VYSOKÉ UČENÍ TECHNICKÉ V BRNĚ
BRNO UNIVERSITY OF TECHNOLOGY

FAKULTA STROJNÍHO INŽENÝRSTVÍ
ÚSTAV MATEMATIKY

FACULTY OF MECHANICAL ENGINEERING
INSTITUTE OF MATHEMATICS

BIFURCATIONS IN A CHAOTIC DYNAMICAL SYSTEM

DIPLOMOVÁ PRÁCE
DIPLOMA THESIS

AUTOR PRÁCE
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VEDOUCÍ PRÁCE
SUPERVISOR

doc. Ing. LUDĚK NECHVÁTAL, Ph.D.

BRNO 2019

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MASTER'S THESIS
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Specification Master's Thesis

Department:	Institute of Mathematics
Student:	**George William Kateregga**
Study programme:	Applied Sciences in Engineering
Study field:	Mathematical Engineering
Supervisor:	**doc. Ing. Luděk Nechvátal, Ph.D.**
Academic year:	2018/19

Pursuant to Act no. 111/1998 concerning universities and the BUT study and examination rules, you have been assigned the following topic by the institute director Master's Thesis:

Bifurcations in a chaotic dynamical system

Concise characteristic of the task:

Chaos is an interesting phenomenon that can occur in a three or higher–dimensional nonlinear dynamical system. It was discovered by E. Lorenz (an American mathematician and meteorologist) around 1963, when trying to model a convection in the atmospehere. Chaotic systems exhibit a very complicated and (from a long–term point of view) unpredictable behavior. A characteristic feature is an enormous sensitivity to disturbances in initial conditions (two trajectories of a system starting close to each other separate exponentialy fast even though remain bounded).

The mentioned complex behavior is commonly accompanied by bifurcations. A bifurcation means a qualitative (or topological) change in the system's behavior when a small change is made to one or more system's parameters.

Goals Master's Thesis:

Theoretical part:
Study of selected topics from theory of nonlinear dynamical systems (equlibrium point, stability, limit cycle, bifurcation, chaos, strange attractor, largest Lyapunov exponent);
Practical part:
Stability and bifurcation analysis of a suitable chaotic system, numerical testing (e.g. the largest Lyapunov exponent, Poincaré section, bifurcation diagram, etc.).

Recommended bibliography:

PERKO, L. Introduction to Applied Nonlinear Dynamical Systems and Chaos, 2nd ed. Springer. 2003. ISBN 978-0387001777.

STROGATZ, S. H. Nonlinear Dynamics and Chaos, 2nd ed. Westview Press. 2015. ISBN 978-0813349107.

HIRSCH, M. W., SMALE, S., DEVANEY, R. L. Differential Equations, Dynamical Systems, and an Introduction to Chaos, Elsevier Science Publishing, 2012. ISBN 978-0123820105.

Deadline for submission Master's Thesis is given by the Schedule of the Academic year 2018/19

In Brno,

L. S.

prof. RNDr. Josef Šlapal, CSc. Director of the Institute	doc. Ing. Jaroslav Katolický, Ph.D. FME dean

Abstract

Dynamical systems possess an interesting and complex behaviour that have attracted a number of researchers across different fields, such as Biology, Economics and most importantly in Engineering. The complex and unpredictability of nonlinear customary behaviour or the chaotic behaviour, makes it strange to analyse them. This thesis presents the analysis of the system of nonlinear differential equations of the so–called Lu–Chen–Cheng system. The system has similar dynamical behaviour with the famous Lorenz system. The nature of equilibrium points and stability of the system is presented in the thesis. Examples of chaotic dynamical systems are presented in the theory. The thesis shows the dynamical structure of the Lu–Chen–Cheng system depending on the particular values of the system parameters and routes to chaos. This is done by both the qualitative and numerical techniques. The bifurcation diagrams of the Lu–Chen–Cheng system that indicate limit cycles and chaos as one parameter is varied are shown with the help of the largest Lyapunov exponent, which also confirms chaos in the system. It is found out that most of the system's equilibria are unstable especially for positive values of the parameters a, b. It is observed that the system is highly sensitive to initial conditions. This study is very important because, it supports the previous findings on chaotic behaviours of different dynamical systems.

keywords

Dynamical Systems, Bifurcation, Chaos, attractor, Lyapunov exponent.

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I declare that I have worked on this thesis independently under a supervision of doc. Ing. Luděk Nechvátal, Ph.D. and using the sources listed in the bibliography.

George William Kateregga

AKNOWLDGEMENT

I thank the Almighty God for giving me life and Knowledge to reach this accomplishment and level of education. My professor and supervisor doc. Ing. Luděk Nechvátal, I can't thank you enough. I am so appreciative to all your support and advice to see to it i produce this beautiful work. I thank my family and friends that have supported me unconditionally. Great thanks to all the professors in Italy and Czech Republic, for the great work done in me. May God bless you abundantly.

George William Kateregga

Contents

1 Introduction

Studying dynamics of systems became a point of interest in the past and in the most recent mathematics and engineering researches. This is due to its fundamental characteristic of being a time–evolutionary process. It can be deterministic or having a pattern of a complex system that can be analysed but unpredictable.

Nonlinear dynamics tries to answer questions of how a deterministic trajectory can be unpredictable, discusses routes to chaotic trajectories and other dynamical behaviours of systems. Over the past studies, interest and progress in nonlinear systems, chaos theory, and fractals have been noted, and this is reflected in many scientific journals.

The essense of this thesis, is to discuss the behaviour of the so-called Lu–Chen–Cheng system. It belongs to the a Lorenz-like family of systems. In 1963, Edward Lorenz worked on the paper "Deterministic nonperiodic flow" that described numerical results obtained by integrating the third-order nonlinear ordinary differential equations. He was trying to model a convection in the atmosphere. He named his findings "Butterfly effect". His work became famous and so influential and set the trend of studying chaotic systems.

In this thesis, we shall perform analysis of the Lu–Chen–Cheng system with respect to the set of values of the system's parameters.

1.1 Motivation

We shall investigate the routes to chaos by some qualitative as well as numerical techniques. The simulations will include e.g., bifurcation diagrams and calculation of the LLE (Largest Lyapunov Exponent) which is the basic indicator of a chaotic behavior. Focus will be put on bifurcations meaning the qualitative change of any system's structure with respect to any control parameter. Chaos is when a very small change may make the system behave completely differently. In other words, chaotic systems have extreme sensitivity to initial conditions.

The content of the thesis is organised as follows:

In Chapter Two, we discuss linear and nonlinear systems. We shall put more emphasis on nonlinear phenomena. These include, e.g., the finite escape time, possible existence of multiple isolated equilibria, existence of limit cycles, chaos, etc. Definition of a dynamical system, flow of a system, limit cycles, attractor and sensitive dependence of initial conditions, linearisation theorem and bifurcation will be presented here as well.

In Chapter Three, we present the system under investigation, computation of the equilibria and discussion of their existence and number depending on the parameters is addressed. We also put emphasis on the stability analysis of the equilibria based on the Routh Hurwitz test. Regions of parameters making the equilibria stable will be discussed. Hopf bifurcation is also discussed in line with the system's stability.

In Chapter Four, we present some simulations supporting the obtained theoretical results.

In Chapter Five; we have main conclusions of the work and for future tasks as open problems.

2 Nonlinear phenomena

2.1 Nonlinear Dynamics

In nonlinear systems, the equations of motion include at least one term that has a square, higher powers or even a product of system variables and more complicated functions. Unlike the linear systems, the addition of two solutions does not yield another valid solution, no matter how the system variables are defined.

All physical systems describable in terms of classical equations of motion are nonlinear. Examples are the pendulum dynamics, climate and other biological problems. The consequences of nonlinearity are profound. They can contain multiple attractors, and each one having its own basin of attraction. Thus nonlinear dynamical systems may be dependent on their initial conditions. The nonlinear phenomena arises when the basins of attraction change due to the variation of parameters [4] .

Autonomous dynamical system

Definition 2.1. An autonomous system is a system of ordinary differential equations of the form

$$\dot{x} = f(x, \mu) \tag{2.1}$$

where $x, \mu \in \Re$ and μ are the system parameters. When $f(x, \mu)$ if zero, the autonomous dynamical system is stationary and at this point, x is called a fixed point. The stationarity property means that no two trajectories cross because at every state x, the change in state determined by $f(x, \mu)$ is fixed. Thus the system does not explicitly depend on the independent variable. When the variable is time, they are also called time-invariant systems [11] .

Consider the initial value problem of a nonlinear autonomous dynamical system

$$\dot{x} = f(x) \tag{2.2}$$

$$x(0) = x_0 \tag{2.3}$$

where $x \in \Re^n$. We assume that $f : \Re^n \to \Re^n$ is a vector field of class C^r with $r \geq 1$, a condition for ensuring the existence and uniqueness theorem of (2.2).

2.2 The fundamental Existence-Uniqueness Theorem

According to [1], If you have a differential equation, the theorem guarantees that the differential equation has a unique solution provided $f \in C^1(E)$ where E is an open subset of $\Re^n$. The theorem is proved by use of Picard's approximations. Consider the initial value problem

$$\dot{y} = f(x, y)$$

$$y(x_0) = y_0,$$

where $(x_0, y_0) \in E$.

Theorem 2.2. *Let E be a domain in $\Re^2$, $f : E \subseteq \Re^2 \to \Re$ and $f : E \to \Re$ be a real function satisfying the following conditions*

1. *f is continuous on E*
2. *$f(x, y)$ is Lipschitz continuous with respect to x on E with positive Lipschitz constant α. Let (x_0, y_0) be an interior point on E and let $a > 0$, $b > 0$ be constants such that the rectangle $R = (x, y) : |x - x_0| \leq a, |y - y_0| \leq b \subset E$ Let*

$$M = \max_{(x,y)\in E} f(x, y)$$

$$h = \min\left(a, \frac{b}{M}\right)$$

then, the initial value problem has a unique solution y on the interval

$$|x - x_0| \leq h.$$

Proof. Since R is a closed rectangle in E, f satisfies all properties inside R If $a < \frac{b}{M}$ then $h = a$. If $\frac{b}{M} < a$then $h = \frac{b}{M}$ thus h is smaller than a, therefore

$$R = (x, y) : |x - x_0| \leq a, |y - y_0| \leq b,$$

$$R_1 = (x, y) : |x - x_0| \leq h, |y - y_0| \leq b$$

this means that if $a < \frac{b}{M}$ then $R_1 = R$ and if $\frac{b}{M} < a \implies R_1 \subset R$. Suppose we have iterants $\phi_1(x), \phi_2(x), \phi_3(x), ...$ on $|x - x_0| \leq h$ and are defined by

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)dt$$

$$\phi_2(x) = y_0 + \int_{x_0}^{x} f(t, \phi_1(t))dt$$

$$\cdots$$

$$\phi_n(x) = y_0 + \int_{x_0}^{x} f(t, \phi_{n-1}(t))dt \tag{2.4}$$

we prove the existence of solution to the IVP on $[x_0, x_0+h]$. Similar arguments hold for $[x_0-h, x_0]$. Then, uniqueness of solution follows from Uniqueness theorem. We divide the proof into 4 steps.
(i) The function ϕ_n defined by (2.4) is
a) well–defined,
b) ϕ_n's have continuous derivatives,
c) $|\phi_n(x) - y_0 \leq b$ on $[x_0, x_0 + h]$
d) $f(x, \phi_n(x))$ is well- defined.
From mathematical induction, we will assume that it is true for n-1, it is true for n and check for it is true for all n+1. Assume that $\phi_{n-1}(x)$ exists, and has a continuous derivative on $[x_0, x_0+h]$ and it satisfies $|\phi_{n-1}(x) - y_0| \leq b$ for $x \in [x_0, x_0 + h] \implies (x, \phi_{n-1}(x)) \in R_1$. At this point we have $f(x_0, \phi_{n-1}(x))$ defined and is continuous on the interval $[x_0, x_0+h]$. Further, $|f(x_0, \phi_{n-1}(x)| \leq M$ on the interval $[x_0, x_0 + h]$. Consider $\phi_n(x) = y_0 + \int_{x_0}^{x} f(t, \phi_{n-1}(t))dt \implies \phi_n(x)$ exists and has continuous derivative on interval $[x_0, x_0 + h]$. Also consider

$$\begin{aligned}
|\phi_{n-1}(x) - y_0| &= \left| \int_{x_0}^{x} f(t, \phi_{n-1}(t))dt \right| \\
&\leq \int_{x_0}^{x} \left| f(t, \phi_{n-1}(t)) \right| dt \\
&\leq \int_{x_0}^{x} M dt = M(x - x_0)h = \min(a, \frac{b}{M}) \\
&\leq Mh \leq b.
\end{aligned}$$

$(x, \phi_n(x))$ lies in the rectangle R, and hence $f(x, \phi_n(x))$ is defined and continuous on the interval $[x_0, x_0 + h]$. When $n = 1$, we have

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)dt$$

f is continuous and phi_1 is defined. Hence, has continuous derivative on $[x_0, x_0 + h]$. Also,

$$\begin{aligned}
|\phi_1(x) - y_0| &\leq \int_{x_0}^{x} \left| f(t, y_0) \right| dt \\
&\leq M(x - x_0) \\
&\leq Mh
\end{aligned}$$

$$\leq b.$$

This implies that $(x, \phi_1(x))$ is in R_1 and hence $f(x, \phi_n(x))$ is continuous on the interval $[x_0, x_0 + h]$, therefore, properties are true for $n = 1$. Thus, by the method of mathematical induction, ϕ_n the sequence of functions defined in (2.8) have all the desired properties in the interval $[x_0, x_0 + h]$. Hence part (i) of the proof.
(ii)The function ϕ_n satisfies the following inequality

$$\phi_n(x) - \phi_{n-1}(x) \leq \frac{M}{\alpha}\frac{(\alpha h)^n}{n!},$$

on $[x_0, x_0 + h]$ we prove this by mathematical induction. Assume that

$$\left|\phi_{n-1}(x) - \phi_{n-2}\right| \leq \frac{M\alpha^{n-2}}{(n-1)!}(x - x_0)^{n-1} \tag{2.5}$$

where $x \in [x_0, x_0 + h]$ Then

$$\left|\phi_n(x) - \phi_{n-1}\right| = \left|\int_{x_0}^{x} f(t, \phi_{n-1}(t)) - \int_{x_0}^{x} f(t, \phi_{n-2}(t))dt\right|$$

By part (i), $|\phi_n \leq b| \;\forall n$ and $x \in [x_0, x_0 + h]$, hence $(x, \phi_{n-1}(x)), (x, \phi_{n-2}(x))$ are in R_1 for $x \in [x_0, x_0 + h]$. By Lipschitz continuity of f, we have

$$\begin{aligned}
\left|\phi_n(x) - \phi_{n-1}\right| &\leq \alpha \int_{x_0}^{x} \left|\phi_{n-1}(t) - \phi_{n-2}(t)\right|dt \\
&\leq \alpha \int_{x_0}^{x} \frac{M\alpha^{n-2}}{(n-1)!(t - x_0)^{n-1}}dt \\
&\leq \frac{M\alpha^{n-1}}{(n-1)!}\left[\frac{(t - x_0)^n}{n}\right]_{x_0}^{x} = \frac{M\alpha^{n-1}}{n!}(x - x_0)^n = \frac{M}{\alpha}\frac{\alpha^n}{n!}h^n \\
&\leq \frac{M}{\alpha}\frac{(\alpha h)^n}{n!}.
\end{aligned}$$

This implies that the inequality is true for n. Let $n = 1$,

$$\begin{aligned}
\left|\phi_1(x) - y_0\right| &\leq \int_{x_0}^{x} \left|f(t, y_0)\right|dt \\
&\leq M(x - x_0) \\
&\leq Mh.
\end{aligned}$$

Therefore by mathematical induction, the inequality os true for all n. This proves part (ii).
(iii) As $n \to \infty$. ϕ_n converges uniformly to a continuous function ϕ on $[x_0, x_0 + h]$
To prove (iii) we need to show that

$$\left|\phi(x) - y_0\right| - \left|\int_{xo}^{x} f(t, \phi(t))dt\right| \leq \left|\phi(x) - \phi_n(x)\right| + \alpha \int_{x_0}^{x} \left|\phi(t) - \phi_{n-1}(t)\right|dt \tag{2.6}$$

$$\left|\phi(x) - y_0\right| - \left|\int_{xo}^{x} f(t, \phi(t))dt\right| \leq \left|\phi(x) - \phi_n(x)\right| + \alpha h \max_{x_0 \leq t \leq x_0 + h} \left|\phi(t) - \phi_{n-1}(t)\right| \tag{2.7}$$

And therefore, the uniform convergence of ϕ_n to ϕ implies that the right hand side of (2.5) tends to zero as $n \to \infty$, but the left hand side is independent of n. Thus ϕ satisfies the required properties of (2.8).

