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FAKULTA STROJNÍHO INŽENÝRSTVÍ
ÚSTAV MATEMATIKY

FACULTY OF MECHANICAL ENGINEERING
INSTITUTE OF MATHEMATICS

BASICS OF QUALITATIVE THEORY OF LINEAR FRACTIONAL DIFFERENCE EQUATIONS

ZÁKLADY KVALITATIVNÍ TEORIE LINEÁRNÍCH ZLOMKOVÝCH DIFERENČNÍCH ROVNIC

DIZERTAČNÍ PRÁCE
DOCTORAL THESIS

AUTOR PRÁCE Ing. TOMÁŠ KISELA
AUTHOR

VEDOUCÍ PRÁCE doc. RNDr. JAN ČERMÁK, CSc.
SUPERVISOR

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Abstrakt

Tato doktorská práce se zabývá zlomkovým kalkulem na diskrétních množinách, přesněji v rámci takzvaného (q, h)-kalkulu a jeho speciálního případu h-kalkulu. Nejprve jsou položeny základy teorie lineárních zlomkových diferenčních rovnic v (q, h)-kalkulu. Jsou diskutovány některé jejich základní vlastnosti, jako např. existence, jednoznačnost a struktura řešení, a je zavedena diskrétní analogie Mittag-Lefflerovy funkce jako vlastní funkce operátoru zlomkové diference. Dále je v rámci h-kalkulu provedena kvalitativní analýza skalární a vektorové testovací zlomkové diferenční rovnice. Výsledky analýzy stability a asymptotických vlastností umožňují vymezit souvislosti s jinými matematickými disciplínami, např. spojitým zlomkovým kalkulem, Volterrovými diferenčními rovnicemi a numerickou analýzou. Nakonec je nastíněno možné rozšíření zlomkového kalkulu na obecnější časové škály.

Abstract

This doctoral thesis concerns with the fractional calculus on discrete settings, namely in the frame of the so-called (q, h)-calculus and its special case h-calculus. First, foundations of the theory of linear fractional difference equations in (q, h)-calculus are established. In particular, basic properties, such as existence, uniqueness and structure of solutions, are discussed and a discrete analogue of the Mittag-Leffler function is introduced via eigenfunctions of a fractional difference operator. Further, qualitative analysis of a scalar and vector test fractional difference equation is performed in the frame of h-calculus. The results of stability and asymptotic analysis enable us to specify the connection to other mathematical disciplines, such as continuous fractional calculus, Volterra difference equations and numerical analysis. Finally, a possible generalization of the fractional calculus to more general settings is outlined.

Klíčová slova

zlomkový kalkulus – časové škály – zlomková diferenční rovnice – Riemannův-Liouvilleův diferenční operátor – stabilita – asymptotické chování – diskrétní Mittag-Lefflerova funkce – Volterrova diferenční rovnice – Laplaceova transformace

Keywords

fractional calculus – time scales – fractional difference equation – Riemann-Liouville difference operator – stability – asymptotic behaviour – discrete Mittag-Leffler function – Volterra difference equation – Laplace transform

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Affirmation

I declare that I have written this doctoral thesis all by myself under the direction of my supervisor doc. RNDr. Jan Čermák, CSc. using the literature listed in the bibliography.

Brno, June 28, 2012 Ing. Tomáš Kisela

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Contents

Introduction

The idea of non-integer order derivatives is almost as old as the notion of the classical derivative itself. Its first mention dates from the end of the 17th century when Leibniz discussed the meaning of derivative of order one half in his list to l'Hospital. Thereafter, derivatives and integrals of arbitrary order attracted an attention of many important mathematicians who contributed to a discussion on this matter (e.g. Euler, Laplace, Fourier, Abel, Liouville, Riemann, Laurent, etc.). At the end of the 19th century, most of definitions and results were unified and the theory, historically established as *fractional calculus*, finally got a solid base.

During the 20th century, the technology development along with higher demands placed on mathematical models created a suitable opportunity for involvement of the fractional calculus into the world of applications and subsequently for its wide theoretical expansion. The true landmark in history of fractional calculus occurred in 1974 when the first international conference specialized on this subject was held in New Haven and the first book presenting a comprehensive survey of both fractional calculus theory and its applications was published by Oldham and Spanier [41]. Since then, the importance of fractional calculus is increasing and the number of related papers is permanently growing. Apart from the study of the fractional calculus itself and its influence on other mathematical disciplines (e.g. theory of functions, probability theory), an extensive research takes place also in applicational fields, such as rheology, biology, electrical engineering, control theory, etc. Hence, numerical methods for solving of differential equations with fractional derivatives (the so-called fractional differential equations, in short FDEs) became one of the main courses in fractional calculus expansion and, consequently, motivated the theoretical development of *discrete fractional calculus*.

The theory of discrete fractional calculus originates from the works by Agarwal [1] and Diaz & Osler [21], where the first definitions of non-integer order differences and sums were introduced for functions considered on the sets of points forming geometric and arithmetic sequences, respectively. Recently, both the cases were unified and generalized, because fractional operators were successfully established on any set of points such that the distance of two neighbours (the so-called graininess function) is given by an appropriate linear function (see [18]). So far, the research in this field is mainly concentrated on methods for solving of difference equations involving fractional differences (the so-called fractional difference equations, in short FdEs; see, e.g. [7, 8, 38, 40]), while qualitative analysis of FdEs is just at the beginning.

Since we discuss fractional calculus on various settings in this work, it is convenient to utilize a notation independent on the underlying set of points. For that purpose, the *time scales theory* turns out to be an excellent tool. This theory concerning joint investigation of continuous and discrete analysis was introduced by Hilger in 1988 and its inspiring ideas resulted in fast development summarized in comprehensive monographs [11, 12]. Up to now, there are no general definitions of fractional operators in the framework of the time scales theory applicable in practice. Nevertheless, the time scales theory allows at least a symbolical comparison of results achieved for particular settings where the fractional calculus is well-established (including the continuous fractional calculus), and, moreover, it provides a suitable starting point for possible further generalizations.

This doctoral thesis is based on some of the author's papers. In particular, it summarizes the papers [14–17] (written jointly with other authors) with regard to [32, 33]. The former group of papers deals mostly with solutions of FdEs (often considered as discretizations of appropriate FDEs), their qualitative properties and potential consequences for some numerical methods, while the latter one concerns with numerical solution of some particular problems.

The work is organized as follows. Chapter 1 consists of three large parts. First, the basics of continuous fractional calculus and some advanced results connected to our research are presented. Further, some necessary notions from the time scales theory are recalled and the generalized Laplace transform is established. The last part contains an overview of discrete fractional calculus in a framework of the time scales theory. Some original results regarding Laplace transform and discrete fractional calculus are also included. Chapter 2 is based on papers [14, 15]. It deals, among others, with the closed form of a solution for higher-order scalar linear FdEs. In particular, a discrete analogy of the Mittag-Leffler function is introduced. All the results of this chapter are formulated on the time scale with a general linear graininess function. In Chapter 3, originating from [16], the stability and asymptotic properties of a scalar two-term linear FdE are studied via its conversion to a Volterra difference equation. Further, Chapter 4 is based on the paper [17] and utilizes the discrete Laplace transform for an investigation of qualitative properties of linear fractional difference systems. The equations considered in Chapters 3 and 4 are studied on time scales with a constant graininess, especially because of possible applications to numerical analysis of corresponding FDEs. At last, the problem of an introduction of fractional operators on a general time scale is discussed and a possible solution with its consequences is sketched in Chapter 5.

1 Preliminaries

In past few decades, several attempts to establish definitions of discrete fractional operators were performed (see, e.g. [1,18,21,27]). Some of these definitions differ only in formal details, some of them in underlying set of points and some of them are equivalent. Our approach to discrete fractional calculus is closest to the one presented in [18]. It takes into account the characteristics and results of the continuous fractional calculus and tries to consider them as paradigms. For that purpose, the notation known from the time scales theory seems to be very convenient.

This chapter includes a variety of preliminary results. It refers mainly to some classical monographs, but presents also several assertions established in the author's joint papers [14, 15, 17]. First, we recall the well-established notation, definitions and statements of continuous fractional calculus (Section 1.1) and the time scales theory (Section 1.2 up to Subsection 1.2.4). Then we summarize some recent results regarding generalized Laplace transform and formulate a few related statements useful for our further investigation. The last section is mainly devoted to the discrete fractional calculus, where several original results on this matter are presented.

1.1 Basics of continuous fractional calculus

As it has been outlined in the Introduction, the fractional calculus appeared as a theoretical concept without any instant practical application, and therefore without any possible objective evaluation of proposed definitions. Such an origin predetermined fractional calculus for an interesting history embracing a parallel evolution of various approaches (see, e.g. [39, 41]).

Nowadays, the relations between the approaches are clarified which makes the theory of fractional calculus broad and unitary at the same time. In this work, we employ mostly the so-called *Riemann-Liouville* approach, which is the most important one from the historical and theoretical point of view. In applications its position is not so unquestionable, since there is a growing influence of the so-called *Caputo* fractional derivative, which will be also mentioned in this thesis several times.

1.1.1 Differintegration operators and their properties

Although many current definitions of fractional calculus have a rich historical background originating from complex analysis, it is quite customary to introduce them more straightforwardly (see, e.g. in modern monographs [31, 39, 43]). As a bridge between classical and fractional calculus we can consider the Cauchy formula for mth integral

$$
{}_aI^m f(t) = \underbrace{\int_a^t \int_a^{\tau_{m-1}} \dots \int_a^{\tau_1}}_{m} f(\tau_0)\mathrm{d}\tau_0 \dots \mathrm{d}\tau_{m-1} = \int_a^t \frac{(t-\tau)^{m-1}}{(m-1)!} f(\tau)\mathrm{d}\tau\,, \tag{1.1}
$$

where $m \in \mathbb{Z}^+$, $a \in \mathbb{R}$, $t > a$ and $f(t)$ is a real function integrable on $[a,t]$. In other words, the mth integral is represented as a single convolution integral with a polynomial kernel dependent on m.

The generalization of this formula for non-integer values of m requires only a slight adjustment of the kernel. While the power function of non-integer (real or even complex) order is well-known, the factorial has to be replaced by the Euler Γ-function defined as

$$
\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^{z-1}}{z(z+1)\dots(z+n)}\,, \qquad z \notin \mathbb{Z}_0^-\,,
$$

which is the unique logarithmically convex solution of the factorial equation $\Gamma(z+1) = z\Gamma(z)$ with the normalizing condition $\Gamma(1) = 1$ (see, e.g. [36]). We can see that Γ-function has simple poles at non-positive integers, but its reciprocal $1/\Gamma(z)$, relevant to our study, is analytical at all complex values of the argument. Although in principle we can introduce integrals and derivatives of complex orders (see, e.g. [31, 39]), in this work we concern with real orders only.

These considerations lead us to the following definition of integral ${}_aI^\gamma$, where the order γ is a positive real. For a better consistency, we will denote this integral by a symbol for a derivative of negative order, i.e. ${}_a\mathbf{D}^{-\gamma} \equiv {}_aI^\gamma$.

Definition 1.1. Let $\gamma \in \mathbb{R}_0^+$ and $\tilde{a}, a, b \in \mathbb{R}$ be such that $\tilde{a} \le a < b$. Then for a function $f : (\tilde{a}, b] \to \mathbb{R}$ we define the *fractional integral* of order $\gamma \in \mathbb{R}^+$ with the lower limit a as

$$
{}_a\mathbf{D}^{-\gamma} f(t) = \int_a^t \frac{(t-\tau)^{\gamma-1}}{\Gamma(\gamma)} f(\tau)\,\mathrm{d}\tau\,, \quad t \in [a,b] \tag{1.2}
$$

and for $\gamma = 0$ we put ${}_a\mathbf{D}^0 f(t) = f(t)$.

Remark 1.2. Fractional operators can be considered in various function spaces (see, e.g. [31]). For our purposes we note that the fractional integral given by (1.2) is well-defined for functions being continuous on $(a,b]$ and integrable on any finite subinterval of $[a,b]$. For $t = a$ we consider the fractional integral in a limit sense.

Considering a derivative of arbitrary order, the situation is more complicated since there is no differential analogue of (1.1). A simple extension of (1.2) to negative values of γ can be utilized in the frame of generalized functions theory, but it is not effective in our case due to the loss of integrability. We introduce two most broadly used definitions of fractional derivative, both employing the fractional integral to reach non-integer orders. To keep the harmony in our notation, the usual mth derivative ($m \in \mathbb{Z}^+$) will be denoted by ${}_a\mathbf{D}^m$, although the parameter a has no factual meaning here. Further, we recall the ceiling function defined for every $\xi \in \mathbb{R}$ as $\lceil \xi \rceil = \min\{m \in \mathbb{Z}\,;\, m \geq \xi\}$.

Definition 1.3. Let $\gamma \in \mathbb{R}_0^+$ and $\tilde{a}, a, b \in \mathbb{R}$ be such that $\tilde{a} \leq a < b$. Then for a function $f : (\tilde{a}, b] \to \mathbb{R}$ we define the *Riemann-Liouville fractional derivative* of order α with the lower limit a as

$$ {}_a\mathbf{D}^{\alpha} f(t) = {}_a\mathbf{D}^{\lceil \alpha \rceil}{}_a\mathbf{D}^{-(\lceil \alpha \rceil - \alpha)} f(t)\,, \quad t \in [a, b]\,. $$

Remark 1.4. (i) In the case of α being a positive integer, the definition reduces to an ordinary derivative and the integrability assumption implied by the fractional integration can be removed. Thus, the integer-order derivatives are the only local ones in the family of fractional operators.
(ii) The *Caputo fractional derivative* of order α with the lower limit a differs from the Riemann-Liouville one by reversed sequence of the operators, i.e.

$$ {}_a^{\mathrm{C}}\mathrm{D}^{\alpha} f(t) = {}_a\mathbf{D}^{-(\lceil \alpha \rceil - \alpha)}{}_a\mathbf{D}^{\lceil \alpha \rceil} f(t)\,, \quad t \in [a, b]\,. $$

More information about definitions of fractional operators and their properties, including the ones bellow, can be found, e.g. in [31, 39, 43]. In the sequel, we occasionally use the term fractional derivative even for fractional integral which will be indicated by its negative order.

Besides an apparent linearity of all introduced fractional operators, we should first mention the composition rules. It holds

$$ {}_a\mathbf{D}^{\alpha}{}_a\mathbf{D}^{-\beta} f(t) = {}_a\mathbf{D}^{\alpha-\beta} f(t)\,, \tag{1.3} $$

$$ {}_a\mathbf{D}^{\alpha}{}_a\mathbf{D}^{\beta} f(t) = {}_a\mathbf{D}^{\alpha+\beta} f(t) - \sum_{k=1}^{\lceil \beta \rceil} {}_a\mathbf{D}^{\beta-k} f(t)\big|_{t=a} \frac{(t-a)^{-\alpha-k}}{\Gamma(1-\alpha-k)}\,, \tag{1.4} $$

where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^+$.

In mathematical analysis, the formulas for mth derivative and integral ($m \in \mathbb{Z}^+$) of elementary functions (such as t^{γ}, e^t, $\sin t$, $\cos t$, etc.) are well-known. It can be shown that corresponding formulas for fractional derivatives are generalizations of these relations (see, e.g. [39, 41]). In particular, we present the formula

$$ {}_a\mathbf{D}^{\alpha} \frac{(t-a)^{\beta}}{\Gamma(\beta+1)} = \frac{(t-a)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}\,, \quad \beta \in (-1, \infty)\,,\ \alpha \in \mathbb{R} \tag{1.5} $$

for the Riemann-Liouville derivative of a power function, which will serve as a paradigm in our later considerations. Note that according to (1.5) the Riemann-Liouville derivative of a constant function is nonzero, which is one of the most significant reasons why different definitions of fractional operators, especially the Caputo one, are often preferred in many physical applications.

The Laplace transform (see, e.g. [46]), for a given function $f(t)$ denoted by $\mathcal{L}\{f\}(z)$, is well-known to be a powerful instrument for solving of linear ordinary differential equations. The case of linear FDEs is not different in this respect (see, e.g. [31, 43]). On this account, we conclude this subsection by recalling the Laplace transforms of the fractional operators

$$\mathcal{L}\{{}_0\mathbf{D}^{-\gamma}f\}(z) = z^{-\gamma}\mathcal{L}\{f\}(z)\,, \quad \gamma \geq 0\,, \tag{1.6}$$

$$\mathcal{L}\{{}_0\mathbf{D}^{\alpha}f\}(z) = z^{\alpha}\mathcal{L}\{f\}(z) - \sum_{k=0}^{\lceil\alpha\rceil-1} z^k\,{}_0\mathbf{D}^{\alpha-k-1}f(t)\big|_{t=0}\,, \quad \alpha > 0\,. \tag{1.7}$$

For $\alpha \in \mathbb{Z}^+$, the latter relation is reduced to the formula for the Laplace transform of the classical integer-order derivative.

1.1.2 Linear FDEs

In this work, we discuss mainly linear equations with a few outlines to more general problems. Hence, we recall some known results from the theory of linear FDEs for later comparisons. For more information see, e.g. [31, 43].

In the theory of linear ordinary differential equations, a crucial significance belongs to the exponential function. In the theory of linear FDEs, this role is formally played by the so-called Mittag-Leffler function defined by the series expansion

$$E_{\eta,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\eta k+\beta)}\,, \quad \eta \in \mathbb{R}^+\,,\ \beta \in \mathbb{R}\,. \tag{1.8}$$

We can see that for $\eta = \beta = 1$ (1.8) is simplified to a well-known series expansion of the exponential function. Regarding fractional calculus, the key property of the Mittag-Leffler function is

$${}_0\mathbf{D}^{\alpha}(t^{\beta-1}E_{\eta,\beta}(\lambda t^{\eta})) = \begin{cases} t^{\beta-\alpha-1}E_{\eta,\beta-\alpha}(\lambda t^{\eta})\,, & \beta \neq \alpha\,, \\ \lambda t^{\eta-1}E_{\eta,\eta}(\lambda t^{\eta})\,, & \beta = \alpha\,, \end{cases} \tag{1.9}$$

where $\alpha, \beta, \eta \in \mathbb{R}^+$ and $\lambda \in \mathbb{R}$. For solving of linear FDEs via the Laplace transform it is useful to recall the formula

$$\mathcal{L}\{t^{\beta-1}E_{\eta,\beta}(\lambda t^{\eta})\}(z) = \frac{z^{\eta-\beta}}{z^{\eta}-\lambda}\,. \tag{1.10}$$

As indicated by (1.9) and (1.10), the notation via the Mittag-Leffler functions is utilized rather for historical reasons. The actual "fractional" generalization of the exponential function is

$$\mathcal{E}(t;\lambda,\eta,\beta) = t^{\beta-1}E_{\eta,\beta}(\lambda t^{\eta}) = \sum_{k=0}^{\infty}\frac{\lambda^k t^{\eta k+\beta-1}}{\Gamma(\eta k+\beta)} \tag{1.11}$$

introduced in a slightly more general form in [43, formula (9.8)]. However, we will keep the usual notation and express the results in terms of $E_{\eta,\beta}$.

In many aspects, theory of linear FDEs tracks its integer-order pattern. It can be demonstrated by existence and uniqueness theorems for various types of operators, solving methods and even by the form of a fundamental system for solutions of homogeneous linear equations (when solutions form a vector space). However, a question arises when a choice of initial conditions should be performed. Their number is derived from the number of integer-order derivatives occurring in involved operators, in particular the fractional derivative ${}_0\mathbf{D}^\alpha$ ($\alpha \in \mathbb{R}^+$) generates $\lceil\alpha\rceil$ initial conditions. Their type is specified via (1.7), i.e. the Riemann-Liouville derivative requires the initial conditions in the form of fractional derivatives and integrals. The matter of physical meaning of such initial conditions is a subject of current discussions (see [28]) and it is the other reason for the use of the Caputo derivative in applications (it works with the ordinary initial conditions, see, e.g. [43]).

For our later considerations, it is useful to introduce some initial value problems. First we deal with the initial value problem

$$
\begin{aligned}
&{}_0\mathbf{D}^\alpha x(t) = \lambda x(t)\,, \quad t \in \mathbb{R}^+\,,\\
&{}_0\mathbf{D}^{\alpha-j} x(t)\big|_{t=0} = x_{\alpha-j}\,, \quad j = 1, 2, \ldots, \lceil\alpha\rceil\,,
\end{aligned}
$$

where $\alpha \in \mathbb{R}^+$, $\lambda \in \mathbb{R}$ and $x_{\alpha-j} \in \mathbb{R}$ ($j = 1, \ldots, \lceil\alpha\rceil$). Using (1.7) and (1.10) it can be shown that the solution is given by

$$
x(t) = \sum_{j=1}^{\lceil\alpha\rceil} x_{\alpha-j} t^{\alpha-j} E_{\alpha,\alpha-j+1}(\lambda t^\alpha)\,. \tag{1.12}
$$

Further, we mention some recent stability and asymptotic results derived for the matrix initial value problem

$$
{}_0\mathbf{D}^\alpha x(t) = A\,x(t)\,, \quad t \in \mathbb{R}^+\,, \tag{1.13}
$$

$$
{}_0\mathbf{D}^{\alpha-1} x(t)\big|_{t=0} = x_0\,, \tag{1.14}
$$

where $0 < \alpha \leq 1$, $A \in \mathbb{R}^{d\times d}$ and $x_0 \in \mathbb{R}^d$. The autonomous system (1.13) is said to be asymptotically stable if and only if for any $x_0 \in \mathbb{R}^d$ the solution $x(t)$ of the initial value problem (1.13), (1.14) satisfies $\|x(t)\| \to 0$ as $t \to \infty$.

From this point on, eigenvalues of a matrix A are denoted by $\lambda(A)$ and the function $\mathrm{Arg}(z)$ means the principal argument of z for any complex number z. We conclude this section by the main result of the paper [45].

Theorem 1.5. *If all eigenvalues $\lambda(A)$ satisfy $|\mathrm{Arg}(\lambda(A))| > \frac{\alpha\pi}{2}$, then (1.13) is asymptotically stable. In this case, the components of the state decay towards zero like $t^{-(1+\alpha)}$.*

1.2 Basics of the time scales theory

To keep this thesis self-contained, we have to introduce some basic concepts of the time scales calculus. The time scales theory is divided into delta and nabla calculus according to the utilized definition of a derivative. As the names suggest, the delta calculus is built on a generalization of the forward difference, while the nabla calculus is based on the backward one. Although the delta version is usually preferred in the time scales literature, we will employ the nabla (backward) calculus, which seems to be more suitable for investigation of fractional operators as we will discuss later.

In the sequel, we often use the terms derivative and integral also on discrete settings (in accordance with the time scales terminology).

1.2.1 Elemental definitions of the nabla calculus

By a time scale $\mathbb{T}$ we understand any non-empty closed subset of real numbers with ordering inherited from reals. We start our survey by general notions independent of our choice of the nabla calculus. The presented definitions and properties were adopted, with a few minor adjustments, from [11, 12].

Throughout this thesis, we utilize especially the following particular time scales:

- $\mathbb{T} = h\mathbb{Z} = \{nh\,;\, n \in \mathbb{Z}\}$, where $h > 0$,
- $\mathbb{T}_{(q,h)} = \{t_0 q^n + h\frac{q^n-1}{q-1}\,;\, n \in \mathbb{Z}\} \cup \{\frac{h}{1-q}\}$, where $t_0 > 0$, $q \geq 1$, $h \geq 0$, $q + h > 1$.

The parameter t_0 occurring in the definition of $\mathbb{T}_{(q,h)}$ is not significant for our study, hence, for the sake of simplicity, it is not explicitly included in the symbol for this time scale. Note that if $q = 1$, $h > 0$ and $t_0 = h$, then $\mathbb{T}_{(q,h)} = h\mathbb{Z}$ and the cluster point $h/(1-q) = -\infty$ is not involved. If $h = 0$ and $q > 1$, then $\mathbb{T}_{(q,h)} = q^{\mathbb{Z}} = \{t_0 q^n\,;\, n \in \mathbb{Z}\}$, i.e. we obtain the q-calculus (see, e.g. [30]).

Further, we use the symbols $(a, b]_{\mathbb{T}}$, $[a, b]_{\mathbb{T}}$, etc. for an intersection of a respective interval with the time scale $\mathbb{T}$.

Definition 1.6. Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{\tau \in \mathbb{T}\,;\, \tau > t\}\,,$$

while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{\tau \in \mathbb{T}\,;\, \tau < t\}\,,$$

where we put $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. The jump operators of order $m \in \mathbb{Z}^+$ are defined recursively, i.e. $\sigma^m(t) = \sigma(\sigma^{m-1}(t))$ and $\rho^m(t) = \rho(\rho^{m-1}(t))$, respectively.

The jump operators allow to classify points in $\mathbb{T}$ as follows.

Definition 1.7. Let $\mathbb{T}$ be a time scale and let $t \in \mathbb{T}$. Then t is called

(i) *right-scattered*, if $\sigma(t) > t$,

(ii) *left-scattered*, if $\rho(t) < t$,

(iii) *isolated*, if it is right-scattered and left-scattered at the same time,

(iv) *right-dense*, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$,

(v) *left-dense*, if $t > \inf \mathbb{T}$ and $\rho(t) = t$,

(vi) *dense*, if it is right-dense and left-dense at the same time.

If all points of $\mathbb{T}$ are isolated, we call $\mathbb{T}$ the *isolated time scale.*

The graininess functions given bellow provide a convenient tool for a description of properties of time scales as well as functions defined on them.

Definition 1.8. Let $\mathbb{T}$ be a time scale. The *forward graininess function* $\mu : \mathbb{T} \to \mathbb{R}$ is defined by

$$\mu(t) = \sigma(t) - t$$

and the *backward graininess function* $\nu : \mathbb{T} \to \mathbb{R}$ is defined by

$$\nu(t) = t - \rho(t)\,.$$

Remark 1.9. Considering $\mathbb{T} = \mathbb{T}_{(q,h)}$, the jump operators and graininess functions are linear with respect to t, i.e. $\sigma(t) = qt + h$, $\rho(t) = q^{-1}(t - h)$, $\mu(t) = (q - 1)t + h$ and $\nu(t) = (1 - q^{-1})t + q^{-1}h$. It holds $\nu(t) = q\nu(\rho(t))$ and $\mu(\sigma(t)) = q\mu(t)$.

From this point on, we focus only on notions of the nabla (backward) part of the time scales theory. Regarding discrete sets, it is useful to introduce a truncated time scale $\mathbb{T}_\kappa$ derived from $\mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum $\hat{t}$, then $\mathbb{T}_\kappa = \mathbb{T} \setminus \{\hat{t}\}$, otherwise $\mathbb{T}_\kappa = \mathbb{T}$. Further, the symbol $\mathbb{T}_{\kappa^m}$ for $m \in \mathbb{Z}^+$ is specified via the recurrence $\mathbb{T}_{\kappa^m} = (\mathbb{T}_{\kappa^{m-1}})_\kappa$.

Now we can define the nabla derivative, a corner stone of the nabla calculus. In this thesis, we denote the nabla derivative by $\nabla f(t)$ instead of the usual $f^\nabla(t)$, because it is more suitable for a fractional generalization. If a function of two variables occurs, the symbol $\nabla f(t, s)$ stands for the derivative with respect to the first variable.

