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Asymptotic properties of solutions of the second-order discrete Emden-Fowler equation

Asymptotické vlastnosti řešení diskrétní Emden-Fowlerovy rovnice druhého řádu

Theoretical Electrical Engineering

Short version of PhD Thesis

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1 Introduction

Classical differential equations are widely used in different processes. For example, the input continuous signal of the linear system $x(t)$ and the corresponding output signal $y(t)$ can be connected by some differential equation. But if we want to replace a continuous variable t with a discrete one, it leads to the replacement of the differential equation with a difference equation.

To analyse difference equations, we can also use different analytical methods, most of them using approaches similar to those of the classical differential equation. We can also use numerical methods of solving, obtaining a result in the form of a numerical sequence, therefore, the difference equation in this case is perceived as an algorithm for the functioning of a discrete system for which a suitable computer programs can be devised.

We also mention the contribution of the mathematicians Bohner M., Georgiev, S.G. and Peterson A.C [7], [8] and [9] to the creation of a theory that combines both classical calculus and the theory of difference equations, expanding the scope of application to continuous scales, as well as allowing us to consider both more complex discrete scales or a combination of discrete-continuous time scales.

In the doctoral thesis we discuss the asymptotic properties of the Emden-Fowler discrete equation. This equation is an extension to the theory of difference equation of a well-known Lane-Emden-Fowler differential equation, which has a great deal of applications in physics, cosmology, meteorology and chemistry. In [16], the form of this equation was

$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} + \beta^2 u^n = 0, \tag{1.1}$$

where r is the radius of a polytropic gas sphere, $n = 1/(k-1)$, with k being the polytropic index and β some physical constant.

The change of variables $u = y/r$ transforms (1.1) into the following equation

$$y'' + \beta^2 r^{1-n} y^n = 0.$$

Now we get the form that is often used in mathematical literature:

$$y'' + x^\sigma |y|^{k-1} y = 0,$$

where k and σ are constants. Later, this equation was generalized for the case of n-th order differential equation

$$y^{(n)} + p(x)|y|^k \mathrm{sgn} y = 0, \tag{1.2}$$

where $n > 2$ is an integer, $p(x)$ is a continuous function and k is a constant.

Different properties of the solutions of Emden-Fowler differential equations were investigated by many authors. The R.Bellman's monograph [5] had a great influence on the investigation of the Emden-Fowler equations, where he discussed the asymptotic properties of the solutions tending to infinity. F.V.Atkinson in [4] also made a significant contribution to the theory of Emden-Fowler equations. The list of works devoted to the Emden-Fowler type equations is very wide, we will mention some of them: H. Fowler [18], I.T. Kiguradze, T.A. Chanturia [28], V.A. Kondratev, V.S. Samovol [29], I.V. Astashova [3], H. Goenner, P. Havas [19], S.C. Mancas, H.C. Rost [30], C.M. Khalique [22] and P. Guha [21].

1.1 The current state

In previous chapter, we have already mentioned that there are many papers and books on the Emden-Fowler differential equation. However, turning our attention to the discrete case, we see that there are not so many articles about this type of equation. We can refer to papers by L. Erbe, J. Baoguo and J. Peterson [17] dealing with non-oscillatory solutions of Emden-Fowler type discrete equations providing asymptotic properties of a similar equation on time scales.

V. Kharkov in [23], [24], [25] has also discussed the asymptotic properties of the equation

$$\Delta^2 y_n = \alpha p_n |y_{n+1}|^\sigma \operatorname{sgn} y_{n+1},$$

where $\alpha \in \{-1.1\}$, $\sigma \in \mathbb{R} \setminus \{0, 1\}$ and the sequence p_n satisfies the following condition

$$\lim_{n\to+\infty} \frac{n\Delta p_n}{p_n} = k, \quad k \in \mathbb{R} \setminus \{-2, -1-\sigma\}.$$

In the thesis we will discuss the asymptotic properties of the solutions to the another discrete equivalent of the Emden-Fowler equation. In our case, let k_0 be a natural number. By $\mathbb{N}(k_0)$ we denote the set of all natural numbers greater than or equal to k_0, that is,

$$\mathbb{N}(k_0) := \{k_0, k_0 + 1, ...\}.$$

We will study the asymptotic behaviour of the solutions of a second-order non-linear discrete equation of Emden-Fowler type

$$\Delta u(k) \pm k^\alpha u^m(k) = 0, \tag{1.3}$$

where $u : \mathbb{N}(k_0) \to \mathbb{R}$ is an unknown solution, $\Delta u(k)$ is its first-order forward difference, i.e., $\Delta u(k) = u(k+1) - u(k)$, $\Delta^2(k)$ is its second-order forward difference, i.e., $\Delta^2 u(k) = \Delta(\Delta u(k)) = u(k+2) - 2u(k+1) + u(k)$, and α, m are real numbers. A function $u = u^* : \mathbb{N}(k_0) \to \mathbb{R}$ is called a solution of equation (1.3) if the equality

$$\Delta^2 u^*(k) \pm k^\alpha (u^*(k))^m = 0$$

holds for every $k \in \mathbb{N}(k_0)$.

Equation (1.3) is a discretization of the classical Emden-Fowler second-order differential equation (we refer, e.g., to [5]) $y'' \pm x^\alpha y^m = 0$, where the second-order derivative is replaced by a second-order forward difference and the continuous independent variable is replaced by a discrete one.

One special case of the discrete Emden-Fowler type equation has been discussed in a recent article by Christianen, M.H.M., Janssen, A.J.E.M., Vlasiou, M., and Zwart, B. [11], which describes the charging process of electric vehicles, considering their random arrivals, their stochastic demand for energy at charging stations, and the characteristics of the electricity distribution network. The equation

$$v_{j+1} - 2v_j + v_{j-1} = k/v_j$$

is considered, where $j = 1, 2, ...$; $v_0 = 1$, $v_1 = 1 + k$ and proving that there exists a solution with "logarithmic" asymptotic behaviour, i.e.

$$v_j \sim j(2k \ln(j))^{1/2},$$

when $j \to \infty$.

1.2 Preliminaries

This section introduces the notation, definitions and theorems used in the thesis.

Definition 1. A function $u_{upp} : \mathbb{B} \to \mathbb{R}$ is said to be an *approximate solution* to equation (1.3) of an order g where $g : \mathbb{N}(k_0) \to \mathbb{R}$ if

$$\lim_{k\to\infty} [\Delta^3 u_{upp}(k) \pm k^\alpha u^n_{upp}(k)]g(k) = 0.$$

If the main term (i.e. the term being asymptotically leading) in $u_{upp}(k)$ is a power-type function, we say that it is a *power-type* approximate solution.

Definition 2. We say that a function $x(k)$ is of order $O(y(k))$ if there exists a constant K, such that

$$|x(k)| \le |M(y(k))|$$

on $\mathbb{N}(k_0)$. We use the shorter notation $O(y(k))$.

Definition 3. We say that a function $x(k)$ is of order $o(y(k))$ if $y(k) \neq 0$ for all sufficiently large $k \in \mathbb{N}(k_0)$ and

$$\lim_{k\to\infty} \frac{x(k)}{y(k)} = 0.$$

This property is more simply written as $x(k) = o(y(k))$.

In computations below, we will also use the following modification of the Landau order symbol big "O".

Definition 4. Let $f : \mathbb{N}(k_0) \to \mathbb{R}$, $g : \mathbb{N}(k_0) \to (0, \infty)$. We write $f = O^+(g)$ if there exists an index $k_1 \ge k_0$ such that inequality

$$|f(k)| \leqslant g(k), \quad \forall k \in \mathbb{N}(k_1)$$

holds.

Definition 5. A solution of the equation (1.2) is called a *blow-up* one if there exists some point $x_0 \in \mathbb{R}$, such that

$$\lim_{x\to x_0 - 0} y(x) = \infty.$$

1.2.1 Binomial series

In the proof of the main results, we use the following formula for the decomposition of a binom into a "binomial series".

Let $r \in \mathbb{R}$, $p \in \mathbb{R}$, $k \in \mathbb{N}(k_0)$ and let

$$\left|\frac{r}{k}\right| < 1.$$

Then,

$$\left(1 + \frac{r}{k}\right)^p = 1 + \binom{p}{1}\frac{r}{k} + \binom{p}{2}\frac{r^2}{k^2} + \binom{p}{3}\frac{r^3}{k^3} + ... + \binom{p}{l}\frac{r^l}{k^l} + ... \tag{1.4}$$

where

$$\binom{p}{l} := p(p-1)...(p-l+1)\frac{1}{l!}.$$

1.2.2 Discrete retract principle

In the proofs of the results on the asymptotic behaviour of solutions to equation (1.3), we use an auxiliary apparatus taken from [12, 14] and described below. Consider a system of discrete equations

$$\Delta Y(k) = F(k, Y(k)), \ \ k \in \mathbb{N}(k_0) \tag{1.5}$$

where $Y = (Y_0, \ldots, Y_{n-1})^T$ and

$$F(k, Y) = (F_1(k, Y), \ldots, F_n(k, Y))^T \colon \mathbb{N}(k_0) \times \mathbb{R}^n \to \mathbb{R}^n. \tag{1.6}$$

A solution $Y = Y(k)$ of system (1.5) is defined as a function $Y \colon \mathbb{N}(k_0) \to \mathbb{R}^n$ satisfying (1.5) for each $k \in \mathbb{N}(k_0)$. The initial problem

$$Y(k_0) = Y^0 = (Y_0^0, \ldots, Y_{n-1}^0)^T \in \mathbb{R}^n$$

defines a unique solution to (1.5). Obviously, if $F(k, Y)$ is continuous with respect to Y, then the initial problem (1.5), (1.6) defines a unique solution $Y = Y(k_0, Y^0)(k)$, where $Y(k_0, Y^0)$ indicates a dependence of the solution on the initial point (k_0, Y^0), which depends continuously on the value Y^0. Let $b_i, c_i \colon \mathbb{N}(k_0) \to \mathbb{R}$, $i = 1, ..., n$ be given functions satisfying

$$b_i(k) < c_i(k), \ \ k \in \mathbb{N}(k_0), \ \ \ i = 1, ..., n. \tag{1.7}$$

Define auxiliary functions $B_i, C_i \colon \mathbb{N}(k_0) \times \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, n$ as

$$B_i(k, Y) := -Y_{i-1} + b_i(k), \quad C_i(k, Y) := Y_{i-1} - c_i(k) \tag{1.8}$$

and auxiliary sets

$$\begin{aligned}\Omega_B^i := \{(k, Y) \colon k \in \mathbb{N}(k_0), B_i(k, Y) = 0, B_j(k, Y) \le 0, C_p(k, Y) \le 0, \\ \forall\ j, p = 1, \ldots, n, \ j \neq i\},\end{aligned} \tag{1.9}$$

$$\begin{aligned}\Omega_C^i := \{(k, Y) \colon k \in \mathbb{N}(k_0), C_i(k, Y) = 0, B_j(k, Y) \le 0, C_p(k, Y) \le 0, \\ \forall\ j, p = 1, \ldots, n, \ p \neq i\}\end{aligned} \tag{1.10}$$

where $i = 1, ..., n$.

