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AUTONOMOUS SYSTEMS OF DIFFERENTIAL EQUATIONS – CLASSICAL VS FRACTIONAL ONES

AUTONOMNÍ SOUSTAVY DIFERENCIÁLNÍCH ROVNIC – KLASICKÉ VS ZLOMKOVÉ

BACHELOR'S THESIS
BAKALÁŘSKÁ PRÁCE

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Specification Bachelor's Thesis

Department:	Institute of Mathematics
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Pursuant to Act no. 111/1998 concerning universities and the BUT study and examination rules, you have been assigned the following topic by the institute director Bachelor's Thesis:

Autonomous systems of differential equations – classical vs fractional ones

Concise characteristic of the task:

The filed of differential equations with an operator of non–integer order (the so–called fractional equations) has become quite popular during the last decades due to a large application potential. However, it turns out that the theory of fractional equations is more complex and often requires special approaches. In other words, not all of the standard properties of the classical equations (integer order) can be transfered to the fractional case.

Goals Bachelor's Thesis:

Theoretical part:
Study of selected properties of autonomous systems of differential equations with emphasis put on those that are not analogous to the case of (classical) first order systems and systems of a fractional order (the order is typically considered between zero and one).
Practical part:
The practical part of the thesis is focused on various experiments and computer simulations in order to verify the theoretical results.

Recommended bibliography:

CONG, N. D., TUAN, H. T. Generation of nonlocal fractional dynamical systems by fractional differential equations, J. Integral Equations Appl. 29 (2017), 585-608.

DIETHELM, K. The Analysis of Fractional Differential Equations. An Application-Oriented Exposition Using Differential Operators of Caputo Type, Springer-Verlag Berlin, Heidelberg, 2010. ISBN 978-64--14574-2.

LYNCH, S. Dynamical Systems with Applications Using MATLAB, 2nd ed., Birkhäuser Basel, 2014. ISBN 978-3-319-06819-0.

MILICI, C., DRAGANESCU, G., MACHADO, J. T. Introduction to Fractional Differential Equations, Springer, 2019. ISBN 978-3030008949.

Deadline for submission Bachelor's Thesis is given by the Schedule of the Academic year 2019/20

In Brno,

L. S.

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Abstrakt

Hlavním zaměřením této práce je hlubší studium a porovnání dvou oblastí diferenciálních rovnic, kde důraz je kladen na neceločíselné řády, neboť během posledních desítek let se tato oblast nejenže stala populární, ale dokonce bylo zjištěno, že standardní přístupy řešení nenaplňují očekávání, tudíž jsou vyžadovány speciální postupy.

Práce také obsahuje příklady, experimenty a simulaci pro ověření, případné vyvrácení teoretických výsledků.

Abstract

The main preoccupation of this thesis is an in-depth study and comparison of two fields of differential equations with a greater focus on a non-integer order which during the last decades has proven not only to become more popular because of its applications but also more complex, thus demanding more special approach.

This thesis is also provided with multiple examples, experiments, and simulations in order to verify or invalidate the theoretical results.

klíčová slova

dynamický systém, autonomní systém, zlomkový kalkulus, jednoznačnost, Caputova derivace, matematická biologie, šíření epidemií, SIR model

keywords

dynamical system, autonomous system, fractional calculus, uniqueness, Caputo derivative, mathematical biology, spread of diseases, SIR model

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I, hereby, declare that I wrote this bachelor's thesis all by myself under the direction of my supervisor, doc. Ing. Luděk Nechvátal, Ph.D. All used sources are listed in references.

Anna Glozigová

I wish to express my deepest gratitude to my supervisor doc. Ing. Luděk Nechvátal, Ph.D., especially for the great patience, help and all useful hints and comments. I would like to also pay my special regards to those who tagged along.

Anna Glozigová

Contents

1 Introduction

Dynamics is a time-evolutionary process. However, time itself does not necessarily have an impact on all systems. These systems, which are closely related to dynamical ones, are called autonomous, i.e., they are not explicitly time dependant. Their dynamics may not be that rich, nonetheless, studies have shown their importance in modelling, e.g., prey-predator models, population growth, radioactive decay, electronic circuits, etc.

The history of fractional calculus is as old as the classical calculus itself. Its initiation was in a correspondence between Leibniz and L'Hospital lasting several months in 1695 where a question "*What does* $\frac{d^n}{dx^n}f(x)$ *mean for n=1/2?*" was posed. This issue raised for a fractional derivative was an ongoing topic for incoming decades and its theoretical field developed rather quickly which, on the other hand, cannot be said about practical as, at that time, there was only a handful applications for it. Nevertheless, by the second half of twentieth century, this field expanded to such extent that in 1974 the first conference was held solely focusing on theory and applications. Today, it is a still growing and respected field of mathematics having found its use in real-life modelling thanks to its non-locality, i.e., having "a memory". The fields of application contain biology, engineering, studies of proteins and polymers in chemistry, modelling of viscoelastic and viscoplastic substances in mechanics and last but not least medicine and finances.

The purpose of this thesis is to provide a research of autonomous systems from both points of view, to highlight differences between classical calculus and fractional one and to test the theory on a chosen model, in particular, the epidemiological SIR model with vital dynamics.

The thesis is organized as follows. The second and Section 3, considering the first one being the introduction, provides basic theory of, both, integer order and fractional autonomous systems. The Section 3 also provides necessary fundamental background in order to introduce two the most popular approaches of fractional derivative, namely, the Riemann–Liouville and the Caputo derivative. The Section 4 provides an analysis of the SIR model and the results are discussed in the last Section. The numerical method, which could be considered as fractional generalisation of predictor-corrector method, is being described in the Section 5.

2 Theory of Integer Order

In order to describe dynamics it can be said that it is a time-evolutionary process governed by, e.g., either linear or non-linear differential or difference equations, giving us what we call a dynamical system. We can distinguish between stochastic and deterministic systems, or continuous and discrete. Apart from differential equations, studies have shown that behaviour of dynamical system is not always determinable by analytical solutions. This applies even further for non-linear equations and systems except for certain special cases.

2.1 Classical Autonomous System

Definition 2.1 (Autonomous system)**.** Let $\boldsymbol{f} : \Omega \to \mathbb{R}^n$ be a vector field. The system of ODEs

$$\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y}), \tag{2.1}$$

is called an *autonomous system.*

It is a well-known fact that every non-autonomous system $\boldsymbol{y}' = \boldsymbol{f}(x, \boldsymbol{y})$, where $\boldsymbol{y} \in \mathbb{R}^n$, can be rewritten to an autonomous system (2.1) with $\boldsymbol{y} \in \mathbb{R}^{n+1}$ by setting $y_{n+1} = x$ and $y'_{n+1} = 1$. By relatively weak assumptions on $\boldsymbol{f}$, existence and uniqueness can be obtained.

Let us consider an initial value problem for (2.1), i.e., the pair

$$\begin{aligned} \boldsymbol{y}' &= \boldsymbol{f}(\boldsymbol{y}), \\ \boldsymbol{y}(x_0) &= \boldsymbol{y}_0, \end{aligned} \tag{2.2}$$

(where $\boldsymbol{f} : \Omega \to \mathbb{R}^n$, Ω is an open subset of $\mathbb{R}^n$, $\boldsymbol{f} \in C^1(\Omega)$ and $\boldsymbol{y}_0 \in \Omega$).

Definition 2.2 (Solution of autonomous system)**.** Let $\boldsymbol{f} \in C(\Omega)$ and let Ω be an open subset of $\mathbb{R}^n$. *A solution* of the system (2.1) on an interval I is every function $\tilde{\boldsymbol{y}} : I \to \mathbb{R}^n$ so that $\tilde{\boldsymbol{y}}$ is differentiable on I, for every $x \in I$ condition $\boldsymbol{y}(x) \in \Omega$ is being satisfied, and

$$\boldsymbol{y}'(x) = \boldsymbol{f}(\boldsymbol{y}(x)), \qquad \forall x \in I.$$

Furthermore, if $\boldsymbol{y}_0 \in \Omega$, then $\tilde{\boldsymbol{y}}$ is solution of an initial value problem (2.2) on interval I if $x_0 \in I, \boldsymbol{y}(x_0) = \boldsymbol{y}_0$.

Remark 1. The above notation $\boldsymbol{f} \in C(\Omega)$ is understood component-wise, i.e., $f_i \in C(\Omega), i = 1, \ldots, n$.

Definition 2.3 (Continuous dynamical system)**.** Let $\boldsymbol{\varphi} : \mathbb{R} \times \Omega \to \Omega$ be a continuously differentiable function where $\Omega \subseteq \mathbb{R}^n$. Then $\boldsymbol{\varphi}$ is called *continuous dynamical system* denoting a pair $\{\Omega, \varphi\}$ if the following two conditions are satisfied:

- $\boldsymbol{\varphi}(0, \boldsymbol{y}) = \boldsymbol{y}, \quad \forall \boldsymbol{y} \in \Omega$
- $\boldsymbol{\varphi}(t + u, \boldsymbol{y}) = \boldsymbol{\varphi}(t, \boldsymbol{\varphi}(u, \boldsymbol{y})), \quad \forall t, u \in \mathbb{R}$

where the function $\boldsymbol{\varphi}$ is often called an *evolution operator.*

Remark 2. The function $\boldsymbol{\varphi}$ is sometimes called time-t map, or a flow of a vector field $\boldsymbol{f}$. It is reasonable to expect that $\boldsymbol{\varphi}(t, \boldsymbol{y})$ has $\boldsymbol{\varphi}(-t, \boldsymbol{y})$ as its inverse and, also, this condition guarantees the existence and uniqueness of (2.2).

Theorem 2.4 (Peano existence theorem)**.** *Let $\boldsymbol{f} : \Omega \to \mathbb{R}^n$ be a continuous vector field defined on a neighbourhood of a point $(x_0, y_0) \in \Omega$. Then there exists a solution $\tilde{\boldsymbol{y}}$ of (2.2) in a neighbourhood of x_0.*

Theorem 2.5 (Picard's uniqueness theorem)**.** *Let $\boldsymbol{f} : \Omega \to \mathbb{R}^n$ be defined on Ω and let it be a continuous vector field satisfying Lipschitz condition on the neighbourhood of y_0. Then there exists $\varepsilon > 0$ such that problem (2.2) has a unique solution on interval $(-\varepsilon, \varepsilon)$.*

Remark 3 (Lipschitz condition). Function $\boldsymbol{f}$ satisfies the Lipschitz condition on a set $\Omega \subseteq \mathbb{R}^n$ if a constant $L > 0$ exists with

$$|\boldsymbol{f}(\tilde{\boldsymbol{y}}_1) - \boldsymbol{f}(\tilde{\boldsymbol{y}}_2)| \leq L|\tilde{\boldsymbol{y}} - \tilde{\boldsymbol{y}}_2|, \tag{2.3}$$

whenever $\tilde{\boldsymbol{y}}_1, \tilde{\boldsymbol{y}}_2$ are in Ω. L is a Lipschitz constant.

If (2.3) is satisfied then two trajectories of a system are either disjoint or identical. If for solution $\tilde{\boldsymbol{y}}_1$ on I_1 and $\tilde{\boldsymbol{y}}_2$ on I_2 is satisfied condition $I_1 \subset I_2 (I_1 \neq I_2)$, while $\tilde{\boldsymbol{y}}_1(x) = \tilde{\boldsymbol{y}}_2(x)$ for all $x \in I_1$, then a solution $\tilde{\boldsymbol{y}}_2$ is an *extension* of $\tilde{\boldsymbol{y}}_1$ from the interval I_1 to I_2.

Definition 2.6. Suppose fixed initial point $\boldsymbol{y}_0$ and let $\widehat{I} = \widehat{I}(\boldsymbol{y}_0)$, then the mapping $\varphi(\cdot, \boldsymbol{y}_0) : \widehat{I} \to \Omega$ defines a *trajectory* of the system (2.2) through the point $(\boldsymbol{y}_0 \in \Omega)$.

Definition 2.7. A *phase portrait* of (2.2) is the set of all trajectories in the phase space $\mathbb{R}^n$.

Definition 2.8 (Maximal interval of existence)**.** Consider the initial value problem (2.2). The open interval $\widehat{I}$ is called a *maximal interval of existence* if there exists no further extension of the solution $\tilde{\boldsymbol{y}}$.

Definition 2.9 (Equilibrium point)**.** Let us consider a point $\boldsymbol{y}^* \in \mathbb{R}^n$. The point is called an *equilibrium point* or a *critical point, fixed point* of system (2.1) if it satisfies $\boldsymbol{f}(\boldsymbol{y}^*) = \boldsymbol{0}$.

Generally, there are three main categories of trajectories of the solution $\tilde{\boldsymbol{y}}$:
- *Singular point* – a trajectory of constant solution (corresponding to equilibrium solutions),
- *Cycle* (*closed curve, orbit*) – a trajectory of periodical solution,
- *Curve without self-intersection (open curve)* – other cases of solution.