Definition 1.10. Let $f : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}_\kappa$. We define the *nabla derivative* of f at t, denoted by $\nabla f(t)$, to be the number with the property that given any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f(s) - f(\rho(t)) - \nabla f(t)(s - \rho(t))| \leq \varepsilon |s - \rho(t)| \quad \text{for all } s \in (t - \delta, t + \delta)_{\mathbb{T}}\,.$$

Further, let $m \in \mathbb{Z}^+$ and now let $t \in \mathbb{T}_{\kappa^m}$. We define the mth nabla derivative of f at t recursively by $\nabla^m f(t) = \nabla(\nabla^{m-1} f(t))$.

Obviously, in cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ the nabla derivative corresponds to an ordinary derivative and backward difference, respectively. Considering an isolated time scale (i.e. $\nu(t) \neq 0$ for all $t \in \mathbb{T}_\kappa$) the nabla derivative exists for all $t \in \mathbb{T}_\kappa$ and is given by

$$\nabla f(t) = \frac{f(t) - f(\rho(t))}{\nu(t)} \,.$$

Definition 1.11. Let $a, b \in \mathbb{T}$ be such that $a < b$. Let $f : [a, b]_\mathbb{T} \to \mathbb{R}$ and $F : [a, b]_\mathbb{T} \to \mathbb{R}$ be functions such that $F^\nabla(t) = f(t)$ for all $t \in \mathbb{T}_\kappa$. Then function $F(t)$ is called the *antiderivative* of $f(t)$ and we define the definite *nabla integral* of $f(t)$ over $[a, b]_\mathbb{T}$ as

$$\int_a^b f(t)\nabla t = F(b) - F(a) \,.$$

Further, we put $\int_b^a f(t)\nabla t = -\int_a^b f(t)\nabla t$ and $\int_a^a f(t)\nabla t = 0$.

In the time scales theory, many types of integrals can be introduced (e.g. Riemann's, Lebesgue's, see [12]), but the definition above is sufficient for our concerns. On isolated time scales the definite nabla integral can be calculated as

$$\int_a^b f(t)\nabla t = \sum_{t \in (a,b]_\mathbb{T}} \nu(t) f(t) \,. \tag{1.15}$$

It is known that a sufficient condition for the existence of the antiderivative is a generalized continuity of $f(t)$, the so-called ld-continuity.

Definition 1.12. A function $f : \mathbb{T} \to \mathbb{R}$ is *ld-continuous* (left-dense continuous) provided it is continuous at left-dense points in $\mathbb{T}$ and it has finite right-sided limits at right-dense points in $\mathbb{T}$.

Considering the Laplace transform and fractional calculus on time scales, it is necessary to present also definitions of the improper integral of the first and second kind. Both notions were introduced in [12] in the framework of delta calculus, but their adjustment for the nabla case is straightforward.

Definition 1.13. (i) Let $a \in \mathbb{T}$, $\mathbb{T}$ be unbounded above and let $f : [a, \infty)_\mathbb{T} \to \mathbb{R}$ be ld-continuous. Then we define the *improper integral of first kind* of $f(t)$ over $[a, \infty)_\mathbb{T}$ as

$$\int_a^\infty f(t)\nabla t = \lim_{b \to \infty} \left(\int_a^b f(t)\nabla t \right) \,. \tag{1.16}$$

(ii) Let $\tilde{a}, a, b \in \mathbb{T}$ be such that $\tilde{a} < a < b$ and let $f : (\tilde{a}, b]_\mathbb{T} \to \mathbb{R}$ be ld-continuous on any interval $[a, b]_\mathbb{T}$. Then we define the *improper integral of second kind* of $f(t)$ over $[\tilde{a}, b]_\mathbb{T}$ as

$$\int_{\tilde{a}}^b f(t)\nabla t = \begin{cases} \lim_{a \to \tilde{a}^+} \left(\int_a^b f(t)\nabla t \right) , & \text{if } \tilde{a} \text{ is right-dense,} \\ \int_{\tilde{a}}^b f(t)\nabla t \,, & \text{if } \tilde{a} \text{ is right-scattered.} \end{cases} \tag{1.17}$$

Remark 1.14. (i) If the limit in (1.16) or (1.17) exists (does not exist), we call the corresponding integral convergent (divergent).
(ii) In [12], the improper integral of second kind is defined only for $\tilde{a}$ being right-dense. Our formal extension to right-scattered points simplifies the notation throughout this thesis.

Both nabla derivative and integral are linear operators and many well-known properties, such as relation for derivative of product and ratio or formula for integration by parts, can be derived also in the framework of the time scales theory (see, e.g. [11, 12]). In particular, if we consider the nabla integral as a function of its upper limit $t \in \mathbb{T}$, the nabla derivative (with respect to t) is given by

$$\nabla \int_a^t f(t,\tau)\nabla\tau = f(\rho(t),t) + \int_a^t \nabla f(t,\tau)\nabla\tau \tag{1.18}$$

provided $f(t,s)$ and $\nabla f(t,s)$ are ld-continuous with respect to the second variable.

1.2.2 A dynamic equation on time scales

To prepare a background for the next chapters, especially Chapter 2, we present some results on a nabla dynamic equation introduced for the case of the second order equation in [12]. The generalization of definitions and statements for equations of mth order ($m \in \mathbb{Z}^+$) is based on their delta versions stated in [11].

We discuss the linear initial value problem

$$\sum_{j=0}^{m} p_{m-j}(t)\,\nabla^{m-j} y(t) = 0\,, \quad t \in \mathbb{T}_{\kappa^m}\,, \tag{1.19}$$

$$\nabla^{m-j} y(t)\big|_{t=t_0} = y_{m-j}\,, \quad j = 1,2,\dots,m\,, \tag{1.20}$$

where $t_0 \in \mathbb{T}_{\kappa^m}$, $p_j(t)$ $(j = 1,\dots,m-1)$ are ld-continuous functions for all $t = \mathbb{T}_{\kappa^m}$, $p_m(t) \equiv 1$ and y_{m-j} $(j = 1,\dots,m)$ are arbitrary real scalars.

Regarding the problem of existence and uniqueness of (1.19), (1.20) we recall the notion of ν-regressivity, which is introduced for linear dynamic equations of higher order via the corresponding dynamic system of the first order (for the idea see [11, 12]).

Definition 1.15. A matrix function $A : \mathbb{T} \to \mathbb{R}^{d\times d}$ is called *ν-regressive* if

$$\det(I - \nu(t)A(t)) \neq 0 \quad \text{for all } t \in \mathbb{T}_\kappa\,.$$

Definition 1.16. We say that the equation (1.19) is ν-regressive provided the matrix

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ -p_0(t) & -p_1(t) & \dots & -p_{m-2}(t) & -p_{m-1}(t) \end{pmatrix}$$

is ν-regressive.

Theorem 1.17. *Let (1.19) be ν-regressive. Then the problem (1.19), (1.20) has a unique solution defined for all $t \in \mathbb{T}$.*

Further, we recall the notion of Wronskian (this term is usually utilized in continuous analysis, while in the time scales theory the term Wronski determinant is more common).

Definition 1.18. Let $m \in \mathbb{Z}^+$. Then for $y_j : \mathbb{T} \to \mathbb{R}$ $(j = 1, 2, \ldots, m)$ we define the *Wronskian* $W(y_1, \ldots, y_m)(t)$ for all $t \in \mathbb{T}_{\kappa^m}$ as determinant of the matrix

$$V(y_1, \ldots, y_m)(t) = \begin{pmatrix} y_1(t) & y_2(t) & \ldots & y_m(t) \\ \nabla y_1(t) & \nabla y_2(t) & \ldots & \nabla y_m(t) \\ \vdots & \vdots & \ddots & \vdots \\ \nabla^{m-1} y_1(t) & \nabla^{m-1} y_2(t) & \ldots & \nabla^{m-1} y_m(t) \end{pmatrix}.$$

We recall this section by the following assertion describing the form of a general solution for (1.19), (1.20).

Theorem 1.19. *Let functions $y_1(t), \ldots, y_m(t)$ be solutions of the ν-regressive equation (1.19) and let $W(y_1, \ldots, y_m)(t) \neq 0$ for some $t \in \mathbb{T}_{\kappa^m}$. Then any solution $y(t)$ of (1.19) can be written as $y(t) = \sum_{k=1}^{m} c_k y_k(t)$, where c_k are real constants which can be determined via (1.20).*

1.2.3 Exponential functions, polynomials, power functions

In classical mathematical analysis the definitions of elementary functions were polished during the time and nowadays they are well-established. In the time scales theory we do not have such a historical background and the techniques broadly used in classical analysis often cannot be utilized. In particular, the power series expansion fails as a general definition tool since the variety of $\mathbb{T}$ (and $\nu(t)$) leads to problems with convergence.

We present time scales versions of a few elementary functions to illustrate the idea of the generalization. A common principle of these introductions lies in an identification of a role which should be played by the current function, and its utilization for the definition itself.

To ensure required qualities, the definitions usually appear in two simultaneous versions corresponding to delta and nabla calculus respectively. We discuss only the nabla definitions and omit the term "nabla" in the names of functions.

Exponential function

First, we discuss the time scales generalization of a key function with regard to the theory of differential equations, the exponential function. Among its many properties, its relationship to differential equations turned out to be most suitable for utilization as a paradigm. For more information as well as the proofs of presented assertions we refer to [11, 12].

To simplify the notation we introduce the following classes of functions.

Definition 1.20. The class of all scalar ld-continuous and ν-regressive functions on $\mathbb{T}$ is denoted by $\mathcal{R}_\nu$, i.e. $\mathcal{R}_\nu = \{f : \mathbb{T} \to \mathbb{R}\,;\, f(t)$ is ld-continuous and ν-regressive$\}$. Further, we define $\mathcal{R}_\nu^+ = \{f \in \mathcal{R}_\nu\,;\, 1 - f(t)\nu(t) > 0$ for all $t \in \mathbb{T}_\kappa\}$.

Thus, for any $f \in \mathcal{R}_\nu$ we define the exponential function $\hat{e}_f : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ as a unique solution of the initial value problem $\nabla y(t) = f(t)y(t)$, $y(s) = 1$. It can be written as

$$\hat{e}_f(t,s) = \exp\left(\int_s^t \hat{\xi}_{\nu(\tau)}(f(\tau))\nabla\tau\right), \quad s,t \in \mathbb{T}, \tag{1.21}$$

where $\hat{\xi}_h(z)$, the so-called ν-cylinder transformation, is given by $\hat{\xi}_h(z) = -\frac{1}{h}\mathrm{Log}(1-zh)$ for $h \in \mathbb{R}^+$, $z \in \mathbb{C} \setminus \{1/h\}$ and $\mathrm{Log}(z)$ being the principal logarithm function. This definition is supported by many reasonable properties, some of them are listed bellow.

Theorem 1.21. *Let $f, g \in \mathcal{R}_\nu$ and $s, t, r \in \mathbb{T}$. Then*

(i) $\hat{e}_0(t,s) = 1$ *and* $\hat{e}_f(t,t) = 1$,

(ii) $\frac{1}{\hat{e}_f(t,s)} = \hat{e}_f(s,t)$,

(iii) $\hat{e}_f(t,r)\hat{e}_f(r,s) = \hat{e}_f(t,s)$,

(iv) *if* $f \in \mathcal{R}_\nu^+$ *then* $\hat{e}_f(t,s) > 0$ *for all* $t \in \mathbb{T}$,

(v) *if* $f \in \mathcal{R}_\nu \setminus \mathcal{R}_\nu^+$ *then* $\hat{e}_f(\rho(t),s)\hat{e}_f(t,s) < 0$ *for all* $t \in \mathbb{T}$ *such that* $1 - f(t)\nu(t) < 0$.

Explicit formulas for the exponential function are derived for many special time scales (see [12]). In our investigations we need the following ones.

Theorem 1.22. *Let $s, t \in \mathbb{T}$ and let $\gamma \in \mathbb{R}$ be such that $\gamma \in \mathcal{R}_\nu$.*

(i) *If* $\mathbb{T} = \mathbb{R}$, *then* $\hat{e}_\gamma(t,s) = \mathrm{e}^{\gamma(t-s)}$.

(ii) *If* $\mathbb{T}$ *is isolated and* $t = \sigma^n(s)$, *then* $\hat{e}_\gamma(t,s) = \prod_{k=1}^n (1 - \gamma\nu(\rho^{k-1}(t)))^{-1}$, *in particular for* $\mathbb{T} = h\mathbb{Z}$ *we arrive at* $\hat{e}_\gamma(t,s) = (1-\gamma h)^{-n}$.

Polynomials

The central role in classical analysis is played by polynomials due to their connection to series expansions (namely the Taylor's series expansions). On that account, we focus on polynomials of the type $\frac{(t-s)^m}{m!}$, where $s, t \in \mathbb{R}$ and $m \in \mathbb{Z}_0^+$. It is known that such polynomials can be expressed as the mth integral of the unit function, which is a suitable property for the time scales generalization.

Throughout this thesis, we define the generalized polynomials $\hat{h}_m : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ on an arbitrary time scale recursively by

$$\hat{h}_0(t,s) = 1, \tag{1.22}$$

$$\hat{h}_m(t,s) = \int_s^t \hat{h}_{m-1}(\tau,s)\nabla\tau, \quad m \in \mathbb{Z}^+. \tag{1.23}$$

A proper choice of functions $\hat{h}_m(t,s)$ is supported by their characteristics bellow and also by their use in time scales Taylor's formula (see [12]).

Theorem 1.23. *Let $m \in \mathbb{Z}^+$ and $s,t \in \mathbb{T}$. Then*

(i) $\hat{h}_m(t,t) = 0$,

(ii) $\nabla \hat{h}_m(t,s) = \hat{h}_{m-1}(t,s)$ *for* $t \in \mathbb{T}_\kappa$,

(iii) $\hat{h}_1(t,s) = t - s$.

To discuss the generalized polynomials on $\mathbb{T}_{(q,h)}$, we recall some necessary background of q-calculus (for more information see, e.g. [18, 30]). For any $\xi \in \mathbb{R}$ and $0 < q \neq 1$ we set $[\xi]_q = \frac{q^\xi - 1}{q-1}$. By the continuity, we put $[\xi]_1 = \xi$. Further, the q-Gamma function is defined for $0 < \tilde{q} < 1$ as

$$\Gamma_{\tilde{q}}(\xi) = (1-\tilde{q})^{1-\xi} \prod_{j=0}^{\infty} \frac{1-\tilde{q}^{j+1}}{1-\tilde{q}^{j+\xi}}, \quad \xi \in \mathbb{R} \setminus \mathbb{Z}_0^- .$$

Note that this function satisfies the functional relation $\Gamma_{\tilde{q}}(\xi+1) = [\xi]_{\tilde{q}} \Gamma_{\tilde{q}}(\xi)$ and the condition $\Gamma_{\tilde{q}}(1) = 1$ (compare with the properties of the Euler Γ-function). Using this, the q-binomial coefficients can be introduced as

$$\begin{bmatrix} \xi \\ m \end{bmatrix}_{\tilde{q}} = \frac{\Gamma_{\tilde{q}}(\xi+1)}{\Gamma_{\tilde{q}}(m+1)\Gamma_{\tilde{q}}(\xi-m+1)}, \quad \xi \in \mathbb{R},\ m \in \mathbb{Z}.$$

Although the q-Gamma function is not defined at non-positive integers, we can employ the formula

$$\frac{\Gamma_{\tilde{q}}(\xi+m)}{\Gamma_{\tilde{q}}(\xi)} = (-1)^m \tilde{q}^{\xi m + \binom{m}{2}} \frac{\Gamma_{\tilde{q}}(1-\xi)}{\Gamma_{\tilde{q}}(1-\xi-m)}, \quad \xi \in \mathbb{R},\ m \in \mathbb{Z}^+$$

to calculate this ratio also at such the points. It is well-known that if $\tilde{q} \to 1^-$ then $\Gamma_{\tilde{q}}(z)$ becomes the Euler Gamma function $\Gamma(z)$ (and analogously for the q-binomial coefficients). Among many interesting properties of the q-Gamma function and q-binomial coefficients we mention q-Pascal rules

$$\begin{bmatrix} \xi \\ m \end{bmatrix}_{\tilde{q}} = \begin{bmatrix} \xi-1 \\ m-1 \end{bmatrix}_{\tilde{q}} + \tilde{q}^m \begin{bmatrix} \xi-1 \\ m \end{bmatrix}_{\tilde{q}}, \qquad \xi \in \mathbb{R},\ m \in \mathbb{Z}, \tag{1.24}$$

$$\begin{bmatrix} \xi \\ m \end{bmatrix}_{\tilde{q}} = \tilde{q}^{\xi-m} \begin{bmatrix} \xi-1 \\ m-1 \end{bmatrix}_{\tilde{q}} + \begin{bmatrix} \xi-1 \\ m \end{bmatrix}_{\tilde{q}}, \qquad \xi \in \mathbb{R},\ m \in \mathbb{Z} \tag{1.25}$$

and the q-Vandermonde identity

$$\sum_{j=0}^{m} \begin{bmatrix} \xi \\ m-j \end{bmatrix}_{\tilde{q}} \begin{bmatrix} \zeta \\ j \end{bmatrix}_{\tilde{q}} \tilde{q}^{j^2 - mj + \xi j} = \begin{bmatrix} \xi+\zeta \\ m \end{bmatrix}_{\tilde{q}}, \qquad \xi, \zeta \in \mathbb{R},\ m \in \mathbb{Z}_0^+ \tag{1.26}$$

(see [5]) that turn out to be very useful in our further investigations.

To simplify the notation, we put $\tilde{q} = q^{-1}$ whenever considering the time scale $\mathbb{T}_{(q,h)}$ or a time scale formed by its subinterval.

Theorem 1.24. *Let $s, t \in \mathbb{T}$, $m \in \mathbb{Z}_0^+$.*

(i) *If $\mathbb{T} = \mathbb{R}$, then $\hat{h}_m(t,s) = \frac{(t-s)^m}{m!}$.*

(ii) *If $\mathbb{T} = h\mathbb{Z}$ and $t = \sigma^n(s)$, then*

$$\hat{h}_m(t,s) = h^m \binom{m+n-1}{n-1} = (-1)^{n-1} h^m \binom{-m-1}{n-1}.$$

(iii) *If $\mathbb{T} = \mathbb{T}_{(q,h)}$ and $t = \sigma^n(s)$, then*

$$\hat{h}_m(t,s) = \nu^m(t) \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}_{\tilde{q}} = (-1)^{n-1} \nu^m(\sigma(s))\, \tilde{q}^{\binom{n}{2}} \begin{bmatrix} -m-1 \\ n-1 \end{bmatrix}_{\tilde{q}}.$$

Remark 1.25. Note that for $h = 1$ in (ii) we obtain the polynomials for classical difference calculus ($\mathbb{T} = \mathbb{Z}$) and setting $h = 0$ in (iii) we get the polynomials on the time scale $\mathbb{T} = q^{\mathbb{Z}}$, $q > 1$.

Power functions

As observed in Section 1.1, the power functions enjoy the most importance in continuous fractional calculus. However, we find the lack of well-established formulas allowing their convenient generalization for the time scales theory (similar to (1.21) for exponential function or (1.22) and (1.23) for polynomials). In fact, no essential property of power functions enabling such extension was proposed so far. The most promising attempt was performed in [10], where the authors defined power functions (in delta calculus) via the inverse Laplace transform as $\hat{h}_\beta(t,0) = \mathcal{L}^{-1}\{z^{-\beta-1}\}(t)$, but no explicit formula was derived by this method. Moreover, this approach is not entirely general since, as we will discuss later, the Laplace transform cannot be defined on every time scale.

However, this matter seems to be answerable on some particular time scales, for which an explicit form of $\hat{h}_m(t,s)$ is known when $m \in \mathbb{Z}^+$. Substituting non-integer values of m to the formulas for $\hat{h}_m(t,s)$ yields reasonable relations for power functions. So far, there are only a few time scales enabling this method (see Theorem 1.24), namely $\mathbb{R}$, $\mathbb{T}_{(q,h)}$, $h\mathbb{Z}$, $q^{\mathbb{Z}}$ and their subintervals. Obviously, the power functions on subintervals are inherited from the ones on original time scales.

Definition 1.26. Let $s, t \in \mathbb{T}$, $\beta \in (-1, \infty)$.

(i) If $\mathbb{T} = \mathbb{R}$, then $\hat{h}_\beta(t,s) = \frac{(t-s)^\beta}{\Gamma(\beta+1)}$.

(ii) If $\mathbb{T} = h\mathbb{Z}$ and $t = \sigma^n(s)$, then

$$\hat{h}_\beta(t,s) = h^\beta \binom{\beta+n-1}{n-1} = (-1)^{n-1} h^\beta \binom{-\beta-1}{n-1}. \tag{1.27}$$

(iii) If $\mathbb{T} = \mathbb{T}_{(q,h)}$ and $t = \sigma^n(s)$, then

$$\hat{h}_\beta(t,s) = \nu^\beta(t) \begin{bmatrix} \beta+n-1 \\ n-1 \end{bmatrix}_{\tilde{q}} = (-1)^{n-1}\nu^\beta(\sigma(s))\, \tilde{q}^{\binom{n}{2}} \begin{bmatrix} -\beta-1 \\ n-1 \end{bmatrix}_{\tilde{q}} . \tag{1.28}$$

Remark 1.27. Formulas (1.27) and (1.28) are often employed even for $\beta \le -1$.

Since there are no prescribed properties which are supposed to be met by the power functions, we cannot ultimately decide whether the power function is defined properly. Considering the time scales $\mathbb{R}$, $h\mathbb{Z}$ and $q^{\mathbb{Z}}$, the formulas above are historically established and the behaviour of such functions seems to be reasonable. That is not the case of $\mathbb{T}_{(q,h)}$, since it was introduced only recently (see [18]) and its validity has to be confirmed by verifying the requirements occurring during our investigation.

We show that (1.28) (and consequently (1.27)) extends the fundamental relation (1.23) also to non-integer orders as published in the author's joint paper [15].

Lemma 1.28. *Let $m \in \mathbb{Z}^+$, $\beta \in \mathbb{R}$, $s,t \in \mathbb{T}_{(q,h)}$ and $n \in \mathbb{Z}^+$, $n \ge m$ be such that $t = \sigma^n(s)$. Then*

$$\nabla^m \hat{h}_\beta(t,s) = \begin{cases} \hat{h}_{\beta-m}(t,s)\,, & \beta \notin \{0,1,\dots,m-1\}\,, \\ 0, & \beta \in \{0,1,\dots,m-1\}\,. \end{cases}$$

Proof. First let $m = 1$. For $\beta = 0$ we get $\hat{h}_0(t,s) = 1$ and the first nabla derivative is zero. If $\beta \ne 0$, then by (1.28) and (1.24) we have

$$\begin{aligned} \nabla \hat{h}_\beta(t,s) &= \frac{\hat{h}_\beta(t,s) - \hat{h}_\beta(\rho(t),s)}{\nu(t)} \\ &= \frac{\nu^\beta(\sigma(s))}{\tilde{q}^{-n+1}\nu(\sigma(s))} \left((-1)^{n-1}\tilde{q}^{\binom{n}{2}} \begin{bmatrix} -\beta-1 \\ n-1 \end{bmatrix}_{\tilde{q}} - (-1)^{n-2}\tilde{q}^{\binom{n-1}{2}} \begin{bmatrix} -\beta-1 \\ n-2 \end{bmatrix}_{\tilde{q}} \right) \\ &= (-1)^{n-1}\nu^{\beta-1}(\sigma(s))\tilde{q}^{\binom{n}{2}} \left(\tilde{q}^{n-1} \begin{bmatrix} -\beta-1 \\ n-1 \end{bmatrix}_{\tilde{q}} + \begin{bmatrix} -\beta-1 \\ n-2 \end{bmatrix}_{\tilde{q}} \right) = \hat{h}_{\beta-1}(t,s)\,. \end{aligned}$$

The case $m \ge 2$ can be verified by the induction principle. □

We end this subsection by recalling an asymptotic property of the power function on $\mathbb{T} = h\mathbb{Z}$ revealing its relation to the continuous power functions (see Definition 1.26). We employ a symbol $\sim$ for asymptotic equivalence defined as

$$f \sim g \qquad \text{if and only if} \qquad \lim_{t\to\infty} \frac{f(t)}{g(t)} = 1\,.$$

Corollary 1.29. *Let $\beta \in \mathbb{R} \setminus \mathbb{Z}^-$, $\mathbb{T} = h\mathbb{Z}$, $s,t \in \mathbb{T}$ and let s be fixed. Then it holds*

$$\hat{h}_\beta(t,s) \sim \frac{t^\beta}{\Gamma(1+\beta)} \quad \text{as } t \to \infty\,. \tag{1.29}$$

Proof. Let $n \in \mathbb{Z}^+$ be such that $t = \sigma^n(s)$. Using the asymptotic expansion

$$(-1)^m \binom{\xi}{m} \sim \frac{1}{m^{1+\xi}\Gamma(-\xi)} \quad \text{as } n \to \infty\,, \ \xi \in \mathbb{R} \setminus \mathbb{Z}_0^+, \ m \in \mathbb{Z}^+ \tag{1.30}$$

(see [42, p. 54]) we obtain

$$\hat{h}_\beta(t,s) = (-1)^{n-1} h^\beta \binom{-\beta-1}{n-1} \sim \frac{h^\beta}{n^{-\beta}\Gamma(1+\beta)} = \frac{t^\beta}{\Gamma(1+\beta)} \quad \text{as } n \to \infty\,.$$

□

1.2.4 Laplace transform

This subsection is devoted to the nabla Laplace transform on time scales (shortly the Laplace transform). While its delta (forward) version is extensively studied by many authors (see, e.g. [2, 11, 19]), the basics of the Laplace transform derived from the nabla calculus, i.e. the one we utilize in this thesis, appear only occasionally (see [3]). That is the reason why we discuss this matter in more details. Throughout this subsection we use the symbol $\mathbb{T}_0$ for a time scale such that

$$0 \in \mathbb{T}_0 \qquad \text{and} \qquad \sup \mathbb{T}_0 = \infty\,.$$

Note that the Laplace transform can be, in general, defined with a starting point different from zero, but we are not interested in such a case.

Before we engage in the Laplace transform, we have to establish the notion of a convolution of two functions $(f * g)(t)$, which is essential for our latter investigation. The definitions of a convolution employed in continuous and difference calculus, i.e. $(f*g)(t) = \int_0^t f(t-\tau)g(\tau)\mathrm{d}\tau$ and $(f*g)(n) = \sum_{k=1}^n f(n-k+1)g(k)$, respectively, are not applicable for a general time scale. For the delta calculus the appropriate definition was proposed in [13]. Adapting that approach for the nabla case we obtain the following introductions and assertions.

Definition 1.30. Let $\mathbb{T}$ be such that $\sup \mathbb{T} = \infty$ and fix $t_0 \in \mathbb{T}$. For a given function $f : [t_0, \infty)_{\mathbb{T}} \to \mathbb{C}$, the solution of the *shifting problem*

$$\begin{aligned} \nabla\, u(t, \rho(s)) &= -\tilde{\nabla}\, u(t,s)\,, \quad s,t \in \mathbb{T}\,,\ t > s > t_0\,,\\ u(t,t_0) &= f(t)\,, \quad t \in \mathbb{T}\,,\ t > t_0 \end{aligned}$$

is called the *shift* (or *delay*) of $f(t)$. The symbol $\tilde{\nabla}$ denotes the derivative with respect to the second variable.

