated with a specific point on the manifold. Understanding these mathematical concepts will provide the necessary foundation for comprehending the subsequent discussions and analysis presented in this thesis.

Now, let's proceed with the lemma and example [24].

Lemma 2.3. *If v is a vector in $\mathbb{R}^n$, then*

$$v_p[f] = \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x_i}(p) \tag{2.2}$$

This lemma states that the derivative of f with respect to the vector v at point p can be expressed as the sum of the products of the components of v and the partial derivatives of f with respect to each coordinate x_i evaluated at p.

To illustrate this concept, consider the following example.

Example 2.4. Let $f = xyz^2$, and the vector in $\mathbb{R}^3$ is given by $(1, 2, 4)$. We will compute the directional derivative of f at $p = (-1, 2, 1)$.

First, we calculate the partial derivatives of f:

$$\frac{\partial f}{\partial x} = yz^2$$
$$\frac{\partial f}{\partial y} = xz^2$$
$$\frac{\partial f}{\partial z} = 2xyz$$

Using the formula from the lemma, we can compute the directional derivative:

$$v[f] = 1(yz^2) + 2(xz^2) + 4(2xyz)$$
$$v_p[f] = 1(2) + 2(-1) + 4(-4) = -16$$

This result tells us that, in particular, the function f is initially decreasing as p moves in the direction of the vector v.

These calculations demonstrate how the directional derivative can be computed using the partial derivatives of the function. This understanding will be essential for further analysis and applications in this thesis.

Tangent Map:

The tangent map allows us to analyze the behavior of mappings between tangent spaces and explore the local properties of functions. We define tangent map as follows,

Definition 2.5. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. If v is a tangent vector to $\mathbb{R}^n$ at p, let $f^*(v)$ be the initial velocity of the curve $t \to f(p + tv)$. The resulting function f^* sends tangent vectors from $\mathbb{R}^n$ to tangent vectors to $\mathbb{R}^m$, and is called the tangent map of f.

Corollary 2.6. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a mapping, then at each point p of $\mathbb{R}^n$ the tangent map $f_{*p} : T_p(\mathbb{R}^n) \to T_{f(p)}(\mathbb{R}^m)$ is a linear transformation.*

The tangent map, denoted as f_{*p}, is a linear transformation that best approximates the behavior of f near the point p [24]. The linearity of the tangent map enables us to study the differential properties of functions, such as differentiability and smoothness, by analyzing the corresponding linear transformations.

Inverse Mapping Theorem

And one of the fundamental theorems in differential geometry is the Inverse Mapping Theorem, which establishes the existence of an inverse mapping for a continuously differentiable function in a neighborhood of any point. This theorem provides important insights into the local properties of smooth maps [24].

Theorem 2.7 (Inverse Mapping Theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}^n$ be a C^k map, $k \geq 1$. If the Jacobian matrix $Df(x_0)$ is invertible for $x_0 \in \Omega$, then there exists an open neighborhood U near x_0 such that f is a C^k-diffeomorphism on U.*

The Inverse Mapping Theorem guarantees that if the derivative (Jacobian matrix) of a function is invertible at a point, then the function is locally invertible around that point. Moreover, the theorem asserts that the inverse function is also continuously differentiable.

This theorem allows for the study of local properties of mappings, such as local invertibility and smoothness, which are crucial in understanding manifold structures and transformations between them. The Inverse Mapping Theorem serves as a foundational tool in differential geometry, enabling the investigation of local behavior and providing a bridge between geometric and algebraic aspects of functions.

2.1.1 Manifolds

A manifold is a fundamental concept in mathematics that plays a central role in various areas such as differential geometry, topology, and physics. It provides a framework for studying spaces that locally resemble Euclidean space but may have more complex global structures. Manifolds serve as a foundation for understanding curved spaces, surfaces, and higher-dimensional geometries. In simple terms, a manifold is a topological space that locally resembles Euclidean space. It can have different dimensions, indicating the number of coordinates required to describe points on the manifold. For example, a surface like a sphere or torus is a two-dimensional manifold, while three-dimensional space is a three-dimensional manifold.

Homeomorphisms and open sets:

The concept of "resembles" is formalized through homeomorphisms, which are one-to-one correspondences between subsets of Euclidean spaces that preserve the topological structure. A subset M of $\mathbb{R}^k$ is locally Euclidean of dimension n if each point in M has a neighborhood that is homeomorphic to an open ball in $\mathbb{R}^n$. In this thesis, we explore open sets and their relationship with homeomorphisms. Specifically, we examine how local homeomorphisms can transform open sets while preserving topological properties. These mappings provide a powerful tool for studying functions and transformations in a topological setting.

Let $f : X \to Y$ be a homeomorphism between topological spaces X and Y. Our goal is to prove that for any open set U in X, the image set $f(U)$ is open in Y.

Proof:

Given that f is a homeomorphism, it is a continuous function. Thus, for any open set V in Y, the pre-image $f^{-1}(V)$ is open in X.

Let U be an open set in X. We aim to show that $f(U)$ is open in Y. Consider the pre-image of $f(U)$ in X: $f^{-1}(f(U))$. Since f is a bijection, we have $f^{-1}(f(U)) = U$.

As U is open in X and f is continuous, we conclude that $f(U) = f(f^{-1}(f(U)))$ is open in Y. Thus, a homeomorphism maps open sets to open sets.

Furthermore, we can demonstrate that a homeomorphism preserves the topological properties of the spaces involved. In particular, a homeomorphism maintains properties such as connectedness, compactness, and separation.

Coordinate chart, Atlas and Manifold:

To further develop our understanding, let's introduce the concept of a coordinate chart.

Definition 2.8. (*Chart*) A coordinate chart on a set X is a subset $U \subseteq X$ together with a bijection [26]

$$\phi : U \to \phi(U) \subseteq \mathbb{R}^n \tag{2.3}$$

onto an open set $\phi(U)$ in $\mathbb{R}^n$. The coordinates of a point $p \in U$ in this chart are just the coordinates of $\phi(p) = (x_1(p), ..., x_n(p)) \in \mathbb{R}^n$.
A coordinate chart is also called a local chart and denoted by (U, ϕ), or $(U, \phi; x^i)$ or simply $(U; x^i)$.

In other words, each point is associated with a local set through a coordinate chart. However, a single chart may not cover the entire space, necessitating the need to connect different charts together and perform transformations between them.

Throughout this thesis, we will leverage the concepts of open sets and homeomorphisms to establish connections, compare different charts, and derive insights about the underlying mathematical structures. This approach will enable us to explore the properties of the spaces under consideration and gain a deeper understanding of the topics at hand.

Definition 2.9. (Atlas) [26] An atlas on X is a collection of coordinate charts $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in I}$ such that

- X is covered by $\{U_\alpha\}_{\alpha\in I}$
- for each $\alpha, \beta \in I, \phi_\alpha(U_\alpha \cap U_\beta)$ is open in $\mathbb{R}^n$
- for each $\alpha, \beta \in I$, the transition map

$$g_{\alpha\beta} := \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \phi_\beta(U_\alpha \cap U_\beta)$$

  is a smooth diffeomorphism.

An atlas on X provides a collection of compatible charts that cover the entire space. When two charts overlap on a common domain, we can use transition maps to switch between the coordinate systems. These transition maps are smooth diffeomorphisms, ensuring the smoothness of the manifold and allowing for a smooth transition between charts. A differentiable structure on a manifold refers to a compatible atlas whose transition maps are smooth.

Definition 2.10. An n-**dimensional differentiable manifold** is a space X with a differentiable structure [26].

With a differentiable structure, we can define smooth functions, perform differential calculus, and study differential equations on the manifold.

Examples of manifolds (11):

Example 2.11. 1. The 2-dimensional sphere $\mathcal{S}^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$.
2. The real projective space $\mathbb{P}^n = \{n-$ dimensional subspaces of $\mathbb{R}^{n+1}\}$.
3. Open subsets of a given manifold are called open manifolds, e.g., the general linear group

$$GL(n) = \{A \in M_{n\times n} | det(A) \neq 0\}$$

4. Rotation Matrix: Special Orthogonal Group $SO(3)$

$$SO(3) = \{A \in M_{3\times 3} | det(A) = 1\}$$

There are various types of manifolds, such as smooth manifolds, Riemannian manifolds, Sub-Riemannian, symplectic manifolds, and algebraic manifolds. Each type has its own additional structure and properties that make them suitable for studying specific mathematical and physical phenomena.

Smooth manifolds:

A smooth manifold is a particular type of manifold that possesses additional structure and regularity. It is equipped with a differentiable structure, allowing us to define smooth functions on the manifold and perform differential calculus. The requirement of smoothness ensures that the transition maps between overlapping coordinate charts are themselves smooth, meaning they have derivatives of all orders.

In addition to the differentiable structure, a smooth manifold introduces the concept of tangent spaces at each point. These tangent spaces represent the set of all possible directions or velocities in which one can move from a particular point on the manifold. They capture the local linear behavior of the manifold near the point. The tangent space to a smooth manifold at a given point is completely intrinsic to the manifold itself. It allows us to calculate distances in an intrinsic manner by considering velocity vectors. In other words, to define a metric (inner product) and compute distances within the manifold, we rely on the notion of tangent spaces.

2.1.2 Sub-Riemannian Manifolds

In the context of differential geometry,

- **Vector Field:** A vector field on a smooth manifold $\mathcal{M}$ is a mapping that assigns to each point $p \in \mathcal{M}$ to a tangent vector in the tangent space $T_p\mathcal{M}$. Formally, a vector field X on $\mathcal{M}$ is a smooth assignment of a tangent vector $X(p) \in T_p\mathcal{M}$ to each point $p \in \mathcal{M}$ [11].
- **Distribution:** A distribution on a smooth manifold $\mathcal{M}$ is a sub-bundle Δ of the tangent bundle $T\mathcal{M}$ that assigns a subspace $\Delta(p)$ of the tangent space $T_p\mathcal{M}$ to each point $p \in \mathcal{M}$. Formally, for each $p \in \mathcal{M}$, the distribution $\Delta(p)$ is a subspace of the tangent space $T_p\mathcal{M}$, and the collection $\Delta(p)_{p\in\mathcal{M}}$ forms a smooth sub-bundle of the tangent bundle $T\mathcal{M}$ [11].

The Sub-Riemannian metric captures the geometry and distance properties of the sub-Riemannian manifold, while the distribution Δ specifies the allowed directions of motion within the manifold. The Sub-Riemannian manifold concept is important in the field of control theory and has applications in various areas, including robotics, optimal control, and geometric mechanics.

Definition 2.12. Sub-Riemannian Manifold

A Sub-Riemannian manifold [17] denoted as $(\mathcal{M}, \Delta, g_{SR})$, is a smooth manifold $\mathcal{M}$ equipped with a sub-Riemannian structure (Δ, g_{SR}), where the following conditions hold:

- Δ is a distribution on $\mathcal{M}$, which is a sub-bundle of the tangent bundle $T\mathcal{M}$.
- g_{SR} is a Sub-Riemannian metric associated with (Δ, g_{SR}), defined as a bilinear form

$$g_{SR} : \Delta \times \Delta \to \mathbb{R}.$$

We can extend the sub-Riemannian metric to the entire tangent space $T\mathcal{M}$ by assigning an infinite value to vectors outside the distribution Δ. The metric is defined in the bilinear form $g : T\mathcal{M} \times T\mathcal{M} \to \mathbb{R} \cup \{+\infty\}$ and is given by

$$g(q, v) = \begin{cases} g_{SR}, & \text{if } v \in \Delta(q) \\ +\infty, & \text{otherwise} \end{cases}$$

This reflects the constraints imposed by the sub-Riemannian structure, indicating that motions or vectors outside the allowed directions are not meaningful within the framework of the sub-Riemannian manifold. In a sub-Riemannian manifold, the sub-Riemannian structure provides a restricted framework for studying the geometry and dynamics of the manifold. Sub-Riemannian manifolds find applications in the study of non-holonomic systems, where the non-integrable constraints limit the available motions or controls of the system.

Riemannian manifolds:

In a Riemannian manifold, the structure allows us to define notions of length, angle, and curvature on the manifold. It provides the geometric framework necessary for studying concepts such as geodesics (shortest paths), curvature, and intrinsic geometry of the manifold.

Definition 2.13. Riemannian Manifold [17]
A Riemannian Manifold is a smooth manifold $\mathcal{M}$ equipped with positive inner product

$$g_R : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R} \qquad \forall p \in \mathcal{M}$$

such that $p \longrightarrow g_R$ is smooth

The Riemannian metric g_R associates a symmetric and positive definite bilinear form to each tangent space $T_p\mathcal{M}$ at every point p in the manifold. While both Riemannian and sub-Riemannian manifolds involve the notion of metrics, they differ in terms of the properties and structures they capture. Riemannian manifolds provide a broader framework for studying general smooth manifolds with a positive definite metric, while sub-Riemannian manifolds focus on specific types of manifolds with restricted motion patterns determined by the distribution and metric.

2.2 Lie Algebra

This section aims to offer a concise overview of Lie Algebra and Lie group, and how they relate to the differential structure. It serves as a preliminary introduction to the workings of Lie algebra. However, for a more detailed understanding, the precise definitions and theorems of Lie Algebra will be presented in the following section.

Lie algebra provides a way to study the algebraic structure of smooth vector fields on a manifold and capture the notion of infinitesimal symmetries. A Lie algebra is a vector space equipped with a binary operation called the Lie bracket, which satisfies specific algebraic properties.

Definition 2.14. A Lie algebra [16] is an algebra in which the operation is
1. antisymmetric, $[x, y] = -[y, x]$, and
2. satisfies the Jacobi identity ,

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0 \tag{2.4}$$

Lie Derivative of the vector fields:

By utilizing the 1-parameter group of diffeomorphisms, we can extend the traditional notion of vector field derivation (as seen in Euclidean spaces) to manifolds in a natural way. [26]

Definition 2.15. Let $X, Y \in V(M)$ be two vector fields, and ϕ_t be the 1-parameter group of diffeomorphisms generated by X. The Lie derivative of Y with regard to X is defined by

$$\mathcal{L}_X Y := \left.\frac{d}{dt}\right|_{t=0} (\phi_{-t})_* \circ Y \circ \phi_t = \lim_{t\to 0} \frac{(\phi_{-t})_* \circ Y(\phi_t(x)) - Y(x)}{t}$$

The Lie derivative of a vector field along another vector field is defined as the commutator, or Lie bracket, of the two vector fields. The Lie bracket operation, denoted by [X, Y], takes two vector fields X and Y and produces a new vector field.

Theorem 2.16. *For any vector fields $X, Y \in V(M)$, we have $\mathcal{L}_{XY} = [X, Y]$ [26].*

Proof: In local coordinates, suppose $X = a^i \frac{\partial}{\partial x_i}$ and $Y = b^i \frac{\partial}{\partial x_i}$. Let ϕ_t by the 1-parameter group of diffeomorphisms generated by X and $y = \phi_t(x)$. Then by definition $\partial_t \phi_t|_{t=0} = X$ and

$$\mathcal{L}_X Y = \left.\frac{d}{dt}\right|_{t=0} (\phi_{-t})_* \left(b^i(y) \frac{\partial}{\partial y^i} \right)$$

$$
\begin{aligned}
&= \frac{d}{dt}\bigg|_{t=0}\left(b^i(y)\frac{\partial\phi_{-t}}{\partial y^i}\right) \\
&= \frac{\partial b^i}{\partial y^j}\frac{dy^j}{dt}\frac{\partial\phi_{-t}}{\partial y_i}\bigg|_{t=0} + b^i(y)\frac{d}{dt}\left(\frac{\partial\phi_{-t}}{\partial y_i}\right)\bigg|_{t=0} \\
&= \frac{\partial b^i}{\partial x^j}a^j\frac{\partial}{\partial x^i} - b^i\frac{\partial a^j}{\partial x^i}\frac{\partial}{\partial x^j}
\end{aligned}
$$

Here in the first term, we use the fact that $\phi_0 = id$ *(identity map)*, while we interchange the derivatives of the second term.

One of the key motivations for studying Lie algebras is their connection to infinitesimal symmetries. Vector fields on a manifold describe transformations or flows, and the Lie bracket measures the non-commutativity of these transformations. Lie algebras capture the local behavior of symmetries and provide a framework for studying infinitesimal transformations and Lie groups.

Example: Lie Derivative of the vector fields

Example 2.17. (Martinet case) Consider the following vector fields on $\mathbb{R}^3$,

$$X_1 = \partial_x \qquad \text{and} \qquad X_2 = \partial_y + \frac{x^2}{2}\partial_z$$

The only nonzero brackets are

$$
\begin{aligned}
[X_1, X_2] &= X_1(X_2) - X_2(X_1) \\
&= \partial_x\left(\partial_y + \frac{x^2}{2}\partial_z\right) - \left[\partial_y + \frac{x^2}{2}\partial_z\right](\partial_x) \\
&= 0 + x\partial_z - 0 - 0 \\
&= x\partial_z
\end{aligned}
$$

and,

$$
\begin{aligned}
[X_1, [X_1, X_2]] &= X_1([X_1, X_2]) - [X_1, X_2](X_1) \\
&= \partial_x(x\partial_z) - x\partial_z(\partial_x) \\
&= \partial_z - 0 \\
&= \partial_z.
\end{aligned}
$$

2.2.1 Flow of the vector field

The flow of a vector field plays a fundamental role in differential geometry and dynamical systems. It provides a way to study the evolution of points on a manifold under the influence of a vector field.

The flow of a vector field X on a manifold $\mathcal{M}$ is a one-parameter group of transformations denoted by $\Phi_t : \mathcal{M} \to \mathcal{M}$ [6]. For every point p in $\mathcal{M}$, the flow satisfies the equation:

$$\frac{d\Phi_t}{dt}(p) = X(\Phi_t(p))$$

with the initial condition $\Phi_0(p) = p$.