2.2 Stability

Roughly speaking, stability of solutions to ODEs means that a small perturbation in initial conditions causes a small change in the output, i.e., the new solution remains close to the original one forever.

Definition 2.10 (Stability and attractiveness)**.** A solution $\tilde{\boldsymbol{y}}$ of (2.1) is called
- *stable* if for any $x \geq x_0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for every solution $\tilde{\boldsymbol{y}}$, we have
$$||\boldsymbol{y}_0 - \tilde{\boldsymbol{y}}|| < \delta \implies ||\boldsymbol{y}(x) - \tilde{\boldsymbol{y}}(x)|| < \varepsilon. \tag{2.4}$$
- *unstable* if it is not stable.
- *attractive* if there exists $\delta > 0$ such that for every solution $\tilde{\boldsymbol{y}}$ of system (2.1) following statement holds true
$$||\boldsymbol{y}_0 - \tilde{\boldsymbol{y}}|| < \delta \implies ||\boldsymbol{y}(x) - \tilde{\boldsymbol{y}}|| \to 0 \qquad \text{as} \quad x \to \infty.$$

Remark 4. (i) A solution that is stable and attractive at the same time is called *asymptotically stable*, meaning that it *attracts* nearby solutions.

(ii) The requirement (2.4) describes that if the initial condition changes less than δ at x_0, then the solution on the interval $I = \langle x_0, \infty)$ changes less than ε. This property is also called *Lyapunov stability.*

(iii) Let us note that stable solutions do not necessarily have to be attractive. On the other hand, attractive solutions need not be stable.

Remark 5. Jacobi matrix at an equilibrium point $\boldsymbol{y}^*$ is an $n \times n$ matrix containing partial derivatives, with respect to all variables, of the right-hand side of (2.1) evaluated at $\boldsymbol{y}^*$, i.e.,

$$\boldsymbol{A} = \boldsymbol{J}(\boldsymbol{y}^*) = \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(\boldsymbol{y}^*) & \dots & \frac{\partial f_1}{\partial y_n}(\boldsymbol{y}^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial y_1}(\boldsymbol{y}^*) & \dots & \frac{\partial f_n}{\partial y_n}(\boldsymbol{y}^*) \end{pmatrix}.$$

The eigenvalues of a matrix $\boldsymbol{A}$ are obtained as the roots of characteristic polynomial

$$P(\lambda) = \det(\boldsymbol{A} - \lambda \mathbf{E})$$

where $\mathbf{E}$ represents the identity matrix. The task of finding the roots of a characteristic polynomial can become rather challenging even for dimensions $n \geq 3$. In particular, for dimensions $n = 3$ and 4, complicated formulas can be found, on the other hand, for even higher dimensions they do not exist. However, if we limit ourselves to investigating asymptotic stability, the following criterion is useful:

Theorem 2.11 (Routh–Hurwitz stability criterion)**.** *Let* $P_n(\lambda) = \lambda^n + a_1\lambda^{n-1} + \dots + a_{n-1}\lambda + a_n$ *be a polynomial,* $a_i \in \mathbb{R}, i = 1, \dots, n$ *and let us define Hurwitz's matrix of* P_n

$$\boldsymbol{H}_n = \begin{pmatrix} a_1 & 1 & 0 & 0 & 0 & \dots & 0 \\ a_3 & a_2 & a_1 & 1 & 0 & \dots & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & a_n \end{pmatrix}.$$

Then, all roots of P_n *have negative real parts if and only if the determinants of principal submatrices of* H_n *are all positive. If there exits at least one negative determinant of the sequence, then, the characteristic polynomial has a root with its real part being positive.*

Definition 2.12 (Hyperbolic equilibrium)**.** An equilibrium point of a system is called *hyperbolic* if the eigenvalues of Jacobi matrix at this point satisfy condition $\Re(\lambda_i) \neq 0, \quad i = 1, \dots, n$. Otherwise, if there exists at least one eigenvalue such that $\Re(\lambda_i) = 0, \quad i = 1, \dots, n$, it is *non-hyperbolic.*

Theorem 2.13. *Let us consider a system (2.1) with eigenvalues* $\lambda_1, \dots, \lambda_n$ *of the Jacobi matrix* $\boldsymbol{A}$ *and let* $\boldsymbol{y}^*$ *be an equilibrium. Then,* $\boldsymbol{y}^*$ *is*

- *asymptotically stable if* $\Re(\lambda_i) < 0$*, for all* $i = 1, \dots, n$*;*
- *unstable if there exists at least one eigenvalue* λ *such that* $\Re(\lambda) > 0$*.*
- *non-hyperbolic equilibrium point if there exists at least one eigenvalue* λ *such that* $\Re(\lambda) = 0$*. In this case, its stability cannot be determined.*

Unfortunately, many equilibrium points arising in applications are non-hyperbolic, and as mentioned above, their further investigation requires stronger method.

Remark 6 (Lyapunov function). Consider the non-linear system (2.1) with an equilibrium point $\boldsymbol{y}^* \in \Omega$, let Ω be an open subset of $\mathbb{R}^n$. Suppose that there exists a function $V : \Omega \to \mathbb{R}^n$ which satisfies $V(\boldsymbol{y}^*) = 0$, and $V(\boldsymbol{y}) > 0$ if $\boldsymbol{y} \neq \boldsymbol{y}^*$. Then,

(i) if $\dot{V}(\boldsymbol{y}) \leq 0$ for $\forall \boldsymbol{y} \in \Omega, \boldsymbol{y}^*$ is stable,
(ii) if $\dot{V}(\boldsymbol{y}) < 0$ for $\forall \boldsymbol{y} \in \Omega \setminus \{\boldsymbol{y}^*\}, \boldsymbol{y}^*$ is asymptotically stable,
(iii) if $\dot{V}(\boldsymbol{y}) > 0$ for $\forall \boldsymbol{y} \in \Omega \setminus \{\boldsymbol{y}^*\}, \boldsymbol{y}^*$ is unstable,

The function V is called the *Lyapunov function*. The symbol $\dot{V}(\boldsymbol{y})$ denotes $\dot{V}(\boldsymbol{y}) = \mathbf{f}(\boldsymbol{y}) \cdot \nabla V(\boldsymbol{y})$, where $\nabla V(\boldsymbol{y}) = \left(\frac{\partial V}{\partial y_1}, \ldots, \frac{\partial V}{\partial y_n}\right)$.

Remark 7. (i) If $\dot{V}(\boldsymbol{y}) = 0$ is satisfied for $\boldsymbol{y} \in \Omega$, then trajectories of system (2.1) lie on planes in $\mathbb{R}^n$ (curves in $\mathbb{R}^2$) defined as $V(\boldsymbol{y}) = c$.
(ii) Finding such function is often quite difficult as there exists no general method.

2.3 Linearisation

According to Picard and Peano theorems, a solution of an initial value problem exists on some interval I. Nevertheless, as a contrary to linear systems, we are generally unable to solve the non-linear ones. If so, then only a few of them are solvable analytically. The investigation of non-linear systems is in a neighbourhood of their equilibrium points and it can be shown that their local behaviour near a hyperbolic equilibrium $\boldsymbol{y}^*$ is determinable enough by the behaviour of linearised system.

Consider linearisation of (2.1), i.e., the linear system

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y},$$

where $\boldsymbol{A}$ is an the Jacobi matrix of $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ evaluated at an equilibrium point $\boldsymbol{y}^*$.

Theorem 2.14 (The fundamental solution of linear systems)**.** *Let $\boldsymbol{A}$ be a real $n \times n$ matrix. Then, for given $\boldsymbol{y}_0 \in \mathbb{R}^n$, the initial value problem*

$$\begin{aligned} \boldsymbol{y}' &= \boldsymbol{A}\boldsymbol{y}, \\ \boldsymbol{y}(0) &= \boldsymbol{y}_0 \end{aligned} \tag{2.5}$$

has a unique solution

$$\tilde{\boldsymbol{y}}(x) = e^{\boldsymbol{A}x}\boldsymbol{y}_0,$$

with the exponential of $\boldsymbol{A}$ defined as

$$e^{\boldsymbol{A}} = \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k}{k!}.$$

Let us now focus on the planar systems, i.e., dimension $n = 2$, and let us introduce types of equilibria.

Definition 2.15 (Equilibria in the phase plane of integer order autonomous system)**.** Let us consider linearised system (2.1) with eigenvalues λ_1, λ_2. The equilibrium point $\boldsymbol{y}^*$ is called

- *stable nodal point* if $\lambda_1 < \lambda_2 \leq 0$,
- *unstable nodal point* if $0 \leq \lambda_1 < \lambda_2$,
- *saddle point* if $\lambda_1 < 0 < \lambda_2$,

Furthermore, suppose that matrix $\boldsymbol{A}$ has two complex eigenvalues $\lambda_{1,2} = \mu \pm i\nu$. Then the equilibrium is called

- *stable focus point* if $\lambda_{1,2} = \mu \pm i\nu$, for $\Re(\lambda_{1,2}) > 0$ and $\nu \neq 0$,
- *unstable focus point* if $\lambda_{1,2} = \mu \pm i\nu$, for $\Re(\lambda_{1,2}) < 0$ and $\nu \neq 0$,
- *centre* or a *vortex point* if $\lambda_{1,2} = \pm i\nu$, for $\Re(\lambda_{1,2}) = 0$ and $\nu \neq 0$.

Let us note that in case of centre the equilibrium is stable, the stable nodal point along with stable focus point are asymptotically stable and unstable nodal point along with unstable focus point are unstable.

The eigenvalues λ_1, λ_2 are obtainable from a characteristic equation of the form

$$(\boldsymbol{A} - \lambda \mathbf{E}) = \lambda^2 + a\lambda + b = \lambda^2 - \operatorname{tr}(\boldsymbol{A})\lambda + \det(\boldsymbol{A}) = 0$$

with roots

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}.$$

Remark 8. The type of equilibria are also determinable by coefficients a and b as follows:

(i) stable nodal point if and only if $a > 0, b > 0$ and $a^2 - 4b > 0$,
(ii) unstable nodal point if and only if $a < 0, b > 0$ and $a^2 - 4b > 0$,
(iii) saddle point if and only if $a = 0, b < 0$,
(iv) stable focus point if and only if $a > 0, b > 0$ and $a^2 - 4b < 0$,
(v) unstable focus point if and only if $a < 0, b > 0$ and $a^2 - 4b < 0$,
(vi) centre (vortex point) if and only if $a = 0, b > 0$.

Furthermore, if both eigenvalues are zero, i.e., $\det(\boldsymbol{A}) = 0$, the equilibrium point is called *degenerate*. These points appear in non-hyperbolic systems. Rem. 8 can be visualised by *trace-determinant plane* in following figure where the matrix with trace $\operatorname{tr} \boldsymbol{A}$ and determinant $\det \boldsymbol{A}$ corresponds to the coordinates of a point $(\operatorname{tr}(\boldsymbol{A}), \det(\boldsymbol{A}))$. The geometry of the phase portrait is determined by the location of the point in the plane.

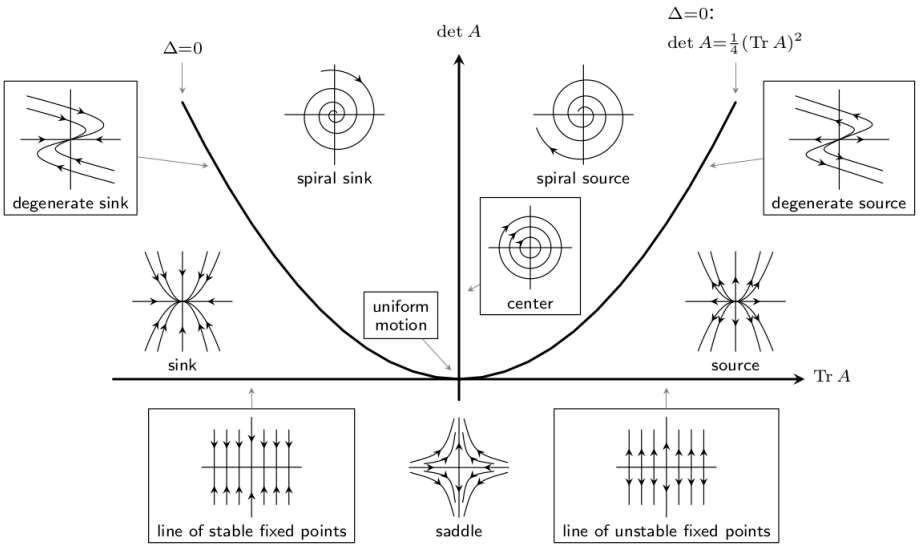

Figure 1: The regions are bounded by the two lines and parabola corresponding to the case of the term in square root equals to zero [19]

3 Theory of Non-Integer Order

3.1 Preliminaries

Quite typical and important feature of fractional calculus is its non-local character, i.e., dependency on initial value which can be useful in systems that work with memory. This allows to describe several phenomena with more precision. Let us recall some higher transcendental functions which play important role in fractional calculus.