Remark 1.31. (i) For $\mathbb{T} = \mathbb{R}_0^+$ and $\mathbb{T} = h\mathbb{Z}_0^+$ the shifting problem has a unique solution $u(t,s) = f(t-s)$ for all $f(t)$.
(ii) The shift of the polynomial $\hat{h}_m(t,r)$ ($m \in \mathbb{Z}_0^+$) is $\hat{h}_m(t,s)$ independent of r.
(iii) Let λ be a ν-regressive constant. Then the shift of the exponential function $\hat{e}_\lambda(t,r)$ is $\hat{e}_\lambda(t,s)$ independent of r.

Definition 1.32. For given functions $f, g : \mathbb{T}_0 \to \mathbb{R}$, their *convolution* is defined by

$$(f * g)(t) = \int_0^t \hat{f}(t, \rho(\tau))g(\tau)\nabla\tau\,, \quad t \in \mathbb{T}_0\,,$$

where $\hat{f}(t,s)$ is the shift of $f(t)$.

Such the definition of convolution keeps the key properties known from the classical analysis, e.g. associativity $(f * g) * u = f * (g * u)$. They can be verified by modifications of proofs in [13]. The most important property, the convolution theorem, is related to the Laplace transform and will be discussed bellow.

In [3] the authors introduced the generalized Laplace transform for the so-called alpha derivatives and the nabla Laplace transform falls within this theory. Setting the alpha function to the backward jump operator ρ yields the following definition and subsequent assertions.

Definition 1.33. Let $f(t)$ be a real function defined at least on $(0,\infty)_{\mathbb{T}_0}$. The *Laplace transform* of $f(t)$ is defined by

$$\mathcal{L}\{f\}(z) = \int_0^\infty f(t)\,\hat{e}_z(0, \rho(t))\nabla t \qquad \text{for } z \in \mathcal{D}(f), \tag{1.31}$$

where $\mathcal{D}(f)$ consists of all complex numbers $z \in \mathcal{R}_\nu$ for which the improper integral exists.

Remark 1.34. It can be proved that the necessary condition for $\mathcal{D}(f)$ being of nonzero measure is existence of $M \in \mathbb{R}^+$ such that $\nu(t) < M$ for all $t \in (0,\infty)_{\mathbb{T}_0}$. Obviously, the constant M does not exist in the case of $\mathbb{T}_{(q,h)}$, while for $\mathbb{T} = h\mathbb{Z}$ we can choose an arbitrary real number greater than h.

Proposition 1.35. *Let $\lambda \in \mathcal{R}_\nu$ be such that $\lim_{t\to\infty} \hat{e}_\lambda(t,0)\hat{e}_z(0,t) = 0$. Then*

$$\mathcal{L}\{\hat{e}_\lambda(\cdot,0)\}(z) = \frac{1}{z-\lambda}\,.$$

Proposition 1.36. *Let $m \in \mathbb{Z}^+$. Then it holds*

(i) $\mathcal{L}\{\hat{h}_m(\cdot,0)\}(z) = z^{-m-1}$,

(ii) $\mathcal{L}\{\nabla^m f\}(z) = z^m\mathcal{L}\{f\}(z) - \sum_{j=0}^{m-1} z^j \nabla^{m-j-1} f(t)\big|_{t=0}$,

(iii) $\mathcal{L}\{\int_0^t f(\tau)\nabla\tau\}(z) = z^{-1}\mathcal{L}\{f\}(z)$.

Finally, the nabla Laplace transform retains the usual form of the convolution theorem and extends its validity on every time scale where the appropriate integrals (1.31) are applicable.

Theorem 1.37. *Let $f(t)$, $g(t)$ be functions such that $\mathcal{L}\{f\}(z)$, $\mathcal{L}\{g\}(z)$ exist. Then*

$$\mathcal{L}\{f * g\}(z) = \mathcal{L}\{f\}(z) \cdot \mathcal{L}\{g\}(z)\,. \tag{1.32}$$

Proof. The assertion can be proved utilizing the nabla analogy of the technique performed in the proof of [13, Theorem 3.2]. □

Now we point our attention at the time scale $h\mathbb{Z}$ and derive some other results related to our later investigations (see the author's joint paper [17]).

h-Laplace transform

Applying the general definition relation (1.31) to $\mathbb{T}_0 = h\mathbb{Z}_0^+$ and considering Theorem 1.22 (ii), we obtain the formula for nabla h-Laplace transform of a function $f(t_n)$ as

$$\mathcal{L}\{f\}(z) = h\sum_{k=1}^{\infty} f(t_k)(1-hz)^{k-1}\,, \quad \text{where } t_n = nh\,. \tag{1.33}$$

The backward h-Laplace transform is given by a power series with the center at $z_0 = h^{-1}$, i.e. it holds $\mathcal{D}(f) = B(h^{-1}, r)$, where we use the notation $B(z_0, r) = \{z \in \mathbb{C};\, |z - z_0| < r\}$. In particular, if the series converges at some $z \neq h^{-1}$, then there exists $r > 0$ such that the series converges locally uniformly (and absolutely) in the open disk $B(h^{-1}, r)$. Moreover, the series expansion is determined uniquely and $\mathcal{L}\{f\}(z)$ is an analytic function on $B(h^{-1}, r)$.

The h-Laplace transform is related to some other known discrete transformations. The connection with the classical $\mathcal{Z}$-transform of $f(t_n)$, defined by $\mathcal{Z}\{f\}(z) = \sum_{k=0}^{\infty} f(t_k)z^{-k}$, is given via the relation

$$\mathcal{L}\{f(\cdot)\}(z) = h\,\mathcal{Z}\{f(\sigma(\cdot))\}\Big(\frac{1}{1-hz}\Big)\,. \tag{1.34}$$

If $h = 1$ then the backward h-Laplace transform agrees with $\mathcal{N}_{t_0}$-transform introduced in [8] for the value $t_0 = 1$.

Now we state the h-Laplace transforms of power functions (1.27) and of an h-analogue of the Kronecker delta (or more generally of Dirac delta function).

Lemma 1.38. *Let $\beta \in \mathbb{R}\setminus\mathbb{Z}^-$ and $\delta_h(t_n)$ be a function defined as $\delta_h(t_n) = h^{-1}$ for $n = 1$ and zero otherwise. Then*

(i) $\mathcal{L}\{\hat{h}_\beta(\cdot, 0)\}(z) = z^{-\beta-1}$,

(ii) $\mathcal{L}\{\delta_h\}(z) = 1$.

Proof. The proof of (i) can be found in [8] (for the case $h = 1$). Here we present an alternative proof based on the binomial theorem. It holds

$$\begin{aligned}\mathcal{L}\{\hat{h}_\beta(\cdot, 0)\}(z) &= h\sum_{k=1}^{\infty} h^\beta(-1)^{k-1}\binom{-\beta-1}{k-1}(1-hz)^{k-1}\\ &= h^{\beta+1}\sum_{k=0}^{\infty}\binom{-\beta-1}{k}(hz-1)^k = h^{\beta+1}(1+hz-1)^{-\beta-1} = z^{-\beta-1}\,,\end{aligned}$$

where the last line holds for $|1 - hz| < 1$, i.e. for each $z \in B(h^{-1}, h^{-1})$.

The part (ii) is an immediate consequence of the definition of $\delta_h(t_n)$. □

The connection of the h-Laplace transform to the theory of power series enables us to determine basic asymptotic properties of $f(t_n)$ from the knowledge of $\mathcal{L}\{f\}(z)$. In Chapter 4 we are going to utilize following assertions (as usual, the symbol ℓ^1 denotes the space of sequences whose series are absolutely convergent).

Proposition 1.39. *Let $f(t_n)$ be such that $\mathcal{L}\{f\}(z)$ converges on $B(h^{-1}, r)$, $r > 0$.*

(i) *If $r > h^{-1}$ then $f(t_n) \in \ell^1$.*

(ii) *If $r < h^{-1}$ then $\limsup_{n\to\infty} |f(t_n)| = \infty$.*

Proof. Assuming $r > h^{-1}$, the Cauchy-Hadamard theorem implies

$$\limsup_{n\to\infty} \sqrt[n]{|f(t_n)|} = \frac{1}{hr} < 1\,,$$

hence $f(t_n) \in \ell^1$. If $r < h^{-1}$, the argumentation is analogous. □

Proposition 1.40. *Let $g(t_n)$ be such that $g(t_n) \in \ell^1$. Then there exists $f(t_n) \in \ell^1$ satisfying $\mathcal{L}\{g\}(z) \cdot \mathcal{L}\{f\}(z) = 1$ if and only if*

$$\inf_{|z-h^{-1}|\leq h^{-1}} |\mathcal{L}\{g\}(z)| > 0\,.$$

Proof. The statement follows from the Wiener theorem (see [20, p. 251]) with respect to (1.34) characterizing the relationship between the nabla h-Laplace transform and $\mathcal{Z}$-transform. □

1.3 Introduction to discrete fractional calculus

Considering the previous two sections we face the question how to establish the fractional calculus in the framework of the time scales theory. Unfortunately, until now this question has no satisfactory answer, at least not in a form exceeding a formal generalization of symbols.

We briefly summarize this situation and comment some of its aspects. Then we focus solely on isolated time scales, where this issue has been overcome, i.e. the time scales with a linear graininess function, namely $\mathbb{T}_{(q,h)}$ and its special case $\mathbb{T} = h\mathbb{Z}$. We recall the relevant definitions and basic properties known from [18], present some original results published in [14, 15] and derive a few useful relations related to our next investigations.

1.3.1 Time scales definition outline

To follow the continuous paradigm, we introduce the nabla Cauchy formula for repeated integration. We state its general form valid on an arbitrary time scale $\mathbb{T}$. Let $a, b \in \mathbb{T}$ and $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ be ld-continuous. We put

$${}_a\nabla^{-1} f(t) = \int_a^t f(\tau)\nabla\tau \quad \text{for } t \in [a, b]_{\mathbb{T}} \tag{1.35}$$

and define recursively

$$ {}_a\nabla^{-m} f(t) = \int_a^t {}_a\nabla^{-m+1} f(\tau)\nabla\tau \tag{1.36} $$

for $m = 2, 3, \ldots$. Then we have

Lemma 1.41. *Let $m \in \mathbb{Z}^+$, $a, b \in \mathbb{T}$ and let $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ be ld-continuous. Then*

$$ {}_a\nabla^{-m} f(t) = \int_a^t \hat{h}_{m-1}(t, \rho(\tau)) f(\tau)\nabla\tau\,, \quad t \in [a, b]_{\mathbb{T}}\,. \tag{1.37} $$

Proof. This assertion can be proved by induction. If $m = 1$, then (1.37) obviously holds. Let $m \geq 2$ and assume that (1.37) holds with m replaced by $m - 1$, i.e.

$$ {}_a\nabla^{-m+1} f(t) = \int_a^t \hat{h}_{m-2}(t, \rho(\tau)) f(\tau)\nabla\tau\,. $$

By the definition, the left-hand side of (1.37) is an integral of ${}_a\nabla^{-m+1} f(t)$. We show that the right-hand side of (1.37) is an integral of $\int_a^t \hat{h}_{m-2}(t, \rho(\tau)) f(\tau)\nabla\tau$. Indeed, it holds

$$ \nabla \int_a^t \hat{h}_{m-1}(t, \rho(\tau)) f(\tau)\nabla\tau = \int_a^t \nabla\hat{h}_{m-1}(t, \rho(\tau)) f(\tau)\nabla\tau = \int_a^t \hat{h}_{m-2}(t, \rho(\tau)) f(\tau)\nabla\tau\,, $$

where we have employed (1.18) and Theorem 1.23 (i). Consequently, the relation (1.37) holds up to a possible additive constant. Substituting $t = a$ we can find this additive constant zero. □

Remark 1.42. In the case of delta calculus, the Cauchy formula can be derived as well. However, there occurs a dependence of the upper integral limit on the order of integration. Hence, generalizing this formula to non-integer orders causes a shift of the domain. In other words, the delta fractional operators are defined on different domain than the original function and this domain does not even belong to the considered time scale. This is one of the main reasons why we prefer the nabla calculus throughout this thesis. For more details we refer to, e.g. [7, 18].

To keep consistency with the notation established by (1.35) and (1.36) and with the continuous case, we denote the usual mth nabla derivative ($m \in \mathbb{Z}^+$) by ${}_a\nabla^m$, although the parameter a has no factual meaning here.

Now we are in a similar position as in the Section 1.1. We have an expression for the mth sum of a function $f(t)$ for $m \in \mathbb{Z}^+$ and we look for its extension to non-integer values of m. Following the path indicated by the continuous fractional calculus (namely Definitions 1.1 and 1.3), the introduction of the fractional operators on time scales seems to be an easy task. In particular, by a generalization of (1.37) we arrive at the expression

$$ {}_a\nabla^{-\gamma} f(t) = \int_a^t \hat{h}_{\gamma-1}(t, \rho(\tau)) f(\tau)\nabla\tau\,, \quad \gamma > 0 \tag{1.38} $$

for the time scales fractional integral of order γ. Thus, we propose

$$
{}_a\nabla^{\alpha} f(t) = {}_a\nabla^{\lceil\alpha\rceil}\, {}_a\nabla^{-(\lceil\alpha\rceil-\alpha)} f(t)\,, \quad \alpha > 0 \tag{1.39}
$$

as the time scales Riemann-Liouville fractional derivative of order α. We note, that these introductions appeared, e.g. in [4].

However, considering a general time scale $\mathbb{T}$, (1.38) (and consequently (1.39)) is nothing but a symbolical expression. Its practical use requires a reasonable and natural extension of a discrete system of polynomials ($\hat{h}_m$, $m \in \mathbb{Z}_0^+$) to a continuous system of power functions ($\hat{h}_\beta$, $\beta \in (-1, \infty)$). As discussed in Subsection 1.2.3, such extensions are available only on some special time scales, namely $\mathbb{R}$ and $\mathbb{T}_{(q,h)}$ (and its special cases $h\mathbb{Z}$, $q^{\mathbb{Z}}$).

In this thesis we deal with the discrete fractional calculus, in particular we focus on the time scales $\mathbb{T}_{(q,h)}$ and $h\mathbb{Z}$ (or their subintervals). Although our considerations do not cover the case of a general scale $\mathbb{T}$, we will consistently use the time scale notation of main procedures and operations to outline a possible way out to further generalization. The unified notation also enables an easier comparison with the results derived on $\mathbb{R}$.

1.3.2 Fractional (q, h)-calculus and h-calculus

In this subsection we discuss more closely the fractional calculus on the time scales $\mathbb{T}_{(q,h)}$ and $h\mathbb{Z}$, the so-called fractional (q, h)-calculus and h-calculus, respectively. Since the time scale $\mathbb{T}_{(q,h)}$ is a generalization of $h\mathbb{Z}$, all results derived for $\mathbb{T}_{(q,h)}$ are also applicable to $\mathbb{T} = h\mathbb{Z}$. For the sake of lucidity, we denote the fractional operators by ${}_a\nabla^{\alpha}_{(q,h)}$ or ${}_a\nabla^{\alpha}_{h}$ whenever we refer specifically to the time scale $\mathbb{T}_{(q,h)}$ or $h\mathbb{Z}$, respectively. For detailed introduction to the fractional (q, h)-calculus we refer to [18]. A survey of basics of the fractional h-calculus for the case of $h = 1$ can be found, e.g. in [29].

The following definitions originate in the approach described in the previous subsection. Consequently, many of the obtained results are not bound to the (q, h)-calculus, but would be actually valid on any other time scale with the well-defined power function.

Definition 1.43. Let $\gamma \in \mathbb{R}_0^+$ and $\tilde{a}, a, b \in \mathbb{T}_{(q,h)}$ be such that $\tilde{a} \le a < b$. Then for a function $f : (\tilde{a}, b]_{\mathbb{T}_{(q,h)}} \to \mathbb{R}$ we define the *fractional integral* of order $\gamma \in \mathbb{R}^+$ with the lower limit a as

$$
{}_a\nabla^{-\gamma}_{(q,h)} f(t) = \int_a^t \hat{h}_{\gamma-1}(t, \rho(\tau)) f(\tau) \nabla\tau\,, \quad t \in [a, b]_{\mathbb{T}_{(q,h)}} \cap (\tilde{a}, b]_{\mathbb{T}_{(q,h)}} \tag{1.40}
$$

and for $\gamma = 0$ we put ${}_a\nabla^{0}_{(q,h)} f(t) = f(t)$.

Remark 1.44. (i) In [18] this definition has been introduced for the case $a = \tilde{a}$. Our extension considers also functions which are undefined at lower limit of the integral. This step is motivated by the continuous case, where such functions form a significant part of

the fractional calculus. Moreover, it turns out that similar phenomenon appears also on discrete settings as we outline in Chapter 4.
(ii) If $a > \tilde{a}$, then (1.40) can be calculated for $t \in [a,b]_{\mathbb{T}_{(q,h)}}$ by an ordinary integration. If $a = \tilde{a}$, we treat (1.40) for $t \in (a,b]_{\mathbb{T}_{(q,h)}}$ as an improper integral of the second kind. Moreover, considering $a > \tilde{a}$ the function $f(t)$ is bounded on $[a,b]_{\mathbb{T}_{(q,h)}}$ and therefore ${}_a\nabla^{-\gamma} f(t)\big|_{t=a} = 0$. The value of ${}_a\nabla^{-\gamma} f(t)\big|_{t=a}$ for $a = \tilde{a}$ will be discussed in Chapter 4.

Definition 1.45. Let $\alpha \in \mathbb{R}^+$ and $\tilde{a}, a, b \in \mathbb{T}_{(q,h)}$, be such that $\tilde{a} \le a < b$. Then for a function $f : (\tilde{a}, b]_{\mathbb{T}_{(q,h)}} \to \mathbb{R}$ we define the *Riemann-Liouville fractional derivative* of order α with the lower limit a as

$$ {}_a\nabla^{\alpha}_{(q,h)} f(t) = {}_a\nabla^{\lceil\alpha\rceil}_{(q,h)} {}_a\nabla^{-(\lceil\alpha\rceil-\alpha)}_{(q,h)} f(t)\,, \quad t \in [\sigma(a), b]_{\mathbb{T}_{(q,h)}} \cap (\sigma(\tilde{a}), b]_{\mathbb{T}_{(q,h)}}\,. \tag{1.41} $$

Remark 1.46. (i) Analogously to the continuous case (see Remark 1.4 (ii)), the *Caputo fractional derivative* of order α with the lower limit a is introduced as

$$ {}^{\mathrm{C}}_{\ a}\nabla^{\alpha}_{(q,h)} f(t) = {}_a\nabla^{-(\lceil\alpha\rceil-\alpha)}_{(q,h)}\, {}_a\nabla^{\lceil\alpha\rceil}_{(q,h)} f(t)\,, \quad t \in [\sigma(a), b]_{\mathbb{T}_{(q,h)}}\,. $$

(ii) To emphasize the discrete settings, the operators ${}_a\nabla^{\alpha}_{(q,h)}$ and ${}_a\nabla^{\alpha}_{h}$ are called fractional differences instead of fractional derivatives throughout Chapters 2-4.

To obtain a representation of fractional (q,h)-integral more convenient for calculations, we expand the definition (1.40) with respect to (1.15) and (1.28). It yields

$$ {}_a\nabla^{-\gamma}_{(q,h)} f(t) = \sum_{k=1}^{n} \nu(\sigma^k(a)) \hat{h}_{\gamma-1}(\sigma^n(a), \sigma^{k-1}(a)) f(\sigma^k(a)) \tag{1.42} $$

$$ = \sum_{k=1}^{n} (-1)^{n-k} \nu^{\gamma}(\sigma^k(a)) \begin{bmatrix} -\gamma \\ n-k \end{bmatrix}_{\tilde{q}} \tilde{q}^{\binom{n-k+1}{2}} f(\sigma^k(a))\,, \tag{1.43} $$

where $\gamma \in \mathbb{R}^+$, $t = \sigma^n(a)$ and $n = 1, 2, \ldots$. These relations along with the definition formula (1.41) provide a solid tool for evaluation of fractional derivatives. Nevertheless, sometimes it is suitable to utilize directly an expansion of fractional derivative, in particular

$$ {}_a\nabla^{\alpha}_{(q,h)} f(t) = \sum_{k=1}^{n} \nu(\sigma^k(a)) \hat{h}_{-\alpha-1}(\sigma^n(a), \sigma^{k-1}(a)) f(\sigma^k(a)) \tag{1.44} $$

$$ = \sum_{k=1}^{n} (-1)^{n-k} \nu^{-\alpha}(\sigma^k(a)) \begin{bmatrix} \alpha \\ n-k \end{bmatrix}_{\tilde{q}} \tilde{q}^{\binom{n-k+1}{2}} f(\sigma^k(a))\,, \tag{1.45} $$

where $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, $t = \sigma^n(a)$ and $n = \lceil\alpha\rceil + 1, \lceil\alpha\rceil + 2, \ldots$ (for more details we refer to [18, Propositions 1 and 3] with respect to (1.28)). If $\alpha \in \mathbb{Z}^+$ then the lower limits of the sums are transformed to $k = n - \alpha$ which leads to known formulas for integer-order differences. This matter is broadly discussed in [18]. Note that the validity of (1.45) can be extended to $\alpha \in \mathbb{Z}^+$ if we put

$$ \begin{bmatrix} \xi \\ m \end{bmatrix}_{\tilde{q}} = 0 \quad \text{for } \xi, m \in \mathbb{Z}^+ \text{ such that } \xi < m\,. $$

Remark 1.47. Obviously there is a formal agreement between formulas for fractional operators so we can write them jointly. For illustrative reasons, we rearrange these formulas on $\mathbb{T} = h\mathbb{Z}$ to the form of a quite familiar relation

$$ {}_a\nabla_h^\alpha f(t_n) = \sum_{k=0}^{n-1} \frac{(-1)^k}{h^\alpha} \binom{\alpha}{k} f(t_{n-k}) \,, \quad \alpha \in \mathbb{R} \,, $$

where $t_n = a + nh$ and $\binom{\xi}{m} = 0$ for $\xi, m \in \mathbb{Z}^+$ such that $\xi < m$.

Next, we recall some assertions presented in the author's joint paper [15]. In particular, we perform an extension of the power rule stated in Lemma 1.28 for fractional operators of (q,h)-calculus.

Lemma 1.48. *Let $\gamma \in \mathbb{R}^+$, $\beta \in \mathbb{R} \setminus \mathbb{Z}^-$ and $a, t \in \mathbb{T}_{(q,h)}$ be such that $t > a$. Then it holds*

$$ {}_a\nabla_{(q,h)}^{-\gamma} \hat{h}_\beta(t,a) = \hat{h}_{\gamma+\beta}(t,a) \,. $$

Proof. Let $t = \sigma^n(a)$ for some $n \in \mathbb{Z}^+$. We have

$$ \begin{aligned} {}_a\nabla_{(q,h)}^{-\gamma} \hat{h}_\beta(t,a) &= \sum_{k=1}^{n} (-1)^{n-k} \nu^\gamma(\sigma^k(a)) \begin{bmatrix} -\gamma \\ n-k \end{bmatrix}_{\tilde{q}} \tilde{q}^{\binom{n-k+1}{2}} \hat{h}_\beta(\sigma^k(a), a) \\ &= (-1)^{n-1} \nu^{\gamma+\beta}(\sigma(a)) \sum_{k=1}^{n} \tilde{q}^{\gamma(1-k) + \binom{n-k+1}{2} + \binom{k}{2}} \begin{bmatrix} -\gamma \\ n-k \end{bmatrix}_{\tilde{q}} \begin{bmatrix} -\beta-1 \\ k-1 \end{bmatrix}_{\tilde{q}} \\ &= (-1)^{n-1} \nu^{\gamma+\beta}(\sigma(a)) \sum_{k=0}^{n-1} \tilde{q}^{k^2 - (n-1)k - \gamma k + \binom{n}{2}} \begin{bmatrix} -\gamma \\ n-k-1 \end{bmatrix}_{\tilde{q}} \begin{bmatrix} -\beta-1 \\ k \end{bmatrix}_{\tilde{q}} \\ &= (-1)^{n-1} \nu^{\gamma+\beta}(\sigma(a)) \begin{bmatrix} -\gamma-\beta-1 \\ n-1 \end{bmatrix}_{\tilde{q}} \tilde{q}^{\binom{n}{2}} = \hat{h}_{\gamma+\beta}(t,a) \,, \end{aligned} $$

where we have used (1.26) on the last row. □

Further, we formulate the assertion dealing with the Riemann-Liouville fractional derivative of the power function.

Corollary 1.49. *Let $\alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R} \setminus \mathbb{Z}^-$ and $a, t \in \mathbb{T}_{(q,h)}$ be such that $t > \sigma^{\lceil \alpha \rceil}(a)$. Then*

$$ {}_a\nabla_{(q,h)}^{\alpha} \hat{h}_\beta(t,a) = \begin{cases} \hat{h}_{\beta-\alpha}(t,a) \,, & \beta - \alpha \notin \{-1, \dots, -\lceil \alpha \rceil\} \,, \\ 0 \,, & \beta - \alpha \in \{-1, \dots, -\lceil \alpha \rceil\} \,. \end{cases} $$

Proof. Lemma 1.48 implies that

$$ {}_a\nabla_{(q,h)}^{\alpha} \hat{h}_\beta(t,a) = {}_a\nabla_{(q,h)}^{\lceil \alpha \rceil} \big({}_a\nabla_{(q,h)}^{-(\lceil \alpha \rceil - \alpha)} \hat{h}_\beta(t,a) \big) = {}_a\nabla_{(q,h)}^{\lceil \alpha \rceil} \hat{h}_{\lceil \alpha \rceil + \beta - \alpha}(t,a) \,. $$

Then the statement is an immediate consequence of Lemma 1.28. □

Now, we are in a position to discuss the (q,h)-analogue of the composition rules (1.3), (1.4). This matter was already studied in [18], where the authors assumed the function $f(t)$ to be defined on the whole $\mathbb{T}_{(q,h)}$ and extended the validity of (1.43) to $t \leq a$ (i.e. $n \leq 0$). Consequently, the derived composition rules have a simple form ${}_a\nabla^{\alpha}_{(q,h)}{}_a\nabla^{\beta}_{(q,h)}f(t) = {}_a\nabla^{\alpha+\beta}_{(q,h)}f(t)$ for all $\alpha, \beta \in \mathbb{R}$.

We choose to hold onto the Definitions 1.43 and 1.45 and not to extend the domain of $f(t)$. This approach results in rules corresponding to (1.3), (1.4). We note that similar results were derived for the case $\mathbb{T} = \mathbb{Z}$ in [29].