(iv) The limit function satisfies the give IVP on $[x_0, x_0 + h]$. We prove that the limit function has a unique solution to the IVP. Let $\overline{\phi}$ and ϕ satisfy (2.8) which yields,

$$\overline{\phi}(x) - \phi(x) \leq \int_{x_0}^{x} \Big| f(t, \overline{\phi}(t) - f(t, \phi(t)) \Big| dt.$$

Both $\overline{\phi}(x)$ and $\phi(t)$ lie in R_1 for all $t \in [x_0, x_0 + h]$ and hence it follows that

$$\overline{\phi}(x) - \phi(x) \leq \alpha \int_{x_0}^{x} \Big| \overline{\phi}(t) - \phi(t) \Big| dt.$$

Thus, from inequality (2.5),

$$\overline{\phi}(x) - \phi(x) \equiv 0.$$

on the interval $[x_0, x_0 + h]$ which means

$$\overline{\phi}(x) = \phi(x).$$

This proves (iv) and hence completes the proof [1].

□

2.3 Linearisation

In complex nonlinear systems, it is nearly impossible to solve the problems analytically. Linearisation is a natural simplification of the original system. It is qualitatively effective in predicting the patterns of the solution of the original system.

Definition 2.3. Suppose we have a linearised system of (2.2)

$$\dot{x} = Ax \tag{2.8}$$

where $A = Df(x_0)$ where x_0 is a critical point. If the dynamics of a system is described by a differential equation, then equilibria can be obtained by setting the derivative to zero.

Definition 2.4. Equilibrium points are zeros of the vector function $f(x)$ of system (2.2).

Remark. The point x^* is an equilibrium point if $f(x^*) = 0$.

In other words, solve the system, we equate the left hand side of the equations (2.2) to zero.

Two autonomous systems of differential equations such as (2.2) and (2.8) are said to be topologically equivalent in a neighborhood of the origin or to have the same qualitative structure near the origin if there is a homeomorphism H mapping to an open set U containing the origin onto an open set V and preserves their orientation by time in the sense that if a trajectory is directed from x_1 to x_2 in U, then its image is directed from $H(x_1)$ to $H(x_2)$ in V. If the homeomorphism H preserves the parametrization by time, then the systems (2.2) and (2.8) are said to be topologically conjugate in the neighbourhood of the origin [1].

Theorem 2.5 (The Hartman–Grobman theorem)**.** *Let E be an open subset of $\Re^n$ containing the origin, let $f \in C^1(E)$, and let ϕ_t be the flow of the nonlinear system* (2.2)*. Suppose that $f(0) = 0$, and that the matrix $A = Df(0)$ has no eigenvalues with zero real part. Then there exists a homeomorphism H of an open set U containing the origin onto an open set V containing the origin such that for every $x_0 \in U$, there is an open interval $I_0 \subset \Re$ containing zero such that for all $x_0 \in U$ and $t \in I_0$*

$$H \circ \phi_t(x_0) = e^{At} H(x_0) :$$

; i.e., H maps trajectories of (2.2) *near the origin onto trajectories of* (2.8) *near the origin and preserves the parametrisation by time.*

Outline of the proof [1]. Consider the nonlinear system (2.2) with $f \in C^1(E)$, $f(0) = 0$ and $A = Df(0)$.
1. Suppose that the matrix A is written in the form

$$A = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \tag{2.9}$$

where the eigenvalues of P have negative real part and the eigenvalues of Q have positive real part.
2. Let ϕ_t be the flow of the nonlinear system (2.2) such that the solution

$$x(t, x_0) = \phi_t(x_0) = \begin{pmatrix} y(t, y_0, z_0) \\ z(t, y_0, z_0) \end{pmatrix} \tag{2.10}$$

where $x_0 = \begin{pmatrix} x_0 \\ z_0 \end{pmatrix} \in \Re^n$, $y_0 \in E^s$, the stable subspace of A and $z_0 \in E^u$, the unstable subspace of A.
3. We define the functions

$$\overline{Y}(y_0, z_0) = y(1, y_0, z_0) - e^P y_0 \tag{2.11}$$

and

$$\overline{Z}(y_0, z_0) = z(1, y_0, z_0) - e^Q z_0 \tag{2.12}$$

such that $\overline{Y}(0) = \overline{Z}(0) = D\overline{Z}(0) = 0$. Since $f \in C^1(E)$, $\overline{Y}(y_0, z_0)$ and $\overline{Z}(y_0, z_0)$ are continuously differentiable. Thus,
$||D\overline{Y}(y_0, z_0)|| \leq a$ and $||D\overline{Z}(y_0, z_0)|| \leq a$ on the compact set $|y_0|^2 + |z_0|^2 \leq S_0^2$. The constant a can be taken as small as we like by choosing s_0 sufficiently small. We let $Y(y_0, z_0)$ and $Z(y_0, z_0)$ be smooth functions which are equal to $\overline{Y}(y_0, z_0)$ and $\overline{Z}(y_0, z_0)$ for $|y_0|^2 + |z_0|^2 \leq \left(\frac{S_0}{2}\right)^2$ and zero for $|y_0|^2 + |z_0|^2 \geq S_0^2$. Then by mean value theorem

$$|Y(y_0, z_0)| \leq a\sqrt{|y_0|^2 + |z_0|^2} \leq a(|y_0| + |z_0|) \tag{2.13}$$

and

$$|Z(y_0, z_0)| \leq a\sqrt{|y_0|^2 + |z_0|^2} \leq a(|y_0| + |z_0|)$$

for all $(y_0, z_0) \in \Re^n$. We next let $B = e^P$ and $C = e^Q$. Then assuming that we have carried out the normalization, $b = ||B|| \leq 1$ and $C = ||C^-1|| \leq 1$.

4. For $x = \begin{pmatrix} y \\ z \end{pmatrix} \in \Re^n$ we define the transformation $L(y, z) = \begin{pmatrix} By \\ Cz \end{pmatrix}$ and

$$T(y, z) = \begin{pmatrix} By + Y(y, z) \\ Cz + Z(y, z) \end{pmatrix};$$

i.e., $L(x) = e^A x$ and locally $T(x) = \phi_1(x)$.

Lemma 2.6. *There exists a homeomorphism H of an open set U containing the origin onto an open set V containing the origin such that*

$$H \circ T = L \circ H.$$

Proof. see [1] □

The lemma above is established using the method of successive approximations.
For $x \in |\Re^n$, let

$$H(x) = \begin{pmatrix} \phi(y,z) \\ \phi(y,z). \end{pmatrix}.$$

Then $H \circ T = L \circ H$ is equivalent to the pair of equations

$$B\phi(y,z) = \phi(By + Y(y,z), Cz + Z(y,z)), \tag{2.14}$$
$$C\psi(y,z) = \psi(By + Y(y,z), Cz + Z(y,z)). \tag{2.15}$$

First of all, we define successive approximations for the second equation by

$$\psi_0(y,z) = z, \tag{2.16}$$
$$\psi_{k+1}(y,z) = C^{-1}\psi_k(By + Y(y,z), Cz + Z(y,z)). \tag{2.17}$$

It then follows by an induction argument that for $k = 0, 1, 2, 3, \dots$, the $\psi_k(y,z)$ are continuous and satisfy $\psi_k(y,z) = z$ for $|y| + |z| \geq 2s_0$. We prove by induction that for $j = 1.2, \dots$

$$|\psi_j(y,z) - \psi_{j-1}(y,z)| \leq Mr^j(|y| + |z|)^\sigma \tag{2.18}$$

where $r = C[2\max(a,b,c)]^\sigma$ with $\sigma \in (0,1)$ chosen sufficiently smooth so that $r \leq 1$ (which is possible since $C \leq 1$) and $M = ac(2s_0)^{1-\frac{\sigma}{r}}$. First of all for $j = 1$

$$\begin{aligned} |\psi(y,z) - \psi_0(y,z)| &= z|C^-1\psi_0(By + Y(y,z), Cz + Z(y,z)) - z| \\ &= |C^-1(Cz + Z(y,z)) - z| \\ &= ||C^{-1}z(y,z)|| \\ &\leq ||C^-1|||z(y,z)| \\ &\leq ca(|y| + |z|) \\ &\leq Mr(|y| + |z|)^\sigma \end{aligned} \tag{2.19}$$

since $z(y,z) = 0$ for $|y| + |z| \geq 2s_0$, and then assuming that the induction hypothesis holds for $j = 1, \dots, k$ we have

$$\begin{aligned} |\psi_{k+1}(y,z) - \psi_k(y,z)| &= |C^{-1}\psi_k(By + Y(y,z), Cz + Z(y,z)) - C^{-1}\psi_{k-1}(By + Y(y,z), Cz + Z(y,z))| \\ &\leq ||C^{-1}|||\psi_k(By + Y(y,z), Cz + Z(y,z)) - \psi_{k-1}(By + Y(y,z), Cz + Z(y,z))| \\ &\leq cMr^k||By + Y(y,z)|| + |Cz + Z(y,z)| \\ &\leq cMr^k[|By + Y(y,z)| + |cz + z(y,z)|]^\sigma \\ &\leq cMr^k[b|y| + 2a(|y| + |z|) + c|z|]^\sigma \\ &\leq cMr^k[2\max(a,b,c)]^\sigma(|y| + |z|)^\sigma \\ &= Mr^{k+1}(|y| + |z|)^\sigma. \end{aligned} \tag{2.20}$$

Thus $\psi_k(y,z)$ is a Cauchy sequence of continuous functions which converges uniformly as $k \to \infty$ to a continuous function $\psi(y,z)$. Also, $\psi(y,z) = z$ for $|y| + |z| \geq 2s_0$. Taking limits to (2.12) shows that $\psi(y,z)$ is a solution of the second equation in (2.14) and (2.15).
The equation (2.9) can be written as

$$B^{-1}\psi(y,z) = \psi(B^{-1}y + Y_1(y,z), C^{-1}z + z_1(y,z)) \tag{2.21}$$

where the functions Y_1 and Z_1 are defined by the inverse of T (which exists if the constant a is sufficiently small, i.e., if s_0 is a sufficiently small) as follows;

$$T^{-1}(y,z) = \begin{pmatrix} B^{-1}y + Y_1(y,z) \\ C^{-1}z + Z_1(y,z) \end{pmatrix}$$

then equation (2.15) can be solved by $\pi(y,z)$ by the method of successive approximation exactly as above with $\pi_0(y,z) = y$ since $b = ||B|| < 1$.We therefore obtain the continuous map.

$$H(y,z) = \begin{pmatrix} \pi(y,z) \\ \psi(y,z) \end{pmatrix}.$$

5.Let H_0 be the homeomorphism defined above and let L^t and T^t be the one-parameter families of transformations defined by

$$L^t(x_0) = e^{At}x_0$$

and

$$T^t(x_o) = \pi_t(x_0)$$

Defining

$$H = \int_o^1 L^{-s} H \circ T^s ds;$$

it follows that using the above lemma that there exists a neighborhood of the origin for which

$$\begin{aligned} L^t H &= \int_0^1 L^{-s} H \circ T^{s-t} ds T^{-t} \\ &= \int_{-t}^{1-t} L^s H \circ T^s ds T^t \\ &= \left[\int_{-t}^0 L^{-s} H \circ T^s ds + \int_0^{1-t} L^{-s} H \circ T^s ds T \right] T^t \\ &= \int_0^1 L^{-s} H \circ T^s ds T^t = HT^t. \end{aligned} \tag{2.22}$$

Since by the above Lemma $H_0 = L^{-1} H \circ T$ which implies that

$$\int_t^0 L^{-s} H \circ T^s ds = \int_{1-t}^1 L^s ds. \tag{2.23}$$

And it can be shown that H is a homeomorphism on $\Re^n$. This completes the outline of the proof of the Hartman–Grobman theorem [1]. □

Remark. The linearised matrix A or the Jacobian is always evaluated at a fixed point $x_0 \in \Re^n$. The Jacobian matrix is given as

$$J = Df(x_0) = \begin{pmatrix} \dfrac{f_1(x_0)}{\partial x_1} & \cdots & \dfrac{f_1(x_0)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{f_n(x_0)}{\partial x_1} & \cdots & \dfrac{f_1(x_0)}{\partial x_1}. \end{pmatrix} \tag{2.24}$$

The eigenvalues μ of the Jacobian can be obtained from the characteristic equation

$$P(\mu) = det(J - \mu I), \tag{2.25}$$

where I represents the identity matrix.

Definition 2.7. An equilibrium point x_0 of the system (2.2) is called hyperbolic if none of the eigenvalues of the Jacobian matrix $J = Df(x_0)$ has zero real part.

Otherwise, the equilibrium point is called non hyperbolic. If the fixed point x_0 is hyperbolic, then according to Hartman–Grobman theorem, there exists a neighborhood of this point, in which the nonlinear system (2.2) is topologically conjugate to the system (2.8), where A is the linearisation matrix.

Example 2.8. Compare trajectories of the system

$$\dot{x_1} = -x_1, \quad \dot{x_2} = x_1^2 + x_2,$$

and its linearization around the zero equilibrium.

Taking the equilibrium point $(0, 0)$, we have the linearised system as

$$J = Df(x_0) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.26}$$

Let $\dot{y_1} = y_1$ and $\dot{y_2} = y_2$. be the linearization, with the solution of $y_1(0) = y_1^0, y_2(0) = y_2^0$, calculated as

$$y_1(t) = y_1^0 \exp(-t)$$

and

$$y_2(t) = y_2^0 \exp(t)$$

In this particular case, we can as well calculate the solution of the original system as

$$x_1(t) = x_1^0 e^{-t},$$

and

$$x_2(t) = x_2^0 e^t + (x_1^0)^2 (e^t - e^{-2t})/3.$$

respectively.

The figures 1 and 2 below show the topological equivalence of the two systems. Trajectories of the considered nonlinear system and its linearisation around origin.

2.4 Limit cycles and attractors

In reality, all we need to know to predict the long term behaviour of our nonlinear dissipative system is the position of the attractor, or perhaps which kind of attractor and the basin of attraction. There are four kinds of attractors. The first one is the simplest attractor called a fixed point which we discussed earlier. It describes the stationary longterm behaviour where the system eventually stops evolving. In the phase space, the trajectories around a fixed point are presented basically as an open loop or some trajectory being convergent to a point. The second one is slightly more complicated attractor called a limit cycle (kind of periodic). The tori (kind of quasi–periodic) is the third kind of attractor and it is also complicated. It behaves as a closed surface. Tori normally exist in higher n–dimensions, i.e., ≥ 3.
Finally, the fourth and strange one which is chaotic in nature is the "strange attractor". These attractors are quite remarkable. Strange attractors are famous for being sensitive to initial conditions. It should be noted that only fixed points can be found analytically, and this is only if the system is "nice" (low–dimensional, simple nonlinearity).For limit cycles can be found analytically only for exceptional cases and unfortunately, the tori and chaotic attractors cannot be found analytically [2].

