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Concept analysis of three-way ordinal matrices

∼ Dissertation Thesis ∼

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Declaration

Hereby I declare that the thesis is my original work.

Some parts of this thesis are based on outcomes of the joint scientific work with Radim Bělohlávek (radim.belohlavek@acm.org) (sections 2.1 to 2.4, and chapter 3); and Radim Bělohlávek (radim.belohlavek@acm.org) and Vilém Vychodil (vilem.vychodil@upol.cz) (chapter 4). All authors have even share in the results and findings contained in the respective parts.

Petr Osička

Věnuji svým rodičům

Preface

The goal of the work summarized in the thesis was to extend an already existing method of relational data analysis to a fuzzy setting. The method in question is Triadic Concept Analysis (TCA), a relatively new method aiming at extraction of hierarchy of clusters from a tree-dimensional table representing a relationship between the collection of objects, the collection of attributes, and collection of conditions. In basic setting, the relationship is a yes-no relationship, that is, the input table describes presence/absence of attributes of objects under conditions. Such a setting is suitable for attributes with clearly defined boundaries, but it is not appropriate for attributes described by vague terms, like “tall” or “cheap”. In order to represent such attributes properly one needs to allow the relationship between objects, attributes, and conditions to be a matter of degree rather than a yes-no relationship. Of course, to analyze this data we cannot use methods suited for ordinary bivalent data. Hence, we need to either develop new methods, or appropriately extend methods suited for ordinary data. For this work, I had chosen the later and extended TCA in a way that allows for analysis of graded data.

The results contained in this thesis can be roughly divided into two parts. The first part, contained in the first three chapters, develops TCA in a fuzzy setting. The chapters provide both, a theoretical treatment of the method, and the description of related algorithms accompanied with experiments. The second part, contained in the last chapter, develops a particular application of TCA to a decomposition of three-way ordinal data. I consider this application very interesting and important.

The results summarized in the thesis are outcome of a joint research with my colleagues at the Department of Computer Science, Palacky University. First and foremost I'd like to thank prof. Radim Bělohlávek who introduced me to the topic, encouraged me and kept me on track the whole time. Without his help this thesis would have hardly existed. My thanks also goes to my colleague Vilém Vychodil, for collaborating with me and for his assistance with creation of some pictures included in this thesis. My thanks belong to my friend and colleague Jan Outrata who helped me with typesetting and provided me with feedback on some parts of the first draft.

I'd like to express the warmest thanks to my parents and to my family who supported me in all possible ways during my studies, and to my friends who always believed in me and encouraged me.

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Contents

Chapter 1

Problem setting

1.1 Introduction

There are areas of human activity in which one needs to analyze large amounts of data. Examples of such data are medical records, whole genomes obtained by genome sequencing, data obtained from social networks, or data obtained from surveys. These datasets are too large or disorganized for people to understand them directly. To overcome this difficulty, methods of data mining aim to discover a small amount of the most essential, possibly unknown information or knowledge from such datasets. The newly obtained information is then more comprehensible by humans.

In the thesis we focus on a Triadic Concept Analysis (TCA), a particular method that aims to extract potentially interesting clusters from data describing relations between objects, attributes and conditions. TCA belongs to a broader class of methods, that are to some extent inspired by our understanding of the manner in which human individuals organize the concepts their encounter. We can view TCA as an extension of Formal Concept Analysis (FCA), a method that belongs to this class too. In order to explain the goals of this work and their motivation, we first introduce both aforementioned methods.

Formal concept analysis FCA is a method of data analysis that deals with object-attribute data, i.e. data describing a relationship between collections of objects and attributes. It aims at extraction of a hierarchically ordered set of clusters, called formal concepts, from the input data. Formal concepts are particular pairs $\langle A, B\rangle$ where A is a set of objects and B is a set of attributes. A and B are maximal in the sense that A is a maximal set of objects having all attributes from B, and vice versa. FCA can be seen as a formalization of traditional theory of concepts. Namely, there is a correspondence between formal concepts and the notion of a concept as a unit of thought in Port-Royal logic, where a concept consists of an extent (objects covered by the concept) and an intent (attributes covered by the concept). For example the extent of a concept "fish" consist of all fishes (e.g. gold fish, salmon) while its intent consists of the properties that fishes have (live underwater, have fins, have branchia). Whence the name formal concept. The set of all formal concepts of a given data, called a *concept lattice*, can be ordered by subconcept-superconcept hierarchy according to which a greater, more general concept covers more objects than a smaller, more specific concept. This ordering roughly corresponds to a way humans organize collections of things by their observed features. For example, the term "mammal" is considered more general than the term "cat", because each cat is a mammal. In this sense, the extent of the concept "cat" is a subset of the extent of the concept "mammal" (in data desribing animals and their features). A concept lattice equipped with the subconcept-superconcept order is indeed a complete lattice and can be easily visualized by its Hasse diagram.

The research on FCA started with Wille's paper [52]. Since then, strong mathematical foundations were developed [30]. A survey on algorithms for computation of formal concepts can be found in [42], currently the most efficient family of algorithms was studied in a series of papers [40, 41, 42]. FCA has many applications as a data mining method in various fields, e.g. [11, 25, 28], as a foundation for other methods of data analysis, most notably the Boolean factor analysis [12], or as a preprocessing tool for other data mining methods [47]. Technical details of FCA are summarized in Section 1.2.2.

Triadic concept analysis TCA is an extension of FCA which takes into account conditions (e.g. time instances, weather conditions, different participants in a survey) in addition to objects and attributes. Thus, instead of two-dimensional tables TCA is concerned with three-dimensional tables that capture a to-have-under relation (objects have attributes under conditions). Triadic concepts are triplets $\langle A, B, C\rangle$, where A is a set of object, B is a set of attributes, and C is a set of conditions, such that A, B, C are maximal in the sense that all objects of A have all attributes of B under all conditions of C. The set of all triadic concepts of a given data, called a *concept trilattice*, can be structured by three quasiorders induced by inclusion of object, attribute, and condition sets and forms a complete trilattice, for details see Section 1.2.3.

The developement of TCA was inspired by works of a philosopher Charles S. Peirce on pragmatism and by his system of categories [43]. TCA was studied in a series of papers [20, 43, 53]. Interesting connections to modal logic were described in [27]. In a recent paper, an application to decomposition of tree-way Boolean matrices was proposed [16].

In basic setting, the input tables to both FCA and TCA contain bivalent attributes, i.e. each table entry is either 1 or 0. The attributes are considered qualitative rather than quantitative. That is, we are concerned with attributes like "being tall" rather than "speed in km/h". In FCA, more general attributes (mostly quantitative and categorical) can be handled using so-called conceptual scaling [30]. However, there are qualitative attributes for which the bivalent approach is not sufficient. For example, consider the attribute "being tall". Such attribute is vague by nature. If we characterize someone as being tall we do not specify precisely how tall that person is. In order to classify people into two sets, the set of tall people (for which "being tall" has value 1) and the set of people that are not tall (and therefore "being tall" has value 0) one has to select a precise threshold. But that is not very appropriate and leads to absurd situations. For example, let's say that the threshold is 180 cm. Then a person with 180.1 cm is classified as tall, but a person with 179.9 cm is classified as not tall. This does not correspond with the meaning of attribute "being tall" at all. The right model for such attributes is a fuzzy set [54]. Using a fuzzy set, one can model "being tall" more smoothly, e. g. a person 180.1 cm tall has this attribute to the degree 0.9 and a person 179.9 cm tall to the degree 0.85. To summarize the previous discussion, in order to work with vague attributes, one has to allow the relationship between objects and attributes (and conditions) to be a matter of degree. For FCA, several generalizations that allowed graded attributes were proposed. We will follow the approach developed independently by Belohlavek [3, 4] and Pollandt [49]. For survey of some of the other approaches, see [10]. For TCA such an extension does not exist at the moment. To develop this extension is the main purpose of this work.

While developing TCA in fuzzy setting we had one its particular application in mind — a decomposition of three-way ordinal data. There has been a growing interest in decompositions of tree-way (or more generally, multiway) data recently. A good overview with many references is [37]. Methods of three-way data decomposition are considered to be important and have applications in many areas, including psychometrics, chemometrics, signal processing, computer vision, neuroscience, and data mining. However, many known methods were designed to work with numeric data (e.g. real numbers) and are not applicable to ordinal data because they distort the meaning of the data [45]. Our main motivation was to contribute to the field of multiway data decomposition with an attempt to fill this gap. Inspired by the approach of utilizing formal concepts as factors in decomposition of ordinal matrices [15], we developed TCA in fuzzy setting in a hope that fuzzy triadic concepts can be used as factors in decomposition of three-way ordinal matrices (and thus provide a generalization of [16]). It turned out that this is indeed possible. The associated results are summarized in Chapter 4.

The work is organized as follows. In the rest of this chapter, we give an overview of the fundamentals of fuzzy sets and fuzzy logic, formal concept analysis, and triadic concept analysis. We do so in order to make the work more self-contained and to unify the notation.

In Chapter 2 we develop triadic concept analysis in fuzzy setting. We study the notions of triadic context, concept forming operators, and triadic concept, and their properties, investigate the structure of concept trilattices and prove a generalization of Theorem 2, and provide an illustrative example. Then we present a theorem dealing with a connection to ordinary TCA and discuss some of its consequences. In the rest of the chapter we develop two algorithms for computation of the set of all triadic fuzzy concepts of a given input data. We conclude with remarks on future research.

In Chapter 3 we study triadic fuzzy Galois connections, the basic mathematical structures behind TCA of data with fuzzy attributes. We present their axiomatization and a representation theorem describing one-to-one relationship between triadic fuzzy Galois connections and ternary fuzzy relations. In the rest of the chapter we focus on representation of triadic fuzzy Galois connections by ordinary triadic Galois connections.

Chapter 4 contains material on decompositions of three-way matrices. First, we describe how triadic fuzzy concepts can be utilized as factors. Then we provide a strong theorem on optimality of decomposition, initial results regarding algorithms, and an illustrative example.

1.2 Preliminaries

1.2.1 Fuzzy sets and fuzzy logic

In this section we recall the fundamental notions from fuzzy logic and fuzzy sets theory. For a more detailed treatment on the material contained in this section we refer the reader to [8, 35].

A concept central to fuzzy logic is the concept of *graded truth.* In fuzzy setting, we allow logical propositions to not only be fully true or fully false, but also partially true. The set L of all truth degrees, which we allow the logical propositions to take as their truth value, needs to be well structured. This is usually done by equipping L with certain operators which play a role of truth functions of logical connectives. The properties of such connectives are then set according to a simple natural requirements having its roots in ordinary mathematical logic and expressing the way multivalued logic should behave [8, 35]. In this work we assume that the truth degrees form a *complete residuated lattice* [51]. A complete residuated lattice $\mathbf{L}$ is a tuple $\langle L, \wedge, \vee, \otimes, \rightarrow, 1, 0\rangle$ such that

(i) $\langle L, \wedge, \vee, 1, 0\rangle$ is a complete lattice with the greatest element 1 and the least element 0, i.e. a partially ordered set, where infima ($\wedge$) and suprema ($\vee$) of arbitrary subset of L exist,

(ii) $\langle L, \otimes \rangle$ is a commutative monoid with the neutral element 1, i.e. $\otimes$ is associative and commutative, and $a \otimes 1 = a$ holds for all $a \in L$,

(iii) the *adjointness property* $a \otimes b \leq c$ iff $b \leq a \rightarrow c$ holds for all $a, b, c \in L$.

The operations $\otimes$ and $\rightarrow$ are taken as truth functions of conjunction and implication, respectively, $\wedge$ and $\vee$ are semantical counterparts of universal and existential quantifiers, respectively. Truth functions of other logical connectives are defined in terms of operations of residuated lattices, e. g. $\leftrightarrow$ (the truth function of the equivalence connective) is defined as

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a).$$

By adjointness property, many important properties of $\otimes$ and $\rightarrow$ can be obtained. As an example we provide the following:

Lemma 1. *Let* $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 1, 0 \rangle$ *be a residuated lattice,* $a, b, c \in L$, *and* I *be an index set. Then it holds:*

- $\otimes$ *is isotone in both arguments,*
- $\rightarrow$ *is antitone in first and isotone in second argument,*
- $a \rightarrow b = 1$ *if and only if* $a \leq b$,
- $a \otimes (a \rightarrow b) \leq b$, $b \leq a \rightarrow (a \otimes b)$,
- $a \otimes b \leq a$, $a \leq b \rightarrow a$,
- $(a \otimes b) \rightarrow c = a \rightarrow (b \rightarrow c)$,
- $a \rightarrow \bigwedge_{i \in I} b_i = \bigwedge_{i \in I} (a \rightarrow b_i)$,
- $\bigvee_{i \in I} a_i \rightarrow b = \bigwedge_{i \in I} (a_i \rightarrow b)$.

Although, we will depend on properties of (complete) residuated lattices thorough this work, presenting their complete list is out of its scope. Instead, we refer the reader to [8].

Complete residuated lattices cover a wide range of structures of truth degrees, including all of the most widely used ones. Perhaps the most common example is the unit interval $L = [0, 1]$ with $\wedge, \vee$ being maximum and minimum, respectively, $\otimes$ being a (left-continuous) t-norm, and $\rightarrow$ given by $a \rightarrow b = \vee\{c \mid a \otimes c \leq b\}$. Particular cases (and in a sense the important ones [8, 35]) are standard Łukasiewicz, Gödel, and product algebras where $\otimes$ is defined by

$$a \otimes b = \begin{cases} \max(0, a+b-1) & \text{(Łukasiewicz algebra)} \\ \min(a, b) & \text{(Gödel algebra)} \\ a \cdot b & \text{(product algebra)} \end{cases}$$

In applications, another common choice of $\mathbf{L}$ is a finite chain equipped with a restriction of a t-norm. For example, $L = \{a_1 = 0, \ldots, a_n = 1\} \subseteq [0, 1]$ with $\otimes$ defined either by $a_i \otimes a_j = a_{\max(i+j-n,0)}$ (Łukasiwicz chain) or as a restriction of Gödel t-norm to L. Residuated lattices are used in several areas of mathematics, most notably in mathematical fuzzy logic [35].

Now we recall the notions related to fuzzy sets and fuzzy relations. A $\mathbf{L}$-set (fuzzy set) A in a universal set X is a map $A : X \to L$. For $x \in X$, $A(x)$ is the degree to which x belongs to A. The set of all $\mathbf{L}$-sets over X is denoted by L^X. A fuzzy set A is also denoted by $\{A(x)/x \mid x \in X\}$, we do not enumerate elements $x \in X$ such that $A(x) = 0$. If there is only one $x \in X$ such that $A(x) \neq 0$, i.e. $A = \{A(x)/x\}$, we call A a *singleton.*

The operations with $\mathbf{L}$-sets are defined componentwise. For example, the union of $\mathbf{L}$-sets $A, B \in L^X$ is defined as $\mathbf{L}$-set $(A \cup B) \in L^X$ such that

$$(A \cup B)(x) = A(x) \vee B(x)$$

for all $x \in X$.

An $\mathbf{L}$-set A is an ordinary subset of an $\mathbf{L}$-set B if $A(x) \leq B(x)$ for all $x \in X$, denoted by $A \subseteq B$. In fuzzy setting, subsethood of $\mathbf{L}$-sets should be a matter of degree. Indeed, the degree $S(A, B)$ to which A is a subset of B is defined as

$$S(A, B) = \bigwedge_{x \in X} A(x) \to B(x) \tag{1.1}$$

It is easy to see (by basic properties of residuated lattices, cf. [8]), that $A \subseteq B$ if and only if $S(A, B) = 1$. The degree of equality of A and B is defined as

$$A \approx B = \bigwedge_{x \in X} A(x) \leftrightarrow B(x). \tag{1.2}$$

For $a \in L$, an *a-cut* of an fuzzy set A is a crisp set ${}^aA \subseteq X$, such that $x \in {}^aA$ if $A(x) \geq a$.

An n-ary $\mathbf{L}$-relation (fuzzy relation) R between sets $U_1, \ldots, U_n$ is an $\mathbf{L}$-set in $U_1 \times \ldots U_n$. A binary fuzzy relation $R \in L^{X \times X}$ is an $\mathbf{L}$-*equivalence* (fuzzy equivalence), if the the following conditions hold for all $x, y, z \in X$:

$$\begin{aligned} &R(x, x) = 1 \quad \text{(reflexivity)}, \\ &R(x, y) = R(y, x) \quad \text{(symmetry)}, \\ &R(x, y) \otimes R(y, z) \leq R(x, z) \quad \text{(transitivity)}. \end{aligned}$$

Moreover, if $R(x, y) = 1$ implies $x = y$, R is an $\mathbf{L}$-*equality* (fuzzy equality). A binary fuzzy relation Q on a set X equipped with an $\mathbf{L}$-equivalence relation $\approx$ is an $\mathbf{L}$-*quasiorder* (fuzzy quasiorder) if it is reflexive and transitive. Note that the 1-cut of fuzzy equality is an ordinary equivalence relation, and that the 1-cut of a fuzzy quasiorder is an ordinary quasiorder relation.

The Cartesian product of fuzzy sets $A_1, \ldots, A_n$ in $X_1, \ldots, X_n$, respectively, is the n-ary fuzzy relation $A_1 \otimes \cdots \otimes A_n$ in $X_1 \times \cdots \times X_n$ defined by

$$(A_1 \otimes \cdots \otimes A_n)(x_1, \ldots, x_n) = A_1(x_1) \otimes \cdots \otimes A_n(x_n) \tag{1.3}$$

For example, the Cartesian product of fuzzy sets $A \in L^X$ and $B \in L^Y$ is the binary relation $A \otimes B$ on $X \times Y$ given by $(A \otimes B)(x, y) = A(x) \otimes B(y)$. It is well known that for fuzzy sets A_1, A_2, B_1, B_2 the following inequality holds

$$S(A_1, A_2) \otimes S(B_1, B_2) \leq S(A_1 \otimes B_1, A_2 \otimes B_2). \tag{1.4}$$

If we take $L = \{0, 1\}$ with $\otimes$ being minimum (the truth function of ordinary conjunction), adjointness yields that $\rightarrow$ is the truth function of ordinary implication. Thus, in this case, $\mathbf{L}$ is the two-element Boolean algebra and all of the above notions coincide with their ordinary counterparts, i.e. fuzzy sets become ordinary sets, fuzzy relations become ordinary relations, and similarly for other notions.

1.2.2 Formal concept analysis

In this section we give an overview of the notions from formal concept analysis and its extension to fuzzy setting. For a more complete treatment on the material contained in this section we refer the reader to [8, 30].

An input to FCA is a two-dimensional table representing the relation between objects and their attributes. An entry in the table contains 1 (or a cross) if the object corresponding to the entry has the attribute corresponding to the entry, otherwise the entry is 0 (or empty space). Technically, the input is represented by a *formal context*. A formal context is a tuple $\langle X, Y, I\rangle$ where X and Y are non-empty sets and I is a binary relation between X and Y. X is then interpreted as the set of objects, Y as the set of attributes. If $\langle x, y\rangle \in I$ we say that object x has attribute y. The relation I induces a pair of *concept forming* (or arrow) operators, $^{\uparrow} : 2^X \leftarrow 2^Y$ and $^{\downarrow} : 2^Y \rightarrow 2^X$, defined for $A \subseteq X$, $B \subseteq Y$ by

$$A^{\uparrow} = \{y \in Y \mid \langle x, y\rangle \in I \text{ for all } x \in A\}$$
$$B^{\downarrow} = \{x \in X \mid \langle x, y\rangle \in I \text{ for all } y \in B\}$$

That is, $^{\uparrow}$ assigns to a set A of objects the set of all attributes shared by all objects from A. In similar manner, $^{\downarrow}$ assigns to a set B of attributes the set of all objects that have all attributes from B. $^{\uparrow}$ and $^{\downarrow}$ form a Galois connection, turning $^{\uparrow\downarrow}$ and $^{\downarrow\uparrow}$ into closure operators on X and Y, respectively [30].

A *formal concept* is a pair $\langle A, B\rangle$, $A \in X$, $B \in Y$ such that $A = B^{\downarrow}$ and $B = A^{\uparrow}$. A and B are called the *extent* and *intent* of $\langle A, B\rangle$. The set of all formal concepts of a formal context $\langle X, Y, I\rangle$ is called the *concept lattice* of $\langle X, Y, I\rangle$, denoted by $\mathcal{B}(X, Y, I)$. Each concept lattice can be partially ordered by a natural concept order (see Section 1.1) defined as

$$\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle \text{ iff } A_1 \subseteq A_2 \text{ iff } B_2 \subseteq B_1, \tag{1.5}$$

A concept lattice ordered by (1.5) is indeed a complete lattice. In the opposite direction the relationship also holds, i.e. each complete lattice is isomorphic to a concept lattice of some formal context [30].

In the literature, several generalizations of FCA that allowed it to handle data with fuzzy attributes have been proposed. We follow the approach developed independently in [3, 4] and [49].

While in the ordinary case the relationship between objects and attributes is a yes-no relationship, in the fuzzy setting, this relationship is a matter of degree. Given a residuated lattice $\mathbf{L}$, an $\mathbf{L}$-*context* (fuzzy context) is a tuple $\langle X, Y, I\rangle$, where X and Y are non-empty sets and I is an $\mathbf{L}$-relation between X and Y. The sets X and Y are interpreted as sets of objects and attributes, respectively. For each $x \in X$ and $y \in Y$, the degree $I(x, y)$ is the degree to which object x has attribute y, or alternatively, the truth degree of proposition "object x has attribute y". A fuzzy context induces a pair of concept-forming operators, $^{\uparrow} : L^X \to L^Y$ and $^{\downarrow} : L^Y \to L^X$ defined for $A \in L^X$ and $B \in L^Y$ by

$$A^{\uparrow}(y) = \bigwedge_{x\in X} A(x) \to I(x, y),$$
$$B^{\downarrow}(x) = \bigwedge_{y\in Y} B(y) \to I(x, y).$$

Now, $A^{\uparrow}(y)$ is the degree to which each object of A has the attribute y, $B^{\downarrow}(x)$ is the degree to which all attributes from B are shared by object x.

An $\mathbf{L}$-concept (fuzzy concept) is a pair $\langle A, B\rangle$, $A \in L^X$ and $B \in L^Y$, such that $A = B^{\downarrow}$ and $B = A^{\uparrow}$. As in the ordinary case, A is called the extent, and B is called the intent of $\langle A, B\rangle$. The set of all fuzzy concepts forms an $\mathbf{L}$-concept lattice (fuzzy concept lattice) $\mathcal{B}(X, Y, I)$. The natural concept order is defined using the ordinary subsethood of extents and intents:

$$\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle \text{ iff } A_1 \subseteq A_2 \text{ iff } B_2 \subseteq B_1,$$

Each fuzzy concept lattice ordered by concept order is a complete lattice. However, the relation in the opposite direction does not hold in general [8].

1.2.3 Concept trilattices

I this section we recall the basics of triadic concept analysis. For more details see [20, 22, 53].

A *triadic context* is a quadruple $\langle X, Y, Z, I\rangle$ where X, Y, and Z are non-empty sets, and I is a ternary relation between X, Y, and Z. X, Y, and Z are interpreted as the sets of objects, attributes, and conditions, respectively; I is interpreted as the "to have-under relation". That is, if $\langle x, y, z\rangle \in I$ we say that object x has attribute y under condition z. For convenience, a triadic context is denoted by $\langle X_1, X_2, X_3, I\rangle$.

From a triadic context, a number of formal (dyadic) contexts can be derived. For $\{i, j, k\} = \{1, 2, 3\}$ and a set $C_k \subseteq X_k$ we obtain a dyadic context

$$\mathbf{K}^{ij}_{C_k} = \langle X_i, X_j, I^{ij}_{C_k}\rangle$$

defined by

$$\langle x_i, x_j\rangle \in I^{ij}_{C_k} \quad \text{iff} \quad \text{for each } x_k \in C_k : x_i, x_j, x_k \text{ are related by } I. \tag{1.6}$$

A triadic context is usually depicted as $|X_3|$ tables corresponding to dyadic contexts $K^{12}_{\{x_3\}}$ for $x_3 \in X_3$, see Table 1.1

We denote the concept forming operators induced by $K^{ij}_{C_k}$ by $^{(i,j,C_k)}$. A *triadic concept* is a triplet of sets $\langle C_1, C_2, C_3\rangle$, $C_1 \subseteq X_1$, $C_2 \subseteq X_2$, and $C_3 \subseteq X_3$, such that for every $\{i, j, k\} = \{1, 2, 3\}$ we have $C_i = C_j^{(j,i,C_k)}$; C_1, C_2, and C_3 are called the *extent*, *intent*, and *modus* of $\langle C_1, C_2, C_3\rangle$, respectively. The set of all triadic concepts is called the *concept trilattice* of $\langle X_1, X_2, X_3, I\rangle$ and denoted by $\mathcal{T}(X_1, X_2, X_3, I)$ or $\mathcal{T}(\mathbf{K})$.

Let $\mathfrak{b}_{ik} : 2^{X_i} \times 2^{X_k} \to \mathcal{T}(\mathbf{K})$ be a map which to a pair of sets $C_i \subseteq X_i$ and $C_k \subseteq X_k$ assigns a triple $\langle A_1, A_2, A_3\rangle$ such that $A_j = C_i^{(i,j,C_k)}$, $A_i = A_j^{(i,j,C_k)}$, and $A_k = A_i^{(i,k,A_j)}$. Then $\langle A_1, A_2, A_3\rangle$ is a triadic fuzzy concept. That is, starting with a pair of sets C_i, C_k, one can obtain a triadic concept using concept forming operators three times.

Now we recall the basic notions and results regarding triordered sets and trilattices [53, 22].

A *triordered set* is a quadruple $\mathbf{V} = \langle V, \lesssim_1, \lesssim_2, \lesssim_3\rangle$, where V is non-empty set, for $i \in \{1, 2, 3\}$ $\lesssim_i$ is a quasiorder on V, i.e. a reflexive and transitive binary relation on V, and for $\sim_i = \lesssim_i \cap \gtrsim_i$, and all $v, w \in V$ the following two conditions hold for all assignments $\{i, j, k\} \in \{1, 2, 3\}$:

$$v \sim_i w \text{ and } w \sim_j v \text{ implies } v = w \tag{1.7}$$

$$v \lesssim_i w \text{ and } v \lesssim_j w \text{ implies } v \gtrsim_k w \tag{1.8}$$

(1.8) is called the *antiordinal dependence* property. Since the relation $\sim_i$ is an equivalence on V, $\lesssim_i$ induces an order relation $\leq_i$ on the equivalence classes of $V/\sim_i$, turning $\langle V/\sim_i, \leq_i\rangle$ into a partially ordered set.