Definition 3.1 (The Gamma function)**.** The function defined by

$$\Gamma(z) := \int_0^\infty \tau^{z-1} e^{-\tau} d\tau, \qquad \Re(z) > 0, \tag{3.1}$$

is called *Gamma function.*

An important property of this function is

$$\Gamma(z+1) = \Gamma(z) z,$$

which, along with $\Gamma(1) = 1$, shows a factorial property

$$\Gamma(n) = (n-1)! \quad \forall n \in \mathbb{N}.$$

Definition 3.2 (Mittag-Leffler function)**.** A function defined as

$$E_{\alpha,\beta}(z) = \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \qquad \alpha > 0, \beta \in \mathbb{R}, \quad z \in \mathbb{C}$$

is a *two-parameter Mittag-Leffler function.* If $\beta = 1$, then we simply write $E_\alpha(z) = E_{\alpha,1}(z)$.

Remark 9. The Mittag-Leffler function is a generalisation of the exponential function e^z which can be written in a form of series

$$e^z = \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(n+1)} = E_1(z).$$

The core idea of fractional calculus is closely related to classical one where a fundamental theorem of calculus is an important outcome which shows us a relation between derivatives and integrals.

Theorem 3.3 (Fundamental theorem of calculus)**.** *Let $f : \langle 0, T\rangle \to \mathbb{R}$ be a continuous function and $F : \langle 0, T\rangle \to \mathbb{R}$ is defined*

$$F(x) := \int_0^x f(\tau) d\tau.$$

Then F is differentiable on $\langle 0, T\rangle$ and

$$F' = f.$$

One of the objectives of fractional calculus is, in some way, to preserve this feature. For the sake of clarity let us rewrite derivatives and integrals as operators

$$Df := f',$$

with f being a differentiable function and

$$If(x) := \int_0^t f(\tau)d\tau, \qquad 0 < x < T,$$

where f is a Riemann-integrable function on $\langle 0, T\rangle$. For $n \in \mathbb{N}$, let us generalise repeated derivatives and integrals as D^n and I^n, i.e. $D^1 := D, I^1 := I, D^n := DD^{n-1}$ and $I^n := II^{n-1}$ for $n \geq 2$.

By rewriting Thm.3.3 in the operator form we obtain

$$DIf = f.$$

Which for $n \in \mathbb{N}$ implies

$$D^n I^n f = f.$$

Theorem 3.4 (Cauchy's formula for repeated integration)**.** *Let f be a Riemann-integrable function on $\langle 0, T\rangle$. Then for $0 \leq x \leq T$, $n \in \mathbb{N}$, following statement holds true*

$$I^n f(x) = \frac{1}{(n-1)!}\int_0^x (x-\tau)^{n-1} f(\tau)d\tau.$$

This formula can be further extended even for $n \notin \mathbb{N}$ by replacing n-th power in $(x-\tau)^{n-1}$ by arbitrary positive real number and by using (3.1) (the factorial is now replaced by the Gamma function).

In this thesis we shall focus on a model with initial point $x_0 = 0$ but generally both further mentioned operators can be defined for any arbitrary initial point of $x_0 = a$, where $a \in \mathbb{R}$.

Definition 3.5 (Riemann–Liouville integral)**.** Let $\alpha \in \mathbb{R}^+$ and $0 < x < T$. Then

$$I^\alpha f(x) := \frac{1}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1} f(\tau)d\tau \tag{3.2}$$

is called the *Riemann-Liouville fractional integral of order α at x.*

Definition 3.6 (Riemann–Liouville derivative)**.** Let $\alpha \in \mathbb{R}^+$ and $m \in \mathbb{N} : \alpha \in (m-1, m\rangle$. Then operator D^α defined as

$$D^\alpha := D^\alpha I^{m-\alpha} f, \qquad 0 \leq x \leq T \tag{3.3}$$

is called the *Riemann-Liouville fractional operator of order α.*

Remark 10. We shall refer to (3.2) as RL integral and to (3.3) RL derivative.

It turns out that the RL-derivatives are not quite convenient in real-world modelling with FDE's. There are several alternative definitions of fractional derivative, however, the following one seems to fit better and shall be used in our model in Section 4.

Let us now preliminarily define the following operator ${}^*D^\alpha$ by

$${}^*D^\alpha f(x) := I^{m-\alpha} D^m f$$

for $\alpha \geq 0, m \in \mathbb{N}, \alpha \in (m-1, m\rangle$ and $D^m f \in L_1(\langle 0, T\rangle)$. For the case $\alpha \in \mathbb{N}$, thus $m = \alpha$, we obtain

$${}^*D^\alpha f = I^0 D^\alpha f = D^\alpha f,$$

which is identical to the classical case of D^n. The key idea of construction of the desired operator involves RL derivatives with following identity on one side and the preliminarily defined operator on the other one.

Theorem 3.7. *Let $\alpha \geq 0$ and $m \in \mathbb{N} : \alpha \in (m-1, m\rangle$. Moreover, assume $f \in AC^m(\langle 0, T\rangle)$. Then,*

$${}^*D^\alpha f = D^\alpha (f - T_{m-1}(f; 0)) \tag{3.4}$$

almost everywhere. $T_{m-1}(f; 0)$ denotes the Taylor polynomial of degree $m-1$ for the function f, centered at 0. For the case $m = 0$ we naturally define $T_{m-1}(f; 0) = 0$.

Definition 3.8 (Caputo derivative)**.** Assuming $\alpha \geq 0$ and such function f that $D^\alpha(f - T_{m-1}(f; 0))$ exists, where $m \in \mathbb{N} : \alpha \in (m-1, m\rangle$. Then the operator ${}^C D^\alpha$ defined as

$${}^C D^\alpha f := D^\alpha (f - T_{m-1}(f; 0))$$

is called the *Caputo differential operator of order α*.

For $\alpha \in N$, thus $m = \alpha$, we obtain the usual differential operator D^n "annihilating" the Taylor polynomial which is now of degree $m-1$, and, in particular, ${}^C D^0$ is the identity operator.

Theorem 3.9. *If f is a continuous function and $\alpha \geq 0$, then*

$${}^C D^\alpha I^\alpha f = f.$$

At this point, we have already introduced all the necessary theoretical background of fractional calculus for this work. Let us remind, that, from now on, we shall consider $\alpha \in (0, 1)$.

3.2 Autonomous System of Non-Integer Order

We shall use the Caputo derivative in this thesis as it is more convenient while stating the initial conditions which are integer order ones. On the other hand, FDE's with RL derivatives demand conditions with non-integer order which, in terms of real world modelling, are hardly applicable.

Definition 3.10 (Autonomous system of non integer order)**.** As an *autonomous system of non-integer order* if is considered a system of fractional differential equations (FDE's) as follows

$${}^C D^\alpha \boldsymbol{y} = \boldsymbol{f}(\boldsymbol{y}), \tag{3.5}$$

where $\boldsymbol{f} : \Omega \to \mathbb{R}^n, \Omega \subseteq \mathbb{R}^n$.

Similarly, as in Subsection 2.1, let us consider an initial value problem of autonomous FDEs

$$\begin{aligned} {}^{C}D^{\alpha}\boldsymbol{y}(x) &= \boldsymbol{f}(\boldsymbol{y}(x)), \\ \boldsymbol{y}(0) &= \boldsymbol{y}_0. \end{aligned} \tag{3.6}$$

where $\boldsymbol{y}_0 \in \mathbb{R}^n$ is an initial value.

Definition 3.11 (Solution of autonomous system of non-integer order)**.** Let $\alpha \in (0,1)$, $\boldsymbol{f} \in C(\Omega)$, $\Omega \subseteq \mathbb{R}^n$ is an open subset and $I = \langle 0, T\rangle$. *A solution* of (3.5) is every function $\tilde{\boldsymbol{y}} \in C(I)$ which satisfies $\boldsymbol{y}(x) \in \Omega$ and

$${}^{C}D^{\alpha}\boldsymbol{y}(x) = \boldsymbol{f}(\boldsymbol{y}(x)) \qquad \forall x \in I. \tag{3.7}$$

Furthermore, if $\boldsymbol{y}_0 \in \Omega$, then $\tilde{\boldsymbol{y}}$ is a solution of (3.6) on interval I.

We shall now focus on existence and uniqueness of solutions, let us first present a theorem corresponding to Peano existence theorem for first order ODE's.

Theorem 3.12 (Existence of solution)**.** *Let $\alpha \in (0,1), \boldsymbol{y}_0 \in \mathbb{R}^n, K > 0$ and $\widehat{h} > 0$. Define $G := \langle y_1^0 - K, y_1^0 + K\rangle \times \cdots \times \langle y_n^0 - K, y_n^0 + K\rangle$ and let function $\boldsymbol{f} : G \to \mathbb{R}^n$ be continuous. Moreover, let us define $M := \sup_{\boldsymbol{\zeta} \in G} |f(\boldsymbol{\zeta})|$ and*

$$h := \begin{cases} \widehat{h} & \text{for } M = 0, \\ \min\{\widehat{h}, (K\Gamma(\alpha+1)/M)^{1/\alpha}\} & \text{else.} \end{cases}$$

Then there exists a function $\tilde{\boldsymbol{y}} \in C(\langle 0, h\rangle)$ which solves the initial value problem (3.6).

Remark 11. As Diethelm [7] states in Rem. 6.3, in the Caputo-type case it turns out that continuity of the function $\boldsymbol{f}$ implies continuity of the solution $\tilde{\boldsymbol{y}}$ throughout the closed interval $\langle 0, h\rangle$.

The following lemma states that the differential formulation of initial value problem is equivalent to integral one.

Lemma 3.13. *Supposing assumptions from Thm 3.16, the function $\tilde{\boldsymbol{y}} \in C(\langle 0, h\rangle)$ is a solution of initial value problem (3.6) if and only if it is a solution of non-linear Volterra integral equation of the second kind*

$$\boldsymbol{y}(x) = \boldsymbol{y}_0 + \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} \boldsymbol{f}(\boldsymbol{y}(\tau)) d\tau$$

with $\alpha \in (0,1)$.

Theorem 3.14 (Uniqueness of solution)**.** *Let $\alpha \in (0,1), \boldsymbol{y}_0 \in \mathbb{R}^n, K > 0$ and $\widehat{h} > 0$. Let us define set G same as in Thm 3.12 and let the function $\boldsymbol{f} : G \to \mathbb{R}^n$ be continuous fulfilling Lipschitz condition with respect to the second variable*

$$|\boldsymbol{f}(\tilde{\boldsymbol{y}}_1) - \boldsymbol{f}(\tilde{\boldsymbol{y}}_2)| \leq L|\tilde{\boldsymbol{y}}_1 - \tilde{\boldsymbol{y}}_2|$$

with suitable constant $L > 0$ and let us define h as in Thm 3.12. Then, the initial value problem (3.6) has a unique solution $\tilde{\boldsymbol{y}} \in C(\langle 0, h\rangle)$.

Following theorem states that, under certain assumptions, the solution exists on the entire interval $\langle 0, \infty \rangle$.

Theorem 3.15. *Let $\alpha \in (0,1), \boldsymbol{y}_0 \in \mathbb{R}^n$, $G := \mathbb{R}^n$ and let function $\boldsymbol{f}$ be a continuous on G. Then, there exists a uniquely defined function $\tilde{\boldsymbol{y}} \in C(\langle 0, \infty \rangle)$ solving the initial value problem (3.6).*

Theorem 3.16 (Global existence of solution)**.** *Assuming the hypotheses from Thm 3.14, except for set G being now defined as $G := \mathbb{R}^n$. Moreover, $\boldsymbol{f}$ is a continuous function on G and there exist such constants $c_1 \geq 0, c_2 \geq 0$ and $0 \leq \mu < 1$ that*

$$|\boldsymbol{f}(\tilde{\boldsymbol{y}})| \leq c_1 + c_2 |\tilde{\boldsymbol{y}}|^{\mu} \qquad \forall \tilde{\boldsymbol{y}} \in G.$$

Then, there exists a function $\tilde{\boldsymbol{y}} \in C(\langle 0, \infty)$ which solves the initial value problem (3.6).

As already mentioned in Section 2, two trajectories of integer order equations do not intersect. If so, they are identical. This property is, in general, not true unless we consider special cases of FDEs. This holds true for one-dimensional FDE and triangular system of FDEs in the form

$$\begin{aligned} {}^{C}D^{\alpha}\boldsymbol{y}_1(x) &= \boldsymbol{f}_1(x, \boldsymbol{y}_1(x)), \\ {}^{C}D^{\alpha}\boldsymbol{y}_2(x) &= \boldsymbol{f}_2(x, \boldsymbol{y}_1(x), \boldsymbol{y}_2(x)), \\ &\dots \\ {}^{C}D^{\alpha}\boldsymbol{y}_2(x) &= \boldsymbol{f}_2(x, \boldsymbol{y}_1(x), \boldsymbol{y}_2(x), \dots, \boldsymbol{y}_n(x)). \end{aligned} \tag{3.8}$$

This triangular system can be solved coordinate-wise, meaning that we consecutively solve just a one-dimensional equation. Therefore, the triangular system inherits many features of the one-dimensional FDEs. Another feature that these systems show is, that in both cases, they generate non-local dynamical systems but not in the sense of Def. 2.3.