The flow Φ_t has several important properties. At each fixed value of t, the flow Φ_t is a local diffeomorphism of $\mathcal{M}$, meaning that it preserves local smoothness and invertibility. Furthermore, the composition property holds, stating that the composition of flows Φ_t and Φ_s is equal to the flow Φ_{t+s}.

The flow has desirable properties such as preserving local smoothness and composition, making it a useful tool for studying the behavior of vector fields and their effects on the manifold. Understanding the flow of a vector field is crucial for analyzing dynamical systems, studying the behavior of the system along trajectories, and investigating the geometric properties of manifolds. It provides a framework for describing the dynamics and transformations occurring on the manifold under the influence of the vector field.

Example: Flow of the vector fields

Example 2.18. Consider a chained system with three states,

$$\dot{x}_1 = u_1 \quad \dot{x}_2 = u_2 \quad \dot{x}_3 = x_2 u_1$$

The vector fields of the system are,

$$\begin{aligned} X_1 &= \partial_{x_1} + x_2 \partial_{x_3} \\ X_2 &= \partial_{x_2} \end{aligned}$$

We compute the flow of the vector field of X_1,

$$\begin{aligned} \dot{X} = X_1 &= \partial_{x_1} + x_2 \partial_{x_3} \\ \dot{x_1} &= 1 \qquad \dot{x_2} = 0 \qquad \dot{x_3} = z_1 x_2 \end{aligned}$$

The integral with respect to time t,

$$x_1(t) = t + x_1(0) \qquad x_2(t) = x_2(0) \qquad x_3(t) = x_2(0)t + x_3(0)$$

Consider the initial conditions to be the origin,

$$\begin{aligned} x_1(0) = 0 \qquad & x_2(0) = 0 \qquad x_3(0) = 0 \\ x_1 = t \qquad & x_2 = 0 \qquad x_3 = 0 \end{aligned}$$

Flow of the vector field X_1,

$$\Phi_t^{X_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} t$$

Similarly, flow of the vector field X_2,

$$\Phi_t^{X_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} t$$

Understanding the concept of the flow of a vector field will provide valuable insights for the subsequent proof and analysis. These concepts will be instrumental in establishing the arguments presented in this thesis.

3 Non-Holonomic System

In this section, we will introduce several concepts of control theory and explore their relation to Sub-Riemannian Manifolds. Control theory is a field that focuses on analyzing and designing control systems to influence the behavior of dynamic systems. Sub-Riemannian Manifolds provide a mathematical framework for studying systems with non-integrable constraints on their motion. By investigating the connection between control theory and Sub-Riemannian Manifolds, we can gain valuable insights into the geometric and structural aspects of these systems. Understanding this relationship is crucial for developing effective control strategies that account for the specific characteristics and constraints of non-integrable systems.

3.1 Introduction to Control theory

Consider a non-linear control system in $\mathbb{R}^n$ given by the equation:

$$\dot{x} = f(x, u), \quad x \in \mathcal{M}, \quad u \in \mathbb{R}^m \tag{3.1}$$

where x represents the state of the system and u represents the control input. The vector fields that define the dynamics of the system are given by the function $f(x, u)$. Typically, these non-linear systems are assumed to operate in the Euclidean setting. However, there are cases where the operating range is constrained to a specific manifold based on the underlying model.

A simple example is a Servo control system, where the operating manifold is $\mathcal{S}^1$ (the unit circle). In many engineering applications, $\mathcal{S}^1$ is approximated as $\mathbb{R}$ (the real line) for convenience. However, this approximation leads to the loss of intrinsic properties associated with the true manifold structure.

It is important to recognize that by treating $\mathcal{S}^1$ as $\mathbb{R}$, certain characteristics and geometric properties inherent to the manifold are disregarded. The intrinsic structure of the operating manifold, such as periodicity and the wrapping behavior of angles, should be taken into account for a more accurate analysis and control design in these scenarios.

The Importance of Manifold Structure: An Example

To illustrate the significance of considering the intrinsic properties of a manifold, let's examine the difference in the shortest path between two points on $\mathcal{S}^1$ (the unit circle) when approximating it as $\mathbb{R}$ (the real line).

Consider the scenario where $\mathcal{S}^1$ is approximated as $\mathbb{R}$. We want to find the shortest distance between 0 and $\frac{3\pi}{2}$ measured in radians.

In the Euclidean setting, the shortest distance between these two points is given by:

$$d_E = \left|\frac{3\pi}{2} - 0\right| = \left|\frac{3\pi}{2}\right| = \frac{3\pi}{2}$$

However, when considering $\mathcal{S}^1$, which accounts for the periodic nature of angles, the shortest distance is different. We can find the shortest path on $\mathcal{S}^1$ by considering the difference between the endpoints, taking into account the periodicity.

Starting from 0 and moving counterclockwise, the shortest distance on $\mathcal{S}^1$ is given by:

$$d_S = \left|\frac{3\pi}{2} - 2\pi\right| = \left|-\frac{\pi}{2}\right| = \frac{\pi}{2}$$

Thus, by treating $\mathcal{S}^1$ as $\mathbb{R}$, the Euclidean approximation yields a shortest distance of $\frac{3\pi}{2}$, while the true shortest distance on $\mathcal{S}^1$ is $\frac{\pi}{2}$. This example highlights the importance of considering the intrinsic properties of the manifold, such as periodicity, to accurately determine distances and paths.

Controllability and Stabilizability:

In the field of control systems, two fundamental questions that arise are controllability and stabilizability.

Controllability addresses the existence of a control input $u(t)$ that can steer the control system from an initial state A to a desired state B within a finite time period T. In other words, controllability examines whether it is possible to manipulate the system's inputs in a way that allows us to control its behavior and reach a specific target state [4],[3].

Stabilizability focuses on the existence of a control input, represented as a function of the system's state $u = u(x(t))$, such that the resulting control system $\dot{x} = f(x, u)$ becomes stable. Here, stability refers to the property where the system's state converges to a desired equilibrium or reference point [4],[3].

For nonlinear systems, these questions are commonly addressed by linearizing the system around an equilibrium point. By linearizing the system, we obtain a linear approximation that facilitates the analysis of controllability and stabilizability properties. If the linearized system is found to be controllable at a particular point, it implies that the nonlinear system is controllable in the vicinity of that point. Linear control techniques, such as eigenvalue analysis and controllability matrices, are often employed to determine the controllability of linear systems.

While various techniques exist to assess the controllability of linear systems, it is important to maintain a broader perspective on the overall concepts of controllability and stabilizability. These concepts provide fundamental insights into the behavior and control of dynamic systems, forming the basis for control system design and understanding the limitations of controlling nonlinear systems.

Linearization and Manifold Structure

Consider the system described by Equation 3.1, and let (x^*, u^*) denote an equilibrium point of the system. The first-order approximation of the system around (x^*, u^*) is given by [17]:

$$\delta\dot{x} = \left.\frac{\partial f}{\partial x}\right|_{(x^*,u^*)} \delta x + \left.\frac{\partial f}{\partial u}\right|_{(x^*,u^*)} \delta u \tag{3.2}$$

where $\delta\dot{x}$ represents the perturbation in the state, δx and δu represent the perturbations in the state and control input, respectively, and $\left.\frac{\partial f}{\partial x}\right|_{(x^*,u^*)}$ and $\left.\frac{\partial f}{\partial u}\right|_{(x^*,u^*)}$ denote the partial derivatives of f with respect to x and u evaluated at (x^*, u^*).

If the linearized system described by Equation 3.2 is controllable, it implies that the original nonlinear system described by Equation 3.1 is controllable in the vicinity of the equilibrium point (x^*, u^*). This observation holds in the Euclidean setting, where the concept of controllability based on linearization is widely applicable.

However, it is important to note that this idea may not hold true in certain manifolds. The assumption of controllability based on linear approximation fails when dealing with systems operating on specific manifolds. The geometry and intrinsic properties of the manifold can significantly impact the controllability characteristics of the system, rendering the linearized analysis insufficient. In such cases, alternative methods and techniques that account for the manifold structure need to be employed to accurately analyze the controllability of the system.

Example 3.1. The Importance of Weighted Pseudo-Norm in Control Analysis [17]
Consider the Brockett integrator system given by:

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 u_2 \end{aligned} \tag{3.3}$$

where $x \in \mathcal{M}_1 \subset \mathbb{R}^3$ and $u \in \mathbb{R}^2$. The vector fields of the system in Equation 3.3 are:

$$V_1(x) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad V_2(x) = \begin{bmatrix} 0 \\ 1 \\ x_1 \end{bmatrix}$$

When we linearize the system at the origin, the linearized system is given by:

$$\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \\ \delta\dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \delta u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \delta u_2 \tag{3.4}$$

From Equation 3.4, it is evident that the linearized system is uncontrollable in the x_3 direction.
However, if we desire to move the system in the x_3 direction, we can employ the following control law:

$$\bar{u}(t) = \begin{cases} (1,0) & \text{if } t \in [0,\epsilon] \\ (0,1) & \text{if } t \in [\epsilon, 2\epsilon] \\ (-1,0) & \text{if } t \in [2\epsilon, 3\epsilon] \\ (0,-1) & \text{if } t \in [3\epsilon, 4\epsilon] \end{cases} \tag{3.5}$$

This control law enables the system described by Equation 3.3 to move from the origin $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$ to $\begin{bmatrix} 0 & 0 & \epsilon^2 \end{bmatrix}^T$ in a time span of $t = 4\epsilon$. Although the linearized system is uncontrollable in the x_3 direction, this specific control law allows for control over the system and achieves the desired motion.

The minimal time required to reach a point X from the origin using controls u such that $||u|| \leq 1$ is denoted as $\bar{T}(X)$. The bounds on the minimal time can be expressed as follows:

$$\frac{1}{3}(|x_1| + |x_2| + |x_3|^{\frac{1}{2}}) \leq \bar{T}(X) \leq 4(|x_1| + |x_2| + |x_3|^{\frac{1}{2}})$$

This implies that the minimal time $\bar{T}(X)$ needs to be compared with the weighted pseudo-norm $(|x_1|+|x_2|+|x_3|^{\frac{1}{2}})$ rather than the usual Euclidean norm. It highlights the importance of considering the specific pseudo-norm for the given system.

When making first-order approximations or linearizations, it is crucial to take into account this pseudo-norm. The linearizations should be performed with respect to such a pseudo-norm rather than the Euclidean norm. This is essential to capture the appropriate dynamics and behavior of the system under consideration.

By considering the weighted pseudo-norm and incorporating it into the analysis, a more accurate understanding of the system's controllability and behavior can be obtained.

3.2 Non Holonomic systems

In this section, our focus is on understanding non-holonomic systems from both a mathematical and mechanical perspective. We will start by presenting the mathematical definition of a non-holonomic system, followed by a brief motivation for the necessity of using sub-Riemannian distance when linearizing such systems.

A non-holonomic system refers to a type of mechanical system characterized by constraints on its motion that cannot be fully integrated. These constraints restrict the system's velocity and cannot be expressed in terms of a potential function.

3.2.1 Non-holonomic System

Definition 3.2. A Non-holonomic System on Manifold $\mathcal{M}$ is a control system which is of the form [17]

$$\dot{q} = u_1 X_1(q) + ... + u_m X_m(q) \qquad x \in \mathcal{M}, u \in \mathbb{R}^m \tag{3.6}$$

where $m > 1$ and $X_1, X_2, ..., X_m$ are C^∞ vector fields on $\mathcal{M}$

The distribution at p of Eq. 3.6 is given by

$$\Delta(p) = span\{X_1(p), ..., X_m(p)\} \subset T_p\mathcal{M} \qquad p \in \mathcal{M} \tag{3.7}$$

where, $T_p\mathcal{M}$ is the tangent space of Manifold $\mathcal{M}$ at point p.

Linearizing the system described by Eq. 3.6 at the equilibrium point (x^*, u^*) yields the following linearized system:

$$\dot{\delta x} = \delta u_1 X_1(x^*) + \ldots + \delta u_m X_m(x^*) \tag{3.8}$$

The reachable set of the linearized system given by Eq. 3.8 is determined by the vector fields:

$$\Delta(x^*) = span\{X_1(x^*), \ldots, X_m(x^*)\}$$

If the $dim(\Delta(x^*)) = n$, then the linearized system is controllable. In this case, the Euclidean norm is suitable for linearization.

However, if the $dim(\Delta(x^*)) \leq n$, then the linearized system is uncontrollable. It is important to note that the actual system described by Eq. 3.6 may still be controllable. In such scenarios, linearization using the Euclidean norm becomes ineffective. Therefore, first-order approximations should be taken with respect to a suitable pseudo-norm instead [17].

It is worth mentioning that the linearization mentioned above corresponds to a first-order approximation based on Euclidean or Riemannian distances. However, nonholonomic systems rely on a sub-Riemannian distance, which exhibits different behavior compared to the Euclidean distance. Consequently, understanding the local behavior of nonholonomic systems requires studying first-order approximations relative to the sub-Riemannian distance, rather than relying solely on the linearized system.

3.2.2 Non-holonomic constraints

Consider a mechanical system described by a state vector $q \in \mathbb{R}^n$, where the velocity of the states is denoted by $\dot{q}$ (i.e., the tangent vector).

In the context of a mechanical system, there can be geometric constraints that depend solely on the states and are expressed as [21]:

$$h_i(q) = 0 \qquad \text{for } i = 1, \ldots, k$$

These constraints restrict the motion of the system to a sub-manifold of dimension $(n-k)$. Such constraints are known as *holonomic constraints*.

Another type of constraint is based on the kinematics of the mechanical system and involves both the states and their velocities:

$$a(q,\dot{q}) = 0$$

In many cases, these constraints are linear with respect to the velocities, taking the form:

$$a(q)\dot{q} = 0 \tag{3.9}$$

These are referred to as *Paffian constraints*.

If the Paffian constraint given by Eq. 3.9 is integrable, it is equivalent to having holonomic constraints. Mathematically, this means that the vector fields in Eq. 3.7 satisfy $\Delta(p) = T_p\mathcal{M}$ for all $p \in \mathcal{M}$, indicating the presence of holonomic constraints.

On the other hand, if the Paffian constraint Eq. 3.9 is not integrable, it is classified as a non-holonomic constraint. Similarly, if the vector fields in Eq. 3.7 satisfy $dim(\Delta(p)) \leq n$, it indicates the presence of non-holonomic constraints in the system.

3.2.3 Examples of non-holonomic systems

Example 3.3. Wheeled mobile robot

Consider a wheeled mobile robot with the state vector $q = \begin{bmatrix} x & y & \theta \end{bmatrix}^T$, where x represents the robot's position along the x-axis, y represents the position along the y-axis, and θ represents the heading angle of the robot. The corresponding velocity is denoted as $\dot{q} = \begin{bmatrix} \dot{x} & \dot{y} & \dot{\theta} \end{bmatrix}^T$. The input to the system is given by $u = [u_1, u_2]$, where u_1 represents the linear velocity of the robot and u_2 represents the angular velocity of the robot.

The Paffian constraint of the system is defined by $\dot{x}\sin(\theta) - \dot{y}\cos(\theta) = 0$. Using Eq. 3.9, we can express this constraint as:

$$a(q) = \begin{bmatrix} \sin(\theta) & -\cos(\theta) & 0 \end{bmatrix} \qquad ker(a^T(q)) = span\left\{ \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \Delta(q)$$

The tangent vector fields of the wheeled robot system are defined as:

$$X_1(q) = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{bmatrix} \qquad X_2(q) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence, the mathematical model of the wheeled robot can be represented as:

$$\dot{q} = X_1u_1 + X_2u_2 \qquad \Longrightarrow \qquad \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2$$

Example 3.4. Game of Chess

The game of chess serves as an intriguing example of a non-trivial system in physics that can be analyzed using the principles of non-holonomic mechanics. In chess, specific constraints govern the movement of each piece. For instance, bishops are restricted to diagonal paths, knights move in an L-shaped pattern, and pawns can only advance one or two squares on their initial move.

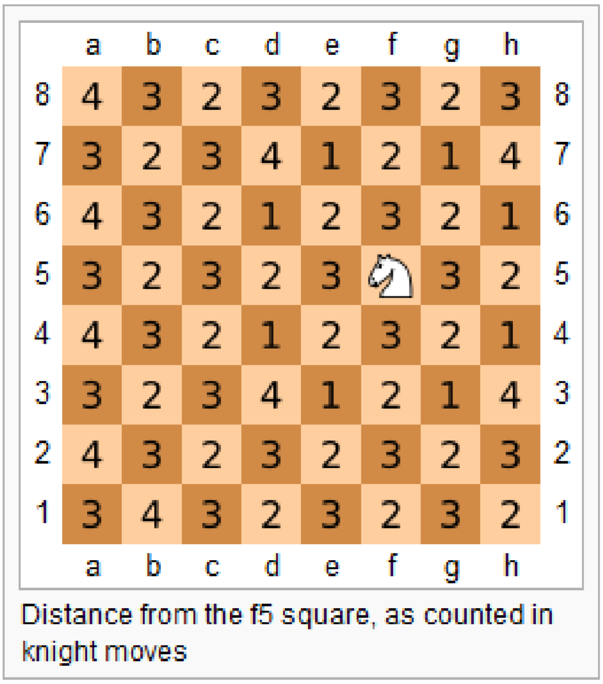

Figure 3.1: Number of moves required for a knight to reach each square starting from f5.

It's important to note that chess is a discrete system where movement is not solely determined by position or velocity. Let's consider the example of a knight. While its movement is confined to an L-shaped pattern, it still has the ability to traverse the entire board. Figure 3.1 illustrates the possible movements of a knight from the starting position f5 and indicates the number of moves required to reach each square [1].