Playing a crucial role in the proofs and being suitable for applications, the following lemma is a slight modification of [12, Theorem 1] (see [14, Theorem 2] also).

1.2.3 Auxiliary result of a Liapunov type

A result formulated below is proved in [13] by Liapunov-like reasonings.

Definition 6. The set Ω is called the *regular polyfacial set* with respect to the discrete system (1.5) if

$$b_i(k+1) - b_i(k) < F_i(k, Y) < c_i(k+1) - b_i(k), \tag{1.11}$$

for every $i = 1, \ldots, n$ and every $(k, Y) \in \Omega_B^i$ and if

$$b_i(k+1) - c_i(k) < F_i(k, Y) < c_i(k+1) - c_i(k), \tag{1.12}$$

for every $i = 1, \ldots, n$ and every $(k, Y) \in \Omega_C^i$.

To formulate the following theorem, we need to define sets

$$\Omega(k) = \{(k, Y), Y = (Y_1, \ldots, Y_n) \in \mathbb{R}^n, b_i(k) < Y_i < c_i(k), i = 1, \ldots, n\},$$

$$\Omega_i(k) = \{(Y) \colon Y \in \mathbb{R}, b_i(k) < Y_i < c_i(k), \ \ i = 1, \ldots, n\}.$$

Theorem 1. *[13, Theorem 4] Let* $F \colon \mathbb{N}(k_0) \times \overline{\Omega} \to \mathbb{R}^n$*. Let, moreover,* Ω *be regular with respect to the discrete system* (1.5)*, and let the function*

$$G_i(w) := w + F_i(k, Y_1, \ldots, Y_{i-1}, w, Y_{i+1}, \ldots, Y_n)$$

be monotone on $\overline{\Omega}_i(k)$ *for every fixed* $k \in \mathbb{N}(k_0)$*, each fixed* $i \in \{1, \ldots, n\}$*, and every fixed*

$$(Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n)$$

such that $(k, Y_1, \ldots, Y_{i-1}, w, Y_{i+1}, \ldots, Y_n) \in \Omega$*. Then, every initial problem* $Y(k_0) = Y^*$ *with* $Y^* \in \Omega(k_0)$ *defines the solution* $Y = Y^*(k)$ *of the discrete system* (1.5) *satisfying the relation*

$$Y^*(k) \in \Omega(k)$$

for every $k \in \mathbb{N}(k_0)$*.*

1.2.4 Auxiliary results of an Anti-Liapunov type

Now we formulate a result which is in [12] proved by a retract method sometimes called an Anti-Liapunov method due to the assumptions used being often an opposite to those used when Liapunov method is applied (such an approach goes back to Ważewski, who formulated his topological method formulated for ordinary differential equations). The following theorem is a slight modification of [12, Theorem 1] (see [14, Theorem 2] also).

Theorem 2. *Assume that the function* $F(k, Y)$ *satisfies* (1.5) *and is continuous with respect to* Y*. Let the inequality*

$$F_i(k, Y) < b_i(k + 1) - b_i(k) \tag{1.13}$$

hold for every $i = 1, \ldots, n$ *and every* $(k, Y) \in \Omega^i_B$*. Let, moreover, inequality*

$$F_i(k, Y) > c_i(k + 1) - c_i(k) \tag{1.14}$$

hold for every $i = 1, \ldots, n$ *and every* $(k, Y) \in \Omega^i_C$*. Then, there exists a solution* $Y = Y(k)$*,* $k \in \mathbb{N}(k_0)$ *of system* (1.5) *satisfying the inequalities*

$$b_i(k) < Y_{i-1}(k) < c_i(k)$$

for every $k \in \mathbb{N}(k_0)$ *and* $i = 1, \ldots, n$*.*

2 Preliminary calculations and theorems

2.1 Constructing an asymptotic power-type solution.

In this chapter we will construct an approximate solution to equation (1.3) in a power form.

Let us define

$$s = (\alpha + 2)/(m - 1)\,, \tag{2.1}$$

$$a = [\mp s(s+1)]^{1/(m-1)} \,, \tag{2.2}$$

$$b = (as(s+1))/(s+2-ms). \tag{2.3}$$

Remark 1. We need to assume $m \neq 0$, $m \neq 1$, $s+2 \neq 0$, and $s+2-ms \neq 0$, that is, $m \neq 0$, $m \neq 1$, $\alpha \neq -2$, and $\alpha \neq -2m$.

Remark 2. If in formula (2.2) either the upper variant of sign is in force (i.e. $-$) and $s(s+1) > 0$ or the (2.2) lower variant of sign in force (i.e. $+$) and $s(s+1) < 0$, then the constant m has the form of a ratio m_1/m_2 of relatively prime integers m_1, m_2, and is m_2 is odd, the difference $m_1 - m_2$ is odd as well. If this convention holds, the formula (2.2) defines two or at least one value.

As equation (1.3) splits into two equations, when formulating the results, we assume that a concrete variant is fixed (either with the sign $+$ or with the sign $-$).

Theorem 3. *Let a, b and s be defined by the formulas* (2.1) – (2.3)*. Then, the function*

$$u_{app}(k) \propto a \cdot k^{-s} + b \cdot k^{-(s+1)} \tag{2.4}$$

is an approximate power-type solution of equation (1.3) *of order* $g(k) = k^{s+3}$.

2.2 System of difference equations equivalent to a differential equation

Below, rather than of equation (1.3), we will analyse an equivalent system of two difference equations. This system will be constructed using the below auxiliary transformations

$$u(k) = a \cdot k^{-s} + b \cdot k^{-(s+1)}(1 + Y_0(k)), \tag{2.5}$$

$$\Delta u(k) = \Delta\left(a \cdot k^{-s}\right) + \Delta\left(b \cdot k^{-(s+1)}\right)(1 + Y_1(k)), \tag{2.6}$$

$$\Delta^2 u(k) = \Delta^2\left(a \cdot k^{-s}\right) + \Delta^2\left(b \cdot k^{-(s+1)}\right)(1 + Y_2(k)). \tag{2.7}$$

where s, a and b are defined by formulas (2.1) – (2.3), and $Y_i(k)$, $i = 0, 1, 2$ are new dependent functions. Below, we derive relations connecting them. Recall a useful known formula (we refer, e.g., to [15]), used in computations. If x and y are defined on $\mathbb{N}(k_0)$, then

$$\Delta(x(k)y(k)) = x(k+1)\Delta y(k) + (\Delta x(k))y(k), \quad k \in \mathbb{N}(k_0).$$

Taking the first differences of the left-hand and right-hand sides of (2.5), we derive

$$\Delta u(k) = \Delta\left(a \cdot k^{-s}\right) + b \cdot (k+1)^{-(s+1)}\Delta Y_0(k) + \Delta\left(b \cdot k^{-(s+1)}\right)(1 + Y_0(k)).$$

Comparing the result with (2.6), we get the equation

$$b \cdot (k+1)^{-(s+1)}\Delta Y_0(k) + \Delta\left(b \cdot k^{-(s+1)}\right)(1 + Y_0(k)) = \Delta\left(b \cdot k^{-(s+1)}\right)(1 + Y_1(k)),$$

which is equivalent with

$$\Delta Y_0(k) = (k+1)^{s+1}\Delta\left(k^{-(s+1)}\right)(-Y_0(k) + Y_1(k)). \tag{2.8}$$

Taking the first differences of the left-hand and right-hand sides of (2.6), we obtain

$$\Delta^2 u(k) = \Delta^2\left(a \cdot k^{-s}\right) + \Delta\left(b \cdot (k+1)^{-(s+1)}\right)\Delta Y_1(k) + \Delta^2\left(b \cdot k^{-(s+1)}\right)(1 + Y_1(k)).$$

Comparing the result with (2.7), we get

$$\Delta\left(b\cdot(k+1)^{-(s+1)}\right)\Delta Y_1(k)+\Delta^2\left(b\cdot k^{-(s+1)}\right)(1+Y_1(k))=\Delta^2\left(b\cdot k^{-(s+1)}\right)(1+Y_2(k)),$$

and an equivalent equation is

$$\Delta Y_1(k)=\frac{\Delta^2\left(k^{-(s+1)}\right)}{\Delta\left((k+1)^{-(s+1)}\right)}(-Y_1(k)+Y_2(k)). \tag{2.9}$$

The derived system of difference equations (2.8), (2.9) defines the relationships between $Y_i(k)$, $i=0,1,2$ implied by transformations (2.5)–(2.7). Next, we will get a system equivalent with equation (1.3). To do this, we must express $Y_2(k)$ in (2.9) in terms of $Y_0(k)$ using initial equation (1.3). After some cumbersome calculations we get

$$\Delta Y_0(k)=\left(-\frac{s+1}{k}+O\left(\frac{1}{k^2}\right)\right)(-Y_0(k)+Y_1(k)), \tag{2.10}$$

$$\Delta Y_1(k)=\left(-\frac{s+2}{k}+O\left(\frac{1}{k^2}\right)\right)\left(\frac{ms}{s+2}Y_0(k)-Y_1(k)+O\left(\frac{1}{k}\right)\right). \tag{2.11}$$

3 Power-type asymptotic behaviour in case of constant upper and lower functions

The aim of this chapter is to find conditions for existence of the solution (1.3) with the power-type asymptotic behaviour when Theorem 2 is applied with constant upper and lower functions $b_1(k)$, $b_2(k)$, $c_1(k)$ and $c_2(k)$. We use the approximate power-type solution described by formula (2.4), where s, a and b are defined by formulas(2.1), (2.2) and (2.3). The results of this chapter were published in [1].