Lemma 1.50. *Let $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^+$. Then it holds*

$$
{}_a\nabla^{\alpha}_{(q,h)}{}_a\nabla^{-\beta}_{(q,h)}f(t) = {}_a\nabla^{\alpha-\beta}_{(q,h)}f(t)\,, \tag{1.46}
$$

$$
{}_a\nabla^{\alpha}_{(q,h)}{}_a\nabla^{\beta}_{(q,h)}f(t) = {}_a\nabla^{\alpha+\beta}_{(q,h)}f(t) - \sum_{j=1}^{\lceil\beta\rceil} \hat{h}_{-\alpha-j}(t,a)\,{}_a\nabla^{\beta-j}_{(q,h)}f(t)\big|_{t=a}\,. \tag{1.47}
$$

Proof. First we prove (1.46). In fact it consists of two cases differing in the sign of α. Let $\alpha, \beta \in \mathbb{R}^+$. Then

$$
\begin{aligned}
&{}_a\nabla^{-\alpha}_{(q,h)}{}_a\nabla^{-\beta}_{(q,h)}f(t) = \int_a^t \hat{h}_{\alpha-1}(t,\rho(\tau)) \int_a^\tau \hat{h}_{\beta-1}(\tau,\rho(\psi))f(\psi)\nabla\psi\,\nabla\tau \\
&= \int_a^t f(\psi) \int_{\rho(\psi)}^t \hat{h}_{\alpha-1}(t,\rho(\tau))\hat{h}_{\beta-1}(\tau,\rho(\psi))\nabla\tau\,\nabla\psi \\
&= \int_a^t f(\psi)\,{}_{\rho(\psi)}\nabla^{-\alpha}_{(q,h)}\hat{h}_{\beta-1}(t,\rho(\psi))\nabla\psi = \int_a^t \hat{h}_{\alpha+\beta-1}(t,\rho(\psi))f(\psi)\nabla\psi = {}_a\nabla^{-\alpha-\beta}_{(q,h)}f(t)\,,
\end{aligned}
$$

where we utilized the relation $\int_a^t \int_a^\tau g(\tau,\psi)\nabla\psi\nabla\tau = \int_a^t \int_{\rho(\psi)}^t g(\tau,\psi)\nabla\tau\nabla\psi$ valid on an arbitrary time scale.

Now we show the case of a fractional derivative applied to a fractional integral. By (1.41) and by the above proved rule we get

$$
{}_a\nabla^{\alpha}_{(q,h)}{}_a\nabla^{-\beta}_{(q,h)}f(t) = {}_a\nabla^{\lceil\alpha\rceil}_{(q,h)}{}_a\nabla^{-\lceil\alpha\rceil+\alpha}_{(q,h)}{}_a\nabla^{-\beta}_{(q,h)}f(t) = {}_a\nabla^{\lceil\alpha\rceil}_{(q,h)}{}_a\nabla^{-\lceil\alpha\rceil+\alpha-\beta}_{(q,h)}f(t) = {}_a\nabla^{\alpha-\beta}_{(q,h)}f(t)
$$

which concludes the proof of (1.46).

To verify (1.47), we start with the rule for a fractional integral of an ordinary mth derivative ${}_a\nabla^{-\alpha}_{(q,h)}{}_a\nabla^{m}_{(q,h)}f(t)$ ($m \in \mathbb{Z}^+$). For that purpose, we need the relation

$$
\hat{h}_\gamma(\sigma^n(a),\sigma^{n-1}(a)) = \nu(\sigma^n(a))\hat{h}_{\gamma-1}(\sigma^n(a),\sigma^{n-1}(a)) \tag{1.48}
$$

following from (1.28), and the derivative of power function

$$
{}_a\tilde{\nabla}\hat{h}_\gamma(t,s) = -\hat{h}_{\gamma-1}(t,\rho(s))\,, \tag{1.49}
$$

where ${}_a\tilde{\nabla}$ denotes the derivative with respect to the second variable (the proof analogous as for Lemma 1.28, compare with Remark 1.31 (ii)). Using these auxiliary results we have

$$
\begin{aligned}
{}_a\nabla^{-\alpha}_{(q,h)}{}_a\nabla^{m}_{(q,h)}f(t) &= \sum_{k=1}^{n}\hat{h}_{\alpha-1}(\sigma^n(a),\sigma^{k-1}(a))\big({}_a\nabla^{m-1}_{(q,h)}f(\sigma^k(a)) - {}_a\nabla^{m-1}_{(q,h)}f(\sigma^{k-1}(a))\big)\\
&= \hat{h}_{\alpha-1}(\sigma^n(a),\sigma^{n-1}(a)){}_a\nabla^{m-1}_{(q,h)}f(\sigma^n(a)) - \hat{h}_{\alpha-1}(\sigma^n(a),a){}_a\nabla^{m-1}_{(q,h)}f(t)\big|_{t=a}\\
&\quad + \sum_{k=1}^{n-1}(\hat{h}_{\alpha-1}(\sigma^n(a),\sigma^{k-1}(a)) - \hat{h}_{\alpha-1}(\sigma^n(a),\sigma^k(a))){}_a\nabla^{m-1}_{(q,h)}f(\sigma^k(a))\\
&= \sum_{k=1}^{n}\nu(\sigma^k(a))\hat{h}_{\alpha-2}(\sigma^n(a),\sigma^{k-1}(a)){}_a\nabla^{m-1}_{(q,h)}f(\sigma^k(a)) - \hat{h}_{\alpha-1}(\sigma^n(a),a){}_a\nabla^{m-1}_{(q,h)}f(t)\big|_{t=a}\\
&= {}_a\nabla^{1-\alpha}_{(q,h)}{}_a\nabla^{m-1}_{(q,h)}f(t) - \hat{h}_{\alpha-1}(t,a){}_a\nabla^{m-1}_{(q,h)}f(t)\big|_{t=a}.
\end{aligned}
$$

Repeating this procedure we obtain

$$
{}_a\nabla^{-\alpha}_{(q,h)}{}_a\nabla^{m}_{(q,h)}f(t) = {}_a\nabla^{m-\alpha}_{(q,h)}f(t) - \sum_{j=1}^{m}\hat{h}_{\alpha-j}(t,a)\,{}_a\nabla^{m-j}_{(q,h)}f(t)\big|_{t=a}. \tag{1.50}
$$

Now let $\alpha,\beta\in\mathbb{R}^+$. The relation (1.47) now follows from (1.46), (1.50) using the expansions ${}_a\nabla^{-\alpha}_{(q,h)}{}_a\nabla^{\beta}_{(q,h)}f(t) = {}_a\nabla^{-\alpha}_{(q,h)}\,{}_a\nabla^{\lceil\beta\rceil}_{(q,h)}\,{}_a\nabla^{\beta-\lceil\beta\rceil}_{(q,h)}f(t)$ and ${}_a\nabla^{\alpha}_{(q,h)}{}_a\nabla^{\beta}_{(q,h)}f(t) = {}_a\nabla^{\lceil\alpha\rceil}_{(q,h)}\,{}_a\nabla^{\alpha-\lceil\alpha\rceil}_{(q,h)}\,{}_a\nabla^{\beta}_{(q,h)}f(t)$, which completes the proof. □

Remark 1.51. (i) We can see that the technique for proving (1.46) uses only general time scales tools and Lemma 1.48. Hence, we proved that (1.46) is valid on every time scale with the well-defined power function satisfying the property of Lemma 1.48.
(ii) Similarly, the rule (1.47) was proved for every isolated time scale where the power function is well-defined and the formulas (1.46), (1.48) and (1.49) hold.

At last, we state the assertion of the utmost importance for Chapter 4. We formulate the relations for h-Laplace transform of fractional operators, i.e. the analogues of (1.6), (1.7). These, or similar relations have been derived in [7,8] or in the author's joint paper [17].

Lemma 1.52. *Let $\alpha,\gamma\in\mathbb{R}^+$ and let $f(t_n)$ be such that its h-Laplace transform $\mathcal{L}\{f\}(z)$ exists. Then it holds*

(i) $\mathcal{L}\{{}_0\nabla_h^{-\gamma}f\}(z) = z^{-\gamma}\mathcal{L}\{f\}(z)$,

(ii) $\mathcal{L}\{{}_0\nabla_h^{\alpha}f\}(z) = z^{\alpha}\mathcal{L}\{f\}(z) - \sum_{j=0}^{\lceil\alpha\rceil-1} z^j\,{}_0\nabla_h^{\alpha-j-1}f(t_n)\big|_{n=0}$.

Proof. Since the fractional integral (1.40) is essentially a convolution, the property (i) is a direct consequence of Theorem 1.37 and Lemma 1.38 (i).

The property (ii) follows from the definition formula (1.41) via the property (i) and Proposition 1.36 (ii). □

2 Basic theory of higher-order linear FdEs on $\mathbb{T}_{(q,h)}$

In this chapter we deal with foundations of the theory of linear FdEs on $\mathbb{T}_{(q,h)}$. We recall that it is the most general discrete setting with well-established fractional calculus. Derived conclusions can be applied to $\mathbb{T} = h\mathbb{Z}$ (for $q = 1$) and $\mathbb{T} = q^{\mathbb{Z}}$ (for $h = 0$) as well. The presented results were published in [15], some of them for $\mathbb{T} = h\mathbb{Z}$ also in [14].

We introduce here some linear FdEs with the Riemann-Liouville difference operator and investigate their basic properties, such as the existence and uniqueness (Theorem 2.4) and the form of a general solution (Theorem 2.8). Further, we focus on a special two-term equation and describe the base of its solution space by the use of eigenfunctions of the operator ${}_a\nabla^{\alpha}_{(q,h)}$ (Theorem 2.15). We show that these eigenfunctions can be taken for discrete analogues of the Mittag-Leffler functions.

For the sake of simplicity, we introduce a restriction of $\mathbb{T}_{(q,h)}$ by

$$\widetilde{\mathbb{T}}^a_{(q,h)} = \{t \in \mathbb{T}_{(q,h)}\,;\, t \ge a > h/(1-q)\}\,, \quad \text{where } a \in \mathbb{T}_{(q,h)}.$$

Obviously $\widetilde{\mathbb{T}}^a_{(q,h)}$ is a time scale with power functions inherited from $\mathbb{T}_{(q,h)}$, because it holds $\widetilde{\mathbb{T}}^a_{(q,h)} = [a,\infty)_{\mathbb{T}_{(q,h)}}$.

2.1 An initial value problem

In this section, we are going to discuss the linear initial value problem

$$\sum_{j=1}^{\lceil\alpha\rceil} p_{\lceil\alpha\rceil-j+1}(t)\,{}_a\nabla^{\alpha-j+1}_{(q,h)}y(t) + p_0(t)\,y(t) = 0\,, \quad t \in \left(\widetilde{\mathbb{T}}^a_{(q,h)}\right)_{\kappa^{\lceil\alpha\rceil+1}}\,, \tag{2.1}$$

$${}_a\nabla^{\alpha-j}_{(q,h)}y(t)\big|_{t=\sigma^{\lceil\alpha\rceil}(a)} = y_{\alpha-j}\,, \quad j = 1,2,\dots,\lceil\alpha\rceil\,, \tag{2.2}$$

where $\alpha \in \mathbb{R}^+$. Further, we assume that $p_j(t)$ $(j = 1,\dots,\lceil\alpha\rceil-1)$ are arbitrary real functions on $\left(\widetilde{\mathbb{T}}^a_{(q,h)}\right)_{\kappa^{\lceil\alpha\rceil+1}}$, $p_{\lceil\alpha\rceil}(t) \equiv 1$ on $\left(\widetilde{\mathbb{T}}^a_{(q,h)}\right)_{\kappa^{\lceil\alpha\rceil+1}}$ and $y_{\alpha-j}$ $(j = 1,\dots,\lceil\alpha\rceil)$ are arbitrary real scalars.

If α is a positive integer, then (2.1), (2.2) corresponds to the standard initial value problem (1.19), (1.20). If α is not an integer, then applying (1.44) we can observe that the equation (2.1) is of the general form

$$\sum_{k=0}^{n-1} a_k(t)y(\rho^k(t)) = 0\,, \quad t \in \left(\widetilde{\mathbb{T}}^a_{(q,h)}\right)_{\kappa^{\lceil\alpha\rceil+1}}\,,\ n \text{ being such that } t = \sigma^n(a)\,,$$

which is usually referred to as the equation of Volterra type. If such an equation has two different solutions, then their values differ at least at one of the points $\sigma(a), \sigma^2(a), \dots, \sigma^{\lceil\alpha\rceil}(a)$. In particular, if $a_0(t) \neq 0$ for all $t \in \left(\widetilde{\mathbb{T}}^a_{(q,h)}\right)_{\kappa^{\lceil\alpha\rceil+1}}$, then arbitrary values of $y(\sigma(a))$, $y(\sigma^2(a)), \dots, y(\sigma^{\lceil\alpha\rceil}(a))$ determine uniquely the solution $y(t)$ on $\left(\widetilde{\mathbb{T}}^a_{(q,h)}\right)_{\kappa^{\lceil\alpha\rceil+1}}$. We show that the values $y_{\alpha-1}, y_{\alpha-2}, \dots, y_{\alpha-\lceil\alpha\rceil}$, introduced by (2.2), keep the same properties.

Proposition 2.1. *Let $\alpha \in \mathbb{R}^+$ and $y : \left(\widetilde{\mathbb{T}}^a_{(q,h)}\right)_\kappa \to \mathbb{R}$ be a function. Then (2.2) represents a one-to-one mapping between the vectors $(y(\sigma(a)), y(\sigma^2(a)), \dots, y(\sigma^{\lceil\alpha\rceil}(a)))$ and $(y_{\alpha-1}, y_{\alpha-2}, \dots, y_{\alpha-\lceil\alpha\rceil})$.*

Proof. The case $\alpha \in \mathbb{Z}^+$ is well-known from the literature. Let $\alpha \notin \mathbb{Z}^+$. We wish to show that the values of $y(\sigma(a))$, $y(\sigma^2(a)), \dots, y(\sigma^{\lceil\alpha\rceil}(a))$ determine uniquely the values of

$$ {}_a\nabla^{\alpha-1}_{(q,h)} y(t)\big|_{t=\sigma^{\lceil\alpha\rceil}(a)}\,,\ {}_a\nabla^{\alpha-2}_{(q,h)} y(t)\big|_{t=\sigma^{\lceil\alpha\rceil}(a)}\,, \dots,\ {}_a\nabla^{\alpha-\lceil\alpha\rceil}_{(q,h)} y(t)\big|_{t=\sigma^{\lceil\alpha\rceil}(a)} $$

and vice versa. Utilizing the relation

$$ {}_a\nabla^{\alpha-j}_{(q,h)} y(t)\big|_{t=\sigma^{\lceil\alpha\rceil}(a)} = \sum_{k=1}^{\lceil\alpha\rceil} \nu(\sigma^{\lceil\alpha\rceil-k+1}(a))\hat{h}_{j-1-\alpha}(\sigma^{\lceil\alpha\rceil}(a), \sigma^{\lceil\alpha\rceil-k}(a)) y(\sigma^{\lceil\alpha\rceil-k+1}(a)) $$

following from (1.44), we can rewrite (2.2) as the linear mapping

$$ \sum_{k=1}^{\lceil\alpha\rceil} r_{jk} y(\sigma^{\lceil\alpha\rceil-k+1}(a)) = y_{\alpha-j}\,, \quad j = 1, \dots, \lceil\alpha\rceil\,, \tag{2.3} $$

where $r_{jk} = \nu(\sigma^{\lceil\alpha\rceil-k+1}(a))\hat{h}_{j-1-\alpha}(\sigma^{\lceil\alpha\rceil}(a), \sigma^{\lceil\alpha\rceil-k}(a))$ are elements of the transformation matrix $R_{\lceil\alpha\rceil}$. We show that $R_{\lceil\alpha\rceil}$ is regular. Obviously,

$$ \det(R_{\lceil\alpha\rceil}) = \left(\prod_{k=1}^{\lceil\alpha\rceil} \nu(\sigma^k(a))\right) \det(H_{\lceil\alpha\rceil})\,, $$

where

$$ H_{\lceil\alpha\rceil} = \begin{pmatrix} \hat{h}_{-\alpha}(\sigma^{\lceil\alpha\rceil}(a), \sigma^{\lceil\alpha\rceil-1}(a)) & \cdots & \hat{h}_{-\alpha}(\sigma^{\lceil\alpha\rceil}(a), a) \\ \vdots & \ddots & \vdots \\ \hat{h}_{\lceil\alpha\rceil-1-\alpha}(\sigma^{\lceil\alpha\rceil}(a), \sigma^{\lceil\alpha\rceil-1}(a)) & \cdots & \hat{h}_{\lceil\alpha\rceil-1-\alpha}(\sigma^{\lceil\alpha\rceil}(a), a) \end{pmatrix}. $$

To calculate $\det(H_{\lceil\alpha\rceil})$, we employ some elementary operations preserving its value. Using the properties

$$ \hat{h}_{i-\alpha}(\sigma^{\lceil\alpha\rceil}(a), \sigma^{\ell}(a)) - \nu(\sigma^{\lceil\alpha\rceil}(a))\hat{h}_{i-\alpha-1}(\sigma^{\lceil\alpha\rceil}(a), \sigma^{\ell}(a)) = \hat{h}_{i-\alpha}(\sigma^{\lceil\alpha\rceil-1}(a), \sigma^{\ell}(a)) $$

$(i = 1, 2, \dots, \lceil\alpha\rceil - 1,\ \ell = 0, 1, \dots, \lceil\alpha\rceil - 2)$ and

$$ \hat{h}_{i-\alpha}(\sigma^{\lceil\alpha\rceil}(a), \sigma^{\lceil\alpha\rceil-1}(a)) - \nu(\sigma^{\lceil\alpha\rceil}(a))\hat{h}_{i-\alpha-1}(\sigma^{\lceil\alpha\rceil}(a), \sigma^{\lceil\alpha\rceil-1}(a)) = 0\,, $$

which follow from Lemma 1.28, we multiply the i-th row ($i = 1, 2, \dots, \lceil\alpha\rceil - 1$) of $H_{\lceil\alpha\rceil}$ by $-\nu(\sigma^{\lceil\alpha\rceil}(a))$ and add it to the successive one. We arrive at the form

$$ \left(\begin{array}{c|ccc} \hat{h}_{-\alpha}(\sigma^{\lceil\alpha\rceil}(a), \sigma^{\lceil\alpha\rceil-1}(a)) & \hat{h}_{-\alpha}(\sigma^{\lceil\alpha\rceil}(a), \sigma^{\lceil\alpha\rceil-2}(a)) & \cdots & \hat{h}_{-\alpha}(\sigma^{\lceil\alpha\rceil}(a), a) \\ \hline 0 & & & \\ \vdots & & H_{\lceil\alpha\rceil-1} & \\ 0 & & & \end{array}\right). $$

Then we apply repeatedly this procedure to obtain the triangular matrix

$$\begin{pmatrix} \hat{h}_{-\alpha}(\sigma^{\lceil\alpha\rceil}(a), \sigma^{\lceil\alpha\rceil-1}(a)) & \hat{h}_{-\alpha}(\sigma^{\lceil\alpha\rceil}(a), \sigma^{\lceil\alpha\rceil-2}(a)) & \cdots & \hat{h}_{-\alpha}(\sigma^{\lceil\alpha\rceil}(a), a) \\ 0 & \hat{h}_{1-\alpha}(\sigma^{\lceil\alpha\rceil-1}(a), \sigma^{\lceil\alpha\rceil-2}(a)) & \cdots & \hat{h}_{1-\alpha}(\sigma^{\lceil\alpha\rceil-1}(a), a) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{h}_{\lceil\alpha\rceil-1-\alpha}(\sigma(a), a) \end{pmatrix}.$$

Since $\hat{h}_{i-\alpha}(\sigma^k(a), \sigma^{k-1}(a)) = \nu^{i-\alpha}(\sigma^k(a))$ $(i = 0, 1, \ldots, \lceil\alpha\rceil - 1)$, we get

$$\det(H_{\lceil\alpha\rceil}) = \prod_{k=1}^{\lceil\alpha\rceil} \nu^{\lceil\alpha\rceil-k-\alpha}(\sigma^k(a)), \quad \text{i.e.} \quad \det(R_{\lceil\alpha\rceil}) = \prod_{k=1}^{\lceil\alpha\rceil} \nu^{\lceil\alpha\rceil-k-\alpha+1}(\sigma^k(a)) \neq 0 .$$

Thus the matrix $R_{\lceil\alpha\rceil}$ is regular, hence the corresponding mapping (2.3) is one-to-one. □

2.2 Existence, uniqueness and structure of the solutions

As demonstrated in Subsection 1.2.2, the key notion connected to the problem of existence and uniqueness of solutions of dynamic equations on time scales is ν-regressivity, in particular ν-regressivity of a matrix related to the solved problem (see Definitions 1.15 and 1.16). We are going to follow this pattern and generalize this notion for the linear FdE (2.1).

Definition 2.2. Let $\alpha \in \mathbb{R}^+$. Then the equation (2.1) is called ν-regressive provided the matrix

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -\frac{p_0(t)}{\nu^{\lceil\alpha\rceil-\alpha}(t)} & -p_1(t) & \cdots & -p_{\lceil\alpha\rceil-2}(t) & -p_{\lceil\alpha\rceil-1}(t) \end{pmatrix} \tag{2.4}$$

is ν-regressive.

Remark 2.3. The explicit expression of the ν-regressivity for (2.1) can be read as

$$1 + \sum_{j=1}^{\lceil\alpha\rceil-1} p_{\lceil\alpha\rceil-j}(t)\nu^j(t) + p_0(t)\nu^\alpha(t) \neq 0 \quad \text{for all } t \in \left(\widetilde{\mathbb{T}}^a_{(q,h)}\right)_{\kappa^{\lceil\alpha\rceil+1}} .$$

If $\alpha \in \mathbb{Z}^+$, then both these introductions agree with the definition of ν-regressivity of a higher order linear dynamic equation presented in Subsection 1.2.2.

Theorem 2.4. *Let (2.1) be ν-regressive. Then the problem (2.1), (2.2) has a unique solution defined for all $t \in \left(\widetilde{\mathbb{T}}^a_{(q,h)}\right)_\kappa$.*

Proof. The conditions (2.2) enable us to determine the values of $y(\sigma(a)),\dots,y(\sigma^{\lceil\alpha\rceil}(a))$ by use of (2.3). To calculate the values of $y(\sigma^{\lceil\alpha\rceil+1}(a))$, $y(\sigma^{\lceil\alpha\rceil+2}(a)),\dots$, we perform the transformation

$$w_j(t) = {}_a\nabla^{\alpha-\lceil\alpha\rceil+j-1}_{(q,h)}y(t)\,,\quad t\in\left(\widetilde{\mathbb{T}}^a_{(q,h)}\right)_{\kappa^j},\ j=1,2,\dots,\lceil\alpha\rceil$$

which allows us to rewrite (2.1) into a matrix form. Before doing this, we need to express $y(t)$ in terms of $w_1(t), w_1(\rho(t)),\dots,w_1(\sigma(a))$. Applying ${}_a\nabla^{\lceil\alpha\rceil-\alpha}_{(q,h)}\,{}_a\nabla^{-(\lceil\alpha\rceil-\alpha)}_{(q,h)}y(t) = y(t)$ (special case of (1.46)) and expanding the fractional difference by (1.44), we arrive at

$$y(t) = {}_a\nabla^{\lceil\alpha\rceil-\alpha}_{(q,h)}w_1(t) = \frac{w_1(t)}{\nu^{\lceil\alpha\rceil-\alpha}(t)} + \sum_{k=1}^{n-1}\nu(\sigma^k(a))\hat{h}_{\alpha-\lceil\alpha\rceil-1}(\sigma^n(a),\sigma^{k-1}(a))w_1(\sigma^k(a))\,, \quad (2.5)$$

where $t=\sigma^n(a)$. Therefore the problem (2.1), (2.2) can be rewritten to the vector form

$$\begin{aligned}{}_a\nabla_{(q,h)}w(t) &= A(t)w(t)+b(t)\,,\quad t\in\left(\widetilde{\mathbb{T}}^a_{(q,h)}\right)_{\kappa^{\lceil\alpha\rceil+1}}\,,\\ w(\sigma^{\lceil\alpha\rceil}(a)) &= (y_{\alpha-\lceil\alpha\rceil},\dots,y_{\alpha-1})^T\,,\end{aligned}$$

where

$$\begin{aligned}w(t) &= (w_1(t),\dots,w_{\lceil\alpha\rceil}(t))^T\,,\\ b(t) &= \Big(0,\dots,0,\ -p_0(t)\sum_{k=1}^{n-1}\nu(\sigma^k(a))\hat{h}_{\alpha-\lceil\alpha\rceil-1}(\sigma^n(a),\sigma^{k-1}(a))w_1(\sigma^k(a))\Big)^T\end{aligned}$$

and $A(t)$ is given by (2.4). The ν-regressivity of the matrix $A(t)$ enables us to write

$$w(t) = (I-\nu(t)A(t))^{-1}(w(\rho(t))+\nu(t)b(t))\,,\quad t\in\left(\widetilde{\mathbb{T}}^a_{(q,h)}\right)_{\kappa^{\lceil\alpha\rceil+1}}\,,$$

hence, using the value of $w(\sigma^{\lceil\alpha\rceil}(a))$, we can solve this system by the step method starting from $t=\sigma^{\lceil\alpha\rceil+1}(a)$. The solution $y(t)$ of the original initial value problem (2.1), (2.2) is then given by the formula (2.5). □

Remark 2.5. The previous assertion on the existence and uniqueness of the solution can be easily extended to the initial value problem involving non-homogeneous linear equations as well as some non-linear equations.

The final goal of this section is to investigate the structure of the solutions of (2.1). We start with the following notion generalizing the Wronskian (see Definition 1.18).

Definition 2.6. Let $m\in\mathbb{Z}^+$ and $\gamma\in[0,1)$. For m functions $y_j:\left(\widetilde{\mathbb{T}}^a_{(q,h)}\right)_\kappa\to\mathbb{R}$ ($j=1,2,\dots,m$) we define the γ-Wronskian $W_\gamma(y_1,\dots,y_m)(t)$ for all $t\in\left(\widetilde{\mathbb{T}}^a_{(q,h)}\right)_{\kappa^m}$ as determinant of the matrix

$$V_\gamma(y_1,\dots,y_m)(t) = \begin{pmatrix} {}_a\nabla^{-\gamma}_{(q,h)}y_1(t) & {}_a\nabla^{-\gamma}_{(q,h)}y_2(t) & \cdots & {}_a\nabla^{-\gamma}_{(q,h)}y_m(t)\\ {}_a\nabla^{1-\gamma}_{(q,h)}y_1(t) & {}_a\nabla^{1-\gamma}_{(q,h)}y_2(t) & \cdots & {}_a\nabla^{1-\gamma}_{(q,h)}y_m(t)\\ \vdots & \vdots & \ddots & \vdots\\ {}_a\nabla^{m-1-\gamma}_{(q,h)}y_1(t) & {}_a\nabla^{m-1-\gamma}_{(q,h)}y_2(t) & \cdots & {}_a\nabla^{m-1-\gamma}_{(q,h)}y_m(t)\end{pmatrix}.$$

Remark 2.7. Note that $W_\gamma(y_1,\dots,y_m)(t)$ coincides for $\gamma=0$ with the classical Definition 1.18. Moreover, it holds $W_\gamma(y_1,\dots,y_m)(t)=W_0({}_a\nabla^{-\gamma}_{(q,h)}y_1,\dots,{}_a\nabla^{-\gamma}_{(q,h)}y_m)(t)$.

Theorem 2.8. *Let functions $y_1(t),\dots,y_{\lceil\alpha\rceil}(t)$ be solutions of the ν-regressive equation (2.1) and let $W_{\lceil\alpha\rceil-\alpha}(y_1,\dots,y_{\lceil\alpha\rceil})(\sigma^{\lceil\alpha\rceil}(a))\neq 0$. Then any solution $y(t)$ of (2.1) can be written in the form*

$$y(t)=\sum_{k=1}^{\lceil\alpha\rceil}c_k y_k(t)\,,\quad t\in\left(\widetilde{\mathbb{T}}^a_{(q,h)}\right)_\kappa, \tag{2.6}$$

where $c_1,\dots,c_{\lceil\alpha\rceil}$ are real constants.