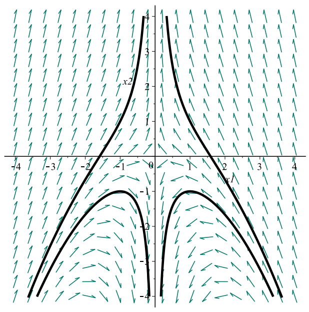


Figure 1: Phase Portrait of the original nonlinear system

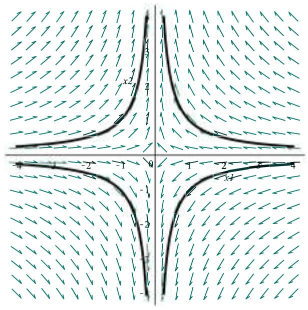


Figure 2: Behaviour of trajectories after linearization around the origin

Definition 2.9. A limit cycle is an isolated closed trajectory that attracts at least one other trajectory.

This means that its neighbouring trajectories are not closed – they spiral either towards or away from the limit cycle. Thus, limit cycles can only occur in nonlinear systems.

Remark. A stable limit cycle is one which attracts all neighbouring trajectories. A system with a stable limit cycle can exhibit self-sustained oscillations, most of the biological processes of interest are of this kind.

The neighbouring trajectories are repelled from unstable limit cycles. Half–stable limit cycles are, of course, ones which attract trajectories from one side and repel those on the other.

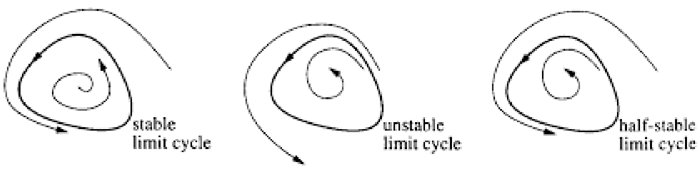


Figure 3: Limit cycles

Example 2.10. Consider the Van der Pol oscillator. It has a stable limit cycle as shown in Figure 4.

$$\dot{x_1} = x_2$$
$$\dot{x_2} = \mu(1 - x_1^2)x_2 - x_1$$

where $\mu > 0$ is a damping parameter.

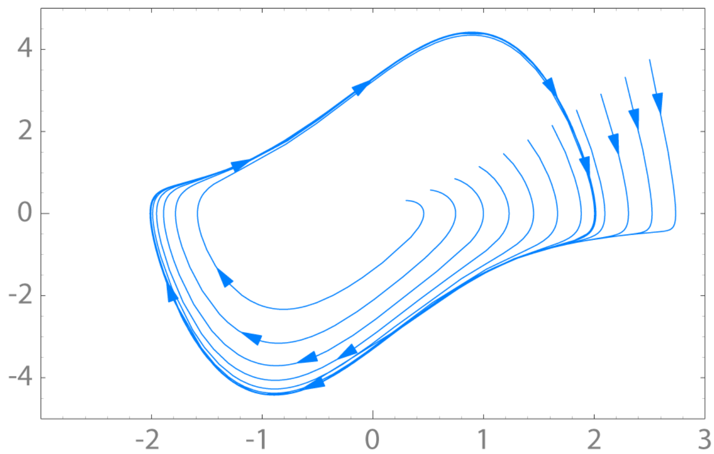


Figure 4: A stable limit cycle of the Van der Pol oscillator

Theorem 2.11 (Poincaré–Bendixson theorem)**.** *Suppose that: R is a closed bounded subset of the plane;* $\frac{dx}{dt} = f(x)$ *is a continuously differentiable vector field on an open set containing R;R does not contain any fixed points; There exists a trajectory C that is confined in R, in the sense that it starts in R and stays in R for all future time. Then either C is the closed orbit, or it spirals towards closed orbit as* $t \to \infty$*. So, R contains a closed orbit.*

Proof. The proof of this theorem is subtle, and requires some advanced ideas from Perko (1991), Coddington and Levinson (1955), Hurewicz (1958), or Cesari (1963). □

Trajectories from the limit cycle's basin of attraction tend toward the limit cycle either in forward or backward time. Limit cycle corresponds to a periodic behaviour. For a system:

$$\dot{x} = P(x, y) \tag{2.27}$$

$$\dot{y} = Q(x, y) \tag{2.28}$$

$x(t + T) = x(t)$ and $y(t + T) = y(t)$: periodic movement with period $T > 0$.

The Poincare–Bendixson theorem is one of the central results of nonlinear dynamics. It reveals that the dynamical possibilities in the phase plane are very limited:if a trajectory is confined to a closed, bounded region that contains no fixed points, then the trajectory must eventually approach a closed orbit. Nothing more complicated is possible. This result depends crucially on the two-dimensionality of the plane. In higher dimensional systems ($n \geq 3$), the Poincare–Bendixson theorem no longer applies, and something radically new can happen: trajectories may wander around forever in a bounded region without settling down to a fixed point or a closed orbit. In some cases, the trajectories are attracted to a complex geometric object called a strange attractor. As discussed earlier that strange attractors are highly sensitive to initial conditions, this sensitivity makes the motion unpredictable in the long run. We are now face to face with chaos. We'll discuss this fascinating topic soon enough.
The following definition of an attracting set is an adaptation of the rigorous definition given by Milnor [5]. While Milnor's is very generalized and applies in a wide variety of mathematical contexts, the one given here has been simplified to deal only in Euclidean space (i.e., $\Re^n$).

Definition 2.12. A set $A \subset \Re^n$ will be called an attracting set if the following two conditions are true:
1. The basin of attraction $B(A)$, consisting of all points whose orbits converge to A, has strictly positive measure.
2. For any closed proper subset $A' \subset A$, the set difference $B(A) \setminus B(A')$ also has strictly positive measure.

Since we are restricted to $\Re^n$, it should be easy to gain an intuitive understanding of what the above definition of an attracting set means. The first condition basically states that our basin of attraction must be, in some sense, tangible. It cannot be a single point or a set of discontinuous points (sets whose measures are zero). Our basin of attraction must consist of some sort of n–dimensional interval. If we consider the space $\Re^2$, the first condition states that any basin of attraction must have some positive area. If we consider the space $\Re^3$, any basin of attraction must have some positive volume.The second condition is slightly more nuanced. It says that if we were to change our attracting set at all, then the measure of our basin of attraction would also change.
In $\Re^2$, for example, if we were to remove a single point from the attracting set, the basin of attraction would lose area. Alternatively, if we remove points from A by defining a new, closed set $A' \subset A$, and it turns out that the basin does not significantly change (the set difference $B(A) \setminus B(A')$ has measure zero), then the implication would be that the removed points were not actually attracting a significant portion of the basin. In this case, A would not be considered an attracting set. Milnor gives the a nice explanation of the two conditions for an attracting set [5]. : "the first condition says that there is some positive possibility that a randomly chosen point will be attracted to A, and the second says that every part of A plays an essential role."
In order for an attracting set to be considered an attractor, it must satisfy a third condition The term strange is usually used for attractors that exhibit chaotic behavior – i.e., sensitivity to initial conditions. Though it is true, the use of the term is somehow misleading. It is important to clarify that strangeness is not dependent on the existence of chaos. Though attractors showing extreme sensitivity to initial conditions are indeed strange, strange attractors need not be chaotic [6].

Definition 2.13. An attractor is strange if its attracting set is fractal in nature.

While the term chaotic is meant to convey a loss of information or loss of predictability, the term strange is meant to describe the unfamiliar geometric structure on which the motion moves in phase space [6]. In a chaotic regime, orbits on an attractor are non-periodic. Thus, any given point in the attracting set is never visited more than once, and there are entire regions of points that are never visited. Such sets of points are fractal in nature and usually have non-integer dimension. It follows that if an attractor exhibits chaotic behavior, then it is a strange attractor.[2]. Suppose A is a closed set, an attract has the given properties:

1. A is an invariant set; any trajectory $x(t)$ starting in A stays in A all the time.
2. An open set of initial conditions are are attracted by A and there exists an open set U containing A such that $x(0) \in U$, then the distance from $x(t)$ to A attracts all trajectories that start sufficiently close to it. It is noted that the largest U is the basis of attraction os A.
3. A is minimal; there exists no proper subset of A that satisfies conditions 1 and 2 [2].

Example 2.14. In 1963, Edward Lorenz (1917–2008), studied convection in the Earth's atmosphere. As the Navier–Stokes equations that describe fluid dynamics are very difficult to solve, he simplified them drastically. The model he obtained probably has little to do with what really happens in the atmosphere. It is a toy-model, but Lorenz soon realised that it is very interesting in a mathematical sense. There are only three parameters in the model so that each point (x, y, z) symbolises a state of the atmosphere.

Consider the Lorenz system of differential equations; which was his model of convection in the atmosphere.

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = x(\rho - z) - y$$
$$\dot{z} = xy - \beta z$$

where $\sigma, \rho, \beta > 0$ and α is the Prandtil number representing the ratio of the fluid viscosity to its thermal conductivity, ρ represents the difference in temperature between the top and bottom of the system, and β is the ratio of the width to height of the box used to hold the system. The choice of parameter values Lorenz used are $\sigma = 10, \rho = 28, \beta = 8/3$. Basically the equations model the flow of fluid (particularly air) from hot area to cold area. On the surface these three equations seem simple to solve. However, they represent an extremely complicated dynamical system. If one plots the results in three dimensions the following figure, called the Lorenz attractor, is obtained[8].

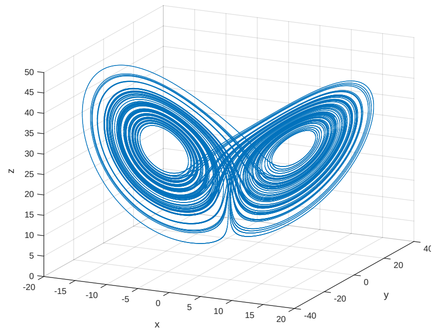


Figure 5: The Lorenz strange attractor

The evolution of the weather thus boils down to following trajectories in a vector field. Again, it is a toy model, and the objective is to try and understand some very complex behaviour. Two conditions of the atmosphere that are extremely close will rapidly evolve quite differently: after a while they represent conditions that are wide apart. Lorenz discovered this sensitivity to initial conditions in his model.

If we take a large number of different initial conditions, then after a while they all land on the same object in the shape of a butterfly: the Lorenz attractor. As we said already that the Lorenz attractor is an example of a strange attractor. Strange attractors are unique from other phase-space attractors in that one does not know exactly where on the attractor the system will be. Two points on the attractor that are near each other at one time will be arbitrarily far apart at later times. The only restriction is that the state of system remain on the attractor. Strange attractors are also unique in that they never close on themselves — the motion of the system never repeats (non-periodic). The motion we are describing on these strange attractors is what we mean by chaotic behavior [3]. The Lorenz attractor was the first strange attractor, but there are many systems of equations that give rise to chaotic dynamics. Examples of other strange attractors include the Rössler, Chua and Hénon attractors among others. see [14].

Example 2.15. The Chua's Dynamical system: The circuit diagram of the Chua Circuit is shown in Figure 4. It contains 5 circuit elements. The first four elements on the left are standard off-the-shelf linear passive electrical components; namely, inductance $L > 0$, resistance $R > 0$, and two capacitances $C_1 > 0$ and $C_2 > 0$. They are called passive elements because they do not need a power supply (e.g., battery). Interconnection of passive elements always leads to trivial dynamics, with all element voltages and currents tending to zero (Chua, 1969).

The Chua's Circuit is described by the equations;

$$\dot{x} = \alpha(y - \phi(x))$$

$$\dot{y} = x - y + z$$
$$\dot{z} = -\beta y.$$

where α andβ are real numbers, and $\phi(x)$ is a scalar function of the single variable x. The

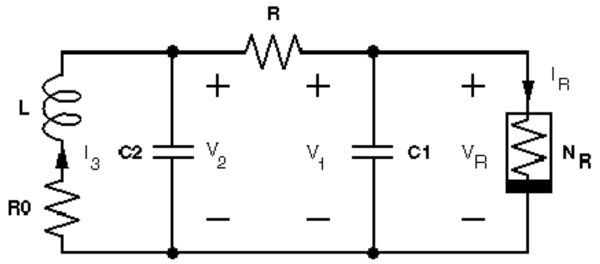


Figure 6: The Chua's circuit.

Chua equations are simpler than the Lorenz equations in the sense that it contains only one scalar nonlinearity, whereas the Lorenz equations contains 3 nonlinear terms, each consisting of a product of two variables (Pivka et al, 1996). In the original version studied in-depth in (Chua et al, 1986), $\phi(x)$ is defined as a piecewise-linear function;

$$\phi(x) \triangleq x + g(x) = m_1 x + \frac{1}{2}(m_0 - m_1)[|x+1| - |x-1|] \tag{2.29}$$

where m_0 and m_1 denote the slope of the inner and outer segments of the piecewise-linear function in the circuit.
Although simpler smooth scalar functions, such as polynomials, could be chosen for $\phi(x)$ without affecting the qualitative behaviors of the Chua equations, a continuous (but not differentiable) piecewise-linear function was chosen strategically from the outset in (Chua et al, 1986) in order to devise a rigorous proof showing the experimentally and numerically derived double scroll attractor is indeed chaotic [13].

Fractal Geometry of the Double Scroll Attractor: Based on an in-depth analysis of the phase portrait located in each of the 3 linear regions of the $x - y - z$ state space, as well as from a specific numerical values of parameters of the double scroll attractor shown in Figure 5, the geometrical structure of the double scroll attractor is found to consist of a juxtaposition of infinitely many thin, concentric, oppositely-directed fractal-like layers. The local geometry of each cross section appears to be a fractal at all cross sections and scales[2]. Chaotic systems have an interesting complex non linear phenomenon which has been intensively studied in the last four decades within the science, Mathematics and engineering communities. The term chaos is also difficult to define in a rigorous way. Loosely speaking, chaos is the science of surprises, of the nonlinear and the unpredictable systems. Transient chaos shows that a deterministic system can be unpredictable, even if its final states are very simple. The high sensitivity of chaotic dynamical systems brings the unpredictable behavior in the system[8]. As shown below in Figure fig. 8 , two trajectories start close and diverge exponentially; A defining attribute of an attractor on which the dynamics is chaotic is that it displays exponentially sensitive dependence on initial conditions. Consider two nearby initial conditions $x_1(0)$ and $x_2(0) = x_1(0) + \Delta(0)$, and imagine that they have evolved in time by a continuous time dynamical system yielding orbits $x_1(t)$ and $x_2(t)$ as shown in Figure 6 above. At time t, the separation between the two orbits is

$$\Delta(t) = x_2(t) - x_1(t).$$

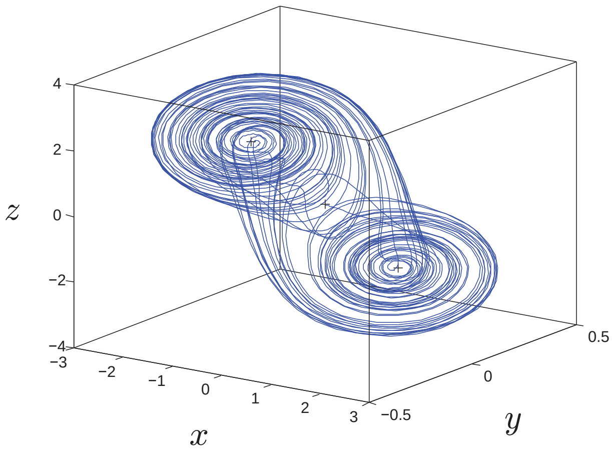


Figure 7: Chua's circuit: $\alpha = 10$ and $\beta = 15$ [13]

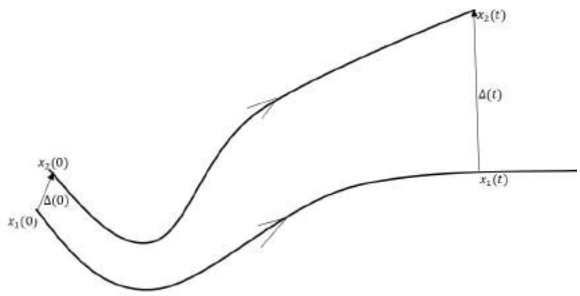


Figure 8: Divergence of orbits in a phase space [5].