Let $V_i, V_k \subseteq V$. An element $v \in V$ is called an *ik-bound* of $\langle V_i, V_k\rangle$ if $v_i \lesssim_i v$ and $v_k \lesssim_k v$ for every $v_i \in V_i$ and $v_k \in V_k$. An ik-bound v is called an *ik-limit* of $\langle V_i, V_k\rangle$ if $u \lesssim_j v$ for every ik-bound u of $\langle V_i, V_k\rangle$. In every triordered set $\langle V, \lesssim_1, \lesssim_2, \lesssim_3\rangle$ there is at most one ik-limit v of $\langle V_i, V_k\rangle$ satisfying $v \lesssim_k u$ for every ik-limit u of $\langle V_i, V_k\rangle$. If such v exists, we call v an *ik-join* of $\langle V_i, V_k\rangle$ and denote it $\nabla_{ik}(V_i, V_k)$. A triordered set $(V, \lesssim_1, \lesssim_2, \lesssim_3)$ in which the ik-join exists for all $i \neq k$ $(i, k \in \{1, 2, 3\})$ and all pairs $\langle V_i, V_k\rangle$ of subsets of V is called a *complete trilattice.*

In what follows we will recall the structural properties of concept trilattices. It is easy to see, that for $i \in \{i, j, k\}$ a relation $\lesssim_i$ on $\mathcal{T}(\mathbf{K})$ defined

$$\langle A_1, A_2, A_3\rangle \lesssim_i \langle B_1, B_2, B_3\rangle \text{ iff } A_i \subseteq B_i$$

is a quasiorder and that $\langle \mathcal{T}(\mathbf{K}), \lesssim_1, \lesssim_2, \lesssim_3\rangle$ is a triordered set. An *order filter* in the quasiordered set $\langle V, \lesssim_i\rangle$ is a subset $F \subseteq V$ for which $v \in F$ whenever $u \in F$ and $u \lesssim_i v$, for every $u, v \in V$. The set of all order filters of $\langle V, \lesssim_i\rangle$ is denoted by $\mathcal{F}_i(\mathbf{V})$. A *principal filter* of $\langle V, \lesssim_i\rangle$ generated by $v \in V$ is the order filter $[v)_i = \{u \in V \mid v \lesssim_i u\}$. A subset $\mathcal{X} \subseteq \mathcal{F}_i(\mathbf{V})$ is called *i-dense* with respect to $\mathbf{V}$ if each principal filter of $\langle V, \lesssim_i\rangle$ is the intersection of some order filters from $\mathcal{X}$. The following theorem proved in [53] establishes that concept trilattice is indeed a complete lattice and that each complete trilattice is isomorphic to a concept trilattice of some triadic context.

Theorem 2 (basic theorem of triadic concept analysis)**.**
Let $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$ *be a triadic context. Then:*
(1) $\mathcal{T}(\mathbf{K})$ *is a complete trillatice for which the ik-joins can be described as follows* $(\{i, j, k\} \in \{1, 2, 3\})$*:*

$$\nabla_{ik}(\mathcal{X}_i, \mathcal{X}_k) = \mathfrak{b}_{ik}(\bigcup\{A_i \mid \langle A_1, A_2, A_3\rangle \in \mathcal{X}_i\}, \bigcup\{A_k \mid \langle A_1, A_2, A_3\rangle \in \mathcal{X}_k\}),$$

(2) a complete trilattice $\mathbf{V} = \langle V, \lesssim_1, \lesssim_2, \lesssim_3\rangle$ *is isomorphic to* $\mathcal{T}(\mathbf{K})$ *if and only if there exist mappings* $\tilde{\kappa}_i : X_i \to \mathcal{F}_i(\mathbf{V})$ $(i = 1, 2, 3)$ *such that:*

(i) $\tilde{\kappa}_i$ *is i-dense with respect to* V*,*

(ii) $A_1 \times A_2 \times A_3 \subseteq I$ *iff* $\displaystyle\bigcap_{i=1}^{3} \bigcap_{a_i \in A_i} \tilde{\kappa}(a_i) \neq \emptyset$ *for all* $A_i \subseteq X_i$*.*

PROOF. The proof can be found in [53]. □

z_1	y_1	y_2
x_1	1	0
x_2	1	0

z_2	y_1	y_2
x_1	1	1
x_2	0	1

Table 1.1: A triadic context depicted as two dyadic contexts

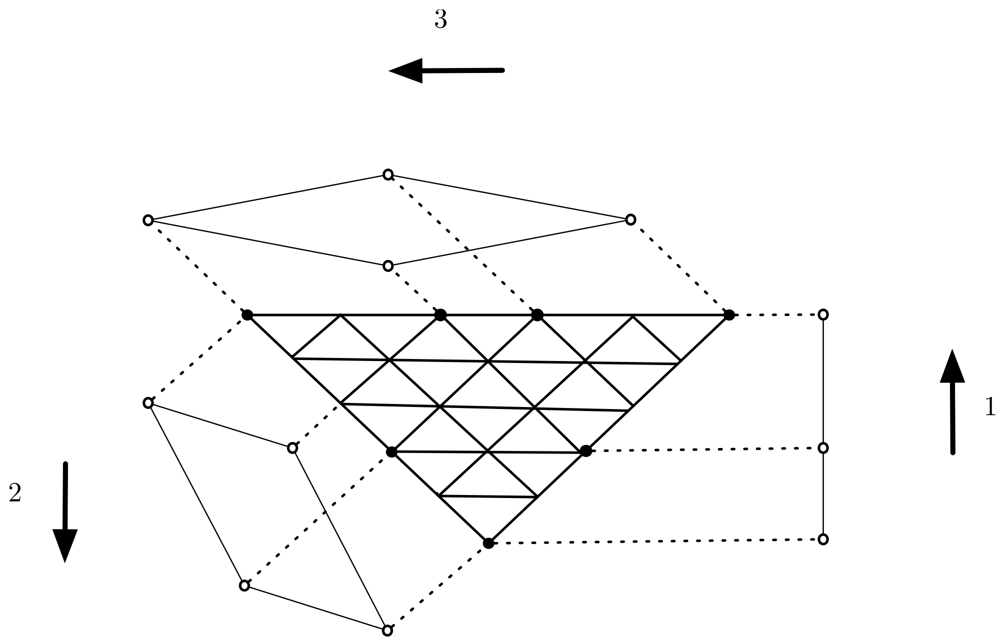


Figure 1.1: Triadic diagram of a concept trilattice corresponding to the triadic context from Table 1.1

Using so called *triadic diagrams*, it is possible to visualize some triordered sets (and thus concept trilattices). However, there are triordered sets which cannot be depicted using only straight lines, see [20, 53]. A diagram of a small triordered set is depicted in Figure 1.1. The triangular area consists of three systems of parallel lines corresponding to three equivalence relations $\sim_1$, $\sim_2$, and $\sim_3$. The circles at the intersections depict the elements of V. The side diagrams are Hasse diagrams of ordered sets $\langle V/\sim_1, \leq_1\rangle$, $\langle V/\sim_2, \leq_2\rangle$, and $\langle V/\sim_3, \leq_3\rangle$. That is, all elements lying on the same line belong to the same equivalence class of the corresponding equivalence. If we depict a concept trilattice, the side diagrams can be labeled by objects, attributes, or conditions. Particular components of a triadic concept can be then read off the labels of side diagrams by including all labels of elements, that are under the corresponding equivalence class.

Chapter 2

Triadic concept analysis in fuzzy setting

2.1 Introduction

In this chapter we proceed with a development of triadic concept analysis in fuzzy setting. As discussed above, our main motivation is to allow the method to deal with graded relational data. We consider it to be important because, first, there are relational data with vague attributes in practice, and second, existing methods that deal with three-way matrices are designed for numerical data. When applied to relational data, their output is hard to interpret in relational context, which we may understand as that these methods distort the intended meaning of the data. First of all, the entries in relational data are truth degrees and as such they can be of purely symbolic nature (i.e. not numbers). In such a case methods designed for numerical data are clearly not applicable. Even if the truth degrees are expressed as numbers (e.g. unit interval $[0, 1]$ or its subset), the use of such methods remains inappropriate. There are several reasons. Firstly, it is well known, that the operations on the set of truth degrees are semantical counterparts (i.e truth functions) of logical connectives. Thus, they have specific properties coming from the requirements on behavior of logical connectives. In general, these properties are not shared by numerical operations (e.g. arithmetic operations) which renders them inappropriate for working with truth degrees. Secondly, outputs of methods designed for numerical data may be hard to interpret as truth degrees. For example, it may happen that the numbers are negative or, more generally, that they do not belong into unit interval $[0, 1]$.

In formal (dyadic) concept analysis, the notion of a formal concept is a simple formalization of the notion of a concept as understood by Port-Royal logic. For triadic concept analysis the inspiration comes from Peirce's system of categories. To emphasis the intended interpretation of triadic contexts and triadic concepts, in their paper [43] Lehmann and Wille first enumerate the Peirce's categories.

> *In his lecture on pragmatics (1903) he [Peirce] gave the following description of his categories:*
>
> - *Category the First is the Idea of that which is such as it is regardless of anything else. That is to say, it is a quality of feeling.*
> - *Category the Second is the Idea of that which is such as it is as being Second to some First, regardless of anything else, and in particular regardless of any Law, although it may conform to a law. That to say, it is a Reaction as an element of the phenomenon.*
> - *Category the Third is the Idea of that which is such as it is as being a Third, or Medium, between a Second and a First. That is to say, it is Representation as an element of the phenomenon.*

Then they argue

> *The... description of the three categories allows us to interpret the triadic relationship as follows: the object g is a First as some suchness to which the attribute m is a Second as some accident while the condition b is a Third as some medium between g and m. In different situations, the Third as medium may be understood more specifically as relation, mediation, representation, interpretation, evidence, evaluation, modality, meaning, reason, purpose, condition etc. concerning a present connection between and object and an attribute.*

In this interpretation, the objects are considered as entities that exists by themselfs and the attributes are viewed as determining objects (i.e. as object descriptions) that cannot exist without them. Conditions are then understood as mediating relations between objects and attributes, such as different evidences, conditions, opinions etc. under which objects have attributes. In graded setting, this interpretation remains valid. We only allow attributes to be vague, that is to describe object in degrees.

This chapter is organized as follows. The notions of fuzzy triadic context and concept-forming operators are studied in Section 2.2. Section 2.3 contains material on fuzzy triadic conceps and their properties. The basic theorem and its proof are contained in Section 2.4. Connections to triadic concept analysis in ordinary case are studied in Section 2.5. Section 2.6 contains material on algorithms for computation of all triadic fuzzy concepts contained in a triadic context. In Section 2.7 we present an illustrative example. We conclude with remarks on future research and open problems in Section 2.8.

This chapter is based on results originally published in the following papers:

- R. Belohlavek, P. Osicka. *Triadic concept analysis of data with fuzzy attributes* Proc. of The 2010 IEEE International Conference on Granular Computing (GrC 2010), 2010, San José, USA
- R. Belohlavek, P. Osicka. *Triadic concept lattices of data with graded attributes* International Journal of General Systems 41 (5) (2012), 93-108
- P. Osicka *Algorithms for computation of concept trilattice of triadic fuzzy context.* Proceedings of 14th International Conference on Information Processing and Management of Uncertainty in Knowledge Based Systems. (accepted in March 2012).

2.2 Concept forming operators

Definition 3. A *triadic* $\mathbf{L}$*-context* (triadic fuzzy context, or just triadic context) is a quadruple $\langle X, Y, Z, I\rangle$ where X, Y, and Z are non-empty sets, and I is a ternary fuzzy relation between X, Y, and Z, i.e. $I : X\times Y\times Z \to L$.

The sets X, Y, and Z are interpreted as sets of objects, attributes, and conditions, respectively. The relation I is a "to-have-under" relation, that is $I(x, y, z) = a$ means that "object x has attribute y under condition z to the degree a". For convenience, we will use the notation $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$, i.e. we use subscripts $_1, _2$, and $_3$ to distinguish between objects, attributes, and conditions. Moreover, if the ordering of x_1, x_2, x_3 is not relevant, we denote $I(x_1, x_2, x_3)$ also by $I\{x_2, x_3, x_1\}$, or $I\{x_3, x_1, x_2\}$, or $I\{x_1, x_3, x_2\}$, etc.

From a triadic context $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$ one can obtain a number of dyadic fuzzy contexts. Given a fuzzy set $C_k \in L^{X_k}$, $\mathbf{K}$ induces a dyadic fuzzy context $\mathbf{K}^{ij}_{C_k} = \langle X_i, X_j, I^{ij}_{C_k}\rangle$, where $I^{ij}_{C_k}$ is defined by

$$I^{ij}_{C_k}(x_i, x_j) = \bigwedge_{x_k\in X_k} C_k(x_k) \to I\{x_i, x_j, x_k\} \tag{2.1}$$

The previous definition is a generalization of (1.6). Namely, (1.6) expresses the ordinary (over a two-element Boolean algebra) semantics of the following formula of predicate logic:

The pair $\langle x_i, x_j\rangle$ belongs to $I^{ij}_{C_k}$ iff for each $x_k \in X_k$ the fact that x_k belongs to C_k implies that the triple $\langle x_i, x_j, x_k\rangle$ belongs to I.

If we write down the semantics over residuated lattices of this formula, we obtain precisely (2.1).

We denote the concept-forming operators induced by a dyadic context $\mathbf{K}^{ij}_{C_k}$ by ${}^{(i,j,C_k)}$. That is, for a fuzzy set $C_i \in L^{X_i}$ we define a fuzzy set $C_i^{(i,j,C_k)} \in L^{X_j}$ by

$$C_i^{(i,j,C_k)}(x_j) = \bigwedge_{x_i \in X_i} C_i(x_i) \to I^{ij}_{C_k}(x_i, x_j). \tag{2.2}$$

The following lemma describes basic properties of concept-forming operators.

Lemma 4. *Let $\{i, j, k\} = \{1, 2, 3\}$. Then*

(a) $A_i^{(i,j,C_k)}(x_j) = \bigwedge_{\langle x_i, x_j\rangle \in X_i \times X_k}((A_i(x_i) \otimes C_k(x_k)) \to I\{x_i, x_j, x_k\})$,

(b) $A_i^{(i,j,C_k)} = C_k^{(k,j,A_i)}$,

(c) $S(A_i, B_i) \otimes S(C_k, D_k) \leq S(B_i^{(i,j,D_k)}, A_i^{(i,j,C_k)})$,

(d) *if $A_i \subseteq B_i$ and $C_k \subseteq D_k$ then $B_i^{(i,j,D_k)} \subseteq A_i^{(i,j,C_k)}$,*

for any $A_i, B_i \in \mathbf{L}^{X_i}$, $C_k, D_k \in \mathbf{L}^{X_k}$.

PROOF. (a)

$$\begin{aligned} A_i^{i,j,C_k}(x_j) &= \textstyle\bigwedge_{x_i \in X_i} A_i(x_i) \to I^{ij}_{C_k}(x_i, x_j) = \\ &= \textstyle\bigwedge_{x_i \in X_i} \Big(A_i(x_i) \to \bigwedge_{x_k \in X_k}(C_k(x_k) \to I\{x_i, x_j, x_k\})\Big) = \\ &= \textstyle\bigwedge_{x_i \in X_i} \bigwedge_{x_k \in X_k} (A_i(x_i) \to (C_k(x_k) \to I\{x_i, x_j, x_k\})) = \\ &= \textstyle\bigwedge_{\langle x_i, x_k\rangle \in X_i \times X_k} (A_i(x_i) \otimes C_k(x_k) \to I\{x_i, x_j, x_k\}) \end{aligned}$$

(b) Follows from (a) by commutativity of $\otimes$.

(c) Consider the dyadic fuzzy context $\langle X_i \times X_k, X_j, I_{ik,j}\rangle$ defined by

$$I_{ik,j}(\langle x_i, x_k\rangle, x_j) = I\{x_i, x_j, x_k\},$$

and fuzzy sets $A_i \otimes C_k$ and $B_i \otimes D_k$ in $X_i \times X_k$ defined by (1.3). According to (a), $A_i^{(i,j,C_k)}(x_j) = (A_i \otimes C_k)^{I_{ik,j}}$ and $B_i^{(i,j,D_k)}(x_j) = (B_i \otimes D_k)^{I_{ik,j}}$. Using (1.4) $S(A_i, B_i) \otimes S(C_k, D_k) \leq S(A_i \otimes C_k, B_i \otimes D_k)$. Now, [6] implies $S(A_i \otimes C_k, B_i \otimes D_k) \leq S((B_i \otimes D_k)^{I_{ik,j}}, (A_i \otimes C_k)^{I_{ik,j}})$ from which the assertion readily follows.

(d) A consequence of (c). Namely, $A \subseteq B$ is equivalent to $S(A, B) = 1$; therefore, if $A_i \subseteq B_i$ and $C_k \subseteq D_k$ then $1 = 1 \otimes 1 \leq S(B_i^{(i,j,D_k)}, A_i^{(i,j,C_k)})$, whence $B_i^{(i,j,D_k)} \subseteq A_i^{(i,j,C_k)}$.

□

Remark 5. (1) Let $\mathbf{K}^{(j)} = \langle X_i \times X_k, X_j, I^{(j)} \rangle$ be a dyadic fuzzy context defined by $I^{(j)}((x_i, x_k), x_j) = I\{x_i, x_j, x_k\}$, and denote the concept forming operators induced by $\mathbf{K}^{(j)}$ by $^{(j)}$. Then Lemma 4 (a) states, that for all $A_i \in L^{X_i}$ and $A_k \in L^{X_k}$ the equality $A_i^{(i,j,A_k)} = (A_i \otimes A_k)^{(j)}$ holds.

(2) It is well known that L^{X_i} equipped with an order relation $\leq$ defined for all $A, B \in L^{X_i}$ as $A \leq B$ iff $A \subseteq B$ is a partially ordered set (in fact, it is a complete lattice, cf [35]). Lemma 4 (d) implies that if we look at $^{(i,j,C_k)}$ as at a map assigning to a pair of fuzzy sets $A_i \in L^{X_i}$ and $A_k \in L^{X_k}$ an fuzzy set in X_j, then this map is antitone in both arguments.

(3) As in ordinary case, a triadic fuzzy context can be depicted as $|X_k|$ dyadic contexts $\mathbf{K}^{ij}_{C_k}$, where C_k iterates over singletons in X_k, i.e. over fuzzy sets $C_k = \{1/x_k\}$ for all $x_k \in X_k$.

2.3 Triadic fuzzy concepts

Definition 6. Let $\mathbf{K} = \langle X_1, X_2, X_3, I \rangle$ be a triadic fuzzy context. Then a *triadic* $\mathbf{L}$*-concept* (triadic fuzzy concept) of $\mathbf{K}$ is a triplet $\langle A_1, A_2, A_3 \rangle$ consisting of fuzzy sets $A_1 \in L^{X_1}$, $A_2 \in L^{X_2}$, and $A_3 \in L^{X_3}$, such that for every $\{i, j, k\} = \{1, 2, 3\}$ we have $A_i = A_j^{(i,j,A_k)}$, $A_j = A_k^{(j,k,A_i)}$, and $A_k = A_i^{(k,i,A_j)}$. The set of all triadic fuzzy concepts of $\mathbf{K}$ is called the *concept trillatice* of $\langle X_1, X_2, X_3, I \rangle$, and is denoted by $\mathcal{T}(\mathbf{K})$.

Remark 7. According to Lemma 4 (a) (see also Remark 5), one can alternatively define a triadic fuzzy concept as a triple $\langle A_1, A_2, A_3 \rangle$ such that $(A_i \otimes A_j)^{(j)} = A_k$ for all assignments $\{i, j, k\} = \{1, 2, 3\}$. Note that in Wille's paper [53], the triadic concepts are defined using $^{(j)}$ operators. For $\mathbf{L} = \mathbf{2}$ ($\mathbf{L}$ is a two-element boolean algebra), the $^{(j)}$ operators coincide with ordinary $^{(j)}$ operators of [53], and triadic fuzzy concepts are just ordinary triadic concepts as defined in [53].

Now, we establish fundamental structural properties of $\mathcal{T}(\mathbf{K})$. For fuzzy triadic concepts $\langle A_1, A_2, A_3 \rangle$, and $\langle B_1, B_2, B_3 \rangle \in \mathcal{T}(\mathbf{K})$, $\{i, j, k\} \in \{1, 2, 3\}$, we define fuzzy relations on $\mathcal{T}(\mathbf{K})$

$$\langle A_1, A_2, A_3 \rangle \precsim_i \langle B_1, B_2, B_3 \rangle = S(A_i, B_i) \tag{2.3}$$

$$\langle A_1, A_2, A_3 \rangle \approx_i \langle B_1, B_2, B_3 \rangle = A_i \approx B_i \tag{2.4}$$

For definitions of $S(A,B)$ and $\approx$ see (1.1) and (1.2). It easy to observe that $\approx_i = \precsim_i \wedge \succsim_i$, and that $\precsim_i$ and $\approx_i$ are **L**-quasiorder and **L**-equality relations, respectively. The 1-cuts of $\precsim_i$ and $\approx_i$, denoted by $\lesssim_i$ and $\eqsim_i$, are ordinary equivalence and quasiorder relations, respectively. Since $A \subseteq B$ iff $S(A,B) = 1$, and $A \approx B = 1$ iff $A = B$, we have

$$\langle A_1, A_2, A_3 \rangle \lesssim_i \langle B_1, B_2, B_3 \rangle \quad \text{iff} \quad A_i \subseteq B_i, \tag{2.5}$$

$$\langle A_1, A_2, A_3 \rangle \eqsim_i \langle B_1, B_2, B_3 \rangle \quad \text{iff} \quad A_i = B_i. \tag{2.6}$$

It is immediate that $\eqsim_i = \lesssim_i \cap \gtrsim_i$.

The following theorem elaborates on the connection between the fuzzy quasiorders $\precsim_1$, $\precsim_2$, and $\precsim_3$.

Theorem 8. *Let $\{i,j,k\} = \{1,2,3\}$. Then*

$$\begin{aligned}&(\langle A_1, A_2, A_3 \rangle \precsim_i \langle B_1, B_2, B_3 \rangle) \otimes (\langle A_1, A_2, A_3 \rangle \precsim_j \langle B_1, B_2, B_3 \rangle) \leq \\ &\leq (\langle B_1, B_2, B_3 \rangle \precsim_k \langle A_1, A_2, A_3 \rangle),\end{aligned}$$

*for all triadic fuzzy concepts $\langle A_1, A_2, A_3 \rangle$ and $\langle B_1, B_2, B_3 \rangle$ from $\mathcal{T}(\mathbf{K})$. Furthermore, $\approx_i \cap \approx_j$ is an **L**-equality on $\mathcal{T}(\mathbf{K})$.*

PROOF. As $\langle A_1, A_2, A_3 \rangle, \langle B_1, B_2, B_3 \rangle \in \mathcal{T}(\mathbf{K})$, we have $A_i^{(i,k,A_j)} = A_k$ and $B_i^{(i,k,B_j)} = B_k$. Lemma 4 (c) therefore yields

$$\begin{aligned}&(\langle A_1, A_2, A_3 \rangle \precsim_i \langle B_1, B_2, B_3 \rangle) \otimes (\langle A_1, A_2, A_3 \rangle \precsim_j \langle B_1, B_2, B_3 \rangle) = \\ &= S(A_i, B_i) \otimes S(A_j, B_j) \leq S(B_i^{(i,k,B_j)}, A_i^{(i,k,A_j)}) = \\ &= S(B_k, A_k) = \langle A_1, A_2, A_3 \rangle \precsim_k \langle B_1, B_2, B_3 \rangle.\end{aligned}$$

Since $\approx_i \cap \approx_j$ is an **L**-equivalence (an intersection of two **L**-equivalences), it suffices to show that if $\langle A_1, A_2, A_3 \rangle (\approx_i \cap \approx_j) \langle B_1, B_2, B_3 \rangle = 1$ then $\langle A_1, A_2, A_3 \rangle = \langle B_1, B_2, B_3 \rangle$. If $\langle A_1, A_2, A_3 \rangle (\approx_i \cap \approx_j) \langle B_1, B_2, B_3 \rangle = 1$ then $A_i \approx B_i = 1$ and $A_j \approx B_j = 1$, whence $A_i = B_i$ and $A_j = B_j$. But then $A_k = A_i^{(i,k,A_j)} = B_i^{(i,k,B_j)} = B_k$ because both $\langle A_1, A_2, A_3 \rangle$ and $\langle B_1, B_2, B_3 \rangle$ are triadic concepts. □

We immediately obtain the following corollary.

Corollary 9. *For all assignments $\{i,j,k\} = \{1,2,3\}$, and all triadic fuzzy concepts $\langle A_1, A_2, A_3 \rangle$ and $\langle B_1, B_2, B_3 \rangle$ from $\mathcal{T}(\mathbf{K})$, it holds that*

if $\langle A_1, A_2, A_2 \rangle \lesssim_i \langle B_1, B_2, B_3 \rangle$ and $\langle A_1, A_2, A_2 \rangle \lesssim_j \langle B_1, B_2, B_3 \rangle$,
then $\langle B_1, B_2, B_3 \rangle \lesssim_k \langle A_1, A_2, A_3 \rangle$.

Furthermore, $\eqsim_i \cap \eqsim_j$ is the identity on $\mathcal{T}(\mathbf{K})$.

PROOF. Since $\lesssim_i$s are 1-cuts of $\precsim_i$s and $\eqsim_i$s are 1-cuts of $\approx_i$s, we get the assertion by Theorem 8. □

Remark 10. By Corollary 9 we can see that $(\mathcal{T}(\mathbf{K}), \lesssim_1, \lesssim_2, \lesssim_3)$ is a tri-ordered set. Indeed, the first assertion is the antiordinal property (1.8). The second assertion clearly implies (1.7).

The following theorem shows how to compute a triadic fuzzy concept. Starting with two sets, $C_i \in L^{X_i}$ and $C_k \in L^{X_k}$, we can obtain a triadic fuzzy concept $\langle A_1, A_2, A_3\rangle$ using the concept-forming operators three times. First we compute A_i from C_i and C_k, then we compute A_j from A_i and C_k, and finally, we obtain A_k from A_i and A_j. Moreover, $\langle A_1, A_2, A_3\rangle$ has convenient properties which we utilize in Section 2.4.

