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DISERTAČNÍ PRÁCE

Multivalued fractals and hyperfractals

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Prohlášení

Prohlašuji, že jsem disertační práci zpracoval samostatně pod vedením prof. RNDr. dr hab. Jana Andrese, DSc. s použitím uvedené literatury.

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Poděkování

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Contents

Abstract

Fractals generated by iterated function systems are systematically studied from 1981 (see [Hu]). The theory of multivalued fractals is developed from 2001 (see [AG1]) almost separately. In the thesis, we treat multivalued fractals and structures supported by them by means of hyperfractals. We deal with structure, self-similarity of multivalued fractals and their visualization. We discuss relationship of multivalued fractals to fractals generated by iterated function systems. We also extend the theory of hyperfractals.

We proceed in the following way. First, we review hyperspaces, maps and hypermaps, since fractals are fixed points of hypermaps in hyperspaces. We also remind iterated function systems and related notions of code space, self-similarity and invariant measure.

Although we usually construct fractals by means of the Banach theorem, we discuss other fixed point theorems, which can be applied in hyperspaces. We describe generalizations of the Banach (metric) and Schauder (topological) fixed point theorems. Existence results for metric and topological multivalued fractals and hyperfractals are supplied. We study multivalued fractals and associated hyperfractals generated by the same iterated multifunction system. We prove that they possess the same address structure.

Since we can also regard fractals generated by iterated function systems as attractors of chaotic dynamical systems, we can draw them by means of the chaos game. We remind the theory related to the chaos game, particularly the ergodic theory and chaos.

Next, we investigate visualization and dimension of hyperfractals. Hyperfractals lie in hyperspaces, which are nonlinear and infinite-dimensional spaces. For a particular class of hyperfractals, we construct projections of their structure by means of support functions. We apply the Moran formula to calculate dimension of self-similar hyperfractals. We also show that self-similar fractals form a subset of shadows of self-similar hyperfractals.

Since hyperfractals are attractors of iterated function systems, we visualize also an invariant measure. Moreover, we construct a shadow of the invariant measure supported by the underlying multivalued fractal by means of ergodic theorem.

Finally, our results are generalized to spaces of fuzzy sets. We remind the theory of fuzzy sets. Then, we define fuzzy fractals and fuzzy hyperfractals, which are related in the same way as multivalued fractals and hyperfractals. We find their address structure, which helps us to visualize these fractals and calculate their Hausdorff dimension.

1. Introduction

1.1. Current state of the art

Fractals are extensively studied objects without an exact definition (see [Ma1], [Ma2], [Ma3]). However, fractals have usually some of the following features [Fa1]

- Fractals have a “natural” appearance.
- Fractals have a fine structure, that is irregular detail at arbitrarily small scales.
- Fractals are too irregular to be described by calculus or traditional geometrical language, either locally or globally.
- Fractals have often some sort of self-similarity or self-affinity, perhaps in a statistical or approximate sense.
- The “fractal dimension” (defined in some way) is strictly greater than the topological dimension of a fractal.
- In many cases of interest, fractals have a very simple, perhaps recursive, definition.

We will meet all the features of fractals but the last three are crucial for us. We will consider self-similar fractals with a noninteger Hausdorff (=fractal) dimension generated by iterated function systems (cf. [Hu], [Ba1]). An *iterated function system* (IFS) is usually a system of a finite number of contractions on a complete metric space $\{X; f_i, i = 1, 2, \dots, n\}$. Then the transformation $F : K(X) \to K(X)$ defined by

$$F(B) := \bigcup_{i=1}^{n} f_i(B),$$

for all $B \in K(X)$, is a contraction mapping on the complete metric space $(K(X), d_H)$, i.e. the space of compact subsets of X. Its unique fixed point $A^* \in K(X)$ obeys

$$A^* = \bigcup_{i=1}^{n} f_i(A^*).$$

The fixed point A^* is called an attractor of the IFS and it is a union of its contracted copies. If the copies are separated and contractions are in addition similitudes, we talk about self-similarity of an attractor. For self-similar attractors, the Hausdorff dimension can be calculated by means of the Moran formula. These results were stated by Hutchinson [Hu] (see also [B], [BG], [BK], [E], [Fa1], [Fa2], [PJS], [Sc], [Wc] and [Wi]).

Barnsley explores more ways of drawing fractals and invariant measures (see [Ba1], [BD]). He also came out with the idea of fractal image compression. Iterated function systems enable us to store image-like data in a few parameters of functions comprising the IFS whose attractor is close to the image. Barnsley developed techniques for image encoding and decoding by means of IFS (cf. [BEHL], [BH]). His method provides a great compression ratio but it is very time consuming. Jacquin developed the approach based on the idea of IFS which use domain and range blocks of a picture (cf. [Ja1]–[Ja4]). See also [F] or [WJ] for further references.

Iterated function systems were extended in many ways, for instance to $\mathbb{R}^\infty$ in [CR]. Infinite IFS were described in [GJ] and [L3]. One of the most significant generalizations of fractals generated by IFSs are *multivalued fractals.* They are generated by *iterated multifunction systems* (shortly IMSs). By multivalued fractals, we understand the fixed points of operators $F : K(X) \to K(X)$, such that

$$F(A) = \bigcup_{i}^{n} F_i(A),$$

where F_i are induced by continuous multivalued maps $\mathcal{F}_i : X \to K(X)$ from an IMS $\{(X, d), \mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n\}$.

Multivalued fractals were extensively investigated in the last ten years. They were developed independently by three groups. Andres and co-workers represent one group which also introduced the terminology [A1], [A2], [AF], [AFGL], [AG1], [AG2], [AV], [Fi]. Their work was inspired by relationships between maps, multivalued maps and hypermaps. Problems in differential inclusions motivated Petruşel and co-workers ([LPY], [P1], [P2], [PR1], [PR2]). For the similar approach see also [BBP], [CP], [CL], [GG], [KLV1], [KLV2], [LtM], [Mh], [MM], [Ok], [SPK]. The articles are devoted mainly to existence results by means of the fixed point theorems and structure of attractors. Drawing and approximation of attractors is also investigated. Lasota and Myjak studied the related notion of semifractals in [L1]-[L4], [LM1]–[LM4].

Our work was inspired by superfractals [Ba2] (see also [BHS1]–[BHS4]). Barnsley found out that it is more effective to treat some sets like sets of fractal sets. Thus, he investigated iterated function systems of iterated function systems. In the easiest case, he considered IFSs $F_1, F_2, \ldots, F_m$,

$$F_i = \{(X, d), f_1^i, f_2^i, \ldots, f_{i_n}^i\},$$

where $f_j^i : X \to X$ are contractions. The operators $F_i : K(X) \to K(X)$,

$$F_i(A) = \bigcup_{j} f_j^i(A),$$

comprise the IFS $\{(K(X),\, d_H),\, F_1,\, F_2,\, \ldots,\, F_m\}$ and generate the contracting operator $\phi : K(K(X)) \to K(K(X))$,

$$\phi(\alpha) = \bigcup_j \bigcup_{A\in\alpha} \{F_j(A)\}.$$

The fixed point of ϕ in a hyperhyperspace is called *a superfractal.*

At the end of [BHS1], it is suggested to investigate the previous case with multivalued mappings f_j^i. Attractors of such systems will be called hyperfractals (see [AR1] and [AR2]). Hyperfractals are fixed points of operators ϕ in $K(K(X))$. Given a system $\{(X,\, d),\, \mathcal{F}_1,\, \mathcal{F}_2,\, \ldots,\, \mathcal{F}_n\}$, of multivalued maps $\mathcal{F}_i : X \to K(X)$ we induce the maps to $F_i : K(X) \to K(X)$,

$$F_i(A) = \bigcup_{x\in A} \mathcal{F}_i(x).$$

The mappings $F_i : K(X) \to K(X)$ comprise an IFS $\phi = \{(K(X),\, d_H),\, F_1,\, F_2,\, \ldots,\, F_n\}$, like in the case of superfractals. Then we obtain the contracting operator $\phi : K(K(X)) \to K(K(X))$,

$$\phi(\alpha) = \bigcup_{i=1}^{n} \phi_i(\alpha) = \bigcup_{i=1}^{n} \bigcup_{A\in\alpha} \{F_i(A)\}.$$

If $\mathcal{F}_i$ are contractions in a complete metric space, ϕ is also a contraction. The fixed point of the contraction is called a hyperfractal.

Hyperfractals are attractors of IFS in hyperspaces. Generalizing IFS to fuzzy sets, fuzzy fractals can be obtained (see e.g. [CFMV], [FLV], [FMV] and [DK]). This research is also motivated by image compression.

1.2. Aims of the thesis

The aim of the thesis is to understand multivalued fractals in a better way and to explore structures supported by them. Multivalued fractals and fractals generated by iterated function systems were developed separately. We are concerned with the address structure, self-similarity and dimension when talking about attractors of IFSs. On the other hand, we mention the existence, drawing and structure of attractors of IMS. We need to investigate the relationship between multivalued fractals and classical fractals.

We also want to explain self-similarity and complexity of multivalued fractals. In the book [PJS], authors discuss different types of self-similarity (see Figure 1). Only the first set called the Sierpiński triangle is self-similar according to the classical Hutchinson theory (see [Hu]). The Sierpiński triangle consists of its three contracted copies. These copies are images of the Sierpiński triangle in three similitudes. These three similitudes comprise an IFS in a complete metric

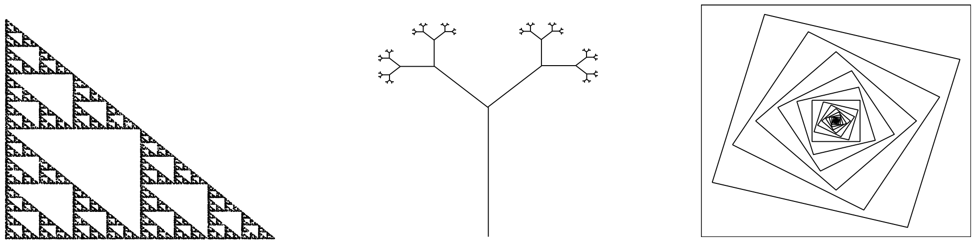

Figure 1: Sets with different kinds of self-similarity

space ($\mathbb{R}^2$, d_{Eucl}). Observe also that in a neighbourhood of each point of the Sierpiński triangle, we can find small copies of the whole set.

Remaining two sets in Figure 1 contain their copies but they do not consist only of them. We say that sets like these are visually self-similar. We distinguish in addition a kind of visual self-similarity. We can find copies of the tree only in neighbourhoods of its leafs and copies of the embedded squares only in neighbourhoods of the middle point. We see self-similarity in infinite number of points and in one point.

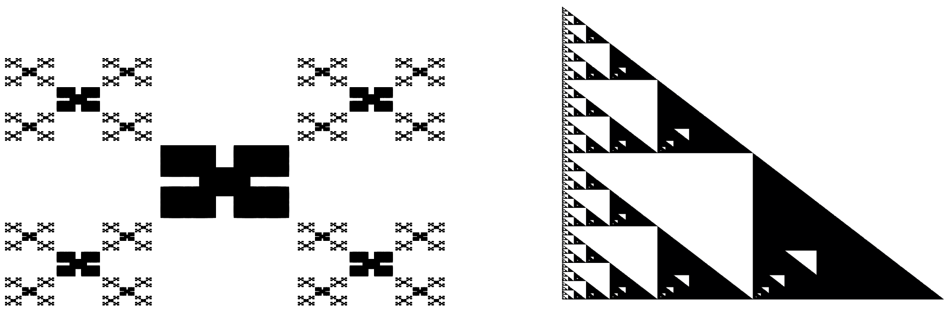

Figure 2: Examples of multivalued fractals

The sets in Figure 1 can be regarded as multivalued fractals (further examples can be found in the Figure 2). This means we are interested in self-similarity of multivalued fractals.

Since we treat the complexity and structure of multivalued fractals by means of hyperfractals, we would also like to extend our results on hyperfractals (cf. [AR2]), particularly their dimension and visualization.

Moreover, we would like to process images with grey levels which are represented in a better way by measures or fuzzy sets. Therefore, we want to develop suitable structures supported by multivalued fractals, i.e. measures on multivalued fractals, fuzzy fractals and fuzzy hyperfractals generated by iterated function systems.

1.3. Theoretical framework

There are two approaches to fractals generated by means of iterated systems. Fractals can be obtained as fixed points of maps or invariant sets of chaotic dynamical systems. We prefer the fixed point approach to investigate the existence of fractals. On the other hand, chaotic approach enables us to draw fractals and approximate integrals on them in a simple way. Multivalued fractals have been studied only by means of the fixed point theory. Having developed the address structure of multivalued fractals, we apply chaotic approach to draw them and construct a measure on them.

1.4. Applied methods

We employ a wide range of the mathematical theory in the work. The first part of the thesis is devoted to the existence results. We apply fixed point theorems. Since we treat fixed points in hyperspaces, we exploit properties of maps, hypermaps and structure of hyperspaces. Except of the theory of IFS and the measure theory, we need basics of dynamical systems, the ergodic theory and chaos. These enable us to draw fractals and invariant measures. Moreover, the theory of convex sets and the Rådström results turn out to be necessary for visualization of hyperfractals and calculation of their dimension. Since fuzzy fractals and fuzzy hyperfractals are only generalization of multivalued fractals and hyperfractals, we apply still the theory of fuzzy sets.

1.5. Main results

The thesis are based on our article [AR2], where we supply the existence results on multivalued fractals and hyperfractals and results following from their address structure. In the article, we studied mainly the existence results on multivalued fractals and hyperfractals. In addition, the Hausdorff dimension of hyperfractals was calculated there. We adopted a part of the article in the thesis. Sections 2 and 4 were written by Professor Andres and completed with a few remarks here. A part of subsection 5.1 was also developed by Professor Andres and it differs slightly in the notation from the article.

Then, we restrict ourselves to compact fractals and study mainly properties of multivalued fractals and hyperfractals related to their address structure. We show the relationship between multivalued fractals and hyperfractals. Self-similarity of multivalued fractals is explained. We plot multivalued fractals by means of the chaos game for hyperfractals. We extend our results on the Hausdorff dimension of hyperfractals. The properties of support functions help us to visualize structure of hyperfractals. Next, we construct a shadow of an invariant measure on hyperfratals with the help of the ergodic theorem. These results are also generalized for spaces of fuzzy sets.

For a better comprehension and a general picture, we state the basic results of the related theory. In order to distinguish our own results, we mark them out with *.

2. Spaces and hyperspaces, maps and hypermaps

Remark 1. Let us note that the most of the section can be found in [AR2] and it was collected and developed by Professor Andres.

In the entire text, all topological spaces will be at least metric. Hence, let (X, d) be a metric space. By the *hyperspace* $(H(X), d_H)$, we will understand as usual a certain class $H(X)$ of nonempty subsets of X endowed with the induced Hausdorff metric d_H, i.e.[1]

$$d_H(A, B) := \inf\{r > 0 | A \subset O_r(B) \text{ and } B \subset O_r(A)\},$$

where $O_r(A) := \{x \in X | \exists a \in A : d(x, a) < r)\}$ and $A, B \in H(X)$. The second possible definition is

$$d_H(A, B) := \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}$$

$$= \max\{\sup_{a\in A}(\inf_{b\in B} d(a, b)), \sup_{b\in B}(\inf_{a\in A} d(a, b))\}.$$

Remark 2. The Hausdorff metric is described in a neat way with the help of indistinguishability in [Wc]. For $A, B \in K(X)$ and $\delta \in [0, \infty)$ we will say A and B are $\delta-$indistinguishable if every element of A is within a distance δ of some element of B and every element of B is within distance of some element of A. You can think of this as meaning that if cannot resolve points distance $\leq \delta$ apart then A and B are visually indistinguishable. The Hausdorff distance between A and B is going to be the least such δ, hence the critical resolution level beyond which A and B can be distinguished.

The following lemma will be frequently used.

Lemma 1. *(*[Hu, p. 719]*) Let $A_i, B_i \in B(X)$ for all $i \in J$. Then*

$$d_H\left(\cup_{i\in J} A_i, \cup_{i\in J} B_i\right) \leq \sup_{i\in J} d_H(A_i, B_i).$$

The following typical classes

$$\begin{aligned} C(X) &:= \{A \subset X | A \text{ is nonempty and closed}\}, \\ B(X) &:= \{A \subset X | A \text{ is nonempty, closed and bounded}\}, \\ K(X) &:= \{A \subset X | A \text{ is nonempty and compact}\} \end{aligned}$$

satisfy the obvious inclusions

$$X \subset K(X) \subset B(X) \subset C(X).$$

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[1] For the hyperspace $(C(X), d_H)$, we in fact often employ the metric $\min\{1, d_H\}$ (cf. e.g. [BV]) but, for the sake of simplicity, we will use the same notation here.

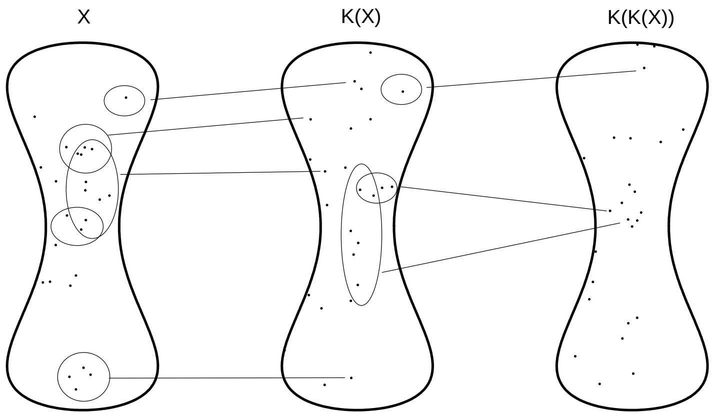

Figure 3: Structure of hyperspaces

If $(X, ||.||)$ is a Banach space, then we can also consider their subclasses, namely

$$C_{Co}(X) := C(X) \cap Co(X),$$
$$B_{Co}(X) := B(X) \cap Co(X),$$
$$K_{Co}(X) := K(X) \cap Co(X),$$

where

$$Co(X) := \{A \subset X | A \text{ is nonempty and convex}\},$$

and obviously

$$X \subset K_{Co}(X) \subset B_{Co}(X) \subset C_{Co}(X) \subset Co(X).$$

The hyperspace $(B(X), d_H)$ is a closed subset of $(C(X), d_H)$ (cf. e.g. [HP, Proposition 1.7]) and if (X, d) is complete, then also $(K(X), d_H)$ is a closed subset of $(C(X), d_H)$ (cf. e.g. [KT, Theorem 4.3.9], [HP, Proposition 1.6], [Be, Exercise 3.2.4 (b)]). Thus, for a complete (X, d), $(C(X), d_H)$ is complete (cf. e.g. [Be, Theorem 3.2.4]), and $(K(X), d_H) \subset (B(X), d_H)$ are both complete subspaces of $(C(X), d_H)$ (cf. e.g. [HP, Propositions 1.6 and 1.7]).