3.3 Sub-Riemannian Geometry

The motivation behind incorporating sub-Riemannian distance when linearizing non-holonomic systems stems from the need to capture their intrinsic geometric structure. Sub-Riemannian geometry provides a framework for studying the geometry of systems with non-integrable constraints, where the notion of distance is defined in relation to admissible curves that adhere to the imposed motion constraints. This appropriate linearization techniques facilitates the study of controllability, stabilization, and other significant properties of non-holonomic systems.

This mathematical framework provides a powerful tool for understanding and analyzing the behavior of non-holonomic systems, enabling the development of effective control strategies and theoretical investigations.

3.3.1 Sub-Riemannian Distance

A non-holonomic system 3.6 induces a distance on $\mathcal{M}$ with the metric $g_{SR} : T\mathcal{M} \times T\mathcal{M} \to \mathbb{R}$ given by [17]

$$g_{SR}(q,v) = \begin{cases} u_1^2 + \ldots + u_m^2 & \text{if } \sum_{i=1}^m u_i X_i(q) = v \\ +\infty & \text{otherwise} \end{cases}$$

$$d_{SR}(q,v) = \inf\{g_{SR}\} \tag{3.10}$$

for $q \in \mathcal{M}$ and $v \in T_q\mathcal{M}$.

In this induced metric, two cases can be distinguished:

- If $v \notin \Delta(q)$, where $\Delta(q)$ is the set of feasible velocities at point q, then $g_{SR}(q, v) = +\infty$. This indicates that the velocity v is not achievable within the constraints of the non-holonomic system.
- If $v \in \Delta(q)$, then there exists a unique minimal control $u^* \in \mathbb{R}^m$ that achieves the velocity v, and the metric is given by $g_{SR}(q, v) = ||u^*||^2$. This corresponds to feasible velocities within the constraints of the non-holonomic system.

Another way to define the sub-Riemannian distance is as the minimal time required for the non-holonomic control system to traverse from point p to q under bounded controls:

$$d(p, q) = inf\left\{T \geq 0 : \exists \text{ a path with } \gamma(0) = p, \gamma(T) = q \text{ and } ||u(t)|| \leq 1 \forall \text{ a.e. } t \in [0, T]\right\}$$

where $\gamma(t)$ is the trajectory of the non-holonomic system it is defined by one parameter representation of time.

For example, consider the Brockett integrator described by Eq. 3.3. Using the control in Eq. 3.5, we can move in the positive x_3 direction to a point $q = \begin{bmatrix} 0 & 0 & \epsilon^2 \end{bmatrix}$. In this case, the sub-Riemannian distance $d(0, q)$ is equal to 4ϵ.

Any reparameterization of a minimizing trajectory remains minimizing. In other words, if we have a trajectory that minimizes a certain quantity, such as distance or energy, any change in the parameterization of that trajectory will still result in a trajectory that minimizes the same quantity. This property allows us to construct minimizing trajectories between any pair of sufficiently close points. Specifically, given two points that are close enough to each other on a manifold, we can find a trajectory that connects them and minimizes the desired quantity. In the case of distance minimization, this trajectory is chosen to have a unit velocity, meaning that the inner product between the position vector and the velocity vector along the trajectory is equal to one for almost every t.

As a consequence of this property, we can associate a control function $u(t)$ with the trajectory γ such that the norm of $u(t)$ is equal to one almost everywhere, i.e., $||u(t)|| = 1$ almost everywhere. This control function captures the direction in which the trajectory moves at each point in time. In other words, the distance between the point p and the trajectory γ increases linearly with the parameter t along the trajectory. This property of minimizing trajectories and their associated controls is crucial in the study of optimization problems on manifolds, where the goal is to find trajectories or paths that optimize certain criteria. By ensuring that reparameterizations of minimizing trajectories remain minimizing, we can construct and analyze optimal paths between points on a manifold.

From the above definitions, it can be inferred that there is a local one-to-one correspondence between sub-Riemannian structures and non-holonomic systems, where the dimension of $\Delta(q) = span\{X_1(q), \ldots, X_m(q)\}$ remains constant and equal to m. However, globally, the dimension of $\Delta(q)$ may vary [17].

3.3.2 Distribution in Non-Holonomic System

Let $VF(\mathcal{M})$ denote the set of smooth vector fields on the manifold $\mathcal{M}$, and let Δ^1 be the linear subspace of $VF(\mathcal{M})$ generated by the vector fields $X_1, \ldots, X_m$ associated with the non-holonomic system [17]. In other words, Δ^1 is the distribution formed by the vector fields $X_1, \ldots, X_m$,

$$\Delta^1 = span\{X_1, \ldots, X_m\}$$

For $s \geq 1$, define $\Delta^{s+1} = \Delta^s + [\Delta^1, \Delta^s]$ where, the Lie bracket of two vector fields X and Y, denoted by $[X, Y]$, is another vector field defined as the commutator of the two vector fields:

$$[X, Y] = XY - YX.$$

Thus, we have:

$$[\Delta^1, \Delta^s] = span\{[X, Y] : X \in \Delta^1, Y \in \Delta^s\}$$

In this way, Δ^{s+1} is generated by Δ^1 and the Lie brackets of elements from Δ^1 with elements from Δ^s. The Lie algebra generated by $X_1, \ldots, X_m$ is defined as the union of all the subspaces Δ^s for $s \geq 1$:

$$Lie(X_1, ..., X_m) = \bigcup_{s\geq 1} \Delta^s$$

In other words, $Lie(X_1, \ldots, X_m)$ is the smallest linear subspace of $VF(\mathcal{M})$ that contains $X_1, \ldots, X_m$ and is closed under the Lie brackets.

Example: Lie Algebra

Example 3.5. Consider a chained system defined by the following differential equations:

$$\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= x_2 u_1 \\
\dot{x}_4 &= x_3 u_1 \\
&\vdots \\
\dot{x}_n &= x_{n-1} u_1
\end{aligned}$$

Let $e_1 = \begin{bmatrix} 1 & 0 & 0 & . & . & 0 \end{bmatrix}^T$, $e_2 = \begin{bmatrix} 0 & 1 & 0 & . & . & 0 \end{bmatrix}^T$, and so on, up to $e_n = \begin{bmatrix} 0 & 0 & . & . & . & 1 \end{bmatrix}^T$. The tangent vector fields of the chained system can be expressed as:

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ x_2 \\ . \\ . \\ x_{n-1} \end{bmatrix}, X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ . \\ . \\ 0 \end{bmatrix} = e_2$$

Now, let's construct the linear subspaces using the Lie algebra evaluated at the origin:

$$\begin{aligned}
\Delta^1(p) &= span\{X_1(p), X_2(p)\} = span\{e_1, e_2\} \\
X_3(p) &= [X_1(p), X_2(p)] = -e_3 \\
\Delta^2(p) &= \Delta^1(p) + \{X_3(p)\} = span\{e_1, e_2, e_3\} \\
X_4(p) &= [X_1(p), X_3(p)] = e_4 \\
\Delta^3(p) &= \Delta^2(p) + \{X_4(p)\} = span\{e_1, e_2, e_3, e_4\} \\
X_n &= [X_1, X_{n-1}] = (-1)^n e_n \\
\Delta^{n-1}(p) &= \Delta^{n-2}(p) + \{X_n(p)\} = span\{e_1, e_2, \ldots, (-1)^n e_n\}
\end{aligned}$$

The Lie algebra generated by X_1 and X_2 is given by:

$$Lie(X_1, X_2) = \Delta^1 \cup \Delta^2 \cup \Delta^3 \cup \ldots \cup \Delta^{n-1} = span\{e_1, e_2, .., (-1)^n e_n\}$$

Length of an element:

Now, we define the length of an element X_I of $\mathcal{L}(X_1,\ldots,X_m)$, denoted by $|X_I|$, inductively as follows [17]:

$|X_I| = 1$ for $X_I = X_1,\ldots,X_m$, and

$$|X_I| = |X_1| + |X_2| \quad \text{for} \quad X = [X_1, X_2].$$

For the example given earlier, the lengths of the elements in $\mathcal{L}(X_1, X_2)$ are as follows:

$$\begin{aligned}
&|X_1| = 1, \qquad |X_2| = 1\\
&|X_3| = |[X_1,X_2]| = |X_1| + |X_2| = 1+1 = 2\\
&|X_4| = |[X_1,[X_1,X_2]| = |X_1| + |[X_1,X_2]| = |X_1| + |X_1| + |X_2| = 1+1+1 = 3\\
&\quad .\\
&\quad .\\
&\quad .\\
&|X_n| = |[X_1,X_{n-1}]| = |X_1| + |X_{n-1}| = 1 + n - 2 = n - 1
\end{aligned}$$

Verifying the Jacobi Identity in Eq. 2.4 for the given example:

$$\begin{aligned}
[[X_1,X_2],X_3] + [[X_2,X_3],X_1] + [[X_3,X_1],X_2] &= [X_3,X_3] + [0,X_1] + [X_4,X_2]\\
&= 0+0+0 = 0
\end{aligned}$$

Due to the Jacobi identity, we have

$$\Delta^s = span\{X_I : |I| \le s\}$$

Therefore, $\Delta^{n-1} = span\{X_I : |X_I| \le n-1\}$.

For $q \in \mathcal{M}$, the lie algebra $Lie(X_1,X_2)(q) = \{X(q) : X \in Lie(X_1,X_2)\}$, and, for $s \ge 1$, $\Delta^s(q) = \{X(q) : X \in \Delta^s\}$.

By definition these sets are linear subspaces of

$$\Big\{\{e_1,e_2\},\{e_1,e_2,e_3\},...,\{e_1,..,e_n\}\Big\} \subseteq T_q\mathcal{M}.$$

Therefore, for the chained system the Lie algebra generated by $X_1, X_2, \ldots, X_n$ spans the entire tangent space $T_q\mathcal{M}$ at every point q in the manifold.

3.3.3 Growth Vector

Consider the distribution of a non-holonomic system in Eq. 3.6, that gives rise to a flag of subspaces of $T_p\mathcal{M}$ at point p:

$$\Delta^1(p) \subset \Delta^2(p) \subset \ldots \subset \Delta^{r-1}(p) \subset \Delta^r(p) = T_p\mathcal{M}, \tag{3.11}$$

Here, $r = r(p)$ is referred to as the *degree of nonholonomy* at p [17].

Let $n_i(p) = \dim(\Delta^i(p))$. The r-tuple of integers

$$(n_1(p),\ldots,n_r(p))$$

is known as the growth vector at p. In this vector,

$$n_1(p) = m \quad \text{and} \quad n_r(p) = n = \dim(\mathcal{M})$$

Example 3.6. In the case of the chained system example, the growth vector is given by

$$G = (2,3,\ldots,n) \in \mathbb{R}^{n-1}, \qquad r = n-1$$

Regular and Singular points in a Non-holonomic system

Definition 3.7. A point p is considered a regular point if the growth vector of a non-holonomic system remains constant in a neighborhood of p. Otherwise, p is referred to as a singular point [17].

The growth vector provides important information about the behavior of a non-holonomic system at a particular point. It represents the dimensions of the subspaces $\Delta^i(p)$, which form a flag of nested subspaces. For a regular point p, the growth vector remains constant in a neighborhood, indicating a consistent pattern of subspace dimensions.

On the other hand, for a singular point p, the growth vector varies within the neighborhood, indicating a lack of regularity in the system's behavior. Singular points often correspond to critical or special configurations where the system exhibits distinct characteristics.

Weights of the Non-holonomic system

Definition 3.8. The weights at p are denoted as

$$w_i = w_i(p), \qquad i = 1, \ldots, n,$$

and are defined as

$$w_j = s \quad \text{if} \quad n_{s-1}(p) < j \leq n_s(p),$$

where $n_0 = 0$. In other words, we have

$$\begin{aligned} w_1 = \ldots = w_{n_1} &= 1, \\ w_{n_1+1} = \ldots = w_{n_2} &= 2, \\ &\ldots \\ w_{n_{r-1}+1} = \ldots = w_n &= r \end{aligned}$$

The weights provide a way to categorize the dimensions of the subspaces in the flag. Each weight w_i represents a distinct level or stratum within the flag. For example, if $w_i = 1$, it means that the subspace $\Delta^i(p)$ has dimension $n_1(p)$, while $w_{i+1} = 2$ indicates that the subspace $\Delta^{i+1}(p)$ has dimension $n_2(p)$, and so on.

Examples of Regular and Singular Non-Holonomic Systems

Let's consider some examples of regular and singular non-holonomic systems to illustrate these concepts:

Example 3.9. Examples of Regular and Singular Non-Holonomic Systems

1. **Regular: The Chained System** The chained system is regular for all $p \in \mathcal{M}$. The growth vector is given by $G = (2, 3, \ldots, n)$, and the weights of the system are

$$w_1 = w_2 = 1, \quad w_3 = 2, \quad w_4 = 3, \ldots, w_n = n - 1$$

2. **Singular: The Martinet Case**
   Consider the non-holonomic system with vector fields

$$X_1 = \partial_x, \qquad X_2 = \partial_y + \frac{x^2}{2}\partial_z$$

The Lie brackets are given by

$$[X_1, X_2] = x\partial_z, \qquad [X_1, [X_1, X_2]] = \partial_z.$$

Consequently, the growth vector is equal to

$$G(p) = \begin{cases} (2,2,3) & \text{if } \forall p \in \{x=0\} \\ (2,3) & \text{otherwise} \end{cases}$$

In this case, the set of singular points is the plane $\{x = 0\}$. Hence, the weights of the system are given by

$$w(p) = \begin{cases} w_1 = w_2 = 1, w_3 = 2 & \text{if } \forall p \in \{x=0\} \\ w_1 = 1, w_2 = 2 & \text{otherwise} \end{cases}$$

These examples illustrate the distinction between regular and singular points in a non-holonomic system. Regular points exhibit a constant growth vector and have a consistent pattern of subspace dimensions. Singular points, on the other hand, deviate from this regular pattern and indicate points of special behavior.

Properties of the Growth Vector:

It is worth noting some important properties of the Growth Vector:

- At a regular point, the growth vector forms an ordered and non-decreasing sequence, and the difference between two consecutive weights is either 0 or 1. When the regular point p locally exhibits an involutive distribution, the set of vector fields in Δ^s shares the same weight w_i, and the successive linear subspace Δ^{s+1} will have a weight $w_{i+1} = w_i + 1$.
- The weight w_i is an upper semi-continuous function. Moreover, if Chow's condition is satisfied, it has an upper bound equal to $dim(\mathcal{M})$.

Understanding the growth vector and weights provides valuable insights into the local behavior and structure of non-holonomic systems, allowing for the identification of regular and singular points and characterizing their properties.

3.3.4 Adapted Frames

To establish a comprehensive understanding of privileged coordinates, we first need to introduce the concept of adapted frames. These frames provide us with valuable insights into the Lie brackets and associated vector fields of the non-holonomic system.

In the previous section, we defined the linear subspaces of vector fields on the manifold as follows:

$$\begin{aligned} \Delta^1 &= \{X_1, \dots, X_{m_1}\} \\ \Delta^2 &= \Delta^1 + [\Delta^1, \Delta^1] = \{X_1, \dots, X_{m_1}\} + \{X_{m_1+1}, \dots, X_{m_2}\} \\ &\vdots \\ \Delta^{s+1} &= \Delta^s + [\Delta^1, \Delta^s] = \{X_1, \dots, X_{m_s}\} + \{X_{m_s+1}, \dots, X_n\} \end{aligned}$$

Recalling the definition of weights and the degree of non-holonomy, we can state that:

$$\begin{cases} X_1, \dots, X_n & \text{is a basis of } T_p\mathcal{M}. \\ X_i \in \Delta^{w_i} & i = 1, \dots, s \end{cases} \tag{3.12}$$

A family of n vector fields satisfying (3.12) is known as an adapted frame at point p. In other words, these vector fields are "adapted to the flag (3.11)" and provide a suitable basis for studying the system's behavior [17].

By establishing an adapted frame, we can gain a better understanding of the system's structure and dynamics. These adapted frames pave the way for constructing privileged coordinates that simplify the analysis and approximation of non-holonomic systems. In the following sections, we will explore the detailed explanation of privileged coordinates and the techniques of nilpotent approximation, that estimates represent the local behavior of non-holonomic systems.

3.4 Reachability in Non-holonomic system

In this section, we explore the concept of reachability in non-holonomic systems and examine Chow's condition, a fundamental result that characterizes the reachable set of states in these systems. Chow's condition provides valuable insights into the controllability and limitations of non-holonomic systems.

3.4.1 Chow's Condition

Definition 3.10. Chow's Condition

The Non-Holonomic system Eq. 3.6 (or the vector fields $X_1, ..., X_m$) satisfies Chow's Condition if

$$Lie(X_1, ..., X_m)(q) = T_q\mathcal{M}, \qquad \forall q \in M. \tag{3.13}$$

Equivalently, Chow's Condition can be expressed as follows: For every point q in the manifold $\mathcal{M}$, there exists an integer $r = r(q) \in \mathbb{Z}$ such that the dimension of the r-th iterated commutator space $[\Delta^r(q)]$ (where $\Delta^r(q)$ is defined as before) is equal to the dimension of the tangent space at that point, i.e.,

$$\forall q \in \mathcal{M}, \quad \exists r = r(q) \in \mathbb{Z} : \dim[\Delta^r(q)] = n,$$

where n is the dimension of the manifold $\mathcal{M}$.

This property is also known as the Lie algebra rank condition (LARC) in control theory, and it is analogous to the Hormander condition in the context of partial differential equations (PDEs) [17].