We will prove the theorem, formulated below. Here we deal only with the case $s+1>0$.

Theorem 4. *Let $s>-1$, $m\neq 0$ and $m\neq 1$. Assume that there exist positive numbers ε_i, $i=1,\dots,4$, such that either*

$$ms>0,\ \ \varepsilon_3<\varepsilon_1,\ \ \varepsilon_2>\varepsilon_4,\ \ \varepsilon_3>\frac{ms}{s+2}\varepsilon_1,\ \ \varepsilon_4>\frac{ms}{s+2}\varepsilon_2, \tag{3.1}$$

or

$$ms<0,\ \ \varepsilon_3<\varepsilon_1,\ \ \varepsilon_2>\varepsilon_4,\ \ \varepsilon_3>-\frac{ms}{s+2}\varepsilon_2,\ \ \varepsilon_4>-\frac{ms}{s+2}\varepsilon_1. \tag{3.2}$$

Then, for a sufficiently large fixed k_0, there exists a solution $u\colon\mathbb{N}(k_0)\to\mathbb{R}$ of equation (1.3) such that, for every $k\in\mathbb{N}(k_0)$,

$$-\varepsilon_1<\left[u(k)-a\cdot k^{-s}-b\cdot k^{-(s+1)}\right]\left[b\cdot k^{-(s+1)}\right]^{-1}<\varepsilon_2, \tag{3.3}$$

$$-\varepsilon_3<\left[\Delta u(k)-\Delta\left(a\cdot k^{-s}\right)-\Delta\left(b\cdot k^{-(s+1)}\right)\right]\left[\Delta\left(b\cdot k^{-(s+1)}\right)\right]^{-1}<\varepsilon_4, \tag{3.4}$$

$$-\varepsilon_1+O\left(k^{-1}\right)<\left[\Delta^2u(k)-\Delta^2\left(a\cdot k^{-s}\right)-\Delta^2\left(b\cdot k^{-(s+1)}\right)\right]\left[\Delta^2\left(b\cdot k^{-(s+1)}\right)ms(s+2)^{-1}\right]^{-1}<\varepsilon_2+O\left(k^{-1}\right). \tag{3.5}$$

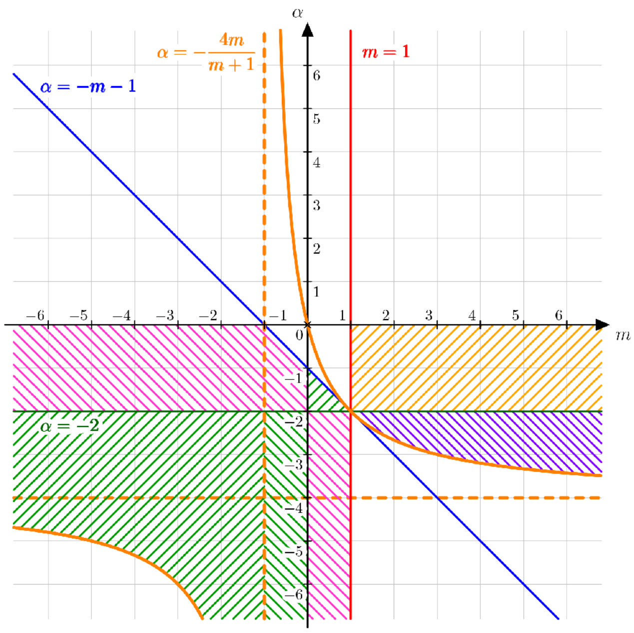

Figure 3.1: Summary of admissible values

Remark 3. In the proof of Theorem 4 we will apply Theorem 2 from Chapter 1, where system (2.10), (2.11) is considered instead of a system of discrete equations (1.5). That is, in system (1.5) we set $n = 2$ and

$$F_1(k, Y_0(k), Y_1(k)) := \left(-\frac{s+1}{k} + O\left(\frac{1}{k^2}\right)\right)(-Y_0(k) + Y_1(k)),$$
$$F_2(k, Y_0(k), Y_1(k)) := \left(-\frac{s+2}{k} + O\left(\frac{1}{k^2}\right)\right)\left(\frac{ms}{s+2}Y_0(k) - Y_1(k) + O\left(\frac{1}{k}\right)\right).$$

The core of the proof consists of verifying inequalities (1.13), (1.14) estimating functions F_1 and F_2 for properly defined functions $b_i, c_i\colon \mathbb{N}(k_0) \to \mathbb{R}$, $i = 1, 2$ (see (1.7)) satisfying $b_i(k) < c_i(k)$, $k \in \mathbb{N}(k_0)$, $i = 1, 2$. By b_i and c_i, $i = 1, 2$ functions $B_i(k, Y)$ and $C_i(k, Y)$, $i = 1, 2$ in (1.8) and sets Ω_B^i, Ω_C^i, $i = 1, 2$ in (1.9), (1.10) are defined.

All particular cases are highlighted in Figure 3.1 in (m, α)-plane in corresponding colours. If a fixed (m, α) belongs to the domain of admissible values, all hypotheses of Theorem 4 are true and, for a sufficiently large fixed k_0, there exists a solution $u\colon \mathbb{N}(k_0) \to \mathbb{R}$ of equation (1.3) satisfying, for every $k \in \mathbb{N}(k_0)$, inequalities (3.3)–(3.5).

4 Power-type asymptotic behaviour in case of tending to zero upper and lower functions

In this chapter, we will show that the areas of possible coefficient values for which equation (1.3) has solutions asymptotically expressed by a power-type function may change depending on the type of the upper and lower functions. We will search for the conditions such that there exists a solution to equation (1.3) with the following asymptotic behaviour:

$$u(k) = a \cdot k^{-s} + b \cdot k^{-(s+1)} + O\left(k^{-(\gamma+s+1)}\right), \tag{4.1}$$

where a, b and s are defined in (2.2), (2.3) and (2.1) and γ is some positive constant.

In this chapter, we have chosen power-type upper and lower functions $b_1(k)$, $b_2(k)$, $c_1(k)$ and $c_2(k)$ tending to zero.

The idea of the proof is similar to the one in the previous chapter, while requiring more complex calculations. The scheme of all investigations is the following. The transformations (2.5)–(2.7), where $a_\pm$, $b_\pm$ are computed by formulas (2.2), (2.3), are used to transform the equation (1.3) into an auxiliary system of two equations (2.10), (2.11).

Then, some particular results of those published in [12, 14]) are applied to investigate system (2.10), (2.11). A correct use of Theorem 2 necessitates the proper choice of functions $b_i(k)$, $c_i(k)$, $i = 1, 2$. In this chapter, we will assume

$$b_1(k) := -\varepsilon_1 \cdot k^{-\gamma}\,, \quad c_1(k) := \varepsilon_2 \cdot k^{-\gamma}\,, \quad b_2(k) := -\varepsilon_3 \cdot k^{-\beta}\,, \quad c_2(k) := \varepsilon_4 \cdot k^{-\beta}, \tag{4.2}$$

where ε_j, $j = 1, \dots, 4$ are positive constants.

This chapter is divided into 4 parts depending on the values $s + 1$ and ms, where s is defined in (2.1).

To prove all the below theorems we need to define some auxiliary sets and functions identical for all four cases.

Let $\varepsilon_i > 0$, $i = 1, \dots, 4$ and let β and γ be fixed. Assuming k_0 positive and sufficiently large such that the asymptotic computations in the proof are correct for every $k \in \mathbb{N}(k_0)$, define functions b_i, c_i, $i = 1, 2$, satisfying (1.7), by formulas

$$b_1(k) := -\varepsilon_1 \cdot k^{-\gamma}\,, \quad c_1(k) := \varepsilon_2 \cdot k^{-\gamma}\,, \quad b_2(k) := -\varepsilon_3 \cdot k^{-\beta}\,, \quad c_2(k) := \varepsilon_4 \cdot k^{-\beta}.$$

Then,

$$\begin{aligned} B_1(k, Y) &:= -Y_0 + b_1(k) = -Y_0 - \varepsilon_1\,, \quad B_2(k, Y) := -Y_1 + b_2(k) = -Y_1 - \varepsilon_3\,, \\ C_1(k, Y) &:= \;\; Y_0 - c_1(k) = \;\; Y_0 - \varepsilon_2\,, \quad C_2(k, Y) := \;\; Y_1 - c_2(k) = \;\; Y_1 - \varepsilon_4 \end{aligned}$$

and

$$\Omega_B^1 = \left\{(k, Y)\colon k \in \mathbb{N}(k_0),\ Y_0 = -\varepsilon_1 \cdot k^{-\gamma},\ -\varepsilon_3 \cdot k^{-\beta} \le Y_1 \le \varepsilon_4 \cdot k^{-\beta}\right\}, \tag{4.3}$$
$$\Omega_B^2 = \left\{(k, Y)\colon k \in \mathbb{N}(k_0),\ Y_1 = -\varepsilon_3 \cdot k^{-\beta},\ -\varepsilon_1 \cdot k^{-\gamma} \le Y_0 \le \varepsilon_2 \cdot k^{-\gamma}\right\}, \tag{4.4}$$
$$\Omega_C^1 = \left\{(k, Y)\colon k \in \mathbb{N}(k_0),\ Y_0 = \;\; \varepsilon_2 \cdot k^{-\gamma},\ -\varepsilon_3 \cdot k^{-\beta} \le Y_1 \le \varepsilon_4 \cdot k^{-\beta}\right\}, \tag{4.5}$$
$$\Omega_C^2 = \left\{(k, Y)\colon k \in \mathbb{N}(k_0),\ Y_1 = \;\; \varepsilon_4 \cdot k^{-\beta},\ -\varepsilon_1 \cdot k^{-\gamma} \le Y_0 \le \varepsilon_2 \cdot k^{-\gamma}\right\}. \tag{4.6}$$

For later formulation, we will need to verify four differences: $b_1(k+1) - b_1(k)$, $b_2(k+1) - b_2(k)$, $c_1(k+1) - c_1(k)$ and $c_2(k+1) - c_2(k)$. As functions $b_1(k)$, $b_2(k)$, $c_1(k)$ and $c_2(k)$ are similar, we will show the calculation for only one case using the binomial formula (1.4):

$$b_1(k+1) - b_1(k) = -\varepsilon_1 \cdot (k+1)^{-\gamma} + \varepsilon_1 \cdot k^{-\gamma} = \varepsilon_1 \gamma \cdot k^{-\gamma+1}\left(1 + O\left(k^{-1}\right)\right). \tag{4.7}$$

To apply Theorem 1.13, inequalites (1.13) and (1.14) must hold.