It is possible to show that there exists a map

$$\varphi_{x,\widehat{x}}(\cdot) : \mathbb{R}^n \to \mathbb{R}^n, \qquad \forall x, \widehat{x} \in I$$

called *two-parameter flow* in $\mathbb{R}^n$ satisfying following three properties

(i) $\varphi_{x,\widehat{x}}$ is continuous as a function of three variables $x, \widehat{x} \in I$ and $\boldsymbol{y} \in \mathbb{R}^n$,
(ii) the function $\varphi_{x,\widehat{x}}(\cdot)$ is a homeomorphism of $\mathbb{R}^n$ for any fixed $x, \widehat{x} \in I$,
(iii) the flow property $\varphi_{x,\widehat{x}} \circ \varphi_{\bar{x},x} = \varphi_{\bar{x},\widehat{x}}$ for all $\bar{x}, \widehat{x}, x \in I$.[1]

With these properties, in particular cases, it is possible to define fractional dynamical system. These cases are the same, as mentioned above, either one-dimensional or triangular and therefore the two-parameter flow generated by them (3.7) or (3.8) is a non-local dynamical system.

In other cases, for $n \geq 2$, the FDE (3.7) does not, in general, generate a two-parameter flow in $\mathbb{R}^n$. Therefore, it does not, in general, generate non-local dynamical system. More details can be found in [3].

3.3 Stability of Fractional Order Systems

Let us now focus on stability of FDEs. We shall not redefine notions such as the stability itself and the equilibrium point as they do not change irregardless of the order of the system. If so, we shall modify them.

[1] symbol $\circ$ denotes composition of maps

Definition 3.17 (Hyperbolic point of equilibrium in fractional system)**.** An equilibrium point $\boldsymbol{y}^*$ of system (3.5) is called *hyperbolic* if all of the eigenvalues λ_i $(i = 1, \ldots, n)$, of the matrix $\boldsymbol{A}$ at the point of equilibrium $\boldsymbol{y}^*$ are non-zero and $|\mathrm{Arg}(\lambda)| \neq \alpha\pi/2$.

Similarly, as in integer order case, we can linearise (3.5) as follows

$$ {}^{C}D^{\alpha}\boldsymbol{y} = \boldsymbol{A}\boldsymbol{y} \tag{3.9} $$

with $\boldsymbol{A} = D\boldsymbol{f}(\boldsymbol{y}^*)$. Again, as in classical case both systems have the same qualitative behaviour in the neighbourhood of the hyperbolic point of equilibrium. The following theorem specifies the form of solution (3.9):

Theorem 3.18 (Solution of homogeneous linear fractional system)**.** *Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ and eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$, where, the eigenvalues have equal algebraic and geometric multiplicity. Then, the general solution of (3.9) can be written as*

$$ \boldsymbol{y}(x) = \sum_{i=1}^{n} c_i \boldsymbol{v}_i E_\alpha(\lambda_i t^\alpha) $$

with constants $c_i \in \mathbb{C}$.

As shown in [4], similarly as in the classical integer-order case, there is a close relationship between the stability of the original non-linear system and its linearisation. More precisely, assume that (3.5) has an equilibrium $\boldsymbol{y}^*$. Then the problem of stability of this equilibrium can be converted to the problem of stability of zero solution to the following system:

$$ \boldsymbol{\eta}' = \boldsymbol{A}\boldsymbol{\eta} + \boldsymbol{g}(\boldsymbol{\eta}), $$

where $\boldsymbol{A} = D\boldsymbol{f}(\boldsymbol{y}^*), \quad \boldsymbol{g}(\boldsymbol{0}) = (\boldsymbol{0})^2$.

Indeed, the substitution $\boldsymbol{y} = \boldsymbol{\eta} + \boldsymbol{y}^*$ yields

$$ \boldsymbol{y}' = \boldsymbol{\eta}', $$

thus

$$ \boldsymbol{\eta}' = \boldsymbol{f}(\boldsymbol{\eta} + \boldsymbol{y}^*) = \boldsymbol{f}(\boldsymbol{\eta} + \boldsymbol{y}^*) + \boldsymbol{A}\boldsymbol{\eta} - \boldsymbol{A}\boldsymbol{\eta}. $$

Therefore,

$$ \boldsymbol{\eta}' = \boldsymbol{A}\boldsymbol{\eta} + \boldsymbol{g}(\boldsymbol{\eta}), $$

where

$$ \boldsymbol{g}(\boldsymbol{\eta}) = \boldsymbol{g}(\boldsymbol{\eta} + \boldsymbol{y}^*) - \boldsymbol{f}(\boldsymbol{y}^*) - \boldsymbol{A}\boldsymbol{\eta}. $$

Following holds true

$$ \begin{aligned} \boldsymbol{g}(\boldsymbol{0}) &= \boldsymbol{f}(\boldsymbol{y}^*) - \boldsymbol{A}\boldsymbol{0} = \boldsymbol{0} \\ \boldsymbol{g}'(\eta) &= D\boldsymbol{f}_y(\boldsymbol{\eta} + \boldsymbol{y}^*) - \boldsymbol{A}. \end{aligned} $$

In order to $\boldsymbol{g}'(\boldsymbol{0}) = \boldsymbol{0}$, $\boldsymbol{A} = D\boldsymbol{f}(\boldsymbol{y}^*)$, i.e., $\boldsymbol{A}$ is equal to Jacobi matrix of vector field $\boldsymbol{f}$ in the equilibrium $\boldsymbol{y}^*$.

Remark 12. Analogously, by substituting $\boldsymbol{y}'$ by the Caputo derivative ${}^{C}D^{\alpha}\boldsymbol{y}$.

[2] $\boldsymbol{g} \in C^1$ in the neighbourhood of origin

This allows us to state following theorem.

Theorem 3.19 (Linearised asymptotic stability for FDEs)**.** *Let* $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ *and let* $\lambda_i (i = 1, \ldots, n)$ *be its eigenvalues. Then, the trivial solution, with* $\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0}$ *and* $D\mathbf{f}(\boldsymbol{0}) = \boldsymbol{0}$*, is*

(i) is asymptotically stable if and only if all eigenvalues λ_i *satisfy* $|Arg(\lambda_i)| > \alpha\pi/2$.

(ii) Unstable if at least one of the eigenvalues λ_i *of the Jacobi matrix satisfies* $|Arg(\lambda_i)| < \alpha\pi/2$.

With respect to the above note we immediately have:

Corollary 3.20. *Let* $\boldsymbol{y}^*$ *be an equilibrium of (3.5) and let* $\boldsymbol{f}$ *be a* C^1*-vector field with* $D\boldsymbol{f}(\boldsymbol{y}^*)$ *having the eigenvalues satisfying* $|Arg(\lambda_i)| > \alpha\pi/2$*. Then the* $\boldsymbol{y}^*$ *is asymptotically stable. If there is at least one eigenvalue such that* $|Arg(\lambda_i)| < \alpha\pi/2$*, then it is unstable.*

Remark 13. If $|\mathrm{Arg}(\lambda_i)| \geq \alpha\pi/2$ and the equality shall occur for at least one eigenvalue λ_i, then we are unable to determine - the equilibrium point is non-hyperbolic.

To classify equilibria for the planar system we can use Def.2.15, considering that the stability is again determined by eigenvalues λ_1, λ_2 with following differences:

(i) There is no singular point which would correspond to centre which is a consequence of FDEs not being able to have periodic solutions.

(ii) The stability domain, in case of stable focus points, widens. I.e., as stable focus points are even considered those with $\Re(\lambda_{1,2}) > 0$ despite satisfying $|\mathrm{Arg}(\lambda_{1,2})| > \alpha\pi/2$. This means that the border between stability and instability is no longer an imaginary axis as we can see on following figure 2.

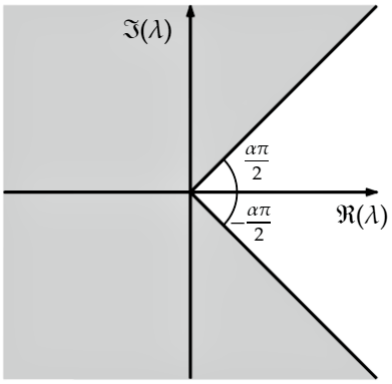

Figure 2: Stability region of ODEα is also referred to as Matignon sector [1]

Remark 14. Equilibria, for $n = 2$, can be classified with a use of trace and determinant of a matrix. Using the same notation as in Rem. 8, we show that the equilibrium point $\boldsymbol{y}^*$ is asymptotically stable if and only if $b > 0$ and $a + 2\sqrt{b}\cos(\alpha\pi/2) > 0$. These inequalities can be obtained by considering the fact that

$$|\mathrm{Arg}(\lambda_{1,2})| = \frac{|\Im(\lambda_{1,2})|}{|\Re(\lambda_{1,2})|}.$$

As [1] states, and as already mentioned, if the eigenvalues of linear system (3.9) are at the boundary of stable region, they shall not generate a periodic solution in fractional order systems, however following theorem shows that the solutions of this system converge asymptotically to a closed orbit shown in Fig. 3.

Theorem 3.21. *The trajectory of the system*

$$\begin{aligned} {}^{C}D^{\alpha}\boldsymbol{y}(x) &= \boldsymbol{A}\boldsymbol{y}(x), \\ \boldsymbol{y}(0) &= (K_1, K_2), \end{aligned} \tag{3.10}$$

where $\alpha \in (0,1)$ *and* $n = 2$ *with eigenvalues* $\lambda_{1,2} = re^{\pm i\frac{\alpha\pi}{2}}$ *and* $K_1, K_2 \in \mathbb{R}$ *converges to a closed orbit,* $y_1^2(x) + y_2^2(x) = \frac{K_1^2+K_2^2}{\alpha^2}$ *in a phase plane.*

However, we note that this is not a solution. On the other hand, some recent studies discuss existence of asymptotically periodic solutions, see, e.g., [10].

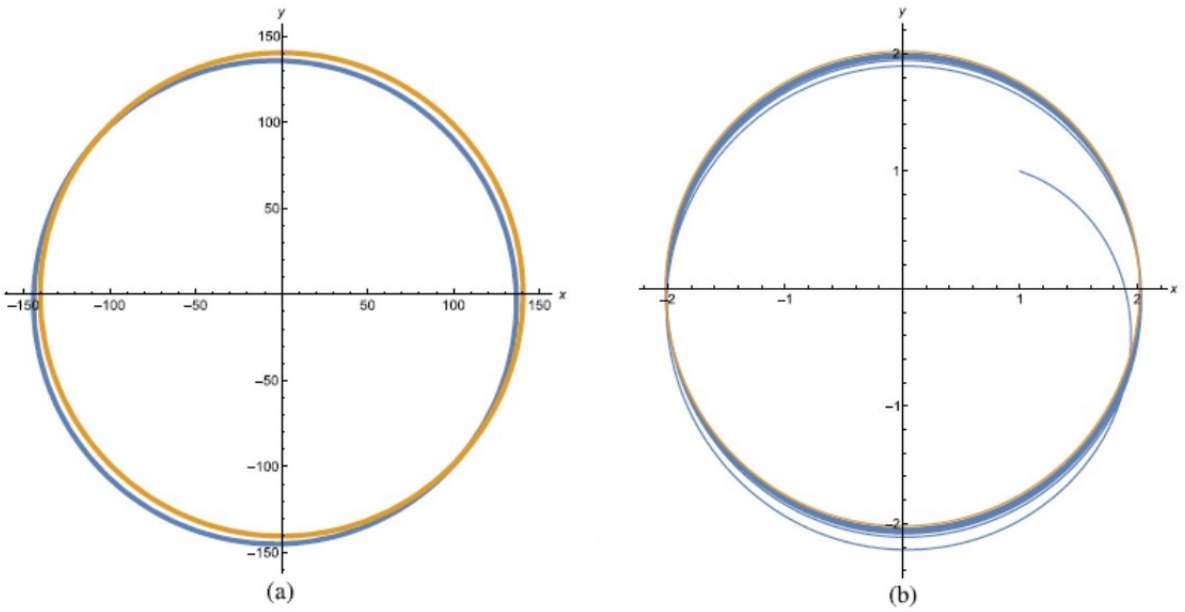

Figure 3: $\lambda = e^{i\frac{\alpha\pi}{2}}$ and $y(0) = (1,1)$ (a) $\alpha = 0.01$, (b) $\alpha = 0.7$ [1]

Another important and unique phenomenon of planar fractional autonomous systems is that if the eigenvalue is close to the borderline of stable region it may generate a self–intersecting trajectory, i.e., an occurrence of equilibria in solution trajectories happens if the trajectory $\boldsymbol{y}(x)$ is not smooth in the neighbourhood of point $\boldsymbol{y}(0)$. Examples of such equilibria are either *cusps* or *multiple points* as can be seen on following figures.