If in the limit $\left|\Delta(t)\right| \to 0$, and large t, orbits remain bounded and the difference between the solutions $\left|\Delta(t)\right|$ grows exponentially for typical orientation of the vector $\Delta(0)$;

$$\left|\frac{\Delta(t)}{\Delta(0)}\right| \quad \exp(ht), h > 0,$$

then we write that the system displays sensitive dependence on initial conditions and is chaotic [2].

By bounded solutions, we mean that there is some ball in phase space, $|x| < R < \infty$, which solutions never leave[2]. This means that if the motion is on an attractor, then the attractor lies in $|x| < R$. The reason of imposing the restriction that orbits remain bounded is that, if orbits go to infinity, it is relatively simple for their distances to diverge exponentially.

The exponential sensitivity of chaotic solutions means that as time goes on, small errors in the solution can grow very rapidly. On the existence of chaos,the bifurcation diagrams and nature the of Lyapunov Exponents give the confirmation whether chaos exits or not.

2.5 Bifurcation

The qualitative behaviour of system (2.1) changes as we change the function or vector field $\boldsymbol{f}$ in (2.1).

Definition 2.16. Bifurcation is the qualitative change in behavior of the solution set of a system

$$\dot{x} = f(x, \mu)$$

depending on a system parameter. The behavior changes as the vector field f passes through a point in the bifurcation set or as the parameter varies through a bifurcation value μ_0.

[9].

The qualitative structure of the flow can change as parameters are varied. In particular, fixed points can be created or destroyed, or their stability can change. These qualitative changes in the dynamics are called bifurcations, and the parameter values at which they occur are called bifurcation points. If the qualitative behavior remains the same for all nearby vector fields, then the system (2.2) or the vector field $\boldsymbol{f}$ is said to be structurally stable.

The qualitative structure of the solution set or of the global phase portrait of (2.2) changes as the vector field $\boldsymbol{f}$ passes through a point in the bifurcation set.
Consider the equations;(2.1) and

$$\dot{x} = g(x) \tag{2.30}$$

$$\dot{x} = f(x, \mu) \tag{2.31}$$

Definition 2.17. Let E be an open subset of $\mathbb{R}$. A vector field $\boldsymbol{f} \in C^1(E)$ is said to be structurally stable if there is an $\epsilon > 0$ such that for all $\boldsymbol{g} \in C^1(E)$ with

$$||\boldsymbol{f} - \boldsymbol{g}||_1 < \epsilon$$

f and g are topologically equivalent on E; i.e., there is a homeomorphism $H : E \mapsto E$ which maps trajectories of (2.1) and preserves their orientation by time.

Structural stability is typical of any dynamical system modeling a physical problem. Considering an example of a damped pendulum. Mass, length and friction are changed in the pendulum by just a small value ϵ. This means that the qualitative behavior of the solution remains un changed; i.e., the global phase geometrical structures of the two systems (2.1) and (2.32) modeling the two pendula will be topologically equivalent. Thus, the dynamical system (2.1) modeling the physical system consisting of a damped pendulum is structurally stable [8].

Remark. Given $E = \Re^n$, then the ϵ-perturbations of f in the above definition, i.e., the functions $g \in C^1(E)$ satisfying

$$||\boldsymbol{f} - \boldsymbol{g}||_1 < \epsilon$$

include the C^1-ϵ-perturbations. Also if K is a compact subset of E and if $g \in C^1(K)$ satisfies

$$\max_{x \in K} |f(x) - g(x)| + \max_{x \in K} ||Df(x) - Dg(x)|| \tag{2.32}$$

then there exists a compact subset k of E containing K and a function $g \in C^1(E)$ such that $\overline{g}(x) = g(x)$ for all $x \in \overline{K}$, $\overline{g}(x) = f(x)$ for all $x \in E \sim K$ and $||\boldsymbol{f} - \overline{\boldsymbol{g}}||_1 < \epsilon$. Thus, in order to show that $f \in C^l(\Re^n)$ is not structurally stable on $\Re^n$, it suffices to show that f is not structurally stable on some compact $K \subset \Re^n$ with nonempty interior.

Definition 2.18. A point $x \in E$ is a non-wandering point of the flow ϕ_t defined by (2.32) if for any neighborhood U of x and for any $T > 0$ there is a $t > T$ such that

$$\phi_t(U) \cap U \neq 0.$$

The non-wandering set Ω of the flow ϕ_t is the set of all non-wandering points of $\phi_t \in E$. Any point $x \in E \equiv \Omega$ is called a wandering point of ϕ_t

Non-wandering points of a flow are the equilibrium points and points on periodic orbits. These points are for a relatively-prime, planar, analytic flow. The only non-wandering points are critical points, points on cycles and points on graphics that belong to the ω-limit set of a trajectory or the limit set of a sequence of periodic orbits of the flow (on $\Re^2$ or on the Bendixson sphere)[1].

3 Analysis of Lu-Chen-Cheng system

In Chapter two, we discussed the nonlinear phenomena of dynamical systems in relation to chaos and several chaotic models have been presented. As we said before, in this chapter, we focus directly on the analysis of the Lu-Chen-Cheng autonomous nonlinear system.
We investigate the local behavior of the equilibria in their neighborhood, and the behavior of the trajectories depending on the parameter variations. Phase portraits and Bifurcations in chapter will also follow as per the numerical analysis of the system.
According to [12], Vanecek and Celikovsky [1999] introduced the so-called generalized Lorenz system. More recently in 2004, Lu, Chen and cheng; discovered some similar but different chaotic systems. Below is one of those systems that is under our analysis. The system's equations are;

The system has some similarities with the famous Lorenz system, but has a richer dynamics depending on the parameters

$$\begin{cases} \dot{x} = -\frac{ab}{a+b}x - yz + c \\ \dot{y} = ay + xz \\ \dot{z} = bz + xy, \end{cases} \tag{3.1}$$

where a, b, c are real constants. In our analysis, we assume 3 cases of the nature of the parameters in the system. The system shows chaotic behaviour for different values of c and a as the mostly varied parameters in this work, i.e., for $a = -10, \quad b = -4$ and $c = 19, 8, 2\text{and} - 60$ as shown in the figures below.

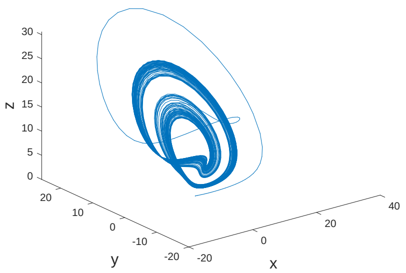


Figure 9: Attractor with respect to parameters $a = -10, b = -4$ and $c = 19$ with $(1, -1, 1)$, as initial condition.

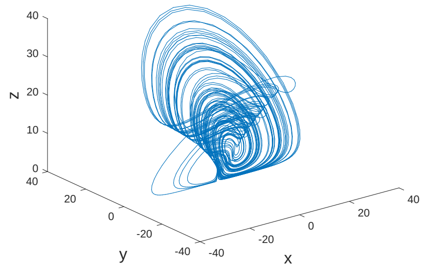


Figure 10: Attractor with respect to parameters $a = -10, b = -4$ and $c = 8$ with $(1, -1, 1)$, as initial condition.

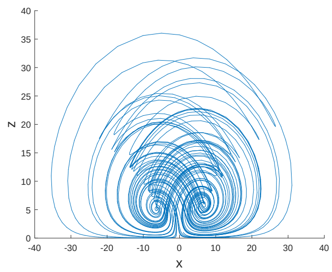


Figure 11: Attractor with respect to parameters $a = -10, b = -4$ and $c = 2$ with $(1, -1, 1)$, as initial condition.

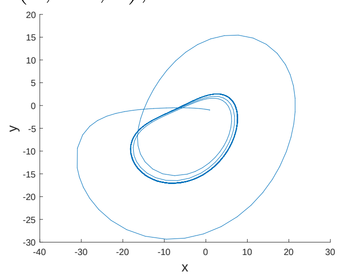
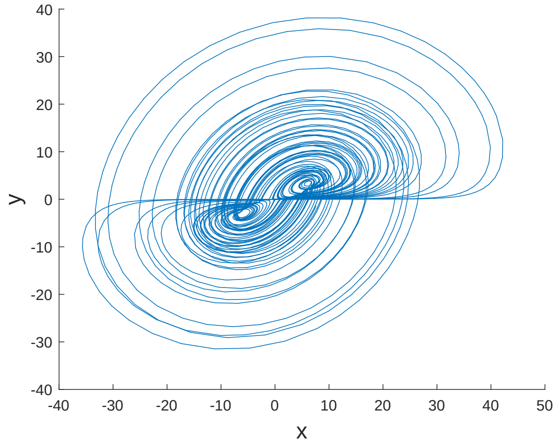
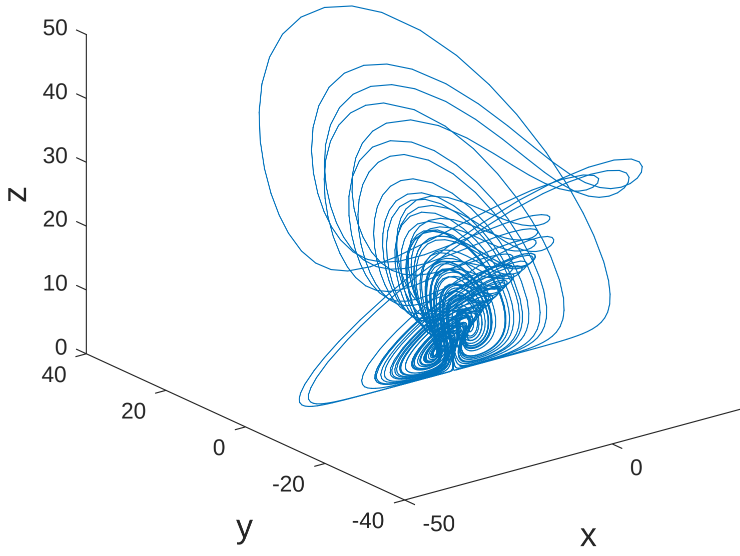


Figure 12: Attractor with respect to parameters $a = -10, b = -4$ and $c = -60$ with $(1, -1, 1)$, as initial condition.

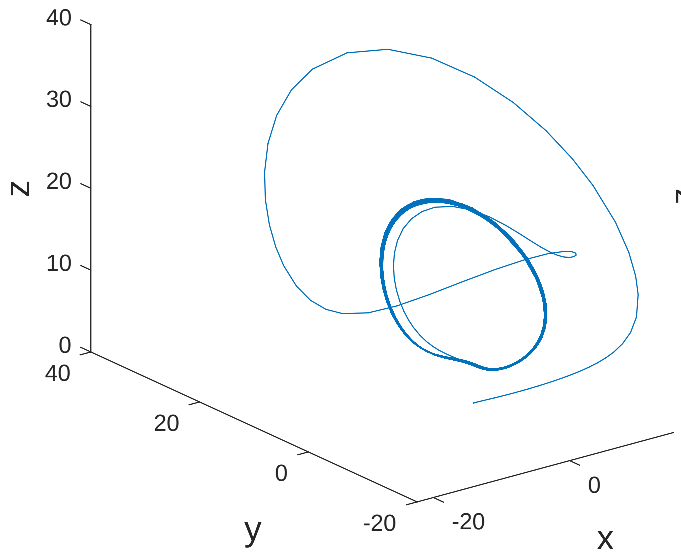
Pictures in fig. 9 to fig. 12 are generated by one initial condition, but what brings the difference in their structures is the change in parameter c. It is observed that the system is chaotic in several regions, for example, when $|c| = 19$, we have 2 two chaotic attractors with dense orbits. The system produces 2 periods of two limit cycles when the parameter c is between 22.4 and 31.2 as shown in 13c. The system also appears to have attractors when the varying parameter is from $8.0 \leq |c| \leq 19.1$. we have partial and bounded attractors whenever the system parameter c is such that $2.0 \leq |c| \leq 8.3$ In the interval of $11.8 \leq |c| \leq 14.8$, the system has a periodic window.



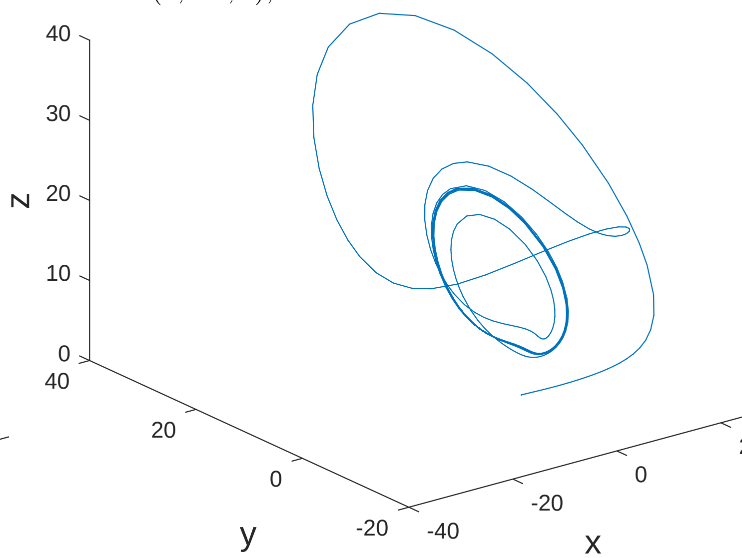
(a) The xy view of the periodic window with respect to parameters $a = -10, b = -4$ and $c = 10$ with $(1, -1, 1)$, as initial condition.



(b) the xyz view of the attractor with respect to parameters $a = -10, b = -4$ and $c = 0$ with $(1, -1, 1)$, as initial condition.



(c) Limit cycles of the system with respect to parameters $a = -10, b = -4$ and $c = 30$ with $(1, -1, 1)$, as initial condition.



(d) Attractor with respect to parameters $a = -10, b = -4$ and $c = 45$ with $(1, -1, 1)$, as initial condition.

Figure 13: Chaotic attractors and limit cycles of the system

3.1 Phase portraits of the system

The following dynamical structures of the system show limit cycles of the system, attractors and chaotic regions depending on the variation of system parameters and change in initial conditions. Figures fig. 14 and Figure fig. 14 show a 3-D visualisation of the system with $2-D$ views of $x-y, y-z$ and $x-z$ planes respectively. Other topological structures are also observed with respect to different axes. These diagrams are generated from two initial conditions $[5, 1, 5]$ and $[5, 1, -5]$, showing the upper attractor and lower attractor respectively. The pictures in Figure 12 are generated from the initial conditions $[1, -1, 1]$ and $[2, 0, -2]$ with parameter c changed to zero. In future of this thesis, we shall show these behaviours from the bifurcation diagram.