Since inequality (1.13) assumes $(k, Y) \in \Omega_B^i$, $i = 1, \dots, n$ and inequality (1.14) assumes $(k, Y) \in \Omega_C^i$, $i = 1, \dots, n$, we need to verify (taking into account specifications (4.3)–(4.6)) and (4.7) the following:

$$F_1(k, b_1(k), Y_1)\big|_{(k, Y_0, Y_1) \in \Omega_B^1} = F_1\left(k, -\varepsilon_1 \cdot k^{-\gamma},\ \ Y_1\right)\big|_{b_2(k) \le Y_1 \le c_2(k)}$$

$$< b_1(k+1) - b_1(k) = \quad \varepsilon_1\gamma \cdot k^{-(\gamma+1)} \left(1 + O\left(k^{-1}\right)\right), \quad (4.8)$$

$$F_1(k, c_1(k), Y_1)|_{(k,Y_0,Y_1)\in\Omega_C^1} = F_1\left(k, \quad \varepsilon_2 \cdot k^{-\gamma}, \quad Y_1\right)\big|_{b_2(k)\le Y_1\le c_2(k)}$$
$$> c_1(k+1) - c_1(k) = -\varepsilon_2\gamma \cdot k^{-(\gamma+1)} \left(1 + O\left(k^{-1}\right)\right), \quad (4.9)$$

$$F_2(k, Y_0, b_2(k))|_{(k,Y_0,Y_1)\in\Omega_B^2} = F_2\left(k, \quad Y_0, -\varepsilon_3 \cdot k^{-\gamma}\right)\big|_{b_1(k)\le Y_0\le c_1(k)}$$
$$< b_2(k+1) - b_2(k) = \quad \varepsilon_3\beta \cdot k^{-(\beta+1)} \left(1 + O\left(k^{-1}\right)\right), \quad (4.10)$$

$$F_2(k, Y_0, c_2(k))|_{(k,Y_0,Y_1)\in\Omega_C^2} = F_2\left(k, \quad Y_0, \quad \varepsilon_4 \cdot k^{-\gamma}\right)\big|_{b_1(k)\le Y_0\le c_1(k)}$$
$$> c_2(k+1) - c_2(k) = -\varepsilon_4\beta \cdot k^{-(\beta+1)} \left(1 + O\left(k^{-1}\right)\right) \quad (4.11)$$

whenever

$$-\varepsilon_3 \cdot k^{-\beta} \le Y_1 \le \varepsilon_4 \cdot k^{-\beta}, \quad -\varepsilon_1 \cdot k^{-\gamma} \le Y_0 \le \varepsilon_2 \cdot k^{-\gamma}$$

in (4.8), (4.9) and in (4.10), (4.11).

The scheme of each of the following fourth sections (sections 4.1–4.4) is similar. In each part, we give two theorems on the existence of a power-type solution. The first theorem considers the conditions, including the values and variables not defined in the formulation of the equation (1.3). The second theorem will define the strict values of m and α and will be represented in the plane.

4.1 The case of $ms > 0$ and $s + 1 > 0$

Theorem 5. *Let either*

$$s > 0, \quad m > 0 \tag{4.12}$$

or

$$-1 < s < 0, \quad m < 0. \tag{4.13}$$

Assume that there exists a constant γ satisfying $0 < \gamma < 1$ and positive numbers ε_i, $i = 1, 2, 3, 4$, such that

$$\varepsilon_3 < \varepsilon_1 \frac{\gamma + s + 1}{s + 1}, \quad \varepsilon_4 < \varepsilon_2 \frac{\gamma + s + 1}{s + 1}, \quad \varepsilon_1 < \varepsilon_3 \frac{\gamma + s + 2}{ms}, \quad \varepsilon_2 < \varepsilon_4 \frac{\gamma + s + 2}{ms}.$$

Then, for a sufficiently large fixed $k_0 > 0$, there exists a solution $u\colon \mathbb{N}(k_0) \to \mathbb{R}$ of equation (1.3) such that, for every $k \in \mathbb{N}(k_0)$ asymptotic representation (4.1) holds or, more presisely, this solution satisfies

$$-\varepsilon_1 \cdot k^{-\gamma} < \left[u(k) - a \cdot k^{-s} - b \cdot k^{-(s+1)}\right] \left[b \cdot k^{-(s+1)}\right]^{-1} < \varepsilon_2 \cdot k^{-\gamma}, \tag{4.14}$$

$$-\varepsilon_3 \cdot k^{-\gamma} < \left[\Delta u(k) - \Delta\left(a \cdot k^{-s}\right) - \Delta\left(b \cdot k^{-(s+1)}\right)\right] \left[\Delta\left(b \cdot k^{-(s+1)}\right)\right]^{-1} < \varepsilon_4 \cdot k^{-\gamma}, \tag{4.15}$$

$$-\varepsilon_1 \cdot k^{-\gamma} + O\left(k^{-1}\right) < \left[\Delta^2 u(k) - \Delta^2\left(a \cdot k^{-s}\right) - \Delta^2\left(b \cdot k^{-(s+1)}\right)\right]$$
$$\cdot \left[\Delta^2\left(b \cdot k^{-(s+1)}\right) ms \cdot (s+2)^{-1}\right]^{-1} < \varepsilon_2 \cdot k^{-\gamma} + O\left(k^{-1}\right). \quad (4.16)$$

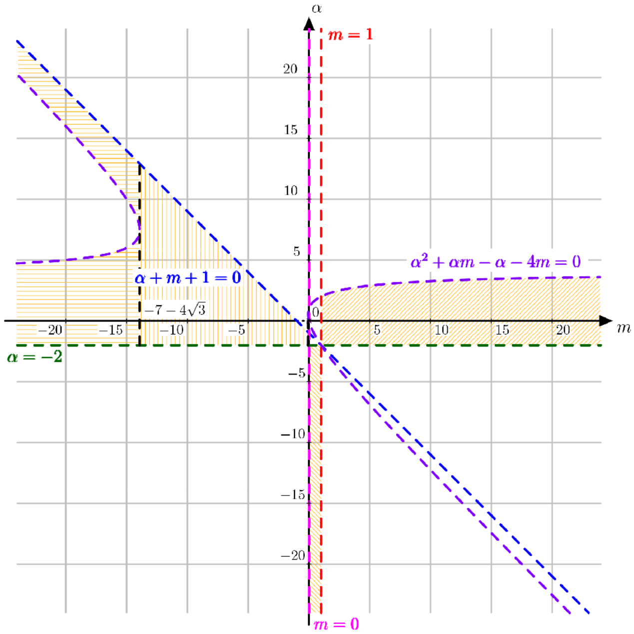

Figure 4.1: Summary of admissible values (Theorem 6)

Theorem 6. *Let at least one of following assumptions hold:*

$$m \in \left(-7-4\sqrt{3}, -7+4\sqrt{3}\right), \quad -2 < \alpha < -m-1, \tag{4.17}$$

$$0 < m < 1, \quad \alpha < -2, \tag{4.18}$$

$$m > 1, \quad -2 < \alpha < \frac{1}{2}\left(-(m-1) + \sqrt{(m-1)^2 + 16m}\right), \tag{4.19}$$

$$-2 < \alpha < -m-1, \quad m < 0, \quad (m-1)^2 + 16m > 0 \tag{4.20}$$

and either

$$\alpha < \frac{1}{2}\left(-(m-1) - \sqrt{(m-1)^2 + 16m}\right)$$

or

$$\alpha > \frac{1}{2}\left(-(m-1) + \sqrt{(m-1)^2 + 16m}\right).$$

Then, the conclusion of Theorem 5 holds.

All suitable areas on the (α, m)-plane indicated in Theorem 6 are visualized on the figures 4.1, 4.2.

4.2 The case of $ms < 0$ and $s + 1 > 0$

Theorem 7. *Let either*

$$s > 0, \quad m < 0$$

or

$$-1 < s < 0, \quad m > 0.$$

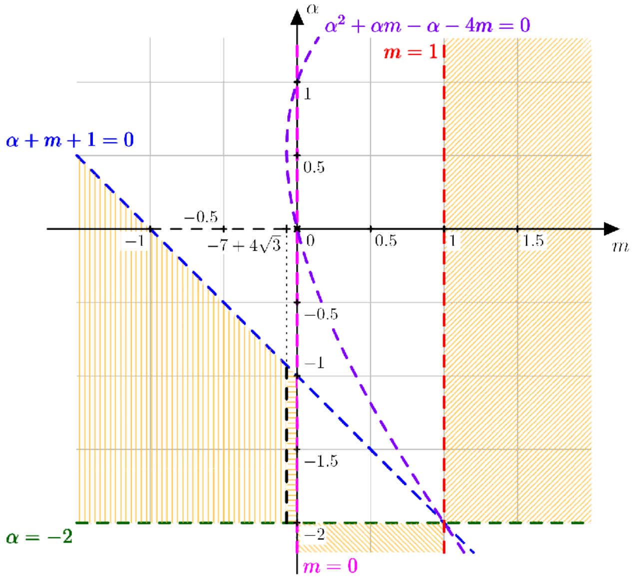

Figure 4.2: Summary of admissible values - zoom (Theorem 6)