Theorem 11. *For $C_i \in L^{X_i}, C_k \in L^{X_k}$ with $\{i,j,k\} = \{1,2,3\}$, let $A_j = C_i^{(i,j,C_k)}$, $A_i = A_j^{(i,j,C_k)}$, and $A_k = A_i^{(i,k,A_j)}$. Then $\langle A_1, A_2, A_3\rangle$ is a triadic fuzzy concept, denoted by $\mathfrak{b}_{ik}(C_i, C_k)$.*

$\mathfrak{b}_{ik}(C_i, C_k)$ has the smallest k-th component among all triadic fuzzy concepts $\langle B_1, B_2, B_3\rangle$ with the greatest j-th component satisfying $C_i \subseteq B_i$ and $C_k \subseteq B_k$. In particular, $\mathfrak{b}_{ik}(A_i, A_k) = \langle A_1, A_2, A_3\rangle$ for each triadic fuzzy concept $\langle A_1, A_2, A_3\rangle$.

PROOF. First, observe that $C_i \subseteq A_i$ and $C_k \subseteq A_k$. Indeed, $C_i \subseteq A_i$ holds true because A_i is the closure of C_i w.r.t. the closure operator on $\mathbf{K}^{ij}_{C_k}$. $C_k \subseteq A_k$ because by definition of A_k, Lemma 4(a) yields that the inclusion is equivalent to $C_k(x_k) \leq A_i(x_i) \otimes A_j(x_j) \to I\{x_i, x_j, x_k\}$ being true for every $\langle x_i, x_k\rangle$, which is equivalent to $A_i(x_i) \otimes A_j(x_j) \otimes C_k(x_k) \leq I\{x_i, x_j, x_k\}$. The last inequality holds true because

$$\begin{aligned}
&A_i(x_i) \otimes A_j(x_j) \otimes C_k(x_k) \leq \\
&\leq (A_j(x_j) \otimes C_k(x_k) \to I\{x_i, x_j, x_k\}) \otimes A_j(x_j) \otimes C_k(x_k) \leq \\
&\leq I\{x_i, x_j, x_k\}.
\end{aligned}$$

Next, we prove that $\langle A_1, A_2, A_3\rangle$ is a triadic fuzzy concept. $A_k = A_i^{(i,k,A_j)}$ is satisfied by definition. Consider A_j. Due to Lemma 4(d), $A_j = C_i^{(i,j,C_k)} \supseteq A_i^{(i,j,A_k)}$ and $A_j \subseteq (A_j^{(j,k,A_i)})^{(j,k,A_i)} = A_k^{(j,k,A_i)} = A_i^{(i,j,A_k)}$, thus $A_j = A_i^{(i,j,A_k)}$. The proof for A_i is similar.

Let $\langle B_1, B_2, B_3\rangle$ be a triadic fuzzy concept with $C_i \subseteq B_i$ and $C_k \subseteq B_k$. Then $B_j = B_i^{(i,j,B_k)} \subseteq C_i^{(i,j,C_k)} = A_j$. This shows that $\langle A_1, A_2, A_3\rangle$ has the greatest j-th component among all concepts $\langle B_1, B_2, B_3\rangle$ that satisfy $C_i \subseteq B_i$ and $C_k \subseteq B_k$. Let now $B_j = A_j$. Then $A_i = A_j^{(i,j,C_k)} \supseteq B_j^{(i,j,B_k)} = B_i$ thus $B_i \subseteq A_i$, whence $A_k = A_i^{(i,k,A_j)} \subseteq (B_i^{(i,k,B_j)} = B_k$.

Finally, if $\langle A_1, A_2, A_3\rangle$ is a triadic fuzzy concept, then $A_j = A_i^{(i,j,A_k)}$, $A_i = A_j^{(i,j,A_k)}$, and $A_k = A_i^{(i,k,A_j)}$ by definition. Hence, $\mathfrak{b}_{ik}(A_i, A_k) = \langle A_1, A_2, A_3\rangle$.

$\square$

The following theorem enables us to look at triadic fuzzy concepts as at maximal cubicals contained in data. Its consequences are of importance in a particular application of TCA, decompositions of three-way ordinal matrices, see Chapter 4.

Theorem 12 (geometrical interpretation of triadic concepts). *For every triadic fuzzy context* $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$*:*

(a) *If* $\langle A_1, A_2, A_3\rangle \in \mathcal{T}(\mathbf{K})$ *then* $A_1 \otimes A_2 \otimes A_3 \subseteq I$. *Moreover,* $\langle A_1, A_2, A_3\rangle$ *is maximal with respect to pointwise set inclusion, i.e. there does not exist* $\langle B_1, B_2, B_3\rangle \in \langle \mathbf{L}^{X_1}, \mathbf{L}^{X_2}, \mathbf{L}^{X_3}\rangle$ *other than* $\langle A_1, A_2, A_3\rangle$ *such that* $A_i \subseteq B_i$ *for every* $i = 1, 2, 3$.

(b) *If* $A_1 \otimes A_2 \otimes A_3 \subseteq I$ *then there exists* $\langle B_1, B_2, B_3\rangle \in \mathcal{T}(\mathbf{K})$ *such that* $A_i \subseteq B_i$ *for every* $i = 1, 2, 3$.

PROOF. (a) Let $\{i, j, k\} = \{1, 2, 3\}$. From $\langle A_1, A_2, A_3\rangle \in \mathcal{T}(\mathbf{K})$ it follows that $A_k(x_k) = A_i^{(i,j,k)} = \bigwedge_{(x_i,x_j)\in X_i\times X_j} A_i(x_i) \otimes A_j(x_j) \to I\{x_i, x_j, x_k\}$. Furthermore,

$$
\begin{aligned}
&A_i(x_i) \otimes A_j(x_j) \otimes A_k(x_k) = \\
&= A_i(x_i) \otimes A_j(x_j) \otimes \textstyle\bigwedge_{x_i\in X_i, x_j\in X_j} (A_i(x_i) \otimes A_j(x_j) \to I\{x_i, x_j, x_k\}) \leq \\
&\leq A_i(x_i) \otimes A_j(x_j) \otimes (A_i(x_i) \otimes A_j(x_j) \to I\{x_i, x_j, x_k\}) \leq \\
&\leq I\{x_i, x_j, x_k\}.
\end{aligned}
$$

Let $\langle A_1, A_2, A_3\rangle$ and $\langle B_1, B_2, B_3\rangle$ be triadic fuzzy concepts with $A_i \subseteq B_i$ for every $i = 1, 2, 3$. Applying Corollary 9 to $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$ we get $B_3 \subseteq A_3$; in a similar manner, $B_1 \subseteq A_1$ and $B_2 \subseteq A_2$, hence $\langle A_1, A_2, A_3\rangle = \langle B_1, B_2, B_3\rangle$, proving maximality of $\langle A_1, A_2, A_3\rangle$.

(b) Let $\{i, j, k\} = \{1, 2, 3\}$ and $\mathfrak{b}_{ik}(A_i, A_k) = \langle B_1, B_2, B_3\rangle$. Due to Theorem 11, $A_i \subseteq B_i$ and $A_k \subseteq B_k$. Moreover

$$
\begin{aligned}
&B_j(x_j) = A_i^{(i,j,A_k)}(x_j) = \\
&= \textstyle\bigwedge_{x_i\in X_i, x_k\in X_k} (A_i(x_i) \otimes A_j(x_k) \to I\{x_i, x_j, x_k)\} \geq \\
&\geq \textstyle\bigwedge_{x_i\in X_i, x_k\in X_k} (A_i(x_i) \otimes A_k(x_k) \to A_i(x_i) \otimes A_k(x_k) \otimes A_j(x_j)) \geq \\
&\geq A_j(x_j),
\end{aligned}
$$

thus $A_j \subseteq B_j$, finishing the proof. □

2.4 Basic theorem

In this section we broaden the material on the structure of $\mathcal{T}(\mathbf{K})$. Our goal is to prove a generalization of the basic theorem of ordinary triadic concept analysis. After we do so, we show that the new theorem is indeed a proper generalization, i.e. that for $\mathbf{L} = \mathbf{2}$ both theorems coincide.

Let $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$ be a triadic fuzzy context and $\mathcal{T}(\mathbf{K})$ the corresponding concept trilattice. Recall, that $(\langle \mathcal{T}(\mathbf{K}), \lesssim_1, \lesssim_2, \lesssim_1\rangle$ is a triordered set (Remark 10), and thus we can proceed with the following construction.

Consider the mappings $\kappa_i : X_i \times L \to \mathcal{T}(\mathbf{K})$ defined by

$$\kappa_i = \{\langle A_1, A_2, A_3\rangle \in \mathcal{T}(\mathbf{K}) \mid A_i(x_i) \geq a\} \tag{2.7}$$

for every $i \in \{1, 2, 3\}$, $x_i \in X_i$ and $a \in L$. It is easy to check, that $\kappa_i(x_i, a)$ is an order filter in $\langle \mathcal{T}(\mathbf{K}), \lesssim_i\rangle$. Moreover, for each triadic fuzzy concept $\langle A_1, A_2, A_3\rangle \in \mathcal{T}(\mathbf{K})$, the principal filter $[\langle A_1, A_2, A_3\rangle)_i$ can be obtained as intersection of all filters $\kappa_i(x_i, A_i(x_i))$ over all $x_i \in X_i$, i. e.

$$[\langle A_1, A_2, A_3\rangle)_i = \bigcap_{x_i \in X_i} \kappa_i(x_i, A_i(x_i)). \tag{2.8}$$

Therefore, the set $\kappa_i(X_i \times L)$ is i-dense w.r.t. $(\langle \mathcal{T}(\mathbf{K}), \lesssim_1, \lesssim_2, \lesssim_1\rangle$. Moreover, $a \leq b$ implies $\kappa_i(x_i, b) \subseteq \kappa_i(x_i, a)$.

Theorem 13 (basic theorem of triadic concept analysis in fuzzy setting)**.** *Let $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$ be a triadic fuzzy context.*

(1) $\mathcal{T}(\mathbf{K})$ is a complete trilattice for which the ik-joins are defined for every $i, k \in \{1, 2, 3\}$, $i \neq k$, by:

$$\nabla_{ik}(\mathcal{X}_i, \mathcal{X}_k) = \mathfrak{b}_{ik}\left(\bigcup\{A_i | \langle A_1, A_2, A_3\rangle \in \mathcal{X}_i\}, \bigcup\{A_k | \langle A_1, A_2, A_3\rangle \in \mathcal{X}_k\}\right).$$

(2) A complete trilattice $\mathbf{V} = \langle V, \lesssim_1, \lesssim_2, \lesssim_3\rangle$ is isomorphic to $\mathcal{T}(\mathbf{K})$ if and only if there are mappings $\tilde{\kappa}_i : X_i \times L \to \mathcal{F}_i(\mathbf{V})$, $i = 1, 2, 3$, such that

- (i) *$\tilde{\kappa}_i(X_i \times L)$ is i-dense with respect to $\mathbf{V}$;*
- (ii) *$A_1 \otimes A_2 \otimes A_3 \subseteq I$ iff $\bigcap_{i=1}^{3} \bigcap_{x_i \in X_i} \tilde{\kappa}_i(x_i, A_i(x_i)) \neq \emptyset$, for every $A_i \in L^{X_i}$, $i = 1, 2, 3$;*
- (iii) *$a \leq b$ implies $\tilde{\kappa}_i(x_i, b) \subseteq \tilde{\kappa}_i(x_i, a)$ for every $a, b \in L$, $x_i \in X_i$, $i = 1, 2, 3$.*

PROOF.

(1): Corollary 9 implies that $\mathcal{T}(\mathbf{K})$ is a triordered set. Moreover, $\mathcal{T}(\mathbf{K})$ is a complete trilattice due to Theorem 11.

(2): "$\Rightarrow$": Let φ be an isomorphism between $\mathbf{V}$ and $\mathcal{T}(\mathbf{K})$. Define a mapping $\tilde{\kappa}_i : X_i \times L \to V$ by $\tilde{\kappa}_i(x_i, b) = \varphi(\kappa(x_i, b))$. As observed above, $\kappa_i(X_i \times L)$ is i-dense w.r.t. $\mathcal{T}(\mathbf{K})$ and $a \leq b$ implies $\kappa_i(x_i, b) \subseteq \kappa_i(x_i, a)$. Therefore, $\tilde{\kappa}_i$ satisfies (i) and (iii). If $A_1 \otimes A_2 \otimes A_3 \subseteq I$, Theorem 12

(b) yields a concept $\langle B_1, B_2, B_2\rangle \in \mathcal{T}(\mathbf{K})$ for which $A_i \subseteq B_i$ for all $i \in \{1,2,3\}$. Clearly, $\langle B_1, B_2, B_2\rangle \in \bigcap_{i=1}^{3}\bigcap_{x_i\in X_i}\kappa_i(x_i, A_i(x_i))$. Conversely, if $\langle B_1, B_2, B_2\rangle$ is an element of $\bigcap_{i=1}^{3}\bigcap_{x_i\in X_i}\kappa_i(x_i, A_i(x_i))$, one has $A_i \subseteq B_i$ for all $i \in \{1,2,3\}$. Monotony of $\otimes$ and Theorem 12 (a) thus yield $A_1\otimes A_2\otimes A_3 \subseteq B_1\otimes B_2\otimes B_3 \subseteq I$. Therefore, $A_1\otimes A_2\otimes A_3\subseteq I$ iff $\bigcap_{i=1}^{3}\bigcap_{x_i\in X_i}\kappa_i(x_i, A_i(x_i)) \neq \emptyset$ iff $\bigcap_{i=1}^{3}\bigcap_{x_i\in X_i}\tilde{\kappa}_i(x_i, A_i(x_i)) \neq \emptyset$, proving (ii).

"$\Leftarrow$": Let ψ be a mapping $\psi : V \to L^{X_1}\times L^{X_2}\times L^{X_3}$ defined by $\psi(v) = \langle A_1^v, A_2^v, A_2^v\rangle$ where

$$A_i^v(x_i) = \bigvee L_{i,x_i}^v.$$

where $L_{i,x_i}^v = \{a \in L \mid v \in \tilde{\kappa}_i(x_i, a)\}$. Since $\mathbf{V}$ is a triordered set, $[v)_1 \cap [v)_2 \cap [v)_3 = \{v\}$. (i) implies that $[v)_i$ is the intersection of all $\tilde{\kappa}_i(x_i, a)$ that contain v, i.e. $[v)_i = \bigcap_{x_i\in X_i}\bigcap_{a\in L_{i,x_i}^v}\tilde{\kappa}_i(x_i, a)$. Therefore,

$$\bigcap_{i=1}^{3}\bigcap_{x_i\in X_i}\bigcap_{a\in L_{i,x_i}^v}\tilde{\kappa}_i(x_i, a) = \{v\} \neq \emptyset.$$

In particular, for every collection of $a_{i,x_i} \in L_{i,x_i}^v$ ($i = 1,2,3$ and $x_i \in X_i$), $\bigcap_{i=1}^{3}\bigcap_{x_i\in X_i}\tilde{\kappa}_i(x_i, a_{i,x_i}) \neq \emptyset$, and (ii), applied to fuzzy sets A_i defined by $A_i(x_i) = a_{i,x_i}$ for each $x_i \in X_i$, thus yields

$$a_{1,x_1}\otimes a_{2,x_2}\otimes a_{3,x_3} \leq I(x_1, x_2, x_3).$$

Using $a\otimes(\bigvee_j b_j) = \bigvee_j(a\otimes b_j)$ (identity of complete residuated lattices), one therefore gets

$$\begin{aligned}
&A_1^v(x_1)\otimes A_2^v(x_2)\otimes A_2^v(x_2) = \\
&= (\bigvee_{a_{1,x_1}\in L_{1,x_1}^v} a_{1,x_1})\otimes(\bigvee_{a_{2,x_2}\in L_{2,x_2}^v} a_{2,x_2})\otimes(\bigvee_{a_{3,x_3}\in L_{3,x_3}^v} a_{3,x_3}) = \\
&= \bigvee_{a_{1,x_1}\in L_{1,x_1}^v}\ \bigvee_{a_{2,x_2}\in L_{2,x_2}^v}\ \bigvee_{a_{3,x_3}\in L_{3,x_3}^v} a_{1,x_1}\otimes a_{2,x_2}\otimes a_{3,x_3} \leq I(x_1, x_2, x_3),
\end{aligned}$$

verifying $A_1^v\otimes A_2^v\otimes A_2^v(x_2) \subseteq I$. (ii) then yields $\bigcap_{i=1}^{3}\bigcap_{x_i\in X_i}\tilde{\kappa}_i(x_i, A_i^v(x_i)) \neq \emptyset$. Due to (iii),

$$\bigcap_{i=1}^{3}\bigcap_{x_i\in X_i}\tilde{\kappa}_i(x_i, A_i^v(x_i)) \subseteq \bigcap_{i=1}^{3}\bigcap_{x_i\in X_i}\bigcap_{a\in L_{i,x_i}^v}\tilde{\kappa}_i(x_i, a) = \{v\},$$

whence

$$\bigcap_{i=1}^{3}\bigcap_{x_i\in X_i}\tilde{\kappa}_i(x_i, A_i^v(x_i)) = \{v\}. \tag{2.9}$$

Using adjointness, one easily verifies that for $\hat{A}_3^v = A_1^{v(1,3,A_2^v)}$ we also have $A_3^v \subseteq \hat{A}_3^v$ and $A_1^v\otimes A_2^v\otimes \hat{A}_3^v \subseteq I$. Due to (ii) and (iii), the latter inclusion and (2.9) imply $\bigcap_{i=1}^{2}\bigcap_{x_i\in X_i}\tilde{\kappa}_i(x_i, A_i^v(x_i)) \cap \bigcap_{x_3\in X_3}\tilde{\kappa}_3(x_3, \hat{A}_3^v(x_3)) = \{v\}$. In particular, $\hat{A}_3^v(x_3) \in L_{3,x_3}^v$ which implies $\hat{A}_3^v \subseteq A_3^v$ because as $A_3^v(x_3) = \bigvee L_{3,x_3}^v \geq \hat{A}_3^v(x_3)$. To sum up, $\hat{A}_3^v = A_3^v$. The same way, one proves $\hat{A}_1^v = A_1^v$ and $\hat{A}_2^v = A_2^v$.

This shows $\psi(v) \in \mathcal{T}(\mathbf{K})$.

If $v_1 \lesssim_i v_2$ for $v_1, v_2 \in V$ then clearly, $L_{i,x_i}^{v_1} \subseteq L_{i,x_i}^{v_2}$ for every $x_i \in X_i$, whence $A_i^{v_1} \subseteq A_i^{v_2}$, showing that ψ preserves $\lesssim_i$.

Let $\langle A_1, A_2, A_3\rangle \in \mathcal{T}(\mathbf{K})$. Theorem 12 (a) and (ii) imply that there exists $v \in \bigcap_{i=1}^{3}\bigcap_{x_i\in X_i}\tilde{\kappa}_i(x_i, A_i(x_i))$ and thus $v \in \bigcap_{x_i\in X_i}\tilde{\kappa}_i(x_i, A_i(x_i))$. The key observation is that $v \notin \tilde{\kappa}_i(x_i, d)$ for all $d \nleq A(x_i)$ for all $x_i \in X_i$. In order to prove it, assume by contradiction that there are $x_i' \in X_i$ and $d \nleq A_i(x_i')$ such that $v \in \tilde{\kappa}_i(x_i', d)$. Then (ii) implies that for all $x_j \in X_j, x_k \in X_k$ we have $d \otimes A_j(x_j) \otimes A_k(x_k) \leq I(x_i', x_j, x_k)$ and thus $(d \vee A_i(x_i)) \otimes A_j(x_j) \otimes A_k(x_k) \leq I(x_i', x_j, x_k)$, a contradiction to the maximality of $\langle A_1, A_2, A_3\rangle$ (Theorem 12). Therefore $A_i(x_i) = \bigvee\{a \mid v \in \tilde{\kappa}_i(x_i, a)\}$ for each $x_i \in X_i$, $i = 1, 2, 3$. This proves $\psi(v) = \langle A_1, A_2, A_3\rangle$, whence the surjectivity of ψ.

(2.9) implies that if $v \neq w$, then $\psi(v) \neq \psi(w)$. Therefore ψ is injective.

If $\psi(v_1) = \langle A_1, A_2, A_3\rangle \lesssim_i \langle B_1, B_2, B_3\rangle = \psi(v_2)$ then $A_i \subseteq B_i$. (iii) implies that for each $x_i \in X_i$ we have $\tilde{\kappa}_i(x_i, B_i(x_i)) \subseteq \tilde{\kappa}_i(x_i, A_i(x_i))$ which implies $[v_2)_i \subseteq [v_1)_i$ and therefore $v_1 \lesssim_i v_2$. Thus $\psi^{-}1$ preserves $\lesssim_i$.

□

We need the following lemma.

Lemma 14. *If $\tilde{\kappa}_i$ satisfy (i)–(iii) of Theorem 13, then $\tilde{\kappa}_i(x_i, 0) = V$ for each $x_i \in X_i$, $i = 1, 2, 3$.*

PROOF. Assume $i = 1$. Consider fuzzy sets $0_1 \in \mathbf{L}^{X_1}$, $1_2 \in \mathbf{L}^{X_2}$, and $1_3 \in \mathbf{L}^{X_3}$ defined for any $x_1 \in X_1$, $x_2 \in X_2$, and $x_3 \in X_3$ by $0_1(x_1) = 0$, $1_2(x_2) = 1$, and $1_3(x_3) = 1$. Due to (ii),

$$0 \otimes 1_2 \otimes 1_3 \subseteq I \text{ iff } \bigcap_{x_1\in X_1}\tilde{\kappa}_1(x_1, 0) \cap \bigcap_{x_2\in X_2}\tilde{\kappa}_2(x_2, 1) \cap \bigcap_{x_3\in X_3}\tilde{\kappa}_3(x_3, 1) \neq \emptyset$$

Let $v = \nabla_{23}(V, V)$. Then v is an 23-bound of $\langle V, V\rangle$ and, therefore, $w \lesssim_2 v$ and $w \lesssim_3 v$ for each $w \in V$. Applying (1.7) we get that $v \lesssim_1 w$ for each $w \in V$.

We now claim that v is the only member of

$$\bigcap_{x_2\in X_2}\tilde{\kappa}_2(x_2, 1) \,\cap\, \bigcap_{x_3\in X_3}\tilde{\kappa}_3(x_3, 1).$$

Indeed, assume by contradiction that there is $w \neq v$ such that

$$w \in \bigcap_{x_2 \in X_2} \tilde{\kappa}_2(x_2, 1) \cap \bigcap_{x_3 \in X_3} \tilde{\kappa}_3(x_3, 1).$$

Then $w \in \tilde{\kappa}_2(x_2, 1)$ for each $x_2 \in X_2$ and by (iii) $w \in \tilde{\kappa}_2(x_2, a)$ for each $x_2 \in X_2$ and $a \in L$. (i) implies that $v \sim_2 w$ (if $w <_2 v$ then $[v)_i$ cannot be obtained as an intersection of some subset of $\tilde{\kappa}_2(X_2 \times L)$). Similarly, we get $w \sim_3 v$. Therefore, (1.7) and (1.8) imply $v = w$, a contradiction.

Moreover, we have

$$\begin{aligned}
&\bigcap_{x_1 \in X_1} \tilde{\kappa}_1(x_1, 0) \cap \bigcap_{x_2 \in X_2} \tilde{\kappa}_2(x_2, 1) \cap \bigcap_{x_3 \in X_3} \tilde{\kappa}_3(x_3, 1) \neq \emptyset \text{ iff} \\
\text{iff } &\bigcap_{x_1 \in X_1} \tilde{\kappa}_1(x_1, 0) \cap \{v\} \neq \emptyset \text{ iff} \\
\text{iff } &v \in \bigcap_{x_1 \in X_1} \tilde{\kappa}_1(x_1, 0) \text{ iff} \\
\text{iff } &v \in \tilde{\kappa}_1(x_1, 0) \text{ for each } x_2 \in X_2.
\end{aligned}$$

Since $v \lesssim_1 w$ for each $w \in V$ and since $\tilde{\kappa}_1(x_1, 0)$ is an order filter, it follows that $\tilde{\kappa}_1(x_1, 0) = V$. The proofs for $i = 2, 3$ are analogous.

□

Remark 15. Let us see that Theorem 13 indeed generalizes Wille's basic theorem of triadic concept analysis [53]. This is clear for (1) because Wille's is a particular case of (1) for $L = \{0, 1\}$. For part (2), we show that for $L = \{0, 1\}$, the existence of mappings $\tilde{\kappa}_i$ satisfying (i), (ii), and (iii), is equivalent to Wille's conditions, i.e. to the existence of mappings $\tilde{\kappa}'_i : X_1 \to \mathcal{F}_i(\mathbf{V})$ satisfying: (i') $\tilde{\kappa}'_i(X_i)$ is i-dense in $\mathbf{V}$ and (ii') $A_1 \times A_2 \times A_3 \subseteq I$ iff $\bigcap_{i=1}^{3} \bigcap_{x_i \in A_i} \tilde{\kappa}'_i(x_i) \neq \emptyset$. In doing so, we identify sets and relations (used in the ordinary setting) with their characteristic functions (i.e. with fuzzy sets, used in a fuzzy setting).

Let $\tilde{\kappa}_i$ satisfy (i)–(iii). Define $\tilde{\kappa}'_i(x_i) = \tilde{\kappa}_i(x_i, 1)$. To see that $\tilde{\kappa}'_i$ satisfy (i') and (ii'), it is sufficient to observe that $\tilde{\kappa}_i(x_i, 0) = V$ for every $i = 1, 2, 3$ and $x_i \in X_i$. The fact $\tilde{\kappa}_i(x_i, 0) = V$ was established in Lemma 14. If $\tilde{\kappa}'_i$ satisfy (i') and (ii') then putting $\tilde{\kappa}_i(x_i, 0) = V$ and $\tilde{\kappa}_i(x_i, 1) = \tilde{\kappa}'_i(x_i)$, $\tilde{\kappa}_i$ clearly satisfy (i)–(iii).

2.5 Reduction to ordinary TCA

Recall that for a fuzzy set $A \in L^X$, we define an ordinary set $\lfloor A \rfloor \subseteq X \times L$ by

$$\lfloor A \rfloor = \{(x, a) \mid x \in X, a \in L, A(x) \geq a\} \tag{2.10}$$

In the opposite direction, given an ordinary set $B \subseteq X \times L$ such that $(x, a) \in B$ implies $(x, b) \in B$ for all $b \leq a$, and the set $\{a \mid (x, a) \in B\}$ has the greatest element, the fuzzy set $\lceil B \rceil$ is defined by

$$\lceil B \rceil(x) = \bigvee \{a \mid (x, a) \in B\}. \tag{2.11}$$

We may thing of $\lfloor A \rfloor$ as the area below A, while of $\lceil B \rceil$ as the upper envelope of B. Initially, these mapping were studied in [7] and further developed in [8] (they were also independently introduced in [49]). In what follows, some of the properties of $\lfloor\ \rfloor$ and $\lceil\ \rceil$ are used. An interested reader can find their detailed description in [8].

