UNIVERZITA PALACKÉHO V OLOMOUCI
Přírodovědecká fakulta
Katedra optiky

VOJTĚCH KUPČÍK
III. ročník - prezenční studium

Obor: Optika a optoelektronika

KVANTOVÁ PROVÁZANOST SVĚTLA A MECHANICKÉHO OSCILÁTORU

Bakalářská práce

Vedoucí práce: Doc. Mgr. Radim Filip, Ph.D.

OLOMOUC 2012

PALACKÝ UNIVERSITY OLOMOUC
Faculty of Science
Department of Optics

VOJTĚCH KUPČÍK
3^{rd} grade - Full-time study

Study Program: Optics and optoelectronics

QUANTUM ENTANGLEMENT OF LIGHT AND MECHANICAL OSCILLATOR
Bachelor thesis

Supervisor: Doc. Mgr. Radim Filip, Ph.D.

OLOMOUC 2012

Tímto prohlašuji, že jsem tuto bakalářskou práci vypracoval samostatně a použil jen uvedených zdrojů a literatury. Tímto také souhlasím se zveřejněním práce.

V Olomouci dne 31.7.2012 Vojtěch Kupčík

I hereby declare that this bachelor thesis is completely my own work and that I used only the cited sources. I also agree with publishing of this thesis.

Olomouc, July 31, 2012 Vojtěch Kupčík

Tímto bych chtěl poděkovat Doc. Mgr. Radimu Filipovi, Ph.D., vedoucímu bakalářské práce, za mnoho užitečných rad, trpělivost a podporu.

I would like to thank Doc. Mgr. Radim Filip, Ph.D., the supervisor of this bachelor thesis for numerous useful advices, patience and support.

Contents

Chapter 1

ABSTRACT

Quantum correlations between two physically different systems are an interesting phenomenon to study. They can provide an operational tools to manipulate limitedly accessible experimental platforms, such as noisy macroscopic atomic or mechanical systems, at quantum noise level using well controllable strongly quantum probes. A very popular probe is light, mainly because of its nonclassical state preparation, basic measurement-induced manipulations and detection, that have been experimentally achieved very successfully. If we consider squeezing of light as a basic resource, it enables to study an interesting interface between a non-classical light and a very classical matter system. The classicality of matter system is based on a classical thermal noise at a room temperature. In principal, any system can be even cooled down to start detecting quantum features, but in this work we keep a noisy character of the matter as a cardinal point of our analysis.

Chapter 2

THEORY

2.1 INTRODUCTION TO THE PROBLEM

One of the most popular ideas of experiments in physics is the Schrödinger´s cat, which was devised by Austrian physicist Erwin Schrödinger in 1935. It is a thought experiment and because of the meaning that this experiment illustrates, it is sometimes described as a paradox.

To clarify the problem we follow the popular interpretation which can be found in [1]. It is useful to imagine a non-transparent box. Inside of it we place a cat along with a device that contains radioactive source, a hammer and a flask which contains a poison. The experiment is constructed in the way that after some time there is a 50 percent probability of decaying the radioactive atom. If an internal counter detects radiation, the hammer breaks the flask with the poison and the cat is killed. Because of the non-transparency of our box an observer does not know what it is happening inside, whether or not an atom has decayed, and consequently, whether the poison has been released, and the cat killed. According to the law of quantum physics and because we have no inmorfation about the inside, the cat is both dead and alive. And this we call a *superposition* of states between atom and cat. In the momment when we open the box, the cat becomes one or the other (alive or dead) and the superposition is lost. We influence the state of the system by opening the box, and therefore we make the wave function collapse. On the other hand, by a measurement on the atom we can prepare the cat in a superposition of both alive and dead being strongly non-classical from classical world perspective.

If we look at the experiment in detail, there is an exciting fact to think about. It contains a connection between a macrosystem described by classical physics and a microsystem described by quantum laws. Actually, it enables

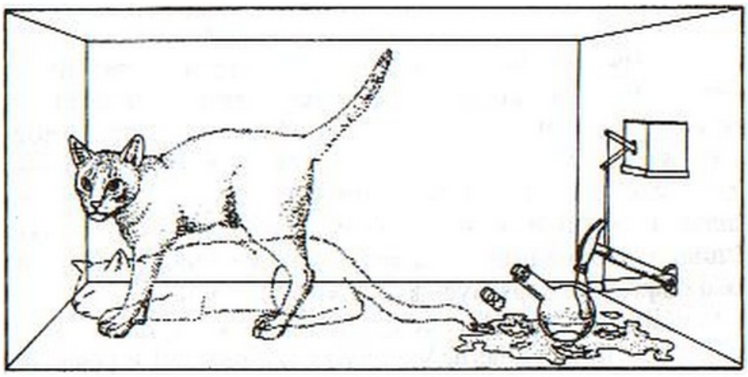

Figure 2.1: Schrödinger´s cat in a superposition state - dead or alive. [1]

to transpose the very non-classical character to the very classical system. And I would like to stress this idea as the main motivation and a starting point for this work. The term entanglement plays a huge role in describing two-system states after an interaction, and especially in this work, we use it very often. And it is an interesting thing that during the development of this experiment, Schrödinger coined this term(in German - Verschränkung).

We would also like to mention here another thought experiment which is called EPR paradox. It has some interesting features that we discussed during working up this thesis and it is undoubtedly useful to include it in the introduction. As it was not our main motivation, we introduce this very briefly. EPR is an abbreviation of three authors Einstein, Podolsky and Rosen. It is called paradox because of the meaning of the experiment, it should have shown that the theory of quantum mechanics is incomplete. They tried to show that it is possible to measure position and momentum of a quantum particle at the same time, which is opposite to what Heisenberg´s uncertainty principal says. Although, the EPR paradox has been fully explained, it stimulated a birth of the method of remote state preparation. The remote state preparation is able to prepare a set of eigenstates of complementary variables (like position and momentum) on system A by an adjustable measurement on system B, if A and B are quantum-mechanically correlated. By this method we will check the applicability of states that can be understood as the analogues of the Schrödinger´s cat states in real physics. We will focus on quantum states of light and matter system (mechanical oscillator, spin- oscillator of ensemble of atoms).

2.2 QUANTUM DESCRIPTION

2.2.1 CLASSICAL HARMONIC OSCILLATOR

The harmonic oscillator is one of the most prominent examples of a classical mechanical system. Mechanical oscillations can be seen all around us, they are a common form of motion in nature. For example, we can find them in microscopic objects such as molecules or in a more common example in engines, clocks and the sound that a quitar makes is also the consequence of the string oscillations. If we look at the principle, it is always the same: harmonic oscillations are some kind of repetitive variation of a variable around a value, that the system is getting closer and closer to ,in a period of time. The simple harmonic oscillator is described by the equation of motion

$$x(t) = Asin(\omega t + \varphi). \tag{2.1}$$

Here A is the amplitude, $\omega = 2\pi f$ is the oscillator´s eigenfrequency and φ is the phase factor related to the point of origin at $t = 0$. This is a description of an ideal classical harmonic oscillator, but we have to take into account fluctuations of the quadratures. Then both the position $x(t)$ and the conjugate momentum are stochastic variables. If the classical oscillator is described by classical physics, only technical thermal noise limits its dynamics.

2.2.2 QUANTUM MECHANICAL HARMONIC OSCILLATOR

In this introductory paragraph it is closely followed the standart derivation that can be found in [2]. Quantum mechanics changes the view on the classical harmonic oscillator. It is one of the simpliest physical modes that can be solved analytically, but already here we face some of the peculiar quantum features that make the quantum mechanics so interesting and also different from the classical physics. It is convenient and also usual to start with the classical Hamiltonian function from classical mechanics. We replace the classical variables x and p with their corresponding quantum operators $x \rightarrow \hat{x}$ and $p \rightarrow \hat{p} = -2i\frac{d}{dx}$ (for simplicity we use the notation without a hat for the quantum operators again). The quantum mechanical Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{m\omega_m^2 x^2}{2}, \tag{2.2}$$

where ω_m is the angular frequency of a classical oscillator. Both operators x and p are Hermitian. We can also use the notation of two non-Hermitian operators that are called creation a^+ and annihilation a operators

$$a = \sqrt{\frac{m\omega_m}{2\hbar}}(x + \frac{ip}{m\omega_m}) \tag{2.3}$$

$$a^+ = \sqrt{\frac{m\omega_m}{2\hbar}}(x - \frac{ip}{m\omega_m}). \tag{2.4}$$

As x and p fullfil the commutation relation $[x, p] = i\hbar$, a and a^+ obey the following relations

$$[a, a^+] = 1 \tag{2.5}$$

$$[a, a] = \left[a^+, a^+\right] = 0. \tag{2.6}$$

The Hamiltonian can be now expressed in terms of these operators

$$H = \hbar\omega_m(a^+a + \frac{1}{2}). \tag{2.7}$$

We denote an energy eigenstate of a^+a by its eigenvalue n, so

$$a^+a|n\rangle = n|n\rangle. \tag{2.8}$$

Using (2.7) and (2.8) we obtain

$$H|n\rangle = \hbar\omega_m(n + \frac{1}{2})|n\rangle, \tag{2.9}$$

therefore

$$E_n = \hbar\omega_m(n + \frac{1}{2}). \tag{2.10}$$

We now determine the term *phonon*. By a phonon we mean a quantum on energy, or more precisely it is a quantum of mechanical oscillations, vibrations of a solid state system. The energy is $E = \hbar\omega_m$ with ω_m being the frequency of the oscillator and $\hbar$ stands for the reduced Planck´s constant. We also introduce Fock states that are eigenstates of the Hamiltonian of the quantum mechanical oscillator with the notation $|n_M\rangle$ with n_M an integer value. It means that there are n_M quanta of mechanical oscillations. For the EM fiels we use the term *photon*, it is quite analogous to this case.

2.3 QUANTIZATION OF THE FREE EM FIELD. COHERENT, SQUEZZED AND THERMAL STATES

2.3.1 QUANTIZATION OF THE EM FIELD

The classical description of the free electromagnetic field is given by Maxwell´s equations and it is also convenient to start with them here. These equations represent the relation between the electric and magnetic field vectors ($\mathbf{E}$, $\mathbf{H}$) and the displacement and inductive vectors ($\mathbf{D}$, $\mathbf{B}$), which is illustrative from the form that they have:

$$\begin{aligned} \bigtriangledown \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t}, \\ \bigtriangledown \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \bigtriangledown \bullet \mathbf{B} &= 0, \\ \bigtriangledown \bullet \mathbf{D} &= 0. \end{aligned} \tag{2.11}$$

There are two other useful ralations connected with Maxwell´s equations, we call them the constitutive relations:

$$\begin{aligned} \mathbf{B} &= \mu_0 \mathbf{H}, \\ \mathbf{D} &= \epsilon_0 \mathbf{E}, \end{aligned} \tag{2.12}$$

where μ_0 and ϵ_0 stand for the free space permeability and permitivity, respectively. These two constants satisfy $\mu_0 \epsilon_0 = \frac{1}{c^2}$ where c is the speed of light in vacuum. Using Maxwell´s equations and constitutive relations we derive, that $\mathbf{E}(\mathbf{r}$,t) satisfies the wave equation

$$\bigtriangledown^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \tag{2.13}$$

where we also used $\bigtriangledown \times (\bigtriangledown \times \mathbf{E}) = \bigtriangledown(\bigtriangledown \bullet \mathbf{E}) - \bigtriangledown^2 \mathbf{E}$. The electric and the magnetic fields can be associated with a vector potencial $\mathbf{A}$

$$\mathbf{B} = \bigtriangledown \times \mathbf{A} \tag{2.14}$$

$$\mathbf{E} = \frac{\partial \mathbf{A}}{\partial t} \tag{2.15}$$

that also satisfies the wave equation

$$\bigtriangledown^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0. \tag{2.16}$$