Proof. Let $y(t)$ be a solution of (2.1). By Proposition 2.1, there exist real scalars $y_{\alpha-1},\dots,y_{\alpha-\lceil\alpha\rceil}$ such that $y(t)$ is satisfying (2.2). Now we consider the function

$$u(t)=\sum_{k=1}^{\lceil\alpha\rceil}c_k y_k(t)\,,$$

where the $\lceil\alpha\rceil$-tuple $(c_1,\dots,c_{\lceil\alpha\rceil})$ is the unique solution of

$$V_{\lceil\alpha\rceil-\alpha}(y_1,\dots,y_{\lceil\alpha\rceil})(\sigma^{\lceil\alpha\rceil}(a))\cdot\begin{pmatrix}c_1\\ c_2\\ \vdots\\ c_{\lceil\alpha\rceil}\end{pmatrix}=\begin{pmatrix}y_{\alpha-\lceil\alpha\rceil}\\ y_{\alpha-\lceil\alpha\rceil+1}\\ \vdots\\ y_{\alpha-1}\end{pmatrix}.$$

The linearity of (2.1) implies that $u(t)$ has to be its solution. Moreover, it holds

$${}_a\nabla^{\alpha-j}_{(q,h)}u(t)\big|_{t=\sigma^{\lceil\alpha\rceil}(a)}=y_{\alpha-j}\,,\quad j=1,2,\dots,\lceil\alpha\rceil\,,$$

hence $u(t)$ is a solution of the initial value problem (2.1), (2.2). By Theorem 2.4, it must be $y(t)=u(t)$ for all $t\in\left(\widetilde{\mathbb{T}}^a_{(q,h)}\right)_\kappa$ and (2.6) holds. □

Remark 2.9. The formula (2.6) is essentially an expression of the general solution of (2.1).

2.3 Eigenfunctions of the Riemann-Liouville difference operator

Our main interest in this section is to find eigenfunctions of the fractional operator ${}_a\nabla^\alpha_{(q,h)}$, $\alpha\in\mathbb{R}^+$. In other words, we wish to solve the equation (2.1) in a special form

$${}_a\nabla^\alpha_{(q,h)}y(t)=\lambda y(t)\,,\quad \lambda\in\mathbb{R},\ t\in\left(\widetilde{\mathbb{T}}^a_{(q,h)}\right)_{\kappa^{\lceil\alpha\rceil+1}}. \tag{2.7}$$

Throughout this section we assume that the ν-regressivity condition is ensured, i.e.

$$\lambda\nu^\alpha(t)\neq 1\,.$$

Discussions on methods for solving of FdEs are just at the beginning. Some techniques how to explicitly solve these equations (at least in particular cases) are exhibited, e.g. in [7, 8, 38], where a discrete analogue of the Laplace transform turns out to be the most developed method.

In this section, we describe the technique not utilizing the transform method, but directly originating from the role which is played by the Mittag-Leffler function (1.8) in the continuous fractional calculus (see, e.g. [43]). More precisely, we introduce a (q,h)-analogue of the modified Mittag-Leffler function (1.11), which satisfies under special choices of parameters a continuous analogy of the equation (2.7). Also, the form (1.11) seems to be much more convenient for discrete extensions than (1.8), which requires, among others, the validity of the law of exponents. These our results generalize and extend those derived in [40] and [14].

Definition 2.10. Let $\eta, \beta, \lambda \in \mathbb{R}$. We introduce the (q,h)-Mittag-Leffler function $E^{s,\lambda}_{\eta,\beta}(t)$ by the series expansion

$$E^{s,\lambda}_{\eta,\beta}(t) = \sum_{k=0}^{\infty} \lambda^k \hat{h}_{\eta k+\beta-1}(t,s)\,, \quad s,t \in \widetilde{\mathbb{T}}^a_{(q,h)},\ t \ge s\,.$$

It is easy to check that the series on the right-hand side converges (absolutely) if $|\lambda|\nu^\eta(t) < 1$. As it might be expected, the particular (q,h)-Mittag-Leffler function

$$E^{a,\lambda}_{1,1}(t) = \prod_{k=1}^{n} \frac{1}{1-\lambda\nu(\rho^{k-1}(t))}\,,$$

where $n \in \mathbb{Z}^+$ satisfies $t = \sigma^n(a)$, is the solution of the equation

$${}_a\nabla_{(q,h)} y(t) = \lambda y(t), \quad t \in \big(\widetilde{\mathbb{T}}^a_{(q,h)}\big)_\kappa\,,$$

i.e. it coincides with the exponential function $\hat{e}_\lambda(t,a)$ from Theorem 1.22 (ii).

The main properties of the (q,h)-Mittag-Leffler function are described by the following assertion.

Theorem 2.11. (i) *Let $\gamma \in \mathbb{R}^+$ and $t \in \big(\widetilde{\mathbb{T}}^a_{(q,h)}\big)_\kappa$. Then*

$${}_a\nabla^{-\gamma}_{(q,h)} E^{a,\lambda}_{\eta,\beta}(t) = E^{a,\lambda}_{\eta,\beta+\gamma}(t)\,.$$

(ii) *Let $\alpha \in \mathbb{R}^+$ and $\eta k + \beta - \alpha \notin \{0,-1,\ldots,-\lceil\alpha\rceil+1\}$ for all $k \in \mathbb{Z}^+$. If $t \in \big(\widetilde{\mathbb{T}}^a_{(q,h)}\big)_{\kappa^{\lceil\alpha\rceil+1}}$ then*

$${}_a\nabla^{\alpha}_{(q,h)} E^{a,\lambda}_{\eta,\beta}(t) = \begin{cases} E^{a,\lambda}_{\eta,\beta-\alpha}(t)\,, & \beta-\alpha \notin \{0,-1,\ldots,-\lceil\alpha\rceil+1\}\,, \\ \lambda E^{a,\lambda}_{\eta,\beta-\alpha+\eta}(t)\,, & \beta-\alpha \in \{0,-1,\ldots,-\lceil\alpha\rceil+1\}\,. \end{cases} \tag{2.8}$$

Proof. The part (i) follows immediately from Lemma 1.48. Considering the part (ii) we can write

$$
{}_a\nabla^{\alpha}_{(q,h)} E^{a,\lambda}_{\eta,\beta}(t) = {}_a\nabla^{\alpha}_{(q,h)} \sum_{k=0}^{\infty} \lambda^k \hat{h}_{\eta k+\beta-1}(t,a) = \sum_{k=0}^{\infty} \lambda^k {}_a\nabla^{\alpha}_{(q,h)} \hat{h}_{\eta k+\beta-1}(t,a)
$$

due to the absolute convergence property.

If $k \in \mathbb{Z}^+$ then Corollary 1.49 implies the relation

$$
{}_a\nabla^{\alpha}_{(q,h)} \hat{h}_{\eta k+\beta-1}(t,a) = \hat{h}_{\eta k+\beta-\alpha-1}(t,a) \tag{2.9}
$$

due to the assumption $\eta k + \beta - \alpha \notin \{0, -1, \ldots, -\lceil\alpha\rceil + 1\}$. If $k = 0$ then two possibilities may occur. If $\beta - \alpha \notin \{0, -1, \ldots, -\lceil\alpha\rceil + 1\}$ we get (2.9) with $k = 0$ which implies the validity of $(2.8)_1$. If $\beta - \alpha \in \{0, -1, \ldots, -\lceil\alpha\rceil + 1\}$, the fractional difference of this term is zero and by shifting the index k we obtain $(2.8)_2$. □

Remark 2.12. The assumption $\eta k + \beta - \alpha \notin \{0, -1, \ldots, -\lceil\alpha\rceil + 1\}$ for all $k \in \mathbb{Z}^+$ in Theorem 2.11 (ii) may seem to be quite restrictive. Note that it is satisfied trivially for $\beta \in \mathbb{R}^+$ and $\eta + \beta > \alpha$ and, as shown in the following assertion, this is the case we are interested in.

Corollary 2.13. *Let $\alpha \in \mathbb{R}^+$. Then the functions*

$$
E^{a,\lambda}_{\alpha,\beta}(t), \quad \beta = \alpha - \lceil\alpha\rceil + 1, \ldots, \alpha - 1, \alpha \tag{2.10}
$$

define eigenfunctions of the Riemann-Liouville fractional difference operator ${}_a\nabla^{\alpha}_{(q,h)}$ on each set $[\sigma(a), b] \cap \left(\widetilde{\mathbb{T}}^a_{(q,h)}\right)_{\kappa}$, where $b \in \left(\widetilde{\mathbb{T}}^a_{(q,h)}\right)_{\kappa^{\lceil\alpha\rceil+1}}$ is satisfying $|\lambda|\nu^{\alpha}(b) < 1$.

Proof. The assertion follows from Theorem 2.11 (ii) by use of $\eta = \alpha$. □

Our final aim is to show that any solution of the equation (2.7) can be written as a linear combination of (q,h)-Mittag-Leffler functions (2.10).

Lemma 2.14. *Let $\alpha \in \mathbb{R}^+$ and $\lambda \in \mathbb{R}$ be such that $|\lambda|\nu^{\alpha}(\sigma^{\lceil\alpha\rceil}(a)) < 1$. Then*

$$
W_{\lceil\alpha\rceil-\alpha}(E^{a,\lambda}_{\alpha,\alpha-\lceil\alpha\rceil+1}, E^{a,\lambda}_{\alpha,\alpha-\lceil\alpha\rceil+2}, \ldots, E^{a,\lambda}_{\alpha,\alpha})(\sigma^{\lceil\alpha\rceil}(a)) = \prod_{k=1}^{\lceil\alpha\rceil} \frac{1}{1 - \lambda\nu^{\alpha}(\sigma^k(a))} \neq 0.
$$

Proof. The case $\lceil\alpha\rceil = 1$ is trivial. For $\lceil\alpha\rceil \geq 2$, we can formally write $\lambda E^{a,\lambda}_{\alpha,\alpha-\ell}(t) = E^{a,\lambda}_{\alpha,-\ell}(t)$ for all $t \in \left(\widetilde{\mathbb{T}}^a_{(q,h)}\right)_{\kappa^{\lceil\alpha\rceil}}$ $(\ell = 0, \ldots, \lceil\alpha\rceil - 2)$. Consequently, applying Theorem 2.11, the Wronskian can be expressed as

$$
W_{\lceil\alpha\rceil-\alpha}(E^{a,\lambda}_{\alpha,\alpha-\lceil\alpha\rceil+1}, E^{a,\lambda}_{\alpha,\alpha-\lceil\alpha\rceil+2}, \ldots, E^{a,\lambda}_{\alpha,\alpha})(\sigma^{\lceil\alpha\rceil}(a)) = \det(M_{\lceil\alpha\rceil}(\sigma^{\lceil\alpha\rceil}(a)))\,,
$$

where

$$M_{\lceil\alpha\rceil}(\sigma^{\lceil\alpha\rceil}(a)) = \begin{pmatrix} E^{a,\lambda}_{\alpha,1}(\sigma^{\lceil\alpha\rceil}(a)) & E^{a,\lambda}_{\alpha,2}(\sigma^{\lceil\alpha\rceil}(a)) & \dots & E^{a,\lambda}_{\alpha,\lceil\alpha\rceil}(\sigma^{\lceil\alpha\rceil}(a)) \\ E^{a,\lambda}_{\alpha,0}(\sigma^{\lceil\alpha\rceil}(a)) & E^{a,\lambda}_{\alpha,1}(\sigma^{\lceil\alpha\rceil}(a)) & \dots & E^{a,\lambda}_{\alpha,\lceil\alpha\rceil-1}(\sigma^{\lceil\alpha\rceil}(a)) \\ \dots & \dots & \ddots & \dots \\ E^{a,\lambda}_{\alpha,2-\lceil\alpha\rceil}(\sigma^{\lceil\alpha\rceil}(a)) & E^{a,\lambda}_{\alpha,3-\lceil\alpha\rceil}(\sigma^{\lceil\alpha\rceil}(a)) & \dots & E^{a,\lambda}_{\alpha,1}(\sigma^{\lceil\alpha\rceil}(a)) \end{pmatrix}.$$

Using the q-Pascal rule (1.24) we obtain the equality

$$E^{a,\lambda}_{\alpha,i}(\sigma^{\lceil\alpha\rceil}(a)) - \nu(\sigma(a))E^{a,\lambda}_{\alpha,i-1}(\sigma^{\lceil\alpha\rceil}(a)) = E^{\sigma(a),\lambda}_{\alpha,i}(\sigma^{\lceil\alpha\rceil}(a))\,, \quad i\in\mathbb{Z},\ i\geq 3-\lceil\alpha\rceil\,. \tag{2.11}$$

Starting with the first row, $\binom{\lceil\alpha\rceil}{2}$ elementary row operations of the type (2.11) transform the matrix $M_{\lceil\alpha\rceil}(\sigma^{\lceil\alpha\rceil}(a))$ into the matrix $\widehat{M}_{\lceil\alpha\rceil}(\sigma^{\lceil\alpha\rceil}(a))$ given by

$$\begin{pmatrix} E^{\sigma^{\lceil\alpha\rceil-1}(a),\lambda}_{\alpha,1}(\sigma^{\lceil\alpha\rceil}(a)) & E^{\sigma^{\lceil\alpha\rceil-1}(a),\lambda}_{\alpha,2}(\sigma^{\lceil\alpha\rceil}(a)) & \dots & E^{\sigma^{\lceil\alpha\rceil-1}(a),\lambda}_{\alpha,\lceil\alpha\rceil}(\sigma^{\lceil\alpha\rceil}(a)) \\ E^{\sigma^{\lceil\alpha\rceil-2}(a),\lambda}_{\alpha,0}(\sigma^{\lceil\alpha\rceil}(a)) & E^{\sigma^{\lceil\alpha\rceil-2}(a),\lambda}_{\alpha,1}(\sigma^{\lceil\alpha\rceil}(a)) & \dots & E^{\sigma^{\lceil\alpha\rceil-2}(a),\lambda}_{\alpha,\lceil\alpha\rceil-1}(\sigma^{\lceil\alpha\rceil}(a)) \\ \dots & \dots & \ddots & \dots \\ E^{a,\lambda}_{\alpha,2-\lceil\alpha\rceil}(\sigma^{\lceil\alpha\rceil}(a)) & E^{a,\lambda}_{\alpha,3-\lceil\alpha\rceil}(\sigma^{\lceil\alpha\rceil}(a)) & \dots & E^{a,\lambda}_{\alpha,1}(\sigma^{\lceil\alpha\rceil}(a)) \end{pmatrix}$$

with the property $\det(\widehat{M}_{\lceil\alpha\rceil}(\sigma^{\lceil\alpha\rceil}(a))) = \det(M_{\lceil\alpha\rceil}(\sigma^{\lceil\alpha\rceil}(a)))$. By Lemma 1.28, we have

$$\begin{aligned} &E^{\sigma^i(a),\lambda}_{\alpha,p}(\sigma^{\lceil\alpha\rceil}(a)) - \nu(\sigma^{\lceil\alpha\rceil}(a))E^{\sigma^i(a),\lambda}_{\alpha,p-1}(\sigma^{\lceil\alpha\rceil}(a)) = E^{\sigma^i(a),\lambda}_{\alpha,p}(\sigma^{\lceil\alpha\rceil-1}(a))\,, \quad i\leq\lceil\alpha\rceil-2\,, \\ &E^{\sigma^i(a),\lambda}_{\alpha,p}(\sigma^{\lceil\alpha\rceil}(a)) - \nu(\sigma^{\lceil\alpha\rceil}(a))E^{\sigma^i(a),\lambda}_{\alpha,p-1}(\sigma^{\lceil\alpha\rceil}(a)) = 0\,, \quad i=\lceil\alpha\rceil-1\,, \end{aligned} \tag{2.12}$$

where $p\in\mathbb{Z},\ p\geq 3-\lceil\alpha\rceil+i$. Starting with the last column, using $\lceil\alpha\rceil-1$ elementary column operations of the type (2.12) we obtain the matrix

$$\left(\begin{array}{c|ccc} E^{\sigma^{\lceil\alpha\rceil-1}(a),\lambda}_{\alpha,1}(\sigma^{\lceil\alpha\rceil}(a)) & 0 & \dots & 0 \\ \hline E^{\sigma^{\lceil\alpha\rceil-2}(a),\lambda}_{\alpha,0}(\sigma^{\lceil\alpha\rceil}(a)) & & & \\ \vdots & & \widehat{M}_{\lceil\alpha\rceil-1}(\sigma^{\lceil\alpha\rceil-1}(a)) & \\ E^{a,\lambda}_{\alpha,2-\lceil\alpha\rceil}(\sigma^{\lceil\alpha\rceil}(a)) & & & \end{array}\right)$$

preserving the value of $\det(\widehat{M}_{\lceil\alpha\rceil}(\sigma^{\lceil\alpha\rceil}(a)))$. Since

$$E^{\sigma^{\lceil\alpha\rceil-1}(a),\lambda}_{\alpha,1}(\sigma^{\lceil\alpha\rceil}(a)) = \sum_{k=0}^{\infty}\lambda^k(\nu(\sigma^{\lceil\alpha\rceil}(a))^{\alpha k} = \frac{1}{1-\lambda\nu^{\alpha}(\sigma^{\lceil\alpha\rceil}(a))}\,,$$

we can observe the recurrence

$$\det(\widehat{M}_{\lceil\alpha\rceil}(\sigma^{\lceil\alpha\rceil}(a))) = \frac{1}{1-\lambda\nu^{\alpha}(\sigma^{\lceil\alpha\rceil}(a))}\det(\widehat{M}_{\lceil\alpha\rceil-1}(\sigma^{\lceil\alpha\rceil-1}(a)))\,,$$

which implies the assertion. □

Now we summarize the results of Theorem 2.8, Corollary 2.13 and Lemma 2.14 to obtain the discrete analogue of (1.12).

Theorem 2.15. *Let $y(t)$ be any solution of the equation (2.7) defined on $[\sigma(a), b] \cap \left(\widetilde{\mathbb{T}}^a_{(q,h)}\right)_\kappa$, where $b \in \left(\widetilde{\mathbb{T}}^a_{(q,h)}\right)_{\kappa^{\lceil\alpha\rceil+1}}$ is satisfying $|\lambda|\nu^\alpha(b) < 1$. Then*

$$y(t) = \sum_{j=1}^{\lceil\alpha\rceil} c_j E^{a,\lambda}_{\alpha,\alpha-\lceil\alpha\rceil+j}(t)\,,$$

where $c_1, \ldots, c_{\lceil\alpha\rceil}$ are real constants.

We conclude this chapter by an illustrating example.

Example 2.16. Consider the initial value problem

$$\begin{aligned}
{}_a\nabla^\alpha_{(q,h)} y(t) &= \lambda\, y(t)\,, \quad \sigma^3(a) \le t \le \sigma^n(a)\,, \quad 1 < \alpha \le 2\,,\\
{}_a\nabla^{\alpha-1}_{(q,h)} y(t)\big|_{t=\sigma^2(a)} &= y_{\alpha-1}\,,\\
{}_a\nabla^{\alpha-2}_{(q,h)} y(t)\big|_{t=\sigma^2(a)} &= y_{\alpha-2}\,,
\end{aligned}$$

where n is a positive integer given by the condition $|\lambda|\nu^\alpha(\sigma^n(a)) < 1$. By Theorem 2.15, its solution can be expressed as a linear combination

$$y(t) = c_1\, E^{a,\lambda}_{\alpha,\alpha-1}(t) + c_2\, E^{a,\lambda}_{\alpha,\alpha}(t)\,.$$

The constants c_1, c_2 can be determined from the system

$$V_{2-\alpha}(E^{a,\lambda}_{\alpha,\alpha-1}, E^{a,\lambda}_{\alpha,\alpha})(\sigma^2(a)) \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_{\alpha-2} \\ y_{\alpha-1} \end{pmatrix}$$

with the matrix elements

$$\begin{aligned}
v_{11} = v_{22} &= \frac{[1]_q + ([\alpha]_q - [1]_q)\lambda\,\nu^\alpha(\sigma(a))}{(1-\lambda\,\nu^\alpha(\sigma(a)))(1-\lambda\,\nu^\alpha(\sigma^2(a)))}\,,\\
v_{12} &= \frac{[2]_q\nu(\sigma(a)) + ([\alpha]_q - [2]_q)\lambda\,\nu^{\alpha+1}(\sigma(a))}{(1-\lambda\,\nu^\alpha(\sigma(a)))(1-\lambda\,\nu^\alpha(\sigma^2(a)))}\,,\\
v_{21} &= \frac{[\alpha]_q\lambda\,\nu^{\alpha-1}(\sigma(a))}{(1-\lambda\,\nu^\alpha(\sigma(a)))(1-\lambda\,\nu^\alpha(\sigma^2(a)))}\,.
\end{aligned}$$

By Lemma 2.14, the matrix $V_{2-\alpha}(E^{a,\lambda}_{\alpha,\alpha-1}, E^{a,\lambda}_{\alpha,\alpha})(\sigma^2(a))$ has a nonzero determinant, hence applying the Cramer rule we get

$$\begin{aligned}
c_1 &= \frac{y_{\alpha-2}v_{22} - y_{\alpha-1}v_{12}}{W_{2-\alpha}(E^{a,\lambda}_{\alpha,\alpha-1}, E^{a,\lambda}_{\alpha,\alpha})(\sigma^2(a))}\,,\\
c_2 &= \frac{y_{\alpha-1}v_{11} - y_{\alpha-2}v_{21}}{W_{2-\alpha}(E^{a,\lambda}_{\alpha,\alpha-1}, E^{a,\lambda}_{\alpha,\alpha})(\sigma^2(a))}\,.
\end{aligned}$$

Now we make a particular choice of the parameters α, a, λ, $y_{\alpha-1}$ and $y_{\alpha-2}$ and consider the initial value problem in the form

$$\begin{aligned}
{}_1\nabla^{1.8}_{(q,h)} y(t) &= -\frac{1}{3}\, y(t)\,, \quad \sigma^3(1) \le t \le \sigma^n(1)\,,\\
{}_1\nabla^{0.8}_{(q,h)} y(t)\big|_{t=\sigma^2(1)} &= -1\,,\\
{}_1\nabla^{-0.2}_{(q,h)} y(t)\big|_{t=\sigma^2(1)} &= 1\,,
\end{aligned}$$

where n is a positive integer satisfying $\nu(\sigma^n(1)) < 3^{5/9}$. If we take the time scale of integers (the case $q = h = 1$), then the solution $y(t)$ of the corresponding initial value problem takes the form

$$y(t) = \frac{14}{5}\sum_{k=0}^{\infty}\left(-\frac{1}{3}\right)^k \frac{\prod_{j=1}^{t-2}(j+1.8k-0.2)}{(t-2)!} - \frac{2}{15}\sum_{k=0}^{\infty}\left(-\frac{1}{3}\right)^k \frac{\prod_{j=1}^{t-2}(j+1.8k+0.8)}{(t-2)!}$$

for $t = 2, 3, \ldots$. Similarly we can determine $y(t)$ for other choices of q and h. For comparative reasons, Figure 1 depicts (in addition to the above case $q = h = 1$) the solution $y(t)$ under particular choices $q = 1.2$, $h = 0$ (the pure q-calculus), $q = 1$, $h = 0.1$ (the pure h-calculus) and also the solution of the corresponding continuous (differential) initial value problem.

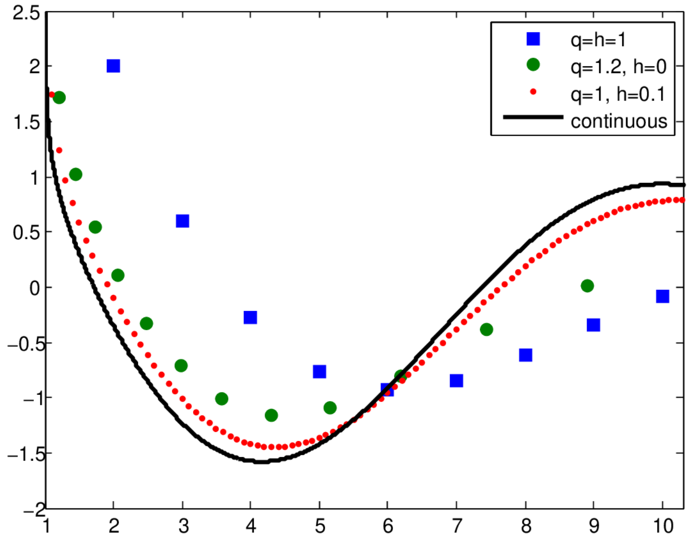

Figure 1: $\alpha = 1.8,\ a = 1,\ \lambda = -\frac{1}{3},\ y_{\alpha-1} = -1,\ y_{\alpha-2} = 1$

3 Qualitative analysis of a scalar linear FdE on $\mathbb{T} = \mathbb{Z}$

In this chapter we utilize the time scale of integers, i.e. $\mathbb{T} = \mathbb{Z}$. We note that a possible extension of the following results to $\mathbb{T} = h\mathbb{Z}$ with arbitrary $h > 0$ is only a technical matter. We prefer the standard difference case due to close relations of studied problems to some parts of the qualitative theory of difference equations.

We investigate here stability and asymptotic properties of the linear FdE

$$_0\nabla^{\alpha}_{h=1} y(t) = \lambda y(t)\,, \quad t = 2, 3, \dots\,, \tag{3.1}$$

where $0 < \alpha < 1$, $\lambda \neq 1$ are real scalars.

By Theorem 2.15 we can write the solution of (3.1) via discrete Mittag-Leffler functions as $y(t) = c\, E^{0;\lambda}_{\alpha,\alpha}(t)$ where c is a real constant. However, the asymptotic behaviour of $E^{0;\lambda}_{\alpha,\alpha}(t)$ has not been described yet, even though the role of this function in discrete fractional calculus was discovered in several papers (see, e.g. [9, 40] and [14, 15]). Moreover, the validity of this solution representation is restricted only for (3.1) with $|\lambda| < 1$. On this account, this result does not seem to be suitable for the qualitative analysis of (3.1).

Hence, we choose a different approach. We consider (3.1) in the form of a Volterra equation of convolution type (see Section 2.1). This enables us to analyze its properties, in particular stability (Theorem 3.10) and asymptotics (Corollary 3.17), by use of tools standardly employed in the qualitative investigation of Volterra difference equations. At the end of this chapter we show that the Volterra equation originating from fractional calculus provides an interesting perspective on some recent observations on the qualitative theory of Volterra difference equations. All the presented results come from [16].

First we introduce a Volterra form of (3.1) which will be studied in the sequel. To agree with the notation used in the theory of difference equations, we denote the independent variable by n instead of t throughout this chapter.

