PALACKÝ UNIVERSITY OLOMOUC
FACULTY OF SCIENCE

BACHELOR'S THESIS

Rational Trigonometry

Department of mathematical analysis and applications of mathematics

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Abstract: This thesis presents and compares mathematical notions from the theories of the rational trigonometry and the classical trigonometry. Theories are firstly compared theoretically then tested practically with a computer program. At the end there are subjective and objective analyses of the results from the tests.

Key words: Rational trigonometry, geometry, quadrance, spread, test, comparison

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Abstrakt: Tato práce prezentuje a srovnává matematické pojmy z teorií racionální trigonometrie a klasickeho přístupu k trigonometrii. Teorie jsou srovnány teoretickým i praktickým způsobem nejen za použití počítačového programu vytvořeného k tomuto účelu. Konec práce obsahuje subjektivní i objektivní shrnutí výsledků programu.

Klíčová slova: Racionální trigonometrie, geometrie, quadrance, spread, test, srovnání

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Statement

I declare that this piece of work is entirely my own, only the references cited have been acknowledged.

Czech Translation:

Prohlašuji, že jsem bakalářskou práci vypracoval samostatně a všechny použité zdroje jsem uvedl.

Date *&* Place: Author's Signature:

Table of Contents

1. Introduction

Introductions do not need to be full chapters necessarily. This introduction however brings more information than usually forewords bring. It is so for a simple reason. No one knows what rational trigonometry is. And that is fair. Therefore, this introductory chapter is enlarged to incorporate the basics of rational trigonometry.

The first aim of this chapter is, however, to state the purpose of this thesis. The second aim is to introduce rational trigonometry in context. But before I do that, I would like to mention these facts about this text.

I first saw the theory of rational trigonometry after I started studying applied mathematics. At that time I liked about it that is seemed novel and highly applicable. Despite not finding a friend to consult the theory with I continued my exploration of rational trigonometry on my own. After a while I slowly gained confidence in the theory, so I figured that it would be good topic for my Bachelor's thesis. In this thesis I would like to present what I have learned about rational trigonometry, using the skills I have acquired so far.

I would also like to clarify why I chose to write this piece of work in English. I have in fact two reasons for doing so. Firstly, the entirety of the theory of rational trigonometry is in English and I am not experienced enough to properly translate all the new notions of rational trigonometry. Secondly, since I am comfortable reading English I would also like to become comfortable writing it.

1.1. Content & Contribution

This subchapter presents a summary of the whole thesis. I hope that through reading the following lines you will gain a clearer idea of what and why is included.

This thesis consists of two digital files — this document and one computer program [7]. I encourage you to read this document first, mainly because it provides an important insights about the inner workings of the program.

In the upcoming subsection there are last introductory words. Chapter two titled Theory offers a theoretical minimum of rational trigonometry useful for later use in this text. Other covered topics include a side-by-side comparison of trigonometric notions and applications of rational trigonometry. The aim is to clarify how rational trigonometry functions and how it differs from classical trigonometry.

Chapters three and four are more practical. Former chapter titled Tests is devoted to the computer program accompanying this text. The aim of the chapter is to convey relevant information about the program and routines it is running. Latter chapter titled Results illustrates the outcomes of the program and analyses them. Last chapter called Conclusion offers a brief conclusion.

This thesis is my attempt to bring understandable information about mathematical theories I am interested in. The attached computer program [7] and the analysis of it's output are my unique contributions.

1.2. Overview

Tigonometry as you know it is not anything new. When I write classical trigonometry in this thesis I mean precisely that what you imagine under the term trigonometry. Rational trigonometry differs from classical trigonometry in that it is made to work over an arbitrary field. This is possible by careful usage of the square root function and avoidance of trigonometric functions.

In this thesis, however, I will not take advantage of this fact. I could work with rational trigonometry over a finite field, but I would not be able to compare the theories easily. My approach is to work over the real numbers field in the theory section and use float number type for all important variables in the program.

The rest of this chapter is about understanding how to charaterize triangles without using angles. In rational trigonometry we characterize a triangle by computing all the quadrances of its sides and all its spreads.

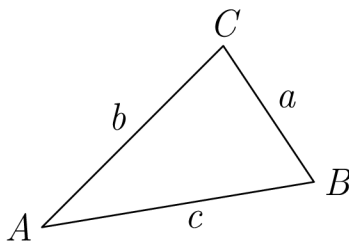If we want to understand it let us start with the quadrance. Imagine a triangle like this one:

Figure 1: $Q(A, B)$ measures the separation of points A, B

The quadrance $Q(A, B)$ is the length of the side c squared, or $Q(A, B) = |c|^2$. Giving us

$$Q(A, B) = Q_C = (x_A - x_B)^2 + (y_A - y_B)^2,$$

for $A = [x_A, y_A]$ and $B = [x_B, y_B]$. Note that by computing quadrance you do less work than by computing length.

Spread is a more subtle notion which measures the separation of two lines. When computing the spread of two lines l and k we can use a simple formula like this:

$$s(l, k) = \frac{(a_l b_k - b_l a_k)^2}{(a_l^2 + b_l^2)(a_k^2 + b_k^2)},$$

for l characterized by $xa_l + yb_l + c_l = 0$, and k by $xa_k + yb_k + c_k = 0$. But what is the interpretation of it?

First let us remember how an angle and the sine of an angle are defined. Angle θ is a length along the unit circle:
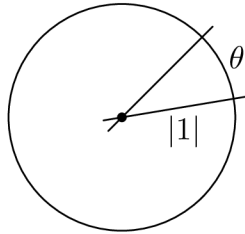

Figure 2: Representation of an angle θ

Sine of θ is the value of the series

$$\sin\theta = \sum_{i=0}^{\infty} \frac{(-1)^i \, \theta^{1+2i}}{(1+2i)!}.$$

On the other hand, the definition of spread does not require prior knowledge of circles or infinite series. In its most basic form it is a ratio of two quadrances R and Q. This is an unintuitive insight that we can visualize.
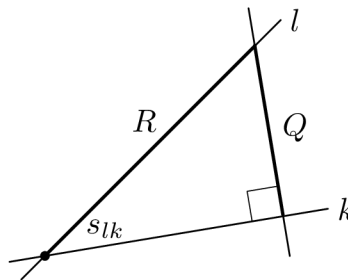

Figure 3: Spread s_{lk} as a ratio of quadrances R and Q

Both spreads can be obtained by constructing a perpendicular line to one of the lines we started with. The perpendicular line can be constructed anywhere along it but not where it meets the other line. Quadrance R can be measured on the unused line and quadrance Q can be measured on the constructed perpendicular line.

From the definition of spread we can see that the spread number of two lines is uniquely determined. Identity bridging both spread and angle is

$$\sin^2(\theta) = s(l, k),$$

where θ denotes the angle at the cross of lines l and k.

2. Theory

This chapter consists of two subchapters. The first one focuses on trigonometry. It mainly contains the theory used later and a few extra counter-examples of classical trigonometry for the purpose of showing the differences between the theories.

Theory for this chapter is taken from [3] and [4]. For consistency, proofs from [3] are translated from Czech to English. If theorems or definitions are labeled, the original label is used.

The second subchapter is devoted to four applications of rational trigonometry. Some of these are relevant in later chapters. I used [2], [3] and [4] as my source material for it.

2.1. Trigonometry

The theory of rational trigonometry consists of many new definitions and theorems. They work best with rational numbers and finite fields. However, for the purpose of this text, we use real numbers field to ensure compatibility for later testing.

The upcomming section is about proving the *Spread law* with theoretical consistency. This is a theorem used later in one of the tests. If not indicated otherwise the theory presented is from [4]. Let us begin with defining fundamental building blocks of rational trigonometry.

Definition: A *3-proportion* is an expression of the form $a : b : c :$ where a, b and c are numbers, not all zero, with the convention that

$$a : b : c = da : db : dc$$

for any non-zero number d •

Definition: A *line* $l = a : b : c$ is a 3-proportion with the property that a and b are not both zero •

Line defined using a 3-poportion ensures that each line is uniquely determined. It also enables us to define a canonical form for each line. We could say than a line is in canonical form if $a = 1$. And we could uniquely determiny every line in a plane except when $a = 0$. These lines can not be expressed using this convention.

Other useful definitions include:

Definition: The line $l = a : b : c$ is a *null line* precisely when

$$a^2 + b^2 = 0$$

•

Definition: The *point* $A = [x, y]$ *lies on the line* $l = a : b : c$ precisely when

$$ax + by + c = 0$$

•

The following definition is repeated here for consitency.

Definition: For points $A = [x_A, y_A]$ and $B = [x_B, y_B]$, quadrance $Q(A, B)$ is the number

$$Q(A, B) = (x_B - x_A)^2 + (y_B - y_A)^2$$

•

The following definition makes it easy for a computer to check whether two given lines are parallel.