If $(X, ||.||)$ is a normed space, then

$$(K_{Co}(X), d_H) \subset (B_{Co}(X), d_H) \subset (C_{Co}(X), d_H)$$

$(X,\, d)$	$(K(X),\, d_H)$	references
compact	compact	[Be], [KT], [Mi]
complete	complete	[Be], [HP], [KT]
separable	separable	[HP], [KT]
Polish	Polish	[HP]
locally compact	locally compact	[Mi]
connected	connected	[Mi]
locally connected	locally connected	[Mi]
locally continuum-connected	ANR	[Cu]
locally continuum-connected and connected	AR	[Cu]

Table 1: Induced properties of $(K(X),\, d_H)$ from $(X,\, d)$.

are all closed subsets of $(C(X),\, d_H)$ (cf. e.g. [HP, Corollary 1.9]), and subsequently, for a Banach space $(X,\, ||.||)$, all the above subsets are complete subspaces of $(C(X),\, d_H)$ (cf. e.g. [HP, Remark 1.10]).

If $X \subset E^n$ is nonempty, compact convex subset of $E^n = \underbrace{E \times \cdots \times E}_{n-\text{times}}$, where $(E,\, ||.||)$ is a Banach space, then $(K_{Co}(X),\, d_H)$ is, according to [HH] and [LFKU], a compact and convex subspace of $(K(X),\, d_H)$.

Besides the mentioned completeness, the list of some further induced properties of $(K(X),\, d_H)$ by those of $(X,\, d)$ can be found in Table 1. Except last two properties in Table 1, the implications hold in both directions, i.e. these properties are in fact equivalent on the lines.

Remark 3. Since the Vietoris topology, called a finite topology in [Mi], coincides in $K(X)$ with the Hausdorff metric topology (cf. e.g. [HP, Theorem 1.30 on p. 14], [Be, Exercise 3.2.9]), we could also employ for Table 1 the equivalences proved in [Mi].

Let us recall that, by a *Polish space*, we understand as usual a complete and separable metric space and that X is an AR (ANR) if, for each Y and every closed $A \subset Y$, every continuous mapping $f : A \to X$ is extendable over Y (a neighbourhood of A in Y). Furthermore, a metric space $(X,\, d)$ is *locally continuum-connected* if, for each neighbourhood U of each point $x \in X$, there is a neighbourhood $V \subset U$ of x such that each point of V can be connected with x by a subcontinuum (i.e. a compact, connected subset) of U.

In locally compact (e.g. Euclidean) spaces $(X,\, d)$, the local continuum-connectedness can be simply replaced by the local connectedness in Table 1. Since the ANRs and ARs are locally continuum-connected and the ARs are still connected, for $(K(X),\, d_H)$ to be an ANR (AR), it is obviously enough to assume that so are $(X,\, d)$, respectively.

(X, d)	$(C(X), d_H)$	references
compact	compact	[Be], [HP]
complete	complete	[Be], [HP]
relatively compact	relatively compact	[Be]

Table 2: Equivalent properties of (X, d) and $(C(X), d_H)$.

For more details concerning the ANRs and ARs, see e.g. [AG2], [AV], [Cu].

It will be also convenient to recall some equivalent properties of (X, d) and $(C(X), d_H)$ in Table 2. However, let us note that, unlike in Table 1, many properties are not induced here from spaces to hyperspaces. For instance, because of a counter-example in [KSY], $(C(\mathbb{R}^n), d_H)$ was shown there to be non-separable and, in particular, not Polish. More precisely, $(C(\mathbb{R}^n), d_H)$ has according to [KSY, Proposition 7.2] uncountably many components (i.e. maximal connected subsets) and $(K(\mathbb{R}^n), d_H)$ is the only separable (connected, closed and open) component of $(C(\mathbb{R}^n), d_H)$. In particular, $(C(\mathbb{R}^n), d_H)$ is disconnected. It is also not locally compact (cf. [CLP]). This demonstrates that connectedness and local compactness are not induced.

On the other hand, according to [BV, Corollary 3.8], a metric locally convex space (X, d) is normable if and only if its hyperspace $(C(X), d_H)$ is an ANR. This implies that $(C(\mathbb{R}^n), d_H)$ is an ANR and, in view of its just mentioned disconnectedness, it cannot be an AR. In particular, $(C(\mathbb{R}^n), d_H)$ is only locally (path-)connected. Furthermore, e.g. the hyperspace $(C(\mathbb{R}^\infty), d_H)$, where $\mathbb{R}^\infty$ is the countable product of lines, is not an ANR (despite $(\mathbb{R}^\infty, |.|)$ is an AR) and, equivalently (see the comments to Diagram 1 in [BV]), it is even not locally connected. Similarly, non-normable Fréchet spaces (X, d) do not induce, unlike Banach spaces, $(C(X), d_H)$ to be locally connected. In [BV], sufficient and necessary conditions were established, for both $(C(X), d_H)$ and $(B(X), d_H)$, to be ANR (AR). Thus, for instance, $(B(X), d_H)$, where $X \subset E$ is a convex subset of a normed linear space $(E, ||.||)$ is an AR (cf. [AC]). Moreover, every component of $(C(E^n), d_H)$, where $(E, ||.||)$ is a Banach space, was shown in [KSY] to be a complete AR.

Hyperspaces of compact convex spaces (in our case $K_{Co}(\mathbb{R}^m)$) were extensively studied in [NQS1], [NQS2] and main results were summed up in [IML]. Let us remind two theorems [IML, Theorems 23, 24, p. 27].

Theorem 1. *If Y is compact and convex and* $\dim(Y) \geq 2$), *then $K_{Co}(Y)$ is homeomorphic to the Hilbert cube.*

Theorem 2. *If Y is either the open unit ball in $\mathbb{R}^n$ or $\mathbb{R}^n$ itself* ($n \geq 2$), *then $K_{Co}(\mathbb{R}^n)$ is homeomorphic to Hilbert cube minus a point.*

Remark 4. Since, $\dim(K_{Co}(\mathbb{R})) = 2$ (see [IML]), we will be able to visualize isometrically hyperfractals in $K_{Co}(\mathbb{R})$.

For more details concerning the relationship between spaces and hyperspaces see e.g. [AC], [Be], [BV], [CLP], [Cu], [HP], [IML], [IN], [KSY], [KT], [Mi], [N1], [NQS1], [NQS2], [Wi].

Now, we proceed to maps. Let us remind at least notions related to contractions.

Definition 1. ([Ba1, Definition 6.1, p. 74]) A transformation $f : X \to X$ on a metric space (X, d) is called *contractive* or a *contraction mapping* if there is a constant $0 \leq r < 1$ such that

$$d(f(x), f(y)) \leq rd(x, y), \forall x, y \in X.$$

Any such number r is called a *contractivity factor* for f.

We will meet often a particular contraction called a similitude.

Definition 2. A transformation $f : X \to X$ on a metric space (X, d) is called a *similitude* if there is a constant $0 \leq r < 1$ such that

$$d(f(x), f(y)) = rd(x, y)\, \forall x, y \in X.$$

Proposition 1. [Hu, p. 717] *A mapping $f : \mathbb{R}^m \to \mathbb{R}^m$ is a similitude if and only if*

$$f = g_r \circ g_b \circ \mathscr{Q},$$

for some homothety g_r, translation g_b and orthonormal transformation $\mathscr{Q}$.

Remark 5. This means that we can write for a similitude $f : \mathbb{R}^m \to \mathbb{R}^m$,

$$f(x) = r\mathscr{Q}x + b,$$

where $r \in [0, 1)$, $\mathscr{Q} \in \mathbb{R}^{m\times m}$ is orthonormal, $b \in \mathbb{R}^m$.

In $\mathbb{R}^2$, each orthonormal transformation $\mathscr{Q}$ can be expressed as rotation (and reflection if $\det(\mathscr{Q}) = -1$),

$$\mathscr{Q} = \mathscr{O} \cdot \mathscr{R},$$

where

$$\mathscr{O} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}, \quad \mathscr{R} = \begin{pmatrix} 1 & 0 \\ 0 & \det(\mathscr{Q}) \end{pmatrix}.$$

Theorem 3. *(Banach theorem* [Ba1, Theorem 6.1, p. 75]*) Let $f : X \to X$ be a contraction mapping on a complete metric space (X, d). Then f possesses exactly one fixed point $x_f \in X$ and, moreover, for any point $x \in X$, the sequence $\{f^0(x), f^1(x), f^2(x), \dots\}$ converges to x_f.*

Remark 6. We can also estimate the distance $d(x, x_f)$ of any $x \in X$,

$$d(x, x_f) \leq \frac{d(x, f(x))}{1 - r}.$$

These are three crucial properties of fractals generated by IFSs. We will obtain a unique fractal for each IFS, every orbit will converge to it and we will be able to estimate the distance of any iteration from the fractal.

Next, let us study multivalued maps and the induced (in hyperspaces) hypermaps.

Definition 3. ([AFGL]) Let (X, d) and (Y, d') be two metric spaces. *A multivalued map* from X to Y is a map

$$\mathcal{F} : X \to 2^Y/\{\emptyset\}.$$

All multivalued maps will have at least closed values. By a *fixed point* of a multivalued map $\mathcal{F} : X \to 2^X/\{\emptyset\}$, we mean $x_{\mathcal{F}} \in X$ with $x_{\mathcal{F}} \in \mathcal{F}(x_{\mathcal{F}})$.

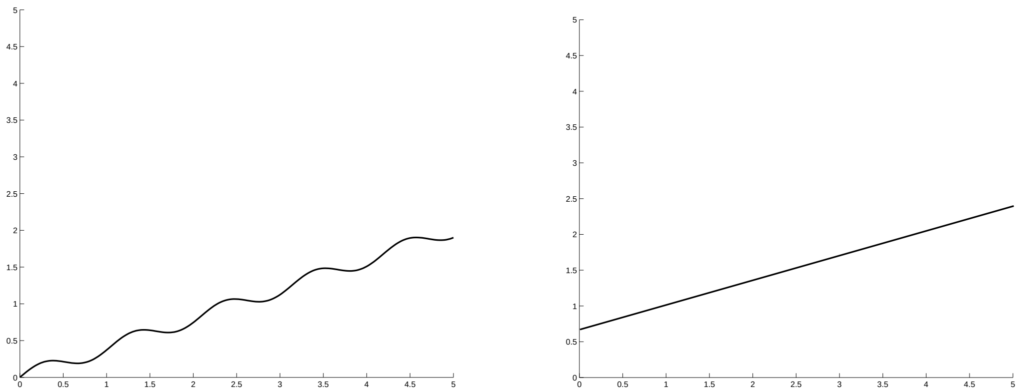

Figure 4: Contraction and similitude in $\mathbb{R}$

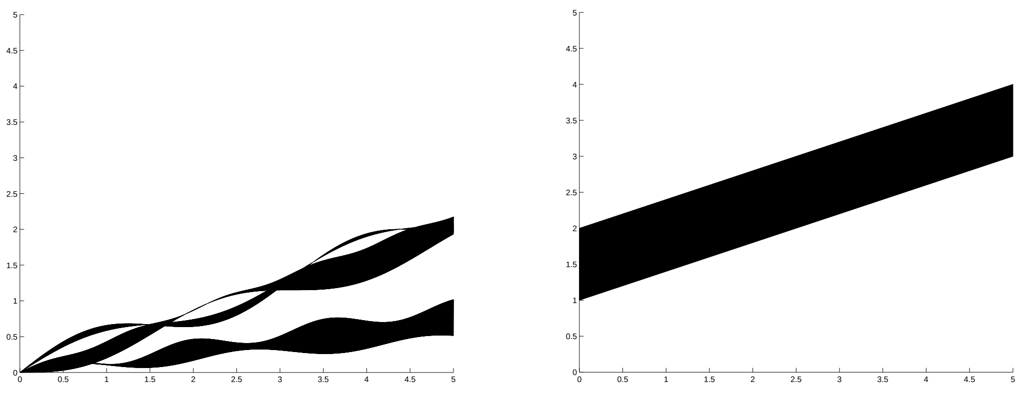

Figure 5: Multivalued contraction and similitude in $\mathbb{R}$

Let us also remind induction of a map. A map $f : X \to X$ is induced to a hypermap $F : 2^X/\{\emptyset\} \to 2^X/\{\emptyset\}$,

$$F(A) = \bigcup_{x \in A} \{f(x)\}.$$

Remark 7. We will often induce contractions $f : X \to X$ to $F : K(X) \to K(X)$,

$$F(A) = \bigcup_{x \in A} \{f(x)\}.$$

Note that $F : K(X) \to K(X)$ is also a contraction. If $f : X \to X$ is a similitude then $F : K(X) \to K(X)$ is also a similitude.

A multivalued map $\mathcal{F} : X \to 2^X/\{\emptyset\}$ is induced to a hypermap $F : 2^X/\{\emptyset\} \to 2^X/\{\emptyset\}$,

$$F(A) = \bigcup_{x \in A} \mathcal{F}(x).$$

In view of applications in Section 4, we would like to have at least continuous hypermaps. There are examples (see e.g. [AF]) that the upper semicontinuity of multivalued maps is insufficient for this aim. It is well-known (see e.g. [AG2], [HP]) that compact-valued upper semicontinuous maps $\mathcal{F} : X \to K(Y)$ induce the (single-valued) hypermaps $\mathcal{F} : K(X) \to K(Y)$ which are continuous only w.r.t. the upper-Vietoris topology, and subsequently the upper-Hausdorff topology, but not necessarily continuous w.r.t. the Hausdorff metric topology. On the other hand, if a compact-valued $\mathcal{F} : X \to K(Y)$ is continuous w.r.t. a metric in X and the Hausdorff metric topology in $K(Y)$, then the induced hypermap is also continuous w.r.t the Hausdorff metric topology. If (X, d_1) and (Y, d_2) are, in particular, compact metric spaces (by which $(K(X), d_{H_1})$ and $(K(Y), d_{H_2})$ become compact as well, see Table 1), then $\mathcal{F} : K(X) \to K(Y)$ with closed (=compact) values is continuous if and only if

$$\mathcal{F}(cl_X(\cup_{A \in K(X)} A)) = cl_Y(\cup_{A \in K(X)} \mathcal{F}(A)).$$

Since continuous w.r.t. d_{H_2} in $C(Y)$ multivalued maps $\mathcal{F} : X \to C(Y)$ with closed but not necessarily compact values are only lower semicontinuous (for the definition, see below) in general (see e.g. [AG2]), i.e. continuous only w.r.t. d_1 in X and the lower-Vietoris topology in $C(X)$, the induced hypermaps are obviously again not necessarily continuous w.r.t. d_{H_1} and d_{H_2}. Thus, in order to preserve continuity by induction from spaces to hyperspaces, its concept must be different here.

Hence, a map $\mathcal{F} : X \to C(Y)$ with closed values is said to be *upper semicontinuous* (u.s.c.) if, for every open $U \subset Y$, the set $\{x \in X | \, \mathcal{F}(x) \subset U\}$ is open in X. It is said to be *lower semicontinuous* (l.s.c.) if, for every open $U \subset Y$, the set $\{x \in X | \, \mathcal{F}(x) \cap U \neq \emptyset\}$ is open in X. If it is both u.s.c. and l.s.c., then it is called *continuous.*

Remark 8. Unlike for multivalued maps with noncompact values, for compact-valued multivalued maps, this continuity concept coincides with the continuity w.r.t. d_1 in X and the Hausdorff metric topology induced by d_{H_2} in $K(Y)$ (cf. e.g. [AG2], [HP]).

The following implications hold for continuous multivalued maps and the induced hypermaps (cf. [AF], [AG2, Appendix 3], [HP]):

$$\begin{aligned} \mathcal{F}: X \to C(X) &\Rightarrow \overline{\mathcal{F}}: C(X) \to C(X), \\ \textit{Lipschitz } \mathcal{F}: X \to B(X) &\Rightarrow \overline{\mathcal{F}}: B(X) \to B(X), \\ \mathcal{F}: X \to K(X) &\Rightarrow \mathcal{F}: K(X) \to K(X), \end{aligned}$$

where $\overline{\mathcal{F}}$ means that

$$\overline{\mathcal{F}}(A) := \overline{\bigcup_{x \in A} \mathcal{F}(x)} = cl_X(\bigcup_{x \in A} \mathcal{F}(x)), \text{ for all } A \in C(X),$$

but obviously

$$\mathcal{F}: E \to C_{Co}(E) \not\Rightarrow \overline{\mathcal{F}}: C_{Co}(E) \to C_{Co}(E),$$

where $(E, ||.||)$ is a Banach space.

Nevertheless, for special maps of the form

$$\mathcal{F}_0 : E^n \to C_{Co}(E^n),\ \mathcal{F}_0(x) := \mathscr{A}x + C,$$

where $\mathscr{A}$ is a real $n \times n-$matrix, $C \in C_{Co}(E^n)$, $E^n = \underbrace{E \times \cdots \times E}_{n-\text{times}}$ and $(E, ||.||)$ is a Banach space, we also have

$$\mathcal{F}_0 : E^n \to C_{Co}(E^n) \Rightarrow \overline{\mathcal{F}_0} : C_{Co}(E^n) \to C_{Co}(E^n).$$

Let us note that although, in vector spaces, the linear combinations of convex sets are convex (see e.g. [Be, Theorem 1.4.1]), they need not be closed (see e.g. [AB, Example 5.3.]) which justifies to use $\overline{\mathcal{F}_0}$ instead of $\mathcal{F}_0$. On the other hand, in vector spaces, scalar multiples of closed sets are closed and the algebraic sum of a compact set and a closed set is closed (see e.g. [AB, Lemma 5.2]). Thus, if $C \in K_{Co}(E^n)$, then the bar need not be used for $\mathcal{F}_0$, in order the last implication to be satisfied. Moreover, an affine function image of a convex set is convex (see e.g. [Be, Theorem 1.4.1]).

Since $\mathcal{F}_0$ is Lipschitz continuous, in view of the above implications, we get still

$$\mathcal{F}_0 : E^n \to B_{Co}(E^n) \Rightarrow \overline{\mathcal{F}_0} : B_{Co}(E^n) \to B_{Co}(E^n),$$

$$\mathcal{F}_0 : E^n \to K_{Co}(E^n) \Rightarrow \mathcal{F}_0 : K_{Co}(E^n) \to K_{Co}(E^n),$$

provided $C \in B_{Co}(E^n)$ and $C \in K_{Co}(E^n)$, respectively. If $C \in K_{Co}(E^n)$, then we can also simply write

$$\mathcal{F}_0 : E^n \to C_{Co}(E^n) \Rightarrow \mathcal{F}_0 : C_{Co}(E^n) \to C_{Co}(E^n),$$

$$\mathcal{F}_0 : E^n \to B_{Co}(E^n) \Rightarrow \mathcal{F}_0 : B_{Co}(E^n) \to B_{Co}(E^n).$$

map	hypermap	references
continuous $\mathcal{F} : X \to K(X)$	continuous $\mathcal{F} : K(X) \to K(X)$	[AF], [AG2, Prop. A 3.43]
compact $\mathcal{F} : X \to K(X)$	compact $\mathcal{F} : K(X) \to K(X)$	[AF], [AG2, Prop. A 3.47]
(weakly) contractive $\mathcal{F} : X \to C(X)$ $\mathcal{F} : X \to B(X)$ $\mathcal{F} : X \to K(X)$	(weakly) contractive $\overline{\mathcal{F}} : C(X) \to C(X)$ $\overline{\mathcal{F}} : B(X) \to B(X)$ $\mathcal{F} : K(X) \to K(X)$	[AF], [AG2, Prop. A 3.20]
contraction $\mathcal{F}_0 : E^n \to C_{Co}(E^n)$ $\mathcal{F}_0 : E^n \to B_{Co}(E^n)$ $\mathcal{F}_0 : E^n \to K_{Co}(E^n)$	contraction $\overline{\mathcal{F}_0} : C_{Co}(E^n) \to C_{Co}(E^n)$ $\overline{\mathcal{F}_0} : B_{Co}(E^n) \to B_{Co}(E^n)$ $\mathcal{F}_0 : K_{Co}(E^n) \to K_{Co}(E^n)$	trivial consequences of the upper equivalences

Table 3: Some equivalent properties of maps and hypermaps.