Chow's Condition plays a crucial role in the study of sub-Riemannian geometry and control for non-holonomic systems. It ensures that the non-holonomic system has sufficient maneuverability and allows for controllability from any initial configuration to any final configuration within the manifold. When Chow's Condition is satisfied, the control system can access any point in the manifold using appropriate control inputs, and the associated sub-Riemannian geometry exhibits interesting properties that can be studied using the tools of geometric control theory.

Remark. The chained system statisfies Chow's condition,

$$Lie(X_1, X_2) = span\{e_1, ..., e_n\} \quad \text{and} \quad span\{e_1, ..., e_n\} = T_q\mathcal{M}$$
$$\text{Equivalently,} \quad \forall q \in \mathcal{M}, r = n-1 : dim[\Delta^{n-1}(q)] = n$$

Lemma 3.11. *If the Non-Holonomic system Eq. 3.6 satisfies Chow's Condition, then for every $p \in M$, the reachable set $\mathbb{R}^p$ is a neighbourhood of p [17].*

The reachability property of the system is directly linked to the fulfillment of Chow's Condition.

We can prove this lemma using the example of the chained system provided earlier, but it is important to note that the result can be generalized to any Non-Holonomic system defined by Eq. 3.6.

In the case of the chained system example, we have shown that the Lie algebra generated by the vector fields $X_1, X_2, \ldots, X_n$ spans the tangent space $T_q\mathcal{M}$ at every point q in M. This means that for any given point p in M, we can construct a trajectory starting from p that can reach any point in a neighborhood of p. The reachable set $\mathbb{R}^p$ of p consists of all the points that can be reached from p by following the trajectories defined by the vector fields $X_1, X_2, \ldots, X_n$. Since the Lie algebra satisfies Chow's Condition, it implies that the reachable set $\mathbb{R}^p$ is a neighborhood of p.

Proof: Take a small neighbourhood $U \subset \mathcal{M}$ of p that we identify with a neighbourhood of 0 in $\mathbb{R}^n$.

Let $\phi_t^{X_i} = \exp(tX_i)$ be the flow of the vector field $X_i, i = 1, 2$. Every curve $t \rightarrow \phi_t^{X_i}(q)$ is a trajectory of the chained system and we have

$$\phi_t^{X_i} = \exp(tX_i) = \sum_{k \geq 0} \frac{1}{k!}(tX_i)^k = id + tX_i + o(t)$$

where For every element $I \in L(1,2)$, we define the local diffeomorphisms $\phi_t^{X_i}$ on U by induction on the length $|X_I|$ of X_I: if $X_I = [X_1, X_2]$, then

$$\phi_t^{X_3} = [\phi_t^{X_1}, \phi_t^{X_2}] := \phi_{-t}^{X_2} \circ \phi_{-t}^{X_1} \circ \phi_t^{X_2} \circ \phi_t^{X_1}$$

By construction, ϕ_t^I may be expanded as a composition of flows of the vector field $X_i, i = 1, 2$. As a consequence, $\phi_t^I(q)$ is the endpoint of a trajectory of the chained system issued from q.

Moreover, on a neighbourhood of p there holds

$$\begin{aligned}
\phi_t^{X_1} &= id + tX_1 + o(t) \\
\phi_t^{X_2} &= id + tX_2 + o(t) \\
\phi_t^{X_3} &= id + t^{|X_3|}X_3 + o(t^{|X_3|}) \\
&= id + t^2 X_3 + o(t^2) \\
\phi_t^{X_4} &= id + t^3 X_4 + o(t^3) \\
&. \\
&. \\
&. \\
\phi_t^{X_n} &= id + t^{n-1} X_3 + o(t^{n-1})
\end{aligned}$$

To obtain a diffeomorphism whose derivative with respect to the time is exactly X_I, we set

$$\psi_t^{X_n} = \begin{cases} \phi_{t^{1/|X_n|}}^{X_n} & \text{if } t \geq 0 \\ \phi_{-|t|^{1/|X_n|}}^{X_n} & \text{if } t < 0 \quad \text{and } |X_n| \text{ is odd} \\ [\phi_{|t|^{1/|X_n|}}^{X_1}, \phi_{|t|^{1/|X_n|}}^{X_{n-1}}] & \text{if } t < 0 \quad \text{and } |X_n| \text{ is even} \end{cases}$$

$$\psi_t^{X_1} = \begin{cases} \phi_t^{X_1} & \text{if } t \geq 0 \\ \phi_{-|t|}^{X_1} & \text{if } t < 0 \quad \text{and} \quad |X_1| = 1 \end{cases}$$

$$\psi_t^{X_2} = \begin{cases} \phi_t^{X_2} & \text{if } t \geq 0 \\ \phi_{-|t|}^{X_2} & \text{if } t < 0 \quad \text{and} \quad |X_2| = 1 \end{cases}$$

$$\psi_t^{X_3} = \begin{cases} \phi_{t^{1/2}}^{X_3} & \text{if } t \geq 0 \\ \phi_{-|t|^{1/2}}^{X_2} \circ \phi_{-|t|^{1/2}}^{X_1} \circ \phi_{|t|^{1/2}}^{X_2} \circ \phi_{|t|^{1/2}}^{X_1} & \text{if } t < 0 \quad \text{and} \quad |X_3| = 2 \end{cases}$$

$$\psi_t^{X_4} = \begin{cases} \phi_{t^{1/3}}^{X_4} & \text{if } t \geq 0 \\ \phi_{-|t|^{1/3}}^{X_4} & \text{if } t < 0 \quad \text{and} \quad |X_4| = 3 \end{cases}$$

where $X_3 = [X_1, X_2]$. Hence, we have the equation

$$\psi_t^{X_I} = id + tX_I + o(t) \tag{3.14}$$

where $\psi_t^{X_I}(q)$ represents the endpoint of a trajectory of the chained system originating from the point q.

By satisfying Chow's Condition, we can select commutators $X_1, ..., X_n$ such that their values at the point p span the tangent space $T_p\mathcal{M}$. This choice is possible due to the linear independence condition provided by Chow's Condition.

Next, we introduce the map φ, defined on a small neighborhood Ω of the origin in $\mathbb{R}^n$, as follows:

$$\varphi(t_1, ..., t_n) = \psi^{X_n}t_n \circ ... \circ \psi^{X_1}t_1(p) \in \mathcal{M}$$

We can conclude from the expression in Eq. 3.14 that φ is continuously differentiable (C^1) in the vicinity of the origin and possesses an invertible derivative at the origin. Consequently, φ is a local C^1-diffeomorphism. Thus, the image $\varphi(\Omega)$ includes a neighborhood of the point p.

For any t belonging to Ω, the point $\varphi(t)$ corresponds to the endpoint of a concatenation of trajectories, with the initial trajectory emanating from p. Therefore, $\varphi(t)$ represents the endpoint of a trajectory that originates from p. Consequently, we can deduce that $\varphi(\Omega) \subset \mathbb{R}^p$, which implies that $\mathbb{R}^p$ is a neighborhood of the point p.

3.4.2 Chow-Rashevsky's theorem

The Chow-Rashevsky's theorem is a fundamental result in the theory of non-holonomic systems. It establishes a key property of connected non-holonomic manifolds satisfying Chow's Condition. The theorem states that [17],

Theorem 3.12. ***(Chow-Rashevsky's theorem)***

If $\mathcal{M}$ is connected and if the non-holonomic system satisfies Chow's Condition, then any two points of $\mathcal{M}$ can be joined by a trajectory of the non-holonomic system.

Proof: Let $p \in \mathcal{M}$. If $q \in \mathbb{R}^p$, then $p \in \mathbb{R}^q$. Consequently, $\mathbb{R}^p = \mathbb{R}^q$ for any $q \in \mathcal{M}$. By applying the lemma mentioned above, we deduce that $\mathbb{R}^p$ is an open set. Thus, the manifold $\mathcal{M}$ is covered by the union of pairwise disjoint open sets $\mathbb{R}^p$. Since $\mathcal{M}$ is connected, there exists only one such open set [17].

Chow-Rashevsky's theorem, also known as Chow's theorem, provides a significant result in the study of non-holonomic systems. The essence of Chow-Rashevsky's theorem is that if a connected manifold satisfies Chow's Condition, indicating that the constraints imposed on the system are appropriate, then it is possible to find a trajectory within the non-holonomic system that connects any two points in the manifold. This result is of great significance as it guarantees the existence of feasible paths for motion planning and control in non-holonomic systems, providing a theoretical basis for designing control strategies and achieving desired behaviors in such systems.

4 Nilpotent Systems and Exponential Coordinates

4.1 Nilpotent Systems

Nilpotent Lie Algebra

A Lie algebra is said to be *nilpotent* if there exists a positive integer k such that the kth iteration of the Lie bracket yields zero [16], i.e.,

$$[x_1, [x_2, [\ldots[x_{k-1}, x_k]\ldots]]] = 0$$

for all elements $x_1, x_2, \ldots, x_k$ in the Lie algebra .

In other words, in a nilpotent Lie algebra, repeated applications of the Lie bracket eventually result in zero. The smallest value of k for which this condition holds is called the *nilpotency class* of the Lie algebra.

Example 4.1. The upper triangular matrices Lie algebra:

Consider the set of $n \times n$ upper triangular matrices with real entries. This forms a Lie algebra under matrix commutation. The Lie bracket of two matrices is given by the matrix commutator. This Lie algebra is nilpotent of class $n - 1$, where n is the dimension of the matrices.

The nilpotent property has important implications for the structure and properties of the Lie algebra. It implies that the Lie algebra has a stratified structure, with each layer obtained by taking the iterated Lie brackets of the previous layer. The number of layers is equal to the nilpotency class of the Lie algebra.

Nilpotent Lie algebras are extensively studied in mathematics and physics, as they arise in various areas, including differential geometry, representation theory, and the theory of dynamical systems. They provide a rich framework for understanding the algebraic and geometric structures underlying many physical phenomena and mathematical objects.

Nilpotent Systems

In the context of control systems, a system that satisfies a nilpotent Lie algebra is referred to as a nilpotent system.

A nilpotent system is characterized by the property that repeated Lie bracket operations of the vector fields associated with the system eventually yield zero. In other words, starting from the vector fields of the system, if we iteratively compute their Lie brackets, the process will terminate after a finite number of steps with the result being the zero vector field.

Example 4.2. The Heisenberg Case: Nilpotent System

Consider the Heisenberg case, where the vector fields are defined as follows:

$$X_1 = \partial_x \qquad X_2 = \partial_y + x\partial_z$$

The Lie brackets are computed as follows:

$$[X_1, X_2] = \partial_z = X_3, \qquad [X_1, X_3] = [X_2, X_3] = 0$$

Here, the Lie bracket of any two given vector fields yields the third vector field, and the third vector field commutes with every other vector field. Consequently, the Lie brackets eventually result in zero. This Lie algebra is nilpotent of class 2.

This example demonstrates the concept of nilpotency in Lie algebras and provides an instance of an algebraic structure where repeated Lie bracket operations eventually result in zero.

Example 4.3. Wheeled Robot Case: Not a Nilpotent System

Consider the dynamics of a wheeled robot, where the vector fields are defined as follows:

$$X_1 = \cos\theta\partial_x + \sin\theta\partial_y \qquad X_2 = \partial_\theta$$

The Lie brackets are computed as follows:

$$\begin{aligned} X_3 &= [X_1, X_2] = \sin\theta\partial_x - \cos\theta\partial_y, \qquad [X_1, X_3] = 0 \\ X_4 &= [X_2, X_3] = \cos\theta\partial_x + \sin\theta\partial_y \\ X_5 &= [X_2, X_4] = -\sin\theta\partial_x + \cos\theta\partial_y \\ X_6 &= [X_2, X_5] = -\cos\theta\partial_x - \sin\theta\partial_y \end{aligned}$$

Here, the Lie bracket of X_1 and X_2 vector fields yields the third vector field X_3, and the third vector field X_3 commutes with the vector field X_1, resulting in zero. However, X_2 does not commute with X_3 and any other derived vector fields such as X_4, X_5, X_6, and so on. Therefore, this system is not a nilpotent system.

The nilpotency condition reflects a specific structural property of the system and has implications for its controllability and behavior. The study of nilpotent systems contributes to various areas of control theory, including nonlinear control, differential geometry, and robotics.

4.2 Exponential coordinate

Given a nilpotent system, we can express a point on a manifold by "exponentiating" tangent vectors from a chosen frame at a reference point. Exponential coordinates are a useful tool in differential geometry for representing points on a manifold in terms of tangent vectors and exponential mappings. This subsection introduces two types of exponential coordinates: the first kind and the second kind.

4.2.1 Exponential Coordinate of the First Kind

Consider an adapted frame $X_1, \ldots, X_n$ at a point p on a manifold. The inverse of the local diffeomorphism

$$(z_1, \ldots, z_n) \mapsto \exp(z_1X_1 + \ldots + z_nX_n)(p) \tag{4.1}$$

defines a canonical coordinates of the first kind [17]. The exponential mapping takes tangent vectors multiplied by corresponding coefficients z_i and generates points on the manifold starting from the reference point p. The inverse of this mapping allows us to express points on the manifold in terms of these coefficients, yielding the canonical coordinates of the first kind. These coordinates are particularly valuable in certain areas of research where the specific properties of hypoelliptic operators and certain types of Lie groups are of interest.

Example 4.4. Consider the Heisenberg case, where the growth vector is $(2, 3)$. We have the following vector fields and coordinate transformations:

Vector fields:

$$\begin{aligned} X_1 &= \partial_{x_1} \\ X_2 &= \partial_{x_2} + x_1\partial_{x_3} \end{aligned}$$

In Step 1, we compute the distribution at point p. Here, at the origin p, the distribution is given by:

$$\begin{aligned}\Delta^1(p) &= \{X_1(p), X_2(p)\} = \{e_1, e_2\}\\ V_3 &= [X_1, X_2] = \partial_{x_3}\\ \Delta^2(p) &= \{X_1(p), X_2(p), X_3(p)\} = \{e_1, e_2, e_3\}\end{aligned}$$

In Step 2, we compute the flow of the vector field at p using exponential coordinates. The vector field V is defined as:

$$\begin{aligned}V &:= z_1X_1 + z_2X_2 + z_3X_3\\ V &= \begin{bmatrix} z_1 \\ z_2 \\ x_1z_2 + z_3 \end{bmatrix}\end{aligned}$$

Rewrite the vector field as the differential equations,

$$V := \dot{x}_1 = z_1 \qquad \dot{x}_2 = z_2 \qquad \dot{x}_3 = x_1z_2 + z_3$$

Computing the flow of the vector field V at p yields:

$$\begin{aligned}\Phi_t^V := x_1(t) &= z_1t + x_1(0)\\ x_2(t) &= z_2t + x_2(0)\\ x_3(t) &= \left(\frac{1}{2}z_1t^2 + x_1(0)t\right)z_2 + z_3t + x_3(0)\end{aligned}$$

Substituting the initial condition $p = (x_1(0), x_2(0), x_3(0)) = (0, 0, 0)$, we get:

$$\begin{aligned}\Phi_t^V := x_1(t) &= z_1t\\ x_2(t) &= z_2t\\ x_3(t) &= \frac{1}{2}z_1z_2t^2 + z_3t\end{aligned}$$

In Step 3, we compute the vector field in the exponential coordinate of the first kind. Substituting $t = 1$ in the flow equations, we get the flow of the vector field in the (x_1, x_2, x_3) coordinate on the $\mathcal{M}$:

$$x_1 = z_1, \quad x_2 = z_2, \quad x_3 = \frac{1}{2}z_1z_2 + z_3$$

Rearranging and substituting, we obtain the inverse mapping to define the flow in the (z_1, z_2, z_3) coordinate on the same manifold $\mathcal{M}$:

$$z_1 = x_1, \quad z_2 = x_2, \quad z_3 = -\frac{1}{2}x_1x_2 + x_3$$

To find the Jacobian between these tensors, we compute the derivatives of (z_1, z_2, z_3) with respect to (x_1, x_2, x_3):

$$\dot{z}_1 = \dot{x}_1, \quad \dot{z}_2 = \dot{x}_2, \quad \dot{z}_3 = -\frac{1}{2}x_1\dot{x}_2 - \frac{1}{2}x_2\dot{x}_1 + \dot{x}_3$$

This can be written in matrix form as:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{x_2}{2} & -\frac{x_1}{2} & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$$

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{x_2}{2} & -\frac{x_1}{2} & 1 \end{bmatrix}$$

With the Jacobian matrix, we can perform the transformation of the vector field defined in the (x_1, x_2, x_3) coordinate to the new coordinate (z_1, z_2, z_3). Let X_1 be the vector field defined in the (x_1, x_2, x_3) coordinate and Z_1 be the vector field defined in the (z_1, z_2, z_3) coordinate:

$$Z_1 = JX_1$$

$$Z_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{x_2}{2} & -\frac{x_1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -\dfrac{x_2}{2} \end{bmatrix}$$

Now we change the coordinates from (x_1, x_2, x_3) to the new coordinate (z_1, z_2, z_3):

$$Z_1 = \begin{bmatrix} 1 \\ 0 \\ -\dfrac{z_2}{2} \end{bmatrix}$$

Similarly, let Z_2 be the vector field defined in the (z_1, z_2, z_3) coordinate:

$$Z_2 = JX_2$$

$$Z_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{x_2}{2} & -\frac{x_1}{2} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -\dfrac{x_1}{2} + x_1 \end{bmatrix}$$

Now we change the coordinates from (x_1, x_2, x_3) to the new coordinate (z_1, z_2, z_3):

$$Z_2 = \begin{bmatrix} 0 \\ 1 \\ \dfrac{z_1}{2} \end{bmatrix}$$

Hence, the system defined in the exponential coordinate is given by:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\dfrac{z_2}{2} & \dfrac{z_1}{2} & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} u_1 \\ u_2 \\ -\frac{z_2}{2}u_1 + \frac{z_1}{2}u_2 \end{bmatrix}$$

The vector fields of the wheeled robot in the first canonical form are:

$$Z_1 = \partial_{z_1} - \frac{z_2}{2}\partial_{z_3} \tag{4.2}$$

$$Z_2 = \partial_{z_2} + \frac{z_1}{2}\partial_{z_3} \tag{4.3}$$

We can verify the distribution in the new coordinate at $p = (0, 0, 0)$:

$$\begin{aligned} Z_1(p) &= \partial_{z_1} \\ Z_2(p) &= \partial_{z_2} \\ Z_3(p) &= [Z_1, Z_2](p) = \frac{1}{2}\partial_{z_3} - \left(-\frac{1}{2}\partial_{z_3}\right) = \partial_{z_3} \end{aligned}$$

The vector fields (4.8) satisfy the Chow's condition.