Definition: The *lines* $l = a_l : b_l : c_l$ and $k = a_k : b_k : c_k$ are *parallel* precisely when

$$a_l b_k - a_k b_l = 0 \qquad \bullet$$

Next two theorems create a nice pair and both are useful later. All other named theorems can be looked-up in the original text [4] as well.

Theorem (Line through two points): For any two distinct points A and B, there is a unique line $l = AB$ passing through them both. If $A = [x_A, y_A]$ and $B = [x_B, y_B]$ then

$$l = AB = y_A - y_B : x_B - x_A : x_A y_B - x_B y_A \qquad \bullet$$

Proof: If $A = [x_A, y_A]$ and $B = [x_B, y_B]$ are distinct points, then

$$l = y_A - y_B : x_B - x_A : x_A y_B - x_B y_A$$

passes through them both, since

$$(y_A - y_B)x_A + (x_B - x_A)y_A + x_A y_B - x_B y_A = 0$$

and

$$(y_A - y_B)x_B + (x_B - x_A)y_B + x_A y_B - x_B y_A = 0 \qquad \bullet$$

Theorem (Point on two lines): If the lines l and k are not parallel, then there is a unique point $A = lk$ which lies on both lines. If $l = a_l : b_l : c_l$ and $k = a_k : b_k : c_k$ then

$$A = lk = \left[\frac{b_l c_k - b_k c_l}{a_l b_k - a_k b_l}, \frac{c_l a_k - c_k a_l}{a_l b_k - a_k b_l} \right] \qquad \bullet$$

Proof: The point $[x, y]$ lies on $l = a_l : b_l : c_l$ and $k = a_k : b_k : c_k$ precisely when

$$a_l x + b_l y + c_l = 0$$
$$a_k x + b_k y + c_k = 0$$

Since l and k are not parallel, $a_l b_k - a_k b_l \neq 0$, so the unique solution to this system of equations is

$$x = \frac{b_l c_k - b_k c_l}{a_l b_k - a_k b_l}$$

and
$$y = \frac{c_l a_k - c_k a_l}{a_l b_k - a_k b_l}$$ •

Next theorems have the property that they seem to simplify the general concept.

Theorem (Parallel to a line): For any point $A = [x, y]$ and any line $l = a : b : c$ there is a unique line k, called the parallel through A to l, which passes through A and is parallel to l, namely
$$k = a : b : -ax - by$$ •

Proof: If $l = a : b : c$ then any line k parallel to l must have the form $k = a : b : d$ for some number d, passing through $A = [x, y]$ precisely when

$$ax + by + d = 0.$$

Thus such a line k is uniquely determined, and has the form

$$k = a : b : -ax - by$$ •

Theorem (Altitude to a line): For any point $A = [x, y]$ and any line $l = a : b : c$ there is a unique line n, called the altitude from A to l, which passes through A and is perpendicular to l, namely
$$n = -b : a : bx - ay$$ •

Proof: If $l = a : b : c$ then any line n perpendicular to l must have the form $n = -b : a : d$ for some number d, passing through $A = [x, y]$ precisely when

$$-bx + ay + d = 0.$$

Thus such a line n is uniquely determined, and has the form

$$n = -b : a : bx - ay$$ •

Theorem (Foot of an altitude): For any point $A = [x, y]$ and any non-null line $l = a : b : c$, the altitude n from A to l intersects l at the point

$$F = \left[\frac{b^2 x - aby - ac}{a^2 + b^2}, \frac{-abx + a^2 y - bc}{a^2 + b^2} \right]$$ •

Proof: By the Altitude to a line theorem, the altitude from $A = [x, y]$ to $l = a : b : c$ is $n = -b : a : bx - ay$. These two lines are not parallel since by assumption $a^2 + b^2 \neq 0$. By the Point on two lines theorem

$$F = nl = \left[\frac{b^2 x - aby - ac}{a^2 + b^2}, \frac{-abx + a^2 y - bc}{a^2 + b^2} \right] \qquad \bullet$$

On the other hand, the following definitions may seem as a theoretical weight up until you see how to apply it.

Definition: For a point A define ρ_A, the *rotation* in A, to be the transformation defined by

$$\rho_A(B) = 2A + (-1)B \qquad \bullet$$

In the definition there is the word transformation in it. The source **[4]** defines transformation to be a function with a point as input and output.

Definition: If l is a non-null line and F the foot of the altitude from a point A to l, then the transformation that sends A to $B = \rho_F(A)$ is the *reflection in the line l* and is dented σ_l $\qquad \bullet$

Some useful applications of rotation and reflection in a line are:

Theorem (Reflection of a point in a line): If $l = a : b : c$ is a non-null line and $A = [x, y]$, then

$$\sigma_l(A) = \left[\frac{(b^2 - a^2)x - 2aby - 2ac}{a^2 + b^2}, \frac{-2abx + (a^2 - b^2)y - 2bc}{a^2 + b^2} \right] \qquad \bullet$$

Proof: The foot F of the altitude from $A = [x, y]$ to $l = a : b : c$ is, by the Foot of an altitude theorem,

$$F = \left[\frac{b^2 x - aby - ac}{a^2 + b^2}, \frac{-abx + a^2 y - bc}{a^2 + b^2} \right].$$

By definition $\sigma_l(A) = \sigma_F(A) = 2F + (-1)A$ so that

$$\sigma_l(A) = \left[\frac{(b^2 - a^2)x - 2aby - 2ac}{a^2 + b^2}, \frac{-2abx + (a^2 - b^2 y - 2bc}{a^2 + b^2)} \right] \qquad \bullet$$

Next proposition is well known result from trigonometry and comes occasionally useful.

Theorem (Pythagors' theorem): Suppose that the triangle ABC has quadrances $Q_A = Q(B, C), Q_B = (A, C)$ and $Q_C = (A, B)$. Then

$$Q_A + Q_B = Q_C$$

precisely when AC and BC are perpendicular ●

Proof: Suppose the points are $A = [x_A, y_A]$, $B = [x_B, y_B]$ and $C = [x_C, y_C]$. Then by the Line through two points theorem

$$AC = y_A - y_C : x_C - x_A : x_A y_C - x_C y_A,$$
$$BC = y_B - y_C : x_C - x_B : x_B y_C - x_C y_B.$$

These lines are perpendicular precisely when

$$(y_A - y_C)(y_B - y_C) + (x_C - x_A)(x_C - x_B) = 0.$$

Now

$$
\begin{aligned}
Q_A + Q_B - Q_C &= (x_B - x_C)^2 + (y_B - y_C)^2 + (x_A - x_C)^2 + (y_A - y_C)^2 \\
&\quad - (x_B - x_A)^2 - (y_B - y_A)^2 \\
&= 2(x_C^2 - x_B x_C - x_A x_C + x_B x_A + y_A y_B - y_A y_C - y_B y_C + y_C^2) \\
&= 2((y_A - y_C)(y_B - y_C) + (x_C - x_A)(x_C - x_B)).
\end{aligned}
$$

Since $2 \neq 0$, AC and BC are perpendicular precisely when $Q_A + Q_B = Q_C$ ●

To compare classical trigonometry and rational trigonometry, I have included the following theorem. It is dragged from the context of the original text [3]. V_a denotes the altitude to the point a in a ABC triangle and $|XY|$ denotes the distance between two poins X and Y.

Theorem 16 (Pythagorova): Let ABC be a right triangle with the right angle at C. Then the sides a, b and the hypotenuse c holds the following identity

$$a^2 + b^2 = c^2$$

●

Proof: The proof follows from both Euclid's theorems (**Theorem 15**) ●

Theorem 15 (Euklidovy věty): Let ABC be a right triangle with the right angle at C. Let $|BV_c| = c_a$ and $|AV_c| = c_b$. Then

$$a^2 = cc_a,$$
$$b^2 = cc_b$$

and

$$v_c^2 = c_a c_b \qquad \bullet$$

Proof: The proof is trivial (Use the property that right triangles are similar) $\qquad \bullet$

I agree that such a proof is trivial but you can see how the author is leaving a gap between the theory and a possible implementation of a computable proof. This is understandable since the source text [3] is not trying to be a gapples resource. In is nice to see rational trigonometry tries to be more elementary in some sense.

Continuing on with our proof of the *Spread law*, let us define spread once again. And defining cross and twist can helps us see spread in a broader context.