Now, let us recall that a multivalued mapping $\mathcal{F} : X \to C(X)$ is said to be *weakly contractive* (cf. [AF], [AG2]) if, for any $x, y \in X$, $d_H(\mathcal{F}(x), \mathcal{F}(y)) \leq h(d(x, y))$, where $h : [0, \infty) \to [0, \infty)$ is a continuous, nondecreasing function such that $h(0) = 0$ and $0 < h(t) < t$, for $t > 0$. For $h(t) := rt$, $t \in [0, \infty)$, ($\Rightarrow r \in [0, 1)$), the mapping $\mathcal{F}$ is obviously a *contraction.* For weakly contractive single-valued maps $\mathcal{F} : X \to X$, it is enough to replace d_H by d.

Remark 9. As pointed out in [AFGL, Remark 1] and [KS, Remark on p. 8], the notion of a weak contraction can be weaken, namely that the function h need not be monotonic and it also suffices to take a right upper semicontinuous (in a single-valued sense) h. The equivalences for weakly contractive maps and hypermaps were proved in [AF], [AG2, Proposition A 3.20], provided still $\lim_{t\to\infty} t - h(t) = \infty$, but this condition does not play any role.

The implications in Table 3 concerning the properties of hypermaps induced by those of maps were proved in [AF], [AG2, Appendix 3]. Because of the trivial reverse implications these properties are, in fact, equivalent. The last equivalences, for the map $\mathcal{F}_0$ defined above, follow directly from the preceding ones, on the basis of the above arguments.

Remark 10. As already pointed out, the bar can be omitted for $\mathcal{F}_0$ in Table 3, provided $C \in K_{Co}(E^n)$ in the definition of $\mathcal{F}_0$.

Remark 11. Because of counter-examples (see e.g. [L4, Examples 1 and 2]), compact hypermaps $\mathcal{F}$ and $\mathcal{F}_0$ (even $\overline{\mathcal{F}}$ and $\overline{\mathcal{F}_0}$ or Lipschitz $\overline{\mathcal{F}}$ and $\overline{\mathcal{F}_0}$), on the hyperspaces $C(X)$, $B(X)$ and $C_{Co}(E^n)$, $B_{Co}(E^n)$, need not imply the compactness of the related maps $\mathcal{F}$ and $\mathcal{F}_0$, on the spaces X and E^n. Nevertheless, compact maps $\mathcal{F}$ and $\mathcal{F}_0$ imply there the compact induced hypermaps $\overline{\mathcal{F}}$ and $\overline{\mathcal{F}_0}$.

Remark 12. In view of the above arguments, we can still add one more assertion. Let $X \subset E^n$ be a nonempty, compact, convex subset of E^n such that $\mathcal{F}_0 : X \to K_{Co}(X)$. Since $\mathcal{F}_0$ is (Lipschitz-)continuous, so must be $\mathcal{F}_0 : K_{Co}(X) \to K_{Co}(X)$ as well, and vice versa. Observe that, in this case, the Lipschitz constant L need not necessarily satisfy $L \in [0, 1)$.

In the following sections, we will define the hyperspaces $(H_1(H_2(X)), d_{H_H})$ as certain classes of nonempty subsets of $H_2(X)$, endowed with the Hausdorff metric d_{H_H}, induced by the metric d_H. Obviously, many properties of these new spaces can be directly induced from the supporting spaces. For instance, $K(K_{Co}(X), d_{H_H})$ is compact if and only if so is (X, d). Similarly, many properties of hypermaps on these new hyperspaces can be induced by those of maps on the supporting spaces. For instance, $\mathcal{F} : K(K(X)) \to K(K(X))$ is compact continuous if and only if so is $\mathcal{F} : X \to K(X)$ or $\mathcal{F} : K(X) \to K(K(X))$ and $\mathcal{F}_0 : K(K_{Co}(E^n)) \to K(K_{Co}(E^n))$ is a contraction if and only if so is $\mathcal{F}_0 : X \to K_{Co}(E^n)$ or $\mathcal{F}_0 : K_{Co}(E^n) \to K(K_{Co}(E^n))$.

3. Iterated function systems and invariant measures

Iterated function systems (IFSs) provide the simplest tool to produce fractals. Their attractors are often self-similar and we can calculate their Hausdorff dimension by means of the Moran formula. Moreover, attractors of IFSs are supports of invariant measures for IFSs with probabilities. Since attractors are fully described by parameters of functions comprising IFSs, Barnsley (cf. [BEHL], [BH]) used IFSs for data compression. Moreover, many current methods of data compression are based on his results.

3.1. Iterated function systems

Let us remind the crucial results for us given mainly by Barnsley [Ba1] and Hutchinson [Hu].

Definition 4. [Ba1, Definition 7.1, p. 80] *A (hyperbolic) iterated function system* consists of a complete metric space (X, d) together with a finite set of contraction mappings $f_i : X \to X$, with respective contractivity factors r_i, for $i = 1, 2, \ldots, n$. The abbreviation "IFS" is used for "iterated function system." The notation for the IFS just announced is $\{X; f_i, i = 1, 2, \ldots, n\}$ and its contractivity factor is $r = \max\{r_i : i = 1, 2, \ldots, n\}$.

Remark 13. We often drop hyperbolic.

Theorem 4. [Ba1, Theorem 7.1, p. 81] *Let $\{X; f_i, i = 1, 2, \ldots, n\}$ be a hyperbolic IFS with contractivity factor r. Then the transformation $F : K(X) \to K(X)$*

defined by

$$F(B) := \bigcup_{i=1}^{n} f_i(B),$$

for all $B \in K(X)$, *is a contraction mapping on the complete metric space* $(K(X),\, d_H)$ *with contractivity factor* r. *That is*

$$d_H(F(B),\, F(C)) \le r d_H(B,\, C),$$

for all $B,\, C \in K(X)$. *Its unique fixed point* $A^* \in K(X)$ *obeys*

$$A^* = \bigcup_{i=1}^{n} f_i(A^*)$$

and is given by

$$A^* = \lim_{k\to\infty} F^k(B)$$

for any $B \in K(X)$.

Proof. Contractivity of F follows from Lemma 1.

$$d_H(F(A),\, F(B)) = d_H(\cup_{i=1}^{m} f_i(A),\, \cup_{i=1}^{m} f_i(B)) \le$$

$$\le \max_{i\in 1,\, 2,\, \ldots,\, m} d_H(f_i(A),\, f_i(B)) \le \max_{i\in 1,\, 2,\, \ldots,\, m} r_i d_H(A,\, B) = r d_H(A,\, B).$$

□

Remark 14. The operator F is called *the Hutchinson-Barnsley operator.* Sometimes, the multivalued map $\mathcal{F} : X \to K(X)$,

$$\mathcal{F}(x) = \bigcup_{i} \{f_i(x)\},$$

is considered. It is called *the Hutchinson-Barnsley map.* Inducing it to a hyperspace, we obtain the Hutchinson-Barnsley operator F.

Definition 5. [Ba1, Definition 7.2, p. 81] The fixed point $A \in K(X)$ described in the theorem is called *an attractor* of the IFS.

Contractions in $K(X)$ need not to be induced only by single-valued mappings.

Definition 6. [Ba1, Definition 9.1, p. 91] Let $(X,\, d)$ be a metric space and let $C \in K(X)$. Define a transformation $f_0 : K(X) \to K(X)$ by

$$f_0(B) = C \text{ for all } B \in K(X).$$

Then f_0 is called *a condensation transformation* and C is called *the associated condensation set.*

Observe that a condensation transformation $f_0 : K(X) \to K(X)$ is a contraction mapping on the metric space $(K(X),\, d_H)$, with contractivity factor equal to zero, and that it possesses a unique fixed point, namely the condensation set.

Definition 7. [Ba1, Definition 9.2, p. 91] Let $\{X; f_i, i = 1, 2, \ldots, n\}$ be a hyperbolic IFS with contractivity factor r. Let $f_0 : K(X) \to K(X)$ be a condensation transformation. Then $\{X; f_i, i = 0, 1, 2, \ldots, n\}$ is called *a hyperbolic IFS with condensation,* with contractivity factor r.

Theorem 5. [Ba1, Theorem 9.1, p. 91] *Let $\{X; f_i, i = 0, 1, 2, \ldots, n\}$ be a hyperbolic IFS with condensation, with contractivity factor r. Then the transformation $F : K(X) \to K(X)$ defined by*

$$F(B) := \cup_{i=0,\,1,\,\ldots,\,n} f_i(B),\ \forall B \in K(X),$$

is a contraction mapping on the complete metric space $(K(X),\, d_H)$ with a contractivity factor r. That is

$$d_H(F(B),\, F(C)) \leq r \cdot d_H(B,\, C),\ \forall B,\, C \in K(X).$$

Its unique fixed point, A^* obeys

$$A^* = F(A^*) = \cup_{i=0,\,1,\,\ldots,\,n} f_i(A^*)$$

and is given by

$$A^* = \lim_{n\to\infty} F^n(B)$$

for any $B \in K(X)$.

Let us introduce the code space (see [Ba1], [Ba2], [BK], [BKS]), which helps us to describe fractals. We denote by Σ'_A the space which consists of all finite strings of symbols from the alphabet A. We denote by Σ_A the space which consists of all infinite strings of symbols from the alphabet A.

Definition 8. [Ba2, Definition 1.4.1, p. 17] Let $\phi : \Sigma \to X$ be a mapping from $\Sigma \subset \Sigma'_A \cup \Sigma_A$ onto a space X. Then ϕ is called *an address function* for X, and points in Σ are called *addresses.* Σ is called a code space. Any point $\sigma \in \Sigma$ such that $\phi(\sigma) = x$ is called an address of $x \in X$. The set of all addresses of $x \in X$ is $\phi^{-1}(\{x\})$.

Remark 15. Σ'_A is countable Σ_A is uncountable.

Remark 16. We denote by $|\sigma|$ the length of $\sigma \in \Sigma'_A$ and $\sigma|k := \sigma_1 \ldots \sigma_k$. From now on, $\Sigma = \Sigma_A$.

We will use the code space to build the address structure for points and subsets of an attractor of an IFS. We can write for each attractor of an IFS

$$A^* = f_1(A^*) \cup f_2(A^*) \cup \cdots \cup f_m(A^*).$$

Let us continue and express each A^* on the right side of the equation by this right side. We obtain

$$A^* = f_1(A^*) \cup f_2(A^*) \cup \cdots \cup f_m(A^*) =$$
$$f_1(f_1(A^*) \cup f_2(A^*) \cup \cdots \cup f_m(A^*)) \cdots \cup f_m(f_1(A^*) \cup f_2(A^*) \cup \cdots \cup f_m(A^*)) =$$
$$= f_1(f_1(A^*)) \cup f_1(f_2(A^*)) \cup \cdots \cup f_1(f_m(A^*)) \cup \ldots f_m(f_m(A^*)).$$

We see that the attractor consists of its contracted images. We assign an address to each of these sets according to applied contractions. Thus, $A^*_1 := f_1(A^*)$, or $A^*_\sigma := f_{\sigma_1}(\ldots f_{\sigma_m}(A^*))$, $\sigma = \sigma_1\sigma_2\ldots\sigma_m$. If we continue this process to infinity, we obtain addresses of the points of the attractor.

Definition 9. [Ba1, Definition 2.1, p. 122] Let $\{X,\, f_1,\, f_2,\, \ldots,\, f_m\}$ be a hyperbolic IFS. The *code space associated with the IFS*, $(\Sigma,\, d_\Sigma)$, is defined to be the code space on m symbols $\{1,\, 2,\, \ldots,\, m\}$, with the metric given by

$$d_\Sigma(\sigma,\, \omega) = \sum_{i=1}^{\infty} \frac{|\sigma_i - \omega_i|}{(m+1)^i},\ \forall \omega,\, \sigma \in \Sigma.$$

After the introduction to addresses, let us provide a few simple facts about attractors of IFSs ([Hu, 3.1 (3), p. 10]).

Proposition 2. *1.* $A^*_{i_1 i_2 \ldots i_p} = \cup_{i_{p+1}=1}^{m} A^*_{i_1 i_2 \ldots i_p i_{p+1}}$.

2. $A^* \supset A^*_{i_1} \supset \cdots \supset A^*_{i_1 i_2 \ldots i_p} \supset \ldots$, *and* $\cap_{p=1}^{\infty} A^*_{i_1 i_2 \ldots i_p}$ *is a singleton whose member is denoted* $a^*_{i_1 i_2 \ldots i_p \ldots}$. A^* *is union of these singletons.*

3. A^* *is the closure of the set of fixed points of the* $f_{i_1 i_2 \ldots i_p}$.

4. $f_{i_1 i_2 \ldots i_p}(A^*_{j_1 j_2 \ldots j_q}) = A^*_{i_1 i_2 \ldots i_p j_1 j_2 \ldots j_q}$.
$f_{i_1 i_2 \ldots i_p}(a^*_{j_1 j_2 \ldots j_q \ldots}) = a^*_{i_1 i_2 \ldots i_p j_1 j_2 \ldots j_q \ldots}$.

5. *If* B *is a nonempty bounded set, then* $d(B_{i_1 i_2 \ldots i_p},\, a^*_{i_1 i_2 \ldots i_p \ldots}) \to 0$ *uniformly as* $p \to \infty$. *In particular,* $F^p(B) \to A^*$ *in the Hausdorff metric.*

Remark 17. Notice that A^*_σ is a set for $|\sigma| = k$, $k \in \mathbb{N}$, and a^*_σ is a point for $|\sigma| = \infty$.

Barnsley proves a very similar claim to 2. Before we state it, let us give supporting proposition and lemmas.

Proposition 3. [Ba1, Theorem 7.1, p. 35] *Let* $(X,\, d)$ *be a complete metric space. Then* $(K(X),\, d_H)$ *is a complete metric space. Moreover, if* $\{A_n\}$, *where* $A_n \in K(X)\, \forall n \in \mathbb{N}$, *is a Cauchy sequence, then*

$$A = \lim_{n\to\infty} A_n \in K(X)$$

can be characterized as follows:

$$A = \{x \in X,\ \text{there is a Cauchy sequence}\, \{x_n \in A_n\}\ \text{that converges to}\ x\}.$$

LEMMA 2. [Ba1, Lemma 2.1, p. 122] *Let $\{X, f_i, i = 1, 2, \ldots, n\}$ be an IFS, where (X, d) is a complete metric space. Let $C \in K(X)$. Then there exists $\widetilde{C} \in K(X)$ such that $C \in \widetilde{C}$ and $f_i : \widetilde{C} \to \widetilde{C}$ for $i = 1, 2, \ldots, n$. In other words, $\{\widetilde{C}, f_i, i = 1, 2, \ldots, n\}$ is an IFS where the underlying space is compact.*

LEMMA 3. [Ba1, Lemma 2.2, p. 123] *Let $\{X, f_i, i = 1, 2, \ldots, n\}$ be an IFS of contractivity factor s, where (X, d) is a complete metric space. Let (Σ, d_Σ) denote the code space associated with the IFS. For each $\sigma \in \Sigma$, $n \in \mathbb{N}$, and $x \in X$, define*

$$\phi(\sigma, n, x) := f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_n}.$$

Let C denote a compact nonempty subset of X. Then there is a real constant D such that

$$d(\phi(\sigma, m, x_1), \phi(\sigma, n, x_2)) \leq D s^{m \wedge n},$$

for all $\sigma \in \Sigma$, $m, n \in \mathbb{N}$, and $x_1, x_2 \in C$.

THEOREM 6. [Ba1, Theorem 2.1, p. 123] *Let (X, d) be a complete metric space. Let $\{X, f_1, f_2, \ldots, f_N\}$ be an IFS. Let A^* denote the attractor of the IFS. Let (Σ, d_Σ) denote the code space associated with the IFS. For each $\sigma \in \Sigma$, $n \in \mathbb{N}$, and $x \in X$, let*

$$\phi(\sigma, n, x) = f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_n}(x).$$

Then

$$\phi(\sigma) = \lim_{n \to \infty} \phi(\sigma, n, x)$$

exists, belongs to A^* and is independent of $x \in X$. If C is a compact subset of X, then the convergence is uniform over $x \in C$. The function $\phi : \Sigma \to A^*$ is continuous and onto.

Proof. Let $x \in X$. Let $C \in K(X)$ be such that $x \in C$. Employing [Ba1, Lemma 2.1, p. 122], we can define $F : K(X) \to K(X)$ in the usual way. F is a contraction mapping on the metric space $(K(X), d_H)$; and we have

$$A = \lim_{n \to \infty} F^n(C).$$

In particular, $\{F^n(C)\}$ is a Cauchy sequence in $(K(X), d_H)$. Notice that $\phi(\sigma, n, x) \in F^n(C)$. It follows from ([Ba1, Theorem 7.1, p. 81]) that if $\lim_{n\to\infty} \phi(\sigma, n, x)$ exists, then it belongs to A^*.

That the latter limit does exist, follows from the fact that, for fixed $\sigma \in \Sigma$, $\{\phi(\sigma, n, x)\}_{n=1}^{\infty}$ is a Cauchy sequence: by Lemma 2.2 ([Ba1, p.123])

$$d(\phi(\sigma, m, x), \phi(\sigma, n, x)) \leq D s^{m \wedge n},$$

for all $x \in C$, and the right hand side here tends to zero as m and n tend to infinity. The uniformity of this convergence follows from the fact that the constant D is independent of $x \in C$.