Proposition 3.1. *Let $0 < \alpha < 1$ and $\lambda \neq 1$. Then $y(n)$ is the solution of (3.1) if and only if $x(n) = y(n+1)$ is the solution of*

$$x(n+1) = \frac{1}{1-\lambda} \sum_{j=0}^{n} (-1)^{n-j} \binom{\alpha}{n-j+1} x(j), \quad n = 0, 1, \dots\,. \tag{3.2}$$

Proof. Rewriting the Riemann-Liouville operator in (3.1) by (1.45) for $q = h = 1$, we have

$$\sum_{k=1}^{n} (-1)^{n-k} \binom{\alpha}{n-k} y(k) = \lambda y(n)\,, \quad n = 2, 3, \dots\,.$$

Rearranging the terms in this equation we arrive at

$$y(n) = \frac{1}{1-\lambda} \sum_{k=1}^{n-1} (-1)^{n-k-1} \binom{\alpha}{n-k} y(k)\,,$$

which after replacing n by $n+2$ and setting $x(n) = y(n+1)$ turns into (3.2). $\square$

Remark 3.2. The existence and uniqueness of the solution is guaranteed since the ν-regressivity of (3.1) is ensured due to the assumption $\lambda \neq 1$. If it is not satisfied, then (3.1) admits only the identically zero solution via the starting value $y(1) = 0$. If $y(1) \neq 0$, then (3.1) has no solution.

3.1 Some preliminaries on Volterra difference equations

The equation (3.2) belongs to an important class of difference equations known as the Volterra difference equations of convolution type. Their general form is

$$x(n+1) = \sum_{j=0}^{n} a(n-j)x(j), \quad n = 0, 1, \dots, \tag{3.3}$$

where in our case we have

$$a(n) = \frac{(-1)^n}{1-\lambda}\binom{\alpha}{n+1}, \quad n = 0, 1, \dots. \tag{3.4}$$

We recall some relevant stability definitions for the equation (3.3). First we mention standard definitions of stability and asymptotic stability adapted to the linear case.

Definition 3.3. Consider (3.3) along with the initial condition $x(0) = \phi_0$. Then (3.3) is said to be

(i) *stable* if for any real ϕ_0 there exists $\varepsilon > 0$ such that the corresponding solution $x(n)$ of (3.3) satisfies $|x(n)| < \varepsilon$ for all $n \in \mathbb{Z}^+$;

(ii) *asymptotically stable* if $x(n) \to 0$ as $n \to \infty$ for any real ϕ_0.

Remark 3.4. The notion of stability and asymptotic stability for the FdE (3.1) is defined quite analogously due to Proposition 3.1.

The relevant stability results concerning (3.3) involve also the stronger notions of uniform stability and uniform asymptotic stability. Their introductions require to consider an arbitrary (finite) number of initial values. The following definitions are taken from [25].

Definition 3.5. Consider (3.3) with the initial conditions $x(k) = \phi_k$ ($k = 0, 1, \dots, m$), m being arbitrary non-negative integer. Then (3.3) is said to be

(i) *uniformly stable* if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if ϕ_k are reals with $|\phi_k| < \delta$, $k = 0, \dots, m$, then the corresponding solution $x(n)$ of (3.3) satisfies $|x(n)| < \varepsilon$ for all $n \in \mathbb{Z}^+$, $n > m$;

(ii) *uniformly asymptotically stable* if it is uniformly stable and if there exists $\eta > 0$ such that, for any $\varepsilon > 0$ there is $N = N(\varepsilon) \in \mathbb{Z}^+$ such that if ϕ_k are reals with $|\phi_k| < \eta$, $k = 0, \dots, m$, then $|x(n)| < \varepsilon$ for all $n \in \mathbb{Z}^+$, $n \geq m + N$.

A very effective method for stability analysis of (3.3) is the $\mathcal{Z}$-transform method. We recall that the $\mathcal{Z}$-transform of a sequence $u : \mathbb{Z}_0^+ \to \mathbb{R}$, is a complex function given by

$$\tilde{u}(z) = \mathcal{Z}\{u\}(z) = \sum_{k=0}^{\infty} u(k) z^{-k},$$

where z is a complex number for which this series converges absolutely. It is known that $\mathcal{Z}$-transforms can be used to solve the linear Volterra convolution equation (3.3) and find its stability or asymptotic stability conditions by analyzing the roots of the associated characteristic equation

$$z - \tilde{a}(z) = 0, \tag{3.5}$$

where $\tilde{a}(z)$ is the $\mathcal{Z}$-transform of $a(n)$ (for more details we refer to [23]). We recall here the result which is the most relevant for our next study.

Theorem 3.6 ([25, Theorem 2])**.** *Consider the equation (3.3). Then the following statements are equivalent:*

(i) *(3.3) is uniformly asymptotically stable;*

(ii) *all the roots of the characteristic equation (3.5) lie inside the unit disk, i.e.*

$$z - \tilde{a}(z) \neq 0 \quad \textit{for all } |z| \geq 1;$$

(iii) $x(n) \in \ell^1$ *for any solution* $x(n)$ *of (3.3).*

3.2 Stability analysis

We start with the formulation of an explicit necessary and sufficient condition for the uniform asymptotic stability of the Volterra equation (3.2). Then we discuss the asymptotic stability and stability of this equation, and summarize obtained results to present the asymptotic stability condition for the original FdE (3.1).

Theorem 3.7. *Let $0 < \alpha < 1$ and $\lambda \neq 1$. Then (3.2) is uniformly asymptotically stable if and only if*

$$\lambda < 0 \qquad \textit{or} \qquad \lambda > 2^{\alpha}. \tag{3.6}$$

Proof. By Theorem 3.6, we have to set up the corresponding characteristic equation (3.5) and analyze the location of its roots with respect to the unit disk.

Taking the $\mathcal{Z}$-transform of (3.4) and using the binomial theorem we get

$$\begin{aligned}\tilde{a}(z) &= \frac{1}{1-\lambda}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k+1}z^{-k} = \frac{1}{1-\lambda}\sum_{k=1}^{\infty}(-1)^{k-1}\binom{\alpha}{k}z^{-k+1}\\ &= -\frac{z}{1-\lambda}\sum_{k=1}^{\infty}\binom{\alpha}{k}\left(-\frac{1}{z}\right)^k = -\frac{z}{1-\lambda}\left[\left(1-\frac{1}{z}\right)^{\alpha}-1\right]\end{aligned}$$

for all $z \in \mathbb{C}$ with $|z| \geq 1$. Consequently, the characteristic equation (3.5) becomes

$$z + \frac{z}{1-\lambda}\left[\left(1 - \frac{1}{z}\right)^{\alpha} - 1\right] = 0 \tag{3.7}$$

and, applying Theorem 3.6, the equation (3.2) is uniformly asymptotically stable if and only if

$$1 + \frac{1}{1-\lambda}\left[\left(1 - \frac{1}{z}\right)^{\alpha} - 1\right] \neq 0 \quad \text{for all } |z| \geq 1\,. \tag{3.8}$$

The nonzero roots z_r of the characteristic equation (3.7) satisfy

$$\left(1 - \frac{1}{z_r}\right)^{\alpha} = \lambda\,. \tag{3.9}$$

We analyze (3.9) with respect to λ. First, let $\lambda < 0$. Then, obviously, (3.9) has no root z_r, hence the condition (3.8) is satisfied trivially. Further, let $\lambda \geq 0$ (we recall that $\lambda \neq 1$). Then

$$z_r = \frac{1}{1 - \lambda^{1/\alpha}}$$

is the unique nonzero (real) root of the characteristic equation (3.7). To satisfy (3.8), we have to require

$$\lambda^{1/\alpha} > 2, \quad \text{i.e.} \quad \lambda > 2^{\alpha}\,.$$

The assertion is proved. □

Lemma 3.8. *Let $0 < \alpha < 1$ and $\lambda = 0$. Then (3.2) is asymptotically stable.*

Proof. Let $\tilde{x}(z)$ be the $\mathcal{Z}$-transform of a solution $x(n)$ of (3.2), i.e.

$$\tilde{x}(z) = \sum_{k=0}^{\infty} x(k) z^{-k}\,. \tag{3.10}$$

By the well-known shift property and convolution property (see, e.g. [23]),

$$\mathcal{Z}\{x(n+1)\}(z) = z\tilde{x}(z) - zx(0) \quad \text{and} \quad \mathcal{Z}\Big\{\sum_{j=0}^{n} a(n-j)x(j)\Big\}(z) = \tilde{a}(z)\tilde{x}(z)\,.$$

The application of the $\mathcal{Z}$-transform to both sides of (3.2) with $\lambda = 0$ then yields

$$\tilde{x}(z) = \frac{x(0)}{(1 - \frac{1}{z})^{\alpha}}\,.$$

Further, by the binomial theorem,

$$\tilde{x}(z) = x(0)\sum_{k=0}^{\infty}\binom{-\alpha}{k}\left(-\frac{1}{z}\right)^{k} = x(0)\sum_{k=0}^{\infty}(-1)^{k}\binom{-\alpha}{k} z^{-k}\,,$$

and comparing this relation with (3.10), we have

$$x(n) = x(0)(-1)^n \binom{-\alpha}{n} . \tag{3.11}$$

Consequently, the binomial asymptotic relation (1.30) implies the asymptotic property $x(n) \to 0$ as $n \to \infty$ for any initial value $x(0)$. □

Lemma 3.9. *Let $0 < \alpha < 1$ and $0 < \lambda < 2^\alpha$, $\lambda \neq 1$. Then (3.2) is not stable.*

Proof. Let $\tilde{x}(z)$ be the $\mathcal{Z}$-transform of the solution $x(n)$ of (3.2) given by (3.10). Analogously as in the previous proof, we get

$$\tilde{x}(z) = \frac{(1-\lambda)x(0)}{\left(1-\frac{1}{z}\right)^\alpha - \lambda} . \tag{3.12}$$

Setting $w = 1/z$ and considering (3.10), (3.12) we can write

$$\frac{(1-\lambda)x(0)}{(1-w)^\alpha - \lambda} = \sum_{k=0}^{\infty} x(k) w^k . \tag{3.13}$$

The function on the left-hand side of (3.13) has the (unique) pole

$$w_r = 1 - \lambda^{1/\alpha} \in (-1, 1) .$$

Consequently, the series on the right-hand side of (3.13) has the radius of convergence $R < 1$. By the Cauchy-Hadamard theorem,

$$\limsup_{n\to\infty} \sqrt[n]{|x(n)|} = \frac{1}{R} > 1 ,$$

hence (3.2) is not stable. □

To summarize this section, we reformulate some of its results for the FdE (3.1). Considering this equation, we are interested especially in its asymptotic stability. Proposition 3.1, Theorem 3.7 and Lemma 3.8 imply the following assertion.

Theorem 3.10. *Let $0 < \alpha < 1$ and $\lambda \neq 1$. Then (3.1) is asymptotically stable if*

$$\lambda \leq 0 \quad \text{or} \quad \lambda > 2^\alpha . \tag{3.14}$$

Remark 3.11. The condition (3.14) for the asymptotic stability of (3.1) is close to be not only sufficient, but also necessary. It remains to discuss the asymptotic stability of (3.1) with $\lambda = 2^\alpha$, which is still an open problem.

3.3 Asymptotic analysis

In this section, we precise some of the stability results derived in the previous part. In particular, we consider the asymptotic stable case, when the solutions $x(n)$ of (3.2) are tending to zero as $n \to \infty$, and describe the exact rate of their decay. An asymptotic result concerning the non-stable case will be derived as well.

First we note that a preliminary information on the decay rate of the solutions of (3.2) follows immediately from Theorems 3.6 and 3.7.

Corollary 3.12. *Let $0 < \alpha < 1$ and let either $\lambda < 0$ or $\lambda > 2^\alpha$. Then*

$$x(n) \in \ell^1 \tag{3.15}$$

for any solution $x(n)$ of (3.2).

To obtain a precise description of asymptotics of (3.2), we employ the following general result, which is due to Appleby et al. [6]. For any finite $r > 0$, the authors introduced a class $W(r)$ of real-valued weight sequences $\gamma(n)$ by the requirements

$$\gamma(n) > 0, \quad n = 1, 2, \ldots, \qquad \lim_{n\to\infty} \frac{\gamma(n-1)}{\gamma(n)} = \frac{1}{r}, \qquad \sum_{k=0}^{\infty} \gamma(k) r^{-k} < \infty$$

and

$$\lim_{m\to\infty} \left(\limsup_{n\to\infty} \frac{1}{\gamma(n)} \sum_{j=m}^{n-m} \gamma(n-j)\gamma(j) \right) = 0\,.$$

We reformulate here the scalar version of the relevant result (originally proved for vector Volterra difference equations), which describes the asymptotics of a solution $x(n)$ of (3.3) in terms of an appropriate sequence $\gamma(n) \in W(r)$.

Theorem 3.13 ([6, Theorem 3.2])**.** *Suppose that, for some sequence $\gamma(n) \in W(r)$, there exists the finite limit*

$$L = \lim_{n\to\infty} \frac{a(n)}{\gamma(n)}$$

and let

$$\sum_{k=0}^{\infty} |a(k)| r^{-k-1} < 1\,. \tag{3.16}$$

Then the solution $x(n)$ of (3.3) satisfies

$$\lim_{n\to\infty} \frac{x(n)}{\gamma(n)} = \frac{Lx(0)}{r(1-S)^2}\,, \quad \text{where} \quad S = \sum_{k=0}^{\infty} a(k) r^{-k-1}\,. \tag{3.17}$$

This assertion turns out to be very useful in analysis of asymptotic properties of (3.2). It implies

Corollary 3.14. *Let* $0 < \alpha < 1$ *and* $|1-\lambda| > 1$. *Then*

$$\lim_{n\to\infty} \frac{x(n)}{n^{-(1+\alpha)}} = \frac{\alpha(1-\lambda)}{\lambda^2\Gamma(1-\alpha)} x(0) \tag{3.18}$$

for any solution $x(n)$ *of (3.2).*

Proof. Consider the equation (3.2), i.e. the Volterra convolution equation (3.3) with coefficients given by (3.4). We put $r = 1$ and introduce the decreasing sequence

$$\gamma(n) = n^{-(1+\alpha)}, \qquad n = 1, 2, \dots .$$

As it was remarked in [6], $n^{-(1+\alpha)} \in W(1)$ provided $0 < \alpha < 1$. We verify the validity of assumptions of Theorem 3.13 and, in particular, specify the values of L and S. The first calculation employs the asymptotic property (1.30). Using this we can write

$$L = \frac{1}{1-\lambda} \lim_{n\to\infty} \frac{(-1)^n \binom{\alpha}{n+1}}{n^{-(1+\alpha)}} = \frac{-1}{(1-\lambda)\Gamma(-\alpha)} .$$

Discussing (3.16), we need to sum the infinite series $\sum_{k=0}^{\infty} |a(k)|$. We have

$$\sum_{k=0}^{\infty} |a(k)| = \frac{1}{|1-\lambda|} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k+1} = \frac{1}{|1-\lambda|} \tag{3.19}$$

by use of the binomial theorem. Consequently, (3.16) holds. Analogously we get

$$S = \sum_{k=0}^{\infty} a(k) = \frac{1}{1-\lambda} .$$

The property (3.18) now follows from (3.17) by use of $\Gamma(1-\alpha) = -\alpha\Gamma(-\alpha)$. □

Remark 3.15. The assumptions of Corollary 3.14 do not cover the cases $\lambda = 0$ and $2^\alpha < \lambda \le 2$, when the equation (3.2) is asymptotically stable as well. If $2^\alpha < \lambda \le 2$ then Corollary 3.12 provides at least a partial information on the behaviour of the solutions of (3.2), namely the asymptotic property (3.15). However, this property is no longer valid if $\lambda = 0$, because the corresponding Volterra equation (3.2) is not uniformly asymptotically stable. Fortunately, earlier we have derived the exact form of the solutions $x(n)$ of (3.2) with $\lambda = 0$ via the relation (3.11). Then, using (1.30), we can easily get the following asymptotic result.

Corollary 3.16. *Let* $0 < \alpha < 1$ *and* $\lambda = 0$. *Then*

$$\lim_{n\to\infty} \frac{x(n)}{n^{-(1-\alpha)}} = \frac{1}{\Gamma(\alpha)} x(0)$$

for any solution $x(n)$ *of (3.2).*

We conclude this part by the summary of the derived asymptotic results and their reformulation for the FdE (3.1).

Corollary 3.17. *Let $0 < \alpha < 1$ and let either $\lambda \le 0$ or $\lambda > 2$. Then*

$$y(n) \sim \begin{cases} \dfrac{K_1}{n^{1-\alpha}} & \text{as } n \to \infty, \qquad K_1 = \dfrac{1}{\Gamma(\alpha)}\, y(1) \qquad \text{if } \lambda = 0\,, \\ \dfrac{K_2}{n^{1+\alpha}} & \text{as } n \to \infty, \qquad K_2 = \dfrac{\alpha(1-\lambda)}{\lambda^2\Gamma(1-\alpha)}\, y(1) \quad \text{otherwise} \end{cases}$$

for any solution $y(n)$ of (3.1).

Now we turn our attention to the unstable case. Recently, Atici and Eloe [9] analyzed the closed form of the solutions of (3.1) based on discrete Mittag-Leffler functions and proved that if $1/2 \le \alpha < 1$ and $0 < \lambda < 1$, then $y(n) \to \infty$ as $n \to \infty$ for any solution $y(n)$ of (3.1) with $y(1) > 0$ (in our notation). We employ our approach based on analysis of the corresponding Volterra difference equation (3.2) to obtain a slightly stronger result.

Theorem 3.18. *Let $0 < \alpha < 1$, $0 < \lambda < 1$ and let $x(n)$ be a solution of (3.2) with $x(0) > 0$. Then*

$$\frac{\lambda^{1/\alpha} x(0)}{(1-\lambda^{1/\alpha})^n} < x(n) < \frac{x(0)}{(1-\lambda^{1/\alpha})^n}\,, \qquad n = 1, 2, \dots\,. \tag{3.20}$$

Proof. First we introduce the function

$$v(n) = \frac{x(0)}{(1-\lambda^{1/\alpha})^n}\,, \qquad n \in \mathbb{Z}$$

defining the solution of the Volterra convolution equation with infinite delay

$$v(n+1) = \sum_{j=-\infty}^{n} a(n-j)v(j)\,, \quad n = 0, 1, \dots\,, \tag{3.21}$$

where

$$a(n) = \frac{(-1)^n}{1-\lambda}\binom{\alpha}{n+1} > 0\,, \quad n = 0, 1, \dots\,.$$

Rewrite (3.21) as

$$v(n+1) = \sum_{j=0}^{n} a(n-j)v(j) + g(n)\,,$$

where $g(n) = \sum_{j=-\infty}^{-1} a(n-j)v(j) > 0$. Using the variation of constants formula (see, e.g. [25]) we have

$$v(n) = x(n) + \frac{1}{x(0)}\sum_{j=0}^{n-1} x(n-j-1)g(j)\,. \tag{3.22}$$

Since $x(n)$ is positive, (3.22) implies

$$x(n) < v(n)\,, \quad n = 1, 2, \dots\,,$$

which proves the right inequality of (3.20).

Further, it follows from (3.22) that

$$\nabla v(n) = x(n) + \left(\frac{g(0)}{x(0)} - 1\right) x(n-1) + \frac{1}{x(0)} \sum_{j=1}^{n-1} x(n-j-1) \nabla g(j)\,.$$

Since

$$\nabla g(j) = \sum_{k=-\infty}^{-1} \frac{(-1)^{j-k}}{1-\lambda} \binom{\alpha+1}{j-k+1} v(k) < 0\,,$$

we get

$$\nabla v(n) < x(n) + p\, x(n-1)\,, \quad \text{where} \quad p = \frac{g(0)}{x(0)} - 1\,. \tag{3.23}$$

Using the binomial theorem we can verify that

$$p = -\frac{\lambda^{1+1/\alpha} - (1+\alpha)\lambda^{1/\alpha} + \alpha}{(1-\lambda)(1-\lambda^{1/\alpha})}\,.$$

We show that $p < 0$, i.e.

$$F_\alpha(\lambda) = \lambda^{1+1/\alpha} - (1+\alpha)\lambda^{1/\alpha} + \alpha > 0$$

for all $0 < \alpha < 1$ and $0 < \lambda < 1$. Indeed, we have

$$F_\alpha(0) = \alpha > 0\,, \quad F_\alpha(1) = 0 \quad \text{and} \quad F'_\alpha(\lambda) = \frac{\alpha+1}{\alpha} \lambda^{1/\alpha}(1-\lambda^{-1}) < 0$$

for all such values of α and λ. Consequently, we can neglect the last term in the inequality (3.23) to obtain

$$\nabla v(n) = \frac{\lambda^{1/\alpha} x(0)}{(1-\lambda^{1/\alpha})^n} < x(n)\,, \quad n = 1, 2, \dots\,.$$

The left inequality of (3.20) is proved. □

Remark 3.19. A reformulation of this asymptotic result for the FdE (3.1) is analogous as in Corollary 3.17.

3.4 A connection to some recent results

Our stability investigation of (3.2) was based on analysis of the roots of the corresponding characteristic equation and their location with respect to the unit disk. In general, this direct approach is not practical just because of difficulties connected with the localization of the roots of a complex function resulting from the utilized $\mathcal{Z}$-transform method. Therefore, the following explicit criterion for the asymptotic stability of (3.3) is usually applied (for other types of stability conditions we also refer to [35]).

Theorem 3.20 ([23, Theorem 6.18]). *Suppose that $a(n)$ does not change sign for $n \in \mathbb{Z}_0^+$ and*

$$\Big|\sum_{n=0}^{\infty} a(n)\Big| < 1\,. \tag{3.24}$$

Then (3.3) is uniformly asymptotically stable.

Till lately, it was an open question whether or not (3.24) is also necessary for the uniform asymptotic stability (or asymptotic stability) of (3.3). Only recently, Elaydi et al. [24] have constructed a class of equations (3.3) that violate the condition (3.24), but they are still asymptotically stable.

We discuss the strictness of (3.24) with respect to the Volterra equation (3.2). First notice that $\operatorname{sgn}(a(n)) = \operatorname{sgn}(1-\lambda)$ for all n, i.e. $a(n)$ does not change sign (we still assume $0 < \alpha < 1$). Furthermore,

$$\Big|\sum_{n=0}^{\infty} a(n)\Big| = \frac{1}{|1-\lambda|}$$

(see (3.19)). Hence, applying (3.24) to (3.2) we get $|1-\lambda| > 1$, which is a weaker result than (3.6) yields. More precisely, if $2^\alpha < \lambda \le 2$, then Theorem 3.7 implies the uniform asymptotic stability of (3.2), although (3.24) does not hold. Actually, using (3.2) we can show something more, namely that the condition (3.24) is not necessary for the uniform asymptotic stability of (3.2) even if we consider it in a weaker form

$$\left|\sum_{n=0}^{\infty} a(n)\right| < M, \quad M > 0 \text{ being large enough}\,.$$

Example 3.21. Let $M > 0$ be arbitrary. Choose α and λ such that

$$0 < \alpha < 1 \quad \text{and} \quad 2^\alpha < \lambda < 1 + 1/M\,.$$

Then, by Theorem 3.7, the equation (3.2) is uniformly asymptotically stable, but

$$\left|\sum_{n=0}^{\infty} a(n)\right| = \frac{1}{\lambda - 1} > M\,.$$

Similarly, using (3.2) we can construct another appropriate counterexamples related to some of open problems posed in [23].

Besides this contribution to the stability theory of Volterra difference equations, we can observe some other specific qualitative properties of the FdE (3.1). It concerns, e.g. the asymptotic stability property of (3.1) with $\lambda = 0$ and arbitrary $0 < \alpha < 1$ (see Lemma 3.8). This result does not agree with the limit (trivial) case $\alpha = 1$, when (3.1) is stable, but not asymptotically stable.

On the other hand, the qualitative behaviour of the FdE (3.1) with $\lambda = 0$ is qualitatively different from the behaviour of (3.1) with other values of λ corresponding to the asymptotic stable case ($\lambda < 0$ or $\lambda > 2^{\alpha}$). Firstly, the decay rate of the solutions is lower than that for $\lambda < 0$ and $\lambda > 2^{\alpha}$ (see Corollaries 3.12 and 3.17). Secondly, the asymptotic stability property for $\lambda = 0$ is not uniform.

Our last remark concerns an algebraic decay of the solutions $y(n)$ of (3.1) such that $|1-\lambda| > 1$. By Corollary 3.17, its order is equal to $1+\alpha$, which is the same as in Theorem 1.5 for the corresponding FDE. This resemblance, as well as the asymptotic stability region (3.14) indicate that the FdEs derived from (3.1) could be a very suitable tool for numerical approximations of the underlying FDE. We discuss this connection in more details in the next chapter.

4 Qualitative analysis of a vector linear FdE on $\mathbb{T} = h\mathbb{Z}$

As we mentioned in the Introduction, the development of numerical methods for solving of FDEs is one of propulsion powers of the discrete fractional calculus. One of the simplest numerical methods, a generalization of the well-known Euler method, consists of reduction of involved functions to

$$\mathbb{T} = h\mathbb{Z}_0^+ = \{t_n = nh\,;\, n \in \mathbb{Z}_0^+\}\,, \quad h > 0 \tag{4.1}$$

and replacing of the continuous fractional operators by the corresponding discrete ones. Obviously, every definition of discrete fractional difference induces its own method. For an illustration we refer to methods described in [22, 44], whose basic properties were studied, e.g. in [37].

The subject of this chapter is closely related to the qualitative analysis of a numerical method utilizing the fractional difference given in Remark 1.47. This method, a generalization of the backward Euler method, was proposed in [33], where its convenience for solving of initial value problems with fractional-order initial conditions was discussed. Its application to the boundary value problems representing anomalous diffusion was introduced in [32].

The results presented in this chapter originate from the paper [17] and some of them can be viewed as a vector extension of the main results of the previous chapter. However, while proof techniques employed in Chapter 3 utilize tools from the theory of Volterra equations, assertions presented in this chapter are derived by original direct methods.

We study the qualitative properties of a system of linear FdEs on the time scale (4.1). We show, among others, that the discrete system can retain the key qualitative properties of the underlying continuous one regardless of the discretization stepsize (this property of backward discretizations is well-known for $\alpha = 1$ and we wish to confirm it also for $0 < \alpha < 1$). Further, we discuss relationships between the qualitative properties of studied fractional systems on $\mathbb{T} = \mathbb{R}_0^+$ and $\mathbb{T} = h\mathbb{Z}_0^+$ with respect to changing h. In particular, we formulate Theorem 4.11, representing a direct discrete counterpart to Theorem 1.5, using the h-Laplace transform as a proof tool.

4.1 Problem formulation and its solution

We are going to discuss some basic qualitative properties of the fractional difference system

$$_0\nabla_h^\alpha y(t_n) = Ay(t_n)\,, \quad 0 < \alpha \le 1\,,\; n = 1, 2, \dots\,, \tag{4.2}$$

where A is a $d \times d$ constant matrix with real entries, $y(t_n)$ is d-vector. Note that we use the same symbols for vector and scalar quantities and their actual meaning will become clear from the context.