Definition: The *spread* $s(l, k)$ between the non-null lines $l = a_l : b_l : c_l$ and $k = a_k : b_k : c_k$ is the number

$$s(l, k) = \frac{(a_l b_k - a_k b_l)^2}{(a_l^2 + b_l^2)(a_k^2 + b_k^2)} \qquad \bullet$$

Definition: The *cross* $c(l, k)$ between the non-null lines $l = a_l : b_l : c_l$ and $k = a_k : b_k : c_k$ is the number

$$c(l, k) = \frac{(a_l b_k + a_k b_l)^2}{(a_l^2 + b_l^2)(a_k^2 + b_k^2)} \qquad \bullet$$

Definition: The *twist* $t(l, k)$ between the non-null lines $l = a_l : b_l : c_l$ and $k = a_k : b_k : c_k$ is the number

$$t(l, k) = \frac{(a_l b_k - a_k b_l)^2}{(a_l b_k + a_k b_l)^2} \qquad \bullet$$

Theorem (Spread plus cross): For non-null lines l and k, let s and c be respectively the spread and cross between them. Then

$$s + c = 1 \qquad \bullet$$

An important identity in rational trigonometry is Fibonacci's identity which appears in the next proof. It has the following form

$$(ad - cb)^2 + (ac + bd)^2 = (a^2 + b^2)(c^2 + d^2).$$

Proof: This follows from the definitions of spread and cross, together with the Fibonacci's identity

$$(a_l b_k - a_k b_l)^2 + (a_l b_k + a_k b_l)^2 = (a_l^2 + b_l^2)(a_k^2 + b_k^2) \qquad \bullet$$

This nice property of rational trigonometry is compensaded by the fact that spread does not have additive properties. For example, all three spreads oflines in a triangle do not add up into a nice constant but there is a formula involving three spreads in a triangle. And since the formula is not needed to prove Spread law let us prove the following conditional theorem instead.

Theorem (Spread ratio): Suppose that ABC is a right triangle with a right angle at C, and quadrances Q_A, Q_B and Q_C. Then the spread at the point A is

$$s_A = s(AB, AC) = Q_A/Q_C \qquad \bullet$$

Proof: Assume the points are $A = [x_A, y_A]$, $B = [x_B, y_B]$ and $C = [x_C, y_C]$. Then

$$AB = y_A - y_B : x_B - x_A : x_A y_B - x_B y_A,$$
$$AC = y_A - y_C : x_C - x_A : x_A y_C - x_C y_A,$$
$$AB = y_B - y_C : x_C - x_B : x_B y_C - x_C y_B,$$

so that

$$s_A = \frac{\big((y_A - y_B)(x_C - x_A) - (y_A - y_C)(x_B - x_A)\big)^2}{\big((y_A - y_B)^2 + (x_B - x_A)^2\big)\big((y_A - y_C)^2 + (x_C - x_A)^2\big)}.$$

Since AC and BC are perpendicular

$$(y_A - y_C)(y_B - y_C) + (x_C - x_A)(x_C - x_B) = 0.$$

Using the following identity

$$(y_A - y_B)(x_C - x_A) - (y_A - y_C)(x_B - x_A)$$
$$=$$
$$(y_A - y_C)(x_C - x_B) - (y_B - y_C)(x_C - x_A)$$

the numerator of s_A is

$$\big((y_A - y_B)(x_C - x_A) - (y_A - y_C)(x_B - x_A)\big)^2$$
$$=$$
$$\big((y_A - y_C)(x_C - x_A) - (y_B - y_C)(x_C - x_A)\big)^2$$
$$+\big((y_A - y_C)(y_B - y_C) + (x_C - x_A)(x_C - x_B)\big)^2$$
$$=$$
$$\big((y_A - y_C)^2 + (x_C - x_A)^2\big)\big((y_B - y_C)^2 + (x_C - x_B)^2\big)$$

where Fibonacci's identity is used in the last step. Hence

$$s_C = (Q_B Q_A)/(Q_C Q_B) = Q_A/Q_C \qquad \bullet$$

To prove the Spread law one more definition is needed.

Definition: A triangle ABC is a *null triangle* precisely when one or more of its lines is a null line $\qquad \bullet$

With all this we can prove a parallel theorem to the law of Sines — the Spread law.

Theorem (Spread law): Suppose three points A, B and C form non-zero quadrances $Q_A = Q(B,C), Q_B = (A,C)$ and $Q_C = (A,B)$. Define the spreads $s_A = s(AB, AC), s_B = s(BA, BC)$ and $s_C = s(CA, CB)$. Then

$$\frac{s_A}{Q_A} = \frac{s_B}{Q_B} = \frac{s_C}{Q_C} \qquad \bullet$$

Proof: If A, B and C are collinear, then $s_A = s_B = s_C = 0$ and the theorem is immediate. Otherwise the three points form a triangle ABC, which is not null, since one of the quadrance is non-zero. Suppose that D is the foot of the altitude from A to BC. Define the quadrances

$$R_A = Q(A,D), \qquad R_B = Q(B,D), \qquad R_C = Q(C,D)$$

Since ABD and ACD are right triangles, with right vertices at D, the Spread ratio theorem asserts that

$$s_B = R_A/Q_C, \qquad S_C = R_A/Q_B.$$

Solve for R_A to get

$$R_A = Q_C s_B = Q_B s_C$$

so that

$$\frac{s_B}{Q_B} = \frac{s_C}{Q_C}.$$

In a similar manner

$$\frac{s_A}{Q_A} = \frac{s_B}{Q_B} \qquad \bullet$$

Now we can compare this proof to the prove of the law Sines given in [3].

Theorem 13 (sinová věta): In every triangle ABC the following holds

$$a \sin \beta = b \sin \alpha,$$

$$b \sin \gamma = c \sin \beta,$$

and

$$c \sin \alpha = a \sin \gamma \qquad\qquad\bullet$$

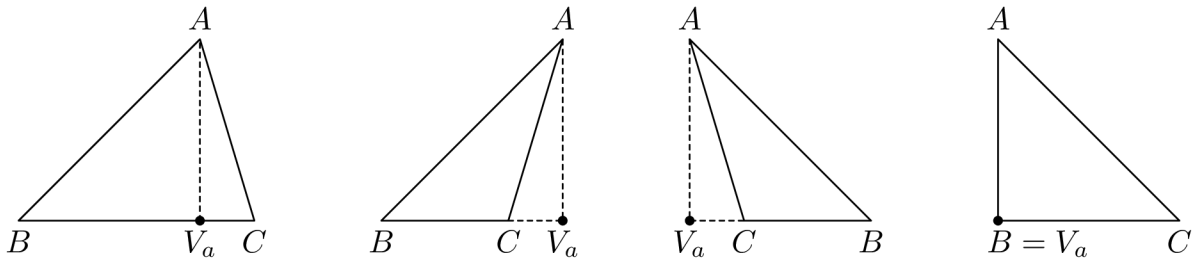**Proof**: Consider the following figure:

Figure 4: triangles (a), (b), (c) and (d)

For the triangles (a), (b) and (c) we get

$$\sin \beta = v_a/c \qquad \sin \beta = v_a/c \qquad \sin(\pi - \beta) = v_a/c.$$

Because $\sin(\pi - \beta) = \sin \beta$ we get $v_a = c \sin \beta$ in all cases. When $\beta = \pi/2$ the identity $v_a = c \sin \beta = c$ holds which you can check above (d). From the identity we also get the formula for area of a triangle ABC namely

$$P_{ABC} = \frac{1}{2}ab \sin \gamma = \frac{1}{2}bc \sin \alpha = \frac{1}{2}ca \sin \beta$$

In all these cases the identity $v_a = c \sin \beta$ holds. Analogously we can obtain the result $v_a = b \sin \gamma$ from the triangle (d). Both formulas give the indentity $b \sin \gamma = c \sin \beta$. This is the first part of the theorem regarding V_a. Remaining two parts can be obtained similarly $\qquad\qquad\bullet$

In this proof a visual component is employed to conceive an elegant solution.

After providing a taste of how rational trigonometry proves it's theorems let us continue this chapter without proofs. Naturally, all proofs can be found in [4]. The Triple spread formula mentioned above is a crucial part of the theory used in later pages.

Theorem (Triple spread formula): Suppose that l, k and j are non-null lines, with $s_l = s(k, j), s_k = s(l, j)$ and $s_j = s(k, l)$. Then

$$(s_l + s_k + s_j)^2 = 2(s_l^2 + s_k^2 + s_j^2) + 4s_l s_k s_j \qquad \bullet$$

Cross mirrors the Spread as Cosine mirrors the Sine functions. Therefore there is also a Cross law and trible cross formula.