Next, we prove that $\phi : \Sigma \to A^*$ is continuous. Let $\epsilon > 0$ be given. Choose n so that $s^n D < \epsilon$, and let $\sigma,\, \omega \in \Sigma$ obey

$$d_\Sigma(\sigma,\, \omega) < \sum_{m=n+2}^{\infty} \frac{N}{(N+1)^m} = \frac{1}{(N+1)^{n+1}}.$$

Then one can verify that σ must agree with ω through n terms: that is, $\sigma_1 = \omega_1,\, \sigma_2 = \omega_2,\, \ldots,\, \sigma_n = \omega_n$. It follows that, for each $m \geq n$, we can write

$$d(\phi(\sigma,\, m,\, x),\, \phi(\omega,\, m,\, x)) = d(\phi(\sigma,\, n,\, x_1),\, \phi(\sigma,\, n,\, x_2)),$$

for some pair $x_1,\, x_2 \in \widetilde{C}$. By Lemma 2.2 [Ba1, p. 123], the right hand side here is smaller than $s^n D$ which is smaller than ϵ. Taking the limit as $m \to \infty$, we find

$$d(\phi(\sigma),\, \phi(\omega)) < \epsilon.$$

Finally, we prove that ϕ is onto. Let $a \in A^*$. Then, since

$$A^* = \lim_{n\to\infty} F^n(\{x\}),$$

it follows from Theorem 7.1 ([Ba1, p. 35]) that there is a sequence $\{\omega^{(n)} \in \Sigma,\, n = 1,\, 2,\, 3,\, \ldots\}$ such that

$$\lim_{n\to\infty} \phi(\omega^{(n)},\, n,\, x) = a.$$

Since $(\Sigma,\, d_\Sigma)$ is compact, it follows that $\{\omega^{(n)}\}$ possesses a convergent subsequence with a limit $\omega \in \Sigma$. Without loss of generality, assume that $\lim_{n\to\infty} \omega^{(n)} = \omega$. Then the number of successive initial agreements between components of $\omega^{(n)}$ and ω increases without limit. That is, if

$$\alpha(n) = \operatorname{card}\{j \in \mathbb{N} : \omega_k^{(n)} = \omega_k \text{ for } 1 \leq k \leq j\},$$

then $\alpha(n) \to \infty$ as $n \to \infty$. It follows that

$$d(\phi(\omega,\, n,\, x),\, \phi(\omega^{(n)},\, n,\, x) \leq D s^{\alpha(n)}.$$

By taking the limit on both sides as $n \to \infty$, we find $d(\phi(\omega),\, a) = 0$, which implies $\phi(\omega) = a$. Hence, $\phi : \Sigma \to A^*$ is onto. This completes the proof. □

3.2. Measure

In the following subsection, we will introduce invariant measures for IFSs with probabilities. The space of normalized Borel measures on a complete metric space equipped with the Hutchinson metric is according to Barnsley [Ba1] “the space where fractals really live”. Before defining a measure, let us remind basic notions.

Definition 10. ([Ba1, Definition 2.1 p. 337]) Let X be a space. Let $\mathcal{M}$ denote a nonempty class of subsets of a space X such that

1. $A,\, B \in \mathcal{M} \Rightarrow A \cup B \in \mathcal{M}$;
2. $A \in \mathcal{M} \Rightarrow X/A \in \mathcal{M}$.

Then $\mathcal{M}$ is called *a field.*

Definition 11. ([Ba1, Definition 2.3]) Let $\mathcal{M}$ be a field such that

$$A_i \in \mathcal{M} \text{ for } i \in \{1,\, 2,\, 3 \ldots\} \Rightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{M}.$$

Then $\mathcal{M}$ is called *$\sigma-$field.*

Given any field there is always a minimal $\sigma-$field which contains it.

Theorem 7. [Ba1, Theorem 2.2 p. 340]*) Let X be a space and let $\mathcal{G}$ be a set of subsets of X. Let $\{\mathcal{M}_\alpha : \alpha \in I\}$ denote the set of all $\sigma-$fields on X which contain $\mathcal{G}$. Then $\mathcal{M} = \cap_\alpha \mathcal{M}_\alpha$ is a $\sigma-$field.*

Definition 12. ([Ba1, Definition 2.4]) Let $\mathcal{G}$ be a set of subsets of a space X. The minimal $\sigma-$field which contains $\mathcal{G}$ from the last theorem is called *$\sigma-$field generated by $\mathcal{G}$.*

Definition 13. ([Ba1, Definition 2.5]) Let $(X,\, d)$ be a metric space. Let $\mathbb{B}(X)$ denote the $\sigma-$field generated by the open subsets of X. $\mathbb{B}(X)$ is called *the Borel field associated with the metric space X*. An element of $\mathbb{B}(X)$ is called a *Borel subset* of X.

Theorem 8. *(*[Ba1, Theorem 2.3]*) Let $(X,\, d)$ be a compact metric space. Then the associated Borel field is generated by a countable set of balls.*

In order to develop the measure theory, we combine [Ba1], [Hu], [Fa1].

Definition 14. Let X be a metric space. We call μ *a measure* on X if μ assigns a non-negative number, possibly ∞, to each subset of X such that

1. $\mu(\emptyset) = 0$,
2. if $A \subset B$ then $\mu(A) \leq \mu(B)$, and
3. if $A_1,\, A_2,\, \ldots$ is a countable sequence of sets then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

Given a measure μ, there is a family of subsets of X on which μ behaves in a nice additive way: a set $A \subset X$ is called $\mu-$measurable (or just measurable it the measure in use is clear) if

$$\mu(E) = \mu(E \cap A) + \mu(E/A),$$

for all $E \subset X$.

We write $\mathcal{M}$ for the family of measurable sets which always form a $\sigma-$field, that is $\emptyset \in \mathcal{M}$, $X \in \mathcal{M}$, and if $A_1,\, A_2,\, \cdots \in \mathcal{M}$ then $\cup_{i=1}^{\infty} A_i \in \mathcal{M}$, $\cap_{i=1}^{\infty} A_i \in \mathcal{M}$ and $A_1/A_2 \in \mathcal{M}$. For reasonably defined measures, $\mathcal{M}$ will be a very large family of sets, and in particular will contain the $\sigma-$field of Borel sets.

Remark 18. What is termed a 'measure' here is often referred to as an outer measure in general texts on the measure theory (see Technical note in [Fa1, p. 8]). Such texts define measure μ only on some $\sigma-$field $\mathcal{M}$, with (1), (2), (3) holding for sets of $\mathcal{M}$, with equality in (3) if the A_i are disjoint sets in $\mathcal{M}$. However, μ can then be extended to all $A \subset X$ by setting

$$\mu(A) = \inf\{\sum_i \mu(A_i),\, A \subset \cup_i A_i,\, A_i \in \mathcal{M}\}.$$

We will consider only Borel measures.

Definition 15. Let $(X,\, d)$ be a metric space. Let $\mathbb{B}(X)$ denote the Borel subsets of X. Let $\mathbb{B}(X)$ be $\mu-$measurable. Then μ is called *a Borel measure.*

We can find out whether a measure is a Borel measure by means of the Carathéodory criterion (see [Fa1]).

Proposition 4. *A measure μ is a Borel measure on $X \subset \mathbb{R}^n$ if and only if*

$$\mu(A \cup B) = \mu(A) + \mu(B),$$

for all $A,\, B \in X$ and dist$(A,\, B) > 0$.

Example 1. [Fa1, Example 1.4, p. 14] One of the most useful measures is the Lebesgue measure $\mathcal{L}^m$. It extends the notion of $n-$dimensional volume to a large collection of subsets in $\mathbb{R}^m$ that includes the Borel sets. If $A = \{(x_1,\, x_2,\, \ldots,\, x_m) \in \mathbb{R}^n :\, a_i \leq x_i \leq b_i\}$ is a "coordinate parallelepiped"(we will also use "blocks") in $\mathbb{R}^m$, the $m-$dimensional volume of A is given by

$$\mathrm{vol}^m(A) = (b_1 - a_1)(b_2 - a_2) \cdots (b_m - a_m).$$

We obtain a measure on $\mathbb{R}^m$ by defining

$$\mathcal{L}^m(A) = \inf\left\{\sum_{i=1}^{\infty} \mathrm{vol}^m(A_i) : A \subset \bigcup_{i=1}^{\infty} A_i\right\},$$

where the infimum is taken over all coverings of A by coordinate parallelepipeds A_i. We get that $\mathcal{L}^m(A)$ =vol$^m(A)$ if A is a coordinate parallelepiped or, indeed, any set for which the volume can be determined by the usual rules of mensuration. Sometimes, we need to define "$k-$dimensional" volume on a $k-$dimensional plane X in $\mathbb{R}^m$; this can be done by identifying X with $\mathbb{R}^k$ and using $\mathcal{L}^k$ on subsets of X in the obvious way.

Let us also supply the definition of the Hausdorff measure and the related Hausdorff dimension (cf. [Hu, p. 720]). In a complete metric space (X, d), for every $s \geq 0$, $\delta > 0$ and $S \subset X$, we can define the *$s-$dimensional Hausdorff measure* of S by

$$H^s(S) := \lim_{\delta \to 0} H^s_\delta(S), \tag{1}$$

where $H^s_\delta(A) := \inf\{\sum_{i=1}^{\infty} (\mathrm{diam} A_i)^s | A \subset \bigcup_{i=1}^{\infty} A_i,\, \mathrm{diam} A_i \leq \delta\}$, and the *Hausdorff dimension* $\dim_H(A)$ of A by

$$\dim_H(A) := \inf\{s \geq 0 | H^s(A) = 0\} = \sup\{s \geq 0 | H^s(A) = \infty\}. \tag{2}$$

the Hausdorff measure is also a Borel measure.

THEOREM 9. *(*[Ba2, Theorem 2.3.19, p. 111]*) Let $\nu \in \mathbb{P}(X)$ be a Borel measure and let $f : X \to X$ be continuous. Then there exists on X Borel measure $\mu \in \mathbb{P}(X)$ such that*

$$\mu(B) = \nu(f^{-1}(B)) \text{ for all } B \in \mathbb{B}(X).$$

We denote this measure μ by $f(\nu)$ and also by $f \circ \nu$.

DEFINITION 16. ([Ba2, Definition 2.3.20, p. 111]) The measure $f(\nu)$ is called *the transformation of the measure ν* by the function f or the transformation f applied to the measure ν.

DEFINITION 17. ([Ba1, Definition 3.3, p. 344]) Let (X, d) be a metric space, and let μ be a Borel measure. Then the *support of* μ is the set $\mathrm{supp}(\mu)$ of points $x \in X$ such that $\mu(O(x, \epsilon)) > 0$, for all $\epsilon > 0$ (where $O(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$).

THEOREM 10. *(*[Ba1, Theorem 3.4, p. 344]*) Let (X, d) be a metric space, and let μ be a Borel measure. Then the support of μ is closed. Let (X, d) be a compact metric space and $\mu(X) > 0$, then* $\mathrm{supp}\, \mu \in K(X)$

DEFINITION 18. ([Ba1, Definition 5.1, p. 349]) Let (X, d) be a compact metric space, and let μ be a Borel measure on X. If $\mu(X) = 1$, then μ is said to be *normalized.*

DEFINITION 19. ([Ba1, Definition 5.2, p. 349]) Let (X, d) be a compact metric space. Let $\mathbb{P}(X)$ denote the set of normalized Borel measures on X. *The Monge-Kantorovich (Hutchinson) metric d_{MK}* on $\mathbb{P}(X)$ is defined by

$$d_{MK}(\mu, \nu) := \sup_{f \in \mathscr{L}(X, \mathbb{R})} \left[\int_X f(x) d\mu - \int_X f(x) d\nu \right],$$

where $\mathscr{L}(X, \mathbb{R}) := \{f : X \to \mathbb{R} | \, |f(x) - f(y)| \leq d(x, y), \text{ for all } x, y \in X\}$.

Let us supply a brief information about the related space of probability measures and the Markov operators acting on it. An operator $M : \mathbb{P}(X) \to \mathbb{P}(X)$ is called the *Markov operator* if it satisfies

1. $M(\lambda_1\mu + \lambda_2\nu) = \lambda_1 M(\mu) + \lambda_2 M(\nu)$, for all $\lambda_1, \lambda_2 \in [0, \infty)$ and $\mu, \nu \in \mathbb{P}(X)$,

2. $M(\mu(X)) = \mu(X)$, for all $\mu \in \mathbb{P}(X)$.

   If there is still a dual operator $U : C(X) \to C(X)$, where $C(X)$ denotes the space of continuous functions $f : X \to \mathbb{R}$ endowed with the sup-norm on X such that

3. $\int_X Uf(x)d\mu = \int_X f(x)Md\mu$, for every $f \in C(X)$ and $\mu \in \mathbb{P}(X)$, then M is called the *Markov-Feller operator.*

It can be proved that every nonexpansive Markov operator is Markov-Feller. For more details and properties of Markov operators, see e.g. [LM1]-[LM4], [MS].

THEOREM 11. *(*[Ba1, Theorem 5.1, p. 349]*) Let (X, d) be a compact metric space. Let $\mathbb{P}(X)$ denote the set of normalized Borel measures on X. Then $(\mathbb{P}(X), d_{MK})$ is a compact metric space.*

REMARK 19. In the same way as we define the hyperspaces $(H_1(H_2(X)), d_{H_H})$ as certain classes of nonempty subsets of $H_2(X)$, endowed with the Hausdorff metric d_{H_H}, induced by the metric d_H, we can define $(\mathbb{P}(H(X)), d_{MK_H})$ as the space of probability measures on $H(X)$, endowed with the Monge-Kantorovich metric d_{MK_H} defined as follows

$$d_{MK_H}(\mu, \nu) := \sup_{f \in \mathscr{L}(H(X), \mathbb{R})} \left[\int_{H(X)} f(x)d\mu - \int_{H(X)} f(x)d\nu \right],$$

where

$$\mathscr{L}(H(X), \mathbb{R}) := \{f : H(X) \to \mathbb{R} |\, |f(x) - f(y)| \le d_H(x, y), \text{ for all } x, y \in H(X)\}.$$

DEFINITION 20. ([Ba1, Definition 6.1, p. 350]) Let (X, d) be a compact metric space, and let $\mathbb{P}(X)$ denote a space of normalized Borel measures on X. Let $\{X, f_1, f_2, \dots, m; p_1, p_2, \dots, p_m\}$ be a hyperbolic IFS with probabilities. *The Markov operator associated with the IFS* is the function $M : \mathbb{P}(X) \to \mathbb{P}(X)$ defined by

$$M(\nu) := p_1\nu \circ f_1^{-1} + p_2\nu \circ f_2^{-1} + \dots + p_m\nu \circ f_m^{-1},$$

for all $\nu \in \mathbb{P}(X)$.

THEOREM 12. *(*[Ba1, Theorem 6.1, p. 351]*) Let (X, d) be a compact metric space. Let $\{X, f_1, f_2, \dots, m; p_1, p_2, \dots, p_m\}$ be a hyperbolic IFS with probabilities. Let $r \in (0, 1)$ be a contractivity factor for the IFS. Let $M : \mathbb{P}(X) \to \mathbb{P}(X)$ be the associated Markov operator. Then M is a contraction mapping, with the contractivity factor r, with respect to the Monge-Kantorovich metric on $\mathbb{P}(X)$. That is*

$$d_H(M(\nu), M(\mu)) \le r d_H(\nu, \mu).$$

Definition 21. ([Ba1, Definition 6.2, p. 352]) Let μ denote the fixed point of the Markov operator, promised by the preceding theorem. μ is called *the invariant measure of the IFS with probabilities.*

Theorem 13. *(*[Ba1, Theorem 6.2, p. 359]*) Let (X, d) be a compact metric space. Let $\{X, f_1, f_2, \dots, m; p_1, p_2, \dots, p_m\}$ $(p_i > 0)$ be a hyperbolic IFS with probabilities. Let μ be the associated invariant measure. Then the support of μ is the attractor of the IFS.*

In order to calculate measure defined by an IFS with probabilities, let us consider the special IFS [BD] $\{\Sigma, s_i, i = 1, 2, \dots, n\}$, where $s_i : \Sigma \to \Sigma$ is defined by

$$s_i(\sigma) := i\sigma \text{ for } \sigma \in \Sigma.$$

It means σ is shifted right by one place and the symbol i is placed as the first component. Note that s_i, $i = 1, 2, \dots, n$, are contractions w.r.t. the code space metric d_Σ. We will use the notation $\mathbb{B}(\Sigma)$ for the Borel subsets of Σ. This $\sigma-$field is generated by the cylinders

$$\{\sigma;\, \sigma_l = i_l,\, o \le l < o + k\},$$

where each $i_l \in \{1, 2, \dots, n\}$. We will define measure ρ for the IFS $\{\Sigma, s_i, i = 1, 2, \dots, n\}$ with p_i, $i = 1, 2, \dots, n$,

$$\rho(\{\sigma \in \Sigma :\, \sigma_l = i_l,\, o \le l < o + k\}) = \prod_{l=o}^{o+k-1} p_{i_l}.$$

We will denote T^* an analogue of the operator M on $\mathbb{P}(X)$ for the space $\mathbb{P}(\Sigma)$. In particular, we have

$$(T^*\nu)(B) := \sum_{i=1}^{n} p_i \nu(s_i^{-1}(B)),$$

for any measure $\nu \in \mathbb{P}(\Sigma)$ and $B \in \mathbb{B}(\Sigma)$.

Remark 20. ([BD, p. 256]) Let us remind that for $\sigma \in \Sigma$, $s_i^{-1}(\sigma) = \emptyset$ for $\sigma_1 \neq i$ and $s_i^{-1}(\sigma) = \omega$ for $\sigma_1 = i$, where $\omega_j = \sigma_j + 1$. For a subset B of Σ, we have $s_i^{-1}(B) = \{s_i^{-1}(\sigma); \sigma \in B\}$.

The following theorem summarizes properties of the IFS $\{\Sigma, s_i, p_i, i = 1, \dots, n\}$.

Theorem 14. *(*[BD, Theorem 4]*) The IFS $\{\Sigma, s_i, p_i, i = 1, 2, \dots, n\}$ with the probability measure defined above have the following properties:*

1. *$\{\Sigma, s_i, i = 1, 2, \dots, n\}$ is a hyperbolic IFS, with attractor Σ;*
2. *ρ is the unique measure for the IFS, in particular, it is the fixed point in $\mathbb{P}(\Sigma)$ of T^*, obeying $T^*(\rho) = \rho$;*

3. *ρ is attractive for any probability measure $\hat{\rho}$ on Σ, namely*

$$\lim_{n\to\infty} T^{*n}(\hat{\rho}) = \rho, \text{ for all } \hat{\rho} \in \mathbb{P}(\Sigma);$$

4. *the support of ρ is Σ, independently of p_i, $p_i > 0$, $i = 1, 2, \ldots, n$;*

5. *for all $B \in \mathbb{B}(\Sigma)$,*

$$\rho(s_i(B)) = p_i\rho(B),\ i = 1, 2, \ldots, n.$$

Using the measure ρ, we can calculate invariant measures for other IFSs.

Theorem 15. *(*[BD, Theorem 5]*) Let $\{X,\ f_i,\ p_i,\ i = 1, 2, \ldots, n\}$ be a hyperbolic IFS. Then there is a unique measure μ, given by $\mu(E) = \rho(\phi^{-1}(E))$ for $E \in \mathbb{B}(X)$; μ is attractive for any probability measure ν on X and the support of μ is the attractor A^* independently of $p_i > 0$, $i = 1, 2, \ldots, n$.*

3.3. Dimension and self-similarity

Let us introduce the essential notions for IFSs. Self-similarity and the open set condition were defined firstly by Hutchinson [Hu]. They are closely related to calculation of the Hausdorff dimension. We will also remind results of Barnsley [Ba1] and Schief [Sc].

Now, let us describe connectedness of fractals applying the open set condition.