Assume that there exists a constant γ *satisfying* $0<\gamma<1$ *and positive numbers* ε_i, $i=1,2,3,4$, *such that*

$$\varepsilon_3<\varepsilon_1\frac{\gamma+s+1}{s+1}\,,\quad \varepsilon_4<\varepsilon_2\frac{\gamma+s+1}{s+1}\quad \varepsilon_1<-\varepsilon_3\frac{\gamma+s+2}{ms}\,,\quad \varepsilon_2<-\varepsilon_4\frac{\gamma+s+2}{ms}\,.$$

Then, for a sufficiently large fixed $k_0>0$, *there exists a solution* $u\colon \mathbb{N}(k_0)\to\mathbb{R}$ *of equation* (1.3) *such that, for every* $k\in\mathbb{N}(k_0)$ *asymptotic representation* (4.1) *holds or, more presisely, this solution satisfies*

$$-\varepsilon_1\cdot k^{-\gamma}<\left[u(k)-a\cdot k^{-s}-b\cdot k^{-(s+1)}\right]\left[b\cdot k^{-(s+1)}\right]^{-1}<\varepsilon_2\cdot k^{-\gamma}\,, \tag{4.21}$$

$$-\varepsilon_3\cdot k^{-\gamma}<\left[\Delta u(k)-\Delta\left(a\cdot k^{-s}\right)-\Delta\left(b\cdot k^{-(s+1)}\right)\right]\left[\Delta\left(b\cdot k^{-(s+1)}\right)\right]^{-1}<\varepsilon_4\cdot k^{-\gamma}\,, \tag{4.22}$$

$$\begin{aligned}-\,\varepsilon_1\cdot k^{-\gamma}+O\left(k^{-1}\right)&<\left[\Delta^2u(k)-\Delta^2\left(a\cdot k^{-s}\right)-\Delta^2\left(b\cdot k^{-(s+1)}\right)\right]\\&\cdot\left[\Delta^2\left(b\cdot k^{-(s+1)}\right)ms\cdot(s+2)^{-1}\right]^{-1}<\varepsilon_2\cdot k^{-\gamma}+O\left(k^{-1}\right)\,. \end{aligned}\tag{4.23}$$

Theorem 8. *Let* m *and* α *satisfy one of the following conditions* (4.24)–(4.26)*:*

$$m<0\quad\wedge\quad \alpha<-2, \tag{4.24}$$
$$0<m<1\quad\wedge\quad -2<\alpha<-m-1, \tag{4.25}$$
$$m>1\quad\wedge\quad -m-1<\alpha<-2, \tag{4.26}$$

and let, moreover,

$$\alpha^2(1+m)+\alpha(m^2+8m-1)+8m^2>0. \tag{4.27}$$

Then, for a sufficiently large fixed $k_0>0$, *there exists a solution* $u\colon\mathbb{N}(k_0)\to\mathbb{R}$ *of equation* (1.3) *such that, for every* $k\in\mathbb{N}(k_0)$, *asymptotic representation* (4.21)–(4.23) *holds.*

In figure 4.3, 4.4 the resulting domain in (m,α)-plane is highlighted in violet.

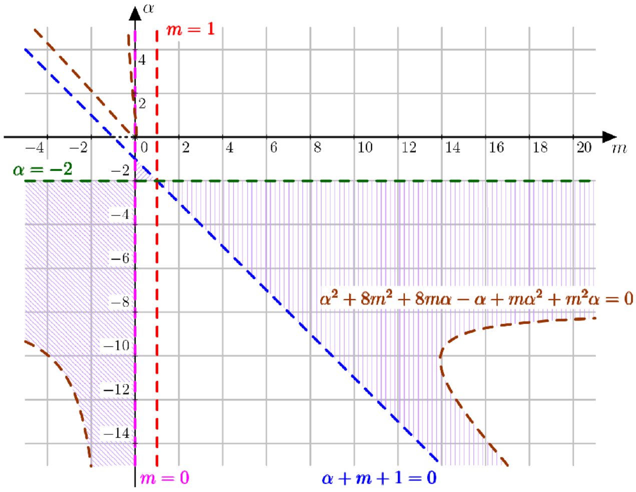

Figure 4.3: Summary of admissible values (Theorem 8)

4.3 The case of $ms < 0$ and $s + 1 < 0$

Theorem 9. *Let $\alpha \neq 0$ and*

$$s < -1, \quad m > 0, \quad s \neq -2$$

Assume that there exists a constant γ, satisfying $0 < \gamma < 1$ and positive numbers ε_i, $i = 1, 2, 3, 4$, such that

$$\varepsilon_4 < -\varepsilon_1 \frac{\gamma + s + 1}{s + 1}\,, \quad \varepsilon_3 < -\varepsilon_2 \frac{\gamma + s + 1}{s + 1}\,, \quad \varepsilon_2 < -\varepsilon_3 \frac{\gamma + s + 2}{ms}\,, \quad \varepsilon_1 < -\varepsilon_4 \frac{\gamma + s + 2}{ms}\,. \tag{4.28}$$

Then, for a sufficiently large fixed $k_0 > 0$, there exists a solution $u\colon \mathbb{N}(k_0) \to \mathbb{R}$ of equation (1.3) such that, for every $k \in \mathbb{N}(k_0)$ asymptotic representation (4.1) holds or, more presisely, this solution satisfies

$$-\varepsilon_1 \cdot k^{-\gamma} < \left[u(k) - a \cdot k^{-s} - b \cdot k^{-(s+1)}\right] \left[b \cdot k^{-(s+1)}\right]^{-1} < \varepsilon_2 \cdot k^{-\gamma}\,, \tag{4.29}$$

$$-\varepsilon_3 \cdot k^{-\gamma} < \left[\Delta u(k) - \Delta\left(a \cdot k^{-s}\right) - \Delta\left(b \cdot k^{-(s+1)}\right)\right] \left[\Delta\left(b \cdot k^{-(s+1)}\right)\right]^{-1} < \varepsilon_4 \cdot k^{-\gamma}\,, \tag{4.30}$$

$$\begin{aligned} -\varepsilon_1 \cdot k^{-\gamma} + O\left(k^{-1}\right) < &\left[\Delta^2 u(k) - \Delta^2\left(a \cdot k^{-s}\right) - \Delta^2\left(b \cdot k^{-(s+1)}\right)\right] \\ &\cdot \left[\Delta^2\left(b \cdot k^{-(s+1)}\right) ms \cdot (s+2)^{-1}\right]^{-1} < \varepsilon_2 \cdot k^{-\gamma} + O\left(k^{-1}\right)\,. \end{aligned} \tag{4.31}$$

Theorem 10. *Let the numbers α and m satisfy*

$$\alpha \neq \{0, -2m\},$$

$$\frac{\alpha + m + 1}{m - 1} < 0\,, \quad m\frac{\alpha + 2}{m - 1} < 0\,, \quad \frac{2\alpha + 5m - 1}{m - 1} > 0\,, \quad (m-1)(\alpha^2 + \alpha m - \alpha - 4m) < 0 \tag{4.32}$$

and

$$\gamma + \frac{\alpha + m + 1}{m - 1} > 0 \tag{4.33}$$

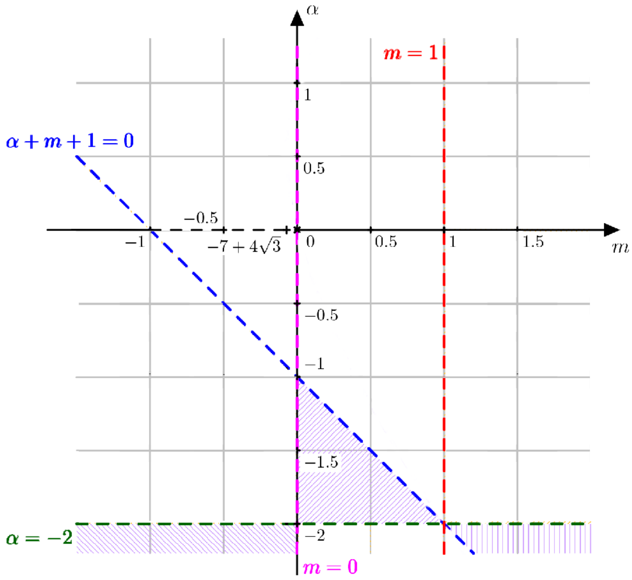

Figure 4.4: Summary of admissible values - zoom (Theorem 8)

where γ *is a fixed number such that* $\gamma \in (\gamma^*, 1)$ *and*

$$\gamma^* = \frac{1}{2}\left(-\frac{2\alpha+3m+1}{m-1} + \sqrt{4m\frac{\alpha+2}{m-1}\cdot\frac{\alpha+m+1}{m-1}+1}\right).$$

Then, the conclusion of Theorem 9 holds.

Remark 4. Note the following. For the solvability of the system of inequalities (4.28), the inequality

$$\gamma + s + 1 > 0 \tag{4.34}$$

is necessary as, in the opposite case, two inequalities from (4.28), cannot be satisfied due to the positivity of ε_i, $i = 1,2,3,4$ and the property $s+1<0$.