Theorem (Cross law): suppose three points A, B and C form quadrances $Q_A = Q(B, C), Q_B = (A, C)$ and $Q_C = (A, B)$, and define the cross $c_C = c(CA, CB)$. Then

$$(Q_A + Q_B - Q_C)^2 = 4Q_A Q_B c_C \qquad \bullet$$

Theorem (Triple cross formula): Suppose that l, k and j are non-null lines, with $c_l = c(k, j), c_k = c(l, j)$ and $c_j = c(k, l)$. Then

$$(c_l + c_k + c_j - 1)^2 = 4c_l c_k c_j \qquad \bullet$$

One last piece of theory used in the next chapter is the following theorem.

Theorem (Triple twist formula): Suppose that l, k and j are non-null lines, with $t_l = t(k, j), t_k = t(l, j)$ and $t_j = t(k, l)$. Then

$$(t_l + t_k + t_j - t_l t_k t_j)^2 = 4(t_l t_k + t_l t_j + t_k t_j + 2t_l t_k t_j) \qquad \bullet$$

2.2. Applications

The theory covered so far would not cover everything needed to understand the next chapter. For this reason the following subchapter contains applications of rational trigonometry which may be used later on.

Area & quadrea

To define area properly the square root function si needed. On the other hand if we stick with quadrance we are able to define quadratic area.

Definition: For three points A, B and C with quadrances $Q_A = Q(B, C), Q_B = (A, C)$ and $Q_C = (A, B)$, the *quadrea* **A** of the set $\{A, B, C\}$ is the number

$$\mathbf{A} = A(Q_A, Q_B, Q_C) = (Q_A + Q_B + Q_C)^2 - 2(Q_A^2 + A_B^2 + Q_C^2) \qquad \bullet$$

And the following theorems are used in later tests.

Theorem (Quadrea): The quadea **A** of $\{A, B, C\}$, for $A = [x_A, y_A]$, $B = [x_B, y_B]$ and $C = [x_C, y_C]$, is

$$\mathbf{A} = 4(x_A y_B - x_A y_C + x_B y_C - x_C y_B + x_C y_A - x_B y_A)^2.$$

In particular the quadrea **A** is a square, and $\{A, B, C\}$ is a triangle precisely when **A** is non-zero $\qquad \bullet$

Theorem (Archimedes' formula): Suppose that a triangle ABC has square quadrances of the form $Q_A = d_A^2, Q_B = d_B^2$ and $Q_C = d_C^2$. Then the quadrea is

$$\mathbf{A} = (d_A + d_B + d_C)(-d_A + d_B + d_C)(d_A - d_B + d_C)(d_A + d_B - d_C) \qquad \bullet$$

Proof:

$$\begin{aligned}
\mathbf{A} &= 4(x_A y_B - x_A y_C + x_B y_C - x_C y_B + x_C y_A - x_B y_A)^2 \\
&= 4d_A^2 d_B^2 - (d_A^2 + d_B^2 - d_C^2)^2 \\
&= \left(2d_A d_B - (d_A^2 + d_B^2 - d_C^2)\right)\left(2d_A d_B (d_A^2 + d_B^2 - d_C^2)\right) \\
&= \left(d_C^2 (d_A - d_B)^2\right)\left((d_A + d_B)^2 - d_C^2\right) \\
&= (d_A + d_B + d_C)(-d_A + d_B + d_C)(d_A - d_B + d_C)(d_A + d_B - d_C)
\end{aligned}$$

gives us the correct formula •

The equivalent to the previous theorem in classical trigonometry is the *Heron's formula*. With it we can compute the area of a triangle A if we know lengths of it's sides a, b and c.

Theorem (Heron's formula): The area of a triangle A with side lenghts a, b and c is

$$A = \sqrt{\big(s(s-a)(s-b)(s-c)\big)},$$

where

$$s = (a+b+c)/2 \qquad\qquad \bullet$$

Spherical *&* polar coordinates

This application helps us to see the parametrization of a circle in a new light. In [4] there are practical examples of this. Here you can only find a brief overview.

The rational theory of measuring spheres in not finished and lacks the richness of classical spherical trigonometry found it [2]. Nonetheless, we are able to use rational trigonometry to represent rational spherical coordinates.

For a point $A = [x, y]$ in Cartesian coordinates, introduce the polar spread s and the quadrance Q by

$$s = x^2/(x^2 + y^2),$$
$$Q = x^2 + y^2.$$

Then $[s, Q]$ are the rational polar coordinates of the point $A = [x, y]$. The spread s is defined between OA and y axis.
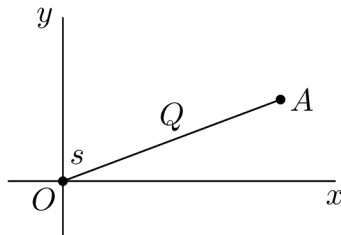
Figure 5: Polar coordinates

To represent a point $A = [x, y, z]$ in three-dimensional space we can use the spherical coordinates $[s, q, R]$ of A

$$s = x^2/(x^2 + y^2),$$
$$q = (x^2 + y^2)/(x^2 + y^2 + z^2),$$
$$R = x^2 + y^2 + z^2.$$

Snell's law

Snell's law states that the refracitive index of two materials can be obtained as the ration of the sines of θ_A and θ_B.

Suppose a particle travels from the point $A = [0, a]$ to the point $B = [c, -b]$ via some point $D = [x, 0]$ on the horizontal axis. The particle speed in the region above and belove the x axis is v_1 and v_2 respectively.
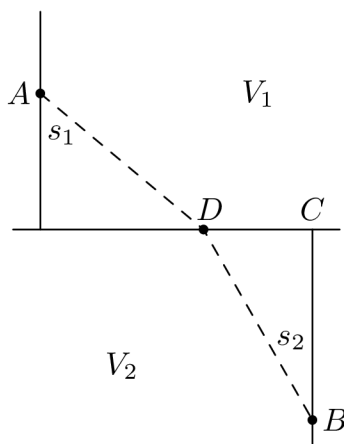

Figure 6: Refractive index

This can be proved purely by using rational trigonometry as demonstrated in [4], page number 239. The rational formula has the form:

$$\frac{V_2}{V_1} = \frac{s_2}{s_1}.$$

And the classical formula has the form:

$$\frac{v_1}{v_2} = \frac{\sin \theta_A}{\sin \theta_B},$$

where V is the square of v.

3. Tests

In this chapter we compare rational trigonometry with classical trigonometry using computers. To do this, [1] was used to help me with the ideas for benchmarks and [4] as usual for the support with rational trigonometry.

The first subsection presents description of the test. Most important notions of rational trigonometry used here are covered in the previous chapter.

Implementation of these tests is made available for you in the form of the attachment [7]. It is a source code of a program I wrote for the purpose of this thesis. The code can be compiled into an executable program which can be run inside your terminal.

The second subsection contains implementation details about the program. And then the source code [7] itself contains many useful comments.

3.1. Routines

Following pages explain each test included in [7]. Angles and spreads are not usually shown in figures. And they are called by a nearby point as usual.

Right triangle

Starting simple, the task of the first test is to solve a right triangle. This test is inspired by a test from [1], page number 424.
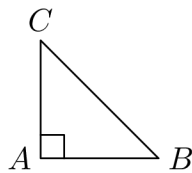
Figure 7: Right triangle ABC

For each repetition of the test lengths of both legs of a right triangle are pseudo-randomly generated. In this test the angle near the point A is always $\pi/2$ and the spread is always 1. The sides b and c can have length anywhere between 0 and 9.9 units. The program then times itself how fast it can solve the generated triangle using classical trigonometry and rational trigonometry.

Let us compare both methods. In the case of classical trigonometry we advance in this fashion:
$$a = \sqrt{b^2 + c^2},$$
$$\theta_B = \arcsin(b/a),$$
$$\theta_C = \arcsin(c/a).$$

Rational trigonometry uses the following steps:
$$Q_b = b^2,$$
$$Q_c = c^2,$$
$$Q_a = Q_b + Q_c,$$
$$s_B = Q_b/Q_a,$$
$$s_C = Q_c/Q_a.$$

Overlapping triangles

A bit more difficult task is to solve the cross-section of two triangles. One vertex of each triangle is generated randomly. The idea for this test is taken from [4], page number 222.
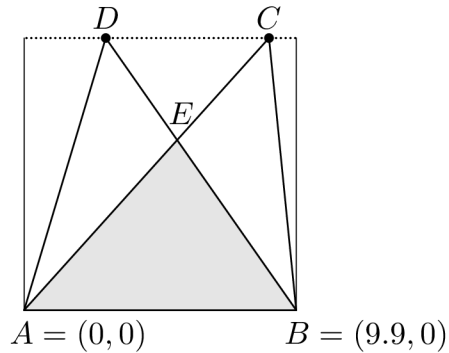

Figure 8: Overlapping triangles

Both triangles share one side. Generated vertices can only change their coordinate in the horisontal direction and can not abandon their half of the line. With this narrowed down scenario it is easy to compute the triangle in question using both theories.