It can be expected that the stability and asymptotic properties do not depend on the form of initial conditions. The technique presented in Chapter 2 suggests ${}_0\nabla_h^{\alpha-1} y(t_n)\big|_{n=1}$ as the starting value, but utilizing the h-Laplace transform implies a different approach. Indeed, by Lemma 1.52 (ii) the initial condition is assumed to take the form

$$ {}_0\nabla_h^{\alpha-1} y(t_n)\big|_{n=0} = y_0\,, \quad y_0 \in \mathbb{R}^d\,, \tag{4.3} $$

which requires some additional comments. Suppose that the solution of (4.2), (4.3) is a function given on $\mathbb{T} = h\mathbb{Z}_0^+$. Then Definition 1.43 (as discussed in Remark 1.44 (ii)) automatically implies the zero value of (4.3), i.e. ${}_0\nabla_h^{\alpha-1} y(t_n)\big|_{n=0} = 0$. Now assume that the solution of (4.2), (4.3) has the domain $h\mathbb{Z}^+$, i.e. its value at point $t_0 = 0$ is undefined. Then Definition 1.43 does not assign any value to the symbol ${}_0\nabla_h^{\alpha-1} y(t_n)\big|_{n=0}$ and the fractional difference ${}_0\nabla_h^{\alpha} y(t_n)$ is not defined for $n = 1$ (see Definition 1.45). Thus, the case, when the system (4.2), (4.3) is considered for $n = 1$, does not seem to be covered. However, we have to keep in mind that Definition 1.45 discusses a stand-alone function $f(t)$ given by its values, while the solution of the initial value problem is specified via its properties. In particular, a prescription of the initial condition by (4.3) provides an additional information allowing to interpret (4.2) even for $n = 1$. Indeed, the sequential expanding of (1.41) yields for (4.2), (4.3)

$$ {}_0\nabla_h^{\alpha-1} y(t_n)\big|_{n=1} - {}_0\nabla_h^{\alpha-1} y(t_n)\big|_{n=0} = hAy(t_1), $$

which utilizing (4.3) and expanding the first term by (1.43) for $q = 1$ leads to

$$ y(t_1) = h^{\alpha-1}(I - h^{\alpha}A)^{-1} y_0\,. \tag{4.4} $$

This relation defines (under the regularity assumption of the matrix $I - h^{\alpha}A$) a one-to-one mapping between y_0 and $y(t_1)$ (and by extension ${}_0\nabla_h^{\alpha-1} y(t_n)\big|_{n=1}$). Summarizing this, in the frame of Riemann-Liouville approach we shall study the initial value problem (4.2), (4.3), where the meaning of (4.2) for $n = 1$ and the meaning of (4.3) are specified via (4.4).

Remark 4.1. (i) In the time scales theory the initial conditions are not usually prescribed at right-scattered minimum of a time scale, but at a "sufficiently distant" point with respect to the order of the equation (see Subsection 1.2.2). However, utilizing the ideas outlined above we can construct a one-to-one mapping between the conventional conditions and the conditions at right-scattered minimum on an arbitrary time scale for every ν-regressive fractional initial value problem.
(ii) The initial condition (4.3) is a formal discrete analogy of (1.14). We recall that any nonzero choice of x_0 in (1.14) implies unboundedness of the solution $x(t)$ of (1.13) in a neighbourhood of zero. Hence, the solutions of both continuous and discrete initial value problem are not defined at zero.

Now, we discuss a condition guaranteeing the existence and uniqueness for (4.2), (4.3). If $\alpha = 1$ then this condition can be expressed via ν-regressivity of A (see Definition 1.15). This property can be extended to $0 < \alpha < 1$ as follows.

Definition 4.2. A matrix function $A : \mathbb{T} \to \mathbb{R}^{d\times d}$ is called ν^α-regressive if

$$\det(I - \nu^\alpha(t)A(t)) \neq 0 \quad \text{for all } t \in \mathbb{T}_\kappa\,. \tag{4.5}$$

Remark 4.3. (i) Considering $\nu(t) \neq 0$, the matrix function $A(t)$ is ν^α-regressive if and only if the function $\nu^{\alpha-1}(t)A(t)$ is ν-regressive.
(ii) We are interested in the case of a constant matrix A on the time scale $\mathbb{T} = h\mathbb{Z}_0^+$, i.e. (4.5) reduces to a single inequality $\det(I - h^\alpha A) \neq 0$.

Proposition 4.4. *Let A be ν^α-regressive. Then the initial value problem (4.2), (4.3) has a unique solution.*

Proof. The proof is a matrix analogue to the proof of Proposition 3.1. Thus, due to the invertibility of $I - h^\alpha A$, the system (4.2) can be written as

$$y(t_{n+1}) = (I - h^\alpha A)^{-1} \sum_{j=1}^{n} (-1)^{n-j} \binom{\alpha}{n-j+1} y(t_j)\,, \quad n = 1, 2, \dots\,,$$

which is the Volterra difference system of convolution type with starting vector $y(t_1)$ uniquely given by (4.4).

The existence and uniqueness of the solution of the initial value problem (4.2), (4.3) now follows from its equivalence with this Volterra system. □

In this chapter, we utilize the h-Laplace transform (1.33) for qualitative analysis of (4.2). Nevertheless, it can serve also as a useful tool for finding the solution of the initial value problem (4.2), (4.3). Doing this, we recall the h-Mittag-Leffler function in the matrix form

$$E^A_{\eta,\beta}(t_n) = \sum_{k=0}^{\infty} A^k \hat{h}_{\eta k+\beta-1}(t_n, 0)\,, \quad \eta, \beta \in \mathbb{R}\,,\ A \in \mathbb{R}^{d\times d},$$

where all eigenvalues $\lambda(A)$ are assumed to lie inside $B(0, h^{-\eta})$ (see, e.g. [9]).

Remark 4.5. If this definition is applied to a scalar case ($d = 1$, matrix A replaced by a scalar λ), it coincides with Definition 2.10 for $q = 1$ and $s = 0$.

Proposition 4.6. *It holds*

$$\mathcal{L}\{E^A_{\eta,\beta}\}(z) = z^{\eta-\beta}(z^\eta I - A)^{-1}\,.$$

Proof. Lemma 1.38 (i) and related definitions imply

$$\begin{aligned}\mathcal{L}\{E^A_{\eta,\beta}\}(z) &= \sum_{k=1}^{\infty} hE^A_{\eta,\beta}(t_k)(1-hz)^{k-1} = h\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} A^j \hat{h}_{\eta j+\beta-1}(t_k,0)(1-hz)^{k-1}\\ &= \sum_{j=0}^{\infty} A^j \mathcal{L}\{\hat{h}_{\eta j+\beta-1}(\cdot,0)\}(z) = \sum_{j=0}^{\infty} A^j z^{-\eta j-\beta} = z^{-\beta}\sum_{j=0}^{\infty}(z^{-\eta}A)^j = z^{\eta-\beta}(z^\eta I-A)^{-1}\,,\end{aligned}$$

where the last equality is true if all eigenvalues $\lambda(A)$ lie in $B(0,|z|^\eta)$. The restrictions on the complex parameter z guarantee that there exists a positive radius of convergence. □

Theorem 4.7. *Assume that all eigenvalues $\lambda(A)$ lie inside $B(0,h^{-\alpha})$. Then the initial value problem (4.2), (4.3) has the unique solution given by*

$$y(t_n) = E^A_{\alpha,\alpha}(t_n)y_0\,.$$

Proof. Applying the backward h-Laplace transform to both sides of (4.2), using the initial condition (4.3) and Lemma 1.52 (ii), we get

$$\mathcal{L}\{y\}(z) = (z^\alpha I - A)^{-1}y_0\,. \tag{4.6}$$

The statement now follows from Proposition 4.6 with respect to the uniqueness of the h-Laplace transform and the uniqueness of the solution $y(t_n)$ of (4.2), (4.3) guaranteed by Proposition 4.4. □

In the continuous case, the representation of solutions for (1.13) via standard Mittag-Leffler functions is well known (see [43, p. 137]), as well as the asymptotic behaviour of these special functions (see [43, p. 29]). Then such a solution representation can provide the key tool for stability analysis of (1.13) (see [45]).

Contrary to this continuous case, the asymptotic behaviour of h-Mittag-Leffler functions is unknown even in the scalar case (see Chapter 3). Besides, the solution representation involved in Theorem 4.7 can be utilized only for system (4.2) with the matrix A having all its eigenvalues inside $B(0,h^{-\alpha})$ (note that this condition becomes trivial in the continuous case). Consequently, the above mentioned method of stability investigations of the fractional differential system (1.13) does not seem to be convenient in the discrete case, hence considering the difference systems (4.2) we shall proceed differently. Although the technique performed in the previous chapter, utilizing of the theory of Volterra difference equations, can be applied in the vector case too, we introduce another approach based mostly on the h-Laplace transform.

To supplement the previous discussion, we note that it is possible to describe asymptotics of corresponding h-Mittag-Leffler functions as a by-product of our results introduced bellow.

4.2 Stability and asymptotics

This section formulates and comments a discrete analogue of Theorem 1.5. First we introduce the stability notions for (4.2).

Definition 4.8. The fractional difference system (4.2) is said to be

(i) *stable* if and only if for any $y_0 \in \mathbb{R}^d$ there exists $K > 0$ such that the solution $y(t_n)$ of (4.2), (4.3) satisfies $\|y(t_n)\| \leq K$ for all $n = 1, 2, \dots$;

(ii) *asymptotically stable* if and only if for any $y_0 \in \mathbb{R}^d$ the solution $y(t_n)$ of (4.2), (4.3) satisfies $\|y(t_n)\| \to 0$ as $n \to \infty$.

Our stability analysis of (4.2) is based on the investigation of the h-Laplace transform of the solution $\mathcal{L}\{y\}(z)$, in particular regarding its series expansion (see Subsection 1.2.4). On that account, we recall the following general result which turns out to be useful in the next procedures (see, e.g. [26, p. 144-146]).

Lemma 4.9. *Let $F(z)$ be a complex function analytic at $z_0 \in \mathbb{C}$. Then the radius of convergence of its power series (centered at z_0) equals to the distance between z_0 and the nearest singular point of $F(z)$.*

Before we formulate the main theorem of this section, we introduce the following preliminary assertion.

Proposition 4.10. *Let $y(t_n)$ be a solution of (4.2), (4.3) and let R be the set of all roots of the equation*

$$\det(z^\alpha I - A) = 0\,. \tag{4.7}$$

(i) *If $\min_{z \in R} |z - h^{-1}| > h^{-1}$, then $y(t_n) \in \ell^1$.*

(ii) *If $\min_{z \in R} |z - h^{-1}| < h^{-1}$, then (4.2) is not stable.*

Proof. Let A be similar to a Jordan canonical form, i.e. there exists an invertible matrix P such that $A = PJP^{-1}$, where $J = \operatorname{diag}(J_1, \dots, J_s)$ and J_ℓ are Jordan blocks of order r_ℓ ($\ell = 1, \dots, s$). Further, let $\lambda_i(A)$ be eigenvalues of A, let $k_i \in \mathbb{Z}^+$ be their algebraic multiplicities and $p_i \in \mathbb{Z}^+$ geometric multiplicities ($i = 1, \dots, m$). The Laplace transform of $y(t_n)$ is

$$\mathcal{L}\{y\}(z) = (z^\alpha I - A)^{-1} y_0 = P^{-1}(z^\alpha I - J)^{-1} P\, y_0 \tag{4.8}$$

by virtue of (4.6). The matrix $(z^\alpha I - J)^{-1}$ is a block matrix. The number of blocks corresponding to $\lambda_i(A)$ is p_i and their form is given by the upper triangular matrix

$$\begin{pmatrix} (z^\alpha - \lambda_i(A))^{-1} & (z^\alpha - \lambda_i(A))^{-2} & \dots & (z^\alpha - \lambda_i(A))^{-r_q} \\ 0 & (z^\alpha - \lambda_i(A))^{-1} & \dots & (z^\alpha - \lambda_i(A))^{-r_q+1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (z^\alpha - \lambda_i(A))^{-1} \end{pmatrix}, \tag{4.9}$$

where r_q $(q = 1, \dots, p_i)$ is the size of the block. On this account, each component of $\mathcal{L}\{y\}(z)$ is formed by a linear combination of the rational functions occurring in (4.9), hence R is the set of all poles of $\mathcal{L}\{y\}(z)$ because of

$$\det(z^\alpha I - A) = \det(z^\alpha I - J) = \prod_{i=1}^{m} (z^\alpha - \lambda_i(A))^{k_i} .$$

Since the case $\alpha = 1$ is elementary, we further consider $0 < \alpha < 1$. Then $\mathcal{L}\{y\}(z)$ has a singular point at zero due to the presence of power function z^α in (4.8).

(i) If $\min_{z \in R} |z - h^{-1}| > h^{-1}$, then the radius of convergence of $\mathcal{L}\{y\}(z)$ is $r = h^{-1}$ by use of Lemma 4.9. Further, as we have already observed, each component of $y(t_n)$ is a linear combination of functions $u_{i,j}(t_n)$ such that $\mathcal{L}\{u_{i,j}\}(z) = (z^\alpha - \lambda_i(A))^{-j}$ $(j = 1, \dots, \hat{r}_q)$, where $\hat{r}_q$ is the maximal size of blocks (4.9) corresponding to $\lambda_i(A)$. To prove the statement (i) it is enough to show that $u_{i,j}(t_n) \in \ell^1$.

First let $j = 1$. We define the auxiliary functions $g_i : h\mathbb{Z}^+ \to \mathbb{C}$ as

$$g_i(t_n) = \hat{h}_{-\alpha-1}(t_n, 0) - \lambda_i(A)\delta_h(t_n) ,$$

where $\delta_h(t_n)$ is given in Lemma 1.38. Lemma 1.38 imply that $\mathcal{L}\{g_i\}(z) = z^\alpha - \lambda_i(A)$. Since

$$\begin{aligned} \sum_{n=1}^{\infty} |g_i(t_n)| &= h^{-1}|h^{-\alpha} - \lambda_i(A)| + h^{-\alpha-1} \sum_{n=2}^{\infty} (-1)^n \binom{\alpha}{n-1} \\ &= h^{-\alpha-1}(1 + |1 - h^\alpha \lambda_i(A)|) , \end{aligned}$$

it holds $g_i(t_n) \in \ell^1$. Moreover, $\mathcal{L}\{g_i\}(z) \cdot \mathcal{L}\{u_{i,1}\}(z) = 1$ and $|\mathcal{L}\{g_i\}(z)| > 0$ on $\mathrm{cl}B(h^{-1}, h^{-1})$. Proposition 1.40 then yields $u_{i,1}(t_n) \in \ell^1$.

Now let $j > 1$. In virtue of Theorem 1.37 we have

$$u_{i,j}(t_n) = (\underbrace{u_{i,1} * \cdots * u_{i,1}}_{j\times})(t_n) .$$

Since the convolution of two elements of ℓ^1 is again an element of ℓ^1 (see [34, pp. 89-91]), the property $u_{i,j}(t_n) \in \ell^1$ follows from the induction principle.

(ii) If $\min_{z \in R} |z - h^{-1}| < h^{-1}$, then the radius of convergence is less than h^{-1} for at least one of components of $\mathcal{L}\{y\}(z)$. By Proposition 1.39 (ii), the system (4.2) is not stable. □

Let $C = (c_{ij})$ be a matrix. By the symbol $|C|$ we shall understand the matrix given by $|C| = (|c_{ij}|)$. Further, we introduce the regions

$$\mathcal{S}_{\alpha,h} = \left\{ z \in \mathbb{C} \,;\, |\mathrm{Arg}(z)| > \frac{\alpha\pi}{2} \text{ or } |z| > \frac{2^\alpha}{h^\alpha} \cos^\alpha \left(\frac{\mathrm{Arg}(z)}{\alpha} \right) \right\}$$

and the interior of its complement in $\mathbb{C}$

$$\mathcal{U}_{\alpha,h} = \left\{ z \in \mathbb{C} \,;\, |\mathrm{Arg}(z)| < \frac{\alpha\pi}{2} \text{ and } |z| < \frac{2^\alpha}{h^\alpha} \cos^\alpha \left(\frac{\mathrm{Arg}(z)}{\alpha} \right) \right\} .$$

Using this notation we have

Theorem 4.11. *Let the matrix A be ν^α-regressive, $0 < \alpha \le 1$ and let $y(t_n)$ be the solution of (4.2), (4.3).*

(i) *If all eigenvalues $\lambda(A)$ satisfy $\lambda(A) \in \mathcal{S}_{\alpha,h}$, then $y(t_n) \in \ell^1$, hence (4.2) is asymptotically stable. Moreover, if all eigenvalues of the matrix $|(I - h^\alpha A)^{-1}|$ lie inside the open unit disk, then each component of $y(t_n)$ tends to zero like $O(n^{-(1+\alpha)})$ as $n \to \infty$.*

(ii) *If there exists an eigenvalue $\lambda(A)$ such that $\lambda(A) \in \mathcal{U}_{\alpha,h}$, then (4.2) is not stable.*

Proof. We have observed that the set R of all roots of (4.7) consists of complex values z_i such that $z_i^\alpha = \lambda_i(A)$ $(i = 1, \dots, m)$. Then

$$|z_i|^\alpha \exp\left(\mathrm{j}\alpha \operatorname{Arg}(z_i)\right) = |\lambda_i(A)| \exp\left(\mathrm{j} \operatorname{Arg}(\lambda_i(A))\right), \quad \mathrm{j} = \sqrt{-1}\,.$$

Since for all complex values z it holds $\operatorname{Arg}(z) \in (-\pi, \pi]$, the roots z_i exist provided $\operatorname{Arg}(\lambda_i(A)) \in (-\alpha\pi, \alpha\pi]$. Assuming this we arrive at

$$z_i = |\lambda_i(A)|^{\frac{1}{\alpha}} \exp\left(\mathrm{j}\, \frac{\operatorname{Arg}(\lambda_i(A))}{\alpha}\right).$$

Further, we show that the conditions $|z_i - h^{-1}| > h^{-1}$ and $\lambda_i(A) \in \mathcal{S}_{\alpha,h}$ are equivalent. Indeed,

$$\left| |\lambda_i(A)|^{\frac{1}{\alpha}} \exp\left(\mathrm{j}\, \frac{\operatorname{Arg}(\lambda_i(A))}{\alpha}\right) - h^{-1} \right| > h^{-1}$$

occurs if and only if

$$|\lambda_i(A)|^{\frac{1}{\alpha}} > \frac{2}{h} \cos\left(\frac{\operatorname{Arg}(\lambda_i(A))}{\alpha}\right). \tag{4.10}$$

If $|\operatorname{Arg}(\lambda_i(A))| > \frac{\alpha\pi}{2}$, then (4.10) is satisfied trivially. If $|\operatorname{Arg}(\lambda_i(A))| \le \frac{\alpha\pi}{2}$, then (4.10) is equivalent to $|\lambda_i(A)| > \frac{2^\alpha}{h^\alpha} \cos^\alpha\left(\frac{\operatorname{Arg}(\lambda_i(A))}{\alpha}\right)$. Summarizing this, $\lambda_i(A) \in \mathcal{S}_{\alpha,h}$.

The stability part of Theorem 4.11 (i) now follows from Proposition 4.10 (i). Analogously, Proposition 4.10 (ii) implies the assertion of Theorem 4.11 (ii).

Now we derive the asymptotic estimate formulated in Theorem 4.11 (i). Doing this, we investigate (4.2) in its equivalent form

$$y(t_{n+1}) = \sum_{j=1}^{n} M(t_{n-j+1}) y(t_j)\,, \quad \text{where} \quad M(t_n) = (-1)^{n-1} \binom{\alpha}{n} (I - h^\alpha A)^{-1}$$

(see the proof of Proposition 4.4). Using the asymptotic property of the power function (1.29) we can verify the existence of a finite limit

$$\lim_{n \to \infty} \frac{M(t_n)}{n^{-1-\alpha}} = \frac{\alpha}{\Gamma(1-\alpha)} (I - h^\alpha A)^{-1}\,,$$

hence we can apply [6, Theorem 3.2]. In particular, if all eigenvalues of the matrix

$$\sum_{n=1}^{\infty}|M(t_n)| = \sum_{n=1}^{\infty}(-1)^{n-1}\binom{\alpha}{n}|(I-h^{\alpha}A)^{-1}| = |(I-h^{\alpha}A)^{-1}|$$

lie inside the unit disk, then the direct application of [6, formula (15)] yields that $y(t_n)/n^{-1-\alpha}$ tends to a non-trivial limit

$$(I-(I-h^{\alpha}A)^{-1})^{-1}\frac{\alpha}{\Gamma(1-\alpha)}(I-h^{\alpha}A)^{-1}(I-(I-h^{\alpha}A)^{-1})^{-1}y(t_1)\,.$$

To simplify this limit value we note that

$$(I-(I-h^{\alpha}A)^{-1})^{-1}(I-h^{\alpha}A)^{-1} = -h^{-\alpha}A^{-1}\,,$$

hence

$$\begin{aligned}\lim_{n\to\infty}\frac{y(t_n)}{n^{-1-\alpha}} &= \frac{\alpha}{\Gamma(1-\alpha)}(-h^{\alpha}A+(I-h^{\alpha}A)^{-1}h^{\alpha}A)^{-1}y(t_1)\\ &= \frac{\alpha}{\Gamma(1-\alpha)}(h^{-2\alpha}A^{-2}-h^{-\alpha}A^{-1})y(t_1)\,.\end{aligned}$$

Notice that thus we have derived a slightly stronger asymptotic result than that formulated in Theorem 4.11. □

Remark 4.12. An alternative expression of the asymptotic stability region $\mathcal{S}_{\alpha,h}$ can be provided by

$$\mathcal{S}_{\alpha,h} = \big\{z\in\mathbb{C}\,;\; z = h^{-\alpha}(1-w)^{\alpha},\, w\in\mathbb{C},\, |w|>1\big\} \tag{4.11}$$

(analogously we can rewrite $\mathcal{U}_{\alpha,h}$). Equivalence of both expressions can be shown by use of some elementary calculations in the complex plain.

Utilizing (4.11) we can directly verify that if all eigenvalues of the matrix $|(I-h^{\alpha}A)^{-1}|$ lie inside the open unit disk, then all eigenvalues of A lie inside $\mathcal{S}_{\alpha,h}$. Indeed, let all eigenvalues of $|(I-h^{\alpha}A)^{-1}|$ lie inside the open unit disk. Then the same is true for all eigenvalues of $(I-h^{\alpha}A)^{-1}$, i.e. all eigenvalues of $I-h^{\alpha}A$ lie outside the unit disk. Equivalently, all eigenvalues $\lambda(A)$ satisfy $|1-h^{\alpha}\lambda(A)|>1$. Considering the form of $\mathcal{S}_{\alpha,h}$ given by (4.11), we write $\lambda(A)=h^{-\alpha}(1-w)^{\alpha}$, i.e. $h^{\alpha}\lambda(A)=(1-w)^{\alpha}$. It remains to show that $|1-(1-w)^{\alpha}|>1$ implies $|w|>1$, or equivalently, $|1-u^{\alpha}|>1$ implies $|1-u|>1$. This can be proved via direct calculations. Thus, we have verified that the supplementary condition of Theorem 4.11 (i) actually poses a restriction of the asymptotic stability region $\mathcal{S}_{\alpha,h}$.

Remark 4.13. Theorem 1.5 implies that the asymptotic stability region $\mathcal{S}_{\alpha}$ of the differential system (1.13) is given by $\mathcal{S}_{\alpha}=\{z\in\mathbb{C}\,;\,|\mathrm{Arg}(z)|>\frac{\alpha\pi}{2}\}$. We can see that $\mathcal{S}_{\alpha}\subset\mathcal{S}_{\alpha,h}$ for any $0<\alpha\le 1$ and any $h>0$. Moreover, by Theorem 4.11, the discretization (4.2) preserves the decay rate of the exact solutions (at least in a part of asymptotic stability region $\mathcal{S}_{\alpha,h}$).

These facts indicate that (4.2) as a very convenient tool for numerical approximations of the fractional differential system (1.13). In particular, the property $\mathcal{S}_\alpha \subset \mathcal{S}_{\alpha,h}$, when asymptotic stability region of the continuous system (1.13) is a subset of the corresponding discrete one for any $h > 0$, is in numerical analysis usually referred to as A-stability of a given numerical method. Consequently, the method originating from (4.2), which is essentially the backward fractional Euler method, is A-stable when applied to (1.13).

It is also interesting to observe the dependence of $\mathcal{S}_{\alpha,h}$ on parameters α and h (in particular, when $\alpha \to 1^-$ and $h \to 0^+$). The Figures 2 and 3 depict these situations (regions $\mathcal{S}_{\alpha,h}$ lie outside the corresponding boundary curves).

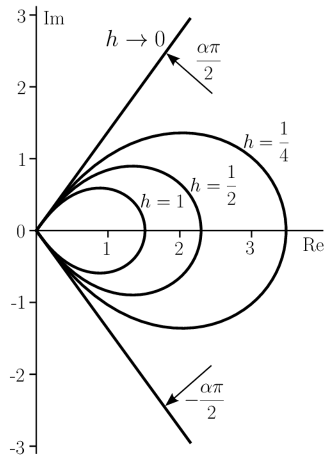

Figure 2: Dependence of the stability domain on the parameter h for $\alpha = 0.6$

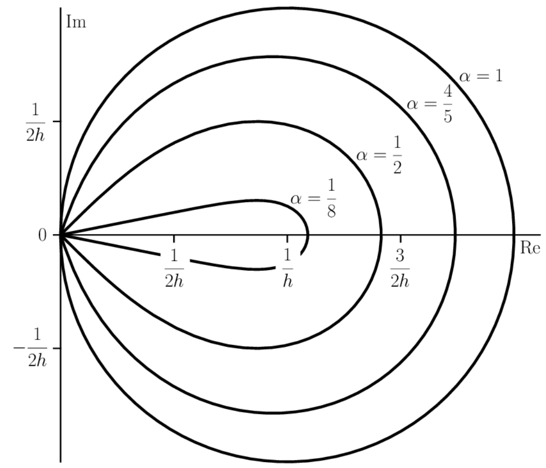

Figure 3: Dependence of the stability domain on the parameter α

Remark 4.14. If we consider the scalar case (with the matrix A replaced by the scalar λ), then Theorem 4.11 describes the asymptotic behaviour of the h-Mittag-Leffler function $E^{0,\lambda}_{\alpha,\alpha}(t_n)$ provided $-h^{-\alpha} < \lambda < 0$.

Theorem 4.11 does not solve the stability problem when some of eigenvalues $\lambda(A)$ lie on the stability boundary. The following assertion demonstrates that all stability variants are possible in such the case.

Theorem 4.15. *Let the matrix A be ν^α-regressive, $0 < \alpha \le 1$, let A has the zero eigenvalue $\lambda_1(A) = 0$ and let all its nonzero eigenvalues belong to $\mathcal{S}_{\alpha,h}$. Denote $\hat{r} \in \mathbb{Z}^+$ the maximal size of the Jordan block corresponding to $\lambda_1(A)$.*

(i) *If $\hat{r} < \alpha^{-1}$, then (4.2) is asymptotically stable. Moreover, each component of all solutions $y(t_n)$ of (4.2) tends to zero like $O(n^{\hat{r}\alpha-1})$ as $n \to \infty$.*
(ii) *If $\hat{r} = \alpha^{-1}$, then (4.2) is stable, but not asymptotically stable.*
(iii) *If $\hat{r} > \alpha^{-1}$, then (4.2) is not stable.*

Proof. Let p_1 be the geometric multiplicity of the zero eigenvalue $\lambda_1(A) = 0$. Analyzing (4.8) and (4.9) we can see that $\mathcal{L}\{y\}(z)$ contains (in addition to terms corresponding to $\lambda_i(A) \in \mathcal{S}_{\alpha,h}$ discussed in the proof of Proposition 4.10) the power functions $z^{-\alpha}, \ldots, z^{-\hat{r}\alpha}$. Lemma 1.38 (i) implies that the eigenvalue $\lambda_1(A) = 0$ contributes to the form of $y(t_n)$ by terms $\hat{h}_{\alpha-1}(t_n, 0), \ldots, \hat{h}_{\hat{r}\alpha-1}(t_n, 0)$. All the assertions of Theorem 4.15 now result from the limit property of power functions

$$\lim_{n\to\infty} \hat{h}_\beta(t_n, 0) = \lim_{n\to\infty} \frac{h^\beta n^\beta}{\Gamma(\beta+1)} = \begin{cases} 0\,, & \beta < 0\,, \\ 1\,, & \beta = 0\,, \\ \infty\,, & \beta > 0 \end{cases}$$

following from the asymptotic expansion (1.29). □

Remark 4.16. The case (i) never occurs when $\alpha = 1$. Similarly, the case (ii) may occur only when α is reciprocal of a positive integer. Finally, if $\hat{r} = 1$, i.e. when algebraic and geometric multiplicities of the zero eigenvalue are equal, then (4.2) is asymptotically stable for all $\alpha \in (0, 1)$ and $y(t_n) = O(n^{\alpha-1})$ as $n \to \infty$ for any solution $y(t_n)$ of (4.2).