Remark 5. The system of inequalities (4.32)–(4.33) is solvable as well as the system of inequalities (4.28). We show that the system of inequalities (4.32)–(4.33) is satisfied, e.g., for the choice $m = 1/2$, $\alpha = -27/20$. In such a case, inequalities (4.32) hold since

$$s = \frac{\alpha+2}{m-1} = -\frac{13}{10}, \quad s+1 = \frac{\alpha+m+1}{m-1} = -\frac{3}{10} < 0,$$

$$ms = m\frac{\alpha+2}{m-1} = -\frac{13}{20} < 0, \quad 2s+5 = \frac{2\alpha+5m-1}{m-1} = \frac{12}{5} > 0,$$

$$(m-1)(\alpha^2+\alpha m-\alpha-4m) = -\frac{199}{800} < 0.$$

Moreover

$$\gamma_2 = \frac{1}{2}\left(-\frac{2\alpha+3m+1}{m-1} + \sqrt{4m\frac{\alpha+2}{m-1}\cdot\frac{\alpha+m+1}{m-1}+1}\right) = -\frac{1}{5}+\frac{1}{2}\sqrt{1.78} \doteq 0.467$$

and inequality (4.33) holds since

$$\gamma + s + 1 = \gamma - \frac{3}{10} > 0,$$

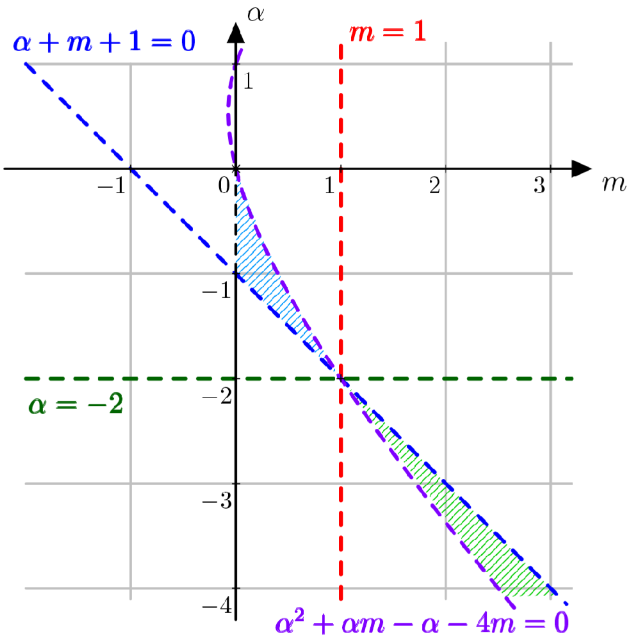

Figure 4.5: Summary of admissible values (Theorem 10)

where γ is a fixed number such that $\gamma \in (\gamma_2, 1)$. Let, e.g., $\gamma = 0.8$. Then the system of inequalities (4.28) equals

$$\varepsilon_4 < -\,\varepsilon_1 \frac{\gamma+s+1}{s+1} = -\varepsilon_1 \frac{0.8-0.3}{-0.3} = \frac{5}{3}\,\varepsilon_1, \tag{4.35}$$

$$\varepsilon_1 < -\,\varepsilon_4 \frac{\gamma+s+2}{ms} = -\varepsilon_4 \frac{0.8-0.3+1}{-13/20} = \frac{30}{13}\,\varepsilon_4. \tag{4.36}$$

The choice, e.g., $\varepsilon_1 = \varepsilon_4 = 1$ solve the sub-system (4.35), (4.36) i.e., solve the sub-system of inequalities (4.28).

Remark 6. The domain defined by inequalities (4.32)–(4.33) in Theorem 10 is visualized in (m, α)-plane by Figure 4.5. This domain splits other two open sub-domains, one of them being blue color and other green.

4.4 The case of $ms > 0$ and $s + 1 < 0$

Theorem 11. *Let $\alpha \neq 0$ and*

$$s < -1, \quad m > 0, \qquad s \neq -2.$$

Assume that there exists a constant γ, satisfying $0 < \gamma < 1$ and positive numbers ε_i, $i = 1, 2, 3, 4$, such that

$$\varepsilon_4 < -\varepsilon_1 \frac{\gamma+s+1}{s+1}\,, \quad \varepsilon_3 < -\varepsilon_2 \frac{\gamma+s+1}{s+1}\,, \quad \varepsilon_1 < \varepsilon_3 \frac{\gamma+s+2}{ms}\,, \quad \varepsilon_2 < \varepsilon_4 \frac{\gamma+s+2}{ms}\,. \tag{4.37}$$

Then, for a sufficiently large fixed $k_0 > 0$, there exists a solution $u \colon \mathbb{N}(k_0) \to \mathbb{R}$ of equation (1.3) such that, for every $k \in \mathbb{N}(k_0)$ asymptotic representation (4.1) holds or, more precisely, this solution satisfies

$$-\varepsilon_1 \cdot k^{-\gamma} < \left[u(k) - a \cdot k^{-s} - b \cdot k^{-(s+1)}\right] \left[b \cdot k^{-(s+1)}\right]^{-1} < \varepsilon_2 \cdot k^{-\gamma}\,, \tag{4.38}$$

$$-\varepsilon_3 \cdot k^{-\gamma} < \left[\Delta u(k) - \Delta\left(a \cdot k^{-s}\right) - \Delta\left(b \cdot k^{-(s+1)}\right)\right]\left[\Delta\left(b \cdot k^{-(s+1)}\right)\right]^{-1} < \varepsilon_4 \cdot k^{-\gamma}\,, \tag{4.39}$$

$$\begin{aligned} -\,\varepsilon_1 \cdot k^{-\gamma} + O\left(k^{-1}\right) < {} & \left[\Delta^2 u(k) - \Delta^2\left(a \cdot k^{-s}\right) - \Delta^2\left(b \cdot k^{-(s+1)}\right)\right] \\ & \cdot \left[\Delta^2\left(b \cdot k^{-(s+1)}\right) ms \cdot (s+2)^{-1}\right]^{-1} < \varepsilon_2 \cdot k^{-\gamma} + O\left(k^{-1}\right)\,. \end{aligned} \tag{4.40}$$

Theorem 12. *Let the numbers α and m satisfy*

$$\alpha \neq \{0, -2m\}$$

$$\frac{\alpha+m+1}{m-1} < 0\,, \quad m\frac{\alpha+2}{m-1} > 0\,, \quad \frac{2\alpha+5m-1}{m-1} > 0\,, \quad \alpha^2 + 8m^2 + 8m\alpha - \alpha + m\alpha^2 + m^2\alpha > 0 \tag{4.41}$$

and

$$\gamma + \frac{\alpha+m+1}{m-1} > 0 \tag{4.42}$$

where γ is a fixed number such that $\gamma \in (\gamma^*, 1)$ and

$$\gamma^* = \frac{1}{2}\left(-\frac{2\alpha+3m+1}{m-1} + \sqrt{1 - 4m\frac{\alpha+2}{m-1} \cdot \frac{\alpha+m+1}{m-1}}\right).$$

Then, the conclusion of Theorem 11 holds.

Remark 7. For the solvability of the system of inequalities (4.37), the inequality

$$\gamma + s + 1 > 0 \tag{4.43}$$

is necessary as, in the opposite case, two inequalities cannot be satisfied due to the positivity of ε_i, $i = 1, 2, 3, 4$ and the property $s + 1 < 0$.

Remark 8. The system of inequalities (4.37) is solvable, e.g., for the choice $m = -2$, $\alpha = 3/2$. In such a case, inequalities (4.41) will hold since

$$s = \frac{\alpha+2}{m-1} = -\frac{7}{6}\,, \quad s+1 = \frac{\alpha+m+1}{m-1} = -\frac{1}{6} < 0,$$

$$ms = m\frac{\alpha+2}{m-1} = \frac{7}{3} > 0\,, \quad 2s+5 = \frac{2\alpha+5m-1}{m-1} = \frac{8}{3} > 0\,,$$

$$\alpha^2 + 8m^2 + 8m\alpha - \alpha + m\alpha^2 + m^2\alpha = \frac{41}{4} > 0.$$

Moreover,

$$\gamma_2 = \frac{1}{2}\left(-\frac{2\alpha+3m+1}{m-1} + \sqrt{1 - 4m\frac{\alpha+2}{m-1}\frac{\alpha+m+1}{m-1}}\right) \doteq 0.46597$$

and inequality (4.42) holds since

$$\gamma + s + 1 = \gamma - \frac{1}{12} > 0,$$

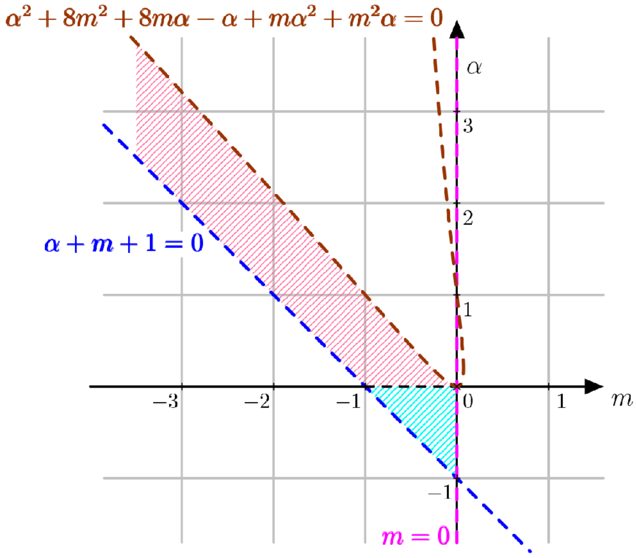

Figure 4.6: Summary of admissible values (Theorem 12)

where γ is a fixed number such that $\gamma \in (\gamma_2, 1)$. Let, e.g., $\gamma = 5/6$. Then, system (4.37) has the form

$$\varepsilon_4 < -\varepsilon_1 \frac{\gamma+s+1}{s+1} = 4\varepsilon_1\,, \quad \varepsilon_3 < -\varepsilon_2 \frac{\gamma+s+1}{s+1} = 4\varepsilon_2,$$
$$\varepsilon_1 < \varepsilon_3 \frac{\gamma+s+2}{ms} = \frac{5}{7}\varepsilon_3\,, \quad \varepsilon_2 < \varepsilon_4 \frac{\gamma+s+2}{ms} = \frac{5}{7}\varepsilon_4.$$

The choice, e.g., $\varepsilon_1 = \varepsilon_2 = 1$, $\varepsilon_3 = \varepsilon_4 = 2$ solves this system.

Remark 9. The domain defined by inequalities (4.41)–(4.42) in Theorem 12 is visualized in (m, α)-plane by Figure 4.6. This domain splits into two open sub-domains, one of them shown in red while the other in blue.

4.5 All the above cases unified and compared with the case of constant upper an lower functions

In this section, we will compare the above results. The results of Theorems 5 – 12 can all be united represented by the below Figures 4.7 and 4.8.