Computation is done analogously in the case of both theories. First angles and spreads of the new triangle are discovered. We can use them together with the knowledge of the side c to determine the final vertex. When solving the $\triangle_{ABE}$ using classical trigonometry we can follow these steps:

$$\theta_A = \arccos\left(\frac{x_C}{\sqrt{x_C^2 + y_C^2}}\right),$$

$$\theta_B = \arccos\left(\frac{x_B - x_D}{\sqrt{(x_B - x_D)^2 + y_D^2}}\right),$$

$$\theta_E = \pi - \theta_A - \theta_B,$$

$$a = \frac{e\sin(\theta_A)}{\sin(\theta_E)},$$

$$b = \frac{e\sin(\theta_B)}{\sin(\theta_E)}.$$

Analogous to the first two calculations we can take the following steps in rational trigonometry:

$$s_A = \frac{y_C^2}{x_C^2 + y_C^2},$$

$$s_B = \frac{y_D^2}{(x_B - x_D)^2 + y_D^2}.$$

Here we used formula for spread in the coordinate system. Triple spread formula helps us with finding the last spread.

$$s_E = s_A + s_B - 2s_A s_B + \sqrt{c_A s_A c_B s_B},$$

$$Q_a = \frac{s_A Q_e}{s_E},$$

$$Q_e = \frac{s_B Q_e}{s_E}.$$

Area of a triangle

Rational trigonometry employs the notion of quadrea. This test is designed to solve the area and the quadrea of a triangle. The vertices of the triangle are randomly generated inside a box with the side length of 9.9. Here *Heron's formula* and *Archimedes' formula* are used for computing the area and the quadrea, and can be found in [1] and [4], page number 437 and 70 respectivelly.
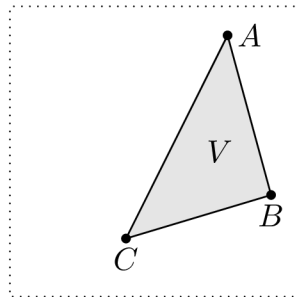
Figure 9: Area V of a triangle

Height of a triangle

Problem is taken from [4] and originally is called Height of object with vertical face. It states that an observer at A measures the vertical spread s_A to the point B directly above C. The quadrance Q_b is known. What is the quadrance Q_a?
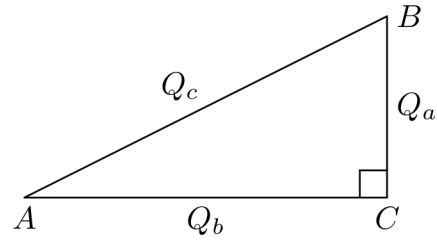
Figure 10: Height of a Triangle

The length of b and θ_A are randomly generated and then converted to Q_b and s_A respectively. But this could be done the other way around. I derived the classical trigonometry solution of this problem.

$$c = \frac{b}{\cos(\theta_A)},$$
$$a = \sqrt{c^2 - b^2}.$$

And the original solution is:

$$Q_a = \frac{s_A Q_b}{c_A.}$$

Snell's Law

Law well known from the theory section is used to design the following test. Beam of light travels through the point A in a material. It reaches different material in point D. Finally it travels through the point B. Supposing the beam of light travels uninterrupted, what is the refractive index r of these materials. This test is partly inspired by a similar problem in [4].
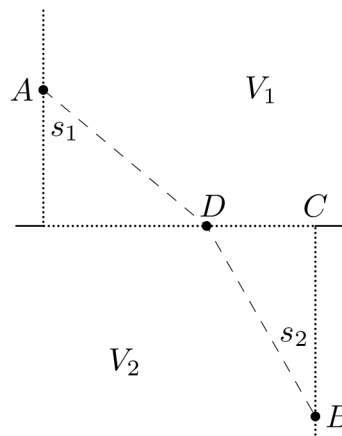

Figure 11: Refractive index

The answer can be derived using classical trigonometry by the following steps:

$$\theta_A = \arctan(x_D/y_A),$$
$$\theta_B = \arctan((x_C - x_D)/y_B),$$
$$r = \sin(\theta_A)/\sin(\theta_B).$$

Rational trigonometry gives the same results despite avoiding inverse trigonometric functions:

$$s_A = \frac{x_D^2}{x_D^2 + y_A^2},$$
$$s_B = \frac{(x_C - x_D)^2}{(x_C - x_D)^2 + y_B^2},$$
$$r = \sqrt{s_A/s_B}.$$

Cumulative Error (CT)

I designed this test to be able to measure other types of characteristics. The goal of the computation described below is to find all the points of a regular n-gon. Length and quadrance from all the vertices $N_i, i = 0, 1, 2, \ldots$, to the center point S of the n-gon are equal to 1. Constructing all points of such n-gon one-by-one leads us back where we started. This property of the test enables us to measure the cumulative error made along the way.
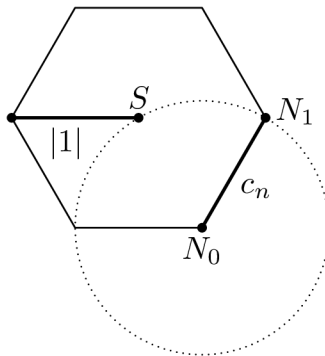

Figure 12: Construction of a regular n-gon

Using classical trigonometry we can proceed with the construction as follows. Length of a side of an n-gon can be obtained with this formula

$$c_n = 2\sin\left(\frac{\pi}{n}\right).$$

Using the short-hand for $x_{N_i} = x_i$ we get

$$x_0 = 0,$$

$$y_0 = 0,$$

$$\theta = \frac{2\pi}{n}.$$

as our initial values of the test. Then we can iterate n-times to get back like this

$$\theta_i = i\theta,$$

$$x_{i+1} = x_i + c_n \cos(\theta_i),$$

$$y_{i+1} = y_i + c_n \sin(\theta_i).$$

Cumulative Error (RT)

Without the usage of trigonometric functions this problem can be solved only for some regular n-gons. The n-gon needs to be constructible with compass and straightedge. For such n-gons we can derive the number x_{N_1}, as in [6]. And for this reason both tests 6 and 7 use only some values for n.

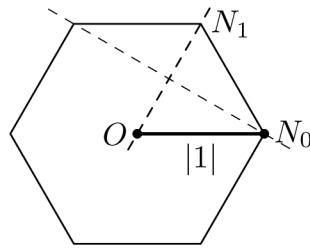Using rational trigonometry I came up with the following solution:

Figure 13: Construction of a regular n-gon

Initial values are set to

$$x_0 = 1,$$

$$y_0 = 0,$$

$$x_1 = c_i,$$

$$t = x_1^{-2} - 1,$$

$$y_1 = x_1\sqrt{t}.$$

Where t is a twist between two lines lying on points O, N_i and O, N_{i+1} respectively. Then we can iterate using the Reflection of a point in a line theorem like this:

$$x_{i+1} = \frac{x_{i-1}(x_i^2 - y_i^2) + 2x_i y_i y_{i-1}}{x_i^2 y_i^2},$$

$$y_{i+1} = \frac{y_{i-1}(y_i^2 - x_i^2) + 2y_i x_i x_{i-1}}{y_i^2 x_i^2}.$$

3.2. Implementation

Comparing theories this way we can not obtain results independent of the computer system running it. It may be possible thanks to the computer science to determine the lower and upper bounds of the time taken by the tests for a hypothetical system, that deos not affect the computation. Another way to aproach the analysis is to concentrate on how the tests run on a real system.

For the purpose of this text I am exploring the latter option and the code was optimized correspondingly. The code of each test in [7] has a three part structure. Initiation leads to allocation of all variables. Last two parts are allways meant to compute equivalent results and are internaly timed. Wherever possible the float number structure is used to retain consistency. This means that both approaches of solving a test deal with similar round-off errors.

Tests 6 and 7 are exceptions. Their structure differs in the number of parts. They have only two of them. Despite both tests taking different aproach to solve the problem, both comprise of a suite that can be thought of as one test.

4. Results

This chapter's focus is on the results obtained by the computations made with the attached program [7]. This data was simulated on a linux machine with default settings of the GNU Compiler Collection.

In the first subchapter there are displayed plots of observed data and the second one contains hypotheses about them.

Figures are made in R using tidyverse packages. Very instructive resource for me was [5]. It walks the reader through variety of plotting methods of the ggplot2 package and this was helpful.