The following theorem establishes a connection between TCA in fuzzy setting and ordinary TCA.

Theorem 16. (crisp representation) *Let* $\mathbf{K} = \langle X_1, X_2, X_3, I \rangle$ *be a fuzzy triadic context and* $\mathbf{K}_{crisp} = \langle X_1 \times L, X_2 \times L, X_3 \times L, I_{crisp} \rangle$ *with* I_{crisp} *defined by* $((x_1, a), (x_2, b), (x_3, c)) \in I_{crisp}$ *iff* $a \otimes b \otimes c \leq I(x_1, x_2, x_3)$ *be a triadic context. Then* $\mathcal{T}(\mathbf{K})$ *is isomorphic to* $\mathcal{T}(\mathbf{K}_{crisp})$.

PROOF. Consider mappings $\varphi : \mathcal{T}(\mathbf{K}) \to \mathcal{T}(\mathbf{K}_{crisp})$ defined by

$$\varphi(\langle A_1, A_2, A_3 \rangle) = \langle \lfloor A_1 \rfloor, \lfloor A_2 \rfloor, \lfloor A_3 \rfloor \rangle, \tag{2.12}$$

and $\psi : \mathcal{T}(\mathbf{K}_{crisp}) \to \mathcal{T}(\mathbf{K})$ defined by

$$\psi(\langle B_1, B_2, B_3 \rangle) = \langle \lceil B_1 \rceil, \lceil B_2 \rceil, \lceil B_3 \rceil \rangle. \tag{2.13}$$

Theorem 12 implies that $\langle \lfloor A_1 \rfloor, \lfloor A_2 \rfloor, \lfloor A_3 \rfloor \rangle \in \mathcal{T}(\mathbf{K}_{crisp})$ for all $\langle A_1, A_2, A_3 \rangle \in \mathcal{T}(\mathbf{K})$, and $\psi(\langle B_1, B_2, B_3 \rangle) \in \mathcal{T}(\mathbf{K})$ for all $\langle B_1, B_2, B_3 \rangle \in \mathcal{T}(\mathbf{K}_{crisp})$.

Namely, let $\langle A_1, A_2, A_3 \rangle \in \mathcal{T}(\mathbf{K})$. Then

$$\begin{aligned} (x_i, b) \in (\lfloor A_j \rfloor^{(i,j,\lfloor A_k \rfloor)} &\quad \text{iff} \\ \text{for all } ((x_j, a), (x_k, c)) \in \lfloor A_j \rfloor \times \lfloor A_k \rfloor & \\ \{(x_i, b), (x_j, a), (x_k, c)\} \in I_{crisp} &\quad \text{iff} \\ \text{for all } x_j \in X_j, x_k \in X_k, a \leq A_j(x_j), b \leq A_k(x_k) & \\ a \otimes b \otimes c \leq I\{x_i, x_j, x_k\} &\quad \text{iff} \\ \text{for all } x_j \in X_j, x_k \in X_k & \\ A_j(x_j) \otimes A_k(x_k) \otimes b \leq I\{x_i, x_j, x_k\} &\quad \text{iff} \\ b \leq A_i(x_i). & \end{aligned}$$

This proves $\langle \lfloor A_1 \rfloor, \lfloor A_2 \rfloor, \lfloor A_3 \rfloor \rangle \in \mathcal{T}(\mathbf{K}_{crisp})$.

For the opposite direction, let $\langle B_1, B_2, B_3\rangle \in \mathcal{T}(\mathbf{K}_{crisp})$. Then

$$
\begin{aligned}
(\lceil A_j\rceil^{(i,j,\lceil A_k\rceil)}(x_i) = b \quad &\text{iff}\\
b = \bigvee\{a \mid \lceil A_j\rceil(x_j) \otimes \lceil A_k\rceil(x_j) \otimes a \leq I(x_i, x_j, x_k), x_j \in X_j, x_k \in X_k\} \quad &\text{iff}\\
b = \bigvee\{a \mid ((x_i, a), (x_j, c), (x_k, d)) \in I_{crisp}, ((x_j, c), (x_k, d)) \in A_j \times A_k\} \quad &\text{iff}\\
b = \bigvee\{a \mid (x_i, a) \in A_j^{(i,j,A_k)}\} = \lceil A_i\rceil(x_i)
\end{aligned}
$$

Therefore $\psi(\langle B_1, B_2, B_3\rangle) \in \mathcal{T}(\mathbf{K})$.

Since $\lceil\lfloor A\rfloor\rceil = A$ for each fuzzy set A, the mappings φ and ψ are mutually inverse and φ is a bijection. Moreover, $\lfloor A\rfloor \subseteq \lfloor B\rfloor$ iff $A \subseteq B$ for all fuzzy sets A and B and thus φ preserves $\lesssim_1, \lesssim_2, \lesssim_3$. □

Theorem 16 can be seen as a way in which one can transfer the results known from ordinary TCA into TCA in fuzzy setting. As an example, we provide an alternative proof of Theorem 13.

PROOF. (An alternative proof of basic theorem of TCA in fuzzy setting) (1) Follows directly from Theorem 16.

(2) We will denote the conditions (i) and (ii) of Theorem 2 by (i)^w and (ii)^w. Let $i \in \{1, 2, 3\}$.

"⇒": Assume that there are mappings $\tilde{\kappa}_i$ such that (i), (ii), (iii) hold. It suffices to show that then there are mappings $\tilde{\kappa}_i^w$ such that (i)^w and $(ii)^w$ hold, because in such case V is isomorphic to $\mathcal{T}(\mathbf{K}_{crisp})$ and by Theorem 16 V is isomorphic to $\mathcal{T}(\mathbf{K})$.

Consider the maps $\tilde{\kappa}_i^w : (X_i \times L) \to \mathcal{F}_i(\mathbf{V})$ defined by $\tilde{\kappa}_i^w((x_i, a)) = \tilde{\kappa}_i(x_i, a)$. It is easy to see that $\tilde{\kappa}_i^w(X_i \times L)$ is i-dense iff $\tilde{\kappa}_i(X_i \times L)$ is i-dense, which proves the condition (i)^w. For all $A_i \in L^{X_i}$ we have that $A_1 \times A_2 \times A_3 \subseteq I$ iff $\lfloor A_1\rfloor \times \lfloor A_2\rfloor \times \lfloor A_3\rfloor \subseteq I_{crisp}$. Since $A_i(x_i) = \vee\{a \mid (x_i, a) \in \lfloor A_i\rfloor\}$ we have $\cap_{i=1}^{3} \cap_{x_i \in X_i} \tilde{\kappa}_i(x_i, A_i(x_i)) \neq \emptyset$ iff $\cap_{i=1}^{3} \cap_{x_i \in X_i} \tilde{\kappa}_i(x_i, \vee\{a \mid (x_i, a) \in \lfloor A_i\rfloor\}) \neq \emptyset$ Using (iii) we obtain $\cap_{i=1}^{3} \cap_{x_i \in X_i} \tilde{\kappa}_i(x_i, \vee\{a \mid (x_i, a) \in \lfloor A_i\rfloor\}) \neq \emptyset$ iff $\cap_{i=1}^{3} \cap_{x_i \in X_i} \cap_{a \leq A_i(x_i)} \tilde{\kappa}_i(x_i, a) \neq \emptyset$ iff $\cap_{i=1}^{3} \cap_{(x_i,a) \in \lfloor A_i\rfloor} \tilde{\kappa}_i^w(x_i, a) \neq \emptyset$ by definition of $\tilde{\kappa}_i^w$. Hence the proof of (ii)^w.

"⇐": Assume that V is isomorphic to $\mathcal{T}(\mathbf{K})$. By Theorem 16 it is also isomorphic to $(\mathbf{K}_{crisp})$ and thus by Theorem 2 there are mappings $\tilde{\kappa}_i^w$ such that (i)^w and (ii)^w hold. It remains to show, that there are $\tilde{\kappa}_i$ that comply with (i), (ii), and (iii).

Denote by $\varphi^w : \mathcal{T}(\mathbf{K}_{crisp}) \to V$ the isomorphism between V and $\mathcal{T}(\mathbf{K}_{crisp})$. Let κ_i^w be mappings $\kappa_i^w : (X_i \times L) \to \mathcal{F}_i(\mathcal{T}(\mathbf{K}_{crisp}))$ defined by

$$\kappa_i((x_i, a)) = \{\langle B_1, B_2, B_3\rangle \mid (x_i, a) \in B_i\}. \tag{2.14}$$

Then the maps $\tilde{\kappa}_i^w((x_i, a)) = \varphi^w(\kappa_i((x_i, a)))$ fulfill (i)^w and (ii)^w (see the proof of Theorem 2). If we define $\tilde{\kappa_i} : X_i \times L \to \mathcal{F}_i(V)$ by

$$\tilde{\kappa_i}(x_i, a) = \tilde{\kappa}_i^w((x_i, a)), \tag{2.15}$$

we immediately obtain that (i) holds. To see that (ii) holds, observe that $A_1 \otimes A_2 \otimes A_3 \in I$ iff $\lfloor A_1 \rfloor \times \lfloor A_2 \rfloor \times \lfloor A_3 \rfloor \subseteq I_{crisp}$ iff $\cap_{i=1}^3 \cap_{(x_i,a) \in \lfloor A_i \rfloor} \tilde{\kappa}_i^w((x_i, a)) \neq \emptyset$ iff $\cap_{i=1}^3 \cap_{(x_i \in X_i} \tilde{\kappa_i}(x_i, a) \neq \emptyset$ for all $A_i \in L^{X_i}$. It remains to prove (iii). (2.13) implies that for all $\langle A_1, A_2, A_3 \rangle \in \mathcal{T}(\mathbf{K}_{crisp})$ if $(x_i, a) \in A_i$ then $(x_i, b) \in A_i$ for all $b \leq a$. By (2.14) and (2.15) we get that $b \leq a$ implies $\tilde{\kappa_i}(x_i, a) \subseteq \tilde{\kappa_i}(x_i, b)$. □

2.6 Algorithms

This section is devoted to algorithms for computation of the set of triadic fuzzy concepts. For Boolean matrices one can compute triadic concepts using Trias algorithm proposed in [36]. However, for matrices with grades no such algorithm exists. We present two ways to compute all triadic fuzzy concepts. The first approach consist in transformation of the matrix with grades into ordinary matrix, computation of the set of ordinary triadic concepts using the Trias algorithm (or any other algorithm for computation of concept trilattice in ordinary setting), and transformation of the result back into fuzzy setting. The second algorithm is an extension of the Trias algorithm to the case of graded data that allows for a direct computation of triadic fuzzy concepts. We prove correctness of the presented algorithms and discuss their computational complexity.

2.6.1 Transformation to the ordinary case

The idea behind algorithmic approach in this Section is a consequence of Theorem 16. It follows that we can compute $\mathcal{T}(\mathbf{K})$ by carrying out the following steps. First, we transform $\mathbf{K}$ into $\mathbf{K}_{crisp}$, i.e. we transform data with fuzzy attributes into data with ordinary attributes. As a second step, we use an already existing algorithm for computation of $\mathcal{T}(\mathbf{K}_{crisp})$ (e.g. Trias algorithm [36]). Finally, to compute $\mathcal{T}(\mathbf{K})$ from $\mathcal{T}(\mathbf{K}_{crisp})$ we use (2.13). From Theorem 16 it follows that this way we indeed obtain $\mathcal{T}(\mathbf{K})$. The algorithm is listed as Algorithm 1.

On lines 1-4, the transformation of the input triadic fuzzy context $\mathbf{K}$ to an ordinary context is carried out. The next step, on line 5, is the computation of concept trilattice using some algorithm for the ordinary case. Finally, on lines 7-8, the ordinary triadic concepts are transformed back into fuzzy triadic concepts (see the map ψ in the proof of the previous theorem).

Algorithm 1: COMPUTECONCEPTS($\langle X_1, X_2, X_3, I\rangle$)

1. $I_{crisp} \leftarrow \emptyset$
2. **foreach** $(x_1, x_2, x_3) \in X_1 \times X_2 \times X_3$:
3.   **foreach** $(a, b, c) \in L \times L \times L$ **such that** $a \otimes b \otimes c \leq I(x_1, x_2, x_3)$:
4. $I_{crisp} \leftarrow I_{crisp} \cup \{((x_1, a), (x_2, b), (x_3, c))\}$
5. $\mathcal{F}_{crisp} \leftarrow$ COMPUTEORDINARYCONCEPTS($\langle X_1 \times L, X_2 \times L, X_3 \times L, I_{crisp}\rangle$)
6. $\mathcal{F} \leftarrow \emptyset$
7. **foreach** $\langle A_1, A_2, A_3\rangle \in \mathcal{F}_{crisp}$:
8. $\mathcal{F} \leftarrow \mathcal{F} \cup \{\langle \lceil A_1 \rceil, \lceil A_2 \rceil, \lceil A_3 \rceil\rangle\}$
9. **return** $\mathcal{F}$

Complexity Since the complexity of the whole algorithm depends on the choice of COMPUTEORDINARYCONCEPTS, we discuss only the complexity of transformations from and to ordinary setting. The cycle on line 2 lasts $|X_1| \cdot |X_2| \cdot |X_3|$ iterations, while the cycle on line 3 lasts $|L|^3$ iterations. The backwards transformation on lines 7-8 takes $|\mathcal{T}(\mathbf{K})| \cdot |L| \cdot (|X_1| + |X_2| + |X_3|)$ operations. Since in the worst case the number of triadic concepts is exponential in the size of its context, the later term dominates the time complexity. Therefore the complexity of the transformations is $O(|\mathcal{T}(\mathbf{K})| \cdot |L| \cdot (|X_1| + |X_2| + |X_3|))$.

2.6.2 TRIAS in fuzzy setting

In this section we show that by a direct fuzzification of TRIAS algorithm [36] we obtain a direct algorithm for computation of the set of triadic fuzzy concepts present in the input data. We call the algorithm FUZZYTRIAS and list it as Algorithm 2.

For an input triadic fuzzy context $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$, FUZZYTRIAS first on lines 2-3 constructs a dyadic fuzzy context $\mathbf{K^{(1)}} = \langle X_1, X_2 \times X_3, I^{(1)}\rangle$ where the binary relation $I^{(1)}$ is defined by $I^{(1)}(x_1, \langle x_2, x_3\rangle) = I(x_1, x_2, x_3)$. Then it calls subroutines FIRSTCONCEPT and NEXTCONCEPT to compute and iterate through the set of formal concepts of $\mathbf{K^{(1)}}$. These subroutines form an interface to some algorithm for computation of concept lattice of dyadic fuzzy context, such are NextClosure [1] or Lindig algorithm [2]. FIRSTCONCEPT returns the first generated concept, NEXTCONCEPT returns the concept generated after the one passed to it as an argument. In the pseudocode we use a convention that NEXTCONCEPT returns *false* if its argument is the last generated concept. Any other returned value is, when interpreted as logical value, considered *true*. It does not matter whether the algorithm first generates all formal concepts and then iterates through them, or it computes formal concepts on demand. On lines 4-11 FUZZYTRIAS iterates through all concepts $\langle A, B\rangle$ of $\mathbf{K}^{(1)}$. The extent A is

Algorithm 2: FuzzyTrias($\langle X_1, X_2, X_3, I\rangle$)

1. **foreach** ($\langle x_1, x_2, x_3\rangle \in X_1 \times X_2 \times X_3$):
2. $I^{(1)}(x_1, \langle x_2, x_3\rangle) \leftarrow I(x_1, x_2, x_3)$
3. $\mathcal{T} \leftarrow \emptyset$
4. $\langle A, B\rangle \leftarrow$ FirstConcept($\langle X_1, X_2 \times X_3, I^{(1)}\rangle$)
5. **do**
6. $\langle C, D\rangle \leftarrow$ FirstConcept($\langle X_2, X_3, B\rangle$)
7.   **do**
8.   **if** $A = C^{(1,2,D)}$
9.   **then** $\mathcal{T} \leftarrow \mathcal{T} \cup \{\langle A, C, D\rangle\}$
10.   **while** $\langle C, D\rangle \leftarrow$ NextConcept($\langle X_2, X_3, B\rangle, \langle C, D\rangle$)
11. **while** $\langle A, B\rangle \leftarrow$ NextConcept($\langle X_1, X_2 \times X_3, I^{(1)}\rangle, \langle A, B\rangle$)
12. **return** $\mathcal{T}$

considered a candidate for an extent of some triconcepts of $\mathcal{T}(\mathbf{K})$, the intent B is in fact a binary fuzzy relation between X_2 and X_3 and thus can be understood as dyadic fuzzy context $\langle X_2, X_3, B\rangle$. On lines 7-10 the algorithm iterates through all formal concepts $\langle C, D\rangle$ of this context. For each $\langle C, D\rangle$ it checks if $A = C^{(1,2,D)}$ (which is in fact a check whether $\langle A, C, D\rangle$ is a triadic concept; see the proof of correctness that follows). If so, $\langle A, C, D\rangle$ is added to the set of triadic fuzzy concepts. At the end, the set of all triadic concepts is returned.

Correctness We need the following lemmas.

Lemma 17. *If $\langle A_1, A_2, A_3\rangle \in \mathcal{T}(\mathbf{K})$, then $\langle A_i, A_j\rangle \in \mathcal{B}(\mathbf{K}^{ij}_{A_k})$ for all $\{i, j, k\} \in \{1, 2, 3\}$*

PROOF. Easy to observe from the definition of triadic fuzzy concept. $\square$

Lemma 18. *Let $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$ be fuzzy triadic context and $\mathbf{K}^{(1)} = \langle X_1, X_2 \times X_3, I^{(1)}\rangle$ a fuzzy dyadic context with $I^{(1)}(x_1, \langle x_2, x_3\rangle) = I(x_1, x_2, x_3)$. Then*

(i) $A_1^{\downarrow_{I^{(1)}}} = I^{23}_{A_1}$ *for all* $A_1 \in L^{X_1}$;

(ii) *if* $\langle A_1, A_2, A_3\rangle \in \mathcal{T}(\mathbf{K})$ *then* A_1 *is an extent of some concept in* $\mathcal{B}(\mathbf{K}^{(1)})$

PROOF.

(i) By an easy computation

$$
\begin{aligned}
A_1^{\downarrow_{I^{(1)}}}(x_2,x_3) &= \bigwedge_{x_1\in X_1} A_1(x_1)\to I^{(1)}(x_1,(x_2,x_3)) = \\
&= \bigwedge_{x_1\in X_1} A_1(x_1)\to I(x_1,x_2,x_3) = I^{23}_{A_1}(x_2,x_3)
\end{aligned}
$$

(ii) Since extents of $\mathcal{B}(\mathbf{K}^{(1)})$ are just fixpoints of the closure operator ${}^{\downarrow_{I^{(1)}}\uparrow_{I^{(1)}}}$ (see [8]), we have to prove that A_1 is a fixpoint of ${}^{\downarrow_{I^{(1)}}\uparrow_{I^{(1)}}}$. Since $A_1 \subseteq A_1^{\downarrow_{I^{(1)}}\uparrow_{I^{(1)}}}$ hold by definition of a closure operator, it suffices to prove $A_1^{\downarrow_{I^{(1)}}\uparrow_{I^{(1)}}} \subseteq A_1$. Since for every fuzzy binary relation I it holds $I(x,y) = \bigvee_{\langle A,B\rangle\in\mathcal{B}(I)} A(x)\otimes B(y)$, we have

$$
\begin{aligned}
&A_1^{\downarrow_{I^{(1)}}\uparrow_{I^{(1)}}}(x_1) = I^{23\,\uparrow_{I^{(1)}}}_{A_1}(x_1) = \\
&= \bigwedge_{(x_2,x_3)\in X_2\times X_3} I^{23}_{A_1}(x_2,x_3)\to I^{(1)}(x1,(x_2,x_3)) = \\
&= \bigwedge_{(x_2,x_3)\in X_2\times X_3} \Big(\bigvee_{\langle B_2,B_3\rangle\in\mathcal{B}(I^{(1)})} B_2(x_2)\otimes B_3(x_3)\Big)\to I^{(1)}(x1,(x_2,x_3)) = \\
&= \bigwedge_{(x_2,x_3)\in X_2\times X_3} \bigwedge_{\langle B_2,B_3\rangle\in\mathcal{B}(I^{(1)})} B_2(x_2)\otimes B_3(x_3)\to I^{(1)}(x1,(x_2,x_3)) \le \\
&\le \bigwedge_{(x_2,x_3)\in X_2\times X_3} A_2(x_2)\otimes A_3(x_3)\to I(x_1,x_2,x_3) = A_1(x_1)
\end{aligned}
$$

$\square$

Remark 19. (a) It is easy to see that similar properties hold for dyadic contexts $\mathbf{K}^{(2)}$ and $\mathbf{K}^{(3)}$.

(b) The opposite direction of (ii) does not hold. Indeed, there is a fuzzy triadic context $\mathbf{K}$ such that there is $A \in Ext(\mathbf{K}^{(1)})$ which is not an extent of any triconcept of $\mathcal{T}(\mathbf{K})$. Namely, let $\mathbf{L}$ be a three-element Łukasiewicz chain and $\mathbf{K} = \langle X,Y,Z,I\rangle$ be given by the following table.

	z_1			z_2		
	y_1	y_2	y_3	y_1	y_2	y_3
x_1	0.5	1	0	0	0.5	1
x_2	0	1	0	1	1	1
x_3	0	0	0.5	1	0	1

Then $\{0/x_1, 0.5/x_2, 1/x_3\}$ is an extent of $\mathcal{B}(\mathbf{K}^{(1)})$, but it is not an extent of $\mathcal{T}(\mathbf{K})$. Note that if we set $\mathbf{L}$ to a two-element Boolean algebra, we can find a counterexample showing that the opposite direction of (ii) does not hold for the ordinary case either.

The following theorem shows that FuzzyTrias is correct.

Theorem 20. *Given a triadic fuzzy context* $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$*,* FuzzyTrias *outputs* $\mathcal{T}(\mathbf{K})$*.*

PROOF. First, observe that the following claims hold.

Claim 1: For every triadic concept $\langle A_1, A_2, A_3\rangle \in \mathcal{T}(\mathbf{K})$ *there is* $B \in L^{X_2\times X_3}$ *such that* $\langle A_1, B\rangle \in \mathcal{B}(X_1, X_2\times X_3, I^{(1)})$ *and* $\langle A_2, A_3\rangle \in \mathcal{B}(X_2, X_3, B)$*.*

Proof of Claim 1. By Lemma 18 (ii) A_1 is an extent of $\mathcal{B}(X_1, X_2\times X_3, I^{(1)})$ and therefore $B = A_1^{\downarrow_{I^{(1)}}}$. Moreover, by Lemma 18 (i), B is precisely the relation $I^{23}_{A_1}$. Lemma 17 then implies that $\langle A_2, A_3\rangle \in \mathcal{B}(X_2, X_3, B)$. *QED Claim 1.*

Claim 2: For every $\langle A, B\rangle \in \mathcal{B}(X_1, X_2 \times X_3, I^{(1)})$ *and every* $\langle C, D\rangle \in \mathcal{B}(X_2, X_3, B)$ *it holds that if* $A = C^{(1,2,D)}$ *then* $\langle A, C, D\rangle$ *is a triadic fuzzy concept.*

Proof of Claim 2: Since by Lemma 18 (ii) $B = I^{23}_A$ and because $\langle C, D\rangle \in \mathcal{B}(X_2, X_3, B)$, we have $C = D^{(2,3,A)}$ and $D = C^{(2,3,A)}$. Since $A = C^{(1,2,D)}$ holds by assumption Theorem 11 yields that $\langle A, C, D\rangle$ is a triadic concept. *QED Claim 2.*

From Claim 2 it follows that each triple $\langle A, C, D\rangle$ that passes the test on line 8 is a triadic concept of $\mathbf{K}$. Claim 1 then implies that every triadic concept of $\mathbf{K}$ is generated on lines 4-11. $\square$

Complexity The time complexity of FuzzyTrias depends on the time complexity of underlying algorithm for computation of dyadic fuzzy concepts. It is well known, that in the worst case the number of dyadic fuzzy concepts is exponential in the size of the input data and in the number of degrees in the residuated lattice, and that the computation of one dyadic fuzzy concept takes polynomial time. The sizes of $\mathbf{K}^{(1)}$ and I^{23}_A (for any $A \in L^{X_1}$) are linear in the size of $\mathbf{K}$. Since FuzzyTrias contains two nested cycles that iterate through all the dyadic fuzzy concepts of $\mathbf{K}^{(1)}$ and I^{23}_A (lines 5-11) we can conclude that the number of iteration the algorithm goes through is exponential in the size of input. Since the complexity of operations done for each iteration of the inner cycle (lines 7-10) and the complexity of the creation of $\mathbf{K}^{(1)}$ (lines 1-2) are polynomial, the complexity of the whole algorithm is dominated by the number of dyadic fuzzy concepts. Therefore, we can conclude that the time complexity of FuzzyTrias is $O(p_1(|X_1|, |X_2|, |X_3|, |L|)\cdot|\mathcal{B}(\mathbf{K}^{(1)})|\cdot p_2(|X_1|, |X_2|, |X_3|, |L|)\cdot \max_{\langle A,B\rangle\in\mathcal{B}(\mathbf{K}^{(1)})}\{|\mathbf{B}(X_2, X_3, B)|\})$, where p_1 and p_2 are polynomials that capture the time of computation of a dyadic concept and their exact form depends on the algorithm we choose for this task.

We can look at FuzzyTrias as at an algorithm that produces one triadic concept at a time, and then proceeds to compute the next one. That is, every time the test on line 9 succeeds, instead of adding $\langle A, C, D\rangle$ into the set $\mathcal{T}$ of so-far computed triadic concepts FuzzyTrias outputs $\langle A, C, D\rangle$ directly. Now, we are interested in the following question: How much time does it take to compute one triadic concept? That is, once FuzzyTrias outputs a triadic concept, how much time does it take to output the next one? To answer this question, recall from Remark 19 (b) that extent A of dyadic concept $\langle A, B\rangle$ produced on line 4 (or line 11 in further iterations) is not necessarily an extent of a triadic concept. In such a case we call A a *false candidate*. For every false candidate A, the test on line 9 fails in every iteration of the cycle on lines 6-10. That is, FuzzyTrias computes all dyadic concepts of $\langle X_2, X_3, B\rangle$, but does not output a triadic concept. Since the number of dyadic concepts of $\langle X_2, X_3, B\rangle$ (and hence the number of iterations of the cycle on lines 6-10) is, in the worst case, exponential in $|X_1|, |X_2|, |X_3|$, and $|L|$, we can conclude that computation of one triadic may take exponential time.