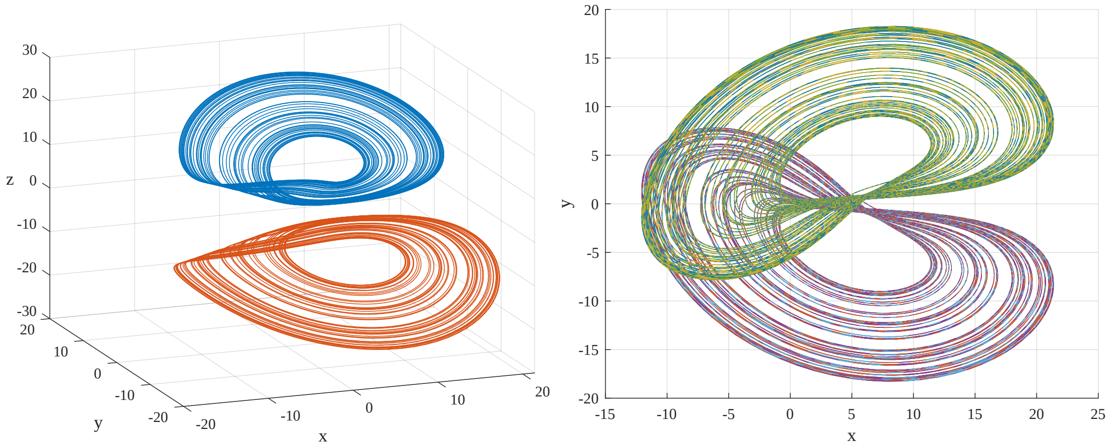


Figure 14: Dynamical behaviors of the system: xyz and $x-y$ planar view with $a = -10, b = -4$ and $c = 18.1$

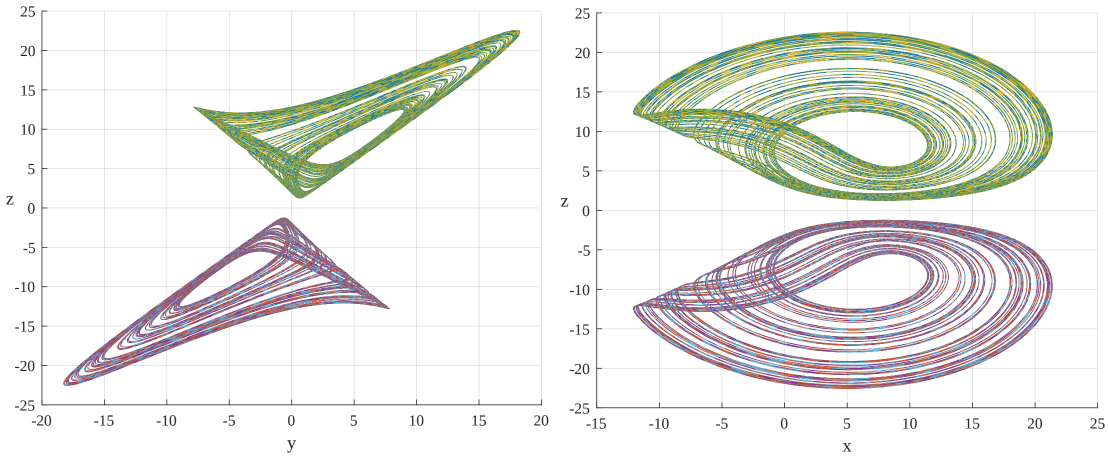


Figure 15: Dynamical behaviors of the system: $y-z$ and $x-z$ planar views

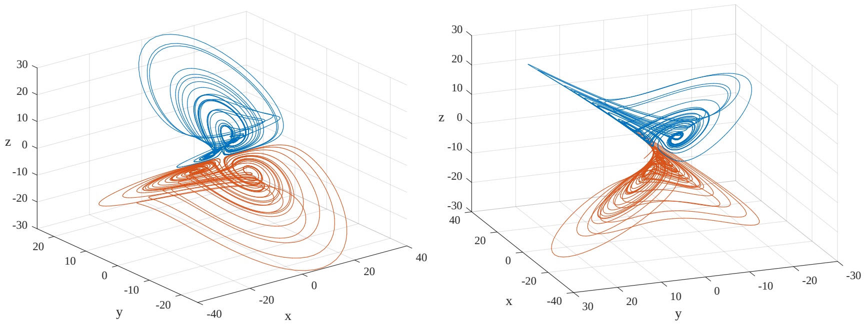


Figure 16: Dynamical behaviors of the system: xyz views, given two initial conditions (1, -1, 1) and (2, 0, -2)

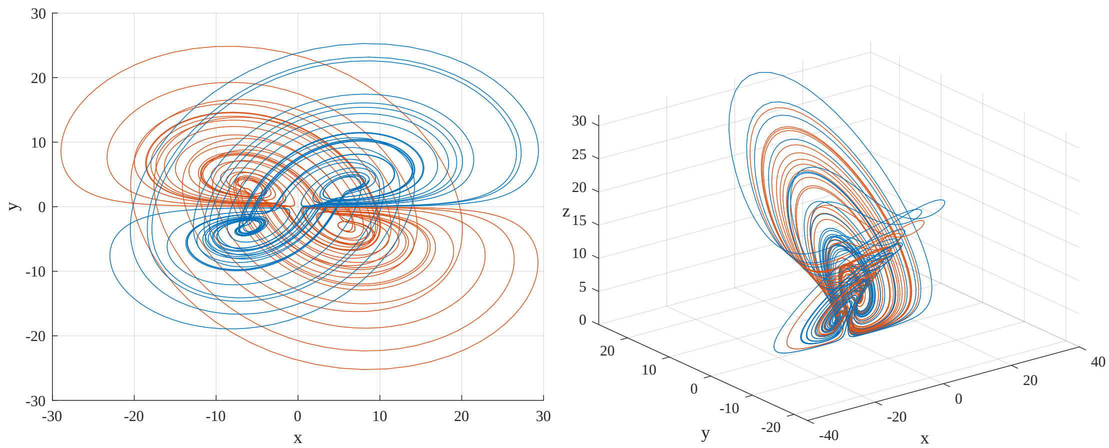


Figure 17: Dynamical behaviors of the system: $x-y$ and xyz views respectively with parameter $c=0$

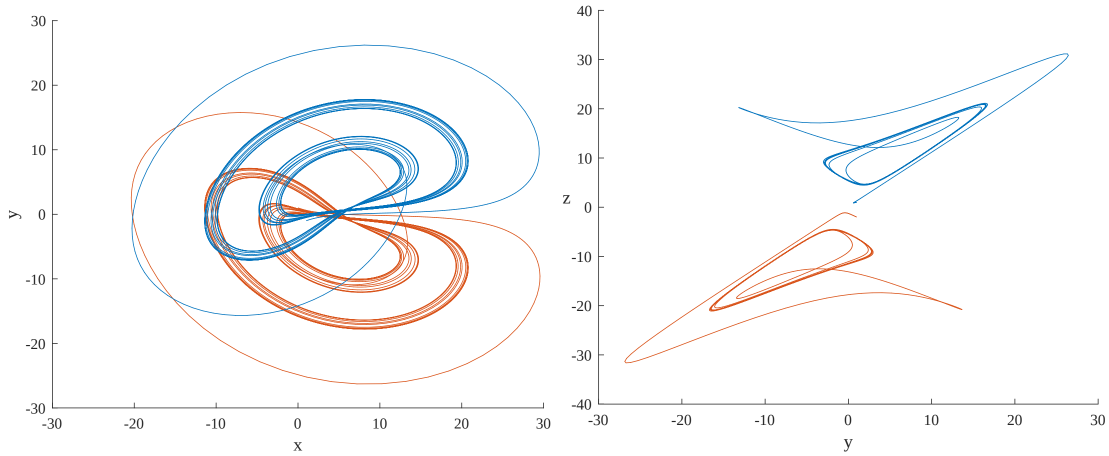


Figure 18: Dynamical behaviors of the system: xy view and yz planar views with $c=20$ and $c=50$ respectively.

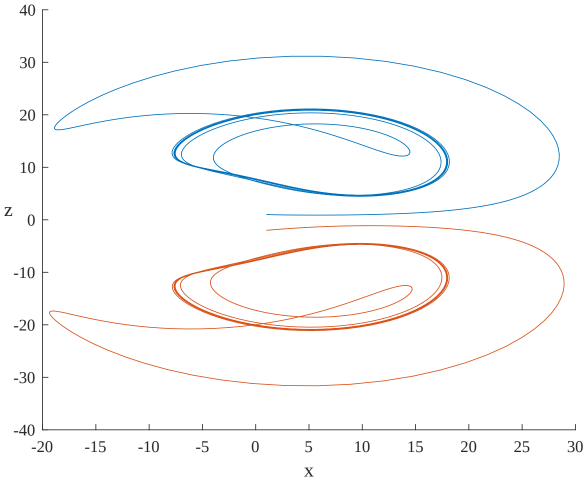


Figure 19: xz view with $a = -10, \quad b = -4$ and $c = 50$ with initial conditions $(1, 1, 1)$ and $(1, 1, -2)$

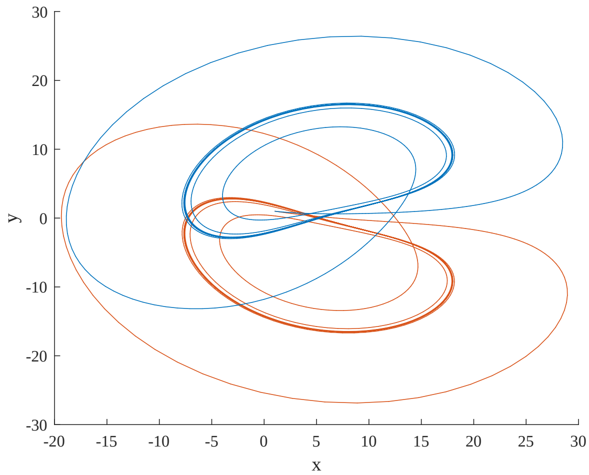


Figure 20: xy view with $a = -10, \quad b = -4$ and $c = 50$ with initial conditions $(1, 1, 1)$ and $(1, 1, -2)$

3.2 Equilibria of the system

As we discussed already, equilibrium means a state of a system which does not change over time. Thus, fixed points of the generalized Lorenz–like system (3.1) are given by considering the nature of its parameters a, b and c as shown in the following cases. Equilibrium points of the system are obtained by the solutions of the equations;

$$-\frac{ab}{a+b}x - yz + c = 0 \tag{3.2}$$

$$ay + xz = 0 \tag{3.3}$$

$$bz + xy = 0, \tag{3.4}$$

The Jacobian of the matrix is given by

$$J = \begin{pmatrix} -\frac{ab}{a+b} & -z & -y \\ z & a & x \\ y & x & b \end{pmatrix}$$

As we discussed already, the Routh–Hurwitz criterion is suitable to use in systems of higher dimension greater than 2. Therefore when the coefficients of the characteristic equation of the polynomial $P(\lambda) = det(A - \lambda I)$ are satisfied, the corresponding equilibrium point will be stable. Otherwise, it will indicate instability of the equilibrium.
Case 1: consider the system parameters

$$a = 0, b \neq 0; \implies$$

$$0 = -yz + c, \quad 0 = xz, \quad 0 = bz + xy$$

Therefore, if $x = 0, z = 0$ and then $c = 0$. Hence, the first case; with $c = 0$, gives infinitely many equilibria, thus the points are $S_1 = (x, 0, 0), S_2 = (0, y, 0)$ for x, y for $x \in K$ and with $c \neq 0$, $S = \phi$ (has no equilibria)

Case 2: For parameters $a, b \neq 0, c = 0$ with $a, b > 0$, We have five equilibria if the above condition is considered.
Let S_i be the equilibria when the parameter $c = 0$. Since a and b are different from zero, for $a, b > 0$ and $c = 0$;

$$S_1 = (0, 0, 0),$$

$$S_2 = \left(\sqrt{ab}, \sqrt{\frac{ab^2}{a+b}}, -\sqrt{\frac{a^2b}{a+b}}\right),$$

$$S_3 = \left(\sqrt{ab}, -\sqrt{\frac{ab^2}{a+b}}, \sqrt{\frac{a^2b}{a+b}}\right),$$

$$S_4 = \left(-\sqrt{ab}, -\sqrt{\frac{ab^2}{a+b}}, \sqrt{\frac{a^2b}{a+b}}\right),$$

$$S_5 = \left(-\sqrt{ab}, \sqrt{\frac{ab^2}{a+b}}, -\sqrt{\frac{a^2b}{a+b}}\right).$$

Let T_i be equilibrium points of Case 2 where $i = 1, 2, 3, 4, 5.$ with $c \neq 0$.

$$T_1 = \left(\frac{a+b}{ab}c, 0, 0\right),$$

$$T_2 = \left(\sqrt{ab}, \sqrt{\frac{ab^2}{a+b} - \frac{bc}{\sqrt{ab}}}, -\sqrt{\frac{a}{b}\left(\frac{ab^2}{a+b} - \frac{bc}{\sqrt{ab}}\right)}\right);$$

$$T_3 = \left(\sqrt{ab}, -\sqrt{\frac{ab^2}{a+b} - \frac{bc}{\sqrt{ab}}}, \sqrt{\frac{a}{b}\left(\frac{ab^2}{a+b} - \frac{bc}{\sqrt{ab}}\right)}\right);$$

$$T_4 = \left(-\sqrt{ab}, \sqrt{\frac{ab^2}{a+b} - \frac{bc}{\sqrt{ab}}}, \sqrt{\frac{a}{b}\left(\frac{ab^2}{a+b} + \frac{bc}{\sqrt{ab}}\right)}\right);$$

$$T_5 = \left(-\sqrt{ab}, -\sqrt{\frac{ab^2}{a+b} - \frac{bc}{\sqrt{ab}}}, -\sqrt{\frac{a}{b}\left(\frac{ab^2}{a+b} + \frac{bc}{\sqrt{ab}}\right)}\right);$$

3.3 Stability analysis of the equilibria

After small perturbations, a stable equilibrium will always go back to its original state and for unstable one, the system will stay away from the equilibrium point.

Definition 3.1. Let ϕ_t denote the flow of the system (3.1) defined for all $t \in \Re$. An equilibrium point x^* is called locally stable if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in N_\delta(x^*)$ and $t > 0$

$$\phi_t(x) \in N_\epsilon(x^*).$$

The equilibrium point x^* is locally asymptotically stable if it is stable and if there exists a $\delta > 0$ such that for all $x \in N_\delta(x^*)$,

$$\lim_{t\to\infty} \phi_t(x) = x^*$$

Stability analysis of equilibrium points is basically done with the method of linearization. The sign of real parts of eigenvalues λ of the Jacobian matrix determines the stability of a fixed point around the origin.

Theorem 3.2. *Let $J = Df(x^*)$ be the Jacobian matrix for the system (3.1) evaluated at a fixed point x^* and let μ_i be its eigenvalues.*
(i) If $\Re(\lambda_i) < 0$ for all λ_i, then the fixed point x^* is asymptotically stable.
(ii) If $\Re(\lambda_i) > 0$ for at least one λ_i, then the fixed point x^* is unstable.
(iii) If $\Re(\lambda_i) = 0$ for at least one λ_i, then the fixed point x^* is non-hyperbolic and its stability cannot be determined by the linearization method [1].

3.4 Lyapunov stability Theorem

Let $D \subset \Re^n$ open and connected.

Theorem 3.3 (Lyapunov)**.** *If the derivative of the Lyapunov function along the trajectories of the system is*
1. negative semi-definite, then the equilibrium is stable in the sense of Lyapunov.
2. negative definite, then the equilibrium is asymptotically stable.
3. positive definite, then the equilibrium is unstable.

The detailed proof of this theorem can be found in [1]. When $\dot{V}(x) \leq 0$, the trajectories start inside a ball and will stay. But for $\dot{V}(x) < 0$, the Lyapunov surface decreases in size. This will continue until the minimum value of $V(x)$is reached. At this point, $V(0) = 0$ which is the equilibrium point, after which the Lyapunov's surface grows along the trajectories which implies that x grows continuously and starts to diverge along any trajectory. This brings instability and at this point the derivative of the Lyapunov function is positive definite.

Remark. In nonlinear systems, it is usually somewhat difficult to find the Lyapunov function to use in the tests and therefore, when the test fails, there is no conclusion done. It is good to keep finding another possible function.

3.5 Routh–Hurwitz Test

In Lu–Chen–Cheng system, we decide to use the Routh–Hurwitz test, since the dynamical system has a dimension greater than two. It is convenient enough to use the method to test for stability of the system.