4.1. Visualization

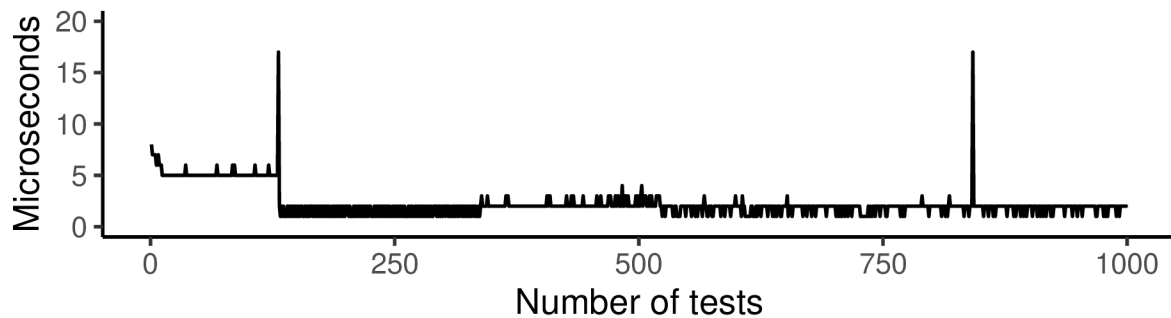Let us visualize how classical trigonometry deals with the first test.

Figure 14: Right triangle (CT)

Iterations in the range zero to a bit over one hundred are considerably slower than the iterations form the set mark and on. There is the exception of two spikes. One spike sits near the iteration one hundred where the time taken to compute one interation gets suddenly shorter. The other spike seems to occur randomly somewhere around the iteration eight hundred. Let us see if the tests computed with rational trigonometry hold the same patterns.
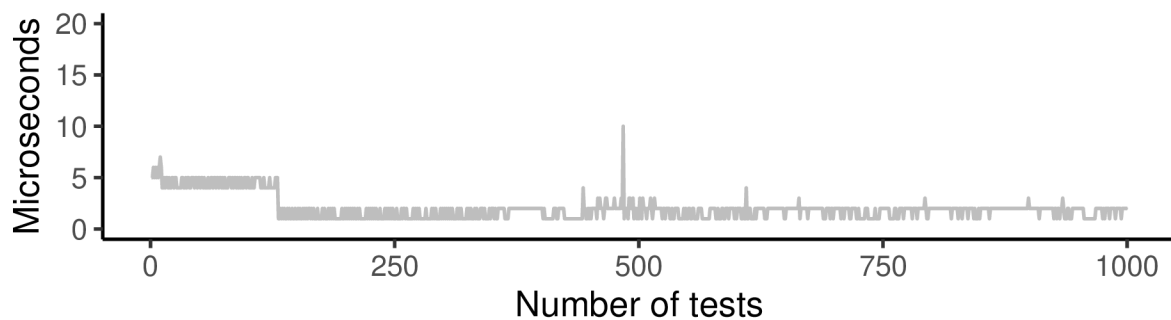

Figure 15: Right triangle (RT)

The one hundred mark still marks a point of a sudden decrease in computation time. Both spikes disappeard but this plot made it clear that around the iteration five hundred some irregularities occur.
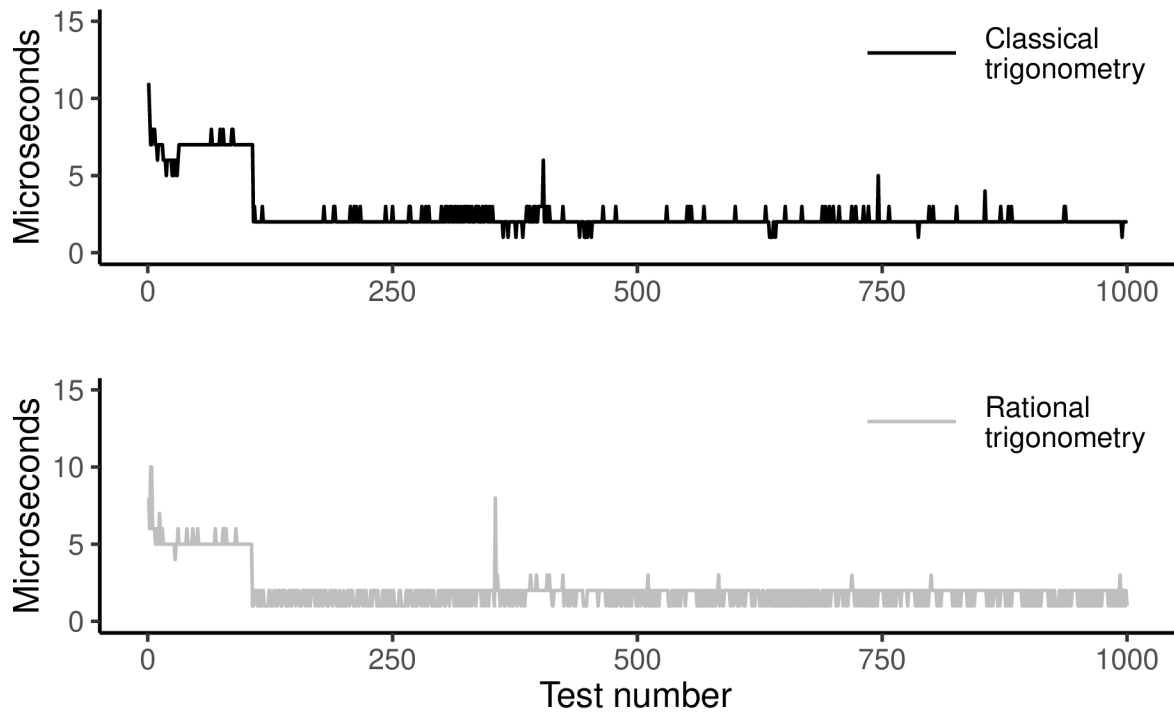
Figure 16: Overlapping triangles

The first hundred iterations keep on having higher time demand. Regular spikes in both tests moved near the first and four-hundredth iteration.
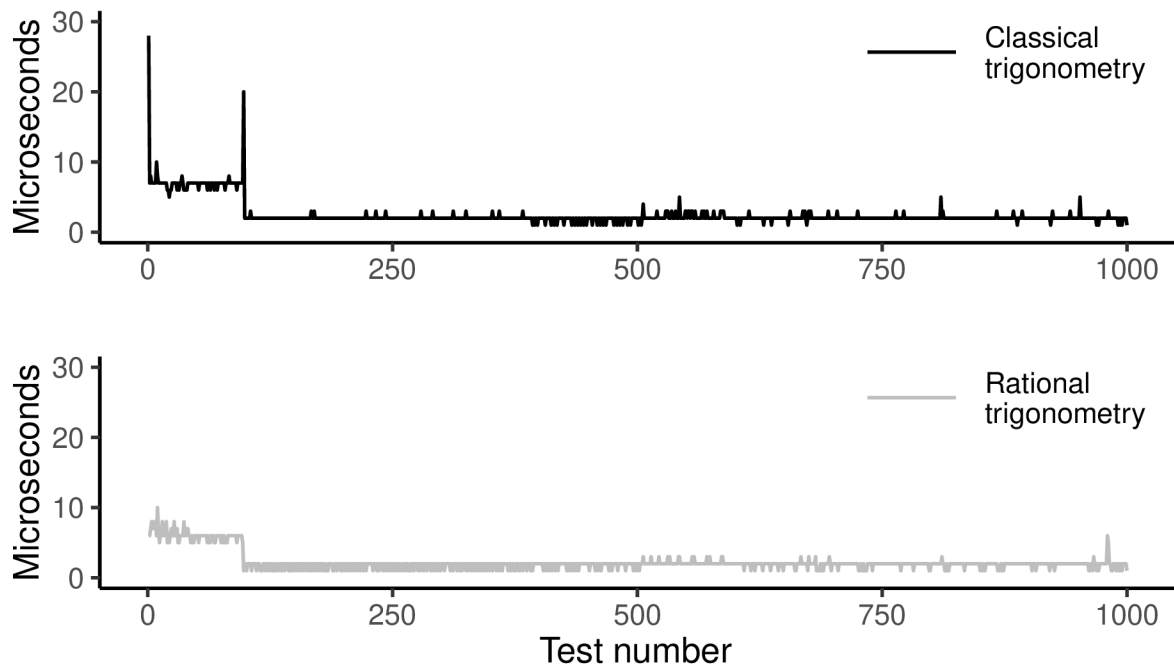