2.6.3 Experiments

In this section we present results obtained from experiments we performed in order to compare algorithms introduced in the previous section and to study their behavior experimentally. Its contents should by no means be understood as a full experimental evaluation of the algorithms. Rather, we illustrate some interesting points about both algorithms and the presented results serve as basis of and motivation for their further improvement.

We implemented both algorithms in ANSI C programing language and ran them on dedicated machine with Intel Xeon 1.6GH CPU and 6 GB of memory. In our implementation of FuzzyTrias we used the fuzzy NextClosure algorithm (see [1]) for computation of dyadic fuzzy concepts. In ComputeConcepts we used Trias ([36]) with Cbo under the hood (see [41]) for the ComputeOrdinaryConcepts subroutine. We performed the experiments on 100 000 randomly generated small triadic contexts. To generate them, we used the following procedure: First, we generated a random permutation of context entries. Then we iterated through this permutation and for each entry we randomly generated (with uniform distribution) a truth degree. This way we ensured a variability of frequencies of degree occurrences among rows, columns, and modi. Unless stated otherwise, we used a three-element Łukasiewicz chain as a scale of truth degrees in all experiments.

In the first set of experiments we compared the two algorithms developed in the previous section. For each input matrix we kept track of the number of concepts, running times of both algorithms and the number of false extent

Table 2.1: Comparison of FuzzyTrias and ComputeConcepts in terms of running time in ms. Each row corresponds to running the algorithms on a 1000 randomly generated triadic contexts of the specified size. The table contains mean values and standard deviations of measured results.

	No. of concepts		FuzzyTrias		Reduction	
	mean	sd	mean	sd	mean	sd
5-5-5	371	63	105	19	697	136
10-5-5	1031	134	1524	343	15142	3562
5-10-5	1025	133	543	62	4211	559
5-5-10	1028	138	766	98	4022	586
19-5-5	2258	252	17074	3988	232057	59920
5-19-5	2261	255	2164	163	17494	1493
5-5-19	2124	222	3706	300	15431	1366

candidates generated during the run of FuzzyTrias. The measurements are for selected sizes of triadic contexts summarized in Table 2.1. Finer visualization of the comparison can be seen in Figure 2.1. For each dimension, we gradually enlarged its size and kept the original size of the two remaining dimensions. This way, we could observe the behavior of algorithms for each dimension in isolation. Clearly, the experiments suggest that FuzzyTrias is in terms of time complexity more efficient than ComputeConcepts. We may also observe, that the difference grows more rapidly if we enlarge the number of objects, than if we enlarge the number of attributes, or conditions. This is caused by the fact, that the number of false extent candidates that Trias produces grows faster if we enlarge the number of objects. This property of Trias is exhibited also by FuzzyTrias, as we can see in Figure 2.2. Enlarging the number of objects leads to faster grow of time complexity than enlarging the number of attributes or conditions, because the number of false extent candidates grows faster for objects than for attributes or conditions. However, we need to keep in mind that we are dealing with data where two dimension are of comparable sizes and the third one is considerably larger. In order to clarify this we run FuzzyTrias on cube shaped triadic contexts with all dimensions set to equal size. The results are depicted on Figure 2.4. The number of false candidates does not grow so fast as if we enlarge only the number of objects, but still considerably faster than if we enlarge only the number of attributes, or conditions.

The last experiment deals with dependency of time complexity of FuzzyTrias on the number of truth degrees. We fixed the size of triadic contexts to $7\times7\times7$, and changed the size of residuated lattices we used. We considered Łukasiewicz and Gödel chains with 3 to 6 elements. The results are depicted in Figure 2.3.

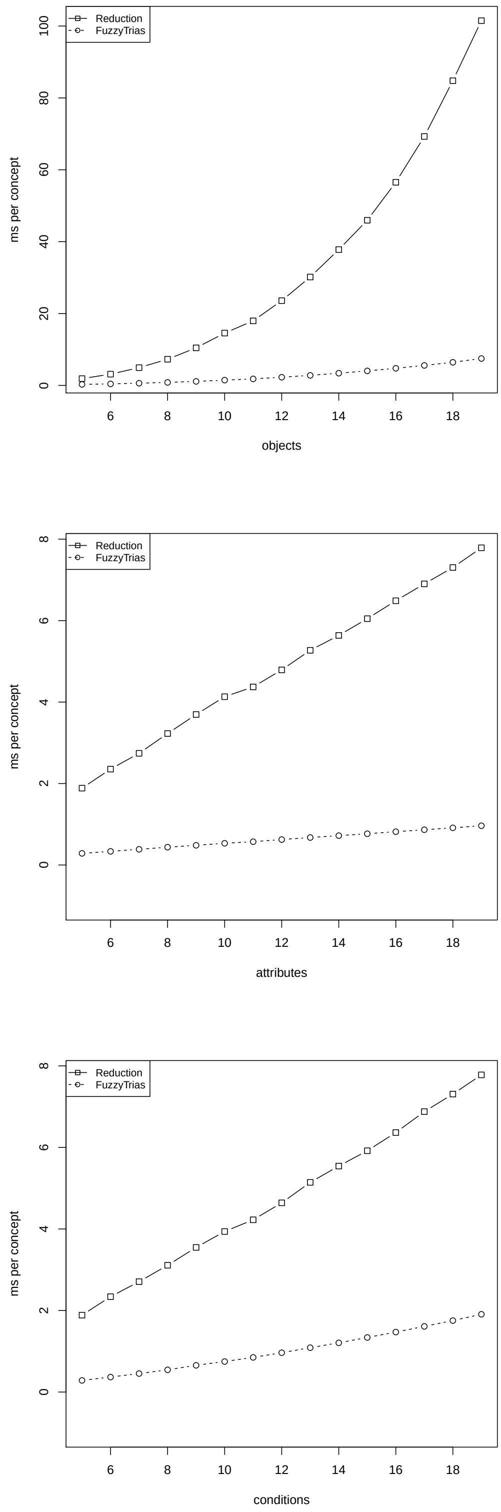
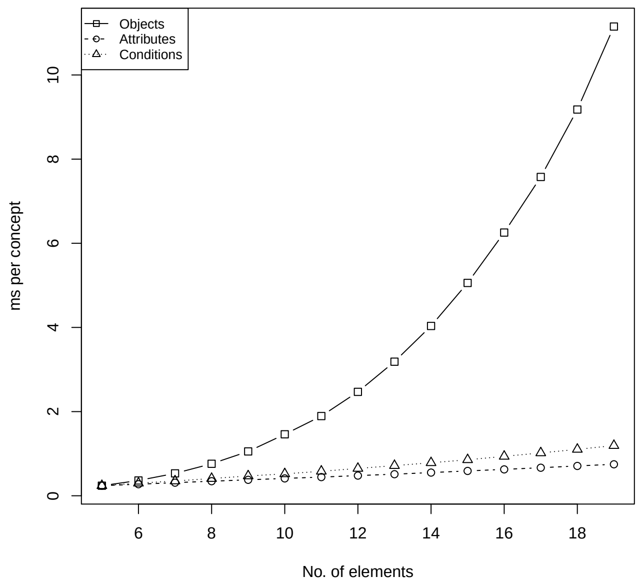


Figure 2.1: Comparison of time efficiencies of FuzzyTrias and ComputeConcepts terms of average time needed to compute one concept. For each graph only one dimension grows, the remaining two are set to 5.



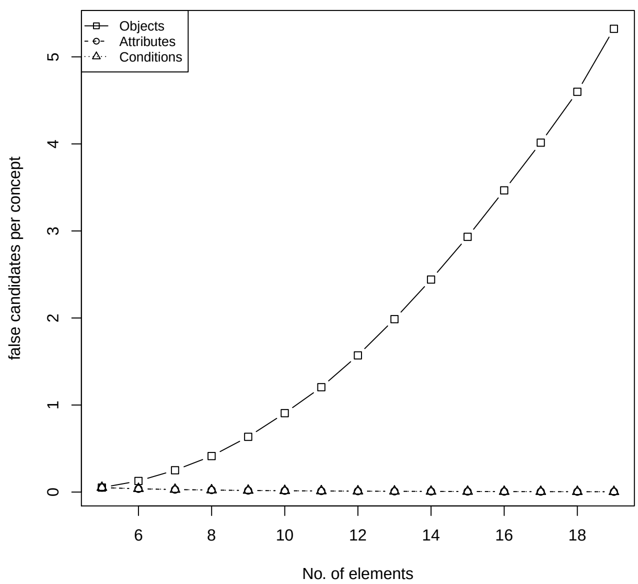


Figure 2.2: Comparison of time complexity and number of false candidates for enlargement of the number of objects, attributes and conditions. In both graphs lines correspond to growth of only one dimension, the remaining two are set to 5.

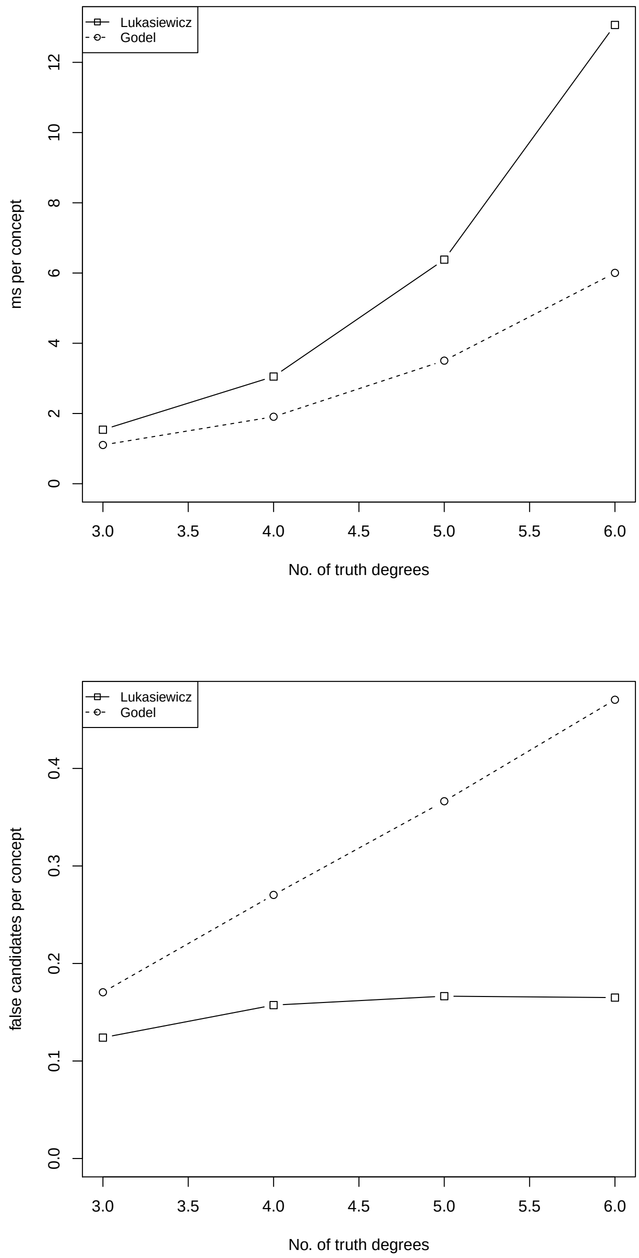


Figure 2.3: Dependency of time complexity and the number of false candidates on the number of truth degrees used.

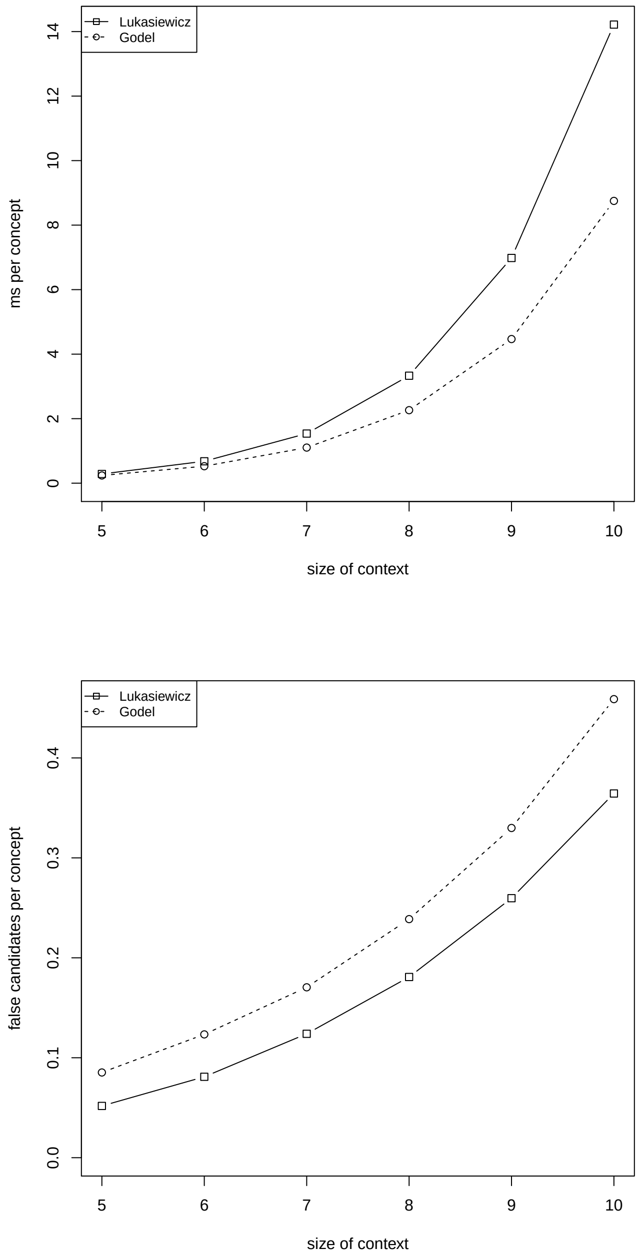


Figure 2.4: Time complexity and false candidates for FuzzyTrias when run on triadic context with all dimensions set to equal size. Łukasiewicz and Gödel three-element chains are considered.

Table 2.2: Triadic fuzzy context “Customers in a restaurant”, (t = taste, a = aroma, l = look, p = price)

	Fry				Bender				Leela				Zoidberg			
	t	a	l	p	t	a	l	p	t	a	l	p	t	a	l	p
steak	1	1	1	0	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	1	1	1	0
salad	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	1	1	1	1	1	1	1	$\frac{1}{2}$
veget.	0	0	$\frac{1}{2}$	1	0	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	1	1	1	0
wings	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	1	0	0	0	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$

2.7 Illustrative examples

In this section, we present for illustration two small examples demonstrating that triadic fuzzy concepts represent interesting patterns in data.

Customer survey One of typical examples of data that are easily transformed into triadic fuzzy context are customer surveys. In such data, we take products as objects, product features as attributes, and customers participating in the survey as conditions. The relation between objects, attributes, and products then captures the degree to which customers regard particular features of products as being of good quality. In our example, we take a customer survey in a hypothetical restaurant as the data we use.

We consider a triadic fuzzy context consisting of a set of four objects, each of which represents a dish in a restaurant (beef steak, cheese salad, vegetable plate and fried chicken wings); a set of four attributes capturing features of the dishes (taste, aroma, look, and price); and a set of four customers who evaluate the dishes (Fry, Bender, Leela, Zoidberg). The context is depicted in Table 2.2.

We use a three element set $\{0, \frac{1}{2}, 1\}$ as a scale of truth degrees with the degrees representing “bad”, “neutral” and “excellent”. A degree to which a dish x, its feature y and a customer z are related is then interpreted as a degree to which according to customer z, x has feature y. For example, the degree 1 to which beef steak, taste and Fry are related is interpreted as Fry considering the beef steak as having excellent taste.

The corresponding fuzzy concept trilattice consists of 112 triadic concepts, therefore we do not comment on the interpretation of all of them. Instead, we present a list of five interesting ones in Table 2.3 to illustrate that triadic concepts are easily interpretable. Namely:

- Concept No. 1 represents a group of customers who find taste and aroma of beaf steak and fried chicken wings excellent and their look at least neutral. We can say that it describes customers who like meat

Table 2.3: Five interesting triadic concepts. The concepts are represented by columns 1, 2, ..., 5.

	1	2	3	4	5
steak	1	0	1	1	1
salad	0	1	1	0	1
veget.	0	0	1	0	1
wings	1	0	1	$\frac{1}{2}$	1
taste	1	1	1	1	1
aroma	1	1	1	$\frac{1}{2}$	1
look	$\frac{1}{2}$	1	1	1	1
price	0	$\frac{1}{2}$	0	0	1
Fry	1	0	0	1	0
Bender	1	0	0	1	0
Leila	0	1	0	0	0
Zoidberg	1	1	1	1	0

dishes for their taste and aroma.

- Concept No. 2 represents customers who like cheese salad for its excellent taste, aroma and look, and partly for its price.

- Concept No. 3 can be interpreted as "customers who have no preferences in food."

- Concept No. 4 represents customers who like beef steak and partly fried chicken wings for their excellent taste and look and at least neutral aroma.

- Concept No. 5 shows that there is no customer who finds all properties, including price, of all dishes excellent.

Let us remark, that the selected concepts are potentially helpful for the management of our imaginary restaurant, because they illustrate the trends in customer behavior and the reasons for their occurrence. For example, concept No. 1 indicates that there is a numerous group of customers who like meat dishes. Moreover, one can see that it is because the customers like the excellent taste and aroma of the dishes.

Students traits In the second example of this Section we consider data describing an evaluation of student's performances in various courses. Such data may be obtained by the way of evaluation forms filled by course tutors. Here the set of objects is the set {a, b, c, d, e, f, g, h } of 8 students. The attributes {in, re, co, cr, id, ca} correspond to students traits: intelligence, responsibility, communication (i.e. the ability to communicate well), creativity, independence, carefulness. The set of conditions is formed by particular courses which students take: Algorithms, Mathematics, Databases, Networks, and Programing. Entries in the table represent degrees to which students exhibit the traits during courses. As a scale of truth degrees we use three-element set $\{0, \frac{1}{2}, 1\}$ with degrees representing "not at all", "partly", and "fully". For example, the degree 1 to which student a, intelligence and Algorithms are related is interpreted as student a fully exhibiting intelligence during the Algorithms course. The corresponding formal context is depicted in Table 2.4.

The concept trilattice computed from the context of Table 2.4 consists of 112 triadic concepts, so again, we do not enumerate and interpret all of them. Instead, in Table 2.5 we list only 4 of the most interesting ones. We can interpret them in the following way:

- Concept No. 1 could be tagged as "practical mindedness" (in a sense this term is understood in Computer Science). Indeed, the concept manifests itself to high degree by carefullness, independence, and communication under Databases, Networks, and Programing. Students who belong to this concept to high degree shows tendencies to become database or network administrators.

- Concept No. 2 can be interpreted as "tendencies to do well as a scientific programer (e. g. in artificial intelligence) or the programing languages theorist".

- Concept No. 3 is manifested by intelligence, creativity, communication and carefulness under Algorithms and Databases. Students who belong to this extent in high degree may have the needed skills to do research in databases, in particular in algorithmic problems from that area.

- Concept No. 4 represents students exhibiting to high degree intelligence, responsibility, and carefulness in Algorithms and Mathematics. We may interpret the concept as "background in formal methods".

Careful analysis of the concepts reveals the structure (or trends) among students.

Table 2.4: Fuzzy triadic context "Students traits".

	Algorithms						Mathematics						Databases						Networks						Programing					
	in	re	co	cr	id	ca	in	re	co	cr	id	ca	in	re	co	cr	id	ca	in	re	co	cr	id	ca	in	re	co	cr	id	ca
a	1	1	0	0	$\frac{1}{2}$	1	1	1	0	0	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
b	1	0	1	1	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	1	0	1	1	1	0	0	0	0	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
c	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	0	1	1	0	$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1
d	1	1	0	0	$\frac{1}{2}$	1	1	1	0	0	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
e	1	0	1	1	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	1	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
f	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{1}{2}$	1	1
g	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	1	0	1	$\frac{1}{2}$	1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0	0	0	1	0	1	$\frac{1}{2}$	1	1
h	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	1	1	0	$\frac{1}{2}$	1	0	1	1	0	$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	0	$\frac{1}{2}$	1

2.8 Summary and topics of future work

We developed a generalization of triadic concept analysis to fuzzy setting. Our motivation was to allow for analysis of graded data. We provided basic notions: triadic fuzzy context, triadic fuzzy concept, and concept forming operators; and studied some of their properties. Next, we examined the structure of the set of all triadic fuzzy concepts of a triadic fuzzy context. We generalized the main structural result of the ordinary TCA — the so-called basic theorem. We illustrated that the theorem we provide is indeed a generalization of the classical one. We answered the natural question whether there is some connection between TCA in fuzzy setting and TCA in ordinary setting apart from the fact that the former is a generalization of the latter. It happens that the theorem in which we established such a connection can be utilized in transferring results known from ordinary TCA into fuzzy setting. As an example, we showed an alternative proof of the basic theorem using this technique. Then we turned to a more practical issue and designed two algorithms for computation of the set of all triadic fuzzy concepts, proved their correctness and analyzed their complexity. We also performed a set of experiments, mainly in order to compare the two proposed algorithms. We conclude with two small examples, whose aim is to illustrate presented notions on particular data, and to some extent the way to interpret them.

In the rest of this section we present topics of TCA of data with fuzzy attributes we consider interesting or important, and we would like to study in the future.

Structure of $\mathcal{T}(\mathbf{K})$ with respect to L-quasiorders In Sections 2.3 and 2.4 we studied a structure of the set of all triadic fuzzy concepts $\mathcal{T}(\mathbf{K})$ with

Table 2.5: Four interesting triadic concepts. The concepts are represented by columns 1, 2, ..., 4.

	1	2	3	4
student a	$\frac{1}{2}$	0	0	1
student b	0	$\frac{1}{2}$	1	0
student c	1	$\frac{1}{2}$	$\frac{1}{2}$	0
student d	$\frac{1}{2}$	0	0	1
student e	$\frac{1}{2}$	$\frac{1}{2}$	1	0
student f	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
student g	0	1	$\frac{1}{2}$	0
student h	1	0	0	$\frac{1}{2}$
inteligence	0	1	1	1
responsibility	1	0	0	1
communication	1	1	1	0
creativity	0	$\frac{1}{2}$	1	0
independence	$\frac{1}{2}$	1	1	$\frac{1}{2}$
carefulness	1	0	0	1
Algorithms	0	$\frac{1}{2}$	1	1
Mathematics	0	1	$\frac{1}{2}$	1
Databases	1	$\frac{1}{2}$	1	0
Networks	1	0	0	0
Programming	1	1	$\frac{1}{2}$	$\frac{1}{2}$

respect to ordinary quasiorder relation (2.5) and ordinary equivalence relation (2.6). It is of mathematical interest to investigate the structure of $\mathcal{T}(\mathbf{K})$ with respect to graded versions of quasiorders and equivalence relations defined by (2.3) and (2.4), respectively. In dyadic case, analogous investigation is contained in [8]. It is immediate, that the proposed research leads to new, potentially interesting concepts. Analogically to the dyadic case, where a notion of completely lattice **L**-ordered set is needed as a generalization of the notion of a lattice, in triadic setting we need to find such an generalization for the notion of a trilattice. Eventually, we might arrive to a more general version of basic theorem (Theorem 13) which takes fuzzy ordering of concepts into account.

Triadic attribute implications In ordinary setting, triadic concept analysis has two outcomes — a hierarchically ordered set of triadic concepts, and a set of triadic attribute implications. In this chapter we studied the former, we consider the latter as an important topic of future research. There are several options how to define the notion of a attribute implication in triadic setting. In [31] the authors studied, among other possibilities, the implications defined as formulas

$$A \stackrel{C}{\Rightarrow} B$$

where A and B are sets of attributes, C is a set of conditions. The intended meaning of the implication in a given triadic context $\mathbf{K}$ is: "For each condition $c \in C$ it holds that if an object $x_1 \in X_1$ has all the attributes in A then it also has all the attributes in B". A graded version of triadic implication has the same form, but A, B, and C are fuzzy sets of attributes and conditions, respectively. We are then interested in grades to which implications hold in triadic context. Moreover, in fuzzy setting, phenomena that are trivial or unnatural in ordinary case can be studied. Similarity can be considered as belonging to them. We may, for example, try to find answer to the question: Given two implications, $A \stackrel{C}{\Rightarrow} B$ and $A \stackrel{D}{\Rightarrow} B$, where C and D are similar, do the implications hold to similar degrees? Other interesting topics include

- *Entailment.* Notions of syntactic and semantic entailment. Is there an axiomatic system and a set of deduction rules that is sound and complete?
- *Nonredundancy.* Study of nonredundancy, stem basis and related issues.
- *Algorithms.* Development of algorithms for triadic implications. These include: computation of degree to which a given implication holds in a triadic context, computation of some canonical representation of all triadic implications holding in a triadic context (e.g. some form of a base) and others.

Reduction of the size of $\mathcal{T}(\mathbf{K})$ As an observant reader surely noticed, even for a small triadic context the number of corresponding triadic concepts is rather large (when compared to size of the context). Indeed, it is well known that in the worst case the number of triadic concepts is exponential in the size of triadic context. Analyzing big number of concepts is time consuming and even impossible sometimes. It is therefore desirable to develop methods that deal with this problem. First step in this direction would be to check, if methods known from dyadic case can be transferred to triadic setting. These methods can be roughly divided into two groups

- *Filtering the interesting concepts.* Methods in this groups require some additional data besides the input formal context, called background knowledge. It can have various forms, e.g. attribute priorities, or attribute dependencies. On basis of this knowledge, the methods then select only those concepts that are compatible with it and therefore potentially interesting.

- *Factorization by similarity.* Methods in this group are based on the idea, that concepts that are sufficiently similar could be, allowing for some degree of imprecision, considered as one concept.

Algorithms and complexity We would like to study the possibility of designing a new algorithm for computation of of all triadic concepts, that avoids the problem of computation of false extent candidates discussed in Section 2.6. Whether there is a way to adjust FuzzyTrias to be more efficient, or we need to use different approach remains to be discovered. Another important, but rather difficult topic, is the analysis of complexity in the average case. This topic seems to be hard because a careful analysis of complexity in average case includes estimation of the average number of triadic concepts present in data.

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Chapter 3

Triadic fuzzy Galois connections

3.1 Introduction

Triadic Galois connections, studied in [21], are basic mathematical structures behind ordinary triadic concept analysis. In this chapter, we study a generalization of triadic Galois connections to a fuzzy setting. We show that fuzzy Galois connections play in TCA of data with fuzzy attributes a role analogical to a role that triadic Galois connections play in ordinary TCA. Moreover, we prove that there is a one-to-one relationship between triadic fuzzy Galois connections and ternary fuzzy relations. Then we turn our attention to the question, whether there is a way to represent triadic fuzzy Galois connection by ordinary Galois connections. We present two such representations, one of which we utilize in providing an alternative proof of the basic theorem of TCA with fuzzy attributes.