The solution of the wave equation takes the form of a Fourier series

$$A = \sum_k (A_k(t) u_k(\mathbf{r}) + A_k^*(t) u_k^*(\mathbf{r})), \tag{2.17}$$

where $u_k(r)$ are the spatial mode functions forming orthogonal basis. Inserting this solution back into the original equation leads us to the fact that each of the Fourier components has to satisfy the harmonic oscillator equation

$$\frac{\partial^2 A_k}{\partial t^2} = -\omega_k^2 A_k. \tag{2.18}$$

To quantize the EM field our purpose is to obtain an expression for a single mode of the field that corresponds to a simple harmonic oscillator in order to be identify with it. First, we express $\mathbf{E}$ and $\mathbf{B}$ using $\mathbf{A}$ and assuming $\mathbf{A}(t) = \mathbf{A}e^{iw_L t}$ and plane wave modes, that is

$$E_k = i\omega_k (A_k e^{-i\omega_k t + ikr} - A_k^* e^{i\omega_k t - ikr}) \tag{2.19}$$

$$B_k = ik \times (A_k e^{-i\omega_k t + ikr} - A_k^* e^{i\omega_k t - ikr}). \tag{2.20}$$

The energy of a single mode k can be now expressed

$$W_k = \frac{1}{2} \int (\epsilon_0 E_k^2 + \mu_0^{-1} B_k^2) dV, \tag{2.21}$$

which can be reduced in terms of A_k to

$$W_k = 2\epsilon_0 V \omega_k^2 A_k A_k^*. \tag{2.22}$$

This looks very much like the energy of a quantum mechanic oscillator (which was shown earlier). To illustrate the exact correspondance it is convenient to define

$$A_k = (4\epsilon_0 V \omega_k^2)^{-\frac{1}{2}} (\omega_k Q_k + iP_k) \epsilon_k, \tag{2.23}$$

where $\epsilon_{\mathbf{k}}$ is the polarization of light. We finally obtain

$$W_{\mathbf{k}} = \frac{1}{2}(P_{\mathbf{k}}^2 + \omega_{\mathbf{k}}^2 Q_{\mathbf{k}}^2), \tag{2.24}$$

which can be identified as a unit mass simmple harmonic oscillator. So far, it was all classical, but we achieved the expression that resembles a simple harmonic oscillator and we know how to quantize a simple harmonic oscillator. For more detailed derivation, see [3] or [4].

Here we introduce the term *photon*. By a photon it is meant a quantum of light or generally a quantum of the eloctromagnetic field. The energy of one photon is $E = \hbar\omega_L$, where $\hbar$ is the reduced Planck´s constant and ω_L stands for the frequency of light. And analogically to the previous case we introduce the notation $|n_L\rangle$ standing for a Fock state.

2.3.2 COHERENT, SQUEEZED AND THERMAL STATES

For our plans, it is convenient to introduce the definitions of classical and nonclassical states. For this purpose it is necessary to know the term *coherent state*. These are states that create a connection between quantum and classical physics in the sense that they are the most appropriate quantum representation of a classical state. Let´s start with the classical oscillator, it can be described by the correlation function for all possible times

$$\langle \alpha^{*n}(t)\alpha^{m}(t')\rangle, \tag{2.25}$$

with an amplitude α. Moving to the quantum oscillator we deal with the creation a^{+} and the annihilation a operators

$$\langle a^{+n}(t)a^{m}(t')\rangle. \tag{2.26}$$

These two expressions should correspond

$$\langle \alpha^{*n}(t)\alpha^{m}(\acute{t})\rangle = \langle \psi|a^{+n}(t)a^{m}(t')|\psi\rangle, \tag{2.27}$$

therefore coherent state $|\alpha\rangle$ is an eigenstate of the annihilation operator $a|\alpha\rangle = \alpha|\alpha\rangle$.

The definition says that classical state is a mixture of coherent states (2.28)

$$\rho = \int P(\alpha)|\alpha\rangle\langle\alpha|d^{2}\alpha, \tag{2.28}$$

whereas non-classical states $\rho \neq \int P(\alpha)|\alpha\rangle\langle\alpha|d^{2}\alpha$ are not compatible with such an expansion.

The huge development of quantum theory together with the invention of laser were the beginning motivation to widely study the states of the field that correspond to the classical oscillations the best. An important consequence of the quantization of the field is the uncertainty relation, which must be satisfied by the conjugate field variables. It is reasonable to propose that the wave function describing the classical oscillations most closely must have minimum uncertainty. From the classical theory we know that classical oscillations have a well-defined amplitude and phase. But when we apply the quantum mechanical approach, this is no more the case. If we study the oscillations in QM way there are fluctuations associated with both amplitude and phase. Assuming the oscillations in a state for a given time with a perfectly kmown amplitude, we have to count with the fact that there is absolutely uncertain phase. Equivalently, we can describe the field in terms of the two conjugate quadrature components. They are limitted by fundamental Heisenberg uncertainty principle which is in the form:

$$V_x V_p \geq 1, \tag{2.29}$$

where V_x and V_p stand for the variance in the position $x$or in the momentum p, respectively. Both variables x and p for both optical and mechanical oscillator is identically decomposing the annihilation operator $a = \frac{x+ip}{2}$ of photons and phonons. A field in a *coherent state* satisfies the minimum-uncertainty equation and also has equal uncertainties in both quadrature components. In principal, there is a possibility of generating a state that satisfies the lowest limit for uncertainty but doesn´t have symmetrical fluctuations. It means that the field in this state has unit variance of the quadrature components reduced below unity. This fact is compensated by increasing variance in the conjugate quadrature, such that the Heisenberg uncertainty relation is not violated. Such states of a radiation field are called *squeezed states*, which can be obtained by unitary trasformation of the coordinate and momentum variables.

$$\begin{aligned} x' &= \lambda x \\ p' &= \lambda^{-1} p; \\ \lambda \in (0, \infty); V_x V_p &= 1 \end{aligned} \tag{2.30}$$

Then there are two cases, if $\lambda < 1$ the state is squeezed in x, or $\lambda > 1$ and the is state squeezed in p. A quadrature component with fluctuations that are below the standard quantum limit, has attractive aplications in gravitational wave detection, optical communication and photon detection techniques. It is already a non-classical state if V_x or V_p becomes less than unity. Recently, the optical process in crystals is able to generate the minimum variance $V \approx 0.1$.

On the other hand, a very classical state corresponding to the oscillator with large thermal fluctuations can be desribed by the form (2.28). The oscillator is said to be in a *thermal state* if it has symmetrical variances in the quadratures and both are greater than one. For more detailed discussion of coherent, squeezed and thermal states, see [3].

It is illustrative to include a dependance of the thermal noise V_N on the ambient temperature and the frequency of the mechanical ocsillator, which is illustrated in FIG.2.6. As phonon statistics is described by Bose-Einstein distribution, we can write

$$a^+ a = \langle n \rangle = \frac{1}{e^{\frac{\hbar \omega_m}{kT}} - 1}, \tag{2.31}$$

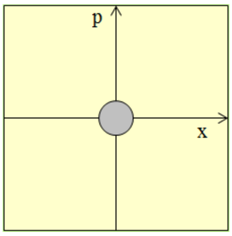

Figure 2.2: vacuum [5]

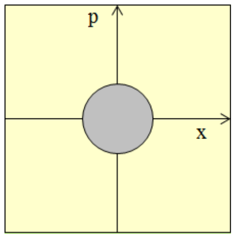

Figure 2.3: a thermal state [5]

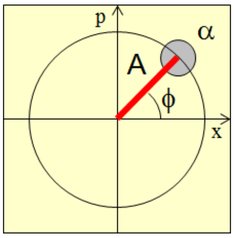

Figure 2.4: acoherent state [5]

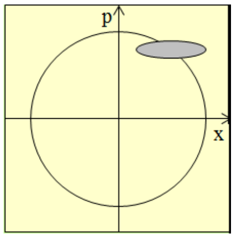

Figure 2.5: a squeezed state [5]

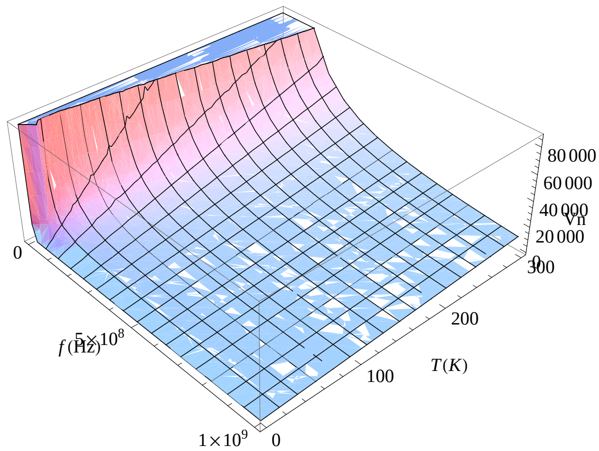

Figure 2.6: The dependance of the thermal noise V_N on the ambient temperature T and the frequency f.

where $\hbar$ is a reduced Planck constant, ω_m is the oscillator frequency, k is a Boltzmann constant and T stands for the ambient temperature. If we use the form of the creation $a^+ = \frac{x-ip}{2}$ and the annihiliation $a = \frac{x+ip}{2}$ operator and consider the commutator $[x,p] = 2i$ and the fact that $\langle x^2 \rangle = \langle p^2 \rangle = V_N$, we obtain the expression that enables to determine the value of V_N

$$V_N = \frac{2}{e^{\frac{\hbar\omega_m}{kT}} - 1} + 1. \tag{2.32}$$

We include this for better imagination of scale we use in graphs that illustrates our results.

2.4 INTERACTION OF LIGHT AND MECHANICAL OSCILLATOR

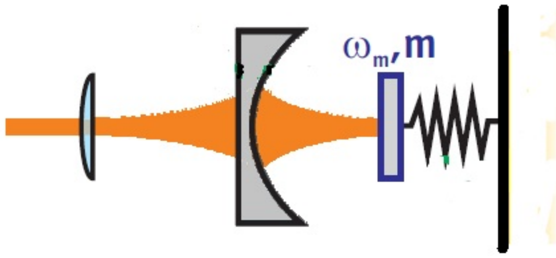

Figure 2.7: Radiation pressure interaction: light is coupled through a input mirror into a resonator with a movable mirror in the back [6].

We now consider a cavity with a movable mirror inside. Let the mirror interact with light. Each photon transfers some kind of energy onto the mirror and thus makes the mirror vibrate (slightly changes the position of the mirror). We say that the light transfers energy through a radiation pressure. The radiation pressure interaction can now be used to modify the dynamics of the mechanical oscillator. We can model this situation as a coupling between a mechanical and an optical oscillator, already described above.

We consider here the case that we are able to detect a single cavity mode and also a single mechanical mode only and that the mechanical modes do not couple to each other, i.e. $\omega_m \ll c/2L$, where L is the length of the cavity and ω_m is the frequency of mecanical oscillator. The full Hamiltonian of the system is

$$H = 2\omega_c a^+ a + \omega_m(p_m^2 + x_m^2) - 2g_0 a^+ a x_m + 2iE(a^+ e^{-i\omega_l t} - a e^{i\omega_l t}). \tag{2.33}$$

Here a and a^+ are the annihilation and creation operators of the cavity field, x_m and p_m stand for position and momentum of the mechanical oscillators,

ω_c represents the frequency of the cavity, ω_m the frequency of the mechanical oscillator and ω_l the frequency of the light. For the purpose of this thesis it is not necessary to express E and g_0 explicitly, but let´s mention that E is related to the input light power and g_0 is the optomechanical coupling rate. Now, let´s look at the full Hamiltonian again and clarify what each part of (2.33) represents. The first term is the energy of the cavity field, while the second expresses the corresponding amount of the mechanical mode. The third part is the one which we concentrate on and it stands for the optomechanical interaction Hamiltonian H_{rp} and the last one characterises the coupling of the light to the cavity.