Figure 17: Area of a triangle

The previous figure depicts data from the test called Area of a triangle. The following figures display in similar fashion data from figures Height of a triangle and Snell's law.
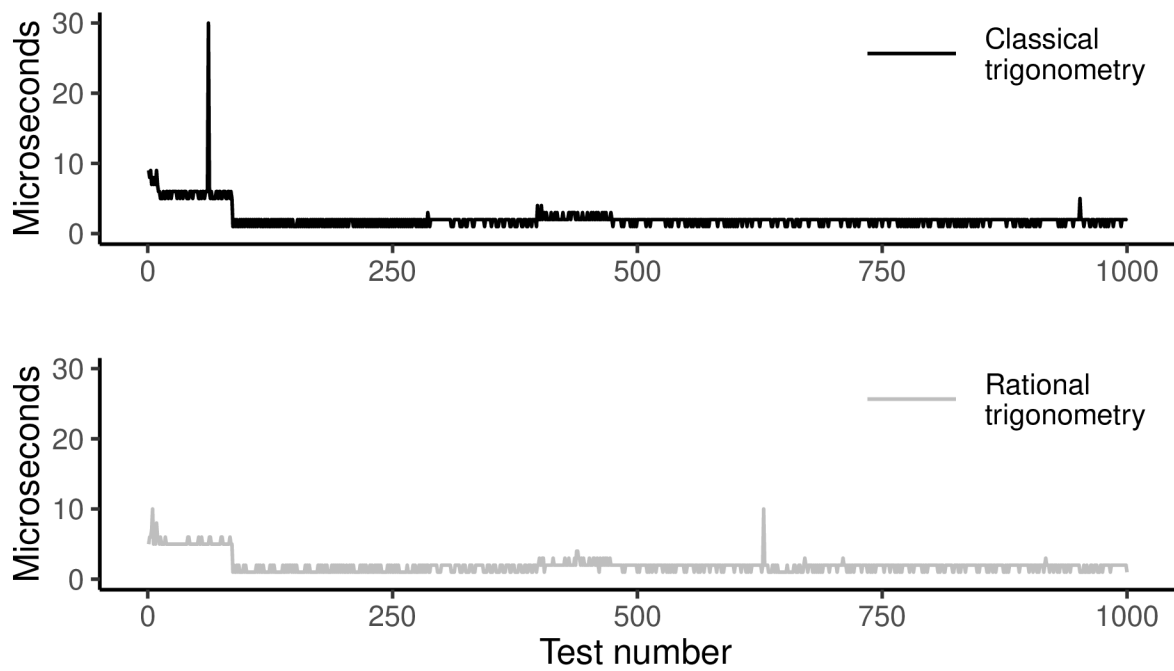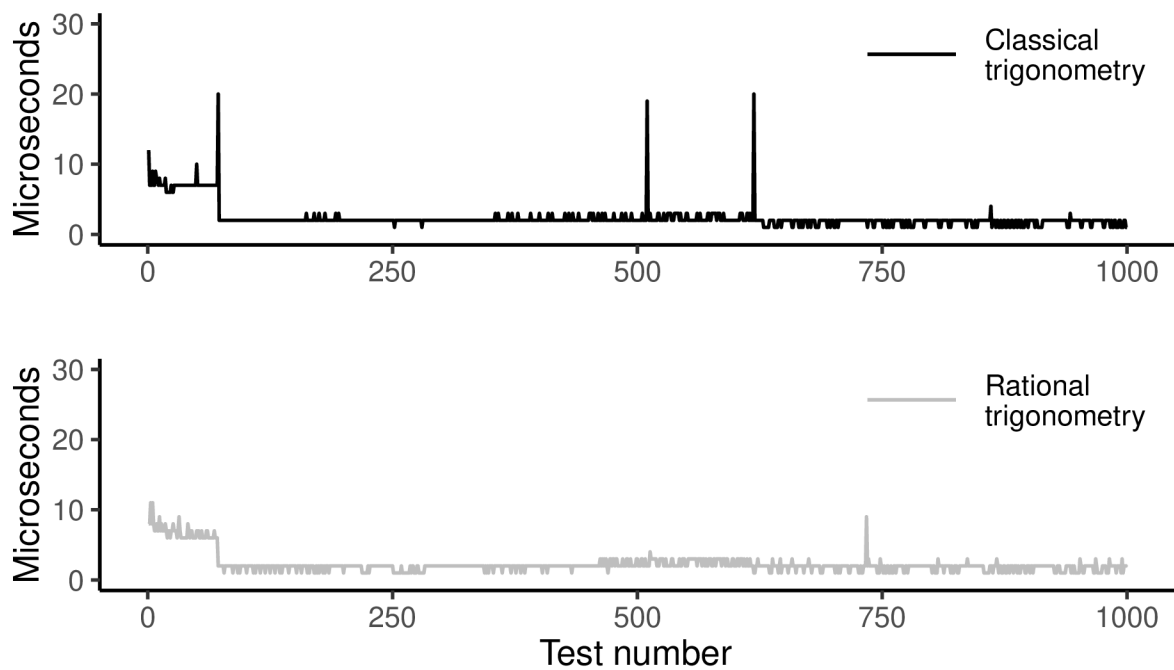

Figure 18: Height of a triangle

Figure 19: Snell's law

Let us also display the time development of the last two tests regarding the construction of a regural n-gon. The usual colouring convention is used and legends are not displayed. Remember that some values are missing due to the fact that rational trigonometry can deal only with computable polygons in the reals number field.
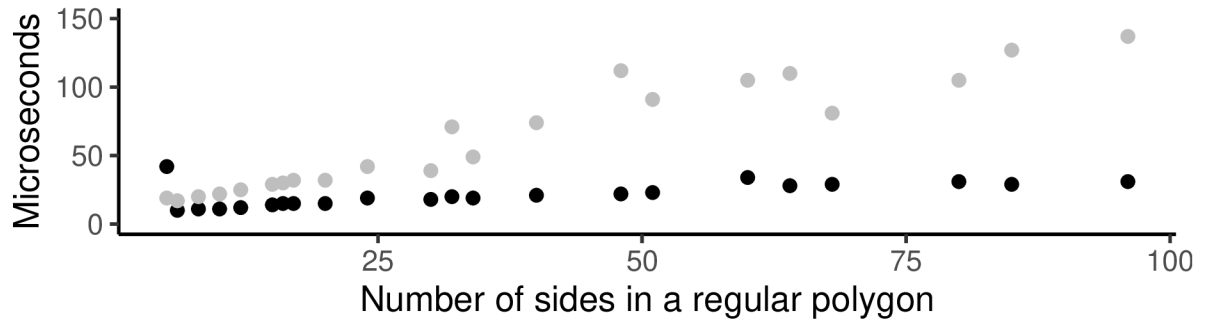

Figure 20: Cumulative error (temporal)

In this figure both theories follow different pattern from each other and from what we have seen so far. These tests also allow us to display cumulative errors gathered along each cycle of the computation.
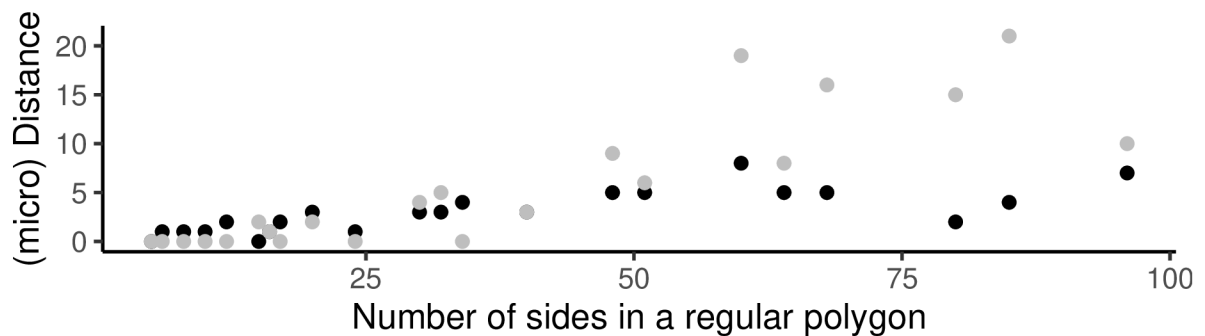

Figure 21: Cumulative error (spatial)

Values for each data point are obtained by adding the absolute values of x and y coordinates in each point respectively. The figure displays length in unspecified units. Since the values are really small they are multiplied by 10^6. That is why the lable of y coordinate deserves the micro prefix.

4.2. Analysis

First five tests share similar properties when it comes to their time demands. Their nature is characterized by a slower start, sudden decrease in computation time aroud the hundredth iteration, and occasional spikes occuring slightly more often and with higher reading in the case of classical trigonometry. First stabilized time after the hundredth iteration usually experiences subtle disturbance that faids away. Around the thousandth iteration the data seems well stabilized.

Last two tests regarding cumulative error show that approach of classical trigonometry is more stable both temporally and spatially. The rational trigonometry approach is slower in all cases except the first one. When it comes to the error in measured distance it is not so simple. The precision of rational trigonometry is better in the beginning but gets worse with time. When the number of sides goes past fourty classical trigonometry is allways faster. Error made by rational trigonometry is more disperse from that point.