This chapter is based on the following paper:

R. Belohlavek, P. Osicka *Triadic fuzzy Galois connections as ordinary connections.* Proceedings of 2012 IEEE International Conference on Fuzzy Systems. (accepted February 2012)

3.2 Axiomatizing Galois connections of triadic fuzzy contexts

Recall from Section 2.2 that a triadic **L**-context $\langle X_1, X_2, X_3, I\rangle$ induces three operators

$$(i)_I : L^{X_j} \times L^{X_k} \to L^{X_i}$$

for $\{i, j, k\} = \{1, 2, 3\}$ which are defined by

$$(A_j, A_k)^{(i)_I} = A_j^{(j,i,A_k)} \tag{3.1}$$

for any $A_j \in L^{X_j}$ and $A_k \in L^{X_k}$. The triplet $\langle {}^{(1)_I}, {}^{(2)_I}, {}^{(3)_I} \rangle$, denoted also just by $\langle {}^{(1)}, {}^{(2)}, {}^{(3)} \rangle$, is axiomatized below.

Remark 21. For convenience, we use also $(A_2, A_1)^{(3)}$ with the meaning $(A_2, A_1)^{(3)} = (A_1, A_2)^{(3)}$; same for ${}^{(1)}$ and ${}^{(2)}$.

Recall that an order filter in a partially ordered set $\langle L, \leq \rangle$ is any subset $K \subseteq L$ for which $a \in K$ and $a \leq b$ imply $b \in K$ for any $a, b \in L$.

Definition 22. Let K be an order filter in $\langle L, \leq \rangle$. A *triadic* $\mathbf{L}_K$*-Galois connection* between sets X_1, X_2, and X_3 is a triplet $\langle {}^{(1)}, {}^{(2)}, {}^{(3)} \rangle$ of mappings ${}^{(1)} : L^{X_2} \times L^{X_3} \to L^{X_1}$, ${}^{(2)} : L^{X_1} \times L^{X_3} \to L^{X_2}$, and ${}^{(3)} : L^{X_1} \times L^{X_2} \to L^{X_3}$, satisfying for every $A_1 \in L^{X_1}$, $A_2 \in L^{X_2}$, and $A_3 \in L^{X_3}$, that if $S(A_3, (A_1, A_2)^{(3)}) \in K$ or $S(A_1, (A_2, A_3)^{(1)}) \in K$ or $S(A_2, (A_1, A_3)^{(2)}) \in K$, then

$$\begin{aligned} S(A_3, (A_1, A_2)^{(3)}) &= S(A_1, (A_2, A_3)^{(1)}) = \\ &= S(A_2, (A_1, A_3)^{(2)}). \end{aligned} \tag{3.2}$$

Remark 23. (a) One can easily see that for $L = \{0, 1\}$, triadic $\mathbf{L}_K$-Galois connections become ordinary triadic Galois connections (observe that in this case, there are only two filters, namely $K = L$ and $K = \{1\}$ and both lead to the same notion of an $\mathbf{L}_K$-Galois connection).

(b) In accordance with [5], we use the term $\mathbf{L}$-Galois connections for $\mathbf{L}_L$-Galois connections.

In the following theorem we provide an alternative characterization of $\mathbf{L}_K$-Galois connections.

Theorem 24. *For* $\{i, j, k\} = \{1, 2, 3\}$*, a triplet* $\langle {}^{(1)}, {}^{(2)}, {}^{(3)} \rangle$ *is a triadic* $\mathbf{L}_K$*-Galois connection iff the following conditions hold for all* $A_i, A_i' \in L^{X_i}$*,* $A_j \in L^{X_j}$*,* $A_k \in L^{X_k}$*:*

(a) $A_i \subseteq (A_j, (A_i, A_j)^{(k)})^{(i)}$ *(extensivity),*

(b) if $S(A_i, A_i') \in K$ *then*
$S(A_i, A_i') \leq S((A_i', A_j)^{(k)}, (A_i, A_j)^{(k)})$ *(antitony).*

PROOF. "⇒" Assume that (3.2) holds for $\langle {}^{(1)}, {}^{(2)}, {}^{(3)} \rangle$. Then

$$\begin{aligned} &S(A_i, (A_j, (A_i, A_j)^{(k)})^{(i)}) = \\ &S((A_i, A_j)^{(k)}, (A_i, A_j)^{(k)}) = 1 \in K, \end{aligned}$$

proving (a).

By (a) and (3.2), we have

$$S(A_i, A_i') \leq S(A_i, (A_j, (A_i', A_j)^{(k)})) = \\ = S((A_i', A_j)^{(k)}, (A_i, A_j)^{(k)})$$

proving (b). □

In the rest of this section, we are to show a bijective correspondence between ternary fuzzy relations and triadic **L**-Galois connections. That is, we prove that triadic **L**-Galois connections are represented by ternary fuzzy relations. In order to do that we need some technical results first.

Lemma 25. *For $\{i,j,k\} = \{1,2,3\}$, index sets P, Q, and fuzzy sets $A_{ip} \in L^{X_i}$, and $A_{jp} \in L^{X_j}$ the following equality holds:*

$$(\bigvee_{p\in P} A_{ip}, \bigvee_{q\in Q} A_{jq})^{(k)} = \bigwedge_{p\in P, q\in Q} (A_{ip}, A_{jq})^{(k)} \tag{3.3}$$

PROOF. We prove $(\bigvee_{p\in P} A_{ip}, A_j)^{(k)} = \bigwedge_{p\in P}(A_{ip}, A_j)^{(k)}$ by proving that for every $A_k \in L^{X_3}$,

$$A_k \leq (\bigvee_{p\in P} A_{ip}, A_j)^{(k)} \text{ iff } A_k \leq \bigwedge_{p\in P} (A_{ip}, A_j)^{(k)}.$$

$A_k \leq (\bigvee_{p\in P} A_{ip}, A_j)^{(k)}$ iff (due to (3.2)) $\bigvee_{p\in P} A_{ip} \leq (A_j, A_k)^{(i)}$ iff for each $p \in P$, $A_{ip} \leq (A_j, A_k)^{(i)}$ iff for each $p \in P$, $A_k \leq (A_{ip}, A_j)^{(k)}$ iff $A_3 \leq \bigwedge_{p\in P}(A_{ip}, A_j)^{(3)}$. The remainder of the proof for j is analogous. □

Lemma 26. *Let $(^{(1)}, ^{(2)}, ^{(3)})$ be a triadic* **L**-*Galois connection. Then for $\{i,j,k\} = \{1,2,3\}$ and $A_i \in L^{X_i}$ let the mappings $\uparrow_{A_i}\colon L^{X_j} \to L^{X_k}$ and $\downarrow_{A_i}\colon L^{X_k} \to L^{X_j}$ be defined as*

$$A_k^{\uparrow_{A_i}} = (A_k, A_i)^{(j)}$$
$$A_j^{\downarrow_{A_i}} = (A_j, A_i)^{(k)}.$$

Then $\langle \uparrow_{A_i}, \downarrow_{A_i} \rangle$ forms a dyadic **L**-*Galois connection between X_j and X_k [5].*

PROOF. The following equality verifies the condition for dyadic **L**-Galois connection:

$$S(A_j, A_k^{\downarrow_{A_i}}) = S(A_j, (A_k, A_i)^{(j)}) = S(A_k, (A_j, A_i)^{(k)}) = S(A_k, A_j^{\uparrow_{A_i}})$$

□

Lemma 27. *For* $\{i, j, k\} = \{1, 2, 3\}$ *it holds*

(a) $a \to (\{1/x_i\}, \{1/x_j\})^{(k)} = (\{a/x_i\}, \{1/x_j\})^{(k)}$

(b) $\bigwedge_{x_i \in X_i} A_i(x_i) \to (\{1/x_i\}, \{1/x_j\})^{(k)} = (A_i, \{1/x_j\})^{(k)}$

PROOF. (a): By Lemma 26 we get that

$$a \to (\{1/x_i\}, \{1/x_j\}) = a \to \{1/x_i\}^{\uparrow_{\{1/x_j\}}}$$

where ${}^{\uparrow_{\{1/x_j\}}}$ is a part of dyadic **L**-Galois connection between X_i and X_k. [5] implies that

$$a \to \{1/x_i\}^{\uparrow_{\{1/x_j\}}} = \{a/x_i\}^{\uparrow_{\{1/x_j\}}}.$$

Finally, by Lemma 26 we have

$$\{a/x_i\}^{\uparrow_{\{1/x_j\}}} = (\{a/x_i\}, \{1/x_j\})^{(k)}.$$

(b): Using (a) and Lemma 25 we get

$$\begin{aligned} &\bigwedge_{x_i \in X_i} A_i(x_i) \to (\{1/x_i\}, \{1/x_j\})^{(k)} = \\ &= \bigwedge_{x_i \in X_i} (\{A_i(x_i)/x_i\}, \{1/x_j\})^{(k)}) = \\ &= (\bigvee_{x_i \in X_i} \{A_i(x_i)/x_i\}, \{1/x_j\})^{(k)}) = \\ &= (A_i, \{1/x_j\})^{(k)} \end{aligned}$$

□

The next theorem shows that triadic **L**-Galois connections are just the mappings obtained from ternary fuzzy relations by (3.1).

Theorem 28. *Let* $I \in L^{X_1 \times X_2 \times X_3}$. *Let* $\langle {}^{(1)}, {}^{(2)}, {}^{(3)} \rangle$ *be a triadic* **L***-Galois connection between* X_1, X_2, *and* X_3 *and define a ternary relation* $I_{\langle (1), (2), (3) \rangle} \in L^{X_1 \times X_2 \times X_3}$ *by*

$$\begin{aligned} I_{\langle (1), (2), (3) \rangle}(x_1, x_2, x_3) &= (\{1/x_1\}, \{1/x_2\})^{(3)} = \\ &= (\{1/x_1\}, \{1/x_3\})^{(2)} = \\ &= (\{1/x_2\}, \{1/x_3\})^{(1)}. \end{aligned}$$

Then

(a) The triplet $\langle {}^{(1)_I}, {}^{(2)_I}, {}^{(3)_I} \rangle$ *forms a triadic* **L***-Galois connection.*

(b) $I = I_{\langle {}^{(1)}I, {}^{(2)}I, {}^{(3)}I \rangle}$.

(c) $\langle {}^{(1)}, {}^{(2)}, {}^{(3)} \rangle = \langle {}^{(1)}I_{\langle {}^{(1)}, {}^{(2)}, {}^{(3)} \rangle}, {}^{(2)}I_{\langle {}^{(1)}, {}^{(2)}, {}^{(3)} \rangle}, {}^{(3)}I_{\langle {}^{(1)}, {}^{(2)}, {}^{(3)} \rangle} \rangle$.

PROOF. (a):

$$
\begin{aligned}
& S(A_i, (A_j, A_k)^{(i_I)}) = \\
&= \bigwedge_{x_i \ X_i} A_i(x_i) \to \bigwedge_{\substack{x_j \in X_j \\ x_k \in X_k}} A_j(x_j) \otimes A_k(k_k) \to I(x_i, x_j, x_k) = \\
&= \bigwedge_{x_j \ X_j} A_j(x_j) \to \bigwedge_{\substack{x_i \in X_i \\ x_k \in X_k}} A_i(x_i) \otimes A_k(k_k) \to I(x_i, x_j, x_k) = \\
&= S(A_j, (A_i, A_k)^{(j_I)})
\end{aligned}
$$

We just checked that (3.2) holds for $\langle {}^{(1)_I}, {}^{(2)_I}, {}^{(3)_I} \rangle$.

(b): We prove that $I_{\langle {}^{(1)}I, {}^{(2)}I, {}^{(3)}I \rangle}$ and I agree for all arguments. For every $x_1 \in X_1, x_2 \in X_2$, and $x_3 \in X_3$ we have

$$
\begin{aligned}
& I_{\langle {}^{(1)}I, {}^{(2)}I, {}^{(3)}I \rangle}(x_1, x_2, x_3) = (\{1/x_1\}, \{1/x_2\})^{(3)_I} = \\
&= 1 \otimes 1 \to I(x_1, x_2, x_3) = I(x_1, x_2, x_3).
\end{aligned}
$$

(c) Using properties of residuated lattices and Lemma 27 (b) we get

$$
\begin{aligned}
& (A_i, A_j)^{{}^{(k)}I_{\langle (1), (2), (3) \rangle}}(x_k) = \\
&= \bigwedge_{x_i \in X_i} A_i(x_i) \to \bigwedge_{x_j \in x_j} A_j(x_j) \to I_{\langle (1), (2), (3) \rangle}(x_i, x_j, x_k) \\
&= \bigwedge_{x_i \in X_i} A_i(x_i) \to \bigwedge_{x_j \in x_j} A_j(x_j) \to (\{1/x_i\}, \{1/x_j\})^{(k)}(x_k) \\
&= \bigwedge_{x_i \in X_i} A_i(x_i) \to (\{1/x_i\}, A_j)^{(k)}(x_k) = \\
&= (A_i, A_j)^{(k)}(x_k)
\end{aligned}
$$

□

As a consequence of the previous theorem we may conclude, that (3.2) provides an axiomatization of the mappings induced by ternary fuzzy relations by (3.1).

3.3 Representation of triadic fuzzy Galois connections

In this section, we investigate the issue of representation of triadic fuzzy Galois connections by ordinary Galois connections. We provide two kinds of such representation. The first one, contained in Section 3.3.1 is based on looking at fuzzy sets A in U as the area bellow the membership function. In the second one, presented in Section 3.3.2, we utilize so-called a-cuts, a way to represent a fuzzy set by a nested system of ordinary sets. Finally, in Section 3.3.3 we show an application of the first representation in proving in a simple way by reduction the basic theorem of fuzzy concept trilattices.

3.3.1 Cartesian representation

For the first type of representation, we utilize the mappings (2.11) and (2.10).

Definition 29. An (ordinary) triadic Galois connection $\langle^{\langle 1\rangle}, ^{\langle 2\rangle}, ^{\langle 3\rangle}\rangle$ between $X_1 \times L$, $X_2 \times L$, $X_3 \times L$ is called *commutative with respect to* $\lfloor\lceil\rceil\rfloor$ iff

$$(\lfloor\lceil A_i\rceil\rfloor, \lfloor\lceil A_j\rceil\rfloor)^{\langle k\rangle} = \lfloor\lceil(A_i, A_j)\rceil\rfloor^{\langle k\rangle} \tag{3.4}$$

holds for any $\{i, j, k\} = \{1, 2, 3\}$ and any sets $A_1 \in X_1 \times L$, $A_2 \in X_2 \times L$, and $A_3 \in X_3 \times L$.

The following definition shows how triplets of mappings on fuzzy sets in X_is may be defined from triplets of mappings on subsets of $X_i \times L$s and vice versa. (By small abuse of notation we utilize $(i)_{\langle i\rangle}$ to denote the mapping induced by (i).)

Definition 30. Let $\{i, j, k\} = \{1, 2, 3\}$. For a triadic Galois connection $\langle^{\langle 1\rangle}, ^{\langle 2\rangle}, ^{\langle 3\rangle}\rangle$ between $X_1 \times L$, $X_2 \times L$, $X_3 \times L$, and fuzzy sets $A_i \in L^{X_i}$, $A_j \in L^{X_j}$, and $A_k \in L^{X_k}$ we define mappings $^{(i)_{\langle i\rangle}} : L^{X_j} \times L^{X_k} \to L^{X_i}$ by

$$(A_j, A_k)^{(i)_{\langle i\rangle}} = \lceil(\lfloor A_j\rfloor, \lfloor A_k\rfloor)^{(i)}\rceil \tag{3.5}$$

Let $\langle^{(1)}, ^{(2)}, ^{(3)}\rangle$ be a triadic **L**-Galois connection between X_1, X_2, and X_3. Then for sets $A_i \in X_i \times L$, $A_j \in X_j \times L$, and $A_k \in X_k \times L$, we define mappings $^{\langle i\rangle_{(i)}} : (X_j \times L) \times (X_k \times L) \to X_i \times L$ by

$$(A_j, A_k)^{\langle i\rangle_{(i)}} = \lfloor(\lceil A_j\rceil, \lceil A_k\rceil)^{(i)}\rfloor \tag{3.6}$$

The following theorem provides the first way to represent triadic fuzzy Galois connections using ordinary connections.

Theorem 31. *Let* $\langle^{(1)}, ^{(2)}, ^{(3)}\rangle$ *be a triadic* L_1*-Galois connection between* X_1*,* X_2*, and* X_3 *and* $\langle^{\langle 1\rangle}, ^{\langle 2\rangle}, ^{\langle 3\rangle}\rangle$ *be a triadic Galois connection. between* $X_1 \times L$*,* $X_2 \times L$*, and* $X_3 \times L$*.*

Then the following holds:

(a) $\langle^{\langle 1\rangle_{(1)}}, {}^{\langle 2\rangle_{(2)}}, {}^{\langle 3\rangle_{(3)}}\rangle$ *is a triadic Galois connection commutative with respect to* $\lfloor\lceil\rceil\rfloor$.

(b) $\langle^{(1)_{\langle 1\rangle}}, {}^{(2)_{\langle 2\rangle}}, {}^{(3)_{\langle 3\rangle}}\rangle$ *is a triadic* $\mathbf{L}_{\{1\}}$*-Galois connection.*

(c) The map $\langle^{(1)}, {}^{(2)}, {}^{(3)}\rangle \mapsto \langle^{\langle 1\rangle}, {}^{\langle 2\rangle}, {}^{\langle 3\rangle}\rangle$ *is an one-to-one map between the set of all triadic* $\mathbf{L}_{\{1\}}$*-Galois connections between* X_1*,* X_2*, and* X_3 *and the set of all triadic Galois connections between* $X_1 \times L$*,* $X_2 \times L$*,* $X_3 \times L$ *that are commutative with respect to* $\lfloor\lceil\rceil\rfloor$.

PROOF. Let $\{i, j, k\} = \{1, 2, 3\}$.

(a): Let $A_i \in X_i \times L, A_j \in X_j \times L, A_k \in X_k$, and let $A_k \in (A_i, A_j)^{\langle k\rangle_{(k)}}$. Then $\lfloor\lceil A_k\rceil\rfloor \subseteq \lfloor\lceil(A_i, A_j)^{\langle k\rangle_{(k)}}\rceil\rfloor$. Since

$$\lfloor\lceil(A_i, A_j)^{\langle k\rangle_{(k)}}\rceil\rfloor = \lfloor\lceil\lfloor(\lceil A_i\rceil, \lceil A_j\rceil)^{(k)}\rfloor\rceil\rfloor = \lfloor(\lceil A_i\rceil, \lceil A_j\rceil)^{(k)}\rfloor,$$

the following chain of implications (which we denote by $\rightsquigarrow$) proves that $\langle^{\langle 1\rangle_{(1)}}, {}^{\langle 2\rangle_{(2)}}, {}^{\langle 3\rangle_{(3)}}\rangle$ is a triadic Galois connection:

$$\begin{aligned}
\lfloor\lceil A_k\rceil\rfloor \subseteq \lfloor(\lceil A_i\rceil, \lceil A_j\rceil)^{(k)}\rfloor \quad &\rightsquigarrow \quad S(\lceil A_k\rceil, (\lceil A_i\rceil, \lceil A_j\rceil)^{(k)}) = 1\\
&\rightsquigarrow \quad S(\lceil A_i\rceil, (\lceil A_j\rceil, \lceil A_k\rceil)^{(i)}) = 1\\
&\rightsquigarrow \quad \lfloor\lceil A_i\rceil\rfloor \subseteq \lfloor(\lceil A_j\rceil, \lceil A_k\rceil)^{(i)}\rfloor\\
&\rightsquigarrow \quad A_i \subseteq (A_j, A_k)^{\langle i\rangle_{(i)}}.
\end{aligned}$$

To prove the commutativity with respect to $\lfloor\lceil\rceil\rfloor$ observe that

$$\begin{aligned}
(\lfloor\lceil A_i\rceil\rfloor, \lfloor\lceil A_j\rceil\rfloor)^{\langle k\rangle_{(k)}} &= \lfloor(\lceil\lfloor\lceil A_i\rceil\rfloor\rceil, \lceil\lfloor\lceil A_j\rceil\rfloor\rceil)^{(k)}\rfloor\\
&= \lfloor(\lceil A_i\rceil, \lceil A_j\rceil)^{(k)}\rfloor\\
&= (A_1, A_2)^{\langle k\rangle_{(k)}}\\
&= \lfloor\lceil(A_1, A_2)^{\langle k\rangle_{(k)}}\rceil\rfloor.
\end{aligned}$$

(b): Let $A_i \in L^{X_i}$, $A_j \in L^{X_j}$, $A_k \in L^{X_k}$, and let $A_i \subseteq (A_j, A_k)^{(k)_{\langle k\rangle}}$. Then $\lfloor A_k\rfloor \subseteq \lfloor(A_j, A_k)^{(k)_{\langle k\rangle}}\rfloor$ implies

$$\lfloor A_k\rfloor \subseteq \lfloor\lceil(\lfloor A_i\rfloor, \lfloor A_j\rfloor)^{\langle k\rangle}\rceil\rfloor = (\lfloor\lceil\lfloor A_i\rfloor\rceil\rfloor, \lfloor\lceil\lfloor A_j\rfloor\rceil\rfloor)^{\langle k\rangle} = (\lfloor A_i\rfloor, \lfloor A_j\rfloor)^{\langle k\rangle}$$

which leads to

$$\begin{aligned}
\lfloor A_i\rfloor \subseteq (\lfloor A_j\rfloor, \lfloor A_k\rfloor)^{\langle i\rangle} &= (\lfloor\lceil\lfloor A_j\rfloor\rceil\rfloor, \lfloor\lceil\lfloor A_k\rfloor\rceil\rfloor)^{\langle i\rangle}\\
&= \lfloor\lceil(\lfloor A_j\rfloor, \lfloor A_k\rfloor)^{\langle i\rangle}\rceil\rfloor\\
&= \lfloor(A_j, A_k)^{(i)^{\langle i\rangle}}\rfloor
\end{aligned}$$

and thus $A_i \subseteq (A_j, A_k)^{(i)_{\langle i\rangle}}$.

(c): We prove that $\langle^{\langle 1\rangle(1)\langle 1\rangle}, ^{\langle 2\rangle(2)\langle 2\rangle}, ^{\langle 3\rangle(3)\langle 3\rangle}\rangle = \langle^{\langle 1\rangle}, ^{\langle 2\rangle}, ^{\langle 3\rangle}\rangle$. Let $A_i \in X_i \times L$, $A_j \in X_j \times L$. First observe that since $A_i \in \lfloor\lceil A_i\rceil\rfloor$ and $A_j \in \lfloor\lceil A_j\rceil\rfloor$, we have

$$(\lfloor\lceil A_i\rceil\rfloor, \lfloor\lceil A_j\rceil\rfloor)^{\langle k\rangle} \subseteq (A_i, A_j)^{\langle k\rangle} \subseteq \lfloor\lceil(A_i, A_j)^{\langle k\rangle}\rceil\rfloor,$$

and the the commutativity with respect to $\lfloor\lceil\rceil\rfloor$ yields

$$(\lfloor\lceil A_i\rceil\rfloor, \lfloor\lceil A_j\rceil\rfloor)^{\langle k\rangle} = (A_i, A_j)^{\langle k\rangle}.$$

Now, by using (3.5) and (3.6) we get

$$(A_i, A_j)^{\langle k\rangle(k)\langle k\rangle} = (\lfloor\lceil A_i\rceil\rfloor, \lfloor\lceil A_j\rceil\rfloor)^{\langle k\rangle}.$$

This together with the previous observation gives the proof.

It remains to prove $\langle^{(1)\langle 1\rangle(1)}, ^{(2)\langle 2\rangle(2)}, ^{(3)\langle 3\rangle(3)}\rangle = \langle^{(1)}, ^{(2)}, ^{(3)}\rangle$. Let $A_i \in L^{X_i}$ and $A_j \in L^{X_j}$. Then by (3.5) and (3.6) we get

$$(A_i, A_j)^{(k)\langle k\rangle(k)} = \lceil\lfloor(\lceil\lfloor A_i\rfloor\rceil, \lceil\lfloor A_j\rfloor\rceil)^{(k)}\rfloor\rceil = (A_i, A_j)^{(k)}$$

completing the proof. □

3.3.2 Cut-like representation

The second representation is based on the notion of an a-cut of a fuzzy set. Recall that for a fuzzy set $A \in L^U$ and a degree $a \in L$, the a-cut ${}^aA$ of A is the ordinary subset of U defined by

$${}^aA = \{u \in U \mid a \leq A(u)\}.$$

It is well known, that every fuzzy set is uniquely represented by a system of its a-cuts. Indeed, we may introduce a notion of a nested system of ordinary subsets of U in a way, that this nested system becomes just the system of a-cut of fuzzy set. More details can be found in [8].

Although the straightforward application of the idea of a-cut turns out to be not sound, e.g. the condition

$$({}^aA_1, {}^aA_2)^{(3)} = {}^a((A_1, A_2)^{(3)})$$

does not hold for triadic fuzzy Galois connections, a cut-like representation of triadic fuzzy Galois connections is possible. Such a representation is shown in the rest of this section.

First, we introduce the appropriate notion of a nested system.

Definition 32. Let $\{i, j, k\} = \{1, 2, 3\}$. A system $\{\langle^{(1_a)}, ^{(2_a)}, ^{(3_a)}\rangle \mid a \in L\}$ of (ordinary) triadic Galois connections is called **L**-*nested* iff

1. for each $a, b \in L$ such that $a \leq b$, and $A_i \in L^{X_i}$, $A_j \in L^{X_j}$ it holds $(A_i, A_j)^{(k_a)} \supseteq (A_i, A_j)^{(k_b)}$
2. for all $x_i \in X_i$, $x_j \in X_j$, $x_k \in X_k$ the set $\{a \in L \mid x_i \in (\{x_j\}, \{x_k\})^{(i_a)}\}$ has a greatest element.

We need the following lemmas.