Definition 22. [Ba1, Definition 2.2, p. 125] The IFS is said to be *totally disconnected* if each point of its attractor possesses a unique address. The IFS is said to be *just-touching* if it is not totally disconnected yet there exists an open set O such that

1. $f_i(O) \cap f_j(O) = \emptyset,\ \forall i, j \in \{1, 2, \ldots, n\}, i \neq j$,
2. $\cup_{i=1}^{n} f_i(O) \subset O$.

The IFS whose attractor obeys 1. and 2. is said to obey *the open set condition.* The IFS is said to be *overlapping* if it is neither just-touching nor disconnected.

Theorem 16. [Ba1, Theorem 2.2, p. 125] *Let $F = \{X; f_1, f_2, \ldots, f_m\}$ be an IFS with an attractor A^*. The IFS is totally disconnected if and only if*

$$f_i(A^*) \cap f_j(A^*) = \emptyset,\ \forall i, j \in \{1, 2, \ldots, m\}, i \neq j.$$

Next, let us introduce self-similarity, which is a characteristic property of attractors of IFSs.

Definition 23. [Hu, 5.1 (1), p. 18] A is *self-similar (with respect to F)* if

1. A is invariant with respect to F, and

2. $H^k(A) > 0$, $H^k(A_i \cap A_j) = 0$ for $i \neq j$, where $k = \dim_H K$.

Self-similarity and the open set condition mean a separation of tiles A_i^* of an attractor A^*, which is essential for the calculation of the Hausdorff dimension of A^*. When talking about dimension of a fractal we always mean the Hausdorff dimension here. Hutchinson considered IFSs consisting of similitudes in $\mathbb{R}^m$

$$\{(\mathbb{R}^m, d_{\text{Eucl}}), f_i, i = 1, 2, \ldots, n\}$$

with contraction factors $0 \leq r_i < 1$, $i = 1, 2, \ldots, n$. In [Hu, Convention 5.1 (2)], it is proved:

LEMMA 4. *There is a unique D such that $\sum_{i=1}^{n} r_i^D = 1$.*

Proof. Let $\gamma(t) = \sum_{i=1}^{n} r_i^t$. Then $\gamma(0) = n$ and $\gamma(t) \to 0$ as $t \to \infty$. The function γ is continuous which implies the statement. $\square$

DEFINITION 24. [Hu, 5.1 (3)] If $\sum r_i^D = 1$, D is called the *similarity dimension* of F.

REMARK 21. The formula $\sum r_i^D = 1$ is called *the Moran* or *Moran-Hutchinson formula* (see also [Mo]).

Hutchinson showed that D is often equal the Hausdorff dimension of fractals.

PROPOSITION 5. [Hu, 5.1 (4), p. 19] *Let A^* be an attractor of IFS and* $\dim_H(A^*) = k$. *Then*

1. *$H^D(A^*) < \infty$ and so $k \leq D$ (this is true for arbitrary contractions f_i),*

2. *$0 < H^k(A^*) < \infty$ implies (A^* is self-similar iff $k = D$).*

The easiest way to calculate the Hausdorff dimension follows from the next proposition.

PROPOSITION 6. [Hu, 5.3 (1), p. 19] *Suppose $F = \{\mathbb{R}^n, f_1, f_2, \ldots, f_m\}$ satisfies the open set condition. Then $0 < H^D(A^*) < \infty$ and A^* is self-similar. In particular,* $\dim_H(A^*) = D$.

It means the Hausdorff dimension of an attractor in $\mathbb{R}^m$ equals the self-similarity dimension if the open set condition is fulfilled.

REMARK 22. It is not easy to find a feasible open set generally. A lot of work in this field was done by Bandt (see e.g. [B], [BG]).

Hutchinson's approach in $\mathbb{R}^m$ was generalized by Schief [Sc] to general complete metric spaces. We need in addition a stronger version of the OSC.

DEFINITION 25. Let $F = \{X; f_1, f_2, \ldots, f_m\}$ be an IFS where f_i are similitudes. We say that F (or for brevity, A^*) fulfills the open set condition (OSC) if there exists a nonempty open set O such that the sets $f_i(O)$, $1 \leq i \leq m$, are pairwise disjoint and all contained in O. If $O \cap A^* \neq \emptyset$, *the strong open set condition* is fulfilled.

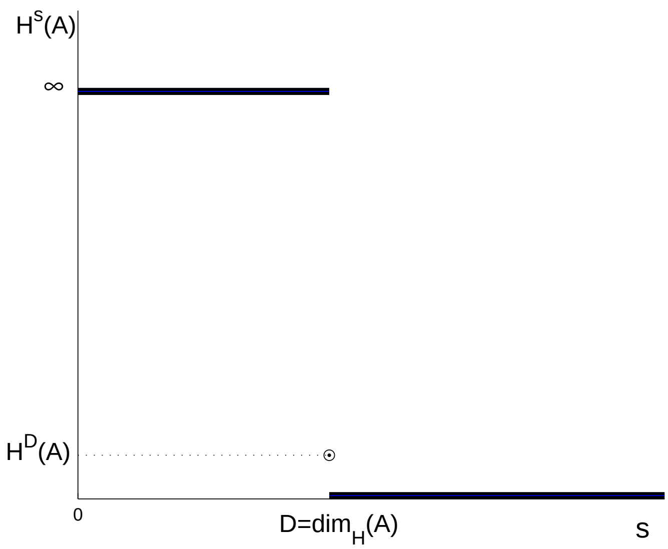

Figure 6: Behavior of the Hausdorff measure of a set

Proposition 7. [Sc, p. 481] *In $\mathbb{R}^n$ and generally in Euclidean case, the following chain of implications holds*

$$SOSC \Leftrightarrow OSC \Leftrightarrow H^D(A^*) > 0 \Rightarrow \dim_H A^* = D.$$

Proposition 8. [Sc, p. 490] *The following chain of implications is valid in complete metric spaces:*

$$A_i^* \cap A_j^* = \emptyset,\ i \neq j \Rightarrow H^D(A^*) > 0 \Rightarrow SOSC \Rightarrow \dim_H A^* = D.$$

We will also use the following proposition to calculate the Hausdorff dimension of fractals.

Proposition 9. *(cf.* [CR] *or* [Fa2, Corollary 2.4, p. 32]*) Assume that (X, d) and (Y, d') are metric spaces, $S \subset X$ and $f : S \to Y$ satisfies the inequalities*

$$a \cdot d(x, y) \leq d'(f(x), f(y)) \leq b \cdot d(x, y), \text{ for all } x, y \in S,$$

with suitable constants $a > 0$ and $b > 0$. Then

$$a^s H^s(E) \leq H^s(f(E)) \leq b^s H^s(E)$$

holds, for every $s \geq 0$.

Remark 23. [Fa2, p. 33] This proposition reveals a fundamental property of the Hausdorff dimension: the Hausdorff dimension is invariant under bi-Lipschitz transformations. Note that two sets are regarded topologically the same if there is a homeomorphism between them. We can regard two fractal sets as the same if there is a bi-Lipschitz mapping between them. Note also that each bi-Lipschitz transformation is a homeomorphism. Therefore, the Hausdorff dimension provides us further distinguishing characteristics between sets.

Remark 24. In other words, metrically equivalent spaces have the same Hausdorff dimension. Note that the graph of the Hausdorff measure (see Figure 6) of metrically equivalent spaces jumps from ∞ to 0 in the same value of s.

For the sake of completeness let us give two definitions.

Definition 26. [Ba1, Definition 2.2, p. 12] Two metrics d_1 and d_2 on a space X are *equivalent* if there exist constants $0 < c_1 < c_2 < \infty$ such that

$$c_1 d_1(x, y) \leq d_2(x, y) \leq c_2 d_1(x, y)\, \forall x, y \in X.$$

Definition 27. [Ba1, Definition 2.3, p. 12] Two metric spaces (X_1, d_1) and (X_2, d_2) are *equivalent* if there is a function $h : X_1 \to X_2$ that is bijective, such that the metric $\hat{d}_1$ on X_1 defined by

$$\hat{d}_1(x, y) := d_2(h(x), h(y)), \forall x, y \in X_1$$

is equivalent to d_1.

We can find more different definitions of self-similarity, for example [Sc, p. 482].

Remark 25. Let $F = \{X; f_1, f_2, \ldots, f_m\}$ be an IFS, where f_i are similitudes. The self-similar set A^* is the unique compact nonempty set such that

$$A^* = \bigcup_{i=1}^{m} f_i(A^*).$$

However, the majority of definitions of self-similarity come from authors, who studied Euclidean spaces. Since we will consider general complete metric spaces, we will use the following definition.

Definition 28. Let $F = \{X; f_1, f_2, \ldots, f_m\}$, where f_i are similitudes, fulfills SOSC. The self-similar set A^* is the unique compact nonempty set such that

$$A^* = \bigcup_{i=1}^{m} f_i(A^*).$$

Remark 26. We define self-similarity in the way that self-similar sets correspond to sets whose dimension can be found by means of the Moran formula.

3.4. Lifted IFS and superfractals

Lifted IFS and superfractals serve as an inspiration how to think of multivalued fractals. Barnsley developed lifted IFS in [Ba1, p. 154], but we will give its more general version from [Ba2, p. 338]. We start with the IFS $F = \{(X,\, d),\, f_1,\, f_2,\, \ldots,\, f_n\}$, where f_i are contractions, and define the lifted IFS

$$\hat{F} = \{X \times \Sigma,\, \hat{f}_1,\, \hat{f}_2,\, \ldots,\, \hat{f}_n\},$$

where

$$\hat{f}_i(x,\, \sigma) := (f_i(x),\, s_i(\sigma))$$

and $s_i(\sigma) := i\sigma := \omega$, with $\omega_1 = i$ and $\omega_{i+1} = \sigma_i$, for $i = 1,\, 2,\, \ldots$ Then $\hat{F}$ is an IFS consisting of contractions with respect to the metric

$$d_{X\times\Sigma}((x,\sigma),\, (y,\, \theta)) := d_X(x,\, y) + d_\Sigma(\sigma,\, \theta),$$

for all $(x,\, \sigma),\, (y,\, \theta) \in X \times \Sigma$. Notice that

$$d_\Sigma(s_i(\sigma),\, s_i(\theta)) \le \frac{1}{2} d_\Sigma(\sigma,\, \theta),$$

for each $i = 1,\, 2,\, \ldots,\, n$, and so it follows that

$$d_{X\times\Sigma}(\hat{f}_i(x,\sigma),\, \hat{f}_i(y,\, \theta)) \le \max\left\{\frac{1}{2},\, r\right\} d_{X\times\Sigma}((x,\sigma),\, (y,\, \theta)).$$

Let $\hat{A}$ denote the set attractor of $\hat{F}$. Then the projections of $\hat{A}$ onto X and Σ are A^* and Σ, respectively.

We can decompose a fractal and also an invariant measure supported on it.

Theorem 17. [Ba2, Theorem 4.9.3, p. 340] *Let $\mu \in \mathbb{P}(X)$ denote the measure attractor of the IFS $F = \{X,\, f_i,\, p_i,\, i = 1,\, 2,\, \ldots,\, n\}$ and let $\mu_\Sigma \in \mathbb{P}(\Sigma)$ denote the measure attractor of the IFS $S = \{\Sigma,\, s_i,\, p_i,\, i = 1,\, 2,\, \ldots,\, n\}$, where $s_i : \Sigma \to \Sigma$ is the transformation defined by $s_i(\sigma) := i\sigma$, for all $\sigma \in \Sigma$. Let $\hat{\mu} \in \mathbb{P}(X\times\Sigma)$ denote the measure attractor of the IFS $\hat{F} = \{X \times \Sigma,\, \hat{f}_1,\, \hat{f}_2,\, \ldots,\, \hat{f}_n,\, p_1,\, p_2,\, \ldots,\, p_n\}$, where $\hat{f}_i = (f_i,\, s_i)$, for $i = 1,\, 2,\, \ldots,\, n$. Then the projections of $\hat{\mu} \in \mathbb{P}(X \times \Sigma)$ onto $\mathbb{P}(X)$ and $\mathbb{P}(\Sigma)$ are μ and $\mu(\Sigma)$, respectively. Moreover,*

$$\mu = \phi(\mu_\Sigma).$$

Example 2. The Sierpiński triangle is an attractor of the IFS $F = \{[0,1]^2,\, f_1,\, f_2,\, f_3\}$,

$$f_i(x) = \frac{x + a_i}{2},$$

where $a_1 = (0,\, 1)'$, $a_2 = (1,\, 0)'$, $a_3 = (0,\, 0)'$. The attractor of the lifted IFS is shown in Figure 7.

Barnsley developed superfractals in [Ba2] and [BHS1]-[BHS4]. We define a compact metric space X with a collection of hyperbolic IFSs $\{F_m : m = 1,\, 2,\, \ldots,\, M\}$ with probabilities, where

$$F_m = \left\{X;\, f_1^m,\, f_2^m,\, \ldots,\, f_{L_m}^m;\, p_1^m,\, p_2^m,\, \ldots,\, p_{L_m}^m\right\}$$

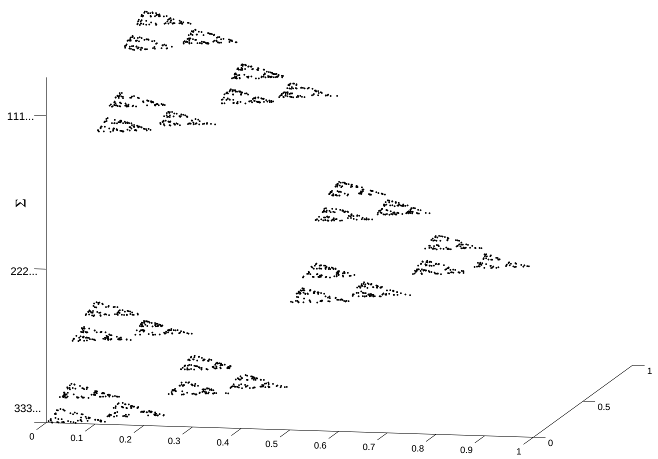

Figure 7: Lifted IFS for the Sierpiński triangle

and $M \geq 1$ is an integer, to be a superIFS. It is denoted in [Ba2] by

$$\{X;\, F_1,\, F_2,\, \ldots,\, F_M\} \text{ or } \{X;\, F_1,\, F_2,\, \ldots,\, F_M;\, P_1,\, P_2,\, \ldots,\, P_M\}\,,$$

where the P_m are probabilities, with

$$\sum_{m=1}^{M} P_m = 1, \text{ for all } m \in \{1,\, 2,\, \ldots,\, M\}.$$

The system is not an IFS but it can be used to define the hyperbolic IFS

$$F^{(1)} = \{K(X);\, F_1,\, F_2,\, \ldots,\, F_M;\, P_1,\, P_2,\, \ldots,\, P_M\}\,.$$

Here, each of the IFSs acts as a transformation

$$F_m : K(X) \to K(X)$$

defined by

$$F_m(B) := \cup_{l=1}^{L_m} f_l^m(B) \text{ for } m = 1,\, 2,\, \ldots,\, M.$$

We denote its attractor by $\alpha^{(1)}$.

Remark 27. Observe that

$$\alpha^{(1)} = \{A_\sigma : \sigma \in \Omega_{1,\, 2,\, \ldots,\, M}\}\,,$$

where

$$F_\sigma(A) = A_\sigma \; \forall A \in K(X).$$

DEFINITION 29. [Ba2, p. 396] The superIFS $\{X;\, F_1,\, F_2,\, \dots,\, F_M;\, P_1,\; P_2,\, \dots,\, P_M\}$ is said to obey *the uniform open set condition* if there exists a nonempty open set $O \subset X$ such that

$$F_m(O) \subset O$$

and

$$f_k^m(O) \cap f_l^m(O) = \emptyset \text{ if } k \neq l, \text{ for all } k,\, l \in \{1,\, 2,\, \dots,\, L_m\}$$

and for all $m \in \{1,\, 2,\, \dots,\, M\}$.

PROPOSITION 10. [Ba2, p, 397] *Let $N > 1$ be a positive integer. Let the superIFS*

$$\left\{\mathbb{R}^N;\, F_1,\, F_2,\, \dots,\, F_M;\, P_1,\, P_2,\, \dots,\, P_M\right\}$$

obey the uniform open set condition. Let the functions that comprise the IFS F_m be similitudes of the form

$$f_l^m(x) = s_l^m O_l^m x + t_l^m,$$

where O_l^m is an orthonormal transformation, $s_l^m \in (0,\, 1)$ and $t_l^m \in \mathbb{R}^N$, for all $l \in \{1,\, 2,\, \dots,\, L_m\}$ and $m \in \{1,\, 2,\, \dots,\, M\}$. Then, for almost all $A_\sigma \in \alpha^{(1)}$,

$$\dim_H A_\sigma = D,$$

where D is the unique solution of

$$\sum_{m=1}^{M} P_m \ln \sum_{l=1}^{L_m} (s_l^m)^D = 0.$$

Next, let us introduce some notation related to the IFS

$$F_{\text{underlying}} = \left\{X;\, f_1^1,\, f_2^1,\, \dots,\, f_{L_1}^1,\, f_1^2,\, f_2^2,\, \dots,\, f_{L_2}^2,\, \dots,\, f_1^M,\, f_2^M,\, \dots,\, f_{L_M}^M\right\}$$

which we call *the underlying IFS*. Its attractor is denoted $A_{\text{underlying}}$ and we have $A_{\text{underlying}} = \cup_{A_\sigma \in \alpha^{(1)}} A_\sigma$.

EXAMPLE 3. Let us consider the superIFS

$$F = \{\mathbb{R}^2,\, f_1^1,\, f_2^1,\, f_1^2,\, f_2^2\}$$

such that $f_j^i(x) = \mathscr{Q}_j^i x + b_j^i$, $i,\, j = 1,\, 2$

$$Q_1^1 = \begin{pmatrix} 0.5 & -0.2887 \\ 0.2887 & 0.5 \end{pmatrix},\; Q_2^1 = \begin{pmatrix} 0.5 & 0.2887 \\ -0.2887 & 0.5 \end{pmatrix},$$

$$Q_1^2 = \begin{pmatrix} 0.5 & 0.4410 \\ -0.4410 & 0.5 \end{pmatrix},\; Q_2^2 = \begin{pmatrix} 0.5 & -0.4410 \\ 0.4410 & 0.5 \end{pmatrix},$$

$b_1^1 = (-0.5,\, 0.2887)$, $b_2^1 = (0.5,\, 0.2887)$, $b_1^2 = (-0.5,\, -0.4410)$, $b_2^2 = (0.5,\, -0.4410)$.

We distinguished different sets of the superfractal in Figure 8 by colours.

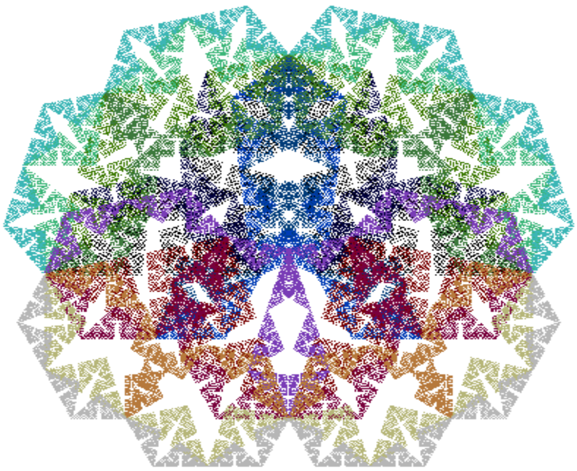

Figure 8: Superfractal from Example 3

4. Fixed point theory in hyperspaces

Remark 28. Let us note first that the most of the section can be found in [AR2] and it was collected and developed by Professor Andres.