Example 4.5. Consider the Heisenberg case, where the growth vector is $(2, 3, 4)$. We have the following vector fields and coordinate transformations:

Vector fields:

$$\begin{aligned} X_1 &= \partial_{x_1} \\ X_2 &= \partial_{x_2} + x_1\partial_{x_3} + \frac{x_1^2}{2}\partial_{x_4} \end{aligned}$$

In Step 1, we compute the distribution at point p. Here, at the origin p, the distribution is given by:

$$\begin{aligned} \Delta^1(p) &= \{X_1(p), X_2(p)\} = \{e_1, e_2\} \\ X_3 &= [X_1, X_2] = \partial_{x_3} + x_1\partial_{x_3} \\ X_4 &= [X_1, X_2] = \partial_{x_3} \\ \Delta^2(p) &= \{X_1(p), X_2(p), X_3(p), X_4(p)\} = \{e_1, e_2, e_3, e_4\} \end{aligned}$$

In Step 2, we compute the flow of the vector field at p using exponential coordinates. The vector field V is defined as:

$$V := z_1X_1 + z_2X_2 + z_3X_3 + z_4X_4$$

$$V = \begin{bmatrix} z_1 \\ z_2 \\ x_1z_2 + z_3 \\ \frac{x_1^2}{2}z_2 + z_4 \end{bmatrix}$$

Rewrite the vector field as the differential equations,

$$V := \dot{x}_1 = z_1 \qquad \dot{x}_2 = z_2 \qquad \dot{x}_3 = x_1z_2 + z_3 \qquad \dot{x}_4 = \frac{x_1^2}{2}z_2 + z_4$$

Computing the flow of the vector field V at p yields:

$$
\begin{aligned}
\Phi_t^V := x_1(t) &= z_1 t + x_1(0) \\
x_2(t) &= z_2 t + x_2(0) \\
x_3(t) &= \left(\frac{1}{2} z_1 t^2 + x_1(0) t\right) z_2 + z_3 t + x_3(0) \\
x_4(t) &= \frac{1}{6} z_2 t^3 z_1^2 + \frac{1}{2}((x_1(0) z_1 z_2 + z_1 z_3)) t^2) + \frac{1}{2}(x_1^2(0) z_2 + 2 x_1(0) z_3) t + t z_4 + x_4(0)
\end{aligned}
$$

Substituting the initial condition $p = (x_1(0), x_2(0), x_3(0), x_4(0)) = (0, 0, 0, 0)$, we get:

$$
\begin{aligned}
\Phi_t^V := x_1(t) &= z_1 t \\
x_2(t) &= z_2 t \\
x_3(t) &= \frac{1}{2} z_1 z_2 t^2 + z_3 t \\
x_4(t) &= \frac{1}{6} z_1^2 z_2 t^3 + \frac{1}{2} z_1 z_3 t^2 + z_4 t
\end{aligned}
$$

In Step 3, we compute the vector field in the exponential coordinate of the first kind. Substituting $t = 1$ in the flow equations, we get the flow of the vector field in the (x_1, x_2, x_3, x_4) coordinate on the $\mathcal{M}$:

$$
x_1 = z_1, \quad x_2 = z_2, \quad x_3 = \frac{1}{2} z_1 z_2 + z_3 \quad x_4 = \frac{1}{6} z_1^2 z_2 + \frac{1}{2} z_1 z_3 + z_4
$$

Rearranging and substituting, we obtain the inverse mapping to define the flow in the (z_1, z_2, z_3, z_4) coordinate on the same manifold $\mathcal{M}$:

$$
z_1 = x_1, \quad z_2 = x_2, \quad z_3 = -\frac{1}{2} x_1 x_2 + x_3 \quad z_4 = \frac{1}{12} x_1^2 x_2 - \frac{1}{2} x_1 x_3 + x_4
$$

To find the Jacobian between these tensors, we compute the derivatives of (z_1, z_2, z_3, z_4) with respect to (x_1, x_2, x_3, x_4):

$$
\begin{aligned}
\dot{z}_1 = \dot{x}_1, \quad \dot{z}_2 = \dot{x}_2, \quad \dot{z}_3 &= -\frac{1}{2} x_1 \dot{x}_2 - \frac{1}{2} x_2 \dot{x}_1 + \dot{x}_3 \\
\dot{z}_4 = \left(\frac{1}{6} x_1 x_2 - \frac{1}{2} x_3\right) \dot{x}_1 &+ \frac{1}{12} x_1^2 \dot{x}_2 - \frac{1}{2} x_1 \dot{x}_3 + \dot{x}_4
\end{aligned}
$$

This can be written in matrix form as:

$$
\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\dfrac{1}{2} x_2 & -\dfrac{1}{2} x_1 & 1 & 0 \\ \frac{1}{6} x_1 x_2 - \frac{1}{2} x_3 & \frac{1}{12} x_1^2 & -\frac{1}{2} x_1 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix}
$$

$$
J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\dfrac{1}{2} x_2 & -\dfrac{1}{2} x_1 & 1 & 0 \\ \frac{1}{6} x_1 x_2 - \frac{1}{2} x_3 & \frac{1}{12} x_1^2 & -\frac{1}{2} x_1 & 1 \end{bmatrix}
$$

With the Jacobian matrix, we can perform the transformation of the vector field defined in the (x_1, x_2, x_3, x_4) coordinate to the new coordinate (z_1, z_2, z_3, z_4). Let X_1 be the vector field defined in the (x_1, x_2, x_3, x_4) coordinate and Z_1 be the vector field defined in the (z_1, z_2, z_3, z_4) coordinate:

$$
\begin{aligned}
Z_1 &= JX_1 \\
Z_1 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\dfrac{1}{2}x_2 & -\dfrac{1}{2}x_1 & 1 & 0 \\ \frac{1}{6}x_1x_2 - \frac{1}{2}x_3 & \frac{1}{12}x_1^2 & -\frac{1}{2}x_1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 0 \\ -\dfrac{x_2}{2} \\ \frac{1}{6}x_1x_2 - \frac{1}{2}x_3 \end{bmatrix}
\end{aligned}
$$

Now we change the coordinates from (x_1, x_2, x_3, x_4) to the new coordinate (z_1, z_2, z_3, z_4):

$$
Z_1 = \begin{bmatrix} 1 \\ 0 \\ -\dfrac{z_2}{2} \\ -\frac{1}{12}x_1x_2 - \frac{1}{2}x_3 \end{bmatrix}
$$

Similarly, let Z_2 be the vector field defined in the (z_1, z_2, z_3, z_4) coordinate:

$$
\begin{aligned}
Z_2 &= JX_2 \\
Z_2 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\dfrac{1}{2}x_2 & -\dfrac{1}{2}x_1 & 1 & 0 \\ \frac{1}{6}x_1x_2 - \frac{1}{2}x_3 & \frac{1}{12}x_1^2 & -\frac{1}{2}x_1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ x_1 \\ \dfrac{x_1^2}{2} \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 1 \\ -\dfrac{x_1}{2} + x_1 \\ \dfrac{x_1^2}{12} - \dfrac{x_1^2}{4} \end{bmatrix}
\end{aligned}
$$

Now we change the coordinates from (x_1, x_2, x_3, x_4) to the new coordinate (z_1, z_2, z_3, z_4):

$$
Z_2 = \begin{bmatrix} 0 \\ 1 \\ \dfrac{z_1}{2} \\ \dfrac{z_1^2}{12} \end{bmatrix}.
$$

Let Z_3 be the vector field defined in the (z_1, z_2, z_3, z_4) coordinate:

$$
Z_3 = JX_3
$$

$$Z_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\dfrac{1}{2}x_2 & -\dfrac{1}{2}x_1 & 1 & 0 \\ \frac{1}{6}x_1x_2 - \frac{1}{2}x_3 & \frac{1}{12}x_1^2 & -\frac{1}{2}x_1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ x_1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ -\frac{1}{2}x_1 + x_1 \end{bmatrix}$$

Now we change the coordinates from (x_1, x_2, x_3, x_4) to the new coordinate (z_1, z_2, z_3, z_4):

$$Z_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{z_1}{2} \end{bmatrix}$$

Hence, the system defined in the exponential coordinate is given by:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\dfrac{z_2}{2} & \dfrac{z_1}{2} & 1 & 0 \\ -\frac{1}{12}z_1z_2 - \frac{1}{2}z_3 & \dfrac{z_1^2}{12} & \frac{1}{2}z_1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ 0 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} u_1 \\ u_2 \\ -\dfrac{z_2}{2}u_1 + \dfrac{z_1}{2}u_2 \\ -\left(\frac{1}{12}z_1z_2 + \frac{1}{2}z_3\right)u_1 + \dfrac{z_1^2}{12}u_2 \end{bmatrix}$$

The vector fields in the first canonical form are:

$$Z_1 = \partial_{z_1} - \frac{z_2}{2}\partial_{z_3} - \left(\frac{z_1z_2}{12} + \frac{z_3}{2}\right)\partial_{z_4} \tag{4.4}$$
$$Z_2 = \partial_{z_2} + \frac{z_1}{2}\partial_{z_3} + \frac{z_1^2}{12}\partial_{z_4} \tag{4.5}$$

We can verify the distribution in the new coordinate at $p = (0, 0, 0, 0)$:

$$\begin{aligned}
Z_1(p) &= \partial_{z_1} \\
Z_2(p) &= \partial_{z_2} \\
Z_3(p) &= [Z_1, Z_2](p) = \frac{1}{2}\partial_{z_3} + \frac{z_1(p)}{6}\partial_{z_4} - \left(-\frac{1}{2}\partial_{z_3} - \frac{z_1(p)}{12}\partial_{z_4} - \frac{z_1(p)}{4}\partial_{z_4}\right) \\
&= \partial_{z_3} + \frac{z_1(p)}{2}\partial_{z_4} = \partial_{z_3} \\
Z_4(p) &= [Z_1, Z_3](p) = \frac{1}{2}\partial_{z_4} - \left(-\frac{1}{2}\partial_{z_4}\right) = \partial_{z_4}
\end{aligned}$$

The vector fields (4.5) satisfy the Chow's condition.

4.2.2 Exponential Coordinate of the Second Kind

Similarly, let us consider an adapted frame $X_1, \ldots, X_n$ at a point p on a manifold. The inverse of the local diffeomorphism

$$(z_1, \ldots, z_n) \mapsto \exp(z_n X_n) \circ \ldots \circ \exp(z_2 X_2) \circ \exp(z_1 X_1)(p) \tag{4.6}$$

defines another system at p, called canonical coordinates of the second kind [17]. In canonical coordinates of the second kind, each exponential mapping $\exp(z_i X_i)$ successively transforms the reference point p along the direction of the corresponding tangent vector X_i multiplied by the coefficient z_i. This sequential application of the exponential mappings results in a clear step-by-step interpretation of the coordinate transformation.

Canonical coordinates of the second kind are often favored due to their simplicity and ease of interpretation, making them valuable in practical applications and intuitive understanding of geometric structures.

Both types of exponential coordinates provide alternative representations of points on a manifold in terms of tangent vectors and exponential mappings, offering distinct advantages depending on the specific context and application.

Example 4.6. Consider the Heisenberg case, where the growth vector is $(2, 3)$. We have the following vector fields and coordinate transformations:

Vector fields:

$$X_1 = \partial_{x_1} - \frac{x_2}{2}\partial_{x_3}$$
$$X_2 = \partial_{x_2} + \frac{x_1}{2}\partial_{x_3}$$

In Step 1, we compute the distribution at point p. Here, at the origin p, the distribution is given by:

$$\Delta^1(p) = \{X_1(p), X_2(p)\} = \{e_1, e_2\}$$
$$X_3 = [X_1, X_2] = \partial_{x_3}$$
$$\Delta^2(p) = \{X_1(p), X_2(p), X_3(p)\} = \{e_1, e_2, e_3\}$$

In Step 2, we compute the flow of the vector field at p using exponential coordinates. The vector field V is defined as:

$$(z_1, z_2, z_3) \rightarrow \exp(z_3 X_3) \circ \exp(z_2 X_2) \circ \exp(z_1 X_1)(p)$$

Computing the flow of the vector field X_1 along z_1 at p yields:

$$X_1 := \dot{x}_1(t) = 1 \qquad \dot{x}_2 = 0 \qquad \dot{x}_3 = -\frac{x_2}{2}$$
$$z_1 X_1 := \dot{x}_1(t) = z_1 \qquad \dot{x}_2 = 0 \qquad \dot{x}_3 = -\frac{x_2 z_1}{2}$$
$$\Phi_t^V := x_1(t) = z_1 t + x_1(0) \quad x_2(t) = x_2(0)$$
$$x_3(t) = -\frac{x_2(0) z_1}{2} t + x_3(0)$$

Substituting the initial condition $p = (x_1(0), x_2(0), x_3(0)) = (0, 0, 0)$, we get:

$$\Phi_t^{X_1} := x_1(t) = z_1 t \qquad x_2(t) = 0 \qquad x_3(t) = 0$$

$$\Phi^{X_1} := x_1 = z_1 \qquad x_2 = 0 \qquad x_3 = 0$$

Computing the flow of the vector field X_2 along z_2 at p yields:

$$X_2 := \dot{x}_1(t) = 0 \qquad \dot{x}_2 = 1 \qquad \dot{x}_3 = \frac{x_1}{2}$$
$$z_2X_2 := \dot{x}_1(t) = 0 \qquad \dot{x}_2 = z_2 \qquad \dot{x}_3 = \frac{x_1z_2}{2}$$
$$\Phi_t^{X_2} := x_1(t) = x_1(0) \quad x_2(t) = z_2t + x_2(0)$$
$$x_3(t) = \frac{x_1(0)z_2}{2}t + x_3(0)$$

Substituting the new initial condition $p = (x_1(0), x_2(0), x_3(0)) = (z_1t, 0, 0)$, we get:

$$\Phi_t^{X_2} := x_1(t) = z_1t \qquad x_2(t) = z_2t \qquad x_3(t) = \frac{1}{2}z_1z_2t^2$$
$$\Phi^{X_2} := x_1 = z_1 \qquad x_2 = z_2 \qquad x_3 = \frac{1}{2}z_1z_2$$

Computing the flow of the vector field X_3 along z_3 at p yields:

$$X_3 := \dot{x}_1(t) = 0 \qquad \dot{x}_2 = 0 \qquad \dot{x}_3 = 1$$
$$z_3X_3 := \dot{x}_1(t) = 0 \qquad \dot{x}_2 = 0 \qquad \dot{x}_3 = z_3$$
$$\Phi_t^V := x_1(t) = x_1(0) \quad x_2(t) = x_2(0)$$
$$x_3(t) = z_3t + x_3(0)$$

Substituting the new initial condition $p = (x_1(0), x_2(0), x_3(0)) = (z_1, z_2, \frac{1}{2}z_1z_2)$, we get:

$$\Phi_t^{X_3} := x_1(t) = z_1t \qquad x_2(t) = z_2t \qquad x_3(t) = z_3t + \frac{1}{2}z_1z_2t^2$$
$$\Phi^{X_3} := x_1 = z_1 \qquad x_2 = z_2 \qquad x_3 = z_3 + \frac{1}{2}z_1z_2$$

In Step 3, we compute the vector field in the exponential coordinate of the second kind. The flow of the vector field in the (x_1, x_2, x_3) coordinate on the manifold $\mathcal{M}$:

$$x_1 = z_1, \quad x_2 = z_2, \quad x_3 = \frac{1}{2}z_1z_2 + z_3$$

Rearranging and substituting, we obtain the inverse mapping to define the flow in the (z_1, z_2, z_3) coordinate on the same manifold $\mathcal{M}$:

$$z_1 = x_1, \quad z_2 = x_2, \quad z_3 = -\frac{1}{2}x_1x_2 + x_3$$

To find the Jacobian between these tensors, we compute the derivatives of (z_1, z_2, z_3) with respect to (x_1, x_2, x_3):

$$\dot{z}_1 = \dot{x}_1, \quad \dot{z}_2 = \dot{x}_2, \quad \dot{z}_3 = -\frac{1}{2}x_1\dot{x}_2 - \frac{1}{2}x_2\dot{x}_1 + \dot{x}_3$$

This can be written in matrix form as:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{x_2}{2} & -\frac{x_1}{2} & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$$

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{x_2}{2} & -\frac{x_1}{2} & 1 \end{bmatrix}$$

With the Jacobian matrix, we can perform the transformation of the vector field in the (x_1, x_2, x_3) coordinate to the new coordinate (z_1, z_2, z_3). Let X_1 be the vector field defined in the (x_1, x_2, x_3) and Z_1 be the vector field defined in the (z_1, z_2, z_3) coordinate:

$$Z_1 = JX_1$$

$$Z_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{x_2}{2} & -\frac{x_1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -\dfrac{x_2}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ -\dfrac{x_2}{2} - \dfrac{x_2}{2} \end{bmatrix}$$

Now we change the coordinates from (x_1, x_2, x_3) to the new coordinate (z_1, z_2, z_3):

$$Z_1 = \begin{bmatrix} 1 \\ 0 \\ -z_2 \end{bmatrix}$$

Similarly, let Z_2 be the vector field defined in the (z_1, z_2, z_3) coordinate:

$$Z_2 = JX_2$$

$$Z_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{x_2}{2} & -\frac{x_1}{2} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \dfrac{x_1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ \dfrac{x_1}{2} - \dfrac{x_1}{2} \end{bmatrix}$$

Now we change the coordinates from (x_1, x_2, x_3) to the new coordinate (z_1, z_2, z_3):

$$Z_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Hence, the system defined in the exponential coordinate is given by:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -z_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} u_1 \\ u_2 \\ -z_2 u_1 \end{bmatrix}$$

The vector fields in the second canonical form are:

$$Z_1 = \partial_{z_1} - z_2 \partial_{z_3} \tag{4.7}$$
$$Z_2 = \partial_{z_2} \tag{4.8}$$

We can verify the order of the privileged coordinate at $p = (0, 0, 0)$:

$$\begin{aligned} Z_1(p) &= \partial_{z_1} - z_2(p)\partial_{z_3} = \partial_{z_1} \\ Z_2(p) &= \partial_{z_2} \\ Z_3(p) &= [Z_1, Z_2](p) = 0 - (-1)\partial_{z_3} = \partial_{z_3} \end{aligned}$$

The vector fields (4.8) satisfy the Chow's condition.