To enhance the strength of the interaction we use a strong classical seed beam with an amplitude α_s To analyze H_{rp} we assume $\alpha_s \gg 1$ and therefore one can write $a \rightarrow \alpha_s + a$ and $a^+ \rightarrow \alpha_s + a^+$, where $a(a^+)$ is now the new associated fluctuation operator. If we neglect higher order terms in the fluctuation operators, we obtain

$$H_{rp} \propto (a + a^+)(b + b^+), \tag{2.34}$$

where we have used the expression of x_m in terms of its operators b and b^+. Based on this reduction of a cubic nonlinearity to a quadratic nonlinearity we can significantly simplify our discussion. The expression (2.34) corresponds to the QND type of interaction. Because we are still considering light and matter as the oscillators we can express them again as waves with the appropriate frequency. Using the so-called rotating wave approximation we can, in principal, achieve $\omega_m = \omega_l$ or $\omega_m = -\omega_l$ and therefore obtain from (2.34) either the interaction Hamiltonian

$$H \propto ab^+ + a^+b, \tag{2.35}$$

or a very different interaction Hamiltonian

$$H \propto ab + a^+b^+. \tag{2.36}$$

They describe two different processes, (2.35) stands for a passive interaction when two systems just change their energy, it corresponds to the Hamiltonian of a beam splitter, whereas (2.36) represents an active process of energy flow through both systems, it represents the Hamiltonian of an amplifier. We have successfully aproached the three interaction Hamiltonians (2.34),(2.35),(2.36) that represent the three prominent types of interactions that we analyze via characteristic features. More detailed analysis can be found in [6].

2.5 ANALOGY TO ATOMIC ENSEMBLES

There is an interesting analogy between the optomechanical interaction described earlier and the interaction between light and atomic ensembles. We describe it here very briefly since it can be another experimental platform for which our results will be applicable. We explain the expressions for the interaction Hamiltonian of the three types of interaction using another approach that correspond to [7]. It is typical and also illustrative to use the Λ-model. In FIG.(2.8)the atoms are prepared in the ground state coupled to

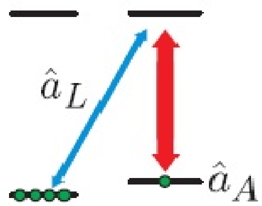

Figure 2.8: beam splitter type of interaction [7]

the quantum field a_L, the bold line represents a strong coupling field. The role of a strong classical coupling field here is to shift the atomic state without a change of any mode of radiation. With a large detunning from the optically excited states, the interaction Hamiltonian for such a system can be written as

$$H \propto a_L a_A^+ + a_L^+ a_A. \tag{2.37}$$

The intuitive picture is clear: if an excitation of a single photon is removed (annihilated)a_L, a single collective atomic excitation is created a_A^+. This Hamiltonian is often reffered as the beam splitter. It is because this type of interaction mixes atom states and light states at the input as the beam splitter do. The refflection coeficient of unity would correspond to a perfect state swapping between the light and atoms. In FIG.(2.9) we can see a very similar structure but now the fields are arranged in a way that is used for a simultaneous generation of photon and atomis excitations. The appropriate Hamiltonian takes the form

$$H \propto a_L a_A + a_L^+ a_A^+. \tag{2.38}$$

This Hamiltonian is formally equivalent to the amplifier type of interaction(2.36). And the last type is illustrated in FIG.(2.10). In principal, it is a combination of the two previous cases with equal coupling constant. The Hamiltonian reads

$$H \propto (a_L + a_L^+)(a_A + a_A^+) \tag{2.39}$$

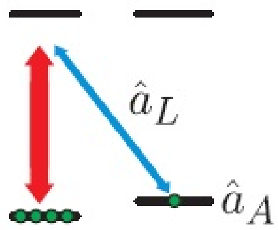

Figure 2.9: amplifier type of interaction [7]

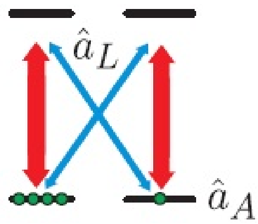

Figure 2.10: quantum non-demolition type of interaction [7]

and this type is called quantum non-demolition interaction. If we look at the results that we derived, they are in a perfect agreement with those derived in previous section, (2.35) corresponds to (2.37), (2.36) is equivalent to (2.38), and (2.34) is the same as (2.43). An ensemble of atoms is a huge amount of atoms where the oscillating factor is the collective spin. The earlier analysis contained in [7] shows that we can treat these ensembles as the linear harmonic oscillator.

2.6 QUADRATURE PICTURES OF INTERACTIONS

Mechanical systems and atomic ensembles can be considered as a quantum harmonic oscillator, similarly to a single mode of light. They are also prominent examples of macroscopic systems. If we assume a regime of a strong optical pumping, when there is enough strong and fast coupling between light and matter to be almost unitary, it still keeps much lower strength than a perfect swap of quantum states between light and matter would require.This coupling can be modificated among very different types of quadratic interactions. It includes all the three typical quadratic interactions: beam-splitter (BS) type, amplifier (AMP) type and quantum non-demolition (QND)type of the couplings.

We describe the interaction between light and matter as the simplest but weak Gaussian interaction between two different linear harmonic quantum oscillators described by X_L, P_L (light) and X_M and P_M (matter) of the complementary quadrature variables. They satisfy $[X, P] = 2i$. Our consideration is based on the fact that light is prepared in a squeezed state. The amount of squeezing in light represents the non-classicality. We denote $V_S < 1$ a variance of squeezing of light and on the other hand, $V_N > 1$ a variance of the thermal noise of matter.

Ideal symmetrical unitary BS interaction can be always tailored by relative phase shifts to transformation

$$\begin{aligned} X_L' &= \sqrt{T}X_L - \sqrt{1-T}X_M, \\ P_L' &= \sqrt{T}P_L - \sqrt{1-T}P_M, \\ X_M' &= \sqrt{T}X_M + \sqrt{1-T}X_L, \\ P_M' &= \sqrt{T}P_M + \sqrt{1-T}P_L, \end{aligned} \tag{2.40}$$

described in the Heisenberg picture, where $T \in (0, 1)$ is the gain of BS interaction. Another type of unitary symmetrical coupling is AMP interaction which can be adjusted by relative phase shifts to transformation described by

$$\begin{aligned} X_L' &= \sqrt{G}X_L - \sqrt{G-1}X_M, \\ P_L' &= \sqrt{G}P_L + \sqrt{G-1}P_M, \\ X_M' &= \sqrt{G}X_M - \sqrt{G-1}X_L, \\ P_M' &= \sqrt{G}P_M + \sqrt{G-1}P_L. \end{aligned} \tag{2.41}$$

The interaction gain $G > 1$ can be connected to the BS type coupling by a relation $G = \frac{1}{T}$, a weak BS or AMP interaction is expressed by the value of T or G close to the unity. On the other hand, asymmetrical unitary QND interaction can be adjusted to the two different transformations, either

$$\begin{aligned} X_L' = X_L \ \ , P_L' = P_L - KP_M, \\ X_M' = X_M + KX_L \ \ , P_M' = P_M \end{aligned} \tag{2.42}$$

transfering X_L to the matter oscillator, or

$$\begin{aligned} X_L' = X_L + KP_M \ \ , P_L' = P_L, \\ X_M' = X_M + KP_L \ \ , P_M' = P_M \end{aligned} \tag{2.43}$$

transferring P_L to the matter system, with QND gain $K > 0$. A weak interaction is represented by the value of K close to the zero.

2.7 THE COVARIANCE MATRIX, ENTANGLEMENT AND CONDITIONAL VARIANCE

To introduce these terms we follow [8].

2.7.1 GAUSSIAN STATES, THE COVARIANCE MATRIX

For describing a continuous variable (CV) system we use a Hilbert space resulting from the tensor product of infinite dimensional Fock spaces. Once again, using the annihilation and the creation operators is convenient here. Let these operators act on each Fock space and define the related quadrature phase operators $\hat{x}_j = (a_j + a_j^+)$, $\hat{p}_j = \frac{(a_j - a_j^+)}{i}$. We denote x_j and p_j the corresponding phase space variables. Let $\hat{X} = \hat{x}_1, \hat{p}_1, ..., \hat{x}_n, \hat{p}_n$ denote the vector of the operators $\hat{x}_j$ and $\hat{p}_j$. In terms of the symplectic form Ω, the commutation relations for the $\hat{X}_j$ take the form

$$[\hat{X}_j, \hat{X}_k] = 2i\Omega_{jk},$$

with

$$\Omega \equiv \bigoplus_{j=1}^{n} \omega, \omega \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Such a CV system could be also described by a positive three-class opertor (the density matrix ρ). In this work, we are interested only in Gaussian states and Gaussian interactions. By definition, the Gaussian state is the one that can be perfectly characterised by the first and the second statistical moments of the field operators. Therefore, an important feature of each system we use here, is the variance (which exactly consists of the first and the second statistal moment). An interaction is called Gaussian if it preserves the Gaussian character of both systems.

As the next important point, we introduce the covariance matrix (CM) formalism. The covariance matrix Γ consists of elements that are defined as

$$\Gamma_{jk} \equiv \frac{1}{2}\langle \hat{X}_j \hat{X}_k + \hat{X}_k \hat{X}_j \rangle - \langle \hat{X}_j \rangle \langle \hat{X}_k \rangle. \tag{2.44}$$

If we look at the definition, it is clear that such a matrix is symmetrical. In terms of the ladder operators and acocording to the earlier definition of the quadrature operators, the entries of CM are real numbers. The canonical commutation relations and the positivity of the density matrix ρ imply

$$\Gamma + i\Omega \geq 0, \tag{2.45}$$

meaning that all the eigenvalues of the matrix $(\Gamma + i\Omega)$ have to be greater or equal than zero. This expression is very powerful because it is a neccessary and sufficient condition the matrix Γ has to fullfil to be the CM corresponding to a physical state. Notice that Ineq.(2.45) is the constraint for any states (not only for the Gaussian states). For pure, uncorrelated states it reduces to a familiar Heisenberg principal. Such a restriction implies $\Gamma \geq 0$.

2.7.2 TWO-MODE STATES

In this work, we focus only on two-mode Gaussian states, therefore it is reasonable to mention some of their basic properties. It is useful to express the CM of two-mode states in terms of the three 2×2 matrices $\alpha, \beta and \gamma$

$$\Gamma \equiv \begin{pmatrix} \alpha & \gamma \\ \gamma^T & \beta \end{pmatrix}. \tag{2.46}$$

In principal, for any two-mode CM Γ we can find local symplectic operations S_1 and S_2 (each S_j acting on one of the two modes) that their direct sum $S_1 \bigoplus S_2$(corresponding to the tensor product of local unitary operations) changes the CM Γ to the so called standard form Γ_{sf}

$$S_l^T \Gamma S_l \begin{pmatrix} a & 0 & c_+ & 0 \\ 0 & a & 0 & c_- \\ c_+ & 0 & b & 0 \\ 0 & c_- & 0 & b \end{pmatrix}. \tag{2.47}$$

The states with $a = b$ are called symmetric. Further, any pure state is symmetric and fullfils $c_+ = -c_- = \sqrt{a^2 - 1}$. Notice, that up to a common sign flip between c_+ and c_-, the standard form associated with any CM is unique. It is so because the correlations a, b, c_+, c_- are determined by the four local symplectic invariants $Det\Gamma = (ab - c_+^2)(ab - c_-^2), Det\alpha = a^2, Det\beta = b^2, Det\gamma = c_+c_-$. Considering two-mode stated, the Ineq.(2.45) can be recast as a constraint $Sp_{(4,R)}$ invariants $Det\Gamma$ and $\Delta(\Gamma) = Det\alpha + Det\beta + 2Det\gamma :$

$$\Delta(\Gamma) \leq 1 + Det\Gamma. \tag{2.48}$$

We denote ν_- and ν_+ the symplectic eigenvalues of a two-mode Gaussian state. With the convention $\nu_- < \nu_+$, the Heisenberg uncertainty relation reducing to

$$\nu_- \geq 1. \tag{2.49}$$

The expression for determining the symplectic eigenvalues is

$$\nu_\mp = \sqrt{\frac{\Delta(\Gamma) \mp \sqrt{\Delta(\Gamma)^2 - 4Det\Gamma}}{2}}. \tag{2.50}$$

The physical meaning of the smallest symplectic eigenvalue appears in a connection with entanglement presented in the state.