The fixed point theory (in spaces) is one of the mostly developed parts of nonlinear analysis. For the metric (Banach-like) theory, see e.g. the handbook [KS] and for the topological (Schauder-like) theory, see e.g. the handbook [BFGJ]. On the other hand, the results concerning the fixed point theory in hyperspaces are rather rare (cf. [A3], [AV], [D1], [D2], [DG], [Ha], [HF], [HH], [IN, Chapter VI], [LFKU], [N2], [RN], [RS] and [Se]).

Everybody knows Banach's (see e.g. [GD, Theorem 1.1]) and Schauder's (see e.g. [GD, Theorem 3.2]) fixed point theorems. In applications, we will need also their generalizations.

The following generalization is a particular case of the Boyd-Wong version of the Banach Theorem (see e.g. [KS, Theorem 3.2, pp. 7–8]).

Lemma 5. *(Boyd-Wong) Let (X, d) be a complete metric space and $f : X \to X$ be a weakly contractive map. Then f has exactly one fixed point.*

The Covitz-Nadler multivalued version of the Banach theorem (see e.g. [GD, Theorem 3.1, p. 28], [KS, Theorem 5.1, pp. 15–16]) reads as follows.

Lemma 6. *Let (X, d) be a complete metric space and $\mathcal{F} : X \to B(X)$ be a contraction. Then $\mathcal{F}$ admits a fixed point, i.e. there exists $x_0 \in X$ such that $x_0 \in \mathcal{F}(x_0)$.*

The following Granas version of the Lefschetz fixed point theorem (see e.g. [GD, Theorem 4.3, p. 425]) is a far reaching generalization of the Schauder theorem.

LEMMA 7. *(Granas) Let X be an ANR-space and $f : X \to X$ be a compact map. Then the generalized Lefschetz number $\Lambda(f)$ is defined and if $\Lambda(f) \neq 0$, then f has a fixed point. In particular, if X is an AR-space, then $\Lambda(f) = 1$, and subsequently a compact map $f : X \to X$ has a fixed point.*

Since the space of probability measures $(\mathbb{P}(X), d_{MK})$ is compact if and only if so is (X, d), as a direct consequence of the Banach fixed point theorem, we can immediately give the following lemma.

LEMMA 8. *Let (X, d) be a complete metric space and $(\mathbb{P}(X), d_{MK})$ be the space of probability Borel measures on X. If the Markov operator $M : \mathbb{P}(X) \to \mathbb{P}(X)$ is a contraction with a constant $r \in [0, 1)$, i.e.*

$$d_{MK}(M(\mu), M(\nu)) \leq r d_{MK}(\mu, \nu), \text{ for all } \mu, \nu \in \mathbb{P}(X),$$

then there exists a unique fixed point $\mu_0 \in \mathbb{P}(X), \mu_0 = M(\mu_0)$, called the invariant measure w.r.t. M.

The first statement for hypermaps is a slight improvement of its analogy in [AF] (see also [AG2, Appendix 3]) in the sense of Remark 9.

PROPOSITION 11. *Let (X, d) be a complete metric space and $\mathcal{F}_1 : X \to C(X)$, $\mathcal{F}_2 : X \to B(X)$, $\mathcal{F}_3 : X \to K(X)$ be (weak) contractions. Then each hypermap $\overline{\mathcal{F}_1} : C(X) \to C(X)$, $\overline{\mathcal{F}_2} : B(X) \to B(X)$, $\mathcal{F}_3 : K(X) \to K(X)$ has exactly one fixed point X_i, $i = 1, 2, 3$. Moreover, each of multivalued maps $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ with bounded values possesses fixed points in $X_1, X_2, X_3 \subset X$.*

Proof. By the above arguments, $(C(X, d_H))$, $(B(X, d_H))$ and $(K(X, d_H))$ are complete hyperspaces (see Table 1 and Table 2) and the induced (single-valued) hypermaps $\overline{\mathcal{F}_1}$, $\overline{\mathcal{F}_2}$, $\mathcal{F}_3$ are self-maps. Moreover, $\overline{\mathcal{F}_1}$, $\overline{\mathcal{F}_2}$, $\mathcal{F}_3$ are (weak) contractions (see Table 3). Hence, applying Lemma 5, resp. Banach's theorem, they have exactly one fixed point X_i, $i = 1, 2, 3$, representing, up to its boundary, positively invariant subset in X. Since X_1, X_2, X_3 are closed subsets of a complete space X, they are also complete. Applying Lemma 6, the multivalued maps $\mathcal{F}_1|_{X_1}$, $\mathcal{F}_2|_{X_2}$, $\mathcal{F}_3|_{X_3}$ with bounded values possess fixed point in X_1, X_2, $X_3 \subset X$. $\square$

By the same arguments, we can give the second metric statement.

PROPOSITION 12. *Let $(E, ||.||)$ be a Banach space. Assume that $|||\mathscr{A}||| < 1$, for the matrix norm of $\mathscr{A}$, and $C_1 \in C(E^n)$, $C_2 \in B(E^n)$, $C_3 \in K(E^n)$, at the affine maps $\mathcal{F}_{01} : E^n \to C_{Co}(E^n)$, $\mathcal{F}_{02} : E^n \to B_{Co}(E^n)$, $\mathcal{F}_{03} : E^n \to K_{Co}(E^n)$. Then each hypermap $\overline{\mathcal{F}_{01}} : C_{Co}(E^n) \to C_{Co}(E^n)$, $\overline{\mathcal{F}_{02}} : B_{Co}(E^n) \to B_{Co}(E^n)$, $\mathcal{F}_{03} : K_{Co}(E^n) \to K_{Co}(E^n)$ has exactly one fixed point X_i, $i = 1, 2, 3$. Moreover, each* *of multivalued maps $\mathcal{F}_{01}$, $\mathcal{F}_{02}$, $\mathcal{F}_{03}$ with bounded values admits a fixed point in $X_i \subset \mathbb{R}^n$, $i = 1, 2, 3$.*

Proof. By the above arguments, $(C_{Co}(E^n), d_H)$, $(B_{Co}(E^n), d_H)$, $(K_{Co}(E^n), d_H)$ are complete hyperspaces. One can readily check that the multivalued maps $\mathcal{F}_{01}$, $\mathcal{F}_{02}$, $\mathcal{F}_{03}$ are contractions. Thus, the induced (single-valued) hypermaps $\overline{\mathcal{F}_{01}}$, $\overline{\mathcal{F}_{02}}$, $\mathcal{F}_{03}$, which are by the above arguments self-maps, must be contractions as well (see Table 3). Hence, applying the Banach fixed point theorem, they have exactly one fixed point X_i, $i = 1, 2, 3$, representing, up to its boundary, a positively invariant subset in E^n. Since X_1, X_2, X_3 are closed subsets of a Banach space E^n, they are also complete. Applying Lemma 6, the multivalued maps $\mathcal{F}_{01}|_{X_1}$, $\mathcal{F}_{02}|_{X_2}$, $\mathcal{F}_{03}|_{X_3}$ with closed bounded values possess fixed points in X_1, X_2, $X_3 \subset E^n$. □

Remark 29. Propositions 11 and 12 can be naturally extended to suitable hyper-hyperspaces $H_1(H_2(X))$, when considering the multivalued (weak) contractions $\mathscr{F} : H_2(X) \to H_1(H_2(X))$. On the other hand, if we consider multivalued (weak) contractions on the supporting space X, as in Propositions 11 and 12, then the unique fixed points of the induced hyper-hypermaps $\mathscr{F} : H(H(X)) \to H(H(X))$ must be the same as those of hypermaps $F : H(X) \to H(X)$.

Remark 30. Condition $|||\mathscr{A}||| < 1$ in Proposition 12 is certainly not necessary. Let, for instance, $E = \mathbb{R}$ and $C \in \mathbb{R}^n$. Then the map $\mathcal{F}_0 : \mathbb{R}^n \to \mathbb{R}^n$ has a unique fixed point $x_0 \in \mathbb{R}^n$ if and only if $(\mathscr{A} - \mathscr{I})$ is regular[2], i.e. $1 \notin \sigma(\mathscr{A})$. The induced hypermap $\mathcal{F}_0 : K(\mathbb{R}^n) \to K(\mathbb{R}^n)$ has exactly the same unique fixed point which can be explicitly calculated as a solution of the algebraic system $(\mathscr{A} - \mathscr{I})x = -C$.

The first topological statement in hyperspaces generalizes its analogy in [AF] (cf. also [AG2, Appendix 3]).

Proposition 13. *Let (X, d) be a locally continuum-connected metric space and $\mathcal{F} : X \to K(X)$ be a compact continuous mapping. Then the induced hypermap $\mathcal{F} : K(X) \to K(X)$ admits a fixed point.*

Proof. If (X, d) is still connected, then $(K(X), d_H)$ is an AR (see Table 1). The induced hypermap $\mathcal{F} : K(X) \to K(X)$ is compact and continuous as well (see Table 3). Thus, applying Lemma 7, there is a fixed point.

If (X, d) is disconnected then, unlike in the supporting space, in the hyperspace $(K(X), d_H)$ which is an ANR (see Table 1), $K(X)$ consists of a finite number of disjoint ARs, and subsequently $\Lambda(\mathcal{F}) \geq 1$ (for more details see [AV]). Applying Lemma 7, $\mathcal{F}$ admits also in this case a fixed point. □

Remark 31. Since in $(K(X), d_H)$, where (X, d) is locally continuum-connected, we get even $N(\mathcal{F}) = \Lambda(\mathcal{F}) \geq 1$, where $N(\mathcal{F})$ denotes the Nielsen number for the lower estimate of fixed points of $\mathcal{F}$, we have in fact to our disposal a multiplicity

[2] $\mathscr{I}$ is the identity matrix and $\sigma(\mathscr{A})$ means a set of eigenvalues of A

result. The problem of calculation of $N(\mathcal{F})$ namely reduces there to a simple combinatorical situation on a finite set (see [AV]).

Remark 32. Propositions 11 and 13 can be extended, by means of degree arguments, to hyperspace continuation principles, where the induced hypermaps and hyperhomotopies need not be self-maps (see [A1], [A2], [AFGL], [RS]). There also exists a fixed point theorem for condensing hypermaps (cf. [L4]), but the nontrivial induction of condesity seems to be a difficult task. On the other hand, there is no direct way for obtaining the hyperspace analogy of the Schauder fixed point theorem, because the hyperspaces of linear spaces have never a linear structure. Despite it, using a Rådström theorem which allows us the embedding of the hyperspace as a positively semilinear subspace, some analogies of the Schauder theorem in terms of the Hausdorff topology were obtained in [D1], [D2], [DG].

Remark 33. We already know from Section 2 that $(B(X),\, d_H)$ is an AR, provided e.g. $X \subset E$ is a convex subset of a normed space $(E,\, ||.||)$ (cf. [AC]). In general, it is according to [BV, Theorem 3.5] an AR if and only if a metric space $(X,\, d)$ is uniformly locally chain equi-connected on each bounded subset of X and that each bounded subset of X lies in a bounded chain equi-connected subspace of X. For the definitions and more details, see [BV]. Proposition 13 can be, therefore, also partially extended to multivalued compact continuous maps $\overline{\mathcal{F}} : X \to B(X)$. Analogous criteria can be also found, in view of [BV, Theorem 3.2], for multivalued compact continuous maps $\overline{\mathcal{F}} : X \to C(X)$ and their inductions on $(C(X),\, d_H)$.

Since $(K_{Co}(X),\, d_H)$ is, according to [HH], [LFKU], a compact convex subset of $(K(X),\, d_H)$, provided $X \subset E^n$ is a compact, convex subset of a Banach space E^n, the following statement can be also regarded as a particular hyperspace version of the Schauder-type theorem.

Proposition 14. *Let $(E^n,\, ||.||)$ be a Banach space and $X \subset E^n$ be a nonempty, compact, convex, subset of E^n. Let $C \in K_{Co}(X)$ be at the affine map $\mathcal{F}_0 : X \to K_{Co}(X)$, defined in the foregoing section. Then the induced hypermap $\mathcal{F}_0 : K_{Co}(X) \to K_{Co}(X)$ admits a fixed point X_0. If still $|||\mathscr{A}||| < 1$ holds, for the matrix norm of $\mathscr{A}$ at $\mathcal{F}_0$, then the fixed point X_0 is unique. Moreover, the multivalued map $\mathcal{F}_0 : X \to K_{Co}(X)$ admits a fixed point in a convex, compact, positively invariant subset $X_0 \subset X \subset E^n$.*

Proof. Since $(K_{Co}(X),\, d_H)$ is convex and compact, it must be also a compact AR. By the above arguments (see Remark 11), the induced hypermap $\mathcal{F}_0$ is a Lipschitz-continuous self-map, i.e. $\mathcal{F}_0 : K_{Co}(X) \to K_{Co}(X)$, which is compact. Applying Lemma 7, there is a fixed point $X_0 \in K_{Co}(X)$. For $|||\mathscr{A}||| < 1$, one can alternatively apply the Banach fixed point theorem to obtain the uniqueness of X_0. The point $X_0 \in K_{Co}(X)$ is at the same time a convex, compact, positively invariant subset of $X \subset E^n$ such that $\mathcal{F}_0|_{X_0} : X_0 \to K_{Co}(X_0)$ is Lipschitz-continuous. Applying a suitable Kakutani-type fixed point theorem whose all assumptions are satisfied (see e.g. [GD, Theorem 8.4, pp. 168–169]), the multivalued map $\mathcal{F}_0$ has a fixed point in $X_0 \subset X \subset E^n$. □

Remark 34. Similarly as for metric statements, Proposition 13 can be naturally extended to the hyperspace $K(K(X))$, when considering the multivalued compact continuous maps $\mathcal{F} : K(X) \to K(K(X))$, provided (X, d) is a locally continuum-connected metric space. Proposition 14 can be extended to the hyper-hyperspace $K(K_{Co}(X))$, when considering the multivalued affine maps $\mathcal{F}_0 : K_{Co}(X) \to K(K_{Co}(X))$, provided $X \subset E^n$ is a nonempty, compact, convex subset of a Banach space E^n.

5. Multivalued fractals and hyperfractals

We are ready to introduce multivalued fractals and hyperfractals. First, we will prove their existence by means of fixed point theorems. Next, we will generalize Barnsley's results ([Ba1]) to multivalued fractals. Finally, we will apply the chaos game to draw multivalued fractals. However, hyperfractals will play a crucial role in our derivations.

5.1. Existence results

Remark 35. Let us note that the most of the subsection can be found in [AR2] and the results were partially developed by Professor Andres.

In our approach to fractals, we follow the classical ideas of J. E. Hutchinson [Hu] and M. F. Barnsley [Ba1] concerning the *iterated function systems (IFSs)* $\{(X, d), f_i : X \to X, i = 1, 2, \ldots n\}$, where (X, d) is a complete metric space and f_i, $i = 1, 2, \ldots n$, are contractions. The prehistory of this approach can be already detected in the paper [Wi] of R. F. Williams.

Replacing single-valued contractions f_i by multivalued ones, we talk about *iterated multifunction systems (IMSs).* This name was used for the first time in the paper [AG1]. In [A1], it was also used for the first time the term *multivalued fractals* for the attractors of IMS. Later on, in [AF], [AFGL], [AG2, Appendix 3], this notion was extended to fixed points in hyperspaces of the related Hutchinson-Barnsley operators determined by multivalued maps. It was also distinguished there between *metric and topological multivalued fractals,* according to the applied metric (Banach-like) and topological (Schauder-like) fixed point theorems. Of course, because of identical images of sets, every set would be a topological fractal, but we always implicitly assumed that there are at least two maps in the generating systems under consideration. Let us note that this terminology seems to be nowadays standard (cf. e.g. [BBP], [CL], [CP], [Fi], [KLV1], [KLV2], [Mh].)

Multivalued fractals were considered for the first time in 2001 in [A1], [AG1] and, independently, by A. Petruşel and I. A. Rus in [P1], [PR1]. At the same time, the similar ideas were also implicitly present in the papers [LM1], [LM2] of A. Lasota and J. Myjak, where an alternative approach was used leading to the notion of *semifractals* (cf. also [LM3], [LM4], [MS] and the references therein).

In Introduction, we already indicated the relationship between *hyperfractals,* defined as fixed points in hyper-hyperspaces of the Hutchinson-Barnsley hyperoperators and determined by a rather general class of multivalued maps, to *superfractals* (cf. [Ba2], [BHS1]–[BHS4], [SJM]) and to multivalued fractals. We also would like to clarify in this section the relationship between hyperfractals and invariant measures, defined on hyperspaces, of the Markov hyperoperators.

Hence, both multivalued fractals and hyperfractals will be investigated here in terms of the fixed point theory from the foregoing section.

We start with a theorem for metric multivalued fractals.

Theorem 18. *Let (X, d) be a complete metric space and $\mathcal{F}_{1i} : X \to C(X)$, $\mathcal{F}_{2i} : X \to B(X)$, $\mathcal{F}_{3i} : X \to K(X)$ be, for all $i = 1, 2, \ldots, n$, weak contractions. Then each Hutchinson-Barnsley operator*

$$\left.\begin{aligned} &\overline{F_1} : C(X) \to C(X),\ \overline{F_1}(A) := \bigcup_{i=1}^{n} cl_{C(X)}\Big(\bigcup_{x\in A} \mathcal{F}_{1i}(x)\Big),\\ &\overline{F_2} : B(X) \to B(X),\ \overline{F_2}(A) := \bigcup_{i=1}^{n} cl_{B(X)}\Big(\bigcup_{x\in A} \mathcal{F}_{2i}(x)\Big),\\ &F_3 : K(X) \to K(X),\ F_3(A) := \bigcup_{i=1}^{n} \bigcup_{x\in A} \mathcal{F}_{3i}(x) \end{aligned}\right\} \tag{3}$$

has exactly one fixed point A_j, $j = 1, 2, 3$, which is at the same time a positively invariant (for $j = 1, 2$, up to its boundary) set w.r.t. the related Hutchinson-Barnsley maps

$$\left.\begin{aligned} &\mathcal{F}_1 : X \to C(X),\ \mathcal{F}_1(x) := \bigcup_{i=1}^{n} \mathcal{F}_{1i}(x),\\ &\mathcal{F}_2 : X \to B(X),\ \mathcal{F}_2(x) := \bigcup_{i=1}^{n} \mathcal{F}_{2i}(x),\\ &\mathcal{F}_3 : X \to K(X),\ \mathcal{F}_3(x) := \bigcup_{i=1}^{n} \mathcal{F}_{3i}(x). \end{aligned}\right\} \tag{4}$$

Moreover, each of the maps $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ possesses fixed points in $A_1, A_2, A_3 \subset X$, provided $\mathcal{F}_{1i}$ have bounded values, for all $i = 1, 2, \ldots, n$.