The first iteration of classical trigonometry in every test is affected by something. Rational trigonometry seems to be affected less. This differece might be caused by the use of trigonometric functions.

The subsequent table displays data of the total time taken by each test. The time is in milliseconds, A and R stand for absolute and relative error respectivelly. A and R are computed as the difference of classical trigonometry relative to rational trigonometry.

	CT	RT	A	R
Right triangle	2.28	2.039	0.241	11.81952 %
Overlapping triangles	2.6	2.031	0.569	28.01576 %
Area of a triangle	2.534	2.235	0.299	13.37808 %
Height of a triangle	2.143	1.998	0.145	7.257257 %
Snell's law	2.386	2.275	0.111	4.879121 %
Cumulative error	0.469	1.369	-0.9	-65.74142 %

Next table displays the time needed for the fist two hundred iterations of the first five tests.

	CT	RT	A	R
Right triangle	0.794	0.692	0.102	14.73988 %
Overlapping triangles	0.933	0.703	0.23	32.71693 %
Area of a triangle	0.91	0.754	0.156	20.68966 %
Height of a triangle	0.698	0.605	0.093	15.3719 %
Snell's law	0.788	0.72	0.068	9.444444 %

All relative errors increased. This indicates that the initial destabilized iterations are calculated faster using rational trigonometry. The following figure shows the one hundred fifty iterations of the first five tests.
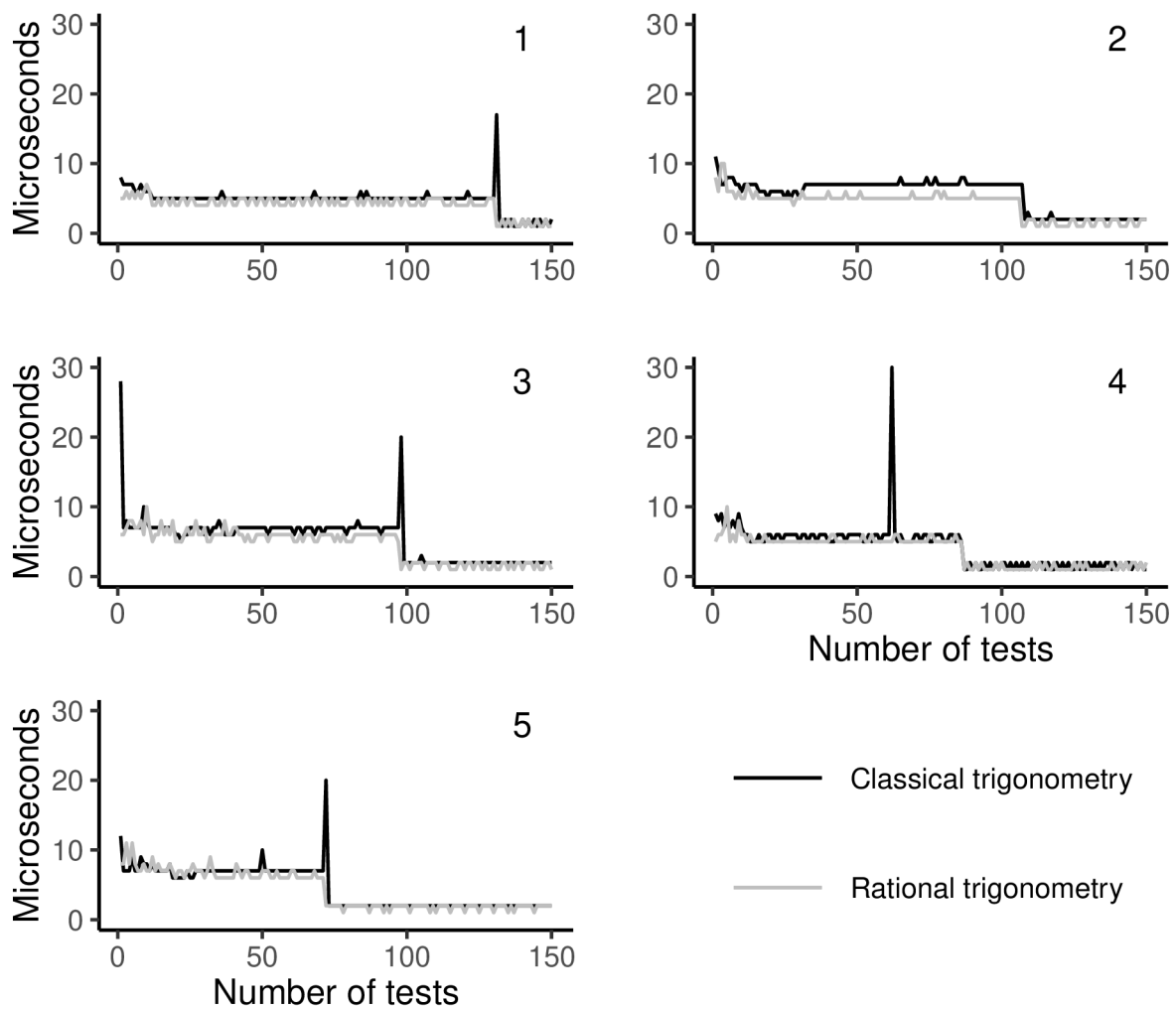

Figure 22: First 150 iterations

Looking at the first two hundred tests it seems to be the case that rational trigonometry usually has lower compuational time for the first five tests.

The temporal visualization of the last two tests is nice and regular but the spatial visualization on the other hand shows unpredictabilities. It seems that both algorithms have the same time complexity but rational trigonometry is just less effective.

The unpredictable differences in the spatial case are maybe caused by the different complexities of expressions encoding values of the x-coordinate of each second vertex in the case of rational trigonometry. These values are encoded only without the use of trigonometric functions. You can see yourself how long the expressions get in [7].
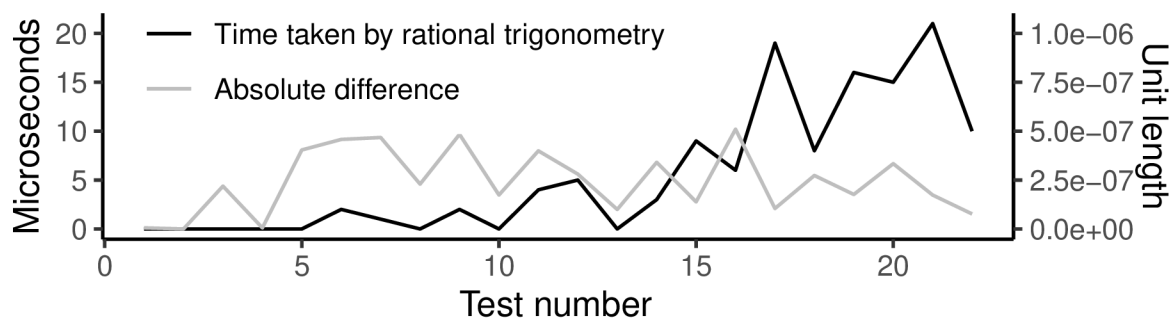

Figure 23: Cumulative error, x-coordinate representaiton

5. Conclusion

All the work I did while working on this text served it's purpose and now I feel like I better understand the gap between classical trigonometry and rational trigonometry. The purpose of this thesis is to form an idea about how much better or worse it is to use rational trigonometry in applied mathematics as well as simply get more familiar with it.

This thesis made me better appreciate the trade-off classical trigonometry undergoes in order to fulfil it's role. For the purposes of applied mathematics it seems to be better suited since it provides better aplicability compared to rational trigonometry in real numbers. This is due to nice mlutiplicative and additive propreties of angles.

To me classical trigonometry does not seem to be well suited as a theory for the first exposure to trigonometry. It requires a leap of faith for people who do not know calculus yet.

Rational trigonometry offers real benefits to classical trigonometry but it is not for free. I think that computer science may benefit greatly from the adoption of this theory. Simplifications that the float number structure brought were immense. Classical trigonometry and float point arithmetics is a great combination that works hand-in-hand. But if rational trigonometry was to be adopted, some other better suited number system sould be developed for it. Since rational trigonometry has the capacity to be precise it is a pitty to lose it with using floats.

Once you get rational trigonometry under your skin, it is fairly intuitive and I recommend you to take the time to learn it if you are interested. Rational trigonometry, I feel like, has the potential to become more general theory as oppose to classical trigonometry which serves mostly it's own purpose.

Acknowledgement

I would like to thank my supervisor RNDr. Pavel Ludvík Ph.D. for his encouragement and patience durig the time this thesis was beeing written.

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Thank you to my educators and to my family.

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Attachments:

[7]   FILIPEC, Viktor. `bachelors_thesis_filipec.c` [computer pragram]