2.7.3 ENTANGLEMENT OF GAUSSIAN STATES

To be able to separate two-mode Gaussian states it is necessary and sufficient to satisfy the positivity of the partially transposed states (PPT criterion). The partial transposition of a bipartite quantum state is defined as a simple transposition but applied on only one of the two subsystems in some given basis. In our case this leads to a sign flip in $Det\gamma$. Therefore $\Delta(\Gamma)$ changes to $\tilde{\Delta}(\Gamma)$, that takes the form

$$\tilde{\Delta}(\Gamma) = Det\alpha + Det\beta - 2Det\gamma. \tag{2.51}$$

Considering this, the symplectic eigenvalues now read

$$\tilde{\nu_\mp} = \sqrt{\frac{\tilde{\Delta}(\Gamma) \mp \sqrt{\tilde{\Delta}(\Gamma)^2 - 4Det\Gamma}}{2}}. \tag{2.52}$$

The PPT criterion thus reduces to a simple inequality that must be satisfied by the smallest symplectic eigenvalue $\tilde{\nu}_-$ of the partially transposed state

$$\tilde{\nu}_- \geq 1, \tag{2.53}$$

which can be equivalently written in the form

$$\tilde{\Delta}(\Gamma) \leq Det\Gamma + 1. \tag{2.54}$$

Looking at the above inequalities in detail, they imply $Det\gamma = c_+c_- < 0$ as the necessary constraint for obtaining entanglement for the two-mode Gaussian state. The quantity $\tilde{\nu}_-$ provides all the information about qualitative characterization of the entanglement for arbitrary two-mode Gaussian states. If the PPT criterion is violated, we automatically know that the systems are entangled which means that we are interested in the states that do not satisfy the Ineq.(2.53).

So far, we were interested only in the qualitative analysis. Now, let´s find out how to characterize entanglement from the quantitative view. A measurement of entanglement is provided by the negativity E. The negativity E of a state ϱ is defined as

$$\mathsf{N}(\varrho) = \frac{\|\tilde{\varrho}\| - 1}{2}, \tag{2.55}$$

where $\tilde{\varrho}$ is the partially transposed density matrix and $\|\hat{o}\| = Tr|\hat{o}|$ represents the trace norm of the hermitian operator $\hat{o}$. Another indicator for studying how much entanglement we get, that plays the priority role in this thesis, is called the logarithmic negativity E_N and it is strictly related to the negativity N. It is defined as $\mathsf{E}_N \equiv log_2\|\tilde{\varrho}\|$. For any two-mode Gaussian states the negativity is a simple decreasing function of $\tilde{\nu}_-$, therefore it is an inverse quantifier of entanglement:

$$\|\tilde{\varrho}\| = \frac{1}{\tilde{\nu}_-} \Rightarrow \mathsf{N}(\varrho) = max[0, \frac{1-\tilde{\nu}_-}{2\tilde{\nu}_-}], \mathsf{E}_N = max[0, -log_2\tilde{\nu}_-]. \tag{2.56}$$

These expressions quantify the amount by which the PPT inequality (2.53) is violated. From these we can say that the symplectic eigenvalue $\tilde{\nu}_-$ completely qualifies and also quantifies the quantum entanglement of a Gaussian state. Another quantity that we introduce here is fidelity. It is not used in this work but we include it here for the completeness. The fidelity sets how much successful a teleportation experiment is, it reaches unity only for a perfect transfer of a state. According to experiments, without using entanglement, by purely classical communication the fidelity of $\mathsf{F}_{CL} = \frac{1}{2}$ is the best that can be achieved. The sufficient fidelity criterion says that if teleportation is made with $\mathsf{F} > \mathsf{F}_{CL}$, then the shared resource is entangled. But converse statement is generally false. The optimal fidelity is given by

$$\mathsf{F}^{OPT} = \frac{1}{1+\nu_-}, \tag{2.57}$$

where ν_- is the smallest symplectic eigenvalue of CM. This equation shows that optimal fidelity of continuous variables teleportation of coherent states depends only on the entanglement quantified by ν_-. More information about fidelity can be found in [9].

2.7.4 CONDITIONAL VARIANCE

A conditional variance means that by measuring light only we try to influence the variance of matter. Furthermore, the main purpose is to prepare a non-classical state of a classical system, here matter. Since we are interested in Gaussian states the only way how to prepare a non-classical state is to squeeze the variance under unity. Therefore we also use the expression the conditional squeezing. Mathematically we express that in this way:

$$V_C = \langle (X_M - gX_L)^2 \rangle, \tag{2.58}$$

where g is a variable gain, which is used to reduce the variance $V_{XM} = \langle X_M^2 \rangle$ to V_C. It applies transformation of X_M to $X_M - g\bar{X}_L$, where $\bar{X}_L$ is a measured value of X_L.

In principal, we distinguish two types of conditional variance, but the difference is only in the preparation how to achieve the required state. If we consider that $\langle X_L \rangle = \langle X_M \rangle = 0$ and minimalizing via g we obtain

$$V_C = \langle X_M^2 \rangle - \frac{|\langle X_M X_L \rangle|^2}{\langle X_L^2 \rangle}, \tag{2.59}$$

which means that in this type the key task is only optimalizing the factor g. In the second type, on the other hand, the principal is in selecting appropriate measurement results. Whenever $X_L \approx 0$ is measured we know that we obtain the minimum conditional variance that is in agreement with (2.59). Any other results that are not in a very small interval around zero are just thrown away. Because of the selection of only a few measurements from a huge number of attempts this method is sometimes referred as probabilistic.

The conditional variance presented simultaneously in both X_M and P_M variables below unity is a direct witness of entanglement and its applicability in a preparation of different non-classical states. It is a positive outcome of a discussion started by the EPR paradox.

Chapter 3

ANALYSIS OF ENTANGLEMENT AND CONDITIONAL VARIANCE

In the following we are interested in deriving conditions that guarantee generation of entanglement between light and matter. We also study the entanglement from the quantitative point of view via the logarithmic negativity. The generated entanglement will be used to prepare conditional variance of matter V_C less than one ($V_C < 1$). The conditional variance is discussed in two ways either in position or in momentum. In general, there is also an interesting possibility of configuration that creates entanglement and $V_C < 1$ together, or creates $V_{CX} < 1$ and $V_{CP} < 1$ by changing measurement of light only, and we try to find out this arrangement. We try to discuss a robustness of each system, in other words how effective the interaction has to be to preserve the existence of entanglement or $V_C < 1$. Each of these questions is discussed for all the three types of interactions.

3.1 ENTANGLEMENT ANALYSIS

The existence of entanglement between a quantum two-level system and a macroscopic object is the characterisic feature of the interpretation of the Schrödinger-cat state. Since we approximate quantum states as Gaussian they are represented by a pure Gaussian quantum states of light. On the other hand, we treat the macroscopic system as a different matter oscillator which is very noisy at a room temperature. Classical continuous evolution of a harmonic oscillator is here replaced by the quantum analogue which is that the high energy of the oscillator can be explained by the mixture of coherent states, and this we mean by its classicality. These two desribed oscillators are coupled by a weak interactions: either BS, or AMP, or QND type of interaction. Conditions for obtaining entangled states are different for each interaction. For the beam splitter type the variance of a quadrature of light is reduced under a limit

$$V_S < \frac{1}{V_N}. \tag{3.1}$$

This inequality means the more thermal noise, the more squeezed light we have to input to obtain entanglement. Advantageously, because of independence (3.1) on T, the condition does not change for low interaction strength.

On the other hand, both amplifier and QND types of interaction are characterized by no restrictions for the interaction strength or squeezing of light to observe entanglement. Entanglement is generated in all considered values of $V_S \in (0,1)$, $V_N > 1$ and $G > 1$ for AMP, or $K > 0$ for QND.

So AMP and QND interactions are more appropriate for generating entanglement than BS, where we have to satisfy restriction for V_S.

Now, we concentrate on the quantitave analysis of Gaussian entanglement and we do it so by using a logarithmic negativity. The logarithmic negativity well measures only Gaussian entanglement which is in our case of Gaussian aproximation of states and interactions the expected one. The definition takes the form:

$$\mathsf{E}_N = max[0, -log_2\nu_-], \tag{3.2}$$

where ν_- is the smallest symplectic eigenvalue of the covariance matrix. Since BS and AMP types can be exchanged without changing a coupling strength it is appropriate to analyse them together. In both cases the function of logarithmic negativity has asymptotic features and it becomes zero in the infinity of V_N. In following, we will look at it in a detail. The most illustrative and simple way how to characterize the realistic change of entanglement is to determine how quickly it increases when the squeezing goes to the threshold $V_S = \frac{1}{V_N}$ for BS type, and $V_S = 1$ for AMP and QND types.

For the beam splitter we used the first derivative of the LN over V_S, then took the limit for $V_S \to \frac{1}{V_N^-}$ (which corresponds to the border of generating entanglement and the minus sign stands for applying the limit from the left side) and finally we used a Taylor series for high values of V_N and a weak interaction. Using this method we obtain

$$\frac{\partial LN}{\partial V_S}|_{V_S=\frac{1}{V_N}} = -\frac{2V_N(1-T)}{ln2} > -\frac{2(1-T)}{V_S ln2} \tag{3.3}$$

for BS. That is a function that characterizes the direction of the logarithmic negativity at $V_S \to \frac{1}{V_N^-}$. The minus sign in (3.3) means that the LN increases with decreasing V_S, therefore the angle between the function and positive axis V_S is $\varphi \in (\frac{\pi}{2}, \pi)$. If we look at (3.3), it is clear that inputting more squeezed V_S makes the LN increase faster. And for $T \to 1$ the LN is characterised by slower increasing. All these findings are illustrated in FIG.3.1. If we look at

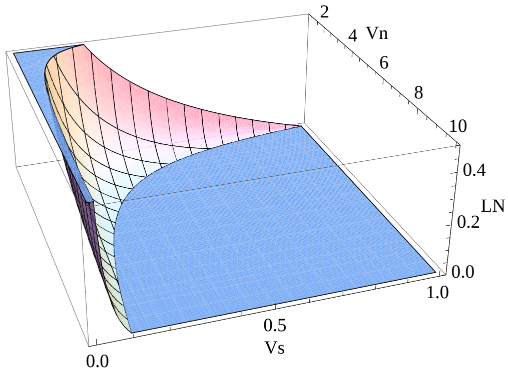

Figure 3.1: BS type of interaction (T=0.95): logarithmic negativity as a function of V_S and V_N.

the graphic illustration of the LN an interesting question appears. How does the function behave in the opposite limit where $V_S \to 0$? If we apply the same analysis to find out what direction the function has in the nearness of maximum squeezing, we obtain the infinity. That means that the logarithmic negativity touches the zero point of V_S in infinity, therefore in the closeness of this point it is parallel to this border.