The Routh–Hurwitz theorem provides a test to determine whether all roots of a given polynomial lie in the left half-plane. As in Subsection 3.2, the stability of the system's equilibria depends on the eigenvalues. To find the eigenvalues λ_i where $i = 1, 2, 3$, it is necessary to solve the characteristic equation;

$$det(J - \lambda I) = 0. \tag{3.5}$$

Theorem 3.4 (Routh–Hurwitz criterion)**.** *Let the polynomial*

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1\lambda+a_n},$$

where a_i are real constants and $i = 1, 2, 3, \ldots, n$ define the n Hurwitz matrices using the coefficients a_i of $P(\lambda)$;

$$H_1 = (a_1), H_2 = \begin{pmatrix} a_1 & 1 \\ a_3 & a_2 \end{pmatrix}, H_3 = \begin{pmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{pmatrix} \tag{3.6}$$

and

$$H_n = \begin{pmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{pmatrix},$$

where $n > 3$. The coefficients $a_j = 0 \quad$ if $j > n$. All of the roots of the polynomial $P(\lambda)$ are negative or have negative real part if and only if all Hurwitz determinants are positive, that is;

$$det(H_j) > 0, forj = 1, 2, \ldots, n.$$

For a particular case say $n = 3$, we have the criteria written such that

$$a_1 > 0, \quad a_3 > 0, and \quad a_1a_2 > a_3.$$

Considering a case of one equilibrium point, we analyze its stability as follows;
From case 2, For $a, b > 0, c = 0$ we have the equilibrium point

$$S_1 = (0, 0, 0),$$

The Jacobian Matrix is given as;

$$J(S_1) = \begin{pmatrix} \frac{-ab}{a+b} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}; \tag{3.7}$$

Thus from Equation 3.6, we have

$$\lambda^3 - \left(\frac{a^2+b^2+ab}{a+b}\right)\mu^2 + \frac{a^2b^2}{a+b} = 0.$$

for stability, the coefficients on λ^2 and λ must satisfy the Routh–Hurwitz test. it is clearly seen that there is no stability since $a_1 < 0$.
Thus, the equilibrium point $(0,0,0)$ is not stable. The characteristic polynomial of S_2 is given as

$$P(\lambda) = \lambda^3 - \left(\frac{a^2+b^2+ab}{a+b}\right)\lambda^2 + \left(\frac{ab^3}{(a+b)^2} + \frac{a^2b^2}{(a+b)^2} - \frac{ab^2}{a+b}\right)\lambda - \left(\frac{3a^3b^2}{(a+b)^2} + \frac{3a^2b^3}{(a+b)^2} + \frac{a^2b^2}{a+b}\right).$$

From Routh–Hurwitz test, thee coefficient on λ^2 is negative, hence no stability.
The equilibria S_3, S_4 and S_5 have the same characteristic polynomial given by

$$P(\lambda) = \lambda^3 - \left(\frac{a^2+b^2+ab}{a+b}\right)\lambda^2 - \frac{4a^2b^2}{a+b}.$$

Hence, all are not stable. They have the same characteristics.

Similarly, for the case where the parameter c is different from zero, and a, b kept positive, we have the linearised system as shown in the matrix below.

Let $J(T_i)$ be the evaluated Jacobian matrix with respect to the equilibria T_i for $i = 1, 2\ldots, 5$. Considering a case of equilibrium point T_1, the Jacobian is given as

$$J(T_1) = \begin{pmatrix} \frac{-ab}{a+b} & 0 & 0 \\ 0 & a & \frac{(a+b)}{ab}c \\ 0 & \frac{(a+b)}{ab}c & b \end{pmatrix}. \tag{3.8}$$

The characteristic polynomial is

$$P(\lambda) = \lambda^3 - \left(\frac{ab+a^2+b^2}{a+b}\right)\lambda^2 - \frac{(a+b)^2c^2}{a^2b^2}\lambda + \frac{a^3b^3 - (-2ab+a^2+b^2)}{ab(a+b)c^2}.$$

It is clearly seen that there is no stability for this equilibrium point.

$$(\frac{a+b}{ab}, 0, 0),$$

Since, the coefficients on λ and λ^2 are negative.

From the system, we note that when $ab \leq 0$, there is only one equilibrium point which is the origin. The symmetry with respect to the x, y and z axes is observed in equilibrium points S_2, S_3, S_4 and S_5. The origin is a saddle point for all $a, b \neq 0$.

When

$$c \geq \frac{-(ab)^(3/2)}{a+b},$$

There are only three equilibrium points namely;

$$T_1 = \left(\frac{a+b}{ab}c, 0, 0\right),$$

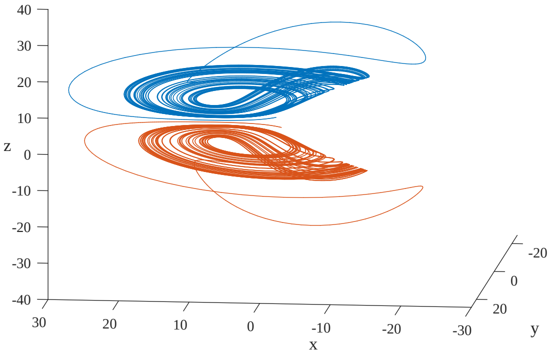
$$T_2 = \left(\sqrt{ab}, \sqrt{\frac{ab^2}{a+b} - \frac{bc}{\sqrt{ab}}}, -\sqrt{\frac{a}{b}\left(\frac{ab^2}{a+b} - \frac{bc}{\sqrt{ab}}\right)}\right);$$

$$T_3 = \left(\sqrt{ab}, -\sqrt{\frac{ab^2}{a+b} - \frac{bc}{\sqrt{ab}}}, \sqrt{\frac{a}{b}\left(\frac{ab^2}{a+b} - \frac{bc}{\sqrt{ab}}\right)}\right);$$

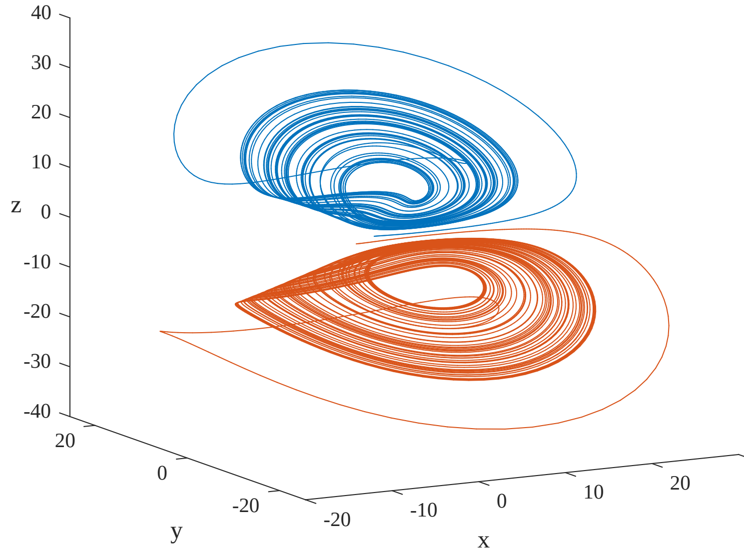
and we have three equilibria only if

$$c \leq \frac{(ab)^(3/2)}{a+b}.$$

If we set the parameter $c = 18.0702$, using two initial conditions we get the following dynamical structures showing three equilibrium points.



(a) The xyz view of the 3 equilibrium points of the system, with 2 initial conditions (x_0, y_0, z_0) and $z_0 > 0$ and $c = 18.1$.



(b) The xyz view with 2 initial conditions (x_0, y_0, z_0) and $z_0 < 0$ and $c = 20$

Figure 21: Chaotic attractors

The system has remarkable interesting behaviour. That is, the system is chaotic for parameters $a = -10, b = -4$ and $7 \leq |c| \leq 22.9$. In this region, the system develops a 4–scroll attractor, and beyond this and not greater than 24.1, the trajectories start converging to a limit cycle. In the interval between 24.1 and 27, the system's trajectories converge to a limit cycle.

Just like the Lorenz attractor, the Lu–Chen–Cheng with system parameter $c = 0$, the system is invariant under the transformation of $(x, y, z) \rightarrow (x, -y, -z)$, $(x, y, z) \rightarrow (-x, -y, z)$ and (x, y, z), i.e.,the system behaves symmetrically about the 3 axes (xy, z) respectively. The symmetry remains as the parameters are varied. Also, the axes are solutions to the system.

Consider the dissipative nature of our dynamical system. Given

$$\begin{cases} \dot{x} = \frac{-ab}{a+b}x - yz + c \\ \dot{y} = xz + ay \\ \dot{z} = bz + xy, \end{cases}$$

we have

$$\nabla V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y}\frac{+\partial \dot{z}}{\partial z} = \frac{-ab}{a+b} + a + b$$
$$= \frac{(a+b)^2 - ab}{a+b}$$

$$= \frac{a^2 + b^2 + ab}{a + b}.$$

Since the divergence is constant, the rate of contraction is exponential from the solution of

$$\dot{V} = \left(\frac{a^2 + b^2 + ab}{a + b}\right)V,$$

which is given by

$$V(t) = V(0)e^{\frac{a^2+b^2+ab}{a+b}t}.$$

This means that whenever $(a+b) < 0$, the volume in the state space shrinks fast to zero. Hence, if we start with a group of initial conditions, it eventually decreases to a limiting set of zero volume, just like air sucked out of a balloon.

At this point, $V(0)$ is contracted by the flow as $t \to \infty$, at the rate $\frac{a^2+b^2+ab}{a+b}$, meaning the trajectories of the system will end up somewhere in a limiting set.

3.6 Hopf Bifurcation

Consider the nonlinear autonomous system;

$$\dot{x} = f(x, \mu)$$

where $\mu \in \Re$ is a parameter and $Df(x, \mu)$ is the Jacobian of the system. Then, there exists a unique equilibrium point x_μ near x_0 such that when the eigenvalues of the Jacobian evaluated at $Df(x_\mu, \mu)$ cross the imaginary axis at the bifurcation value $\mu = \mu_0$, then the stability nature and the topological structure(local phase portraits) will change as μ passes through μ_0 which is the bifurcation value.

Theorem 3.5 (Hopf Bifurcation theorem)**.** *Consider the planar system*

$$\begin{cases} \dot{x_1} = h_\mu(x_1, x_2) \\ \dot{x_2} = g_\mu(x_1, x_2) \end{cases} \tag{3.9}$$

where μ is a parameter. Let the system have a critical point, say the origin. Let also the eigenvalues of the linearized system about the origin be given by $\lambda_{1,2} = \alpha \pm i\beta$. Suppose that $\mu = 0$, then the following conditions are satisfiable;
(i) $\alpha = 0, \beta = \omega \neq 0$ (non-hyperbolicity condition). The conjugate pair of imaginary eigenvalues.
(ii) $\left.\frac{d\alpha(\mu)}{d\mu}\right|_{\mu=0} = d \neq 0$ which is transversality condition. Here, the eigenvalues cross the imaginary axis with non-zero speed.
(iii) $\alpha \neq 0$ where; $\alpha = \frac{1}{16}(h_{x_1x_1x_1} + h_{x_1x_2x_2} + g_{x_1x_1x_2} + g_{x_2x_2x_2}) + \frac{1}{16\omega}(h_{x_1x_2}(h_{x_1x_1} + h_{x_2x_2}) - g_{x_1x_2}(g_{x_1x_1} + g_{x_2x_2}) - h_{x_1x_1}g_{x_1x_1} + h_{x_2x_2}g_{x_2x_2})$, with

$$h_{x_1x_2} = \left.\left(\frac{\partial^2 h_\mu}{\partial_{x_1}\partial_{x_2}}\right)\right|_{\mu=0}(0, 0).$$

(generecity condition)[7].

After which, a unique curve of periodic solutions bifurcates from the origin into the region$\mu >$ 0 if $ad < 0$ or $\mu < 0$ if $ad > 0$. The origin is a stable fixed point for $\mu > 0$ (resp. $\mu < 0$) and an unstable fixed point for $\mu < 0$ (resp. $\mu > 0$) if $d < 0$ (resp. $d > 0$) whilst the periodic solutions are stable (resp. unstable) if the origin is unstable (resp. stable) on the side of $\mu = 0$ where the periodic solutions exist[7].

Suppose that a 2 dimension system has a stable equilibrium point, the point loses stability as μ the varying parameter increases. At this point of time, a periodic orbit emerges out of the stable equilibrium point. The neighbouring trajectories move away from the point and get attracted to the stable limit cycle. The hopf bifurcation has two forms;
(i) supercritical hopf bifurcation and,
(ii) subcritical hopf bifurcation. The nature of eigenvalues is responsible for the occurrence of hopf bifurcation, i.e., when the real part is zero. The super critical bifurcation prevails for $\mu > \mu_0$, when a stable spiral moves into an unstable one.

Example 3.6. consider the system

$$\dot{r} = \mu - r^3,$$
$$\dot{\theta} = \omega + br^2.$$

In this simple system, there are three parameters where $/mu$ is the changing parameter that controls stability of the system at the origin[2].

When $\mu < 0$, the origin is stable and when$\mu = 0$ the origin is still stable but not strong and after this point when $\mu > 0$ there exist an unstable spiral at the origin and a stable circular limit cycle at

$$r = \sqrt{(\mu - \mu_0)}.$$

that attracts neighbouring trajectories.

In the case os subcritical hopf bifurcation, the trajectories jump to an attractor which is a bit distant or somewhere in the phase space.

4 Numerical Bifurcations Analysis

In most cases, it is quite very strange to differentiate between randomness and chaos of oscillations or limit cycles (attractors). But this is clearly shown by the bifurcation analysis of the system's behavior as a varying parameters bifurcates through the equilibria. Bifurcation diagrams also show exactly how the system bifurcates from either stable or unstable structure depending on the system's parameter variations. As we noted earlier, the idea of sensitive dependence is experienced in chaotic nonlinear dynamical systems, in which the smallest error in the change of initial condition grows to become as large as the true and actual value of the state. This makes prediction of the future behavior impossible. This explains why we have sudden bifurcations at un expected points in the bifurcation diagram. However, this does not mean that the system is not deterministic. Numerically, sensitivity is measured by Lyapunov exponent such that a positive value implies the system is really sensitive to initial conditions. This implies that Lyapunov exponents measure the rate of divergence of orbits away from each other

4.1 Bifurcation diagrams

The diagram in figure 22 represents the bifurcation diagram with the change in the values of c and the rest of the system's parameters remain fixed.