5 Approximation Theory

In this section, we define the necessary terminology for approximating the Non-Holonomic system. We will start with the definition of order and how to calculate the order of the system. Based on the definition, we approximate the system locally to that corresponding order.

5.1 Non-Holonomic Order

5.1.1 Non-Holonomic order of a function

Definition 5.1. Let $f : \mathcal{M} \to \mathbb{R}$ be a continuous function. The non-holonomic order of f at p, denoted by $\text{ord}_p(f)$ [17], is the real number defined by

$$\text{ord}_p(f) = \sup\left\{s \in \mathbb{R} : f(q) = \mathcal{O}(d(p,q)^s)\right\} \tag{5.1}$$

This order is always nonnegative. Moreover $\text{ord}_p(f) = 0$ if $f(p) \neq 0$, and $\text{ord}_p(f) = +\infty$ if $f(p) \equiv 0$. Equivalently,

$$\text{ord}_p(f) = \min\left\{s \in \mathbb{N} : \exists i_1, ..., i_s \in \{1, ..., m\}, s.t.(X_{i_1}...X_{i_s}f)(p) \neq 0\right\}, \tag{5.2}$$

For every $f, g \in C^\infty(p)$ and every $\lambda \in \mathbb{R}\backslash 0$,

$$\begin{aligned} \text{ord}_p(fg) &\geq \text{ord}_p(f) + \text{ord}_p(g), \\ \text{ord}_p(\lambda f) &= \text{ord}_p(f), \\ \text{ord}_p(f+g) &\geq \min(\text{ord}_p(f), \text{ord}_p(g)). \end{aligned}$$

Example 5.2. Heisenberg Case

$$\begin{aligned} \dot{x} &= u_1\partial_x \\ \dot{y} &= u_2\partial_y \\ \dot{z} &= \frac{x}{2}u_2\partial_z - \frac{y}{2}u_1\partial_y \\ \text{where,}\quad X_1 &= \partial_x - \frac{y}{2}\partial_z \\ X_2 &= \partial_y + \frac{x}{2}\partial_z \end{aligned}$$

Consider several functions

1. $f_1(x) = x$

$$\begin{aligned} X_1 f_1 &= 1.\frac{\partial f}{\partial x} + 0.\frac{\partial f}{\partial y} - \frac{y}{2}.\frac{\partial f}{\partial z} = 1 + 0 + 0 = 1 \\ X_1 f_1(0) &= 1 \\ X_2 f_1 &= 0.\frac{\partial f}{\partial x} + 1.\frac{\partial f}{\partial y} + \frac{x}{2}.\frac{\partial f}{\partial z} = 0 + 0 + 0 = 0 \end{aligned}$$

From Eq. 5.2 $\implies \text{ord}_p(f_1) = 1$

2. $f_2(y) = y^2/2$

$$\begin{aligned} X_1 f_2 &= 1.0 + 0 - \frac{y}{2}.0 = 0 \\ X_2 f_2 &= 0 + 1.y + \frac{x}{2}.0 = y \\ X_2 f_2(0) &= 0 \\ X_2 X_2 f_2(0) &= 0 + 1.1 + 0 = 1 \end{aligned}$$

From Eq. 5.2 $\implies$ $\text{ord}_p(f_2) = 2$

3. $f_3(z) = z^2$

$$\begin{aligned} X_1 f_3 &= 1.0 + 0 - \frac{y}{2}.2z = -yz \\ X_1 f_3(0) &= 0 \\ X_1 X_1 f_3 &= 1.0 + 0 - \frac{y}{2}.(-y) = \frac{y^2}{2} \\ X_1 X_1 f_3(0) &= 0 \\ X_2 X_2 X_1 X_1 f_3 &= 1 \end{aligned}$$

From Eq. 5.2 $\implies$ $\text{ord}_p(f_3) = 4$

4. $f_4(y) = xy^2/2 = f_1 f_2$

$$\begin{aligned} \text{ord}_p(f_1 f_2) &\geq \text{ord}_p(f_1) + \text{ord}_p(f_2) \\ &\geq 1 + 2 = 3 \\ X_2 X_2 X_1 f_4 &= 1 \end{aligned}$$

From Eq. 5.2 $\implies$ $\text{ord}_p(f_4) = 3$

5.1.2 Non-Holonomic order of Vector fields

Definition 5.3. Let $X \in VF(p)$. The non-holonomic order of X at p, denoted by $\text{ord}_p(X)$ [17], is the real number defined by:

$$\text{ord}_p(X) = \sup\left\{\sigma \in \mathbb{R} : \text{ord}_p(Xf) \geq \sigma + \text{ord}_p(f), \forall f \in C^\infty(p)\right\}. \tag{5.3}$$

For every $X, Y \in VF(p)$ and every $f \in C^\infty(p)$,

$$\begin{aligned} \text{ord}_p([X, Y]) &\geq \text{ord}_p(X) + \text{ord}_p(Y), \\ \text{ord}_p(fX) &\geq \text{ord}_p(f) + \text{ord}_p(X), \\ \text{ord}_p(X) &\leq \text{ord}_p(Xf) - \text{ord}_p(f), \\ \text{ord}_p(X + Y) &\geq min(\text{ord}_p(X), \text{ord}_p(Y)). \end{aligned}$$

As a consequence of the above properties, $X_1, ..., X_m$ are of order ≥ -1, $[X_i, X_j]$ of order ≥ -2, and more generally, every X in the set Δ^k is of order $\geq -k$.

Example 5.4. Chained System

$$\begin{aligned} \text{ord}_p(X_1) &= -1 \\ \text{ord}_p(X_2) &= -1 \end{aligned}$$

$$\mathrm{ord}_p(X_3) = -2$$
$$\vdots$$
$$\mathrm{ord}_p(X_n) = -(n-1)$$

5.2 Privileged Coordinates

In the realm of differential geometry and coordinate systems, the term "privileged coordinates" refers to a specific choice of coordinates that offers simplifications in the analysis or equations describing a given system or geometry. These coordinates are selected based on the particular properties or symmetries exhibited by the system under investigation. The designation of "privileged" signifies that these coordinates possess distinct advantages or special properties compared to other possible choices of coordinates. By employing privileged coordinates, one can exploit the inherent structure of the system to streamline calculations, uncover hidden symmetries, or reveal important relationships between different variables.

Definition 5.5. *Privileged Coordinate*
A system of privileged coordinates at point p [17] is a set of local coordinates $(z_1, \ldots, z_n)$ such that $\mathrm{ord}_p(z_j) = w_j$ for $j = 1, \ldots, n$ that satisfy the following conditions [17]:

$$dz_i(\Delta^{w_i}(p)) \neq 0, \qquad dz_i(\Delta^{w_i-1}(p)) = 0, \qquad i = 1, \ldots, n, \tag{5.4}$$

Alternatively, we can state that $\partial_{z_i}|p$ belongs to $\Delta^{w_i}(p)$ but not to $\Delta^{w_i-1}(p)$. Local coordinates satisfying (5.4) are called *linearly adapted coordinates*.

Example 5.6. Here are a few examples of privileged coordinates:

1. In a planar robotic arm with revolute joints, the joint angles serve as privileged coordinates that are linearly adapted to the rotational distribution of the robotic arm.
2. Exponential coordinates are also regarded as privileged coordinates.

The selection of privileged coordinates relies on the specific problem or context at hand. In some instances, these privileged coordinates may correspond to natural or canonical choices that arise from the geometry or symmetries of the system. In other instances, privileged coordinates can be defined based on physical considerations or mathematical properties that render certain calculations or equations more manageable. For instance, in systems exhibiting translational symmetry, it is advantageous to employ coordinates that align with the directions of translation, as this simplifies the equations of motion.

5.2.1 Algebraic Coordinate

In certain situations, the construction of exponential coordinates may not be computationally efficient or feasible, as it requires integrating flows of vector fields. In such cases, alternative methods for constructing privileged coordinates, known as algebraic privileged coordinates, can be employed. One such effective construction method is Bellaïche's algorithm [17].

1. Choose an adapted frame $Y_1, \ldots, Y_n$ at p.
2. Choose coordinates $(y_1, \ldots, y_n)$ centered at p such that $\partial y_i|_p = Y_i(p)$.

3. For $j = 1, \ldots, n$, set

$$z_j = y_j - \sum_{k=2}^{w_{j-1}} h_k(y_1, \ldots, y_{j-1})$$

Here, the function $h_k(y_1, \ldots, y_{j-1})$ is defined for $k = 2, \ldots, w_{j-1}$ as:

$$h_k(y_1, \ldots, y_{j-1}) = \sum_{\substack{|\alpha|=k, \\ w(\alpha)<w_j}} Y_1^{\alpha_1} \ldots Y_{j-1}^{\alpha_{j-1}} \left(y_j - \sum_{q=2}^{k-1} h_q(y) \right) (p) \frac{y_1^{\alpha_1}}{\alpha_1!} \ldots \frac{y_{j-1}^{\alpha_{j-1}}}{\alpha_{j-1}!}$$

Here, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, and $w(\alpha)$ denotes the weight of the multi-index α.

The resulting privileged coordinates, denoted as $(z_1, z_2, \ldots, z_n)$, are derived from the original coordinates $(y_1, y_2, \ldots, y_n)$ using an expression of the form in step 3 where each z_i is obtained by adding a polynomial function of the preceding variables $y_1, y_2, \ldots, y_{i-1}$ to the corresponding y_i value.

Algebraic privileged coordinates offer an alternative representation of points on a manifold by incorporating polynomial functions of the preceding variables, enabling a different perspective for analyzing and describing the manifold's properties. This construction is particularly useful in scenarios where computing exponential mappings or integrating vector field flows may be computationally intensive or impractical.

Example: Mobile robot with trailer

Example 5.7. In this example, we consider a mobile robot with a trailer [8]. The vector field and algebraic coordinates of the system are derived and analyzed.

The dynamics of the system are given by a set of differential equations,

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \\ -\dfrac{\sin\phi}{l_t} \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 - \dfrac{l_r \cos\phi}{l_t} \end{bmatrix} u_2$$

$$X_1 = \cos\theta \partial_x + \sin\theta \partial_y - \frac{\sin\phi}{l_t} \partial_\phi$$

$$X_2 = \partial_\theta - \left(1 + \frac{l_r \cos\phi}{l_t} \right) \partial_\phi$$

$$\text{where,} \quad X = (x, y, \theta, \phi) \in \mathcal{M} = \mathbb{R}^2 \times \mathcal{S}^1 \times \mathcal{S}^1$$

The vector fields are represented by the differential equations, where the state of the system is denoted by $X = (x, y, \theta, \phi)$. The control inputs of the system are u_1 and u_2, and the lengths of the robot and trailer are denoted by l_r and l_t respectively.

Next, we compute the Lie derivatives and weights of the vector field at the origin. The Lie derivatives help us understand the behavior of the system and identify privileged coordinates. We obtain the Lie derivatives up to the second order, resulting in the vector fields $\Delta^1(p)$, $\Delta^2(p)$, and $\Delta^3(p)$. Consider $l_r = l_t = 1$, the distribution of the vector field at the origin p is

$$\begin{aligned}
\Delta^1(p) &= \{X_1(p), X_2(p)\} = \left\{\partial_x, \partial_\theta - 2\partial_\phi\right\} \\
X_3 &= [X_1, X_2] = \sin\theta\partial_x - \cos\theta\partial_y - (1+\cos\phi)\partial_\phi \\
\Delta^2(p) &= \{X_1(p), X_2(p), X_3(p)\} = \left\{\partial_x, \partial_\theta - 2\partial_\phi, -\partial_y - 2\partial_\phi\right\} \\
X_4 &= [X_1, X_3] = -(1+\cos\phi)\partial_\phi \\
\Delta^3(p) &= \{X_1(p), X_2(p), X_3(p), X_4(p)\} = \left\{\partial_x, \partial_\theta - 2\partial_\phi, -\partial_y - 2\partial_\phi, -2\partial_\phi\right\}
\end{aligned}$$

Based on the given distribution of vector fields at the origin p, we can conclude that the system satisfies Chow's condition. Therefore, we can affirm that the system is reachable locally around the origin. Moreover, by applying Chow-Rashevsky's theorem, we can establish that the system is globally reachable since the underlying manifold $\mathcal{M}$ is connected.
The non-holonomic order of the each coordinate at p is,

$$\mathrm{ord}_p(x) = 1 \qquad \mathrm{ord}_p(y) = 2 \qquad \mathrm{ord}_p(\theta) = 1 \qquad \mathrm{ord}_p(\phi) = 1$$

The weights of this example is,

$$(w_1, w_2, w_3, w_4) = (1, 1, 2, 3)$$

Step 1: The adapted frames X_1, X_2, X_3, X_4 at p,

$$X_1(p) = \partial_x \qquad X_2(p) = \partial_\theta - 2\partial_\phi \qquad X_3(p) = -\partial_y - 2\partial_\phi \qquad X_4(p) = -2\partial_\phi$$

Step 2: Change of coordinates- To analyze the system further, we introduce a change of coordinates using a Jacobian matrix. The new vector fields are expressed in terms of the transformed coordinates. We define $P = [X_1(p), X_2(p), X_3(p), X_4(p)]$,

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & -2 & -2 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & -1 & -\frac{1}{2} \end{bmatrix}$$

We use Jacobian matrix to change the vector field from (x, y, θ, ϕ) coordinate to the new coordinate (z_1, z_2, z_3, z_3).

$$X = P^{-1}Z = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & -1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

$$x_1 = z_1 \qquad x_2 = -z_3 \qquad x_3 = z_2 \qquad x_4 = -2z_1 - z_3 - \frac{1}{2}z_4$$

Thus, the new vector fields are given by

$$Z_1 = \cos z_2 \partial_{z_1} - \sin z_2 \partial_{z_3} + \left(\sin z_2 - \frac{\sin 2(z_2 + z_3 + z_4)}{2}\right)\partial_{z_4}$$

$$Z_2 = \partial_{z_2} + \left(-\frac{1}{2} + \frac{\cos 2(z_2 + z_3 + z_4)}{2}\right)\partial_{z_4}$$
$$Z_3 = \sin z_2 \partial_{z_1} + \cos z_2 \partial_{z_3} - \left(-\frac{1}{2} + \cos(z_2) - \frac{\cos 2(z_2 + z_3 + z_4)}{2}\right)\partial_{z_4}$$
$$Z_4 = \left(\frac{1}{2} + \frac{\cos 2(z_2 + z_3 + z_4)}{2}\right)\partial_{z_4}$$

Step 3: New Algebric coordinates

We then proceed to construct the algebraic coordinates by examining the derivatives of the coordinate function along the new vector fields Z_1 and Z_2. Using the derivatives, we calculate the algebraic coordinates and obtain the transformed vector fields $\bar{Z}_1$, $\bar{Z}_2$, $\bar{Z}_3$, and $\bar{Z}_4$ at the origin.