Proof. Since a finite union of closed sets is closed, of bounded sets is bounded and of compact sets is compact, we can define the maps $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ as in (4). Since the operators $\overline{F_1}$, $\overline{F_2}$, F_3 in (3) can be equivalently defined as $\overline{F_1}(A) := cl_{C(X)}(\bigcup_{x\in A} \mathcal{F}_1(x))$, $\overline{F_2}(A) := cl_{B(X)}(\bigcup_{x\in A} \mathcal{F}_2(x))$, $F_3(A) := \bigcup_{x\in A} \mathcal{F}_3(x)$, they have the same properties as the induced (single-valued) hypermaps $\overline{F_1}$, $\overline{F_2}$, F_3 in Proposition 11. Hence, the application of Proposition 11 completes the proof. □

Remark 36. Because of a weaker notion of a weak contractivity (see Remark 9), the first part of Theorem 18 is slightly more general than its analogies in

[AF], [AG2, Appendix 3]. It is also a generalization for F_3, with multivalued contractions $\mathcal{F}_{3i}$, of the analogous results in [AG1], [KLV1], [P1], [PR1] and for F_3, with single-valued contractions $\mathcal{F}_{3i}$, of the classical results in [Hu], [Ba1].

Remark 37. Since the completeness of $(X,\, d)$ implies the completeness of $(C(X),\, d_H)$, $(B(X),\, d_H)$, $(K(X),\, d_H)$, and subsequently of $(C(C(X)),\, d_{H_H})$, $(B(B(X)),\, d_{H_H})$, $(K(K(X)),\, d_{H_H})$, and since weak contractions are induced on these spaces, the following statement can be regarded as a corollary of Theorem 18 (cf. Remark 29).

Corollary 1. *Let $(X,\, d)$ be a complete metric space and $\mathcal{F}_{1i} : X \to C(X)$, $\mathcal{F}_{2i} : X \to B(X)$, $\mathcal{F}_{3i} : X \to K(X)$ be, for all $i = 1,\, 2,\, \ldots,\, n$, weak contractions. Then each Hutchinson-Barnsley hyperoperator*

$$\begin{aligned}
\overline{\phi_1} : C(C(X)) \to C(C(X)),\, \overline{\phi_1}(\alpha) &:= \bigcup_{A\in\alpha} \bigcup_{i=1}^{n} \left\{ cl_{C(X)}\big(\bigcup_{x\in A} \mathcal{F}_{1i}(x)\big) \right\}, \\
\overline{\phi_2} : B(B(X)) \to B(B(X)),\, \overline{\phi_2}(\alpha) &:= \bigcup_{A\in\alpha} \bigcup_{i=1}^{n} \left\{ cl_{B(X)}\big(\bigcup_{x\in A} \mathcal{F}_{2i}(x)\big) \right\}, \\
\phi_3 : K(K(X)) \to K(K(X)),\, \phi_3(\alpha) &:= \bigcup_{A\in\alpha} \bigcup_{i=1}^{n} \left\{ \bigcup_{x\in A} \mathcal{F}_{3i}(x) \right\}
\end{aligned}$$

has exactly one fixed point α_j^*, $j = 1,\, 2,\, 3$, which is at the same time a positively invariant (for $j = 1,\, 2$, up to its boundary) set w.r.t. the related Hutchinson-Barnsley hypermaps

$$\begin{aligned}
\overline{\mathscr{F}}_1 : C(X) \to C(C(X)),\, \mathscr{F}_1(A) &:= \bigcup_{i=1}^{n} cl_{C(X)} \left\{ \big(\bigcup_{x\in A} \mathcal{F}_{1i}(x)\big) \right\}, \\
\overline{\mathscr{F}}_2 : B(X) \to B(B(X)),\, \mathscr{F}_2(A) &:= \bigcup_{i=1}^{n} \left\{ cl_{B(X)}\big(\bigcup_{x\in A} \mathcal{F}_{2i}(x)\big) \right\}, \\
\mathscr{F}_3 : K(X) \to K(K(X)),\, \mathscr{F}_3(A) &:= \bigcup_{i=1}^{n} \left\{ \bigcup_{x\in A} \mathcal{F}_{3i}(x) \right\}.
\end{aligned}$$

Moreover, each of the hypermaps $\mathscr{F}_1$, $\mathscr{F}_2$, $\mathscr{F}_3$ possesses fixed points in $\alpha_1^* \subset C(X)$, $\alpha_2^* \subset B(X)$, $\alpha_3^* \subset K(X)$, provided F_{1i} have bounded values, for all $i = 1,\, 2,\, \ldots,\, n$.

Definition 30. Fixed points α_j^*, $j = 1,\, 2,\, 3$ of operators ϕ_j^*, $j = 1,\, 2,\, 3$ from the previous corollary are called *hyperfractals*.

Remark 38. Corollary 1 is a generalization for ϕ_3, with special multivalued contractions F_{3i}, of Theorem 8 in [KLV2].

Since the union of convex sets need not be convex, Proposition 12 cannot be applied as Proposition 11 above. Despite this impossibility, Corollary 1 can be still specified as follows.

Theorem 19. *Let $(E,\, ||.||)$ be a Banach space. Consider the affine maps*

$$\begin{aligned}
\mathcal{F}_{1i} : E^n \to C_{Co}(E^n),\, \mathcal{F}_{1i}(x) &:= \mathscr{A}_{1i}x + C_{1i}, C_{1i} \in C_{Co}(E^n), \\
\mathcal{F}_{2i} : E^n \to B_{Co}(E^n),\, \mathcal{F}_{2i}(x) &:= \mathscr{A}_{2i}x + C_{2i}, C_{2i} \in B_{Co}(E^n), \\
\mathcal{F}_{3i} : E^n \to K_{Co}(E^n),\, \mathcal{F}_{3i}(x) &:= \mathscr{A}_{3i}x + C_{3i}, C_{3i} \in K_{Co}(E^n),
\end{aligned}$$

where $\mathscr{A}_{ji}$, $j = 1, 2, 3$, $i = 1, 2, \ldots, n$, *are real* $n \times n-$*matrices. If* $|||\mathscr{A}_{ij}||| < 1$ *holds for the matrix norms of* $\mathscr{A}_{ji}$, $j = 1, 2, 3$, $i = 1, 2, \ldots, n$, *then each Hutchinson-Barnsley hyperoperator*

$$\left.\begin{aligned}
\overline{\phi_{01}} &: C(C_{Co}(E^n)) \to C(C_{Co}(E^n)),\\
\overline{\phi_{01}}(\alpha) &:= \bigcup_{A\in\alpha}\bigcup_{i=1}^{n}\left\{cl_{C_{Co}(E^n)}(\bigcup_{x\in A}\mathcal{F}_{1i}(x))\right\},\\
\overline{\phi_{02}} &: B(B_{Co}(E^n)) \to B(B_{Co}(E^n)),\\
\overline{\phi_{02}}(\alpha) &:= \bigcup_{A\in\alpha}\bigcup_{i=1}^{n}\left\{cl_{B_{Co}(E^n)}(\bigcup_{x\in A}\mathcal{F}_{2i}(x))\right\},\\
\phi_{03} &: K(K_{Co}(X)) \to K(K_{Co}(X)),\\
\phi_{03}(\alpha) &:= \bigcup_{A\in\alpha}\bigcup_{i=1}^{n}\left\{\bigcup_{x\in A}\mathcal{F}_{3i}(x)\right\}
\end{aligned}\right\} \tag{5}$$

has exactly one fixed point α_j^*, $j = 1, 2, 3$.

Proof. Since $(E^n, ||.||)$ is a Banach space, the hyperspaces $(C_{Co}(E^n), d_H)$, $(B_{Co}(E^n), d_H)$, $(K_{Co}(E^n), d_H)$, are, by the above arguments, complete as well as the hyper-hyperspaces $(C(C_{Co}(E^n)), d_{H_H})$, $(B(B_{Co}(E^n)), d_{H_H})$, $(K(K_{Co}(E^n)), d_{H_H})$ (cf. Tables 1 and 2). Since the affine multivalued maps $\mathcal{F}_{ji}$ are, for $|||\mathscr{A}_{ji}||| < 1$, obviously contractions, so are the induced (single-valued) hypermaps $cl_{C_{Co}(E^n)}\mathcal{F}_{1i}, cl_{B_{Co}(E^n)}\mathcal{F}_{2i}, \mathcal{F}_{3i}$, $i = 1, 2, \ldots, n$, (cf. Table 3). Furthermore, since a finite union of these contractions in (5) is a contraction (cf. [AF], [AG2, Appendix 3]), the hyperoperators in (5) must be also contractions (cf. Table 3). Thus, these hyperoperators in (5) have the same properties as the induced (single-valued) hypermaps $\overline{\mathcal{F}_1}$, $\overline{\mathcal{F}_2}$, $\mathcal{F}_3$ in Proposition 11 which completes the proof. □

Remark 39. The unique fixed points α_j^*, $j = 1, 2, 3$, can be only regarded as closed or bounded, closed or compact subsets of closed, convex or bounded, closed, convex or compact, convex subsets of E^n which are positively invariant (for $j = 1, 2$, up to their boundaries) w.r.t. the related Hutchinson-Barnsley hypermaps

$$\mathscr{F}_{01} : C(E^n) \to C(C(E^n)),\ \mathscr{F}_{01}(A) := \bigcup_{i=1}^{n}\left\{cl_{C(E^n)}(\bigcup_{x\in A}\mathcal{F}_{1i}(x))\right\},$$

$$\mathscr{F}_{02} : B(E^n) \to B(B(E^n)),\ \mathscr{F}_{02}(A) := \bigcup_{i=1}^{n}\left\{cl_{B(E^n)}(\bigcup_{x\in A}\mathcal{F}_{2i}(x))\right\},$$

$$\mathscr{F}_{03} : K(E^n) \to K(K(E^n)),\ \mathscr{F}_{03}(A) := \bigcup_{i=1}^{n}\left\{\bigcup_{x\in A}\mathcal{F}_{3i}(x)\right\}.$$

Moreover, each of the hypermaps $\mathscr{F}_{01}$, $\mathscr{F}_{02}$, $\mathscr{F}_{03}$ possesses fixed points in $\alpha_1^* \subset C(E^n)$, $\alpha_2^* \subset B(E^n)$, $\alpha_3^* \subset K(E^n)$, provided $C_{1i} \in B_{Co}(E^n)$, for all $i = 1, 2, \ldots, n$.

Proposition 13 can be applied to obtain the following topological result.

THEOREM 20. *Let* (X, d) *be a locally continuum-connected metric space and* $\mathcal{F}_i : X \to K(X)$ *be, for all* $i = 1, 2, \ldots, n$, *compact continuous mappings. Then the Hutchinson-Barnsley operator*

$$F : K(X) \to K(X),\ F(A) := \bigcup_{i=1}^{n} \left\{ \bigcup_{x \in A} \mathcal{F}_i(x) \right\} \tag{6}$$

admits a fixed point which is at the same time a positively invariant set w.r.t. the Hutchinson-Barnsley map

$$\mathcal{F} : X \to K(X),\ \mathcal{F}(x) := \bigcup_{i=1}^{n} \mathcal{F}_i(x). \tag{7}$$

Proof. Since a finite union of compact sets is compact, we can define the map $\mathscr{F}$ as in (7). Since the operator F in (6) can be equivalently defined as

$$F(A) := \bigcup_{x \in A} \mathcal{F}(x),$$

it has the same properties as the induced (single-valued) hypermap $\mathcal{F}$ in Proposition 13. Hence, the application of Proposition 13 completes the proof. □

REMARK 40. Since (X, d) can be disconnected, it generalizes its analogies in [AF], [AG2, Appendix 3]. In view of Remark 31, we can even obtain in an extremely simple way the lower estimate of the number of fixed points of F in (6). On the other hand, since e.g. $(C(\mathbb{R}), d_H)$ is, according to [BV], only an ANR, but not an AR, it is a difficult task to find sufficient conditions in order $(C(C(X)), d_{H_H})$ or $(B(B(X)), d_{H_H})$ to be ARs. Thus, it seems to be also difficult to extend Theorem 20 to the Hutchinson-Barnsley operators on $(C(X), d_H)$ and $(B(X), d_H)$.

Since the compactness of (X, d) implies the one of $(K(X), d_H)$, and subsequently of $(K(K(X)), d_{H_H})$, and since a continuity is induced on these spaces, the following statement can be regarded as a corollary of Theorem 18 (cf. Remark 29).

COROLLARY 2. *Let* (X, d) *be a locally continuum-connected metric space and* $\mathcal{F}_i : X \to K(X)$ *be, for all* $i = 1, 2, \ldots, n$, *compact continuous mappings. Then the Huchinson-Barnsley hyperoperator*

$$\phi : K(K(X)) \to K(K(X)),\ \phi(\alpha) := \bigcup_{A \in \alpha} \bigcup_{i=1}^{n} \left\{ \bigcup_{x \in A} \mathcal{F}_i(x) \right\}$$

admits a fixed point which is at the same time a positively invariant set w.r.t. the Hutchinson-Barnsley hypermap

$$\mathscr{F} : K(X) \to K(K(X)),\ \mathscr{F}(A) := \bigcup_{i=1}^{n} \left\{ \bigcup_{x \in A} \mathcal{F}_i(x) \right\}.$$

Despite the impossible application of Proposition 14, Corollary 2 can be still specified as follows.

THEOREM 21. *Let $(E, ||.||)$ be a Banach space and $X \in E^n$ be a nonempty, convex, compact subset of E^n. Consider the affine maps*

$$\mathcal{F}_{0i} := X \to K_{Co}(X),\ \mathcal{F}_{0i}(x) := \mathscr{A}_i x + C_i,$$

where $\mathscr{A}_i$ are real $n \times n-$matrices and $C_i \in K_{Co}(X)$, for all $i = 1, 2, \dots, n$. The Hutchinson-Barnsley hyperoperator

$$\phi_0 : K(K_{Co}(X)) \to K(K_{Co}(X)),\ \phi_0(\alpha) := \bigcup_{A \in \alpha} \bigcup_{i=1}^{n} \left\{ \bigcup_{x \in A} \mathcal{F}_{0i}(x) \right\} \tag{8}$$

admits always a fixed point α_0^*. If still $|||\mathscr{A}_i||| < 1$ holds for the matrix norms of $\mathscr{A}_i$, $i = 1, 2, \dots, n$, then the fixed point α_0^* is unique.

Proof. Since $X \subset E^n$ is convex and compact, so is by the above arguments (cf. [HH], [LFKU]) the hyperspace $(K_{Co}(X), d_H)$, by which the hyper-hyperspace $(K(K_{Co}(X)), d_{H_H})$ is a compact AR (cf. Table 1). Furthermore, since the multivalued affine maps $\mathcal{F}_{0i}$ are evidently (Lipschitz-) continuous, so are the induced (single-valued) hypermaps $\mathcal{F}_{0i} : K_{Co}(X) \to K_{Co}(X)$ (cf. Remark 12). Moreover, since a finite union of these hypermaps is also (Lipschitz-) continuous (cf. [AF], [AG2, Appendix 3]), the induced Hutchinson-Barnsley hyperoperator in (8) must be (Lipschitz-) continuous as well (cf. Table 3). Thus, the hyperoperator in (8) has the same properties as the induced (single-valued) hypermap $\mathcal{F}$ in Proposition 13 which completes the first (topological) part of the proof.

For $|||A_i||| < 1$, the multivalued affine maps $\mathcal{F}_{0i}$ are obviously contractions, for all $i = 1, 2, \dots, n$, and so are the induced (single-valued) hypermaps (cf. Table 3). Furthermore, since a finite union of these contractions in (8) is a contraction (cf. [AF], [AG2, Appendix 3]), so must also be the hyperoperator ϕ_0 in (8) (cf. Table 3) which has in this way the same properties as the induced (single-valued) hypermap $\mathcal{F}_3$ in Proposition 11. This completes the second (metric) part of the proof. □

REMARK 41. Similarly as in Remark 38, the fixed points α_0^* can be only regarded as compact subsets of compact, convex subsets of $X \subset E^n$ which are positively invariant w.r.t. the related Hutchinson-Barnsley hypermap

$$\mathscr{F}_0 : K(X) \to K(K(X)),\ \mathscr{F}_0(A) := \bigcup_{i=1}^{n} \left\{ \bigcup_{x \in A} \mathcal{F}_{oi}(x) \right\}.$$

Moreover, for $|||\mathscr{A}_i||| < 1$, $i = 1, 2, \dots, n$, the hypermap $\mathscr{F}_0$ possesses fixed points in $\alpha_0^* \subset K(X)$. Observe that, unlike in Theorem 20, here the matrices $\mathscr{A}_i$, $i = 1, 2, \dots, n$, can be without restrictions.

Now, let (X, d) be a compact metric space and $(\mathbb{P}(X), d_{MK})$ be the space of probability Borel measures on X. Let, for all $i = 1, 2, \ldots, n$, $f_i : X \to X$ be contractions with factors $r_i \in [0, 1)$ and $p_i : X \to [0, 1]$ be the associated continuous probability functions such that $\sum_{i=1}^{n} p_i(x) = 1$, for all $x \in X$.

We can define the *Markov-Feller operators* as follows (cf. e.g. [LM1]–[LM4],[MS]):

$$M : \mathbb{P}(X) \to \mathbb{P}(X),\ M(\mu)(A) := \sum_{i=1}^{n} \int_{f_i^{-1}(A)} p_i(x) d\mu(x), \tag{9}$$

for all $\mu \in \mathbb{P}(X)$ and $A \in \mathbb{B}(X)$, where $\mathbb{B}(X)$ denotes the $\sigma-$algebra of Borel subsets of X. In a particular case of constant probabilities $p_i(x) \equiv p_i$, $i = 1, 2, \ldots, n$, the formula (9) obviously simplifies into

$$M(\mu)(A) := \sum_{i=1}^{n} p_i \mu(f_i^{-1}(A)),\ \mu \in \mathbb{P}(X),\ A \in \mathbb{B}(X). \tag{10}$$

It can be proved (see e.g. [Ba1, Theorem 6.1, p. 351]) that under the above assumptions, M defined by (10) is a contraction, i.e.

$$d_{MK}(M(\mu), M(\nu)) \leq r d_{MK}(\mu, \nu),$$

for all $\mu, \nu \in \mathbb{P}(X)$, where $(1 >) r := \max_{i=1,2,\ldots,n} \{r_i\}$. Thus, applying Lemma 8, there exists a unique fixed point $\mu_0 \in \mathbb{P}(X)$, $\mu_0 = M(\mu_0)$, called the *invariant measure* w.r.t. M.

In this light, for compact metric spaces, a particular case of contractions in Corollary 1 can be extended as follows.