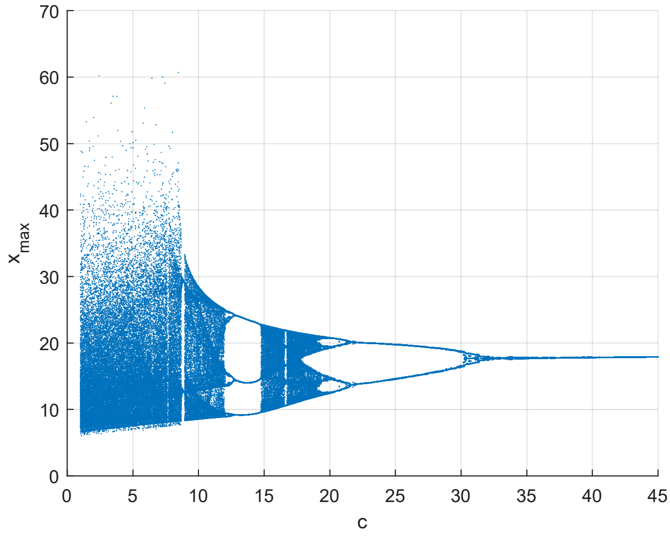
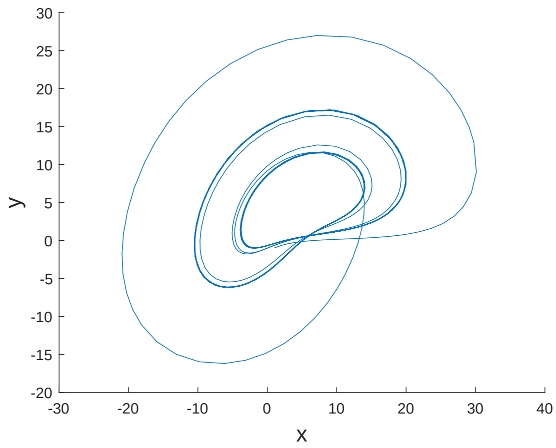


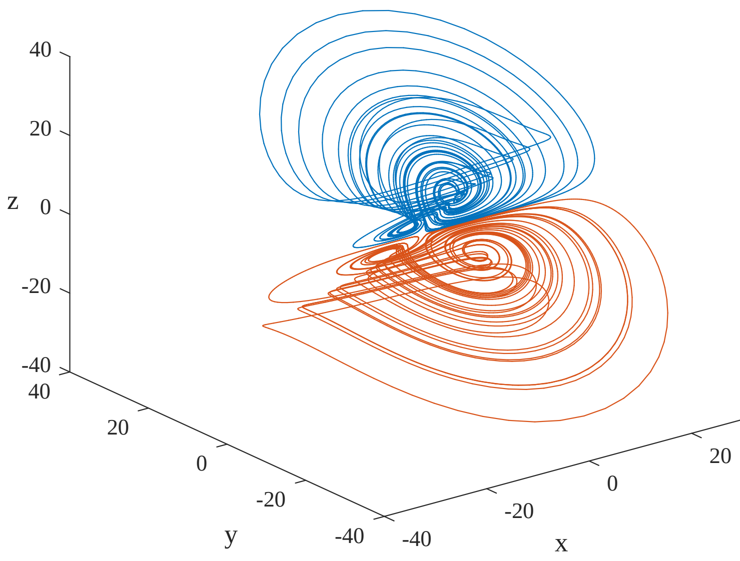
Figure 22: Bifurcation diagram showing chaotic regions as the system parameter c is varied c from values $1 - 45$

As the system parameter c is varied, we observe different chaotic regimes. The bifurcation diagram corresponds with the system attractors as shown already in the diagrams. Starting from $c = 1.9$ to about $c = 8.7$, there is chaos. The system has two big periodic windows between about $12 - 14.7$. There is period doubling bifurcation between $c = 19.1$ to 23, after which it shows 2 limit cycles from $c = 23 - 30$. For values of $c > 30$, the bifurcation diagram indicates that there is only one limit cycle as also portrayed by the dynamical structures of the system. At $c = 23$, there is a bifurcation changing from 4 limit cycles to 2.

The behaviour coincides with the system's structure depending on the value of the varying parameter c as shown in the following figures.



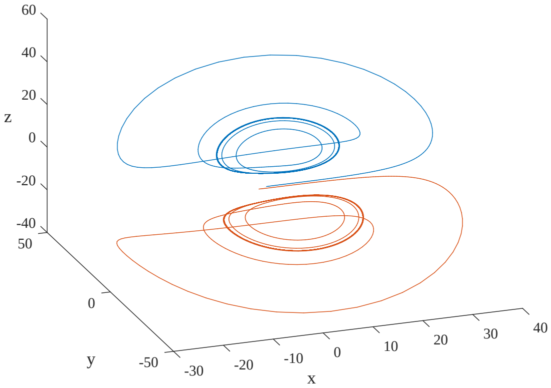
(a) The xy view when c is 23



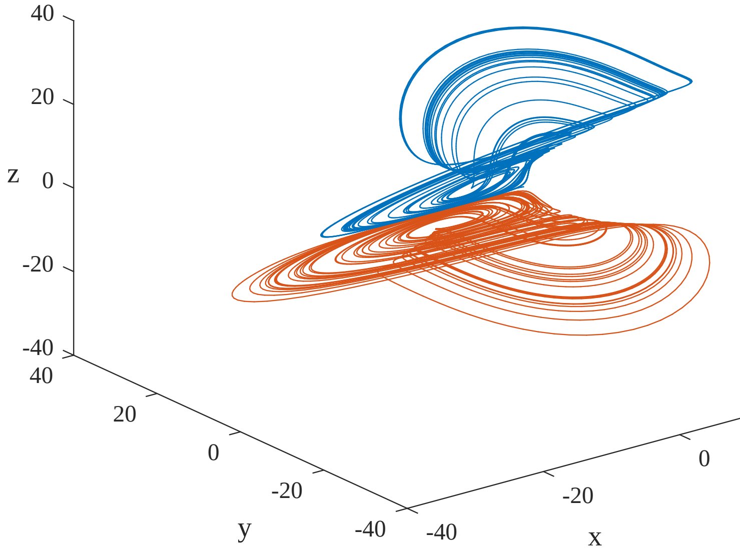
(b) Dynamical structure showing 4 scrolls, when $c = 5$

Figure 23: Limit cycles and Chaotic attractors

The figures fig. 24b behave as mirror images of the others when c is positive and negative respectively.

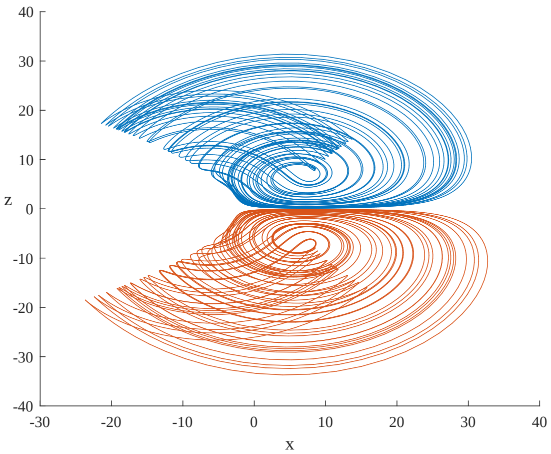


(a) The xyz view of the 3 equilibrium points of the system, with 2 initial conditions $(0, 1, -1)$ and $c = 57$.

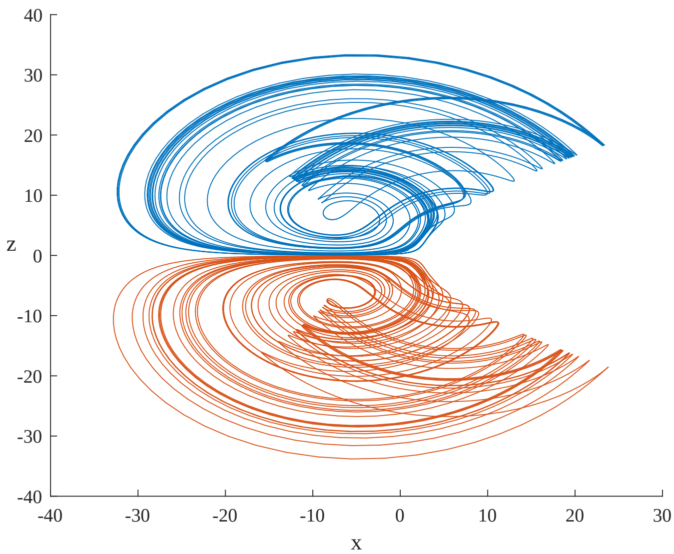


(b) The xyz view of the 4–scroll attractor of 5 equilibrium points of the system, with 2 initial conditions $(1, -1, 1), \text{and}(0, 1, -1)$ with $c = -9$

Figure 24: Topological behaviour of the system



(a) planar view of the 3 equilibrium points of the system, with 2 initial conditions $(1, -1, 1), \text{and}(0, 1, -1)$ and $c = 9$.



(b) Mirror Image of 25a. $c = -9$

Figure 25: Chaotic attractors

4.2 The Largest Lyapunov Exponent (LLE)

The Lyapunov exponent is responsible in the characterisation of the rate of separation of two trajectories in the phase space quantitatively. The trajectories diverge exponentially. Largest Lyapunov exponent is computed at every point for over 10000 initial conditions against the vector of initial conditions $[-1.4916; -0.7076; 20.7295]$ close to the attractor. The maximum Lyapunov exponent is at 7556 initial condition with a value of 1.171817949093744. The positive maximum value of Lyapunov exponent confirms the existence of chaos as shown in fig. 26. This confirms that the Lu–Chen–Cheng system is a chaotic nonlinear dynamical system.

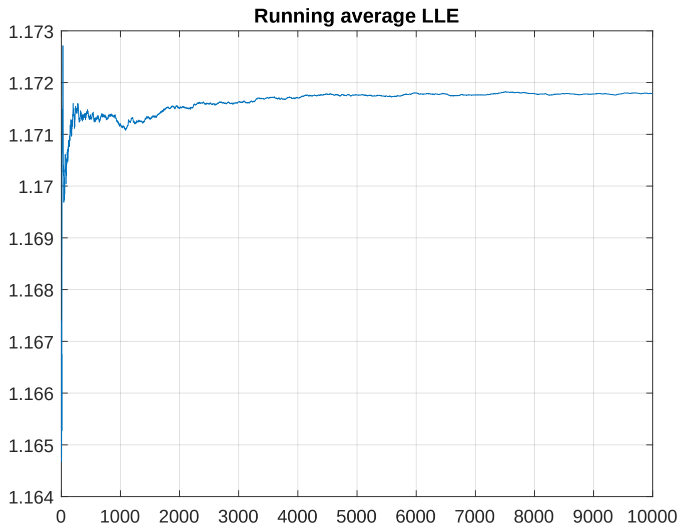
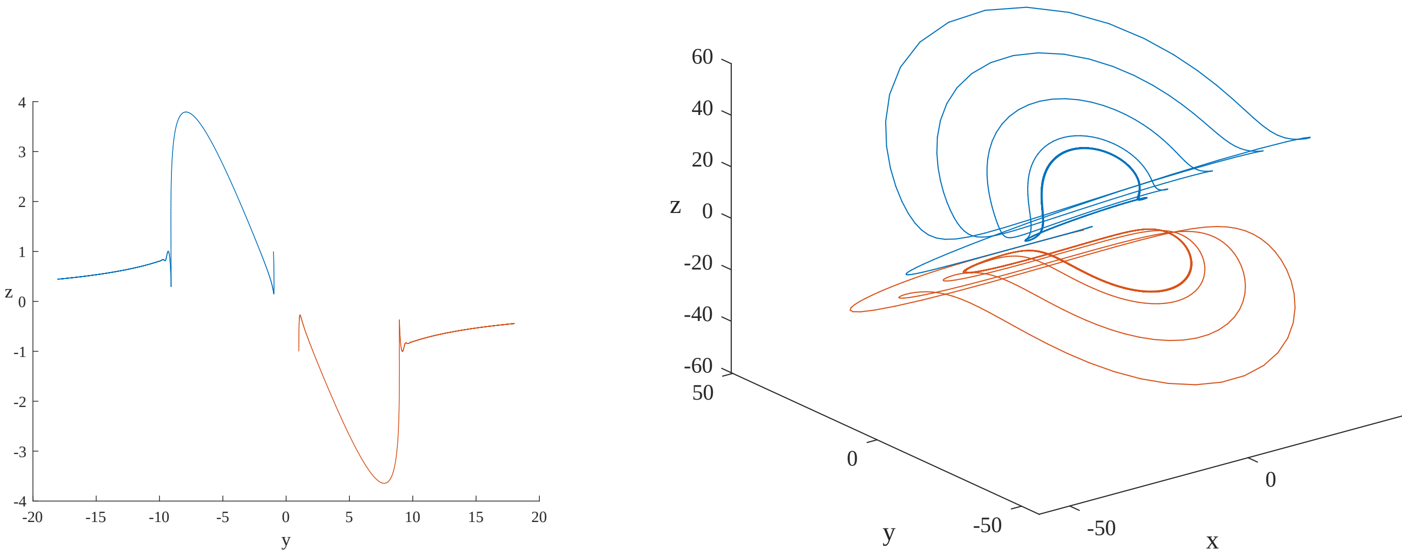


Figure 26: Lyapunov exponent

In the bifurcation diagram where c varies and the rest of the parameters are fixed, there exist several chaotic regions and this is confirmed by the positive values of the maximum Lyapunov exponent.

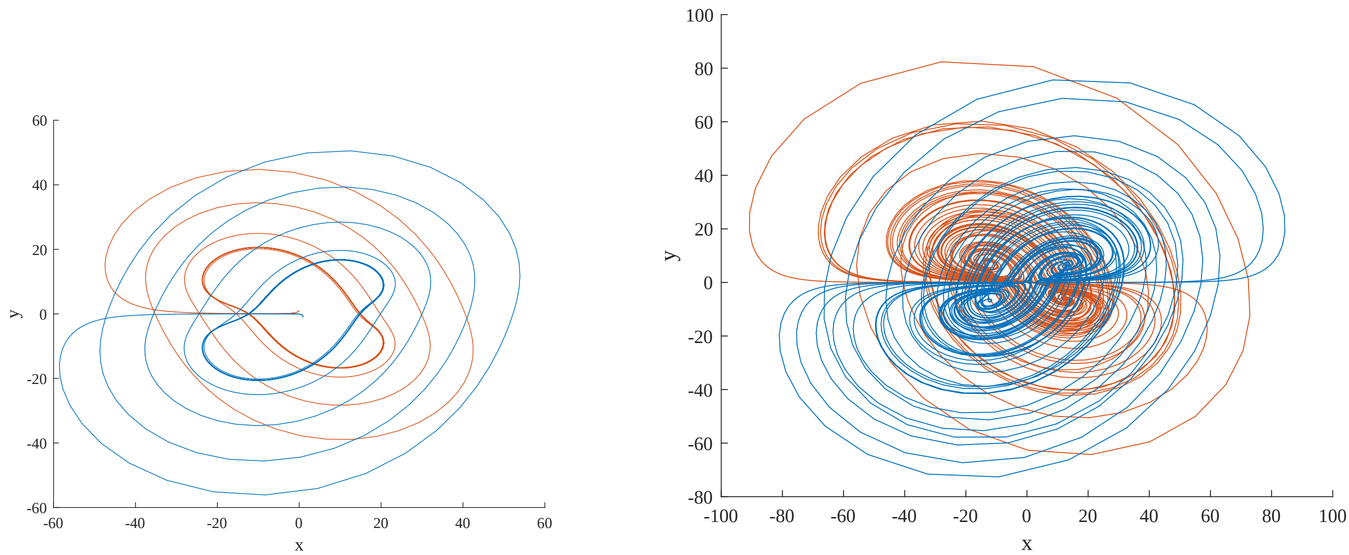
When another parameter a is varied, we obtain similar topological behaviour at specific values (not symmetric like in parameter c). But, there are restrictions for a and b as discussed earlier. The bifurcation diagram 31 also coincides with the already obtained results of chaos. The chaotic regions are seen coinciding also with the diagrams in 28a to 30b and figures from 32a to 35 as shown . We also provide the topological behaviour when the next parameter is varied. Suppose we fix b and c at one value $b = c = -8$ (raised a bit compared to the first value of $b = -4$).

Consider the Bifurcation diagram 31 where c varies directly from $0 - 40$ and parameter a changed to -20 The bifurcation Figure corresponds with the topological structures of the system.