1. The weight of the third coordinate $w(z_3)$ is 2. Hence, we need the first order derivative of z_3 along the vector fields Z_1 and Z_2 to construct the algebraic coordinate,

$$h_1 := Z_1 z_3 = -\sin z_2$$
$$h_2 := Z_2 z_3 = 0$$

At the origin: $h_1(p) = 0 \quad h_2(p) = 0$. Hence, the new coordinate $(\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4) \in \mathcal{P}$ is ,

$$\bar{z}_1 = z_1$$
$$\bar{z}_2 = z_2$$
$$\bar{z}_3 = z_3 + h_1(p)z_1 + h_2(p)z_2$$
$$\implies \quad \bar{z}_1 = z_1, \quad \bar{z}_2 = z_2, \quad \bar{z}_3 = z_3$$

2. The weight of the fourth coordinate $w(z_4)$ is 3. Hence, we need the second order of z_4 along the vector fields Z_1 and Z_2 to construct the algebraic coordinate,

$$h_{11} := Z_1 Z_1 z_3$$
$$= \sin(z_2)\cos 2(z_4 + z_3 + z_2) - \left(\sin(z_2) - \frac{\sin 2(z_4 + z_3 + z_2)}{2}\right)\cos 2(z_4 + z_3 + z_2)$$
$$h_{12} := Z_1 Z_2 z_3$$
$$= \sin(z_2)\sin 2(z_4 + z_3 + z_2) - \left(\sin(z_2) - \frac{\sin 2(z_4 + z_3 + z_2)}{2}\right)\sin 2(z_4 + z_3 + z_2)$$
$$h_{22} := Z_2 Z_2 z_3$$
$$= -\sin 2(z_4 + z_3 + z_2) - \left(\frac{1}{2} + \frac{\cos 2(z_4 + z_3 + z_2)}{2}\right)\sin 2(z_4 + z_3 + z_2)$$

At the origin : $h_{11}(p) = 0, \quad h_{12}(p) = 0, \quad h_{22}(p) = 0$. Hence, the new coordinate $(\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4) \in \mathcal{P}$ is,

$$\bar{z}_1 = z_1$$
$$\bar{z}_2 = z_2$$
$$\bar{z}_3 = z_3 + h_1(p)z_1 + h_2(p)z_2$$
$$\bar{z}_4 = z_3 + \frac{1}{2}(h_{11}(p)z_1^2 + h_{12}(p)z_1 z_2 + h_{22}(p)z_2^2)$$
$$\implies \quad \bar{z}_1 = z_1, \quad \bar{z}_2 = z_2, \quad \bar{z}_3 = z_3 \quad \bar{z}_4 = z_4$$

The new vector fields $\bar{Z}_1, \bar{Z}_2, \bar{Z}_3, \bar{Z}_4$ at the origin p are ,

$$\bar{Z}_1(p) = \partial_{z_1} \qquad \bar{Z}_2(p) = \partial_{z_2} \qquad \bar{Z}_3(p) = \partial_{z_3} \qquad \bar{Z}_4(p) = \partial_{z_4}$$

The non-holonomic order of the new coordinates at p is,

$$\text{ord}_p(\bar{z}_1) = 1 \qquad \text{ord}_p(\bar{z}_2) = 1 \qquad \text{ord}_p(\bar{z}_3) = 2 \qquad \text{ord}_p(\bar{z}_4) = 3$$

The system is defined in the privileged coordinate,

$$\bar{Z}_1 = \cos \bar{z}_2 \partial_{\bar{z}_1} + \sin \bar{z}_2 \partial_{\bar{z}_3} + \left(\sin \bar{z}_2 - \frac{\sin 2(\bar{z}_2 + \bar{z}_3 + \bar{z}_4)}{2} \right) \partial_{\bar{z}_4}$$
$$\bar{Z}_2 = \partial_{\bar{z}_2} - \left(-\frac{1}{2} + \frac{\cos 2(\bar{z}_2 + \bar{z}_3 + \bar{z}_4)}{2} \right) \partial_{\bar{z}_4}.$$

In this example, we consider the dynamics of a robot with a trailer modeled in algebraic coordinates. The choice of algebraic coordinates allows us to describe the system in a privileged coordinate system that simplifies the analysis. By utilizing the privileged coordinate system, we can approximate the system locally, study and analyze the dynamics of the non-holonomic system in a simplified manner.

5.3 First Order Approximation

In the study of vector fields, the concept of first-order approximation plays a crucial role in understanding the behavior of a system near a specific point. It provides an approximation that captures the leading-order terms of the vector fields and enables the analysis of their properties and dynamics.

Definition 5.8. A family of m vector fields $(\hat{X}_1, \dots, \hat{X}_m)$ defined near p is called a *first-order approximation* of $(X_1, \dots, X_m)$ at p if the vector fields $(X_i - \hat{X}_i)$, $i = 1, \dots, m$, are of order ≥ 0 at p.

The definition [17] states that the first-order approximation consists of vector fields that differ from the original vector fields by terms of at least the first order in a neighborhood around the point of interest. It allows us to focus on the dominant contributions of the vector fields and neglect higher-order effects in the analysis.

Example 5.9. Consider the Wheeled mobile robot in Example (3.4). The mathematical model of the wheeled robot is given by

$$\dot{q} = X_1 u_1 + X_2 u_2$$
$$X_1 = \cos \theta \partial_x + \sin \theta \partial_y, \qquad X_2 = \partial_z$$

Using the Taylor series approximation of sin and cos at the origin p:

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$$
$$\cos \theta = 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots$$

We can determine the order of the vector fields X_1 and X_2 at p as follows:

$$\text{ord}_p(X_1) = -\infty$$

$$\text{ord}_p(X_2) \geq -1$$

Now, let's consider the first-order approximation:

$$X_1 = (1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots)\partial_x + (\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots)\partial_y$$

$$\text{Let,} \quad \hat{X}_1 = \partial_x + \theta\partial_y$$

$$X_1 - \hat{X}_1 = (\frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots)\partial_x + (-\frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots)\partial_y$$

$$\text{ord}_p(X_1 - \hat{X}_1) \geq -2$$

Thus, $\hat{X}_1$ serves as the *first-order approximation* of X_1 at p. The first-order approximation allows us to capture the dominant terms of X_1 near p while neglecting higher-order effects, facilitating the analysis of the system's behavior in the vicinity of the point of interest.

The concept of first-order approximation provides a valuable tool for analyzing the behavior of vector fields and understanding their dynamics near specific points. By considering the dominant terms and neglecting higher-order effects, we can gain insights into the essential characteristics of the vector fields and simplify the analysis of complex systems. This approach is widely used in various fields, including control theory, differential geometry, and mathematical physics, to study the behavior of systems in local neighborhoods and derive useful conclusions about their properties.

5.4 Nilpotent Approximation

Let $(z_1, \ldots, z_n)$ be a system of privileged coordinates at point p. In these coordinates, every vector field X_i can be represented as a Taylor expansion of the form:

$$X_i(z) \sim \sum_{\alpha,j} a_{\alpha,j} z^\alpha \partial_{z_j},$$

where $w(\alpha) \geq w_j - 1$ if $a_{\alpha,j} \neq 0$. Here, α represents a multi-index, j ranges from 1 to n, and $a_{\alpha,j}$ are coefficients.

By grouping together the monomial vector fields of the same weighted degree, we can express X_i as a series:

$$X_i = X_i^{(-1)} + X_i^{(0)} + X_i^{(1)} + \ldots,$$

where $X_i^{(s)}$ is a homogeneous vector field of degree s.

Now, consider the family of vector fields $(\hat{X}_1, \ldots, \hat{X}_m)$ defined as $\hat{X}_i = X_i^{(-1)}$ for $i = 1, \ldots, m$. This family serves as a first-order approximation of $(X_1, \ldots, X_m)$ at point p and generates a nilpotent Lie algebra of step $r = w_n$.

In the context of Lie algebras, a Lie algebra $Lie(X_1, \ldots, X_m)$ is said to be nilpotent of step s if all brackets X_I of length $|I|$ greater than s are zero.

Therefore, the family $(\hat{X}_1, \ldots, \hat{X}_m)$ is referred to as the (homogeneous) nilpotent approximation of $(X_1, \ldots, X_m)$ at point p [17], associated with the coordinates z. It provides a useful approximation that simplifies the analysis of the vector fields and their properties in the vicinity of the point p.

We can compute the nilpotent approximation of example of a wheeled mobile robot from Example (4.7) .

Example 5.10. The given mathematical model describes the dynamics of a robot, where the state vector X belongs to $\mathbb{R}^2 \times \mathcal{S}^1$, and the control input vector u belongs to $\mathbb{R}^2$. The dynamics are represented by the equations:

$$\begin{aligned}
\dot{X} &= X_1 u_1 + X_2 u_2 \\
X_1 &= \cos\theta \partial_x + \sin\theta \partial_y \\
X_2 &= \partial_\theta
\end{aligned}$$

Here, θ represents the orientation of the robot, and the vectors X_1 and X_2 are the corresponding vector fields associated with the state components. The Lie bracket $X_3 = [X_1, X_2]$ is defined as $\sin\theta\partial_x - \cos\theta\partial_y$, indicating that the system is regular at every point, and the weights of the vector fields are $(1, 1, 2)$ at every point.

In this system, the coordinate system (x, y, θ) is considered privileged, which allows us to use Taylor series approximations for $\sin\theta$ and $\cos\theta$. By expanding these trigonometric functions, we obtain the homogeneous components of the vector fields:

$$\begin{aligned}
X_1^{(-1)} &= \partial_x + \theta\partial_y, \quad X_1^{(0)} = 0, \quad X_1^{(1)} = \frac{\theta^2}{2!}\partial_x - \frac{\theta^3}{3!}\partial_y, \dots \\
X_2^{(-1)} &= \partial_\theta
\end{aligned}$$

Hence, the homogeneous nilpotent approximation of (X_1, X_2) at the origin in the privileged coordinates (x, y, θ) is given by:

$$\hat{X}_1 = \partial_x + \theta\partial_y \qquad \hat{X}_2 = \partial_\theta$$

It can be verified that the Lie brackets of length 3 of these vector fields are zero, meaning that $[\hat{X}_1, [\hat{X}_1, \hat{X}_2]] = [\hat{X}_2, [\hat{X}_1, \hat{X}_2]] = 0$. Consequently, the Lie algebra $Lie(\hat{X}_1, \hat{X}_2)$ is nilpotent of step 2.

It's important to note that the homogeneous nilpotent approximation is not uniquely defined by the vector field tuple $(X_1, \dots, X_m)$ because it depends on the chosen system of privileged coordinates. However, if $(\hat{X}_1, \dots, \hat{X}_m)$ and $(\hat{X}_1', \dots, \hat{X}_1')$ are the nilpotent approximations associated with two different systems of coordinates, then their Lie algebras $Lie(\hat{X}_1, \dots, \hat{X}_m)$ and $Lie(\hat{X}_1', \dots, \hat{X}_m')$ are isomorphic, meaning they possess the same algebraic structure despite being represented in different coordinate systems.

In summary, the nilpotent approximation is a method that simplifies the analysis of vector fields by providing a first-order approximation using privileged coordinates. It allows expressing vector fields as Taylor expansions and generates a family of vector fields called the homogeneous nilpotent approximation. This approximation captures the essential behavior of the vector fields at the point of interest and facilitates further analysis and computations.

Example 5.11. Robot with a Trailer: Continuation

From the previous example 5.7, the vector fields of the robot with a trailer system in the privileged coordinate are given by:

$$\begin{aligned}
\bar{Z}_1 &= \cos\bar{z}_2\partial_{\bar{z}_1} + \sin\bar{z}_2\partial_{\bar{z}_3} + \left(\sin\bar{z}_2 - \frac{\sin 2(\bar{z}_2 + \bar{z}_3 + \bar{z}_4)}{2}\right)\partial_{\bar{z}_4} \\
\bar{Z}_2 &= \partial_{\bar{z}_2} - \left(-\frac{1}{2} + \frac{\cos 2(\bar{z}_2 + \bar{z}_3 + \bar{z}_4)}{2}\right)\partial_{\bar{z}_4}.
\end{aligned}$$

To approximate the system, we take the Taylor series polynomial based the weights of the vector fields (1, 1, 2, 3). By calculating the nilpotent approximations of the transformed coordinates, we obtain the vector fields $\hat{Z}_1$ and $\hat{Z}_2$.

1. The first-order Taylor series polynomial of the coefficients of the vector fields $\bar{Z}_1$ and $\bar{Z}_2$ in terms of $\bar{z}_1$:

$$\hat{Z}_1 = \partial_{\bar{z}_1} \qquad \hat{Z}_2 = 0$$

2. The first-order Taylor series polynomial of the coefficients of the vector fields $\bar{Z}_1$ and $\bar{Z}_2$ in terms of $\bar{z}_2$:

$$\hat{Z}_1 = 0 \qquad \hat{Z}_2 = \partial_{\bar{z}_2}$$

3. The second-order Taylor series polynomial of the coefficients of the vector fields $\bar{Z}_1$ and $\bar{Z}_2$ in terms of $\bar{z}_3$:

$$\hat{Z}_1 = -\bar{z}_2\partial_{\bar{z}_3} \qquad \hat{Z}_2 = 0$$

4. The third-order Taylor series polynomial of the coefficients of the vector fields $\bar{Z}_1$ and $\bar{Z}_2$ in terms of $\bar{z}_4$:

$$\hat{Z}_1 = -\bar{z}_3\partial_{\bar{z}_4} \qquad \hat{Z}_2 = -\bar{z}_2^2\partial_{\bar{z}_4}$$

Thus the nilpotent approximation of the robot with the trailer is given by:

$$\begin{aligned} \hat{Z}_1 &= \partial_{\bar{z}_1} - \bar{z}_2\partial_{\bar{z}_3} - \bar{z}_3\partial_{\bar{z}_4} \\ \hat{Z}_2 &= \partial_{\bar{z}_2} - \bar{z}_2^2\partial_{\bar{z}_4} \end{aligned} \tag{5.5}$$

This example highlights the process of obtaining nilpotent approximations from the privileged system of coordinates. The nilpotent approximations $\hat{Z}_1$ and $\hat{Z}_2$ provide simplified representations of the dynamics of the robot with a trailer system, allowing for a more manageable analysis of its behavior.

6 Nilpotent Approximation of 3-body Space robot

6.1 Model Setup

For the thesis, we investigate a scenario involving a 3-body robot floating freely, as depicted in Fig. 6.1. In this setup, the body in the middle represents the main space structure (a satellite), while the other two bodies serve as manipulator links. An intriguing control problem arises when the satellite's attitude cannot be controlled using gas jets, and the only available control inputs for reconfiguring the space structure are the manipulator joint torques, which act as internal generalized forces. This approach aims to minimize fuel consumption by avoiding the use of satellite actuators. As we will demonstrate, it is generally feasible to alter the entire structure's configuration solely by manipulating the manipulator joints.

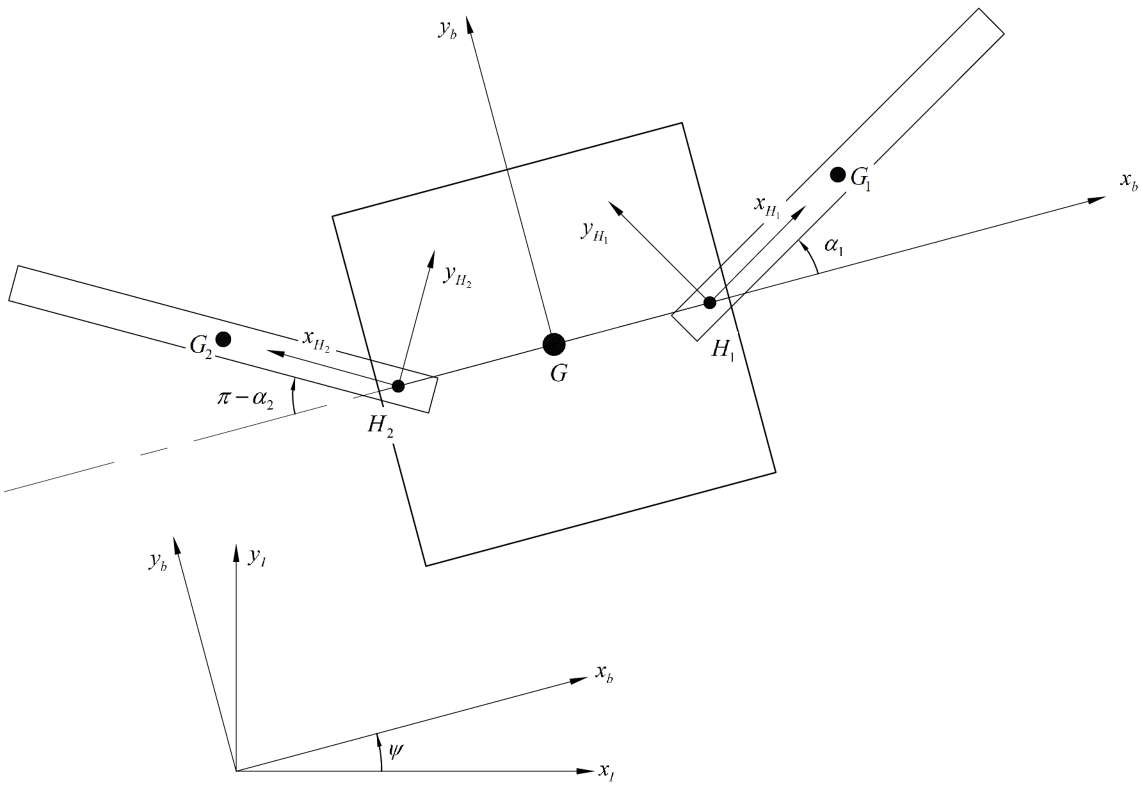

Figure 6.1: Three-body Space Robot

Coordinate Frames

We establish a coordinate system for analysis, assuming an inertial frame at the space station I denoted as (x_I, y_I, z_I). The center of gravity of the entire robot coincides with the center of gravity of the main body, denoted as G. The body frame is defined at the center of gravity C, where the x_b-axis aligns with the right side of the manipulator, the z_b-axis represents the heading angle of the satellite, and the y_b-axis points towards the top of the satellite. Additionally, two hinge points, H_1 and H_2, introduce hinge frames $(x_{H_1}, y_{H_1}, z_{H_1})$ and $(x_{H_2}, y_{H_2}, z_{H_2})$, respectively, which are parallel to the body frame when no rotation occurs in any axis.

Mass and Inertia

The model includes the mass and inertia properties of the main body and the manipulators. The mass of the main body is denoted as m_3, while the masses of the right and left manipulators are represented as m_1 and m_2, respectively. Furthermore, the inertia matrices of the main body, right manipulator, and left manipulator are denoted as I_3, $I_1^{H_1}$, and $I_2^{H_1}$, respectively.