Now, in addition, we need to compare these results with the Theorem 4 of Chapter 3. As the proof of this theorem is structured similarly, it should be mentioned that the crucial role in applying Theorem 2 is played by a proper choice of upper and lower functions $b_i(k)$ and $c_i(k)$, where $i = 1, 2$. Both sets of the upper and lower functions lead to the identical asymptotic relation

$$-\varepsilon_1 \cdot k^{-\gamma} < \left[u(k) - a \cdot k^{-s} - b \cdot k^{-(s+1)}\right] \left[b \cdot k^{-(s+1)}\right]^{-1} < \varepsilon_2 \cdot k^{-\gamma}\,,$$
$$-\varepsilon_3 \cdot k^{-\gamma} < \left[\Delta u(k) - \Delta\left(a \cdot k^{-s}\right) - \Delta\left(b \cdot k^{-(s+1)}\right)\right] \left[\Delta\left(b \cdot k^{-(s+1)}\right)\right]^{-1} < \varepsilon_4 \cdot k^{-\gamma}\,,$$

$$-\,\varepsilon_1 \cdot k^{-\gamma} + O\left(k^{-1}\right) < \left[\Delta^2 u(k) - \Delta^2\left(a \cdot k^{-s}\right) - \Delta^2\left(b \cdot k^{-(s+1)}\right)\right]$$
$$\left[\Delta^2\left(b \cdot k^{-(s+1)}\right) ms \cdot (s+2)^{-1}\right]^{-1} < \varepsilon_2 \cdot k^{-\gamma} + O\left(k^{-1}\right)\,,$$

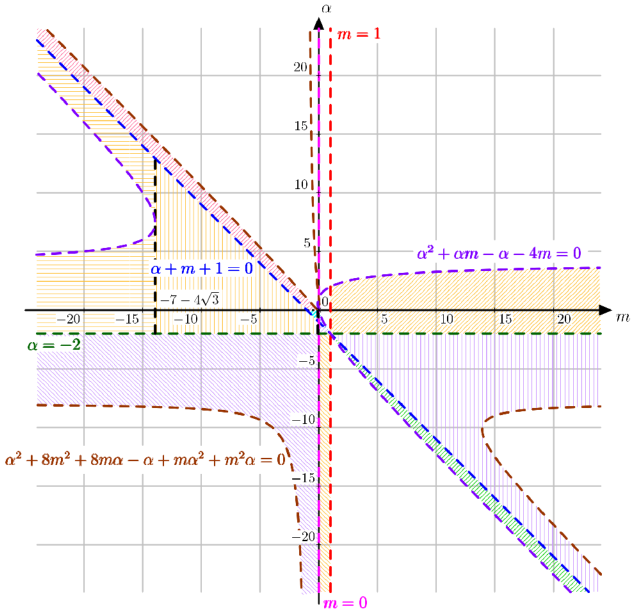

Figure 4.7: Summary of admissible values (Theorems 5–12)

or more precisely

$$\left|u_\pm(k) - a_\pm k^{-s} - b_\pm k^{-s-1}\right| < \frac{\max\{\varepsilon_1, \varepsilon_2\}|b_\pm|}{k^{s+\gamma+1}},$$

$$\left|\Delta u_\pm(k) - a_\pm \Delta k^{-s} - b_\pm \Delta k^{-s-1}\right| < \left|\Delta\left(\frac{b_\pm}{k^{s+1}}\right)\right| \frac{\max\{\varepsilon_1, \varepsilon_2\}}{k^\gamma},$$

$$\left|\Delta^2 u_\pm(k) - a_\pm \Delta^2 k^{-s} - b_\pm \Delta^2 k^{-s-1}\right| < \left|\Delta^2\left(\frac{b_\pm}{k^{s+1}}\right)\right| \left(\max\{\varepsilon_1, \varepsilon_2\}\frac{ms}{k^\gamma |s+2|} + \left|O\left(\frac{1}{k}\right)\right|\right).$$

However, the change of the form of upper and lower functions from constants to power functions extends the set of appropriate conditions reopening the question of the asymptotic behaviour of the Emden-Fowler equation solutions in the case of $s + 1 < 0$.

To illustrate that the set of appropriate conditions has expanded even in the case of $s+1 > 0$ all sets are put in a single Figure 4.9. Here the yellow domain is the summary of the results of this chapter (non-constant case) while the green domain summarises the results of Chapter 3 (constant case).

Remark 10. All the green domains of Figure 4.9 are the subset of the yellow domain.

5 A discrete analogy of the blow-up solution

To illustrate an analogy of blow-up phenomenon for a discrete second-order equation, we will use an autonomous second-order Emden-Fowler type differential equation

$$y''(x) = y^s(x), \tag{5.1}$$

where $s \neq 1$ is a real number.

Let us show that (5.1) can have blow-up solutions.

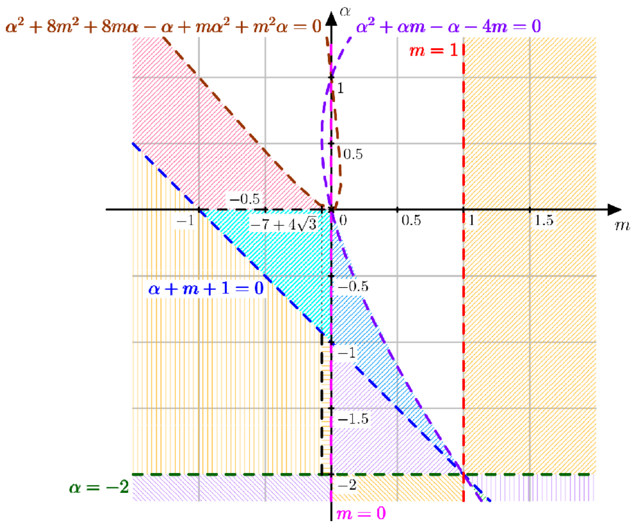

Figure 4.8: Summary of admissible values - zoom (Theorems 5–12)

First, equation (5.1) is solvable and its general solution can be written in the form

$$\int_{y_0}^{y(x)} \frac{dz}{\sqrt{2\int z^s dz + C}} = x - x_0 \tag{5.2}$$

where C is an arbitrary (but admissible) constant and (x_0, y_0) is an arbitrary admissible point. If, for example, $s = 3$ and $C = 0$, then it is easy to derive from (5.2) a class of solutions

$$y(x) = \pm\frac{\sqrt{2}}{x+K} \tag{5.3}$$

where K is an arbitrary constant and one can see the blow-up phenomenon explicitly if $x \to \pm K$.

In directly transferring the above phenomena to discrete equations, there are some circumstances to be taken in consideration because the independent variable in discrete equations is discrete and runs over a set of integers. Therefore, we prove the existence of this phenomenon implicitly as follows. First, we transform equation (5.1) by a transformation

$$x = u(y) \tag{5.4}$$

where u is a new unknown function. This transformation will be such that x tends to a finite limit when y tends to infinity. For example, writing solution (5.3) in the form (5.4), we derive

$$x = u(y) = \pm\frac{\sqrt{2}}{y} - K. \tag{5.5}$$

If $y \to \infty$, then, by (5.5), $x \to -K$. Next, we will compile a differential equation for u in (5.4) and the form of this equation will serve as a motivation for constructing a related discrete equation.

Differentiating the transformation (5.4) with respect to x, we derive

$$1 = u'_y \cdot y'_x. \tag{5.6}$$

Differentiating (5.6) with respect to x again, we have

$$0 = u''_{yy} \cdot (y'_x)^2 + u'_y \cdot y''_{xx}. \tag{5.7}$$

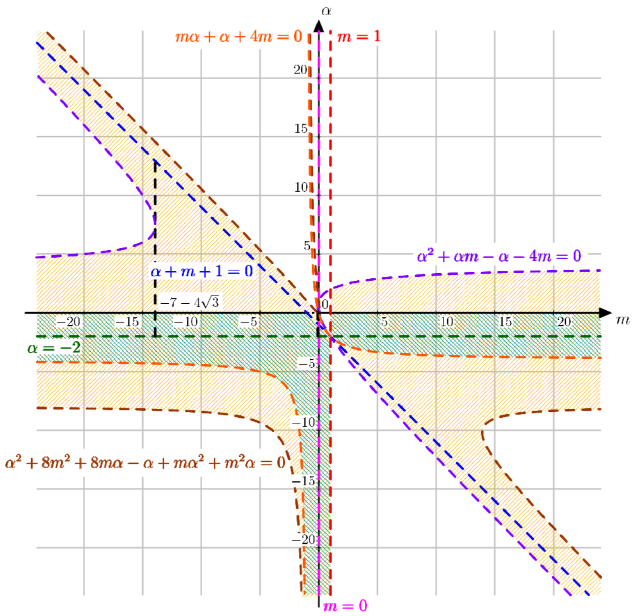

Figure 4.9: Summary of admissible values (Theorems 5–12) and Chapter 3)

Assuming $u'_y \neq 0$, from (5.7), we get

$$y'' = -\frac{u'' \cdot (y')^2}{u'} \tag{5.8}$$

and, using (5.1), (5.6), (5.8)

$$y^s = y'' = -\frac{u'' \cdot (y')^2}{u'} = -\frac{u''}{(u')^3}$$

and, finally, for u we derive

$$u'' = -y^s \, (u')^3 \, . \tag{5.9}$$

Then, a discrete analogy to differential equation (5.9) is the following

$$\Delta^2 v(k) = -k^s \, (\Delta v(k))^3 \, . \tag{5.10}$$

A problem equivalent to blow-up phenomena for differential equation (5.1) is one of proving the existence of a nontrivial solution to equation (5.9) such that limit $\lim_{y\to\infty} u(y)$ exists and is finite. Therefore, we consider the problem to prove the existence of a nontrivial solution to equation (5.10) such that the limit $\lim_{k\to\infty} v(k)$ exists and is finite. More exactly, under condition $s > 1$, we prove the existence of a solution to equation (5.10) such that

$$\lim_{k\to\infty} v(k) = 0. \tag{5.11}$$

5.1 An approximate solution of second-order discrete Emden-Fowler equation (5.10)

We will search for an approximate solution of discrete equation (5.10) with asymptotic behaviour

$$v(k) \sim V(k) := c \cdot k^{-\alpha}$$

as $k \to \infty$ where c and α are constants still unknown. We assume $c \neq 0$, $\alpha \neq 0$ trying to find these constants. To do this, we must replace $\Delta V(k)$ and $\Delta^2 V(k)$ in (5.10). Let us perform, for $k \to \infty$, auxiliary asymptotic computation of $\Delta V(k)$ and $\Delta^2 V(k)$. With the necessary order of accuracy for $\Delta V(k)$, we obtain

$$\Delta V(k) = c(k+1)^{-\alpha} - ck^{-\alpha} = ck^{-\alpha}\left(1+k^{-1}\right)^{-\alpha} - ck^{-\alpha} =$$
$$= -c\alpha \cdot k^{-(\alpha+1)} + c\alpha(\alpha+1)\cdot(1/2)\cdot k^{-(\alpha+2)} - c\alpha(\alpha+1)(\alpha+2)\cdot(1/6)\cdot k^{-(\alpha+3)} + O\left(k^{-(\alpha+4)}\right)$$

and, for $\Delta^2 V(k)$, we have

$$\begin{aligned}\Delta^2 V(k) &= c(k+2)^{-\alpha} - 2c(k+1)^{-\alpha} + ck^{-\alpha}\\ &= c\alpha(\alpha+1)\cdot k^{-(\alpha+2)} - c\alpha(\alpha+1)(\alpha+2)\cdot(1/3)\cdot k^{-(\alpha+3)} + O\left(k^{-(\alpha+4)}\right).\end{aligned}$$