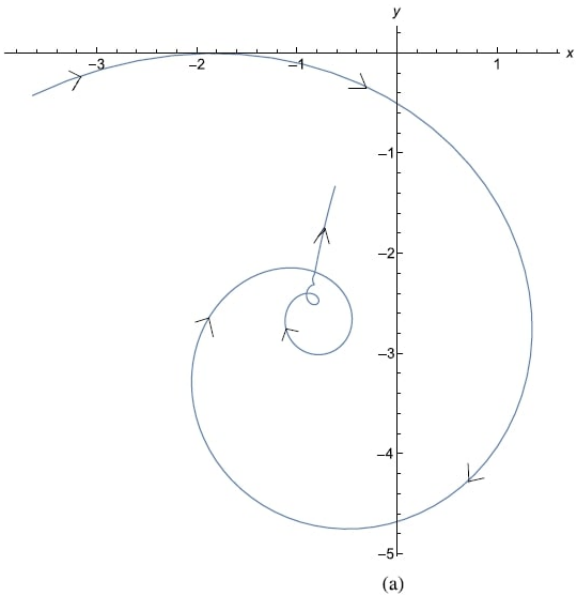

Figure 4: (a) Example of cusp and double points, $\alpha = 0.1, r = 1$ [1]

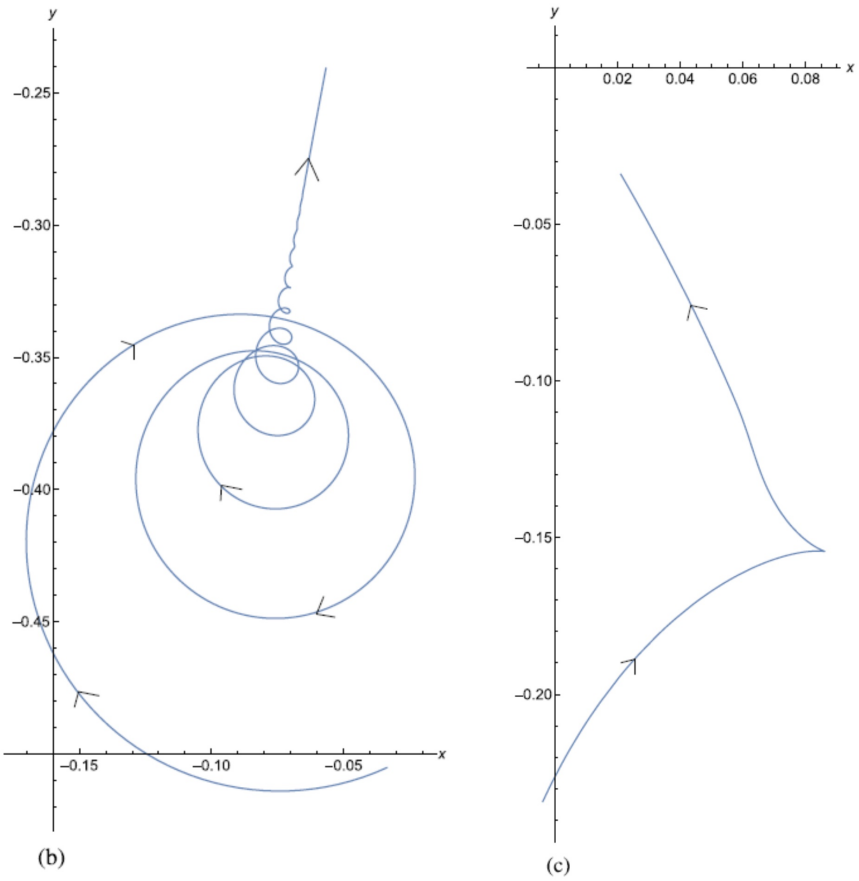

Figure 5: (b) Example of double cusp and multiple points, $\alpha = 0.3, r = 0.5$, (c) Example of single cusp $\alpha = 0.6, r = 1$ [1]

The authors of [1] propose a conjecture along with possible proof that there exist equilibria in the trajectory of planar system (3.9) if and only if the eigenvalues $\lambda = re^{\pm i\nu}$ of A satisfy

$$\frac{\alpha\pi}{2} - \delta_1 < |\text{Arg}(\lambda)| < \frac{\alpha\pi}{2} + \delta_2,$$

with $\delta_1 > 0$ and $\delta_2 > 0$ are sufficient small real numbers.

4 Model analysis

Infectious maladies can be divided into two groups based on their causation. We can distinguish between diseases caused by micro-parasites (viruses or bacteria) and macro-parasites (insects and maggots in some particular phase of their development). In this thesis we shall closely look upon infections with viral or bacterial origin.

When a susceptible individual is infected either directly or indirectly from someone who is already infected – *a source of infection*, at first, he or she shows no symptoms of illness development, this is called *a latent phase*, which is then followed by *an infectious phase* when the individual can infect others who are now susceptible. The individual now shows symptoms of illness which is usually then being followed by isolation until he or she recovers or dies. This time interval from being infected to showing first symptoms can be referred to as *incubation phase.*

In case of recovery, the individual can gain immunity towards the disease either permanently or temporarily.

In this Section we shall describe and analyse the model from classical and fractional point of view.

4.1 Integer Order SIR Model

This model, introduced in 1927 by William Ogilvy Kermack and Anderson Gray McKendrick, is one of the simplest disease models used to explain spread of the contagious disease through a closed population over time. It was proposed for an explanation of the rapid rise and fall in the number of infected in epidemics such as plague in London (1665–1666) or cholera (1865). The assumptions of the model are that the population is fixed (meaning, no births, no deaths due to disease nor due to natural causes) and completely homogeneous with no age or social structure, duration of infectivity is same as length of the disease.

Let S be the number of susceptible individuals, I the number of infected and infectious, R the number of recovered (usually) with lifelong immunity. Then, the model consists of three non-linear ODEs:

$$\begin{aligned} S' &= -\beta SI, \\ I' &= \beta SI - \gamma I, \\ R' &= \gamma I. \end{aligned} \tag{4.1}$$

Let us note that S, I, R are time dependant and the parameters $\beta > 0$ and $\gamma > 0$ of the system represent the contact rate (of ill and healthy) and the recovery rate per capita, respectively.

Remark 15. An average individual meets with βN individuals per time unit, where $\beta \in (0, 1)$. These are enough for disease transmission. Whereas probability of random contact of an individual with a susceptible member is S/N. Since the amount of infected is I, the amount of newly infected individuals per time unit is

$$\beta N \cdot \frac{S}{N} \cdot I = \beta SI.$$

Therefore βSI individuals leave the susceptible compartment and thus $S' = -\beta SI$.

Remark 16. The amount of individuals leaving the infected compartment per time unit is γI. Therefore, βSI individuals shall become infected and γI individuals shall recover, leaving the compartment and thus $I' = \beta SI - \gamma I$. Finally, amount of γI individuals shall move to the recovered compartment, i.e., $R' = \gamma I$.

We assume non-negativity of the three "compartments" and, as mentioned in the introductory part, total number of population $N = S + I + R$ to be fixed, thus $N' = S' + I' + R' = 0$. The system can be translated into a simple scheme as follows:

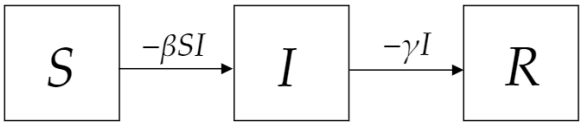

Figure 6: Scheme of the Kermack–McKendrick model. Boxes represent "compartments" and arrows indicate flux between them.

It can be quite easily seen that the value of R is unambiguously determinable by S and I allowing us to omit the third equation leaving us with two-dimensional system

$$S' = -\beta SI, \tag{4.2a}$$
$$I' = (\beta S - \gamma)I \tag{4.2b}$$

with initial conditions $S(0) = S_0 \geq 0, I(0) = I_0 \geq 0$.

At first let us choose

$$S(0) = S_0 > 0, \quad I(0) = 0.$$

In this case we have set of equilibrium points for every S_0 lying on half-line $I = 0, S > 0$. The system has constant solution $(S, I) \equiv (S_0, 0)$. This means that amount of individuals who are susceptible is S_0 and there are no infected or infectious ones. I.e., there is no possibility for the healthy ones to get infected and move to the other "compartment".

Now let us choose

$$S(0) = 0, \quad I(0) = I_0 > 0.$$

In this case there is no set of equilibrium points and as we can see the (4.2a) satisfies $S(t) \equiv 0$ and (4.2b) has following form

$$I' = -\gamma I. \tag{4.3}$$

Along with the initial condition we have a solution

$$I = I_0 \exp(-\gamma t), \quad t > 0. \tag{4.4}$$

Since $\gamma > 0$, then for every $I_0 > 0$ the trajectory curve starts at this point and decreases by $S = 0$ axis to the $(0, 0)$ point. This solution represents a situation of no susceptible individuals, meaning, that everyone is infected. These individuals have no other option but to move to the recovered "compartment", i.e., the amount of infected individuals decreases while the susceptible remain at zero. The last remaining option is

$$S(0) = S_0 > 0, \quad I(0) = I_0 > 0.$$

As stated in Section 2, no trajectories can neither intersect or touch and, moreover, they have to start and remain in the first quadrant for all $t > 0$. We shall now analyse the behaviour of a functions S and I. As we already know $\beta > 0$, therefore we can see that $S' < 0$ for all t, i.e., the function S decreases for all t. Also, $I' > 0$ if and only if $S > \frac{\gamma}{\beta}$.

We shall now prove existence of limits $\lim_{t\to\infty} S(t) = S_\infty$[3], $\lim_{t\to\infty} I(t) = I_\infty$[4] and following statements are true

$$0 < S_\infty < \frac{\gamma}{\beta}, \quad I_\infty = 0.$$

At first, we determine the maximal interval of existence of $((S, I)$ according to Def.2.8

$$J = (a, b).$$

Since we are dealing with real world modelling, we are interested in a development for $t > 0$, thus it is sufficient to determine only the b value and use the interval $[0, b)$. Also, we know that all trajectories must lie in the first quadrant and S is decreasing, therefore

$$\lim_{t\to\infty} S(t) = S_\infty \in [0, \infty)$$

exists.

By adding equations (4.2a),(4.2b) together we have

$$(S + I)' = -\gamma I \tag{4.5}$$

therefore $(S + I)'$ is positive decreasing function with existing limit

$$\lim_{t\to\infty} (S + I)(t) \in [0, \infty).$$

This implies existence of limit

$$\lim_{t\to\infty} I(t) = I_\infty \geq 0,$$

i.e., $b = \infty$.

By analysis of both equations (4.2a),(4.2b) for $t \to \infty$ we obtain

$$\lim_{t\to\infty} S'(t) = -\beta S_\infty I_\infty,$$
$$\lim_{t\to\infty} I'(t) = (\beta S_\infty - \gamma) I_\infty.$$

As both limits S' and I' exist, they must be equal to zero. (In the opposite case, functions S and I would not have finite limit for $t \to \infty$). Therefore

$$-\beta S_\infty I_\infty = 0,$$
$$(\beta S_\infty - \gamma) I_\infty = 0,$$

meaning that (S_∞, I_∞) is equilibrium point for (4.2a),(4.2b), therefore, $I_\infty = 0$ and $S_\infty \geq 0$.

We shall now prove that $S_\infty > 0$. We exclude the t variable from system (4.2a),(4.2b) by dividing as follows

$$\frac{\mathrm{d}I}{\mathrm{d}S} = -1 + \frac{\gamma}{\beta S}.$$

[3] amount of individuals remaining uninfected

[4] amount of individuals remaining uncured

By integrating we obtain

$$I(S) = -S + \frac{\gamma}{\beta} \ln S + c_0, \tag{4.6}$$

where

$$c_0 = I_0 + S_0 - \frac{\gamma}{\beta} \ln S_0 \tag{4.7}$$

and for $t \to \infty$

$$I_\infty = -S_\infty + \frac{\gamma}{\beta} \ln S_\infty + c_0.$$

We already know that $I_\infty = 0$, therefore, the right-hand side must be equal to zero, meaning that S_∞ must not be equal to zero, because in this case $\ln S_\infty$ would be equal to $-\infty$.

The last thing remaining to be proved is $S_\infty < \frac{\gamma}{\beta}$. We already know that I increases while $S > \frac{\gamma}{\beta}, \quad t > 0$. If $S \geq \frac{\gamma}{\beta}$, then $S > \frac{\gamma}{\beta}$ for $t >$ implying that I increases for $t > 0$ which is a contradiction for $I_\infty = 0$. We can now consider two situations by using the fact that I is at its maximum value for $S = \frac{\gamma}{\beta}$ and also S is decreasing function for any t.