Lemma 33. *For $\{i, j, k\} = \{1, 2, 3\}$, let $I \in L^{X_1 \times X_2 \times X_3}$ be an* **L***-relation, $\langle^{(1)}, ^{(2)}, ^{(3)}\rangle$ be the triadic* **L***-Galois connection induced by I and for $a \in L$ let $\langle^{(1_a)}, ^{(2_a)}, ^{(3_a)}\rangle$ be the triadic Galois connections induced by the cuts ${}^aI$. Then*

(a) for every $A_i \in 2^{X_i}$, $A_j \in 2^{X_j}$, and $a \in L$ we have

$$^a(A_i, A_j)^{(k)} = (A_i, A_j)^{(k_a)},$$

(b) for all fuzzy sets $A_i \in L^{X_i}$, $A_j \in L^{X_j}$, and $b, c \in L$ we have

$$^a(A_i, A_j)^{(k)} = \bigcap_{b,c \in L} ({}^bA_i, {}^cA_j)^{(k_{a \otimes b \otimes c})}.$$

PROOF. (a): Let $A_i \in 2^{X_i}, A_j \in 2^{X_j}$, and $a \in L$. Then for any $x_k \in X_k$ we have

$$x_k \in {}^a(A_i, A_j)^{(k)} \quad \text{iff} \quad \bigwedge_{\substack{x_i \in X_i \\ x_j \in X_j}} A_i(x_i) \otimes A_j(x_j) \to I(x_i, x_j, x_k) \geq a.$$

Since A_i and A_j are ordinary sets the following holds

$$\bigwedge_{\substack{x_i \in X_i \\ x_j \in X_j}} A_i(x_i) \otimes A_j(x_j) \to I(x_i, x_j, x_k) =$$

$$= \bigwedge_{\substack{x_i \in A_i \\ x_j \in A_j}} 1 \otimes 1 \to I(x_i, x_j, x_k) = \bigwedge_{\substack{x_i \in A_i \\ x_j \in A_j}} I(x_i, x_j, x_k).$$

To see the claim, observe that $x_k \in (A_i, A_k)^{(k_a)}$ iff $A_i \times A_j \times \{x_k\} \subseteq {}^aI$ iff $a \leq I(x_i, x_j, x_k)$ for all $x_i \in A_i, x_j \in A_j$ iff

$$a \leq \bigwedge_{\substack{x_i \in A_i \\ x_j \in A_j}} I(x_i, x_j, x_k).$$

(b): Let $A_i \in L^{X_i}$, $A_j \in L^{X_j}$. Assume that $x_k \in {}^{a}(A_i, A_j)^k$. Then

$$\bigwedge_{\substack{x_i \in A_i \\ x_j \in A_j}} A(x_i) \otimes A(x_j) \to I(x_i, x_j, x_k) \geq a$$

and thus $a \leq A(x_i) \otimes A(x_j) \to I(x_i, x_j, x_k)$ for all $x_i \in X_i, x_j \in X_j$. By adjunction we get $a \otimes A(x_i) \otimes A(x_j) \leq I(x_i, x_j, x_k)$.

Let $b \in L$. For any $x_i \in {}^{b}A_i, x_j \in {}^{c}A_j$ we have (remember that $A_i(x_i) \geq b, A_j(x_j) \geq c$)

$$a \otimes b \otimes c \leq a \otimes A_i(x_i) \otimes A_j(x_j) \leq I(x_i, x_j, x_k)$$

This implies that ${}^{b}A_i \times {}^{c}A_j \times \{x_k\} \subseteq {}^{a\otimes b\otimes c}I$ and thus $x_k \in ({}^{b}A_i, {}^{c}A_j)^{(k_{a\otimes b\otimes c})}$, which proves ${}^{a}(A_i, A_j)^k \subseteq ({}^{b}A_i, {}^{c}A_j)^{(k_{a\otimes b\otimes c})}$.

To prove the converse, let $b, c \in L$ and $x_k \in ({}^{b}A_i, {}^{c}A_j)^{(k_{a\otimes b\otimes c})}$. Then ${}^{b}A_i \times {}^{c}A_j \times \{x_k\} \subseteq {}^{a\otimes b\otimes c}I$ and therefore $a \otimes b \otimes c \leq I(x_i, x_j, x_k)$ for all $x_i \in {}^{b}A_i, x_j \in {}^{c}A_j$. Using adjunction twice and $A_i(x_i) < b, A_j(x_j) < c$ for any $x_i \notin {}^{b}A_i, x_j \notin {}^{c}A_j$ we obtain

$$a \leq A(x_i) \otimes A(x_j) \to I(x_i, x_j, x_k)$$

for all $x_i \in X_, x_j \in X_j$, and therefore $x_k \in {}^{a}(A_i, A_j)^{(k)}$.

□

Lemma 34. *Let $\langle {}^{(1)_1}, {}^{(2)_1}, {}^{(3)_1} \rangle$ and $\langle {}^{(1)_2}, {}^{(2)_2}, {}^{(3)_2} \rangle$ be triadic* **L**-*Galois connections, let I_1 and I_2 be the corresponding* **L**-*relations between X_1, X_2, and X_3. Then for $\{i, j, k\} = \{1, 2, 3\}$ it holds that $I_1 \subseteq I_2$ iff for each $A_i \in L^{X_i}$, $A_j \in L^{X_j}$ it holds $(A_i, A_j)^{(k)_1} \subseteq (A_i, A_j)^{(k)_2}$.*

PROOF. "⇒": The claim follows from 28, definition of ${}^{(k)}$, and antitony of $\to$ in the second argument.

"⇐": For any $x_i \in X_i, x_j \in X_j, x_k \in X_k$ it holds $I_1(x_i, x_j, x_k) = 1 \otimes 1 \to I_1(x_i, x_j, x_k) = (1/\{x_i\}, 1/\{x_j\})^{(k)_1}(x_k) \leq (1/\{x_i\}, 1/\{x_j\})^{(k)_2}(x_k) = 1 \otimes 1 \to I_2(x_i, x_j, x_k) = I_2(x_i, x_j, x_k)$.

□

The next theorem provides the cut-like representation of triadic fuzzy Galois connections.

Theorem 35. *For a triadic* **L**-*Galois connection $\langle {}^{(1)}, {}^{(2)}, {}^{(3)} \rangle$ between X_1, X_2, and X_3 denote*

$$\mathcal{C}_{\langle {}^{(1)}, {}^{(2)}, {}^{(3)} \rangle} = \{\langle {}^{(1_a)}, {}^{(2_a)}, {}^{(3_a)} \rangle \mid a \in L\}.$$

For an **L***-nested system* $\{\langle^{(1_a)}, {}^{(2_a)}, {}^{(3_a)}\rangle \mid a \in L\}$ *of triadic Galois connections between* X_1*,* X_2*, and* X_3 *denote by* $\langle^{(1)\mathcal{C}}, {}^{(2)\mathcal{C}}, {}^{(3)\mathcal{C}}\rangle$ *the mappings defined for* $\{i,j,k\} = \{1,2,3\}$*, and* $A_i \in L^{X_i}$*,* $A_j \in L^{X_j}$ *by*

$$(A_i, A_j)^{(k)\mathcal{C}}(x_k) = \bigvee\{a \mid x_k \in \bigcap_{b,c\in L} ({}^bA_i, {}^cA_j)^{(k_{a\otimes b\otimes c})}\}.$$

Then it holds

(a) $\mathcal{C}_{\langle^{(1)},^{(2)},^{(3)}\rangle}$ *is an* **L***-nested system of triadic Galois connections,*

(b) $\langle^{(1)\mathcal{C}}, {}^{(2)\mathcal{C}}, {}^{(3)\mathcal{C}}\rangle$ *is a triadic* **L***-Galois connection.*

(c) $\langle^{(1)}, {}^{(2)}, {}^{(3)}\rangle = \langle^{(1)}\mathcal{C}_{\langle^{(1)},^{(2)},^{(3)}\rangle}, {}^{(2)}\mathcal{C}_{\langle^{(1)},^{(2)},^{(3)}\rangle}, {}^{(3)}\mathcal{C}_{\langle^{(1)},^{(2)},^{(3)}\rangle}\rangle$*, and* $\mathcal{C} = \mathcal{C}_{\langle^{(1)\mathcal{C}},^{(2)\mathcal{C}},^{(3)\mathcal{C}}\rangle}$*, i.e. the mappings between the sets of all triadic* **L***-Galois connections and all nested systems of triadic Galois connections are mutually inverse bijections.*

PROOF. (a) It suffices to check the conditions of Definition 32.

To check the first condition, see that if $a \le b$ then ${}^aI \supseteq {}^bI$ and by Lemma 34 it holds $(A_i, A_j)^{(k_a)} \ge (A_i, A_j)^{(k_b)}$ for all $A_i \subseteq X_i, A_j \subseteq X_j$.

The second condition is proved by the following: Since $x_k \in (\{x_i\}, \{x_j\})^{(k_a)}$ iff $(x_i, x_j, x_k) \in {}^aI$ iff $I(x_i, x_j, x_k) \ge a$, the greatest element a such that $x_k \in (\{x_i\}, \{x_j\})^{(k_a)}$ is clearly $I(x_i, x_j, x_k)$.

(b) Let $x_k \in \bigcap_{b,c\in L}({}^bA_i, {}^cA_j)^{(k_{a\otimes b\otimes c})}$. First, we set $I \in L^{X_1\times X_2\times X_3}$ to

$$I(x_i, x_j, x_k) = \bigvee\{a|(x_i, x_j, x_k) \in I_{\langle^{(1_a)},^{(2_a)},^{(3_a)}\rangle}\},$$

where $I_{\langle^{(1_a)},^{(2_a)},^{(3_a)}\rangle}$s are ordinary relations induced by triadic Galois connections in $\mathcal{C}$ (c.f. Theorem 28 for $\mathbf{L} = \mathbf{2}$). The L-nestedness of $\mathcal{C}$ ensures that definition of I is correct. Indeed, (1) of Definition 32 and Lemma 34 yield that $I_{\langle^{(1)_a},^{(2)_a},^{(3)_a}\rangle} \supseteq I_{\langle^{(1)_b},^{(2)_b},^{(3)_b}\rangle}$ whenever $a \le b$, by (2) of Definition 32 we have that $\bigvee\{a|(x_i, x_j, x_k) \in I_{\langle^{(1_a)},^{(2_a)},^{(3_a)}\rangle}\}$ has the greatest element. Therefore $I_{\langle^{(1_a)},^{(2_a)},^{(3_a)}\rangle} = {}^aI$ and by the following well-known property of a-cuts $A(x) = \vee\{a \mid x \in {}^aA\}$ we get that I is defined correctly.

Now, by Lemma 33 $x_k \in \bigcap_{b,c\in L}({}^bA_i, {}^cA_j)^{(k_{a\otimes b\otimes c})}$ is equivalent to $x_k \in {}^a(A_i, A_j)^{(k)}$, where $\langle^{(1)}, {}^{(2)}, {}^{(3)}\rangle$ is induced by I, which proves the claim.

(c) Follows clearly from the proofs of (a) and (b). □

3.3.3 Application of the Cartesian representation

In Section 2.4 we presented a generalization of basic theorem of triadic concept analysis for fuzzy concept trilattices. As an application of the representation provided in Section 3.3.1, we show a simple proof this theorem by a certain reduction utilizing the theorem for ordinary concept trilattices from [53].

We utilize the facts that the set of all triadic (fuzzy) concepts of a triadic (fuzzy) Galois connection $\langle^{(1)},^{(2)},^{(3)}\rangle$ forms a trilattice, denoted here by $\mathcal{T}(X_1, X_2, X_3, \langle^{(1)},^{(2)},^{(3)}\rangle)$.

Theorem 36. *For a triadic $\mathbf{L}_K$-Galois connection $\langle^{(1)},^{(2)},^{(3)}\rangle$ the trilattices $\mathcal{T}(X_1, X_2, X_3, \langle^{(1)},^{(2)},^{(3)}\rangle)$ and $\mathcal{T}(X_1 \times L, X_2 \times L, X_3 \times L, \langle^{\langle 1\rangle(1)},^{\langle 2\rangle(2)},^{\langle 3\rangle(3)}\rangle)$ are isomorphic. Moreover, $\mathcal{T}(X_1 \times L, X_2 \times L, X_3 \times L, \langle^{\langle 1\rangle(1)},^{\langle 2\rangle(2)},^{\langle 3\rangle(3)}\rangle) = \mathcal{T}(X_1 \times L, X_2 \times L, X_3 \times L, I^\times)$, where*

$$\langle (x_1, a), (x_2, b), (x_3, c)\rangle \in I^\times \textit{ iff } c \leq (\{a/x_1\}, \{b/x_2\})^{(3)}.$$

PROOF. To show that the assertion is valid we consider mappings

$$h : \mathcal{T}(\langle^{(1)},^{(2)},^{(3)}\rangle) \to \mathcal{T}(\langle^{\langle 1\rangle(1)},^{\langle 2\rangle(2)},^{\langle 3\rangle(3)}\rangle),$$

$$g : \mathcal{T}(\langle^{\langle 1\rangle(1)},^{\langle 2\rangle(2)},^{\langle 3\rangle(3)}\rangle \to \mathcal{T}(\langle^{(1)},^{(2)},^{(3)}\rangle)$$

defined by

$$\begin{aligned} h(\langle A_1, A_2, A_3\rangle) &= \langle \lfloor A_1\rfloor, \lfloor A_2\rfloor, \lfloor A_2\rfloor\rangle \\ g\langle A_1, A_2, A_3\rangle) &= \langle \lceil A_1\rceil, \lceil A_2\rceil, \lceil A_2\rceil\rangle. \end{aligned}$$

First we show that the mapping are defined correctly. Let $\langle A_1, A_2, A_3\rangle \in \mathcal{T}(\langle^{(1)},^{(2)},^{(3)}\rangle)$ Then by Theorem 31 and Definition 30 for any assignment $\{i, j, k\} = \{1, 2, 3\}$

$$(\lfloor A_i\rfloor, \lfloor A_j\rfloor)^{(k)} = \lfloor(\lceil\lfloor A_i\rfloor\rceil, \lceil\lfloor A_j\rfloor\rceil)^{\langle k\rangle(k)}\rfloor = \lfloor(A_i, A_j)^{\langle k\rangle(k)}\rfloor = \lfloor A_k\rfloor.$$

For $\langle A_1, A_2, A_3\rangle \in \mathcal{T}(\langle^{\langle 1\rangle(1)},^{\langle 2\rangle(2)},^{\langle 3\rangle(3)}\rangle$ we have

$$\begin{aligned} (\lceil A_i\rceil, \lceil A_j\rceil)^{\langle k\rangle(k)} &= \lceil(\lfloor\lceil A_i\rceil\rfloor, \lfloor\lceil A_j\rceil\rfloor)^{(k)}\rceil = \lceil\lfloor\lceil(A_i, A_j)^{(k)}\rceil\rfloor\rceil = \\ &= \lceil\lfloor\lceil A_k\rceil\rfloor\rceil = \lceil A_k\rceil. \end{aligned}$$

Clearly, both g and h are order preserving. Theorem 31 implies that g and h are mutually inverse.

To see that $\mathcal{T}(X_1, X_2, X_3, \langle^{\langle 1\rangle(1)},^{\langle 2\rangle(2)},^{\langle 3\rangle(3)}\rangle) = \mathcal{T}(X_1, X_2, X_3, I^\times)$ it suffices to show that $I^\times$ is precisely the relation $I_{\langle^{\langle 1\rangle(1)},^{\langle 2\rangle(2)},^{\langle 3\rangle(3)}\rangle}$ of Theorem 28 corresponding to $\langle^{\langle 1\rangle(1)},^{\langle 2\rangle(2)},^{\langle 3\rangle(3)}\rangle$. That is we show that $c \leq (\{a/x_1\}, \{b/x_2\})^{(3)}$ iff $(x_3, c) \in (\{(x_1, a)\}, \{(x_2, b)\})^{\langle 3\rangle(3)}$ which is indeed true by Definition 30. □

The following theorem shows an important fact that every fuzzy concept trilattice is isomorphic to a certain concept trilattice.

Theorem 37. *Any* **L***-concept trilattice* $\mathcal{T}(X_1, X_2, X_3, I)$ *is isomorphic to the (ordinary) concept trilattice* $\mathcal{T}(X_1 \times L, X_2 \times L, X_3 \times L, I^{\times})$*, where*

$$\langle (x_1, a), (x_2, b), (x_3, c) \rangle \in I^{\times} \textit{ iff } a \otimes b \otimes c \leq I(x_1, x_2, x_3).$$

PROOF. Let $\langle ^{(1)}, ^{(2)}, ^{(3)} \rangle$ be a triadic **L**-Galois connection induced by I by Theorem 28. Using adjunction two times we get

$$a \otimes b \otimes c \leq I(x_1, x_2, x_3) \text{ iff } c \leq (\{a/x_1\}, \{b/x_2\})^{(3)}.$$

The claim then follows from Theorem 36. □

Remark 38. (a) Note that the previous theorem is essentially the Theorem 16, only this time we provide an alternative way of proving it. Namely, we use a reduction of fuzzy triadic Galois connection to ordinary triadic Galois connection and the fact that the sets of fixpoints they induce are isomorphic.

(b) Using Cartesian representation of fuzzy triadic Galois connections we can provide an alternative proof of Theorem 13. The link is the equivalence of Theorem 16 and Theorem 37 (see the alternative proof of basic theorem in Section 2.4.)

3.4 Summary and topics of future research

We provided an axiomatization of triadic fuzzy Galois connection and established an one-to-one correspondence between them and ternary fuzzy relations. We presented two representation theorems that link triadic fuzzy Galois connections to ordinary triadic Galois connection in two different ways. We demonstrated usefulness of such a representation by an example of carrying over a result known for ordinary TCA to a fuzzy setting.

As a part of future work, we would like to

- Find results that may be automatically carried over from ordinary case to fuzzy case. Investigate if such results can be formally identified.
- Develop other possible types of reduction. Extend the applicability of the presented representation to a wider class of relational methods.

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Chapter 4

Decomposition of three-way ordinal data

4.1 Introduction

Methods of matrix decomposition proved to be applicable to many fields, e.g. psychometrics, chemometrics, signal processing, neuroscience, and data mining [37, 38, 50]. Given an input matrix, the aim of the methods is to find matrices whose product gives back the original matrix, or at least approximate it to some degree. The decomposition can be understood as a discovery of hidden entities in the matrix, usually called factors, features, components and the like, which describe the matrix in a more economical way. The decomposition is then viewed as a mapping of the data from the high dimensional space of directly observable variables into the lower-dimensional space of factors. The number of factors is then interpreted as a dimension of the matrix [46].

For two-way matrices there is a number of decomposition methods such as the most well-known Singular Value Decomposition (SVD), Principal Component Analysis (PCA), and Nonnegative Matrix Factorization (NMF) [34, 44]. The common attribute of these methods is that they are designed to work with matrices whose entries are numerical data (e.g. real numbers). When applied to relational data, that is, to Boolean matrices or to matrices with grades, they distort the intended meaning of the data and factors obtained by the decomposition are extremely hard to interpret [45, 46]. This inappropriateness of known methods for dealing with relational data led to research on methods designed specifically for it. Several of such methods were proposed in the literature [29, 45, 46], some of them utilizing formal concepts as factors [15, 19].

For three-way matrices, the situation is quite similar. The existing methods work well with numerical matrices, but are not well suited for relational

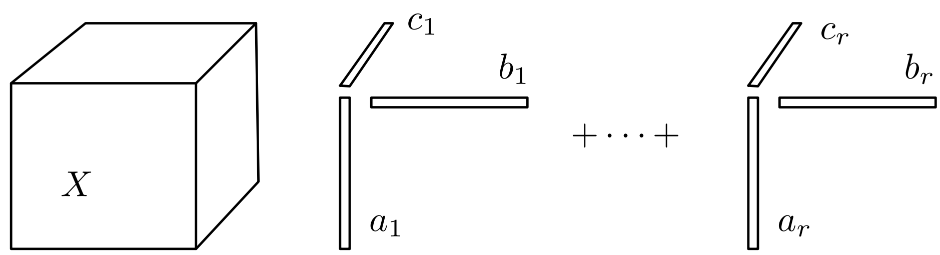


Figure 4.1: Schematic visualization of a CANDECOMP decomposition

ones. A good survey of such methods is [37]. For three-way Boolean matrices, a decomposition method utilizing triadic concepts was proposed recently in [16]. This chapter presents a generalization of this method to a fuzzy setting.

The closest analogue to the method described in this chapter among methods suited to work with numerical data is the Canonical decomposition (CANDECOMP) [23]. Although a process of computation of this decomposition differs significantly from the presented method, from a more general point of view these two methods aim at the same type of decomposition. The CANDECOMP decomposition has the form

$$X = \sum_{r=1}^{R} a_r \odot b_r \odot c_r \tag{4.1}$$

where X is a $n \times m \times p$ three-way matrix of real numbers, a_r, b_r, and c_r are real vectors of sizes n, m, and p, respectively. The operator $\odot$ is an outer vector multiplication, that is, the result of $a_r \odot b_r \odot c_r$ is an $n \times m \times p$ matrix J_r such that

$$(J_r)_{ijt} = (a_r)_i \cdot (b_r)_j \cdot (c_r)_t \tag{4.2}$$

The sum symbol denotes a component-wise sum of three-way matrices. Using (4.2) we can rewrite (4.1) as

$$X_{ijt} = \sum_{r=1}^{R} (a_r)_i \cdot (b_r)_j \cdot (c_r)_t \tag{4.3}$$

The lowest R such that (4.1) exists is a *Shein rank* of matrix X (or a rank, for short), denoted by $\rho(X)$. Clearly, each $J_r = a_r \odot b_r \odot c_r$ has a rank 1. Thus, CANDECOMP aims at decomposition of a matrix into the sum of rank 1 matrices, as depicted visually in Figure 4.1.

Now, lets turn our attention to a method developed in this chapter. We are given a three-way $n \times m \times p$ matrix I with entries taken from a residuated lattice. We interpret this matrix as a fuzzy relation of some triadic context.

That is, the entry I_{ijt} is a degree to which object i has attribute j under condition t. We aim at a decomposition of I into a product

$$I = \circ(A, B, C) \tag{4.4}$$

of a $n \times k$ matrix A, $m \times k$ matrix B, and $p \times k$ matrix C given by

$$\circ(A, B, C)_{ijt} = \bigvee_{l=1}^{k} A_{il} \otimes B_{jl} \otimes C_{jl} \tag{4.5}$$

with k as small as possible. The decomposition (4.5) has the following meaning: The degree in which object i, attribute j, and condition t are related is the truth degree of a proposition:

> "There exists a factor l such that l applies to i, j is a particular manifestation of l, and t is one of the conditions under which l appears."

Matrix A is an object×factor matrix and represents objects in terms of factors. A can be considered as an output of the decomposition. Indeed A is a more economical representation of I. While in I object i is represented by a two-way $n \times p$ attribute-condition matrix $I_{i\_\_}$, in A it is described only by a single row supposedly consisting of a smaller number of entries (this claim is indeed true, as we prove in the next section). The relation between object representations in attribute-condition space and in factor space is studied in Section 4.3. Matrices B and C describe factors in terms of attributes and conditions: B is an attribute-factor matrix whose columns depict degrees to which attributes are manifestations of factors, C is a condition-factor matrix whose columns describe under which conditions factors appear.

In Section 4.2 we show that (4.5) is in fact a decomposition of I into a $\bigvee$-superposition of rank 1 matrices. Indeed, an observant reader certainly noticed that (4.5) and (4.1) are remarkably similar, we just replace the operations of usual multiplication and summation by truth function of conjunction and order-theoretic join operation, respectively. The differences in algebraic properties of these operations (most notably the fact that summation is not idempotent) are part of reasons why (4.1) is inappropriate for relational data.

To illustrate the meaning of decomposition (4.5) consider the following example. Assume that we have data from a customer survey where customers express their motivation for a purchase of a particular type of car. Furthermore, se assume that the data can be transformed into a three-way matrix I whose rows correspond to car types, columns to car features (speed, horse power, etc.) and conditions to customers. That is, I_{ijt} is a degree to which feature j motivates customer t to purchase car type i. Depending on the data, decomposition (4.5) may reveal factors that we can label as sports cars, family cars and the like. In such a case, A characterizes types of cars in the degrees to which they are sport cars, family cars etc.; B describes factors by attributes that are its particular manifestations: e.g. sports car may be to a high degree manifested by speed, acceleration and the like. Finally, C contains degrees to which customers are motivated to buy cars from a particular group, e.g. degrees to which particular customers want sport cars, family cars, etc. This example is elaborated in Section 4.5.

This chapter is organized as follows: in Section 4.2 we describe how to use triadic concepts as factors. We prove that such a decomposition is universal and optimal. In Section 4.3 we study the relations between spaces induced by (4.5). In Section 4.4 we deal with algorithmic and complexity issues related to a computation of (4.5). Section 4.5 contains an illustrative example. We conclude with some remarks on future research directions in Section 4.6.

In this chapter, we use the following notation. We understand a three-way $n \times m \times p$ matrix I as a triadic fuzzy context $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$ with $X_1 = \{1, 2, \ldots n\}$, $X_2 = \{1, 2, \ldots, m\}$, and $X_3 = \{1, 2, \ldots, p\}$ and $I(i, j, t) = I_{ijt}$. We use analogical convention for two-way matrices and dyadic contexts, and vectors and fuzzy sets.

This chapter is based on the following paper:

R. Belohlavek, P. Osicka, V. Vychodil. Factorizing Three-way ordinal data using triadic formal concepts. *Ninth International conference on Flexible Query Answering System*, Ghent, Lecture Notes in Computer Science, 2011, Volume 7022/2011, 400-411, DOI: 10.1007/978-3-642-24764-4_35

4.2 Optimal decomposition using triadic factors

In this section we show how triadic fuzzy concepts of I may be used as factors in decomposition (4.5). First, we observe that (4.5) is a decomposition into rank 1 matrices, which we call cuboids.

Definition 39. A 3-dimensional matrix $J \in L^{n\times m\times p}$ is a *cuboidal matrix* (shortly, a *cuboid*) if there exist vectors $A \in L^n$, $B \in L^m$, and $C \in L^p$ such that $J_{ijt} = A_i \otimes B_j \otimes C_t$, or equivalently $J = \circ(A, B, C)$.

The role of cuboids for decompositions (4.5) is the following:

Lemma 40. *$I = \circ(A, B, C)$ for an $n \times k$ matrix A, $m \times k$ matrix B, and $p \times k$ matrix C iff I is a $\bigvee$-superposition of k cuboids $J_1, \ldots, J_k$, i.e.*

$$I = J_1 \vee \cdots \vee J_k.$$

In addition, for each $l = 1, \ldots, k$, $J_l = \circ(A_{\_l}, B_{\_l}, C_{\_l})$, i.e. each J_l is the product of the l-th columns of A, B, and C.

PROOF. "$\Rightarrow$": We can rewrite (4.5) in the following way

$$I_{ijt} = \bigvee_{l=1}^{k} A_{il} \otimes B_{jl} \otimes C_{tl} = \bigvee_{l=1}^{k} (A_{\_l} \otimes B_{\_l} \otimes C_{\_l})_{ijt}.$$

Clearly, $(A_{\_l} \otimes B_{\_l} \otimes C_{\_l})$ is a cuboid for each $l \in \{1, \ldots, k\}$.