For AMP the only difference in the analysis is that the limit is taken for $V_S \to 1^-$ because generating entanglement here is condition-free, so we are interested in the area of $V_S = 1$. Here we get

$$\frac{\partial LN}{\partial V_S}|_{V_S=1} = -\frac{2(G-1)}{V_N ln2}. \tag{3.4}$$

It is visible that LN after applying BS interaction increases faster but there is a necessity of satisfying (3.1), whereas the amplifier does not need squeezing in light, therefore it is clear to prefer AMP type from the point of view of generating entanglement. As it is shown in FIG.3.1 and FIG.3.2, it is a non-decreasing function of squeezing V_S so it is clear that for getting more entanglement we have to increase squeezing V_S, similarly to the BS case. The LN increases the slower, the higher values of V_N we input. And it also increases the slower, the weaker interaction we consider, which is represented

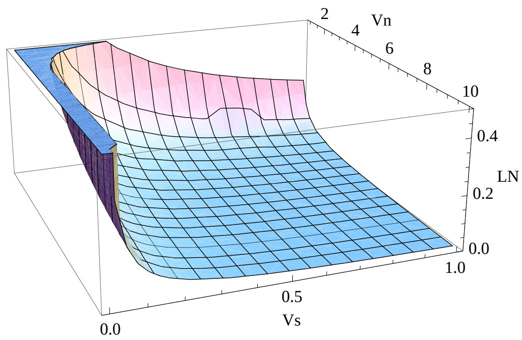

Figure 3.2: AMP type of interaction (G=1.05): logarithmic negativity as a function of V_S and V_N.

by the value of G close to 1. When studying the opposite limit for $V_S \to 0$ we obtain the same result as for BS, which is infinity. The LN of AMP interaction is parallel to the border of $V_S = 0$. We include an explicit form of LN for a weak interaction with high thermal noise and no squeezing which is the most significant configuration. The aproximation takes the form

$$f_{LN} = -log_2(1 - \frac{2(G-1)}{V_N}) \tag{3.5}$$

For QND, we have to realize that the character of this interaction is completely different. In comparison to the first two types, here we are dealing with non-symmetrical interaction. For our type of interaction it is advantageous to consider squeezing of light in the momentum quadrature P_L. In other words, $V_{SX} > 1$(it is a consequence of Heisenberg´s uncertainty relation). Larger values of $V_{SX} > 1$ now correspond to more squeezing in momentum, therefore we identify $V_S \equiv V_{SP}$ for the discussion of LN of QND interaction. The function of the logarithmic negativity again becomes zero in the infinity of V_N. But for QND the LN increases very slowly with increasing V_S, therefore we have to take into account higher orders of a Taylor series. We used the limit $V_S \to 1^+$. The character is descrribed by

$$\frac{\partial LN}{\partial V_S}|_{V_S=1} = \frac{2K^2}{V_N ln2}. \tag{3.6}$$

Here again, the more squeezed V_S(here in momentum), the more entanglement we get, which is obvious if we realize the similarity between (3.4) and (3.6). An interesting fact appears when we study the behaviour of LN from the left side, in the closeness of $V_S = 0$. Oppositely to the two previous cases, here we don´t obtain the infinity. When we apply the first derivative of the

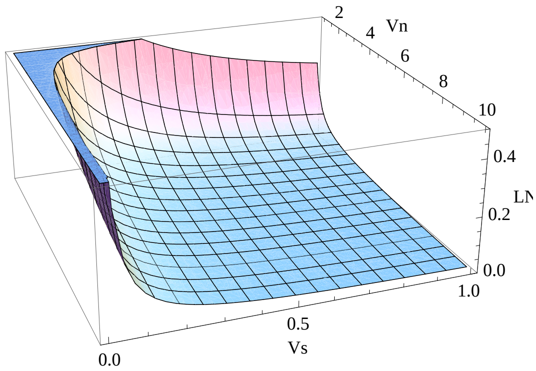

Figure 3.3: QND type of interaction ($K^2 = 0.05$): logarithmic negativity as a function of V_S and V_N.

logarithmic negativity and then take the limit for $V_S \to 0$ we obtain

$$\frac{\partial LN}{\partial V_S}|_{V_S=0} = \frac{2K^2 V_N}{(V_N^2 - 1)ln2}. \tag{3.7}$$

Which has the meaning that in the area of higher thermal noise, the LN is increasing faster in the closeness of $V_S = 0$. Considering high values of V_N, (3.7) reduces to (3.6), which would mean that the derivative does not change with V_S, but this would not correspond to FIG.3.3. Therefore we add here a better approximation of (3.6). We took into account higher orders of Taylor series and obtain $\frac{\partial LN}{\partial V_S}|_{V_S=1} = \frac{2K^2}{V_N ln2} - \frac{8K^4}{V_N^2 ln2}$. If we study the absolute value of entanglement in the area of high V_N with no squeezing for a weak interaction, we obtain

$$f_{LN} = -log_2(1 - \frac{2K^2}{V_N}). \tag{3.8}$$

To illustrate which of the two interactions produces more entangled state we include FIG.3.4, where we can see that amplifier produces more entanglement.

We have shown that generating entanglement between a very classical and a very quantum system is not as hard or even impossible for active interactions as the typical expectations of Schrödinger-cat state predict. The problem may appear when we try to detect it becauce thee amount of entanglement is low and the generated state is very mixed.

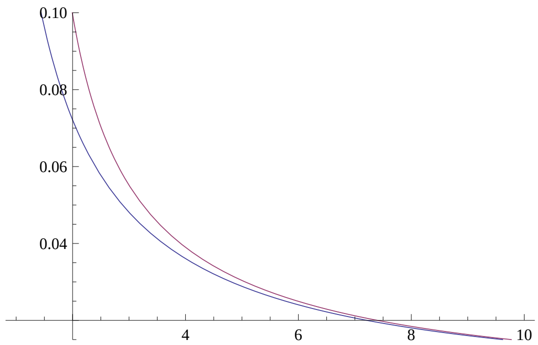

Figure 3.4: A comparison of AMP and QND type of interaction. How much of entanglement each interaction produces ($K^2 = 0.05, G = 1.05$).

3.2 ROBUSTNESS OF ENTANGLEMENT

An important characteristic of any quantum system is its robustness against the most basic imperfection, which is loss of energy. Since we interpret matter being in a classical noisy state (macrosystem) and light being in a quantum state (microsystem), it opens a question whether the entanglement is more sensitive to light or to a matter energy loss. To involve energy loss we use simply the BS type of interaction for both light and matter described previously. For this purpose, we derive a condition for the minimum $\eta_{min} \in (0,1)$, that represents loss in each system. We also include a condition that makes $\eta_{min} \leq 0$ and therefore the entanglement is robust for all values of $\eta > 0$, we say that the robustness is *absolute*. For BS type and loss of energy in matter we derived

$$\eta > 1 + \frac{4T(V_N - V_S)(V_N V_S - 1)}{(V_N^2 - 1)(V_S - 1)^2} \tag{3.9}$$

as the sufficient condition to preserve entanglement. In FIG.3.5 it is clear that squeezing of light is helpful to increase robustness of entanglement. To derive the asymptotic function that characterizes how the entanglement vanishes with increasing thermal noise, we expand the right side of (3.9) using Taylor series for intensive squeezing V_S(represented by small values of V_S) and large V_N and obtain

$$\eta > 1 - \frac{4T}{V_N}. \tag{3.10}$$

For loss in light the condition for robustness is a bit complicated so we directly include only the Taylor expansion for small values of squeezing in light V_S

and large thermal noise V_N to express analytically the asymptotic feature. It takes the form

$$\eta > 1 - \frac{4(1-T)}{V_N}. \tag{3.11}$$

Since $T \in (0,1)$, the condition for the minimum η is more strict for quantum optical oscillator, which is illustrated in FIG.3.6. We use the value of $T = 0.95$ representing a weak interaction. Intensive squeezing in light V_S is obviously needed for generating entanglement under the loss in both matter and light. Considering a weak interaction, unfortunately, the robustness vanishes asymptotically with increasing values of V_N even if we consider massive squeezing of light V_S.

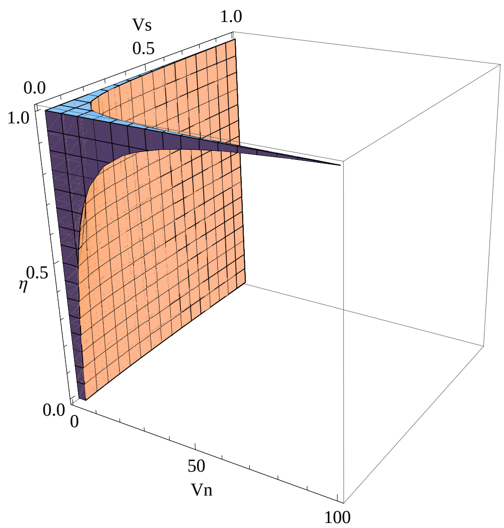

Figure 3.5: BS type of interaction (T=0.95): maximal possible loss of energy in matter to preserve entanglement as a function of V_S and V_N.

Considering a weak interaction, unfortunately, the robustness vanishes asymptotically with increasing values of V_N even if we consider massive squeezing of light V_S. As the V_N gets larger, the acceptable loss η is becoming smaller and smaller which is just a proof of our expectations that BS type is not the appropriate interaction for experimental observation of quantum correlation between classical noisy oscillator and quantum system.

On the other hand, for AMP type we can observe in FIG.3.7 and FIG.3.8 very high robustness that advantageously saturates for high values of V_N. If $G < \frac{(1+V_S)^2}{4V_S}$, then the condition for minimum η is greater than zero and (3.12) and (3.13) stand. But if $G > \frac{(1+V_S)^2}{4V_S}$, then there is no restriction

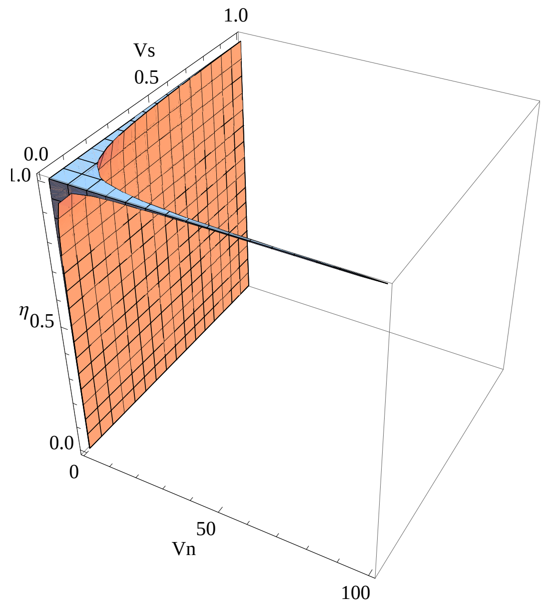

Figure 3.6: BS type of interaction (T=0.95): maximal possible loss of energy in light to preserve entanglement as a function of V_S and V_N.

and the entanglement is generated for all values of $\eta > 0$. It means that if we consider a non-squeezed state ($V_S = 1$) entanglement is always robust because the condition reduces to $G > 1$. The condition also doesn´t depend on the thermal noise V_N which just proves that amplifier is better type than the two others.

It´s immediately visible from FIG.3.7 and FIG.3.8that the non-entangled area corresponds to more squeezed V_S so we are allowed to say that squeezed V_S here is disadvantageous. If we concentrate on high values of V_N and weak interaction and approximate the expression by ignoring members of second or higher order in Taylor series, we obtain analytical form

$$\eta > 1 - \frac{4V_S G}{(1+V_S)^2} \tag{3.12}$$

for loss in matter. The same approximation we use for loss in light and it leads us to

$$\eta > 1 - \frac{4V_S(G-1)}{(V_S-1)^2}. \tag{3.13}$$

The two results (3.12),(3.13)correspond to the 2D FIG.3.9. The result we obtained for BS stands even for AMP. Matter is again more robust than light.

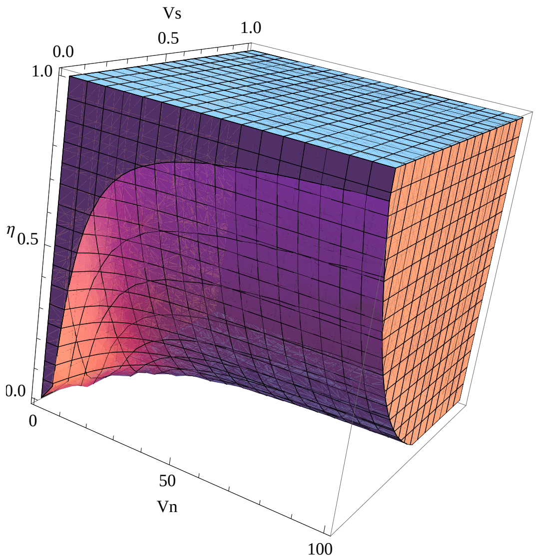

Figure 3.7: AMP type of interaction (G=1.05): maximal possible loss of energy in matter to preserve entanglement as a function of V_S and V_N.