Vector Definitions

We define various vectors within the model. The vector from the center of gravity G to hinge point H_1 is denoted as $r_{h_1}^b$ in the body frame. Similarly, the distance from H_1 to the center of gravity of the right manipulator G_1 is represented as $r_{m_1}^{H_1}$ and defined in the hinge frame $(x_{H_1}, y_{H_1}, z_{H_1})$. Analogously, the vector from G to hinge point H_2 is denoted as $r_{H_2}^b$ in the body frame, and the distance from H_2 to the center of gravity of the left manipulator G_2 is represented as $r_{m_2}^{H_1}$ in the hinge frame $(x_{H_2}, y_{H_2}, z_{H_2})$.

State Variables

The state variables of the model consist of the rotation angles along the (x_I, y_I, z_I) axes, denoted as (ϕ, θ, ψ), respectively. Additionally, (α_1, β_1) represent the rotations of the right manipulator about hinge axis H_1 with respect to the (z_b, y_b) axes, while (α_2, β_2) represent the rotations of the left manipulator about hinge axis H_2 with respect to the (z_b, y_b) axes.

Control Inputs

The model incorporates four control inputs. u_1 and u_2 represent the angular velocities of the right manipulator, while u_3 and u_4 represent the angular velocities of the left manipulator, both with respect to their respective hinge axes.

6.2 Mathematical Model

6.2.1 Assumptions

In the analysis of the multibody system, the following assumptions are made:

No External Forces: In the absence of external forces, and considering no gravity or dissipation forces, the linear and angular momenta of the system are conserved.

Initial Momentum: Initially, the linear and angular momenta of the system are assumed to be zero.

6.2.2 Rotation Matrix

In order to describe the orientations of different frames in the system, we introduce rotation matrices. Specifically, we define the following rotation matrices:

1. The rotation matrix $R_{H_1}^b$ represents the rotation from the first hinge frame $(x_{H_1}, y_{H_1}, z_{H_1})$ to the body frame (x_b, y_b, z_b). This rotation is achieved by first rotating about the y_{H_1} axis, followed by a rotation about the z_{H_1} axis. The resulting rotation matrix $R_{H_1}^b$ can be expressed as:

$$R_{H_1}^b = \begin{bmatrix} \cos\alpha_1 & -\sin\alpha_1 & 0 \\ \sin\alpha_1 & \cos\alpha_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta_1 & 0 & -\sin\beta_1 \\ 0 & 1 & 0 \\ \sin\beta_1 & 0 & \cos\beta_1 \end{bmatrix}$$

2. Similarly, the rotation matrix $R^b_{H_2}$ describes the rotation from the second hinge frame $(x_{H_2}, y_{H_2}, z_{H_2})$ to the body frame (x_b, y_b, z_b). It involves rotating first about the y_{H_2} axis, followed by a rotation about the z_{H_2} axis. The resulting rotation matrix $R^b_{H_2}$ is given by:

$$R^b_{H_2} = \begin{bmatrix} \cos\alpha_2 & -\sin\alpha_2 & 0 \\ \sin\alpha_2 & \cos\alpha_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta_2 & 0 & -\sin\beta_2 \\ 0 & 1 & 0 \\ \sin\beta_2 & 0 & \cos\beta_2 \end{bmatrix}$$

3. Additionally, we define the rotation matrix $R^b_{b_I}$, which represents the rotation from the body frame (x_b, y_b, z_b) to the inertial frame (x_I, y_I, z_I). This rotation is achieved using the Euler $Z-Y-X$ rotation sequence, where we first rotate about the z_b axis, followed by a rotation about the y_b axis, and finally a rotation about the x_b axis. The resulting rotation matrix $R^b_{b_I}$ can be expressed as [25]:

$$R^I_b = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta & 0 \\ 0 & \cos\phi & -\sin\phi & \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta & \end{bmatrix}$$

These rotation matrices play a crucial role in relating the orientations of different frames within the system, allowing us to describe the transformations between them accurately.

6.2.3 Conservation of Linear Momentum

We define the law of conservation of linear momentum with respect to the inertial frame and it can be expressed as:

$$m_a \dot{r}^I_G + m_2 \dot{r}^I_{G_1} + m_2 \dot{r}^I_{G_2} = 0 \tag{6.1}$$

Here, r_G represents the vector from the inertial frame I to the center of gravity G, r_{G_1} represents the vector from I to the center of gravity G_1 of the second body, and r_{G_1} represents the vector from I to the center of gravity G_2 of the third body.

The equation (6.1) is integrable, resulting in:

$$m_a r^I_G + m_2 r^I_{G_1} + m_2 r^I_{G_2} = c \tag{6.2}$$

where c is a constant vector. Consequently, the conservation of linear momentum imposes three holonomic constraints, indicating that the system's center of mass remains stationary.

6.2.4 Conservation of angular momentum

The conservation of angular momentum is another important principle governing the dynamics of the system. It can be expressed as in the body frame as:

$$I^b_1 \omega^b_1 + m_1(r^b_1 \times \dot{r}^b_1) + I^b_2 \omega^b_2 + m^b_2(r^b_2 \times \dot{r}^b_2) + I^b_3 \omega^b_3 = 0 \tag{6.3}$$

In this equation, I^b_1 and I^b_2 represent the inertia matrices of the right manipulators, ω^b_1 and ω^b_2 are their respective angular velocities in the body frame. The terms $m_1(r^b_1 \times r^b_1)$ and $m^b_2(r^b_2 \times \dot{r_2}^b)$ account for the angular momentum contributions due to the linear motions of the manipulators. Here, r^b_1 represents the vector from the center of gravity of the right manipulator G_1 to the center of gravity G of the body, expressed in the body frame. Similarly, r^b_2 represents the vector from the center of gravity of the right manipulator G_2 to the center of gravity G of the body, also written in the body frame.

The conservation of angular momentum, as expressed by Equation (6.3), ensures that the total angular momentum of the system remains constant in the absence of external torques or other dissipative forces. By considering the contributions from the individual manipulators and their relative motions, we can gain insights into the rotational dynamics and stability of the system. The conservation of angular momentum, as expressed in Equation (6.3), introduce non-integrable constraints and make a system non-holonomic.

6.2.5 Mathematical Model

In this section, we present the mathematical model of the space robot, incorporating the conservation of angular momentum. The model is described in the body frame and subsequently transformed to the inertial frame using the rotation matrix R_b^I.

The kinematic equation for the angular velocity of the main body in the body frame is given by:

$$\omega_3^b = -(I_3^b)^{-1}\left(I_1^b R_{H_1}^b \omega_1^{H_1} + m_1(r_1^b \times \dot{r}_1^b) + I_2^b R H_2{}^b \omega_2^{H_2} + m_2^b(r_2^b \times \dot{r}_2^b)\right) \tag{6.4}$$

Here, ω^{H_1} represents the angular velocity control input for the right manipulator, and ω^{H_2} represents the angular velocity control input for the left manipulator. The cross products involve the control input terms associated with them.

To obtain the model in the inertial frame, we use the rotation matrix R_b^I to transform the body frame to the inertial frame. The resulting equations for the Euler angles (ϕ, θ, ψ) in the inertial frame are given by:

$$\dot{q} = \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = -R_b^I (I_3^b)^{-1}\left(I_1^b R_{H_1}^b \omega_1^{H_1} + m_1(r_1^b \times \dot{r}_1^b) + I_2^b R H_2{}^b \omega_2^{H_2} + m_2^b(r_2^b \times \dot{r}_2^b)\right) \tag{6.5}$$

The state vector of the mathematical model for the space robot is defined as:

$$q = \begin{bmatrix} \phi \ \theta \ \psi \ u_1 \ u_2 \ u_3 \ u_4 \end{bmatrix} \in \mathcal{S}^3 \times \mathbb{R}^4$$

Here, ϕ, θ, and ψ represent the Euler angles that describe the orientation of the robot in the inertial frame. The Euler angles provide a way to express the robot's orientation by specifying the rotation angles about three orthogonal axes. The first angle ϕ represents the rotation about the x_I-axis, the second angle θ represents the rotation about the y_I-axis, and the third angle ψ represents the rotation about the z_I-axis.

The control inputs u_1, u_2, u_3, and u_4 correspond to the angular velocities of the robot's right and left manipulator about specific axes. Specifically, u_1 represents the angular velocity about the z_{H_1} axis (right manipulator's first axis of rotation), u_2 represents the angular velocity about the y_{H_1} axis (right manipulator's second axis of rotation), u_3 represents the angular velocity about the z_{H_2} axis (left manipulator's first axis of rotation), and u_4 represents the angular velocity about the y_{H_2} axis (left manipulator's second axis of rotation).

The state vector q belongs to the state space $\mathcal{S}^3 \times \mathbb{R}^4$, which combines the three-dimensional space of possible orientations (represented by ϕ, θ, and ψ) with the space of angular velocities (represented by u_1, u_2, u_3, and u_4). This state space captures the complete information necessary to describe the robot's current orientation and the rates at which it is changing.

By using this mathematical model, it is possible to simulate and analyze the behavior of the space robot's orientation over time. The control inputs u_1, u_2, u_3, and u_4 influence the rotational dynamics of the robot, allowing for the execution of desired maneuvers and tasks. The state vector q represents the current orientation state of the robot, enabling the tracking of its orientation throughout the simulation or control process.

Understanding and utilizing this mathematical model facilitates the study of the space robot's dynamics, evaluation of its performance under different control inputs, and design of control strategies to achieve specific orientations and execute complex tasks in space exploration or other related domains.

6.3 Nilpotent Approximation of the Space Robot

To derive the nilpotent approximation for the space robot, we first need to calculate the privileged coordinates and then determine the corresponding vector fields.

In order to obtain the privileged coordinates, we can use algebraic coordinate to transform the original coordinates into a set of privileged coordinates $(z_1, \ldots, z_n)$ at a given point p. These privileged coordinates provide a convenient representation for analyzing the vector fields near the point p.

Once we have the privileged coordinates, we can express each vector field X_i in terms of a Taylor expansion. Next, we group the monomial vector fields of the same weighted degree and express X_i as a series:

$$X_i = X_i^{(-1)} + X_i^{(0)} + X_i^{(1)} + \ldots,$$

where $X_i^{(s)}$ is a homogeneous vector field of degree s. The degree corresponds to the order of the Taylor expansion, with $X_i^{(-1)}$ representing the linear terms.

Finally, we define the family of vector fields $(\hat{X}_1, \ldots, \hat{X}_m)$ as $\hat{X}_i = X_i^{(-1)}$ for $i = 1, \ldots, m$. This family serves as a first-order approximation of $(X_1, \ldots, X_m)$ at point p and generates a nilpotent Lie algebra of step $r = w_n$.

We compute the approximation in the Maple using Differential Geometry package [6]. The DifferentialAlgebra package builds upon the differential algebra theories developed by Ritt and Kolchin, implementing their ideas in an algorithmic framework. It allows for the elimination of redundant variables and equations from systems of polynomial differential equations, leading to simplified forms that are easier to analyze and manipulate. Additionally, the package facilitates the computation of formal power series solutions for these equations, enabling the study of their behavior and properties [6].

By leveraging the capabilities of the DifferentialAlgebra package, researchers and practitioners can effectively tackle complex systems of polynomial differential equations encountered in various scientific and engineering fields. The package provides powerful tools for understanding the algebraic and differential structure of these equations and extracting meaningful information from them.

6.3.1 Nilpotent Approximation in Maple

Before implementation, download the *DifferentialGeometry* package from the website. Follow these steps to implement the approximation in Maple:

1. **Setting up the environment:**
   The `restart` command clears any previously defined variables and functions.
   The `libname` command sets the path to a specific directory where additional Maple libraries or packages are stored.

Several packages are loaded using the `with` command: `DifferentialGeometry`, `LieAlgebras`, `Tensor`, `LinearAlgebra`, and `MTM`.

2. **Defining the configuration space:**
The variable n is set to 7, representing the number of coordinates in the configuration space. The `DGEnvironment[Coordinate]` command defines the coordinates of the space robot using the symbols `[a, b, c, a1, a2, b1, b2]` and associates them with a manifold M. The vectors q, dq, and Dq are defined as placeholders for the configuration, velocity, and derivative variables, respectively.
3. **Defining parameters and the model:**
Assign specific values to various parameters such as masses (m_1, m_2, m_3) and moments of inertia (I_1, I_2, I_3).
Define the vector fields of the mathematical model in Maple.
4. **Calculating Lie brackets:**
Compute the Lie brackets of the variables $X[1]$ to $X[7]$ using the `LieBracket` function and substitute the variables `[a, b, c, a1, a2, b1, b2]` with the corresponding values `[0, 0, Pi/6, 0, Pi/3, 0, 0]`.
5. **Building a frame:**
Use `DGbasis` to create a frame using the initial basis vectors.
6. **Defining the new coordinate system:**
The command `DGEnvironment[Coordinate]([x1, x2, x3, x4, x5, x6, x7], M1)` creates a new coordinate system with variables `[x1, x2, x3, x4, x5, x6, x7]` associated with the manifold $M1$.
7. **Transforming coordinates:**
Transform the original coordinates `[a, b, c, a1, a2, b1, b2]` to the new coordinates `[x1, x2, x3, x4, x5, x6, x7]` using the Jacobian Matrix:
`Y[i]:=PushPullTensor(A1,InvA1,X[i])`
This calculates the Jacobian and transforms the initial vector field to the new coordinate system.
8. **Algebraic Coordinate:**
Compute the Lie derivatives of the states along the vector fields using `LieDerivative` and compute the polynomial based on the weights. Obtain the privileged coordinate.
9. **Taylor series expansion:**
Expand the vector fields of the privileged coordinate into Taylor series representations using the `mtaylor` command.

Thus we compute the nilpotent system of the complex system in Maple.

6.3.2 Exponential First Kind Coordinate

We use the nilpotent system from Maple and transform it to the exponential first kind coordinate.

1. **Compute the Flow of the Vector Field:**
We can compute the flow of the linear combination of the vector field using the command:
`V = z1*Y1 + z2*Y2 + z3*Y3 + z4*Y4 + z5*Y5 + z6*Y6 + z7*Y7`
`F:=Flow(V,t)`
2. **Defining the New Coordinate System:**
The command `DGEnvironment[Coordinate]([z1, z2, z3, z4, z5, z6, z7], M2)` creates a new coordinate system with variables `[z1, z2, z3, z4, z5, z6, z7]` associated with the manifold $M2$.
3. **Transforming Coordinates:**

Transform the original coordinates `[x1, x2, x3, x4, x5, x6, x7]` to the new coordinates
`[z1, z2, z3, z4, z5, z6, z7]` using the Jacobian Matrix:
`Z[i]:=PushPullTensor(F,InvF,Y[i])`. This calculates the Jacobian and transforms the initial vector field to the new coordinate system.

Thus, we compute the exponential first kind coordinate of the nilpotent system in Maple.

6.3.3 Exponential Second Kind Coordinate

We use the nilpotent system from Maple and transform it to the exponential second kind coordinate.

1. **Compute the Flow of the Vector Field:**
   We can compute the flow of the each vector field using the command:
   `F1:=Flow(z1*Y1,t)` ... `F7:=Flow(z7*Y7,t)`
2. **Composition of the Flow:**
   We can compute the composition of the flow using the command:
   `F:=ComposeTransformations(F7,F6,F5,F4,F3,F2,F1)`
3. **Defining the New Coordinate System:**
   The command `DGEnvironment[Coordinate]([z1, z2, z3, z4, z5, z6, z7], M2)` creates a new coordinate system with variables `[z1, z2, z3, z4, z5, z6, z7]` associated with the manifold $M2$.
4. **Transforming Coordinates:**
   Transform the original coordinates `[x1, x2, x3, x4, x5, x6, x7]` to the new coordinates
   `[z1, z2, z3, z4, z5, z6, z7]` using the Jacobian Matrix:
   `Z[i]:=PushPullTensor(F,InvF,Y[i])`. This calculates the Jacobian and transforms the initial vector field to the new coordinate system.

Thus, we compute the exponential second kind coordinate of the nilpotent system in Maple.

In summary, this section provides a detailed overview of the steps involved in constructing a nilpotent system using algebraic coordinates and subsequently transforming it into exponential coordinate systems within Maple. The process begins with the setup of the Maple environment, followed by the definition of the configuration space and specification of model parameters. Subsequently, the computation of Lie brackets and the creation of a frame using initial basis vectors take place. The next step involves defining a new coordinate system and utilizing the Jacobian matrix to transform the original coordinates to the new system. The algebraic coordinate is then computed, and the system dynamics are approximated using Taylor series. Finally, the computation of the exponential first kind coordinate is performed based on the obtained nilpotent system. This entire process is replicated for the exponential second kind coordinate, involving the computation of vector field flows, composition, and coordinate transformation using the Jacobian matrix.

7 Conclusion

In conclusion, this thesis explores the application of differential geometry, Lie algebra, and control theory in the context of non-holonomic systems. It aims to understand and analyze the behavior of these systems by employing the nilpotent approximation method. The thesis covers fundamental concepts such as manifolds, Lie algebra, and non-holonomic systems with their constraints. It introduces privileged coordinates, including algebraic and exponential coordinates, as useful tools for analyzing and approximating these systems. The thesis also discusses approximation theory, emphasizing the importance of first-order approximations and the concept of nilpotent approximation. Additionally, it applies the developed concepts and techniques to a specific case study involving a 3-body space robot, demonstrating their practical relevance. Overall, this research contributes to the understanding of control theory for constrained systems and provides insights into the analysis and approximation of complex mechanical systems. By bridging the disciplines of differential geometry, Lie algebra, and control theory, this thesis opens up possibilities for further advancements in the field of robotics and control systems.

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