Then, replacing in (5.10) $\Delta v(k)$ and $\Delta^2 v(k)$ with $\Delta V(k)$ and $\Delta^2 V(k)$, after some simplifications we derive

$$\frac{c\alpha(\alpha+1)}{k^{\alpha+2}} = \frac{c^3\alpha^3}{k^{3\alpha+3-s}} + O\left(\frac{1}{k^{3\alpha+4-s}}\right) + O\left(\frac{1}{k^{\alpha+3}}\right). \tag{5.12}$$

Relation (5.12) is satisfied for

$$\begin{cases} \alpha + 2 = 3\alpha + 3 - s,\\ c\alpha(\alpha+1) = c^3\alpha^3. \end{cases} \tag{5.13}$$

The values

$$\alpha = \frac{s-1}{2}, \quad c = \pm\frac{\sqrt{\alpha+1}}{\alpha} = \pm\frac{\sqrt{2s+2}}{s-1} \tag{5.14}$$

solve the system (5.13). Since $V(k)$ can assume two values, we denote

$$V(k) = V_{\pm}(k) = \pm\frac{\sqrt{2s+2}}{s-1}k^{(1-s)/2}.$$

5.2 System equivalent to discrete Emden-Fowler equation (5.10)

Define the following change of variables:

$$v(k) = ck^{-\alpha}(1+Y_1(k)), \tag{5.15}$$
$$\Delta v(k) = (\Delta(ck^{-\alpha}))(1+Y_2(k)), \tag{5.16}$$
$$\Delta^2 v(k) = (\Delta^2\left(ck^{-\alpha}\right))(1+Y_3(k)) \tag{5.17}$$

where $Y_i(k)$, $i = 1,2,3$ are new dependent functions $Y_i\colon \mathbb{N}(k_0) \to \mathbb{R}$, c and α are defined by (5.14). In (5.10) replace $\Delta v(k)$, $\Delta^2 v(k)$ with (5.16), (5.17).

After some cumbersome calculations this change of variables provides us the following system

$$\Delta Y_1(k) = \left(\frac{\alpha}{k} + \frac{\alpha(\alpha-1)}{k^2} + O\left(\frac{1}{k^3}\right)\right)(Y_1(k) - Y_2(k)), \tag{5.18}$$

$$\Delta Y_2(k) = -\left(\frac{\alpha+1}{k} + O\left(\frac{1}{k^2}\right)\right)\left(2Y_2(k) + 3Y_2^2(k) + Y_2^3(k) + O\left(\frac{1}{k}\right)\right). \tag{5.19}$$

Theorem 13. *Let $s > 1$. Let ε_i, γ_i, $i = 1,2$ be fixed positive numbers such that $\varepsilon_2 < \varepsilon_1 < 1$, $\gamma_2 < \gamma_1 < 1$. Then, there exists a solution $Y(k) = Y^*(k) = (Y_1^*(k), Y_2^*(k))$ to the system* (5.18), (5.19) *such that*

$$-\varepsilon_i < Y_i^*(k) < \gamma_i, \;\; i = 1,2, \;\; \forall k \in \mathbb{N}(k_0) \tag{5.20}$$

provided that k_0 is sufficiently large.

5.3 Existence of a nontrivial solution to equation (5.10) with property (5.11)

In this part, we show that Theorem 13 implies the existence of a nontrivial solution to equation (5.10) with property (5.11).

Theorem 14. *Let $s > 1$. Let ε_i, γ_i, $i = 1,2$ be fixed positive numbers such that $\varepsilon_2 < \varepsilon_1 < 1$, $\gamma_2 < \gamma_1 < 1$. Then, there exists a solution $v = v(k)$ to equation (5.10) such that*

$$-\varepsilon_1|c|k^{-\alpha} < v(k) - ck^{-\alpha} < \gamma_1|c|k^{-\alpha},$$

$$-\varepsilon_2\gamma_2\Delta(|c|k^{-\alpha})) < \Delta v(k) - (\Delta(ck^{-\alpha})) < \gamma_2\Delta(|c|k^{-\alpha}))$$

and

$$\Delta^2 v(k) = O(1) \tag{5.21}$$

for all $k \in \mathbb{N}(k_0)$ provided that k_0 is sufficiently large.

6 Conclusion

This doctoral theses studied the asymptotic behaviour of a discrete Emden-Fowler equation. Analysis of the results reveals two different types of asymptotic behaviour.

The first one may be termed a power type. The idea of the proof consists in the retract principle and we see that the choice of different upper and lower functions provides us with different areas of existence of a power-type asymptotic behaviour.

The second one is an analogy for the blow-up solutions. The method of searching for solutions of this type can be applied to other different non linear discrete equations.

Moreover, a little bit more general difference equation than (1.3),

$$\Delta^2 v(k) \pm pk^\alpha v^m(k) = 0, \tag{6.1}$$

where p is a positive constant, can obviously be transformed to the form (1.3) by a transformation $v(k) = qu(k)$ where q is a positive number defined as $q = p^{1/(1-m)}$.

We can also extend the results achieved in Chapters 3 and 4, by adding to the equation (1.3) (or (6.1)) a perturbation - function $\omega(k)$: $\mathbb{N}(k_0) \to \mathbb{R}$ assumed to be sufficiently small. Thus, we can study the equation

$$\Delta u(k) \pm k^\alpha u^m(k) = \omega(k).$$

Here, "sufficiently small" is understood as:

$$\omega(k) = O\left(k^{-(s+4)}\right),$$

where s was defined in (2.1)

From the proofs, we can see that all the calculations can be applied as this “smallness” is hidden in the Landau symbol “big O”.

This thesis includes several theorems on the conditions for the existence of solutions to the Emden-Fowler type difference equations with power-type asymptotic behaviour. Each theorem is supplemented with a figure to be more illustrative. Also, examples are given to show applications of the results achieved.

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Abstract

In the literature a differential second–order nonlinear Emden–Fowler equation

$$y'' \pm x^\alpha y^m = 0,$$

where α and m are constants, is often investigated.

This thesis deals with a discrete equivalent of the second order Emden-Fowler differential equation

$$\Delta^2 u(k) \pm k^\alpha u^m(k) = 0,$$

where $k \in \mathbb{N}(k_0) := \{k_0, k_0+1, ...\}$ is an independent variable, k_0 is an integer and $u\colon \mathbb{N}(k_0) \to \mathbb{R}$ is an unknown solution. In this equation, $\Delta^2 u(k) = \Delta(\Delta u(k))$, $\Delta u(k)$ is the the first-order forward difference of $u(k)$, i.e., $\Delta u(k) = u(k+1) - u(k)$, and $\Delta^2(k)$ is its second–order forward difference, i.e., $\Delta^2 u(k) = u(k+2) - 2u(k+1) + u(k)$, α, m are real numbers. The asymptotic behaviour of the solutions to this equation is discussed and the conditions are found such that there exists a power-type asymptotic: $u(k) \sim 1/k^s$, where s is some constant.

We also discuss a discrete analogy of so-called "blow-up" solutions in the classical theory of differential equations, i.e., the solutions for which there exists a point x^* such that $\lim_{x\to x^*} y(x) = \infty$, where $y(x)$ is a solution of the Emden-Fowler differential equation

$$y''(x) = y^s(x),$$

with $s \neq 1$ being a real number.

The results obtained are compared to those already known and illustrated with examples.

Abstrakt

V literatuře je často studována Emden–Fowlerova nelineární diferenciální rovnice druhého řádu

$$y'' \pm x^\alpha y^m = 0,$$

kde α a m jsou konstanty.

V disertační práci je analyzována diskrétní analogie Emden-Fowlerovy diferenciální rovnice

$$\Delta^2 u(k) \pm k^\alpha u^m(k) = 0,$$

kde $k \in \mathbb{N}(k_0) := \{k_0, k_0+1, ...\}$ je nezávislá proměnná, k_0 je celé číslo a $u\colon \mathbb{N}(k_0) \to \mathbb{R}$ je řešení. V této rovnici je $\Delta^2 u(k) = \Delta(\Delta u(k))$, kde $\Delta u(k)$ je diference vpřed prvního řádu funkce $u(k)$, tj. $\Delta u(k) = u(k+1) - u(k)$ a $\Delta^2(k)$ je její diference vpřed druhého řádu, tj. $\Delta^2 u(k) = u(k+2) - 2u(k+1) + u(k)$, a α, m jsou reálná čísla. Je diskutováno asymptotické chování řešení této rovnice a jsou stanoveny podmínky, garantující existence řešení s asymptotikou mocninného typu: $u(k) \sim 1/k^s$, kde s je vhodná konstanta.

Je také zkoumána diskrétní analogie tzv. "blow-up" řešení (neohraničených řešení) známých v klasické teorii diferenciálních rovnic, tj. řešení pro která v některém bodě x^* platí $\lim_{x\to x^*} y(x) = \infty$, kde $y(x)$ je řešení Emden-Fowlerovy diferenciální rovnice

$$y''(x) = y^s(x),$$

kde $s \neq 1$ je reálné číslo.

Výsledky jsou ilustrovány příklady a porovnávány s výsledky doposud známými.