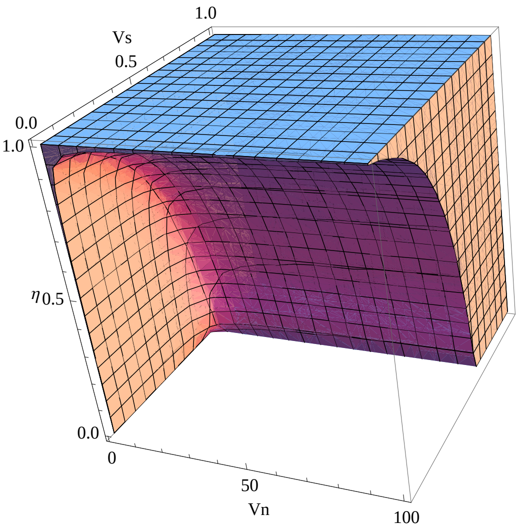

Figure 3.8: AMP type of interaction (G=1.05): maximal possible loss of energy in light to preserve entanglement as a function of V_S and V_N.

As for the amplifier, we also study the case if we obtain entangled state even for loss of energy in both systems. We denote η_M loss in matter and η_L loss in light. We derived earlier that squeezing is not useful for generating entanglement in both cases of loss of energy for AMP. Therefore we simplify our discussion by taking $V_S = 1$. We are interested in a weak interaction

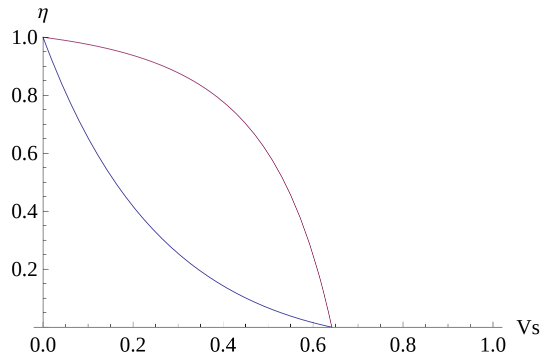

Figure 3.9: AMP type of interaction(G=1.05): loss of energy in matter(the blue curve) and loss of energy in light(the purple curve). Illustrated for high values of V_N

with high values of V_N and the value of G weakly above unity, which leads us to a expression for the smallest symplectic eigenvalue

$$\nu_- = 1 - \frac{(1-\eta_M)(1-\eta_L)}{V_N}. \tag{3.14}$$

The state is entangled, if this eigenvalue ν_- is smaller than 1 and it happens for all possible values of η_M, η_L and V_N. It means that if we input a non-squeezed state, we always obtain entanglement. After looking closer at (3.14), it is clear that for high values of V_N the entanglement decreases.

In comparison with the two previous interactions, robustness of each system after applying QND interaction we describe analytically and also graphically. We again include additional condition that guarantees that the expression for minimum η is larger than zero and therefore (3.15),(3.16),(3.18),(3.19) stand, if it´s not then we obtain entanglement for all values of $\eta > 0$. Under the consideration of loss in matter we find out that

$$\eta > 1 - \frac{4V_N}{V_N^2 - 1}, \tag{3.15}$$

which is independent on K and V_S, therefore if we somehow manage the value of $V_N < 2 + \sqrt{5}$ we automatically get entanglement. This expression corresponds to FIG.3.10. Another important point of the analysis is that we do not need to input squeezed V_S. The area of our main interest is the area of high values of V_N where the condition reduces to

$$\eta > 1 - \frac{4}{V_N}. \tag{3.16}$$

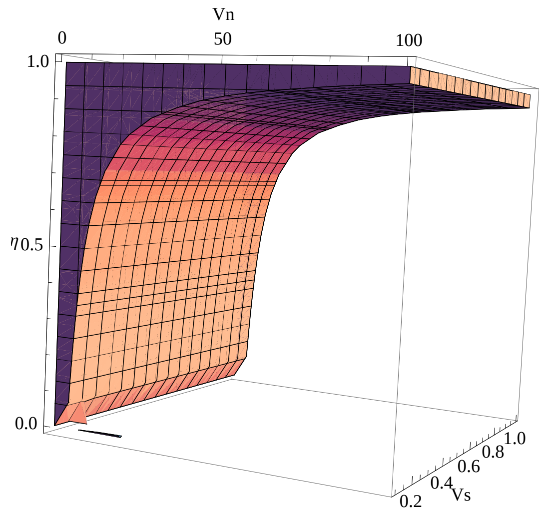

Figure 3.10: QND type of interaction ($K^2 = 0.05$): maximal possible loss of energy in matter to preserve entanglement as a function of V_S and V_N.

For loss in light, if there is satisfied

$$V_S < \frac{1}{1 + \frac{2K^2 V_N}{\sqrt{K^2 V_N (V_N^2 - 1)}}} \tag{3.17}$$

then the following condition stands, but in other case the entanglement is always robust. The condition for minimum η is

$$\eta > 1 - \frac{4K^2 V_S^2 V_N}{(V_N^2 - 1)(V_S - 1)^2}, \tag{3.18}$$

which is illustrated in FIG.3.11. In comparison to (3.15) V_S and K play a substantial role here now. For high values of V_N we obtain from (3.18)

$$\eta > 1 - \frac{\frac{4K^2 V_S^2}{(V_S - 1)^2}}{V_N}, \tag{3.19}$$

which is obviously difficult to achieve. The only way to generate it is to increase V_S towards unity for given η. But the appropriate interval of synchronized values of V_S and η is very thin. So squeezed V_S here is again disadvantageous.

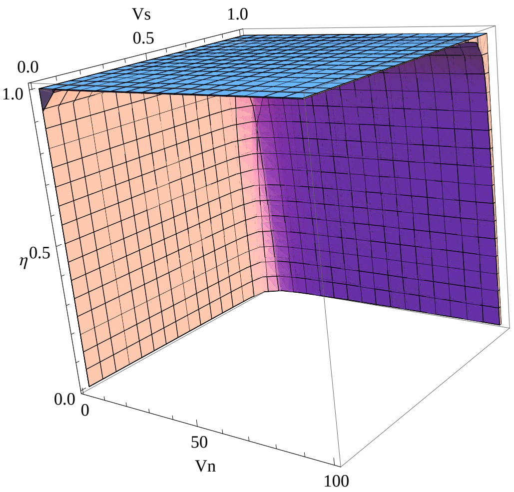

Figure 3.11: QND type of interaction ($K^2 = 0.05$): maximal possible loss of energy in light to preserve entanglement as a function of V_S and V_N.

3.3 CONDITIONAL SQUEEZING

We have shown that after a weak Gaussian interaction it is possible to generate quite robust entanglement even for the interaction between light and a very noisy matter oscillator. In our motivational case of Schrödinger´s cat the macroscopic system can be also conditionally prepared in a non-classical state by a measurement on the quantum system. It can be considered as a transposition of quantum features to the macro-system which was at the beginning in a very classical state. We focus on studying the conditional variance, especially the case when we are able to squeeze the conditional variance under unity which is called conditional squeezing. Our purpose is to find out the configurations that enable the preparation of a non-classical state in matter by measuring light only. To obtain conditional variance of position less than one ($V_{CX} < 1$) the restriction for V_S is in the form

$$V_S < \frac{V_N(1-T)}{V_N - T}. \tag{3.20}$$

for BS. In opposition to the entanglement condition, here it explicitly depends on T. Considering high values of V_N, the condition becomes simplier:

$$V_S < 1 - T. \tag{3.21}$$

The analysis of this expression leads us to conclusion that there is a neccessity of inputting intensively squeezed V_S. Notice, that for weak interactions satisfying (3.21) is nearly impossible - the squeezing has to be massive. Using the knowledge of $V_{CX} < 1$ and from the character of BS interaction it is easy to derive that $V_{CP} < 1$ is impossible to achieve. Asking whether it is possible to obtain entanglement and $V_{CX} < 1$ leads us to conditions that are a bit complicated to express analytically. FIG.3.13 is therefore reasonable here. From this picture it is visible that for obtaining entanglement and

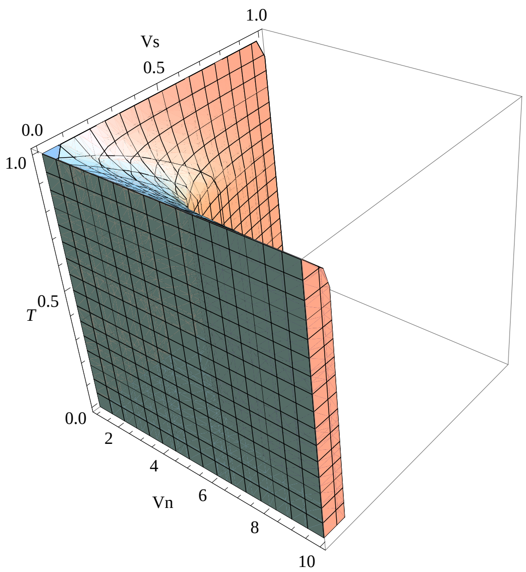

Figure 3.12: BS type of interaction: coexistence of entanglement and conditional variance $V_{CX} < 1$.

$V_{CX} < 1$ simultaneously, it is advantageous to input squeezed V_S. On the other hand, it opens a question, whether there is a configuration of variables that guarantees generating state with $V_{CX} < 1$ but not entanglement. This is a very interesting configuration to study because there is no entanglement, so there exist only classical correlations, but on the other hand because of existence of $V_{CX} < 1$ for matter, the classical correlations are between very non-classical systems. This configuration is represented in (FIG.3.13)

Amplifier is condition-free for generating entanglement but it becomes more interesting when we study the conditional variance. To achieve $V_{CX} < 1$ the system has to satisfy

$$V_S < \frac{V_N(G-1)}{V_N - G}. \qquad (3.22)$$

If we consider high values V_N then it becomes independent on V_N and the condition changes to

$$V_S < G - 1 \qquad (3.23)$$

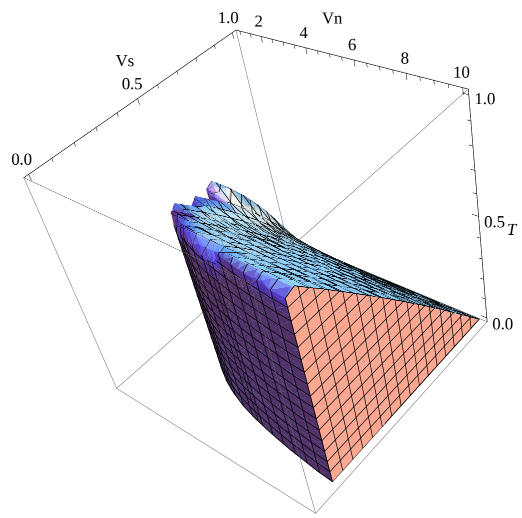

Figure 3.13: BS -configuration that guarantees $V_{CX} < 1$ without generating entanglement.

from where we get the solution that in this case the squeezed V_S is indeed needed for weak interactions. But if we prefer variable gain and $V_S = 1$ then the gain has to be $G > 2$. As for $V_{CP} < 1$, we obtain different condition:

$$V_S > \frac{V_N - G}{(G-1)V_N}. \tag{3.24}$$

For high values of V_N it reduces to

$$V_S > \frac{1}{G-1}, \tag{3.25}$$

which is not a pleasant result because for a weak interaction (represented by values of G close to 1) we are not able to obtain $V_{CP} < 1$ at the output. Using the knowledge that for obtaining entanglement at the output we do not have to satisfy any restrictions, we can immediately derive from it that by satisfying the condition for $V_{CX} < 1$ or $V_{CP} < 1$ we also automatically get entangled state. If we look at the both types of $V_C < 1$ in detail, we find out that coexistence of these two cases is also possible. If we assume high values of V_N the condition for $V_{CP} < 1$ is only a restriction of the condition for $V_{CX} < 1$. Therefore, the squeezing in momentum here implies the squeezing in position. This is illustrated in FIG.3.14.