(i) $S_0 \leq \frac{\gamma}{\beta}$. Then, S decreases to value S_∞ and I decreases immediately to zero, meaning, that there is no epidemic.

(ii) $S_0 > \frac{\gamma}{\beta}$. In this case the S decreases to S_∞ as well but, on the other hand, the I increases to its maximum value and then decreases to I_∞ which is equal to zero meaning the epidemic occurrence.

The key value that governs evolution is so-called *basic reproduction ratio* or *epidemiological threshold*

$$R_0 = \frac{\beta S_0}{\gamma},$$

which, roughly speaking, is derived as the expected number of new infections from a single infection in a population where all subjects are susceptible. Let us note that the choice of the notation R_0 is quite unfortunate and has nothing to do with R and with alternative SIR models has different formulation.

If $R_0 \leq 1$, then each individual who contracted the disease infected less than one person before dying or recovering, meaning the disease shall perish, i.e., $I' < 0$.

If $R_0 > 1$, then each individual who already had the disease infected more than one person resulting in epidemic spread, i.e., $I' > 0$, and the disease shall permanently remain endemic in the population.

We can consider two ways of starting an epidemics:

(i) An epidemic outbreak caused by an individual – *patient zero* travelling to a destination and carrying the disease back to his or hers residual place. Meaning that $I_0 > 0, S_0 + I_0 = N$ and $R_0 = \frac{\beta S_0}{\gamma}$.

(ii) A disease being brought by an individual from different "group". In this case $S_0 \approx N, I_0 \approx 0$ and R_0 is defined as $\frac{\beta N}{\gamma}$.

By integration (4.5) from 0 to ∞ we obtain

$$-\int_0^\infty (S+I)'(t)\mathrm{d}t = S_0 + I_0 - S_\infty = 1 - S_\infty = \gamma \int_0^\infty I(t)\mathrm{d}t. \tag{4.8}$$

From (4.2a) we get

$$\frac{S'}{S} = -\beta I, \qquad t \in [0, \infty).$$

By integration of this equation we get

$$\int_0^\infty \frac{S}{S}(t)\mathrm{d}t = \ln\frac{S_0}{S_\infty} = -\beta\int_0^\infty I(t)\mathrm{d}t.$$

With a use of the result of (4.8) and adjustment of previous equation we get

$$\ln\frac{S_0}{S_\infty} = \beta\left[\frac{N-S_\infty}{\gamma}\right] = R_0\left[1-\frac{S_\infty}{N}\right] \tag{4.9}$$

Remark 17. For $N=1$, $\quad \beta\left[\frac{1-S_\infty}{\gamma}\right] = R_0[1-S_\infty]$.

Equation (4.9) gives us a relation between the reproductive ratio and the extent of epidemics. Let us note that the final extent of epidemics, i.e., the amount of individuals infected during the epidemics, is $N - S_\infty$. The number $(1 - \frac{S_\infty}{N})$ is often being referred to as *affection ratio*.

It is easy to prove that there exists a unique solution S_∞ of (4.9). Let us mark $x = S_\infty$ and define a function

$$f(x) = \ln\frac{S_0}{x} - R_0\left[1-\frac{x}{N}\right], \qquad x > 0. \tag{4.10}$$

Evaluating the function f at points 0 and N

$$f(0_+) = \lim_{x\to 0} f(x) = \infty - R_0 = \infty,$$
$$f(N) = \ln\frac{S_0}{N}.$$

Since $\frac{S_0}{N} < 1$, then $\ln\frac{S_0}{N} < 0$ and therefore

$$f(0_+) > 0, \quad f(N) < 0.$$

We shall now analyse for which values of x the function decreases, i.e., values of x where $f'(x) < 0$. Because

$$f'(x) = -\frac{1}{x} + \frac{R_0}{N}, \qquad x > 0$$

we get

$$f'(0_+) = -\infty,$$
$$f'(x) = 0 \Leftrightarrow x = \frac{N}{R_0},$$

thus $f'(0) < 0$ if and only if

$$0 < x < \frac{N}{R_0}.$$

If $R_0 \leq 1$, then f is monotonically decreasing from $f(0_+) = \infty$ to the negative value at $x = N$. I.e., there exists a unique point S_∞ of function $f(x)$ and following condition $S_\infty < N$ is satisfied.

If $R_0 > 1$, then f monotonically decreases from $f(0_+) = \infty$ to the minimum value of $x = \frac{N}{R_0}$ and then increases to the negative value at the point $x = N$. I.e., there exists such a unique point S_∞ of function $f(x)$ that

$$S_\infty < \frac{N}{R_0}.$$

Also

$$g\left(\frac{S_0}{R_0}\right) = \ln R_0 - R_0 + \frac{S_0}{N} < \ln R_0 - R_0 + 1, \qquad \text{for} \quad S_0 < N.$$

Since $\ln R_0 \leq R_0 - 1$ for $R_0 > 1$, then we obtain

$$g\left(\frac{S_0}{R_0}\right) < 0, \qquad \text{for} \quad R_0 > 1.$$

Therefore following condition is true for $R_0 > 1$

$$S_\infty < \frac{S_0}{R_0}.$$

Generally speaking, it is quite challenging to determine the value of contact rate as it depends on a specific type of the outbreak and social factors. The values of S_0 and S_∞ are determinable, both, before and after the outbreak. By using these data we are able to estimate R_0 value retrospectively, i.e., after the outbreak.

The maximum value of infected individuals at a certain time t is for $S(t) = \frac{\gamma}{\beta}$ when $I'(t) = 0$. Then the maximum value

$$I_{max} = S_0 + I_0 - \frac{\gamma}{\beta}\ln S_0 - \frac{\gamma}{\beta} + \frac{\gamma}{\beta}\ln\frac{\gamma}{\beta}$$

is obtained by substitution $S = \frac{\gamma}{\beta}$ and $I = I_{max}$ from equations (4.6),(4.7).

However, in real world modelling rather complicated versions of the SIR model reflecting the actual biology of a given disease are often being used. Following model could resemble a disease lasting several weeks, even months, which means that during that time many people either die or are being born. In that case, the demographic movement needs to be included as well.

We shall now introduce so-called *SIR model with vital dynamics and constant population* suggested by Herbert W. Hethcote in 1976 as follows

$$\begin{aligned} S' &= \nu N - \mu S - \beta SI, \\ I' &= \beta SI - \gamma I - \mu I, \\ R' &= \gamma I - \mu R. \end{aligned} \tag{4.11}$$

Remark 18. As stated in the introductory part of this Subsection, N stands for fixed population. Therefore, from now on $N = 1$.

We consider same assumptions as in Kermack–McKendrick model. In order to fulfil the constant population requirement, we let death rate be equal to birth rate, i.e., $\mu = \nu > 0$. It can be easily seen that first two equations, again, do not depend on R. Therefore, the

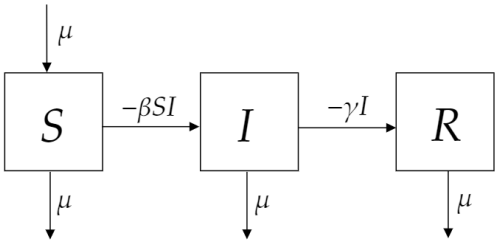

Figure 7: Scheme of the SIR model with vital dynamics.

third equation is omitable, thus obtaining following two-dimensional system of non-linear autonomous equations:

$$\begin{aligned} S' &= \mu - \mu S - \beta SI, \\ I' &= \beta SI - (\gamma + \mu)I, \end{aligned} \tag{4.12}$$

with $R(t) = 1 - S(t) - I(t)$. At first, we shall find the equilibrium points by solving following set of equations

$$\begin{aligned} \mu - \mu S - \beta SI &= 0, \\ \beta SI - (\gamma + \mu)I &= 0 \end{aligned} \tag{4.13}$$

we get

$$E_1 = (1, 0), \qquad E_2 = \left(\frac{\gamma + \mu}{\beta}, \frac{\mu(\beta - \gamma - \mu)}{\beta(\gamma + \mu)} \right).$$

It is evident, that our system is valid only in the first quadrant. Let us now analyse conditions for this requirement.

(i) Equilibrium point E_1 lies in the first quadrant by default means.

(ii) However, the second equilibrium point requires following condition to be satisfied

$$\beta - \gamma - \mu \geq 0$$

while

$$1 \geq \frac{\gamma + \mu}{\beta}.$$

However, in case $E_1 = E_2$ a strict inequality is required

$$1 > \frac{\gamma + \mu}{\beta}.$$

The first equilibrium, also referred to as *disease free equilibrium* (DFE), represents a situation of no outbreak (or "pre-outbreak") and exists every time, whereas the second point, *endemic equilibrium* (EE) exists only if $R_0 > 1$.

Remark 19. For "Hethcote's model" the basic reproduction ratio is $R_0 = \frac{\beta S_0}{\gamma + \mu}$ and similarly, as in SIR model, is derived from susceptible compartment.

The Jacobi matrix of the system has following form:

$$\boldsymbol{J} = D\boldsymbol{f}(S, I) = \begin{pmatrix} -\beta I - \mu & -\beta S \\ \beta I & \beta S - (\gamma + \mu) \end{pmatrix}.$$

The Jacobi matrix for the system evaluated in the disease free equilibrium E_1 becomes

$$\boldsymbol{J}(E_1) = D\boldsymbol{f}(1,0) = \begin{pmatrix} -\mu & -\beta \\ 0 & \beta - (\gamma + \mu) \end{pmatrix},$$

with

$$a = -\operatorname{tr}(\boldsymbol{J}(E_1)) = -2\mu - \gamma + \beta$$

and

$$b = \det(\boldsymbol{J}(E_1)) = \mu(\mu + \gamma - \beta).$$

We are able to determine the eigenvalues $\lambda_{1,2}$

$$\lambda_1 = \beta - \gamma - \mu,$$
$$\lambda_2 = -\mu$$

and analyse the phase portraits by the eigenvalues of the system

(i) Let $\beta - \gamma - \mu > 0$. The singular point in this case is saddle.

(ii) Let $\beta - \gamma - \mu < 0$. The singular point in this case is stable focus.

(iii) Let $\beta - \gamma - \mu = 0$. In this case the phase portrait is a stable line completely built up by fixed equilibrium points.

The stability can be also easily verified with the help of characteristic polynomial

$$P(\lambda) = \det(\boldsymbol{J} - \lambda\mathbf{E}) = \lambda^2 - \operatorname{tr}(\boldsymbol{J})\lambda + \det(\boldsymbol{J}) = 0,$$

and for E_1 we get

$$\lambda^2 - (\beta - \gamma - 2\mu)\lambda + \mu(-\beta + \gamma + \mu) = 0.$$

By using Routh–Hurwitz criterion from Thm.2.11 we are able to determine whether or not the system is asymptotically stable. For a system to be asymptotically stable, following statements must hold true

$$a = -\operatorname{tr}(\boldsymbol{J}(E_1)) = 2\mu + \gamma - \beta > 0 \qquad \text{and} \qquad b = \det(\boldsymbol{J}(E_1)) = \mu(\mu + \gamma - \beta) > 0.$$

Let us also remind that α, β and γ are always positive and it can be verified that for $R_0 < 1$ the singular point is indeed asymptotically stable. However, we cannot tell for $R_0 = 1$ by this method.

Similarly, as in E_1, we shall now analyse stability of the endemic equilibrium E_2. The Jacobi matrix evaluated in E_2 is

$$\boldsymbol{J}(E_2) = D\boldsymbol{f}\left(\frac{\gamma+\mu}{\beta}, \frac{\mu(\beta-\gamma-\mu)}{\beta(\gamma+\mu)}\right) = \begin{pmatrix} \frac{-\mu\beta}{\gamma+\mu} & -\alpha-\mu \\ \frac{\beta\mu-\gamma\mu-\mu^2}{\gamma+\mu} & 0 \end{pmatrix}.$$

with

$$a = -\operatorname{tr}(\boldsymbol{J}(E_2)) = -\frac{\beta\mu}{\gamma+\mu}$$

and

$$\det((\boldsymbol{J}(E_2)) = \mu(\beta - \gamma - \mu).$$

Since we know that

$$1 > \frac{\gamma+\mu}{\beta}$$

the determinant is always greater than zero. The eigenvalues are determined by

$$\lambda_{1,2} = \frac{-\frac{\beta\mu}{\gamma+\mu} \pm \sqrt{(\frac{\beta\mu}{\alpha+\mu})^2 - 4\mu(\beta-\gamma-\mu)}}{2}.$$

By using the substitution

$$\widehat{S} = S - \frac{\gamma+\mu}{\beta}, \qquad \widehat{I} = I - \frac{\mu(\beta-\gamma-\gamma)}{\beta(\gamma+\mu)}$$

the equilibrium point E_2 shall *shift* its position to the origin $\mathbf{0} = (0,0)$. The associated linear equation has form

$$\begin{pmatrix} \widehat{S}' \\ \widehat{I}' \end{pmatrix} = \begin{pmatrix} \frac{-\beta\mu}{\gamma+\mu} & -\gamma+\mu \\ \frac{\beta\mu-\gamma\mu-\mu^2}{\gamma\mu} & 0 \end{pmatrix} \begin{pmatrix} \widehat{S} \\ \widehat{I} \end{pmatrix}. \tag{4.15}$$

We are not able to determine whether the eigenvalues are positive or negative, therefore, we shall analyse the determinant and trace. It can be easily seen that based on default non-negative conditions of β, γ and μ the trace shall always be negative, thus the determinant shall always be positive. The type of a phase portrait is determined with the help of Rem. 8.