5 A possible extension of fractional calculus to general time scales

In this chapter we abandon the study of FdEs on time scales with linear graininess and turn our attention to a wider problem, namely establishing of the fractional calculus in the frame of the time scales theory. As outlined in Chapter 1, there were recently some attempts to resolve this issue.

The approach presented in [10] for the case of the delta calculus is not entirely general since it is applicable exclusively on time scales, where the Laplace transform can be utilized (see Remark 1.34). The suggested definition of the fractional integral rewritten for the nabla case takes the form (in our notation)

$$ {}_0\nabla^{-\gamma} f(t) = \mathcal{L}^{-1}\{z^{-\gamma}\mathcal{L}\{f\}(z)\}(t)\,. \tag{5.1}$$

Obviously, for further calculations it requires to perform an inverse Laplace transform, which may not be an easy task.

In [4] the author claims, among others, to develop the nabla fractional calculus on time scales, but in this context it concerns only a formal introduction of a fractional integral of order larger than one (special case of (1.38)). However, there are stated (without any deeper discussion) some conditions which should be satisfied by the power functions of non-negative orders. Namely, the power functions are introduced as coordinatewise ld-continuous functions $\hat{h}_\beta : \mathbb{T}\times\mathbb{T}\to\mathbb{R}$ ($\beta \geq 0$) such that

$$\hat{h}_{\beta+1}(t,s) = \int_s^t \hat{h}_\beta(\tau,s)\nabla\tau\,, \quad \hat{h}_0(t,s) = 1 \tag{5.2}$$

for all $s,t\in\mathbb{T}$ provided $\mathbb{T}_\kappa = \mathbb{T}$. Furthermore, the equation

$$\int_s^t \hat{h}_\eta(t,\rho(\tau))\hat{h}_\beta(\tau,s)\nabla\tau = \hat{h}_{\eta+\beta+1}(t,s) \tag{5.3}$$

is assumed to be valid for $\eta,\beta > 0$ and for all $s,t\in\mathbb{T}$ such that $s\leq t$.

Our research, performed in the previous chapters, motivates us to improve this approach and provide some new perspectives, relations and comments on this matter. In particular, we give arguments for the essential importance of (5.3) and propose its extension enabling us to define fractional integrals of all positive orders. Further, we outline a connection between this approach and (5.1).

The reasoning

As discussed in Subsection 1.2.3, any key relation nor property determining the power functions on an arbitrary time scale has not been properly established yet. We mentioned the exponential function and the polynomials as examples of time scales introductions of the particular functions. Both the exponential function and polynomials belong to the fundamental elements of the classical mathematical analysis due to their characteristic properties. The corresponding time scales generalizations are required to play an analogous role, therefore they are introduced via these properties, i.e. the exponential function as the solution of a certain initial value problem (see (1.21)) and the polynomials as the result of a repeated integration of the unit function (see (1.22), (1.23)). In the case of power functions, it is questionable to determine their necessary characteristic property in the frame of classical analysis. On the other hand, the power functions form the very nature of the fractional calculus. Hence, we believe that the unifying principle for power functions should be derived from their role in fractional calculus.

Obviously, the importance of power functions in fractional calculus originates in its presence in the formula for fractional integral (1.38). However, this fact itself cannot serve as a base for a definition of power functions. In many ways, power functions in fractional calculus supply the position of polynomials in classical mathematical analysis. Thus, it seems reasonable to expect that the key relation determining the characteristic features of power functions is given by a generalization of (1.22), (1.23). While the former relation remains unchanged, the latter one is transformed into

$$_s\nabla^{-\gamma}\hat{h}_\beta(t,s) = \hat{h}_{\beta+\gamma}(t,s)\,,$$

i.e.

$$\hat{h}_{\beta+\gamma}(t,s) = \int_s^t \hat{h}_{\gamma-1}(t,\rho(\tau))\hat{h}_\beta(\tau,s)\nabla\tau\,, \quad \beta > -1\,,\ \gamma > 0\,. \tag{5.4}$$

Obviously, this formula coincides with (5.3) with wider range of parameters $\eta, \beta > -1$. However, (5.4) is not introduced just formally. If we analyze basic relations of fractional calculus in both continuous and (q,h)-calculus versions, we find that (5.4) often poses as a unifying element of the corresponding proofs. Indeed, the validity of (5.4) induces, e.g. the unitary form of the composition rules (see Remark 1.51). It is also responsible for the special properties of the time scale Mittag-Leffler functions defined by

$$E^{s,\lambda}_{\eta,\beta}(t) = \sum_{k=0}^{\infty} \lambda^k \hat{h}_{\eta k+\beta-1}(t,s)$$

provided the series converges (compare with (1.11) and Definition 2.10). In particular, it enables us to introduce a time scale analogy of Theorem 2.11 and its subsequent results. Apart from the fractional calculus viewpoint, functions satisfying (5.4) automatically meet other demands naturally put on power functions. In particular, it is easy to show that polynomials are a special case of such the power functions and that $(5.2)_1$ and its differential counterpart $\nabla\hat{h}_\beta(t,s) = \hat{h}_{\beta-1}(t,s)$ $(\beta > 0)$ hold.

An introduction of the power functions of negative orders yields a few curious consequences regarding the time scales theory. In particular, the power functions $\hat{h}_{-\gamma}(t,s)$

$(0<\gamma<1)$ have the domain (with respect to t) $(s,\infty)_{\mathbb{T}}$, i.e. the value of $\hat{h}_\beta(s,s)$ is not a real number, but they have to be integrable on $[s,\infty)_{\mathbb{T}}$. Indeed, the convolution relation (5.4) supplied with $\hat{h}_0(t,s)=1$ $(s,t\in\mathbb{T})$ gives

$$\int_s^s \hat{h}_{\gamma-1}(s,\rho(\tau))\hat{h}_{-\gamma}(\tau,s)\nabla\tau = 1\,. \tag{5.5}$$

This formula cannot be correct for $\hat{h}_{-\gamma}(s,s)\in\mathbb{R}$, because Definition 1.11 would imply the zero result. On the other hand the power functions of positive orders can be written as $\hat{h}_\beta(t,s)=\int_s^t \hat{h}_{\beta-1}(\tau,s)\nabla\tau$, therefore the negative-order power functions have to be integrable. Consequently, it is impossible to determine the values $\hat{h}_\beta(s,s)$ $(\beta>0)$. Thus, we add another natural condition for the power functions, namely

$$\hat{h}_\beta(s,s)=\int_s^s \hat{h}_{\beta-1}(\tau,s)\nabla\tau = 0\,,\quad 0<\beta<1\,. \tag{5.6}$$

To the author's knowledge, functions satisfying (5.5) and (5.6) are not well-established in the time scales theory yet. Nevertheless, this proposal agrees with features of $\frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)}$ on $\mathbb{T}=\mathbb{R}$ and is closely related to the work presented in [38], where a similar behaviour was studied at the limit point of $\mathbb{T}=q^{\mathbb{Z}}$. It seems that our considerations lead to an introduction of a class of singular functions in the frame of the time scales theory, but their further investigation exceeds the range of this thesis.

The Laplace transform

In [10], the power functions $\hat{h}_\beta(t,s)$ were proposed as the inverse Laplace transform of $z^{-\beta-1}$. We show that our definition implies the same result at least for $\beta\in\mathbb{Q}$, $\beta>-1$. For the sake of simplicity we demonstrate it only for functions of the type $\hat{h}_{\frac{1}{w}-1}(t,0)$ $(w\in\mathbb{Z}^+$ and $w\geq 2)$. Let $\mathbb{T}$ be such that the Laplace transform can be applied. The previous investigation enables us to write

$$\big(\underbrace{\hat{h}_{\frac{1}{w}-1}*\cdots*\hat{h}_{\frac{1}{w}-1}}_{w\times}\big)(t,0)=1\,.$$

Utilizing $\mathcal{L}\{1\}(z)=z^{-1}$ (see Proposition 1.36 (i)) and Theorem 1.37, we apply the Laplace transform on this relation and arrive at

$$\big(\mathcal{L}\{\hat{h}_{\frac{1}{w}-1}(\cdot,0)\}(z)\big)^w = z^{-1}$$

which yields $\mathcal{L}\{\hat{h}_{\frac{1}{w}-1}(\cdot,0)\}(z)=z^{-\frac{1}{w}}$. In virtue of this relation we can construct the Laplace transform of every power function of rational order and arrive at

$$\mathcal{L}\{\hat{h}_\beta(\cdot,0)\}(z)=z^{-\beta-1}\,,\quad \beta>-1$$

which agrees with [10].

The final proposal and additional comments

Based on the former discussion, we suggest to define the power functions $\hat{h}_\beta(t,s)$ $(\beta > -1)$ in the frame of the time scales theory as a family of functions satisfying

$$(\hat{h}_\beta * \hat{h}_\gamma)(t,s) = \hat{h}_{\beta+\gamma+1}(t,s)\,, \quad t \geq s\,,\ \beta,\gamma > -1\,, \tag{5.7}$$

$$\hat{h}_0(t,s) = 1\,, \quad t \geq s\,, \tag{5.8}$$

$$\hat{h}_\beta(t,t) = 0\,, \quad 0 < \beta < 1\,, \tag{5.9}$$

where $s,t \in \mathbb{T}$ and $(\hat{h}_\beta * \hat{h}_\gamma)(t,s) = \int_s^t \hat{h}_\beta(t,\rho(\tau))\hat{h}_\gamma(\tau,s)\nabla\tau$ by extension of Definition 1.32.

It was outlined above that the system of the conditions (5.7)-(5.9) implies many properties and assertions regarding the fractional calculus as well as basic properties of power functions themselves. Moreover, these conditions are consistent with the definitions presented in [4, 10].

This proposal provides many directions for the future research. Besides a construction of precise proofs of various properties and finding power functions on particular time scales, it is especially important to perform an analysis of existence and uniqueness for the system of conditions (5.7)-(5.9). The introduction of power functions of negative orders opens the discussion on the notion of singular functions on time scales. Finally, establishing of the set of conditions satisfied by power functions on every time scale brings a possibility to incorporate entirely the fractional calculus into the time scales theory.

Conclusions

This doctoral thesis concerns with the fractional calculus on time scales, in particular with the FdEs on the time scale $\mathbb{T}_{(q,h)}$ and its special cases.

The necessary theoretical background, such as basics of continuous fractional calculus, the time scales theory and discrete fractional calculus, is summarized in Chapter 1. It also contains some original preliminary results regarding the power functions in (q,h)-calculus, the h-Laplace transform and properties of fractional operators on the time scale $\mathbb{T}_{(q,h)}$ (we especially refer to (q,h)-version of the power rule established in Lemma 1.48).

Author's main results are presented in Chapters 2-4. The contributions to the field can be summarized into the following points:

- **Basic theory of linear FdEs on $\mathbb{T}_{(q,h)}$** - Basic properties were introduced for a quite general linear initial value problem. In particular, the existence and uniqueness was discussed (Theorem 2.4) and the form of a general solution was given (Theorem 2.8).

- **Eigenfunctions of the fractional difference operator on $\mathbb{T}_{(q,h)}$** - The (q,h)-version of the Mittag-Leffler function was established (Definition 2.10) which enabled to introduce eigenfunctions of the Riemann-Liouville fractional difference operator (Corollary 2.13). Their relation to the solution of a linear two-term FdE was discussed (Theorem 2.15).

- **Qualitative theory** - The stability and asymptotic properties of a scalar linear two-term FdE on $\mathbb{T}=\mathbb{Z}$ were investigated employing a connection to the Volterra difference equations theory (Theorem 3.10 and Corollary 3.17, respectively). A vector analogue of these assertions considering the underlying set $\mathbb{T}=h\mathbb{Z}$ was proven via the properties of h-Laplace transform of the solution (Theorem 4.11).

The thesis is concluded by Chapter 5 which outlines a possible way of an extension of the fractional calculus to the time scales theory. This proposal implies some interesting consequences regarding the time scales theory and generates many other open questions providing many challenges for the future research.

We believe that the main results of this doctoral thesis upgraded the theory of discrete fractional calculus in several directions and thus contributed to its further development. In particular, the foundations of the theory of FdEs in (q,h)-calculus were established and the qualitative theory of FdEs in h-calculus was extended. Moreover, there were brought up some ideas contributing to discussions on some open problems in the theory of Volterra difference equations and the time scales theory.

Apart from the possible usage of our results in further theoretical development, our work can be employed in numerical analysis of FDEs and therefore, by an appropriate extension, used in many applications. It was pointed out that our approach to discrete fractional h-calculus can be taken as a discretization resulting in the backward fractional Euler method. Hence, especially the qualitative investigations of the vector initial value problem on $h\mathbb{Z}$ is, among others, closely related to the numerical analysis of the corresponding continuous initial value problem.

Bibliography

[1] AGARWAL, Ravi P. *Certain fractional q-integrals and q-derivatives.* Mathematical Proceedings of the Cambridge Philosophical Society, 1969, vol. **66**, no. 2, pp. 365-370. doi:10.1017/S0305004100045060.

[2] AHRENDT, Chris R. *Properties of the Generalized Laplace Transform and Transport Partial Dynamic Equation on Time Scales.* Doctoral thesis, University of Nebraska-Lincoln, 2010, 113 pages, supervisors: L. Erbe, A. Peterson.

[3] AKIN-BOHNER, Elvan; BOHNER, Martin. *Exponential functions and Laplace transforms for alpha derivatives.* In Proceedings of the Sixth International Conference on Difference Equations. Augsburg, Germany, 2001, pp. 231-238. ISBN 041-531675-8.

[4] ANASTASSIOU, George A. *Foundations of nabla fractional calculus on time scales and inequalities.* Computers and Mathematics with Applications, 2010, vol. **59**, pp. 3750-3762. doi:10.1016/j.camwa.2010.03.072.

[5] ANDREWS, George E.; ASKEY, Richard; ROY, Ranjan. *Special Functions.* 1st ed. USA: Cambridge University Press, 1999. ISBN 0-521-62321-9.

[6] APPLEBY, John A. D.; GYÖRI, Istvan; REYNOLDS, David W. *On exact convergence rates for solutions of linear systems of Volterra difference equations,* Journal of Difference Equations and Applications, 2006, vol. **12**, no. 12, pp. 1257-1275. doi:10.1080/10236190600986594.

[7] ATICI, Ferhan M.; ELOE, Paul W. *A transform method in discrete fractional calculus.* International Journal of Difference Equations, 2007, vol. **2**, no. 2, pp. 165-176. ISSN 0973-6069.

[8] ATICI, Ferhan M.; ELOE, Paul W. *Discrete fractional calculus with the nabla operator.* Electronic Journal of Qualitative Theory of Differential Equations, 2009, Spec. Ed. I, no. 3, pp. 1-12. ISSN 1417-3875.

[9] ATICI, Ferhan M.; ELOE, Paul W. *Linear systems of fractional nabla difference equations,* Rocky Mountain Journal of Mathematics, 2011, vol. **41**, no. 2, pp. 353-370. doi:10.1216/RMJ-2011-41-2-353.

[10] BASTOS, Nuno R. O. *Fractional Calculus on Time Scales.* Doctoral thesis, University of Aveiro, 2012, 119 pages, supervisor: D. F. M. Torres.

[11] BOHNER, Martin; PETERSON, Allan. *Dynamic Equations on Time Scales: An Introduction with Applications.* 1st ed. Boston: Birkhäuser, 2001. ISBN 0-8176-4225-0.

[12] BOHNER, Martin; PETERSON, Allan. *Advances in Dynamic Equations on Time Scales.* 1st ed. Boston: Birkhäuser, 2003. ISBN 0-8176-4293-5.

[13] BOHNER, Martin; GUSEINOV, Gusein Sh. *The convolution on time scales.* Abstract and Applied Analysis, 2007, vol. **2007**, 24 pages. doi:10.1155/2007/58373.

[14] ČERMÁK, Jan; KISELA, Tomáš. *Note on a discretization of a linear fractional differential equation.* Mathematica Bohemica, 2010, vol. **135**, no. 2, pp. 179-188. ISSN 0862-7959.

[15] ČERMÁK, Jan; KISELA, Tomáš; NECHVÁTAL, Luděk. *Discrete Mittag-Leffler functions in linear fractional difference equations.* Abstract and Applied Analysis, 2011, vol. **2011** (Special Issue), 21 pages, 2011. doi:10.1155/2011/565067.

[16] ČERMÁK, Jan; KISELA, Tomáš; NECHVÁTAL, Luděk. *Stability and asymptotic properties of a linear fractional difference equation,* Advances in Difference Equations, 2012, *to appear.*

[17] ČERMÁK, Jan; KISELA, Tomáš; NECHVÁTAL, Luděk. *The qualitative investigation of linear fractional difference systems.* 2012, *submitted.*

[18] ČERMÁK, Jan; NECHVÁTAL, Luděk. *On (q,h)-analogue of fractional calculus,* Journal of Nonlinear Mathematical Physics, 2010, vol. **17**, no. 1, pp. 1-18. ISSN 1402-9251.

[19] DAVIS, John M.; GRAVAGNE, Ian A.; JACKSON, Billy J.; MARKS II, Robert J.; RAMOS, Alice A. *The Laplace transform on time scales revisited.* Journal of Mathematical Analysis and Applications, 2007, vol. **332**, no. 2, pp. 1291-1307. doi:10.1016/j.jmaa.2006.10.089.

[20] DESOER, Charles A.; VIDYASAGAR, M. *Feedback Systems: Input-Output Properties.* 1st ed. New York: Academic Press, 1975. ISBN 0-12-212050-7.

[21] DÍAZ, Joaquín B.; OSLER, Thomas S. *Differences of fractional order.* Mathematics of Computation, 1974, vol. **28**, no. 125, pp. 185-202. doi:10.1090/S0025-5718-1974-0346352-5.

[22] DIETHELM, Kai; FORD, Neville J.; FREED, Alan D.; LUCHKO, Yury. *Algorithms for the fractional calculus: A selection of numerical methods.* Computer Methods in Applied Mechanics and Engineering, 2005, vol. **194**, no. 6-8, pp. 743-773. doi:10.1016/j.cma.2004.06.006.

[23] ELAYDI, Saber. *An Introduction to Difference Equations.* 3rd ed. New York: Springer Science+Business Media, Inc., 2005. ISBN 0-387-23059-9.

[24] ELAYDI, Saber; MESSINA, Eleonora; VECCHIO, Antonia. *On the asymptotic stability of linear Volterra difference equations of convolution type.* Journal of Difference Equations nad Applications, 2007, vol. **13**, no. 12, pp. 1079-1084. doi:10.1080/10236190701264529.

[25] ELAYDI, Saber; MURAKAMI, Satoru. *Asymptotic stability versus exponential stability in linear Volterra difference equations of convolution type.* Journal of Difference Equations and Applications, 1996, vol. **2**, no. 4, pp. 401-410. doi:10.1080/10236199608808074.

[26] GAMELIN, Theodore W. *Complex Analysis.* 1st ed. New York: Springer Science+Business Media, Inc., 2001. ISBN 0-387-95069-9.

[27] GRAY, Henry L.; ZHANG, Nien F. *On a new definition of the fractional difference.* Mathematics of Computation, 1988, vol. **50**, no. 182, pp. 513-529. doi:10.1090/S0025-5718-1988-0929549-2.

[28] HEYMANS, Nicole; PODLUBNÝ, Igor. *Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives.* Rheologica Acta, 2005, vol. **45**, no. 5, pp. 765-771. doi:10.1007/s00397-005-0043-5.

[29] HOLM, Michael T. *The Theory of Discrete Fractional Calculus: Development and Application.* Doctoral thesis, University of Nebraska-Lincoln, 2011, 116 pages, supervisors: L. Erbe, A. Peterson.

[30] KAC, Victor; CHEUNG, Pokman. *Quantum Calculus.* 1st ed. New York: Springer-Verlag New York, Inc., 2002. ISBN 0-387-95341-8.

[31] KILBAS, Anatoly A.; SRIVASTAVA, Hari M.; TRUJILLO, Juan J. *Theory and Applications of Fractional Differential Equations.* 1st ed. Netherlands: Elsevier B. V., 2006. ISBN 0-444-51832-0.

[32] KISELA, Tomáš. *Application of the fractional calculus: On a discretization of fractional diffusion equation in one dimension.* Communications, 2010, vol. **12**, no. 1, pp. 5-11. ISSN 1335-4205.

[33] KISELA, Tomáš. *On a discretization of the initial value problem for a linear nonhomogeneous fractional differential equation.* In Proceedings of the 4th IFAC Workshop on Fractional Differentiation and Its Applications. Badajoz, Spain, 2010, pp. 264-269. ISBN 978-80-553-0487-8.

[34] KNOPP, Konrad. *Infinite Sequences and Series.* 1st ed. New York: Dover Publications, Inc., 1956. ISBN 486-60153-6.

[35] KOLMANOVSKII, V. B.; CASTELLANOS-VELASCO, E.; TORRES-MUÑOZ, J. A. *A survey: Stability and boundedness of Volterra difference equations.* Nonlinear Analysis: Theory, Methods & Applications, 2003, vol. **53**, no. 7-8, pp. 861-928. doi:10.1016/S0362-546X(03)00021-X.

[36] KUCZMA, Marek; CHOCZEWSKI, Bogdan; GER, Roman. *Iterative Functional Equations.* 1st ed. Cambridge University Press, 1990. ISBN 0-521-35561-3.

[37] LUBICH, Christian. *Discretized fractional calculus.* SIAM Journal on Mathematical Analysis, 1986, vol. **17**, no. 3, pp. 704-719. doi:10.1137/0517050.

[38] MANSOUR, Zeinab S. I. *Linear sequential q-difference equations of fractional order.* Fractional Calculus & Applied Analysis, 2009, vol. **12**, no. 2, pp. 159-178. ISSN 1311-0454.

[39] MILLER, Kenneth S.; ROSS, Bertram. *An Introduction to the Fractional Calculus and Fractional Differential Equations.* 1st ed. New York: John Wiley & Sons, Inc., 1993. ISBN 0-471-58884-9.

[40] NAGAI, Atsushi. *On a certain fractional q-difference and its eigen function,* Journal of Nonlinear Mathematical Physics, 2003, vol. **10**, no. 2, pp. 133-142. doi:10.2991/jnmp.2003.10.s2.12.

[41] OLDHAM, Keith B.; SPANIER, Jerome. *The Fractional Calculus.* 1st ed. USA: Academic Press, Inc., 1974. ISBN 0-12-525550-0.

[42] OLDHAM, Keith B.; MYLAND, Jan; SPANIER, Jerome. *An Atlas of Functions.* 2nd ed. USA: Springer Science+Business Media, 2009. ISBN 978-0-387-48806-6.

[43] PODLUBNÝ, Igor. *Fractional Differential Equations.* 1st ed. USA: Academic Press, 1999. ISBN 0-12-558840-2.

[44] PODLUBNÝ, Igor.; CHECHKIN, Aleksei V.; ŠKOVRÁNEK, Tomáš; CHEN, YangQuan; JARA, Blas M. V. *Matrix approach to discrete fractional calculus II: Partial fractional differential equations.* Journal of Computational Physics, 2009, vol. **228**, no. 8, pp. 3137-3153. doi:10.1016/j.jcp.2009.01.014.

[45] QIAN, Deliang; LI, Changpin; AGARWAL, Ravi P.; WONG, Patricia J. Y. *Stability analysis of fractional differential system with Riemann-Liouville derivative.* Mathematical and Computer Modelling, 2010, vol. **52**, no. 5-6, pp. 862-874. doi:10.1016/j.mcm.2010.05.016.

[46] SCHIFF, Joel L. *The Laplace Transform: Theory and Applications.* 1st ed. USA: Springer-Verlag New York, Inc., 1999. ISBN 0-387-98698-7.

List of Symbols

$\mathbb{Z}$	the set of integers
$\mathbb{Z}^+$	the set of positive integers
$\mathbb{Z}_0^+$	the set of non-negative integers
$\mathbb{Z}_0^-$	the set of non-positive integers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$B(z_0, r)$	an open ball with center $z_0 \in \mathbb{C}$ and radius r (see p. 19)
$\lambda(A)$	an eigenvalue of the matrix A
$\mathbb{T}$	a general time scale (see p. 8)
$\mathbb{T}_\kappa$	the truncation of a time scale $\mathbb{T}$ (see p. 9)
$\mathbb{T}_{(q,h)}$	the underlying set of (q, h)-calculus (see p. 8)
$\widetilde{\mathbb{T}}^a_{(q,h)}$	the restriction of $\mathbb{T}_{(q,h)}$ (see p. 27)
$h\mathbb{Z}$	the set $\{\dots, -2h, h, 0, h, 2h, \dots\}$ (see p. 8)
$q^{\mathbb{Z}}$	the set $\{\dots, q^{-2}, q^{-1}, 1, q, q^2, \dots\}$ (see p. 8)
σ	the forward jump operator (see Definition 1.6)
ρ	the backward jump operator (see Definition 1.6)
μ	the forward graininess function (see Definition 1.8)
ν	the backward graininess function (see Definition 1.8)
$\mathrm{Arg}(z)$	the principal argument of $z \in \mathbb{C}$
$\lceil \cdot \rceil$	the ceiling function (see p. 5)
$\Gamma(z)$	the Euler Gamma function (see p. 4)
$E_{\eta,\beta}$	the classical Mittag-Leffler function (see (1.8))
$E^{a,\lambda}_{\eta,\beta}$	the discrete Mittag-Leffler function (see Definition 2.10)
$E^{A}_{\eta,\beta}$	the matrix Mittag-Leffler function (see p. 50)
$\hat{h}_\beta$	time scales power function of order β (see Definition 1.26)
$\hat{e}_f$	time scales exponential function (see (1.21))
ℓ^1	the space of summable sequences
$f * g$	the convolution of f and g (see Definition 1.32)
$f \sim g$	asymptotic equivalence of f and g (see p. 16)
${}_a\mathbf{D}^\alpha$	fractional derivative or integral (see Definitions 1.1 and 1.3)
${}_a\nabla^\alpha$	time scales fractional derivative or integral (see p. 20-22)
${}_a\nabla^\alpha_{(q,h)}$	fractional difference or sum in (q, h)-calculus (see Definitions 1.43 and 1.45)
${}_a\nabla^\alpha_h$	fractional difference or sum in h-calculus (see p. 22)
$\mathcal{L}\{f\}(z)$	the generalized Laplace transform of f (see Definition 1.33)
$\mathcal{Z}\{f\}(z)$	the $\mathcal{Z}$-transform of f (p. 38)