We have already answered the question when $V_C < 1$. But we haven´t studied what the amount of conditional squeezing in any configuration is. And we´d also like to add the mutual relation between V_{CX} and V_{CP}.

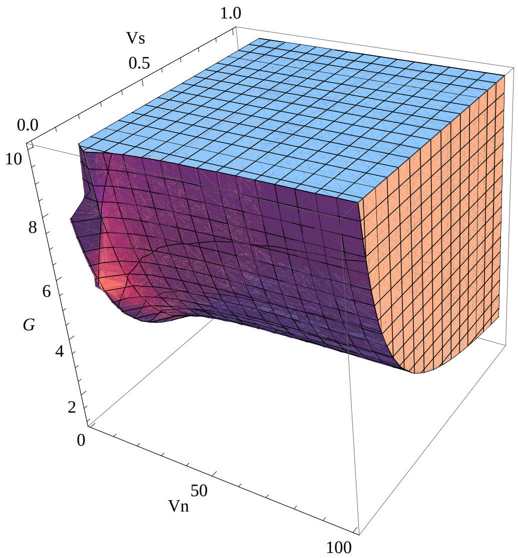

Figure 3.14: AMP type of interaction: coexistence of conditional variances $V_{CP} < 1$ and $V_{CX} < 1$.

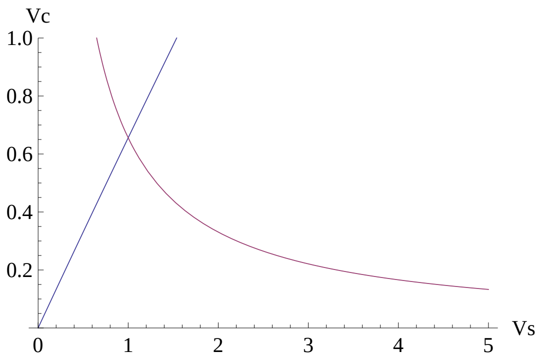

Figure 3.15: AMP type of interaction($G = 2.5$; $V_N = 100$): behaviour of V_{CX}(the blue curve) and V_{CP}(the purple curve).

In FIG.3.15 we demonstrate the behaviour of these conditional variances and we used $V_N = 100$, which represents high values of the thermal noise, and $G = 2.5$ to overcome $G = 2$ but still satisfy a relatively weak interaction. We can see that the conditional variances show a complementary behaviour, one increases whereas the second decreases. In the point of $V_S = 1$ the two functions have identical value, no matter what the gain or V_N are, which could be derived by comparing (3.22) and (3.24). So when there is no squeezing($V_S = 1$), there is no difference between V_{CX} and V_{CP}. Therefore we can inlude only one graph of dependance of the conditional variance on

V_N and G, instead of two, it is illustrated in FIG.3.16. If we look at FIG.3.16, it confirmes that for the non-squeezed state the value of $G = 2$ is a kind of threshold in the sense of generating $V_C < 1$. It also shows that a strong interaction enables generating more conditional squeezing and finally that for high values of V_N the function saturates and takes the form of (3.31) or (3.33) with $V_S = 1$.

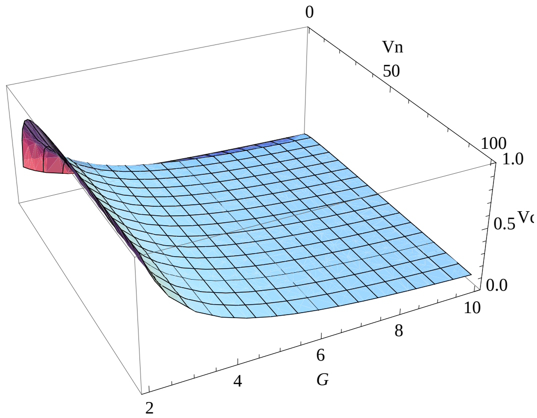

Figure 3.16: AMP type of interaction: behaviour of conditional variance V_C when there is no squeezing represented by $V_S = 1$.

For QND solving the problem of the conditional variance of the position leads us to a result that there is squeezing in V_N needed but this is out of possible values of V_N so there is no chance to obtain $V_{CX} < 1$. On the other hand, $V_{CP} < 1$ is possible to achieve because the condition takes the form

$$V_S < \frac{K^2 V_N}{V_N - 1}. \tag{3.26}$$

In the area of high values of V_N it reduces to

$$V_S < K^2 \tag{3.27}$$

In summary, coexistence of entanglement and $V_{CX} < 1$ is not possible, but entanglement and $V_{CP} < 1$ may be coexisting and that happens if we satisfy the restriction for $V_{CP} < 1$ because generating entanglement for QND is not limitted in our considerated values.

Now, we try to find out the amount of conditional squeezing. In other words, we are interested in how much under unity the conditional variance goes with squeezed V_S. We always start with the BS type of interaction and so we do now. The function that characterizes the conditional squeezing in position takes the form

$$V_{CX} = \frac{V_N V_S}{V_N(1-T) + TV_S}. \tag{3.28}$$

To illustrate better the dependence of V_C on V_S we include FIG.3.17, where it is absolutely clear how important role V_S plays for obtaining squeezed conditional variance. If we consider only high values of V_N by applying

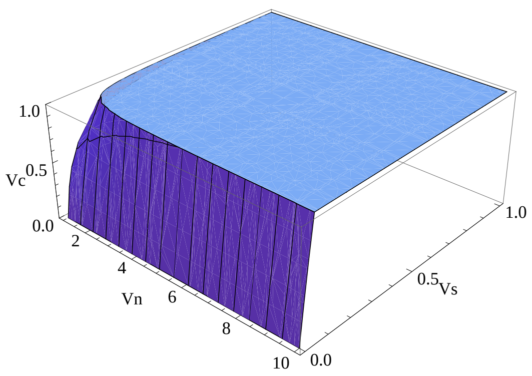

Figure 3.17: BS type of interaction (T=0.95): dependence of the conditional squeezing V_C on V_S and V_N.

Taylor series, Eq.(3.28) becomes simplier and reduces to

$$V_{CX} = \frac{V_S}{1-T}, \tag{3.29}$$

which is a simple linear function of V_S and T. The meaning of V_S for conditional squeezing of BS interaction is therefore huge. There exists a direct connection between V_S and V_C, if we want to achieve more squeezed conditional variance, we have to input more squeezed V_S. In addition, $V_{CP} < 1$ is impossible to achieve in this case as we derived earlier in this paper.

The same analysis we apply on AMP interaction but here we have to discuss both cases V_{CX} and V_{CP} separately. First, we concentrate on V_{CX}. It is described by

$$V_{CX} = \frac{V_N V_S}{V_N(G-1) + GV_S}, \tag{3.30}$$

which is the same as it was for BS if we realize the connection between BS and AMP (G=2-T). Therefore it is not needed to include a graphic illustration because there is no difference between these two figures. The approximation we use for high values of thermal noise stands even here, the function reduces to

$$V_{CX} = \frac{V_S}{G-1}. \tag{3.31}$$

Of course, even the meaning of V_S is in agreement with the previous case. But if we study the conditional variance in momentum, the discribing function is completely different

$$V_{CP} = \frac{V_N}{G + V_S V_N (G-1)}, \tag{3.32}$$

reducing to

$$V_{CP} = \frac{1}{V_S(G-1)} \tag{3.33}$$

for high values of thermal noise. As we derived earlier in the paragraph of qualitative analysis of V_C, for a weak interaction we are not able to get V_{CP} under unity. But it is worth mentioning that the role of V_S here is unexpectable, which can be derived from (3.33) and is illustrated in FIG.3.18. More squeezed V_S at the input influences V_{CP} at the output in a negative

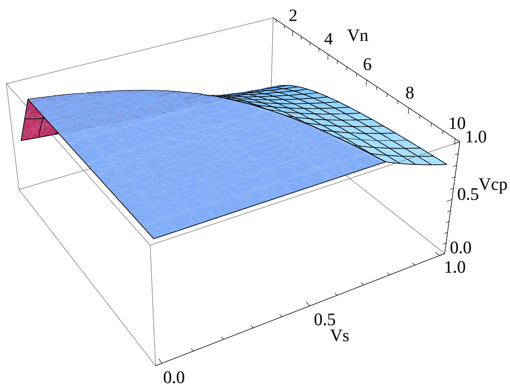

Figure 3.18: AMP type of interaction (G=1.05): dependence of the conditional squeezing V_{CP} on V_S and V_N.

way. For the amplifier we add another point of analysis.

The third type of interaction that we study is QND and from our qualitative analysis we know that conditional squeezing is possible to obtain in momentum, not in position. As it was for logarithmic negativity for QND we identify $V_S \equiv V_{SP}$ here, too. The amount of conditional squeezing can be described by the equation

$$V_{CP} = \frac{V_N V_S}{V_S + K^2 V_N}. \tag{3.34}$$

We again include a graphical illustration of this dependence in FIG.3.19. In

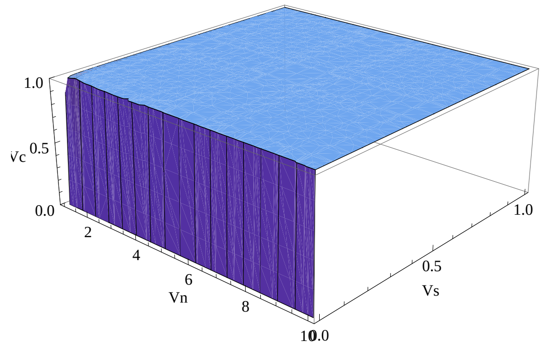

Figure 3.19: QND type of interaction ($K^2 = 0.05$): dependence of the conditional squeezing V_{CP} on V_S and V_N.

the area of high values of V_N the Eq.(3.34) changes to

$$V_{CP} = \frac{V_S}{K^2}. \tag{3.35}$$

From FIG.3.19 and also from (3.35) we can certainly say that V_S plays positive role for obtaining more conditional variance. So, the only case when the squeezing at the input makes less conditional squeezing is AMP with the squeezing in momentum. In all other cases is V_S advantageous for obtaining more conditional squeezing.

3.4 ROBUSTNESS OF CONDITIONAL SQUEEZING

We have discussed the conditional variance in both ways qualitative and quantitative. But what happens with the conditional squeezing if we consider loss of energy? To simulate loss of energy we use the same interaction type (BS) as we did in the section of robustness of entanglement and our purpose is to derive a condition that guarantees that the value of the conditional variance is under unity. Clearly, if the squeezing is induced in matter it is never lost by a pure lost. We therefore focus on deriving condition for minimum η only for the energy loss in light. The condition explicitely depends on η and takes the form

$$\eta > \frac{1 - TV_N - V_S(1-T)}{(V_N - 1)(V_S - 1)}, \tag{3.36}$$

which for high thermal noise it reduces to

$$\eta > \frac{T}{1 - V_S}. \tag{3.37}$$

To complete the analysis of BS we add FIG.3.20, where we can see how important role the squeezing plays. In FIG.3.20 it can be seen that mas-

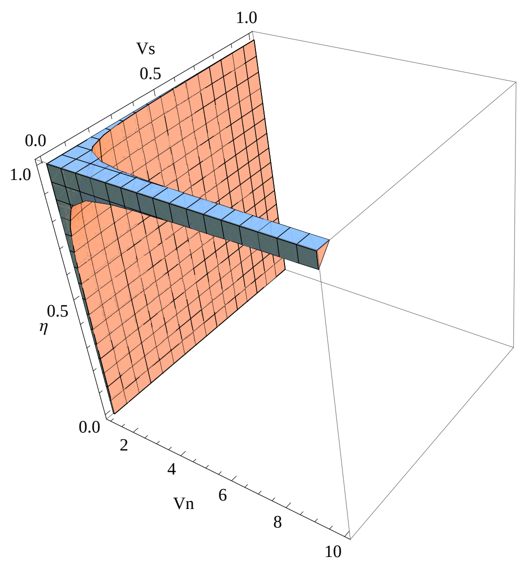

Figure 3.20: BS type of interaction (T=0.95): robustness of conditional squeezing $V_{CX} < 1$ against loss of energy in light.

sively squeezed V_S enables us to generate $V_{CX} < 1$ for all values of V_N From this analysis we can pick up a positive finding that the robustness of conditional squeezing saturates for high thermal noise, which is opposite to the robustness of entanglement that vanishes asymptotically with increasing V_N. Analyzing robustness of V_{CP} is meaningless, because we derived earlier that it is impossible to achieve $V_{CP} < 1$ even in the case without loss of energy.