(i) The condition $\det(\boldsymbol{J}) > \frac{(\mathrm{tr}(\boldsymbol{J})^2}{4}$ is satisfied if and only if $4(\gamma+\mu)^2(\beta-\gamma-\mu)-\beta^2\mu < 0$, hence, E_2 is stable focus.

(ii) The condition $\det(\boldsymbol{J}) = \frac{(\mathrm{tr}(\boldsymbol{J})^2}{4}$ is satisfied if and only if $4(\gamma+\mu)^2(\beta-\gamma-\mu)-\beta^2\mu = 0$, hence, E_2 is stable degenerate focus with one eigenvector.

(iii) The condition $\det(\boldsymbol{J}) < \frac{(\mathrm{tr}(\boldsymbol{J})^2}{4}$ is satisfied if and only if $4(\gamma+\mu)^2(\beta-\gamma-\mu)-\beta^2\mu > 0$, hence, E_2 is stable focus with two eigenvectors.

Since all of the equilibria are stable and attractive, based on Rem. 4. we can state their asymptotic stability.

In conclusion, the singular point E_2 is locally asymptotically stable whenever $\frac{\beta}{\gamma+\mu} > 1$, i.e., the velocity of spreading the disease has to be greater than the recovery rate and "demographic movement" and, as stated in the introduction of this section, does not exist for $\frac{\beta}{\gamma+\mu} \leq 1$.

4.2 Fractional SIR Model

Similarly, as in Subsection 4.1, we shall analyse the SIR mode. However, this time we shall assume a non-integer model with characteristics of fractional derivatives which bring solely new parameter – *order of the differential operator* α and classical derivatives are now substituted by Caputo ones. We shall limit ourselves to the case with identical order of α for both equations.

Remark 20. We shall refer to fractional SIR model as "FSIR" in order to easily to distinguish among them.

In order to have both sides of the same dimension, which is $(time)^{-\alpha}$, the Hethcote's model has following form for the fractional case

$$\begin{aligned} {}^{C}D^{\alpha}S &= \nu^{\alpha} - \mu^{\alpha}S - \beta^{\alpha}SI, \\ {}^{C}D^{\alpha}I &= \beta^{\alpha}SI - \gamma^{\alpha}I - \mu^{\alpha}I, \\ {}^{C}D^{\alpha}R &= \gamma^{\alpha}I - \mu^{\alpha}R. \end{aligned} \tag{4.16}$$

All considered parameters have same meaning as in Subsection 4.1 and the third equation is redundant as well allowing us to similarly modify to

$$\begin{aligned} {}^{C}D^{\alpha}S &= \mu^{\alpha} - \mu^{\alpha}S - \beta^{\alpha}SI, \\ {}^{C}D^{\alpha}I &= \beta^{\alpha}SI - (\gamma^{\alpha} + \mu^{\alpha})I, \end{aligned} \tag{4.17}$$

with $R(t) = 1 - S(t) - I(t)$. It can easily be seen that for $\alpha \to 1^{-}$ the model becomes the classical, already analysed, integer order model (for further details see [8]). By letting the right-hand side equal to zero we obtain the same equilibrium points as in the classical case

$$E_1 = (1, 0) \quad \text{and} \quad E_2 = \left(\frac{\gamma^{\alpha} + \mu^{\alpha}}{\beta^{\alpha}}, \frac{\mu^{\alpha}(\beta^{\alpha} - \gamma^{\alpha} - \mu^{\alpha})}{\beta^{\alpha}(\gamma^{\alpha} + \mu^{\alpha})} \right).$$

After evaluation of these points by Jacobi matrix and then using same substitution as in Subsection 4.1 for E_1 we have

$$\begin{pmatrix} {}^{C}D^{\alpha}\widehat{S} \\ {}^{C}D^{\alpha}\widehat{I} \end{pmatrix} = \begin{pmatrix} -\mu^{\alpha} & -\beta^{\alpha} \\ 0 & \beta^{\alpha} - \gamma^{\alpha} - \mu^{\alpha} \end{pmatrix} \begin{pmatrix} \widehat{S} \\ \widehat{I} \end{pmatrix}.$$

For E_2 analogically

$$\begin{pmatrix} {}^{C}D^{\alpha}\widehat{S} \\ {}^{C}D^{\alpha}\widehat{I} \end{pmatrix} = \begin{pmatrix} \frac{-\beta^{\alpha}\mu^{\alpha}}{\gamma^{\alpha}+\mu^{\alpha}} & -\gamma^{\alpha} + \mu^{\alpha} \\ \frac{\beta^{\alpha}\mu^{\alpha} - \gamma^{\alpha}\mu^{\alpha} - \mu^{2\alpha}}{\gamma^{\alpha}\mu^{\alpha}} & 0 \end{pmatrix} \begin{pmatrix} \widehat{S} \\ \widehat{I} \end{pmatrix}. \tag{4.18}$$

For both points, the eigenvalues are real, thus we analyse, as in classical case, their non-negativity, see Rem. 14. And according to Thm.3.19 and Corr. 3.20, the asymptotic stability of the linearised system implies the asymptotic stability of the trivial solution (3.6).

4.3 Influenza Epidemic in an English Boarding School

For our testing purposes we chose the data of influenza epidemic in an English boarding school in 1978 [20] with artificially declared mortality coefficient $\mu = 0.05$ and above mentioned parameters with fixed values of $\beta = 0.00218, \gamma = 0.44$ with the initial conditions $IC = [762, 1]$ which yields $\frac{\mathrm{d}S}{\mathrm{d}t} \approx -2$ individuals per day i.e., the epidemic commenced with one sick individual, with two more getting infected one day later. For our purposes the observation is on time interval of three hundred sixty days.

Following figures show the S,I and R compartments depending on order α

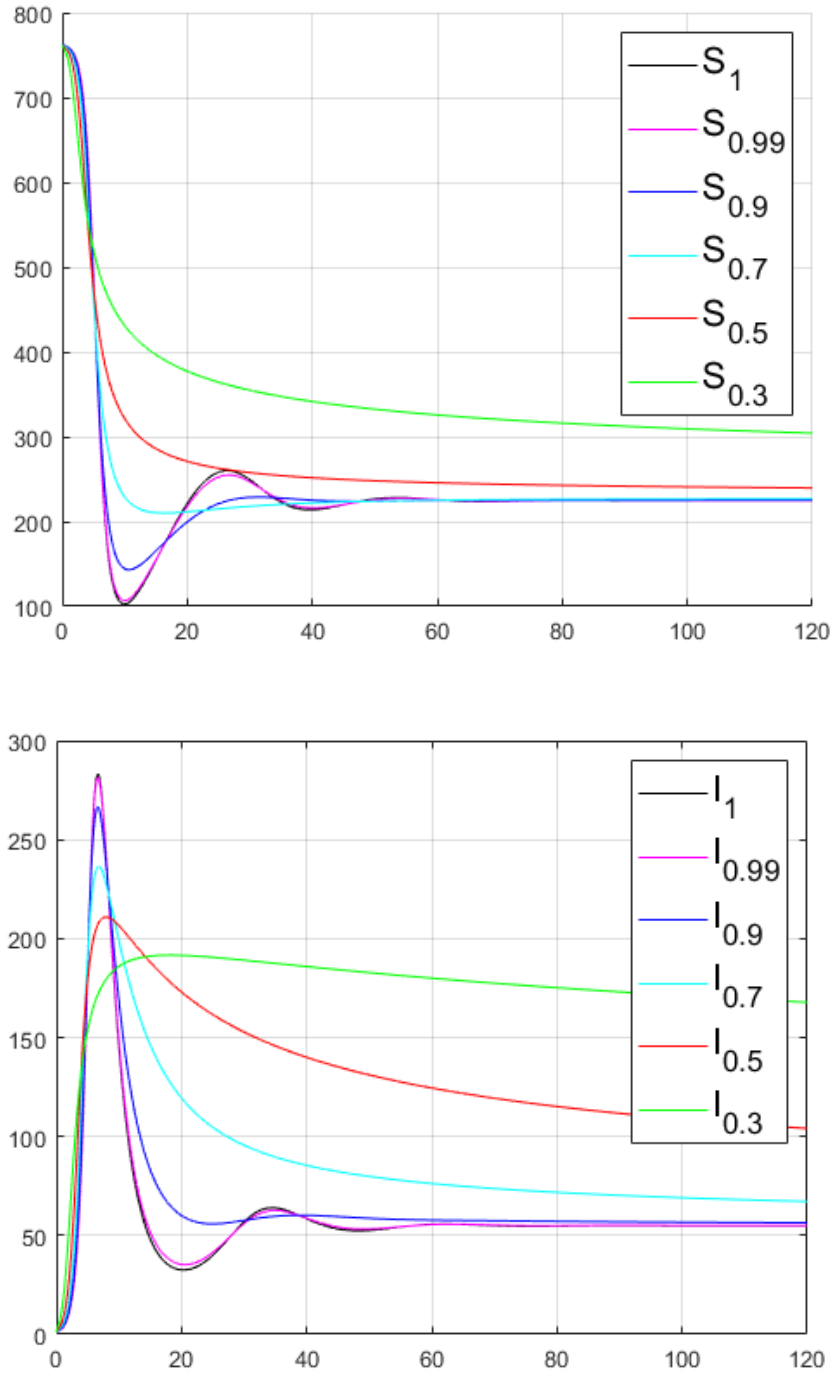

Figure 8: Susceptible and infected compartment, respectively.

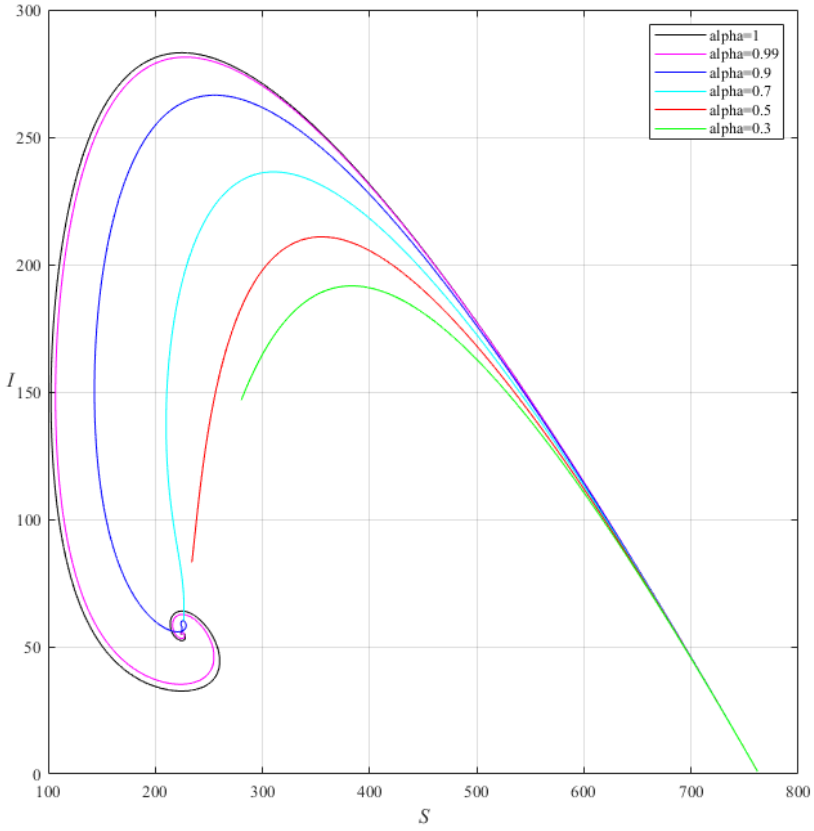

Figure 9: SI phase plane

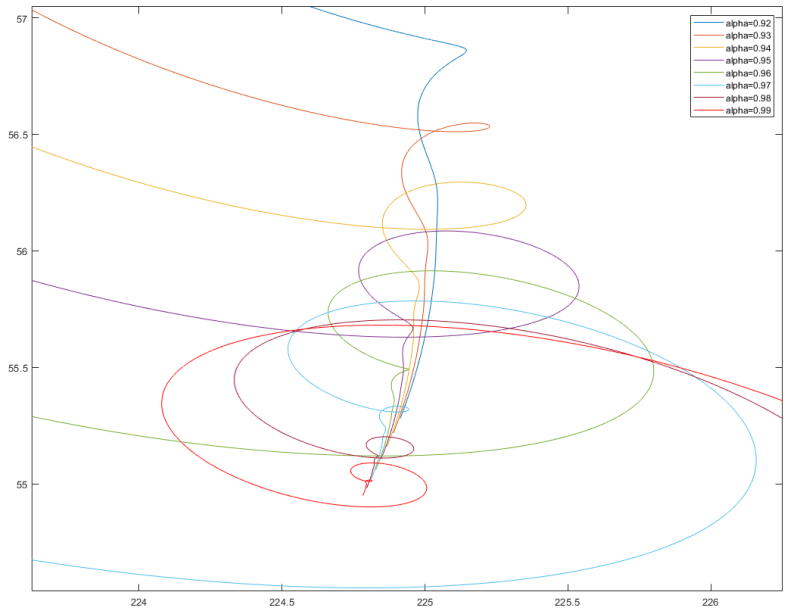

Figure 10: A closer detail of $\alpha \in < 0.99; 0.92 >$

Both, susceptible and infected, "compartments" show that with decreasing order α decreases curvature and "creaseness" of the function for both cases until it smooths out. The recovered compartment does not change irregardless of order α, thus confirming its redundancy. On our plots of the SI phase plane we can observe a transition from stable focus equilibrium point to a stable node like one. Moreover, on the interval $\alpha \in < 0.99; 0.92 >$ we observe self-intersecting trajectories which occur no longer from the order $\alpha = 0.92$ and less.