"$\Leftarrow$": Let $I = J_1 \vee \cdots \vee J_k$. For each J_l there are A^l, B^l, and C^l such that $(J_l)_{ijt} = A^l_i \otimes B^l_j \otimes C^l_t$. If we consider $n \times k$ matrix A such that $A_{\_l} = A^l$, $m \times k$ matrix B such that $B_{\_l} = B^l$, and $p \times k$ matrix C such that $C_{\_l} = C^l$, then we it clearly holds

$$I_{ijt} = (A_{\_1} \otimes B_{\_1} \otimes C_{\_1})_{ijt} \vee \cdots \vee (A_{\_k} \otimes B_{\_k} \otimes C_{\_k})_{ijt} = \bigvee_{l=1}^{k} A_{il} \otimes B_{jl} \otimes C_{tl}.$$

□

As shown above, in order to decompose I using a small number of factors, we need to find a small number of cuboids contained in I whose $\vee$-superposition gives I again (in the following we may use the term "cuboids cover I" for describing this situation). We say that a cuboid J is contained in I if $J_{ijt} \leq I_{ijt}$ for all i, j, t.

In the following lemma we show that triadic fuzzy concepts of I are maximal cuboids in I.

Lemma 41. *$\langle D_1, D_2, D_3 \rangle$ is a triadic concept of I iff $J = \circ(D_1, D_2, D_3)$ is a maximal cuboid contained in I.*

PROOF. "$\Rightarrow$". Let $\langle D_1, D_2, D_3 \rangle$ be a triadic concept. Then by Theorem 12 (a), it is maximal triple of fuzzy sets (w.r.t set inclusion on its components) such that $(D_1 \otimes D_2 \otimes D_3)_{ijt} \leq I_{ijt}$ for all i, j, t (assuming, of course, that $1 \leq i \leq n$, $1 \leq j \leq m$, and $1 \leq t \leq p$). Therefore $J = \circ(D_1, D_2, D_3)$ is a maximal cuboid contained in I.

"$\Leftarrow$": Let $J = \circ(D_1, D_2, D_3)$ be a maximal cuboid contained in I. Then by Theorem 12 (b), there is a triadic concept $\langle E_1, E_2, E_3 \rangle$ such that $D_1 \subseteq E_1$, $D_2 \subseteq E_2$, and $D_2 \subseteq E_2$. Maximality of J then implies $\langle E_1, E_2, E_3 \rangle = \langle D_1, D_2, D_3 \rangle$. □

Let's sum up what we know up to this point: Lemma 40 implies that the decomposition (4.5) is a decomposition into a $\vee$-superposition of cuboids. From Lemma 41 we know, that triadic concepts are cuboids contained in I. This implies, that we can use triadic concepts as factors. In remains to show, how, given a set of triadic concepts, we obtain matrices of the right-hand side of (4.5). For a set

$$\mathcal{F} = \{\langle D_{11}, D_{12}, D_{13} \rangle, \ldots, \langle D_{k1}, D_{k2}, D_{k3} \rangle\}$$

of triadic fuzzy concepts of I (we fix this indexing of concepts, i.e. we speak of the l-th concept in $\mathcal{F}$), we denote by $A_{\mathcal{F}}$ the $n \times k$ matrix in which the l-th column coincides with the extent D_{l1}, by $B_{\mathcal{F}}$ the $m \times k$ matrix in which the l-th column coincides with the intent D_{l2}, $C_{\mathcal{F}}$ the $p \times k$ matrix in which the l-th column coincides with the modus D_{l3} of the l-th concept $\langle D_{l1}, D_{l2}, D_{l3}\rangle$. That is,

$$(A_{\mathcal{F}})_{il} = D_{l1}(i), \quad (B_{\mathcal{F}})_{jl} = D_{l2}(j), \quad (C_{\mathcal{F}})_{tl} = D_{l3}(t).$$

If $I = \circ(A_{\mathcal{F}}, B_{\mathcal{F}}, C_{\mathcal{F}})$, we call the triadic concepts from $\mathcal{F}$ *factor concepts*. Given I, our aim is to find a small set $\mathcal{F}$ of factor concepts.

Let us remark that since triadic concepts tend to be easy to interpret, using them as factors is intuitively appealing. The notion of triadic concept is a simple formal model of human concept considered as an element of thought according to traditional logic approach [43]. Moreover, the extents, intents, and modi of the concepts, i.e. columns of $A_{\mathcal{F}}$, $B_{\mathcal{F}}$, and $C_{\mathcal{F}}$, have a straightforward interpretation: they represent the objects, attributes, and conditions to which the factor concept applies. See Section 4.5 for a particular example.

The next theorem shows that triadic concepts are universal and optimal factors, that is, every 3-dimensional matrix can be factorized using triadic concepts and the factorizations that employ triadic concepts as factors are optimal.

Theorem 42. *Let I be an $n \times m \times p$ matrix with degrees from L.*

(1) $\rho(I) \leq \min(nm, np, mp)$.

(2) *There exists $\mathcal{F} \subseteq \mathcal{T}(X_1, X_2, X_3, I)$ with $|\mathcal{F}| = \rho(I)$ for which*

$$I = \circ(A_{\mathcal{F}}, B_{\mathcal{F}}, C_{\mathcal{F}}).$$

PROOF. (1) Let $\mathcal{F} = \{\mathfrak{b}_{12}(\{1/i\}, \{1/j\}) \mid 1 \leq i \leq n, 1 \leq j \leq m\}$, (see Theorem 11). Due to Lemma 41, every member of $\mathcal{F}$ is a cuboid contained in I. Moreover, for every i, j, t and $\langle D_1, D_2, D_3\rangle = \mathfrak{b}_{12}(\{1/i\}, \{1/j\}$, we have

$$(D_1 \otimes D_2 \otimes D_3)_{ijt} = I_{ijt}.$$

Indeed, by Theorem 11 we have

$$(D_3)_t = \bigwedge_{1 \leq i' \leq n} (D_1)_{i'} \to \bigwedge_{1 \leq j' \leq n} (D_2)_{j'} \to I_{i'j't} = 1 \to (1 \to I_{ijt}) = I_{ijt}.$$

Thus, for each entry I_{ijt} in I there is a cuboid J induced by some triadic concept of $\mathcal{F}$ such that $I_{ijt} = J_{ijt}$ and therefore I is decomposable using $\mathcal{F}$ as factor concepts. Clearly $|\mathcal{F}| \leq mn$ and $\rho(I) \leq nm$. One proves $\rho(I) \leq np$ and $\rho(I) \leq mp$ in a similar way.

(2) Assume that $I = \circ(A, B, C)$ with k factors. Then by Lemma 40 there are k cuboids $J_1, \dots, J_k$ such that $I = J_1 \vee \cdots \vee J_k$. By Theorem 12 (b) any cuboid J_l, $(l = 1, \dots, k)$, is contained in some triadic concept. That is, there is a triadic concept $\langle D_{1l}, D_{2l}, D_{3l} \rangle$ such that $(D_{1l} \otimes D_{2l} \otimes D_{3l})_{ijt} \geq (J_l)_{ijt}$ for all i, j, t. But by Lemma 41 triadic concepts are cuboids contained in I and thus we can substitute J_l with $(D_{1l} \otimes D_{2l} \otimes D_{3l})$ in the decomposition. Thus, for each decomposition with k factors there is a decomposition using k triadic concepts as factors. The claim then follows from the existence of an optimal decomposition. □

4.3 Transformations between induced spaces

As we mentioned in Section 4.1, given a decomposition $I = \circ(A_{\mathcal{F}}, B_{\mathcal{F}}, C_{\mathcal{F}})$ of an $n \times m \times p$ matrix I for some k-element set $\mathcal{F}$ of factor concepts, matrix A is contains descriptions of objects in term of factors.

It is therefore a natural question to ask for a transformation of a description of a given object in attribute-condition space $L^{m \times p}$ into a description in factor space L^k, and vice versa. For the dyadic case, such transformations are described in [19] and were utilized in [47, 48] for improving classification of binary data.

In the attribute-condition space, the object $i \in X_1$ is represented by the $m \times p$ matrix $I_{i\_\_}$ corresponding to the dyadic context $\langle X_2, X_3, I^{23}_{\{1/i\}} \rangle$ (see Section 2.2), called i-th (*object*) *dyadic cut*. In the factor space, i is represented by the i-th row $A_{i\_}$ of A.

Consider the transformations $g : L^{m \times p} \to L^k$ and $h : L^k \to L^{m \times p}$ defined for $P \in L^{m \times p}$ and $Q \in L^k$ by

$$(g(P))_l = \textstyle\bigwedge_{j=1}^{m} \bigwedge_{t=1}^{p} (B_{jl} \otimes C_{tl} \to P_{jt}) \tag{4.6}$$

$$(h(Q))_{jt} = \textstyle\bigvee_{l=1}^{k} (Q_l \otimes B_{jl} \otimes C_{tl}) \tag{4.7}$$

for $l \in \{1, \dots, k\}, j \in \{1, \dots, m\}$ and $t \in \{1, \dots, p\}$.

The previous two operators have the following interpretation. (4.6) says that the degree to which object i applies to factor l equals to a degree to which i has every attribute j under every condition t such that j is a manifestation of l and t is one of the conditions under which l appears; (4.7) says that a degree to which object i has attribute j under condition t equals the degree to which there is a factor l such that l applies to i, j is a manifestation of l, and l is one of the conditions under which l appears.

The suitability of g and h as natural transformations between attribute-condition and factor spaces is demonstrated by the following theorem.

Theorem 43. *For $i \in \{1, \dots n\}$:*

$$g(I_{i\_\_}) = A_{i\_} \text{ and } h(A_{i\_}) = I_{i\_\_}.$$

That is, g maps the object dyadic cuts of I to the rows of A and vice versa, h maps the rows of A to the object dyadic cuts of I.

PROOF. $h(A_{i\_}) = I_{i\_\_}$ follows directly from $I = \circ(A, B, C)$. Since $A = A_{\mathcal{F}}, B = B_{\mathcal{F}}$, and $C = C_{\mathcal{F}}$, the l-th columns of A, B and C coincide with the extent D_{l1}, intent D_{l2}, and modus D_{l3} of a triadic concept $\langle D_{l1}, D_{l2}, D_{l3} \rangle \in \mathcal{F}$, respectively.

$$\begin{aligned}(g(I_{i\_\_}))_l &= \textstyle\bigwedge_{j=1}^{m} \bigwedge_{t=1}^{p} (B_{jl} \otimes C_{tl} \to (I_{i\_\_})_{jt}) = \\ &= \textstyle\bigwedge_{j=1}^{m} \bigwedge_{t=1}^{p} ((D_{l2})_j \otimes (D_{l3})_t \to I_{ijt}) = \\ &= (D_{l2}^{(1,2,D_{l3})})_i = (D_{l1})_i = A_{il}.\end{aligned}$$

□

The following theorem shows that g and h form an isotone Galois connection.

Theorem 44. *For $P, P' \in L^{m \times p}$ and $Q, Q' \in L^k$:*

$$P \leq P' \Rightarrow g(P) \leq g(P'), \tag{4.8}$$

$$Q \leq Q' \Rightarrow h(Q) \leq h(Q'), \tag{4.9}$$

$$h(g(P)) \leq P, \tag{4.10}$$

$$Q \leq g(h(Q)). \tag{4.11}$$

PROOF. (4.8): Follows from monotony of $\to$ in second argument and monotony of $\wedge$

(4.9): Follows from monotony of $\otimes$ and $\vee$

(4.10): Using basic properties of residuated lattices we have

$$\begin{aligned}h(g(P))_j t &= \bigvee_{l=1}^{k} \left(\bigwedge_{j'=1}^{m} \bigwedge_{t'=1}^{p} (B_{j'l} \otimes C_{t'l} \to P_{j't'}) \otimes B_{jl} \otimes C_{tl} \right) \leq \\ &\leq \bigvee_{l=1}^{k} (B_{jl} \otimes C_{tl} \to P_{jt}) \otimes B_{jl} \otimes C_{tl} \leq \\ &\leq \bigvee_{l=1}^{k} P_{jt} = P_{jt}.\end{aligned}$$

(4.11): Using basic properties of residuated lattices we have

$$\begin{aligned}
g(h(Q))_l &= \bigwedge_{j=1}^{m}\bigwedge_{t=1}^{p}\left(B_{jl}\otimes C_{tl}\to(\bigvee_{l'=1}^{k}Q_{l'}\otimes B_{jl'}\otimes C_{tl'})\right)\geq \\
&\geq \bigwedge_{j=1}^{m}\bigwedge_{t=1}^{p}B_{jl}\otimes C_{tl}\to Q_l\otimes B_{jl}\otimes C_{tl}\geq \\
&\geq \bigwedge_{j=1}^{m}\bigwedge_{t=1}^{p}1\to Q_l = Q_l
\end{aligned}$$

□

(4.8)–(4.11) are natural properties of transformations between attributes-condition and factor spaces. For example, (4.8) shows that the higher the degree to which an object has attributes under conditions, the higher the degree to which factors apply to the object, while (4.9) states analogous relationship in the opposite direction.

A geometry behind the transformations is described by the following assertion. For $P \in L^{m\times p}$ and $Q \in L^k$, put

$$\begin{aligned}
g^{-1}(Q) &= \{P \in L^{m\times p} \mid g(P) = Q\}, \\
h^{-1}(P) &= \{Q \in L^{k} \mid h(Q) = P\}.
\end{aligned}$$

Recall that $S \subseteq L^s$ is called *convex* if $V \in S$ whenever $U \leq V \leq W$ for some $U, W \in S$.

Theorem 45. (1) *$g^{-1}(Q)$ is a convex partially ordered subspace of the attribute and condition space and $h(Q)$ is the least element of $g^{-1}(Q)$.*
(2) *$h^{-1}(P)$ is a convex partially ordered subspace of the factor space and $g(P)$ is the largest element of $h^{-1}(P)$.*

PROOF. By standard application of the properties of isotone Galois connections. □

According to Theorem 45, the space $L^{m\times p}$ of attributes and conditions and the space L^k of factors are partitioned into an equal number of convex subsets. The subsets of the space of attributes and conditions have least elements and the subsets of the space of factors have greatest elements. g maps every element of any convex subset of the space of attributes and conditions to the greatest element of the corresponding subset of the factor space, whereas h maps every element of some convex subset of the space of factors to the least element of the corresponding convex subset of the space of attributes and conditions.

Algorithm 3: ComputeFactors(X, Y, Z, I)

```
compute 𝓑(X, Y, Z, I);
set 𝓢 to 𝓑(X, Y, Z, I);
set 𝓕 to ∅;
set U to X × Y × Z;
while U ≠ ∅ do
    select ⟨A, B, C⟩ ∈ 𝓢 which maximizes |U ∩ S_⟨A,B,C⟩|;
    add ⟨A, B, C⟩ to 𝓕;
    set U to U \ S_⟨A,B,C⟩;
    remove ⟨A, B, C⟩ from 𝓢;
return 𝓕
```

4.4 Algorithms

Due to the above results, the problem of finding a minimal decomposition of $\langle X, Y, Z, I\rangle$ can be seen as a problem of finding a minimal subset $\mathcal{F} \subseteq \mathcal{T}(X, Y, Z, I)$ of formal concepts that cover I. We can reduce the problem of finding a matrix decomposition to the Set Cover problem in the following way. The universe U that should be covered corresponds to $X \times Y \times Z$. The family $\mathcal{S}$ of subsets of the universe U that is used for finding a cover contains for each triadic concept in $\mathcal{T}(X, Y, Z, I)$ a set of indices which the triadic concept covers. More precisely, we set

$$\mathcal{S} = \{S_{\langle A,B,C\rangle} \mid \langle A, B, C\rangle \in \mathcal{T}(X, Y, Z, I)\},$$

where

$$S_{\langle A,B,C\rangle} = \{(i, j, k) \mid A_i \otimes B_j \otimes C_k = I_{ijk}\}.$$

In this setting, we are looking for $\mathcal{C} \subseteq \mathcal{S}$ as small as possible such that $\bigcup \mathcal{C} = U$. Thus, finding factor concepts is indeed an instance of the set-covering problem. It is well known that the set covering optimization problem is NP-hard and the corresponding decision problem is NP-complete. However, there exists a greedy approximation algorithm for the set covering optimization problem which achieves an approximation ratio $\leq \ln(|U|) + 1$, see [26]. This gives us a “naive” greedy-approach algorithm for computing all factor concepts.

Algorithm 3, implementing the above-mentioned greedy approach in our setting, first computes a set of all triadic concepts which are stored in $\mathcal{S}$, see lines 1–2. Then it iteratively selects triadic concepts from $\mathcal{S}$, maximizing their overlap with the remaining tuples in U, see lines 5–9. Notice that the size of the overlap of $\langle A, B, C\rangle$ with U is the number of yet uncovered indices at which the cuboid corresponding to $\langle A, B, C\rangle$ has the same value as I. More precisely, it is the number of elements of $U \cap S_{\langle A,B,C\rangle}$.

The drawback of Algorithm 3 is that it first computes a possibly large set of triadic concepts and then it selects a small subset of it as the set of factor concepts. This difficulty can be overcome by computing the factor concepts "on demand". This can be done in a way analogous to the one described in [12]. Development of such an algorithm is an issue for future research.

4.5 Illustrative example

In this section, we present an illustrative example of factorization. We consider input data containing information about potential car buyers and their motivation for the purchase of a particular type of car. Such data is usually obtained via a customer survey. We assume that customers expressed the degrees of their motivation using a 3-element scale (not at all, partly, significantly).

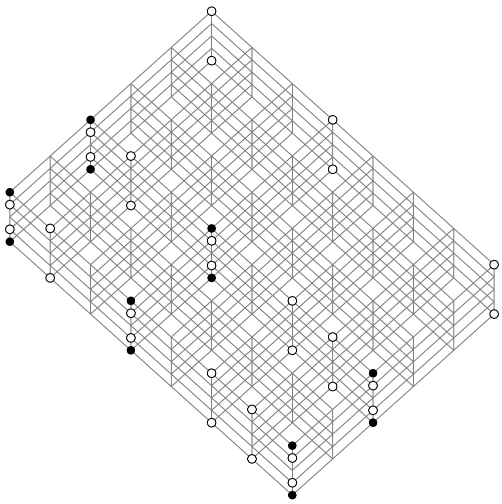
We represent the data by a triadic fuzzy context $\langle X_1, X_2, X_3, I\rangle$, where $X_1 = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{f}, \mathsf{g}, \mathsf{h}\}$ is a set of cars, $X_2 = \{\mathsf{hp}, \mathsf{sp}, \mathsf{ac}, \mathsf{pr}, \mathsf{mc}, \mathsf{sa}\}$ is a set of car characteristics: horse power, space (i.e. the car is spacious), acceleration/speed, price, monthly cost, safety; and $X_3 = \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \mathsf{E}\}$ is a set of customers participating in the survey. The fact that x is related to y under z to the degree $I(x, y, z)$ is interpreted as "the attribute y motivates the customer z for the purchase of x to the degree $I(x, y, z)$". We represent the scale of degrees used in the survey by a 3-element Łukasiewicz chain $\{0, \frac{1}{2}, 1\}$, with the following interpretation:

$$\begin{array}{lcl} 0 & \ldots & \text{not at all,} \\ \frac{1}{2} & \ldots & \text{partly,} \\ 1 & \ldots & \text{significantly.} \end{array}$$

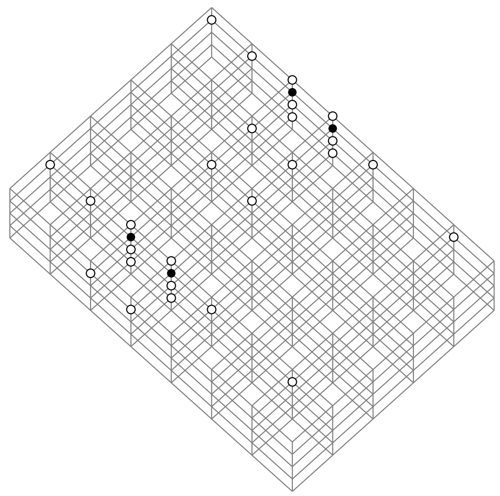
We consider I given the Table 4.1. The rows of the table correspond to cars, the columns correspond to attributes under the various conditions (customers).

In such data, there exists a three-element set $\mathcal{F} = \{F_1, F_2, F_3\}$ of factor concepts. We fix the order of objects, attributes and conditions to the order in which they appear in Table 4.1 in order to represent the extents, intents, and modi of the factor concepts by their characteristic vectors. For example, the characteristic vector of the extent of F_1 is $\langle 1, \frac{1}{2}, \frac{1}{2}, 0, 1, 0, \frac{1}{2}, 1\rangle$ which means that car a belongs to the extent of F_1 to the degree 1, car b to the degree $\frac{1}{2}$ and so forth. The factor concepts in $\mathcal{F}$ are represented by the following triplets of the characteristic vectors of their extents, intents, and modi (the vectors are separated by ;):

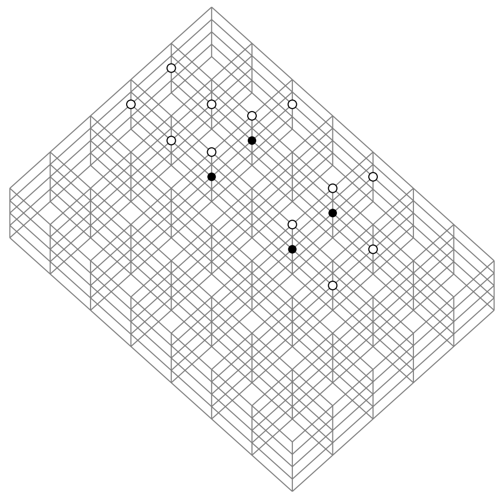
$$\begin{array}{lcl} F_1 & \ldots & \langle 1, \frac{1}{2}, \frac{1}{2}, 0, 1, 0, \frac{1}{2}, 1; 1, 0, 1, 0, 0, \frac{1}{2}; 1, \frac{1}{2}, 0, \frac{1}{2}, 1\rangle, \\ F_2 & \ldots & \langle 0, \frac{1}{2}, 0, \frac{1}{2}, 1, 1, \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, 1, 0, \frac{1}{2}, \frac{1}{2}, 1; 0, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}\rangle, \\ F_3 & \ldots & \langle 0, 0, \frac{1}{2}, 1, 0, 1, \frac{1}{2}, \frac{1}{2}; 0, 0, 0, 1, 1, \frac{1}{2}; 0, 0, 1, 0, \frac{1}{2}\rangle. \end{array}$$



F_1: "ability to go fast"



F_2: "being a family car"



F_3: "cost-effectiveness"

Figure 4.2: Geometric meaning of factors.

Table 4.1: Triadic context

	A						B						C						D						E					
	hp	sp	ac	pr	mc	sa	hp	sp	ac	pr	mc	sa	hp	sp	ac	pr	mc	sa	hp	sp	ac	pr	mc	sa	hp	sp	ac	pr	mc	sa
a	1	0	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	0	0	0	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	1	0	1	0	0	$\frac{1}{2}$
b	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0
c	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0
d	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
e	1	0	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$
f	0	0	0	0	0	0	0	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
g	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0
h	1	0	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	0	1	0	0	$\frac{1}{2}$

Using $\mathcal{F}$, we obtain the following 8×3 object-factor matrix $A_{\mathcal{F}}$, 6×3 attribute-factor matrix $B_{\mathcal{F}}$, and 5×3 conditions-factor matrix $C_{\mathcal{F}}$:

$$A_{\mathcal{F}} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \qquad B_{\mathcal{F}} = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & \frac{1}{2} & 1 \\ \frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}, \qquad C_{\mathcal{F}} = \begin{pmatrix} 1 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & 1 \\ 1 & 1 & \frac{1}{2} \end{pmatrix}.$$

One can check that $I = \circ(A_{\mathcal{F}}, B_{\mathcal{F}}, C_{\mathcal{F}})$, i.e., I decomposes into three (two-dimensional) matrices using three factors. Note that the meaning of the factors can be seen from the extents, intents, and modi of the factor concepts. For instance, F_1 can be interpreted as "the ability to go fast". Indeed, F_1 is manifested by the attributes horse power and speed to the degree 1, and by safety to the degree $\frac{1}{2}$. The factor F_2 is manifested by space and safety to the degree 1, and by horse power, price, and monthly cost to the degree $\frac{1}{2}$. This suggests that F_2 can be interpreted as "being a family car". The high degree manifestations of F_3 are price and monthly cost, leading to a possible interpretation as "cost-effectiveness". As a result, by finding the factors set $\mathcal{F} = \{F_1, F_2, F_3\}$, we have explained the structure of the input data set I using three factors which describe the attractivity of cars to customers in terms of their characteristics.

Let us recall that the factor concepts $\mathcal{F} = \{F_1, F_2, F_3\}$ can be seen as maximal cuboids in I. Indeed, I itself can be depicted as three-dimensional box where the axes correspond to cars, their characteristics, and customers. Figure 4.2 shows the three factors depicted as cuboids. White and black circlets in Figure 4.2 correspond to elements in I. Namely, a white circlet is present on the intersection of $x \in X$, $y \in Y$, and $z \in Z$ in the diagram iff $\circ(A_{\_i}, B_{\_i}, C_{\_i})(x, y, z) = \frac{1}{2}$. Furthermore, the circlet is black iff $\circ(A_{\_i}, B_{\_i}, C_{\_i}) = 1$. That is, for a factor F_i, the circle depicts the degree to which x belongs to the extent of F_i, y belongs to the intent of F_i, and z belongs to the modus of F_i.

4.6 Summary and topics of future research

We presented a method for decomposition of three-way ordinal matrices. The method utilizes triadic fuzzy concepts of the input data as factors. As demonstrated by an small example, a clear interpretation of triadic fuzzy concepts provides to the factor model transparent meaning. We proved that a decompositions using triadic concepts are optimal. We provided natural transformations between descriptions of objects in attribute-condition space and the space of factors. We showed by a reduction to Set Cover problem that the problem of finding an optimal decomposition is NP-hard. We utilized a greedy approach to solving Set Cover problem in an approximation algorithm for finding a decomposition.

The topics left for future work include:

- Design of more efficient algorithm that overcome the necessity of computation of all triadic concepts. Such an algorithm can be based on the idea utilized in decomposition of dyadic matrices [19].

- Further study of algorithms. This includes their computational complexity, approximation factors and performance evaluation.

- Approximate decompositions. The study of a problem of finding a set of factors such that their product equals the original matrix to a given degree.

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