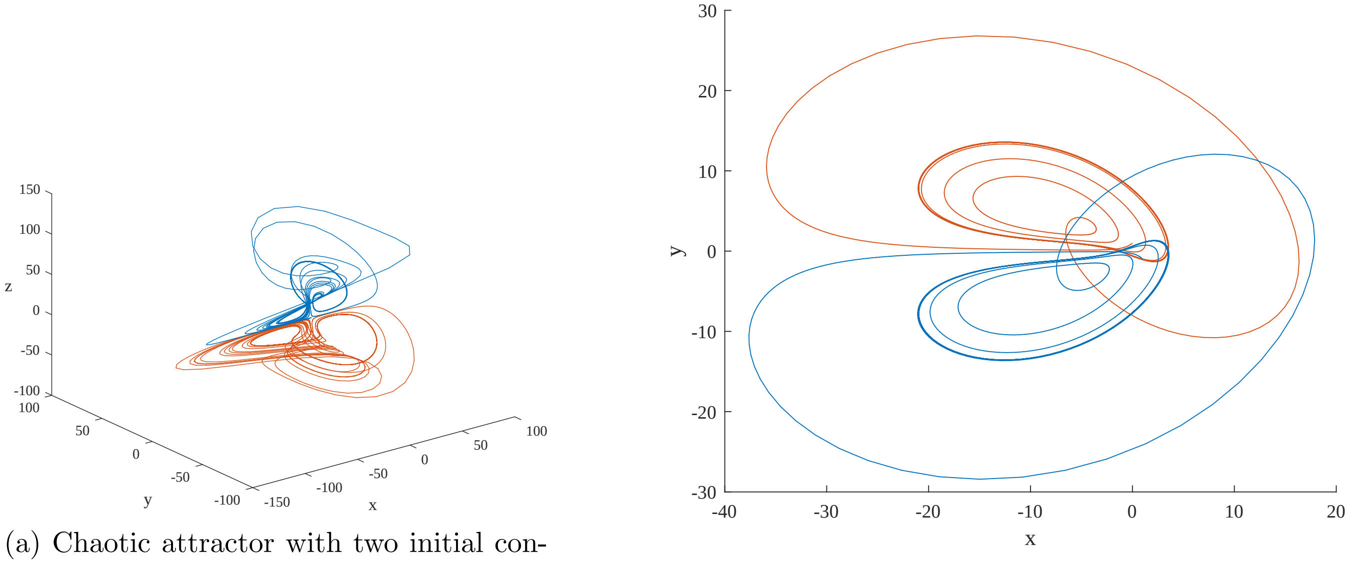
(a) yz view when $a = 0$ (b) he xyz view when$a = -11$

Figure 27: Topological structures of the system



(a) The xy view when$a = -11$ (b) The xy view when$a = -20$

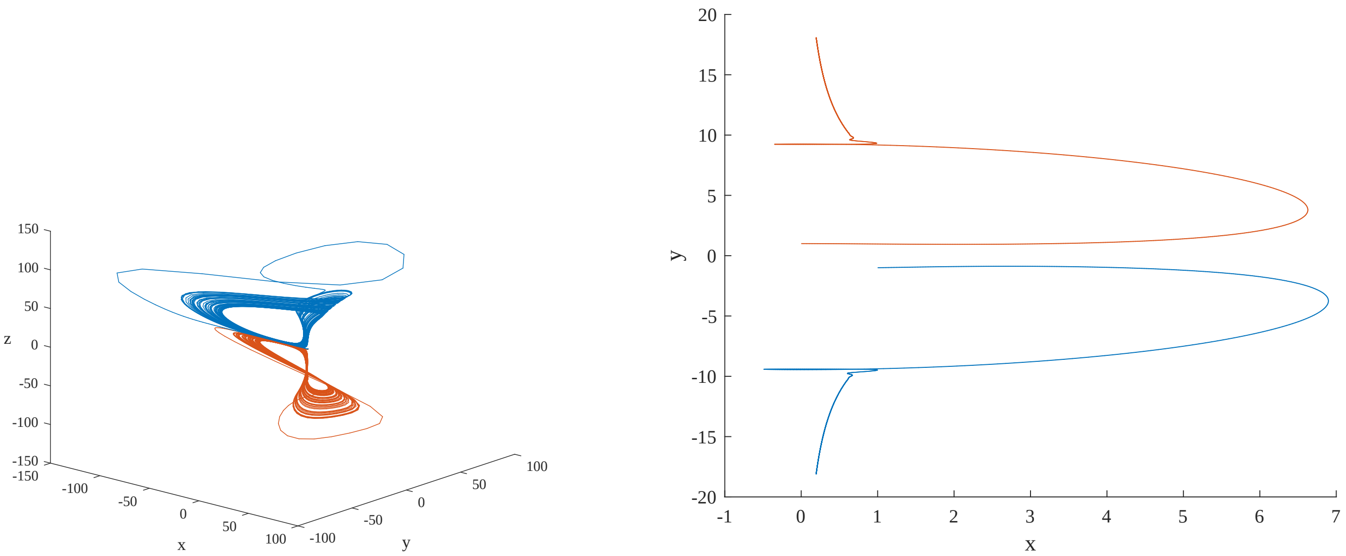
Figure 28: Dynamical behaviour of attractors of the Lu–Chen–Cheng system with two initial conditions where $b = -8$ and $c = -8$



(a) Chaotic attractor with two initial conditions where $a = -40$, $b = -8$ and $c = -8$

(b) $a = -20$, $b = -4$ and $c = -20$

Figure 29: Attractors



(a) The xyz view of 2 initial conditions $(1, -1, 1)$ and $(0, -1, -1)$ for $c = -60$.

(b) $a = 0$, $b = c = -8$ with two initial conditions $(1, -1, 1)$ and $(0, -1, -1)$

Figure 30: Chaotic attractors

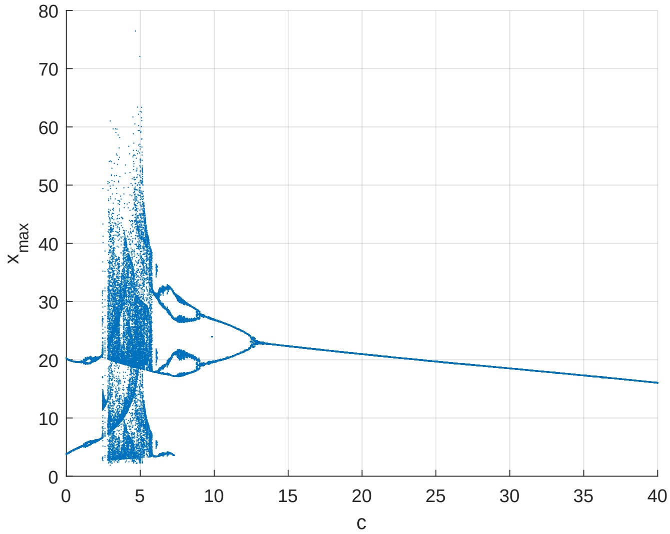
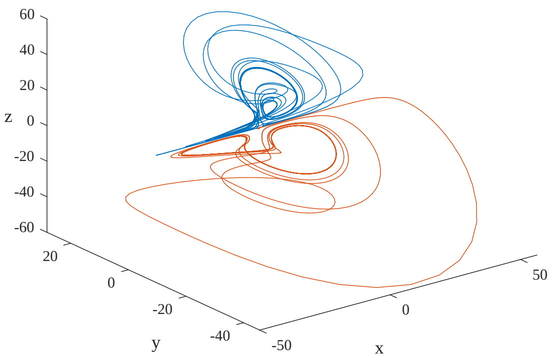
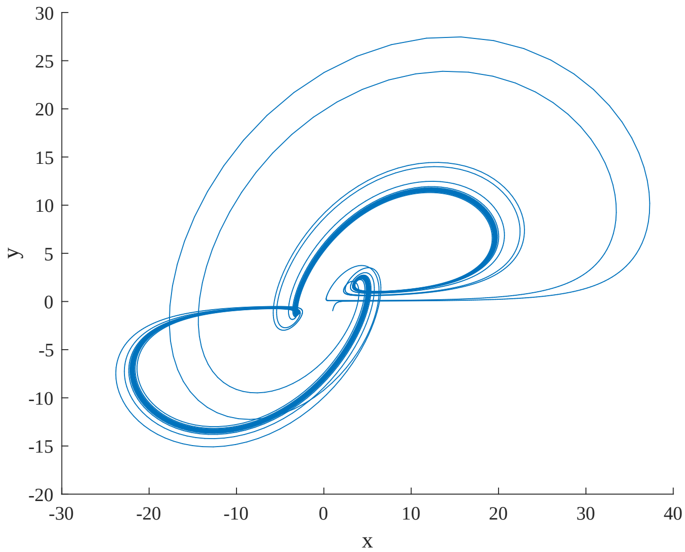


Figure 31: Bifurcation diagram showing points of chaos and limit cycles of the Lu–Chen–Cheng system when the parameter $a = -20, \quad b = -4$ with initial condition $(1, -1, 1)$

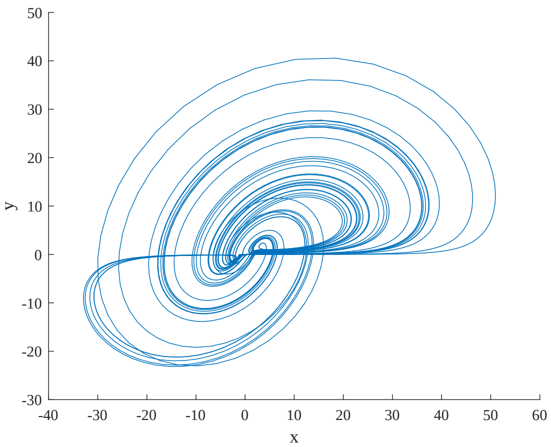


(a) $c = 0$ with two initial conditions $(1, -1, 1)$ and $(1, 0, 0)$

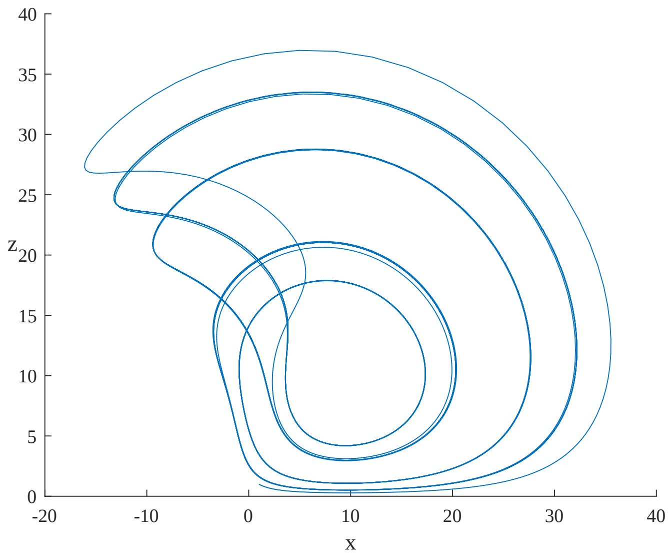


(b) The xy view of initial condition $(1. -1, 1)$ for $c = 1$

Figure 32: Chaotic attractors

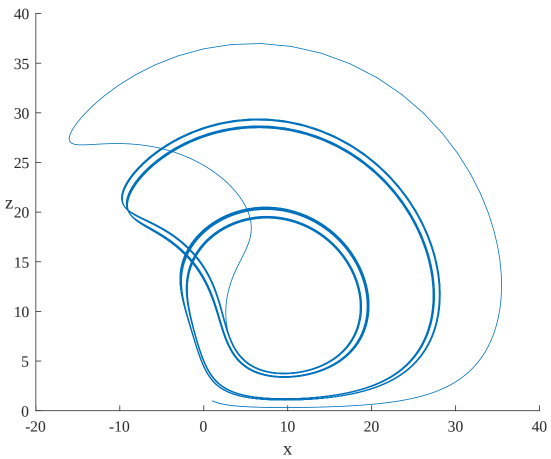


(a) xy view when $c = 4.3$ for initial condition $(1. -1, 1)$

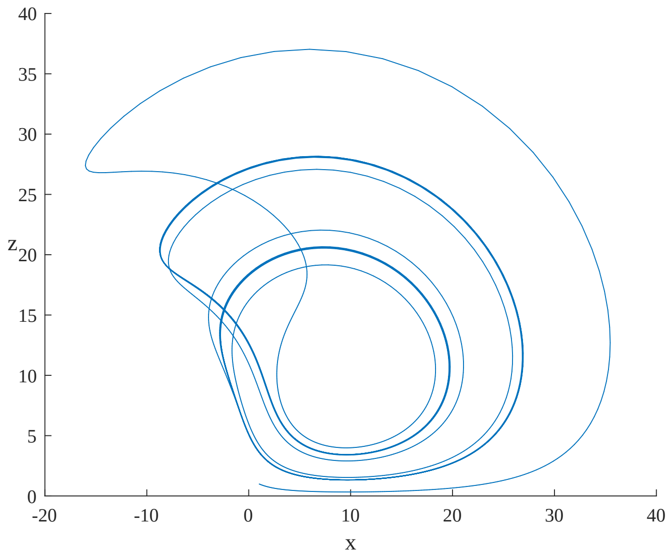


(b) The xz view when $c = 7.1$ for initial condition $(1. -1, 1)$

Figure 33: Attractors

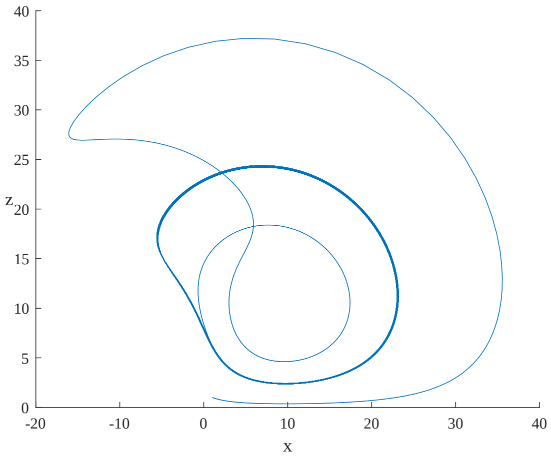


(a) The xz view when $c = 9$ for initial condition $(1. -1, 1)$

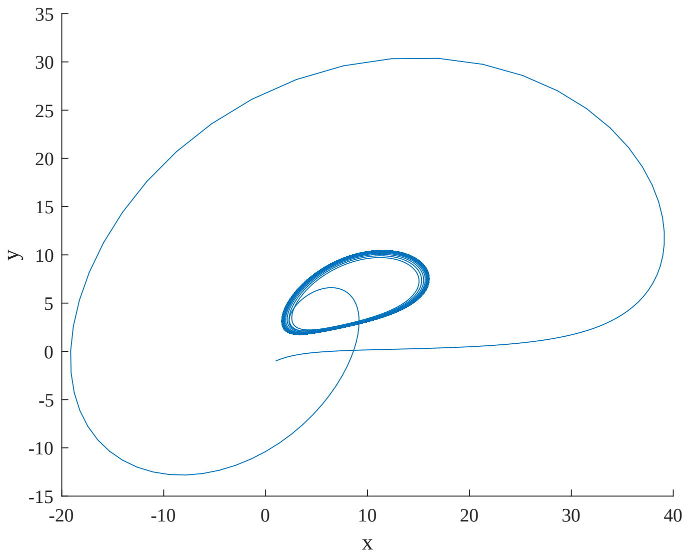


(b) The xz view when $c = 10$ for initial condition $(1. -1, 1)$

Figure 34: Attractors



(a) The xz view of a limit cycle for $c = 12.5$. Initial condition $(1, -1, 1)$.



(b) The xy view of a limit cycle for $c = 40$. Initial condition $(1, -1, 1)$.

Figure 35: Attractors

5 Conclusion

Over the years from 1963, the most exciting and interesting developments in nonlinear systems, is the realisation of the importance of chaos. When one thinks of chaos, it looks not an interesting thing, but yet a number of studies and practical scenarios have been solved by use of chaos. Such applications include the private communication sector, medicine, business and in engineering. It was quite strange to believe that the flapping of wings of a butterfly in Brazil could set off a cascade of atmospheric, events that, weeks later spurs the tornado in Texas. But Edward Lorenz (1963) explained this as a chaotic behaviour that is possessed in most of the nonlinear dynamical systems. Since then, there has been many researches on modeling the nonlinear autonomous differential equations of such systems with a chaotic behaviour.

In this work, the Lu–Chen–Cheng system is one of the many chaotic systems out there, that is analysed. The system has a 1–scroll chaotic attractors on 3 equilibrium points and two 2–scroll chaotic attractors simultaneously on the 5 equilibrium points. The system is found to exhibit a complex behaviour with respect to initial conditions and the parameter variations. Analytical and numerical methods of approach have been exhibited in the research. This work is of full support of the forthcoming investigations of 3 dimensional chaotic systems with three or more than 3 real constant parameters to deduce the system's properties and behaviour of bifurcations and stability structure of such systems.

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