5 Numerical Solution

In this Section we shall introduce a method applicable on wide range of equations due to quite general assumptions. We shall focus on initial value problems with one Caputo derivative not limiting ourselves to only autonomous equations, however, we shall consider dimension $n = 1$ and $\alpha \in (0, 1)$, i.e.,

$$\begin{aligned} {}^{C}D^{\alpha}y(x) &= \mathrm{f}(x, y(x)), \\ y(0) &= y_0 \end{aligned} \tag{5.1}$$

on finite interval $[0, T]$, where T is an adequate positive number.

5.1 The Predictor-Corrector Method

With help of Lem.3.13 we can state that solution of (5.1) is equivalent to the Volterra integral equation

$$y(x) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^x (x - \tau)^{\alpha-1} f(\tau, y(\tau)) \mathrm{d}\tau \tag{5.2}$$

with $\alpha \in (0, 1)$, meaning, that a continuous function is a solution of (5.1) if and only if it is a solution of (5.2).

Let us now recall the classical one-step Adams–Bashforth–Moulton for the first–order equations. Assume an initial-value problem for the first–order differential equation

$$\begin{aligned} y'(x) &= f(x, y(x)), \\ y(0) &= y_0 \end{aligned} \tag{5.3}$$

with such function f that a unique solution exists on some time interval $[0, T]$. We shall also assume a uniform grid $\{\tau_n = nh : n = 0, 1, \ldots, N\}$ where N is an arbitrary integer and $h = \frac{T}{N}$. The core idea is, assuming having already calculated approximations $y_h \approx y(\tau_i)$ for $(i = 1, 2, \ldots, n)$, obtaining the approximation $y_h(\tau_{n+1})$ by means of the following equation

$$y(\tau_{n+1}) = y(\tau_n) + \int_{\tau_n}^{\tau_{n+1}} f(u, y(u)) \mathrm{d}u \tag{5.4}$$

by integrating (5.3) on the interval $[\tau_n, \tau_{n+1}]$. Since we know none of the expressions on the right–hand side of (5.4) precisely, we shall substitute $y(\tau_n)$ by known approximation $y_h(\tau_n)$ instead. The integral is then replaced by trapezoidal quadrature formula thus giving us the equation for the desired approximation as follows

$$y_h(\tau_{n+1}) = y_h(\tau_n) + \frac{h}{2}[f(\tau_n, y(\tau_n)) + f(\tau_{n+1}, y(\tau_{n+1}))], \tag{5.5}$$

where $y(\tau_n)$ and $y(\tau_{n+1}$ are replaced by their approximations $y_h(\tau_n)$ and $y_h(\tau_{n+1})$, respectively. This provides us with with the implicit one-step Adams–Moulton method. However, the unknown $y_h(\tau_{n+1})$ appears on both sides and due to nonlinear characteristics of function f, we are, in general, unable to solve for $y_h(\tau_{n+1})$ directly. Nevertheless, we may use (5.5) in iterative process by inputting an estimate approximation for $y_h(\tau_{n+1})$ in the right-hand side for determination of better approximation which could be used.

The estimate approximation is so-called *predictor* and is obtained similarly with only difference being replacing trapezoidal rule by rectangle one giving us the explicit or forward Euler, or even one-step Adams–Bashforth method

$$y_h^P(\tau_{n+1}) = y_h(\tau_n) + hf(\tau_n, y_h(\tau_n)). \tag{5.6}$$

This process and

$$y_h(\tau_{n+1}) = y_h(\tau_n) + \frac{h}{2}[f(\tau_n, y_h(\tau_n)) + f(\tau_{n+1}, y_h^P(\tau_{n+1}))], \tag{5.7}$$

known as the one–step Adams–Bashforth–Moulton method is convergent of order 2, i.e.,

$$\max_{n=1,2,\ldots,N} |y(\tau_n) - y_h(\tau_n)| = O(h^2).$$

This method is said to be of the PECE (Predict, Evaluate, Correct, Evaluate) kind because, in our case we would, at first, calculate the predictor in equation (5.6), then evaluate $f(\tau_{n+1}, y_h^P(\tau_{n+1}))$, using this for corrector calculation in equation (5.7), and at last evaluating $f(\tau_{n+1}, y_h(\tau_{n+1}))$.

At this point we have introduced the essentials of the classical method. Now, our attempt shall be to carry over these ideas to the fractional problems. Our main focus is to obtain an equation similar to (5.4). Such equation indeed exists, namely the (5.2), however, with one difference being the lower bound of integration starting at zero, not at τ_n which is a consequence of non-local structure of the Caputo differential operator. Nevertheless, this does not cause any serious problems in our generalisation of this method. We shall simply approximate the integral $\int_0^{\tau_{n+1}} (\tau_{n+1} - u)^{\alpha-1} g(u)\mathrm{d}u$ by substitution of the function g with linear interpolant with knots chosen at τ_j $(j = 0, 1, \ldots, n+1)$ and we shall use the product trapezoidal quadrature formula. In other words,

$$\int_0^{\tau_{n+1}} (\tau_{n+1} - u)^{\alpha-1} g(u)\mathrm{d}u \approx \frac{h^\alpha}{\alpha(\alpha+1)} \sum_{j=0}^{n+1} a_{j,n+1} g(\tau_j),$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, & \text{if j=0,} \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & \text{if } 1 \le \text{j} \le \text{n,} \\ 1, & \text{if j=n+1.} \end{cases} \tag{5.8}$$

Now, we obtain the fractional variant of the one-step Adams-Moulton method, i.e., the corrector formula, which is

$$y_h(\tau_{n+1}) = y_0 + \frac{h^\alpha}{\Gamma(n+2)} f(\tau_{n+1}, y_h^P(\tau_{n+1})) + \frac{h^\alpha}{\Gamma(n+2)} \sum_{j=0}^{n} a_{j,n+1} f(\tau_j, y_h(\tau_j)), \tag{5.9}$$

where identity $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ was used and the fact that $a_{n+1,n+1} = 1$.

The remaining problem is to determine the predictor formula in order to evaluate $y_h^P(\tau_{n+1})$. The idea of generalisation of the one–step Adams–Bashforth method is identical to one described above for the Adams–Moulton one, we replace the integral on the right–hand side of the equation (5.2) by the product rectangle rule, where now

$$b_{j,n+1} = \frac{h^\alpha}{\alpha}((n+1-j)^\alpha - (n-j)^\alpha) \tag{5.10}$$

and, therefore, the final formula for the predictor is

$$y_h^P(\tau_{n+1}) = y_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(\tau_j, y_h(\tau_j)). \tag{5.11}$$

The fractional Adams–Bashforth–Moulton method is now fully described by equations (5.11) and (5.9) with weights $a_{j,n+1}$ and $b_{j,n+1}$ defined by (5.8) and (5.10), respectively. The error of this method is expected to behave as

$$\max_{1 \le j \le N} |y(\tau_j) - y_h(\tau_j)| = O(h^p),$$

where

$$p = \min\{2, 1 + \alpha\}.$$

The reason for this specific form of the exponent p is that it may be proved that p must be the minimum of the order of the corrector (2 in our case) and the predictor method (1 in our case) plus the order of the differential operator. For the case $\alpha = 1$, the $p = 2$ which is equivalent to the integer order method mentioned earlier.

Our description of the method can be easily expanded to the higher dimensions, i.e., $n \ge 2$, by replacing $y \in \mathbb{R}$ by the vector $\boldsymbol{y} \in \mathbb{R}^n$ and instead of functions the vector functions are considered.

6 Summary

In this thesis we studied properties of autonomous systems, which are specific class of dynamical ones. Our goal was to point out several differences between the classical, i.e., first order systems and systems of fractional order between zero and one.

At the beginning we recalled some necessary definitions and theorems from the field of classical calculus. Here, we also mentioned techniques used for stability analysis of non-linear systems such as Routh–Hurwitz criterion, linearisation theorem and Lyapunov theorem.

Then, in Section 3, we preliminarily recalled some higher functions and techniques from classical calculus in order to introduce some standard approaches to the definition of fractional derivatives and integrals, namely, the Riemann–Liouville and the Caputo approach. This introduction was then followed by similar description of fractional autonomous systems and their stability. First of all, it is important to point out that in case of fractional system no situation as periodic solution can occur, however, there are studies of existence of asymptotic periodic solutions. Lastly, such phenomenon as self-intersecting trajectories, namely, cusps and multiple points, may occur.

For verification of our theoretical results, the epidemiological SIR model with vital dynamics, also referred to as Hethcote's model, was chosen. In this thesis we focused on its local stability which was determined with help of linearisation theorem, Routh–Hurwitz criterion, eigenvalues of characteristic equation of the system and, for the fractional case, the key factor of order α.

The analysis confirmed our expectation for fractional case that the stability remained. However, it turns out that a type transition of endemic equilibrium occurs, from stable focus to stable node, to be precise, and even a self-intersecting phenomenon can be observed on the interval $\alpha \in < 0.99; 0.93 >$.

This work could be considered as rudimentary for further studies of epidemiological models, e.g., global stability of the SIR model and fractional dynamical systems, not necessarily autonomous ones.

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Appendix

```
%% ''HETHCOTE'S'' FRACTIONAL SIR MODEL
%% input (parameters of the system)
%DATA: Influenza Epidemic in an English Boarding School, 1978
%LINK: https://www.math.unm.edu/¬sulsky/mathcamp/ApplyData.pdf

N=763;          % number of individuals in a population
beta=0.00218;   % proportionality for the disease transmission rate
gam=0.44;       % rate of recovery from the disease
mu=0.05;        % mortality coeffcient
alpha=1;        % order of Caputo derivative

%% output (SIR vector of the state variables)
% S(t) - number of susceptibles at the time t
% I(t) - number of infectives at the time t
% R(t) - number of recovered individuals at the time t (R=N-S-I)

%% numerical solution data
t0=0;           % initial time
T=360;          % final time
h=0.001;        % time step
IC=[762;1];     % initial conditions vector

fde = @(t,y)[-beta*y(1)*y(2)+mu*(N-y(1)) ; beta*y(1)*y(2)-(gam+mu)*y(2)];

%% Fractional PECE method by R. Garrappa (fde12)
[t,SI]=fde12(alpha,fde,t0,T,IC,h);
%% plotting results
%PHASE PORTRAIT
plot(SI(1,:),SI(2,:), '-', 'color', 'black')
legend('alpha=1','alpha=0.99','alpha=0.9','alpha=0.7','alpha=0.5','alpha=0.3')
hold on
grid on
set(gca, 'FontName', 'Times New Roman')
xlabel('S','FontName', 'Times New ...
    Roman','FontAngle','italic','fontsize',12);
ylabel('I','FontName', 'Times New ...
    Roman','FontAngle','italic','fontsize',12);
set(get(gca,'ylabel'),'rotation',0)
hold off

m=size(SI,2);
NN=N*ones(1,m);
R=NN-SI(1,:)-SI(2,:);

% SIR figure
plot(t,SI(1,:),t,SI(2,:),t,R)
legend('S','I','R','Location','northeast','Orientation','vertical')
grid on
set(gca, 'FontName', 'Times New Roman')
xlabel('t','FontName', 'Times New ...
    Roman','FontAngle','italic','fontsize',12);
ylabel('SIR','FontAngle', 'italic','fontsize',12)
```