A similar type of interaction is an amplifier, let´s take a look whether the results are similar too. Since the condition for minimum η is a bit complicated we prefer graphical illustration, which is in FIG.3.21 The behaviour of the function in FIG.3.21 resembles the function in FIG.3.20, which was for beam splitter and loss in light. If we consider high values of V_N only, the condition

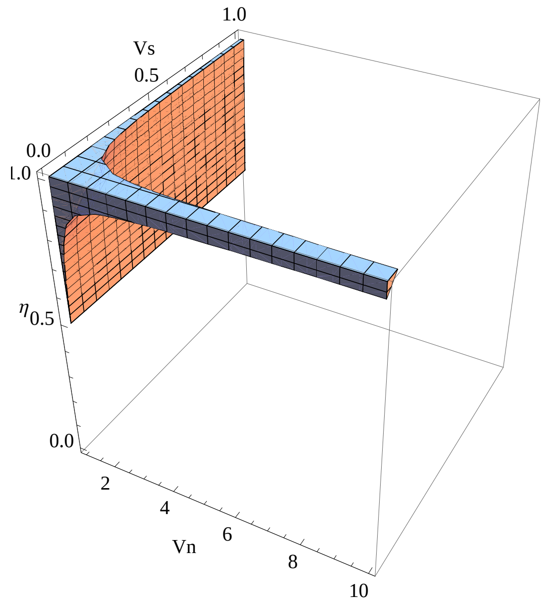

Figure 3.21: AMP type of interaction (G=1.05): robustness of conditional squeezing $V_{CX} < 1$ against loss of energy in light.

becomes simplier

$$\eta > \frac{G}{-1 + 2G - V_S}, \tag{3.38}$$

from where it can be derived that V_S here is really needed to obtain $V_{CX} < 1$. As for V_{CP} for loss in light, we use again only a graphical illustration, which is FIG.3.22, but this time with $G = 2$ because for a weak interaction the area of $V_{CP} < 1$ is hardly visible. In FIG.3.22 it is also easier to see that squeezed V_S is disadvantageous for generating $V_{CP} < 1$. We confirm that by including the reduced condition for large V_N

$$\eta > \frac{V_N}{1 + V_N}. \tag{3.39}$$

So it looks like for the purpose of obtaining $V_{CP} < 1$ it is advantageous not to input squeezed V_S. When we studied the basic conditional variance, we found out that the existence of $V_{CP} < 1$ guaranteed the generation of $V_{CX} < 1$ for AMP interaction. But does this statement stand even if we consider loss of energy? Detailed analysis that we made showed that the FIG.3.22 is a restricion of FIG.3.21, so the statement stands even here. The genereation of $V_{CP} < 1$ guarantees the generation of $V_{CX} < 1$.

QND interaction is a bit more complicated to discuss. As we did earlier, we now again use the formalism that if we discuss V_{CP} we identify $V_S \equiv V_{SP}$,

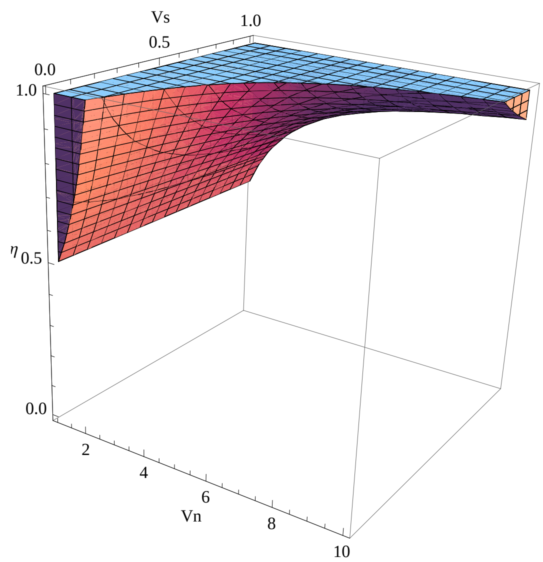

Figure 3.22: AMP type of interaction (G=2): robustness of conditional squeezing $V_{CP} < 1$ against loss of energy in light.

because in principal we can choose if we squeeze the state in position or in momentum. In our analysis we derived that $V_{CX} < 1$ is impossible to achieve no matter where the loss is, which is in agreement with our earlier results that QND is not able to generate $V_{CX} < 1$ at all. So we can focus only on the robustness of conditional variance in momentum. To obtain $V_{CP} < 1$ we have to satisfy

$$\eta > \frac{V_N - 1}{V_N(1 + K^2 - V_S) + V_S - 1}, \tag{3.40}$$

which is illustrated in FIG.3.23 and for high V_N it becomes

$$\eta > \frac{1}{1 + K^2 - V_S}. \tag{3.41}$$

The advantage of squeezed V_S in FIG.3.23 corresponds again to the disadvantage of squeezed V_S in FIG.3.22. Also, FIG.3.23 seems to be very similar to FIG.3.20 and FIG.3.21. So all the three interactions show similar features in loss in in light.

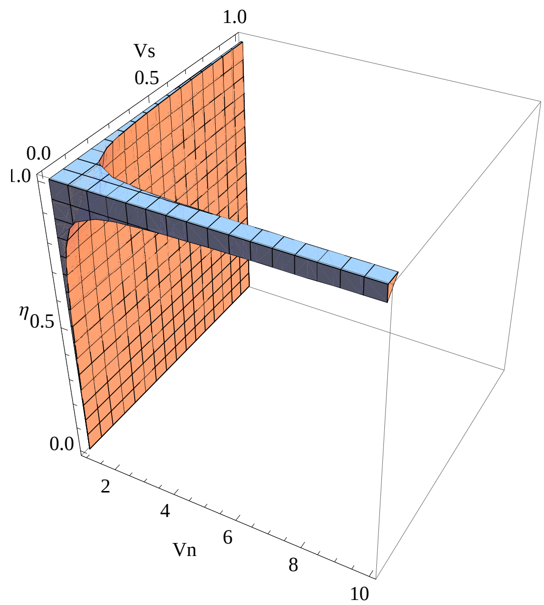

Figure 3.23: QND type of interaction ($K^2 = 0.05$): robustness of conditional squeezing $V_{CP} < 1$ against loss of energy in light.

Chapter 4

CONCLUSION

We have investigated the possibility of using a weak Gaussian interaction between a noisy matter oscillator, considered as a macro-system and a quantum micro-system, here a quantum optical oscillator to obtain a Gaussian version of Schrödinger´s cat state. We have discussed two main properties of three prominent interactions, entanglement generation and the conditional preparation of a matter system in a non-classical state.

As our main goal was to pick the most appropriate intearaction type that would have the best characteristic features, let´s compare the three types of interaction. AMP and QND types of interaction enable to generate entanglement in any points under the consideration, whereas for BS type we have to satisfy a simple condition (3.1). To obtain more entanglement it is necessary to input squeezed light in all the interaction types. The only difference is in how fast entanglement increases with squeezing V_S and how much of entanglement is generated. Since the beam splitter has limitations for generating entanglement, only AMP and QND type of interaction should be compared. FIG.3.4 shows that the amplifier gives a more pleasant result. As for the robustness of entanglement, in all the three possibilities we made the conclusion that matter is more robust than light. The squeezing plays a positive role olny in BS, for AMP and QND it is not needed or even disadvantageous. And considering high values of V_N the condition for obtaining entangled states for BS type is unachievable, for QND type the condition is too strong. Only for AMP type we obtain acceptable condition. Then we focused on the conditional variance. Again, AMP type seemed to be the most appropriate interaction because it is the only type where it is possible to generate $V_{CX} < 1$ and $V_{CP} < 1$ together, which we confirmed later in the discussion of the robustness of V_C. The best of these three types is definitely the amplifier which is stressed here as the main conclusion.

In this paragraph we would like to add a summary of the characteristic features of this interaction that we have found out. Entanglement is generated in all configurations of V_N, V_S, G. The behavior of the logarithmic negativity is illustrated in FIG.3.2. When we discussed the robustness of entanglement, amplifier was the most robust at all. FIG.3.7 and FIG.3.8 show that during this type of interaction the entanglement is more sensitive to light. It also tells us that it is advantageous to input a state of $V_S = 1$, which is pleasant and confirms our conclusion of AMP being the best type. We then focused on the conditional squeezing. AMP was the only interaction that enabled genereating $V_{CX} < 1$ and $V_{CP} < 1$ together. Because of the condition-free entanglement generation, the configuration for coexisting $V_{CX} < 1$ and $V_{CP} < 1$ also guarantees entangled state, it corresponds to FIG.3.14, where we can see the negative role of squeezed V_S and also a very interesting fact that for $V_S = 1$ there exist a threshold $G = 2$. For values $G > 2$ and $V_S = 1$ we obtain the required configuration. If we discuss the role of V_S for the robustness of conditional squeezing, to obtain $V_{CX} < 1$ it is advantageous to input squeezed V_S. But for generating $V_{CP} < 1$ it is better to input $V_S = 1$, these findings can be seen in FIG.3.21 and FIG.3.22.

Chapter 5

OUTLOOK

In this chapter we try to pick up the motivation for future follow-up of this work.

During our analysis we faced some interesting and maybe unexpectable features. AMP type of interaction showed some kind of threshold $G = 2$ when we were discussing the conditional variance. In the future, we will focus on this in a more detailed way and try to break this threshold and if it is somehow possible then analyze what this break costs.

We would also like to add a multiple conditional squeezing and analyze it from the point of view as we did here.

Generally, we will continue to analyze the three types of interaction but in a more detailed way and therefore it is reasonable to apply another, more sofisticated, mathematical aparate of Wigner´s function.

A great motivation for the future work is the opportunity to compare our theoretical findings with practical measurements made at foreign universities.

Bibliography

[1] $http://fyzika.jreichl.com/main.article/view/740-schrodingerova-kocka$

[2] SAKURAI, J. J. Modern Quantum Mechanics(Revised Edition). Addison-Wesley Publishing Company, 1994. 500 p. ISBN 0-201-53929-2

[3] SCULLY, M. O. and ZUBAIRY, M. S. Quantum optics. Cambridge University Press, 1997. 630 p. ISBN 0 521 43595 1

[4] VEDRAL, Vlatko. Modern foundations of quantum optics. Imperial College Press, 2005. 222 p. ISBN 1-86094-553-8

[5] muj.optol.cz/bajer/prezentace/sumy.pps

[6] GRÖBLACHER, Simon. Dissertation-Quantum opto-mechanics with micromirrors: combining nano-mechanics with quantum optics. Wien, 2010

[7] HAMMERER, K., SÖRENSEN, A. S. and POLZIK, E. S. Quantum interface between light and atomic ensembles, Rev. Mod. Phys. 82, 1041 (2010)

[8] LAURAT, Julien, et al. Entanglement of two-mode Gaussian states: characterization and experimental production and manipulation, Journal of Optics B 7 (2005) S577

[9] ADESSO, Gerardo and ILLUMINATI, Fabrizio. Equivalence between entanglement and the optimal fidelity of continuous variable teleportation, Phys. Rev. Lett. 95, 150503 (2005)