Theorem 22. *Let (X, d) be a compact metric space and $\mathcal{F}_i : X \to K(X)$ be Lipschitz-continuous multivalued maps with factors $r_i \geq 0$, for $i = 1, 2, \ldots, n$ (like e.g. $\mathcal{F}_{3i}$, $i = 1, 2, \ldots, n$, in Theorem 19). Let $p_i \in (0, 1]$ be the associated probabilities such that $\sum_{i=1}^{n} p_i = 1$ and $\sum_{i=1}^{n} r_i p_i < 1$. Then the Markov hyperoperator*

$$M : \mathbb{P}(K(X)) \to \mathbb{P}(K(X)),\ M(\mu)(A) := \sum_{i=1}^{n} p_i \mu(\mathcal{F}_i^{-1}(A)), \tag{11}$$

for all $\mu \in \mathbb{P}(K(X))$ and $A \in \mathbb{B}(K(X))$, where $\mathbb{B}(K(X))$ denotes the $\sigma-$algebra of Borel subsets of $K(X)$, has exactly one fixed point $\mu_0 \in \mathbb{P}(K(X))$ such that $supp(\mu_0) := \{x \in K(X) | \mu_0(B(x, r)) > 0,\ for\ every\ r > 0\}$ is the smallest positively invariant set w.r.t. the Hutchinson-Barnsley hypermap

$$\mathscr{F} := \bigcup_{i=1}^{n} \mathcal{F}_i : K(X) \to K(K(X)). \tag{12}$$

Proof. In view of Table 1, $(K(X), d)$ and $(K(K(X)), d_H)$ are compact. Moreover, by the above arguments, Lipschitz-continuous multivalued maps induce (single-valued) Lipschitz hypermaps with the same factors $r_i \geq 0$, $i = 1, 2, \ldots, n$. Therefore, if $\sum_{i=1} nr_i p_i < 1$, then the Markov hyperoperator M in (11), on the hyperspace $(\mathbb{P}(K(X)), d_{MK_H})$, has exactly one fixed point $\mu_0 \in \mathbb{P}(K(X))$ such that $\mathrm{supp}(\mu_0)$ is a semiattractor of the hyperIFS $\{(K(X), d_H), F_i : K(X) \to K(X), i = 1, 2, \ldots, n\}$ (see [LM3, Theorem 3.1], [MS, Fact 3.2 and Corollary 5.6]). At the same time, it is the smallest positively invariant set w.r.t. the hypermap $\mathscr{F}$ in (12) (see e.g. [LM3, Theorem 2.1], [MS, Theorem 5.2]). $\square$

Remark 42. Observe that the factors r_i in Theorem 22 can be greater than 1, for some $i = 1, 2, \ldots, n$. On the other hand, for non-unique positively invariant sets $A \subset K(X)$ w.r.t. $\mathscr{F}$ in (12), we only have that $\mathrm{supp}(\mu_0) \subset A$, but not the equality, as for a uniqueness. Nevertheless, the relationship between invariant measures and topological hyperfractals can be clarified in this way. More precisely, we know from Corollary 2 that if a compact X is still locally connected, then there is always a positively invariant set $A \subset K(X)$ w.r.t. $\mathscr{F}$ in (12). Now, we also know that, under the assumptions of Theorem 22 $\mathrm{supp}(\mu_0) \subset A$. Moreover, in the case of uniqueness, we have that $\mathrm{supp}(\mu_0) = A$.

In order to avoid handicap mentioned in Remark 42, we can give the following corollary of Theorem 22 which already concerns a unique positively invariant set w.r.t. $\mathscr{F}$ in (12).

Corollary 3. *Let (X, d) be a compact space and $\mathcal{F}_i : X \to K(X)$ be, for all $i = 1, 2, \ldots, n$, weak contractions. Moreover, let at least one $\mathcal{F}_i$, say $\mathcal{F}_1$, be a contraction with factor $r_1 < 1$. Let $p_i \in (0, 1]$ be the associated probabilities such that $\sum_{i=1}^{n} p_i = 1$. Let $(\mathbb{P}(K(X)), d_{MK_H})$ be the hyperspace of probability Borel measures on $(K(X), d_H)$. Then the Markov-Feller hyperoperator $M : \mathbb{P}(K(X)) \to \mathbb{P}(K(X))$, which takes the same form as in (11), has exactly one fixed point $\mu_0 \in \mathbb{P}(K(X))$, called the invariant measure w.r.t. the hyperoperator M such that $\mathrm{supp}(\mu_0) = A_3$, where A_3 comes from Corollary 1.*

Proof. Since compact-valued weak contractions are, by definition, nonexpansive, the induced (single-valued) maps must be weakly contractive (see Table 3), and subsequently nonexpansive. Thus, we always have that $\sum_{i=1}^{n} r_i p_i \leq \sum_{i=1}^{n} p_i = 1$. Since $\mathcal{F}_1$ is still a contraction with a constant $r_1 < 1$, so must be the induced (single-valued) map (cf. Table 3), by which $\sum_{i=1}^{n} r_i p_i < 1$. Applying Theorem 22, there is a unique invariant measure μ_0 of the related Markov hyperoperator M whose support is the smallest positively invariant set w.r.t. the Hutchinson-Barnsley hypermap $\mathscr{F}$ in (12). Since this set is, according to Corollary 1 unique, we have that $\mathrm{supp}(\mu_0) = A_3$, where A_3 comes from Corollary 1, as claimed. $\square$

REMARK 43. Corollary 3 is a generalization of its analogy in [KLV1], [KLV2], where all F_i, $i = 1, 2, \dots n$, were strict special contractions. In this case, the proof can be done in a more straightforward way by means of Lemma 8 and multivalued contractions with compact values need not be of a special type as in [KLV1], [KLV2].

REMARK 44. In [LM3], [MS] (cf. also the references therein), the authors considered, for the invariant measures, the supporting Polish space (X, d), the hyperspace $(C(X), d_H)$ and the space $(\mathbb{P}(X), d_{MK})$ of probability Borel measures on X. Although completeness is implied by (X, d) to $(C(X), d_H)$ (cf. Table 2), separability is not preserved in this way, by the arguments explained in Section 2. Thus, we could not directly extend Theorem 22 and Corollary 3, when just replacing a compact (X, d) by a Polish space. Moreover, since the supports of invariant measures in [LM3], [MS] can be noncompact, it follows that, in Theorem 22 and Corollary 3, a compact (X, d) cannot be replaced by a Polish space, when considering the hyperspace $(K(X), d_H)$.

The metric part of Theorem 21 can be generalized in a probabilistic way as follows.

THEOREM 23. *Let $(E^n, ||.||)$ be a real Banach space and $X \subset E^n$ be a nonempty, convex, compact subset of E^n. Consider the affine maps*

$$\mathcal{F}_{0i} : X \to K_{Co}(X),\ \mathcal{F}_{01}(x) := \mathscr{A}_i x + C_i,$$

where $\mathscr{A}_i$ are real $n \times n-$matrices and $C_i \in K_{Co}(X)$, for all $i = 1, 2, \dots, n$. Let $p_i \in [0, 1]$, $i = 1, 2, \dots, n$, be the associated probabilities such that $\sum_{i=1}^{n} p_i = 1$ and $\mathbb{P}(K_{Co}(X), d_{MK_H})$ be the hyperspace of probability Borel measures on $(K_{Co}(X), d_H)$. If $|||\mathscr{A}_i||| < 1$ holds, for the matrix norms of $\mathscr{A}_i$, for all $i = 1, 2, \dots, n$, then the Markov-Feller hyperoperator

$$M_0 : \mathbb{P}(K_{Co}(X)) \to \mathbb{P}(K_{Co}(X)),\ M_0(\mu)(A) := \sum_{i=1}^{n} p_i \mu(F_{0i}^{-1}(A)), \qquad (13)$$

for all $\mu \in \mathbb{P}(K_{Co}(X))$ and $A \in \mathbb{B}(K_{Co}(X))$, has exactly one fixed point $\mu_0 \in \mathbb{P}(K_{Co}(X))$, called the invariant measure w.r.t. the hyperoperator M_0.

Proof. Since $X \subset E^n$ is compact, so is by the above arguments (cf. [HH]) $(K_{Co}(X), d_H)$, and subsequently $(\mathbb{P}(K_{Co}(X)), d_{MK_H})$. Furthermore, because of $|||\mathscr{A}_i||| < 1$, the affine multivalued maps $\mathcal{F}_{0i}$ are obviously contractions, and so are (cf. Table 3) the induced (single-valued) hypermaps $F_{0i} : K_{Co}(X) \to K_{Co}(X)$, for all $i = 1, 2, \dots, n$. Thus, the Markov-Feller hyperoperator defined in (13) is, by the above arguments, a contraction as well. The application of Lemma 8, therefore, completes the proof. $\square$

REMARK 45. For regular matrices $\mathscr{A}_i$, $i = 1, 2, \ldots, n$, the formula (12) takes the more explicit form.

$$M_0 : \mathbb{P}(K_{Co}(X)) \to \mathbb{P}(K_{Co}(X)),\ M_0(\mu)(A) := \sum_{i=1}^{n} p_i \mu[(A - C_i)\mathscr{A}_i^{-1}], \tag{14}$$

for all $\mu \in \mathbb{P}(K_{Co}(X))$ and $A \in \mathbb{B}(K_{Co}(X))$, where $\mathbb{B}(K_{Co}(X))$ denotes the $\sigma-$algebra of Borel subsets of $K_{Co}(X)$. Besides this advantage, the only improvement of Corollary 3 consists in fact that the space of probability Borel measures and the invariant measure μ_0 from it are on $(K_{Co}(X), d_H)$, i.e. on convex, compact subsets of X. Otherwise, Theorem 23 can be only regarded as a consequence of Corollary 3. On the other hand, we still have that $\mathrm{supp}(\mu_0) = A_0$, where A_0 comes from Theorem 21, provided $p_i > 0$, for all $i = 1, 2, \ldots, n$.

We conclude this section by indicating the relationship of the obtained results in terms of fractals.

REMARK 46 (terminological). Fixed points in Theorems 18 and 20 are called *fractals*, while the other fixed points are called *hyperfractals.* To distinguish them still by means of the applied fixed point theorems, we speak about fixed points in Theorems 18, 19 and, for $|||A_i||| < 1$, $i = 1, 2, \ldots, n$, in Theorem 21 and Corollary 1 as *metric,* while about those in Theorems 20, 21 and Corollary 2 as *topological.* Thus, the unique fixed points in Corollary 1 represent metric hyperfractals whose "shadows" (called the underlying fractals in [Ba2]) on the supporting space (X, d) coincide with respective metric multivalued fractals represented by fixed points in Theorem 18. The fixed points in Corollary 2 represent topological hyperfractals whose "shadows" on (X, d) coincide with topological multivalued fractals represented by the fixed points in Theorem 20. The topological hyperfractal $\mathrm{supp}(\mu_0)$ in Theorem 22 can be also called a *hyper-semifractal,* in the lines of [LM3], [MS]. Furthermore, the support of the unique invariant measure in Corollary 3 coincides, for $p_i > 0$, $i = 1, 2, \ldots, n$, with a metric hyperfractal in a particular case of Corollary 1 and the support of the invariant measure in Theorem 23 coincides, for $p_i > 0$ and $|||\mathscr{A}_i||| < 1$, $i = 1, 2, \ldots, n$, with a metric hyperfractal in Theorem 21. The metric hyperfractals in Theorem 19 as well as, for $|||\mathscr{A}_i||| < 1$, $i = 1, 2, \ldots, n$, in Theorem 21, and the topological hyperfractals in Theorem 21 are rather exceptional (cf. Remarks 14 and 16), but their "shadows" on the supporting spaces coincide with special metric multivalued fractals in Theorem 18 and topological multivalued fractals in Theorem 20, respectively.

5.2. Address structure of multivalued fractals

Address structure of multivalued fractals enables us to draw multivalued fractals and measures supported by them efficiently. It is the same as the address structure of associated hyperfractals. We describe the address structure of multivalued fractals in a similar way as the address structure of fractals generated by ordinary IFSs. From

$$A^* = \bigcup_{i=1}^{m} F_i(A^*),$$

we will obtain

$$A^* = \bigcup_{\sigma \in \Sigma} F_\sigma(A^*).$$

Remark 47. In the previous sections we needed to distinguish multivalued maps, hypermaps, multivalued hypermaps and hyper-hypermaps. We used different fonts. From now on, we will immediately induce multivalued maps to hypermaps. Therefore, let us use the same font for a multivalued map and the induced hypermap, for instance, F for $\mathcal{F}$ and F.

Remark 48. Notice that the address structure is usually treated for compact fractals. Hence, from now on, let us restrict ourselves to contractions $F_i : X \to K(X)$.

We need only a slight modification of Barnsley results ([Ba1, Theorem 2.1, p. 123]).

*Proposition 15. *Let (X, d) be a complete metric space. Let $\{K(X), F_1, \ldots, F_N\}$ be a hyperIFS. Let α^* denote the attractor of the hyperIFS. Let (Σ, d_Σ) denote the code space associated with the hyperIFS. For each $\sigma \in \Sigma$, $n \in \mathbb{N}$, and $A \in K(X)$, define*

$$\phi(\sigma, n, A) := F_{\sigma_1} \circ F_{\sigma_2} \circ \cdots \circ F_{\sigma_n}(A).$$

Then

$$\phi(\sigma) = \lim_{n \to \infty} \phi(\sigma, n, A)$$

exists, belongs to A^* and is independent of $A \in K(X)$. If γ is a compact subset of $K(X)$, then the convergence is uniform over $A \in \gamma$. The function $\phi : \Sigma \to \alpha^*$ is continuous and onto.

Remark 49. In order to understand the address structure of multivalued fractals, let us study the formula

$$F(A) = \bigcup_{i=1}^{n} F_i(A),\ A \in K(X).$$

If F_i are contractions for all i, F is also a contraction and it has a fixed point. However, we do not need $F_i : K(X) \to K(X)$ to be induced by single-valued mappings. Let us discuss other two cases. F_i can be induced by multivalued maps $F_i : X \to K(X)$ or general hypermaps $F_i : K(X) \to K(X)$, $i = 1, 2, \ldots, n$. For general contractions $F_i : K(X) \to K(X)$, we can prove only the existence of a fixed point A^*,

$$A^* = F(A^*) = \bigcup_{i=1}^{n} F_i(A^*).$$

Therefore, we will consider mainly contractions $F_i : K(X) \to K(X)$, induced by $\mathcal{F}_i : X \to K(X)$. It assures that

$$F_i(A \cup B) = F_i(A) \cup F_i(B)$$

which is essential for describing the address structure of fractals.

*Theorem 24. *Let (X, d) be a complete metric space. Let $\{(X, d), F_1, \dots, F_N\}$ be such that $F_i : X \to K(X)$, $i = 1, 2, \dots, N$, be an IMS and $\phi = \{(K(X), d_H), F_1, F_2, \dots, F_N\}$ the associated hyperIFS. Let A^* denote an attractor of the IMS. Let (Σ, d_Σ) denote the code space associated with the hyperIFS. For each $\sigma \in \Sigma$, $n \in \mathbb{N}$, and $A \in K(X)$ let*

$$\phi(\sigma, n, A) := F_{\sigma_1} \circ F_{\sigma_2} \circ \cdots \circ F_{\sigma_n}(A).$$

Then

$$\phi(\sigma) = \lim_{n\to\infty} \phi(\sigma, n, A)$$

is a compact subset of A^* and is independent of $A \in K(X)$. If $a \in A^*$, then there exists $\sigma \in \Sigma$ such that $a \in \phi(\sigma)$.

Proof. Let us define $F : K(X) \to K(X)$,

$$F(A) := \cup_i F_i(A).$$

F is a contraction mapping on the metric space $(K(X), d_H)$. We have

$$A^* = \lim_{n\to\infty} F^n(A_0).$$

In particular, $\{F^n(A_0)\}$ is a Cauchy sequence in $(K(X), d_H)$. Notice that

$$\phi(\sigma, n, A_0) \subset F^n(A_0), \ \forall \sigma \in \Sigma.$$

Since limits

$$\phi(\sigma) = \lim_{n\to\infty} \phi(\sigma, n, A_0)$$

and

$$A^* = \lim_{n\to\infty} F^n(A_0)$$

exist, it follows that

$$\phi(\sigma) \subset A^*, \ \forall \sigma \in \Sigma.$$

Next, we prove that each $a \in A^*$ has an address. Consider $A_0 \in K(X)$ and a sequence $\{A_n\}$, $A_n = F^n(A_0)$, $n = 0, 1, 2, \dots$ From [Ba1, Theorem 7.1, p. 35], it follows that we can find a sequence $\{a_n\}$, $a_n \in A_n$, such that $a_n \to a$. There exists a sequence $\{\omega^{(n)} \in \Sigma,\ n = 1, 2, 3, \ldots\}$ such that

$$\lim_{n\to\infty} \phi(\omega^{(n)},\ n,\ A_0) \ni a.$$

(Notice that $a_1 \in \cup F_i(A_0) \Rightarrow \exists i_1 \in \{1, 2, \ldots, N\} : a_1 \in F_{i_1}(A_0)$, $a_2 \in \cup_{ij} F_{ij}(A_0) \Rightarrow \exists i_1',\ i_2' : a_2 \in F_{i_1' i_2'}(A_0)$.)

Since $(\Sigma,\ d_\Sigma)$ is compact, it follows that $\{\omega^{(n)}\}$ possesses a convergent subsequence with a limit $\omega \in \Sigma$. Without loss of generality, assume that $\lim_{n\to\infty} \omega^{(n)} = \omega$. Then the number of successive initial agreements between components of $\omega^{(n)}$ and ω increases without limit. That is, if

$$\alpha(n) = \mathrm{card}\{j \in \mathbb{N} : \omega_k^{(n)} = \omega_k \text{ for } 1 \le k \le j\},$$

then $\alpha(n) \to \infty$ as $n \to \infty$. It follows that

$$d(\phi(\omega,\ n,\ A_0),\ \phi(\omega^{(n)},\ n,\ A_0)) \to 0.$$

From

$$a \in \lim_{n\to\infty} \phi(\omega^{(n)},\ n,\ A_0),$$

it follows that

$$a \in \lim_{n\to\infty} \phi(\omega,\ n,\ A_0).$$

□

The theorem implies the following corollary. Each address point of a hyperfractal equals an address set of the underlying multivalued fractal.

*Corollary 4. *Let* $\{(X,\ d),\ F_1,\ F_2,\ \ldots,\ F_N\}$ *be an IMS and* $\{(K(X),\ d_H),\ F_1,\ F_2,\ \ldots,\ F_N\}$ *the induced hyperIFS. Attractors* A^* *and* α^* *of these iterated systems possess the same address structure,*

$$A^* = \bigcup_{\sigma\in\Sigma} A^*_\sigma,$$

$$\alpha^* = \bigcup_{\sigma\in\Sigma} \{A^*_\sigma\}.$$

We can visualize address sets of multivalued fractals by means of *lifted IMSs.* We will construct lifted IMSs in the similar way as lifted IFSs. Let $\{(X,\ d),\ F_1,\ F_2,\ \ldots,\ F_m\}$ be an IMS where $F_i : X \to K(X)$ are contractions. We define an IMS

$$\hat{F} = \{(X \times \Sigma,\ d_{X\times\Sigma}),\ \hat{F}_1,\ \hat{F}_2,\ \ldots,\ \hat{F}_n\},$$

where $\hat{F}_i : (X \times \Sigma,\ d_{X\times\Sigma}) \to (K(X \times \Sigma),\ d_{X\times\Sigma_H})$,

$$\hat{F}_i(x,\ \sigma) = F_i(x) \times \{s_i(\sigma)\}, \tag{15}$$

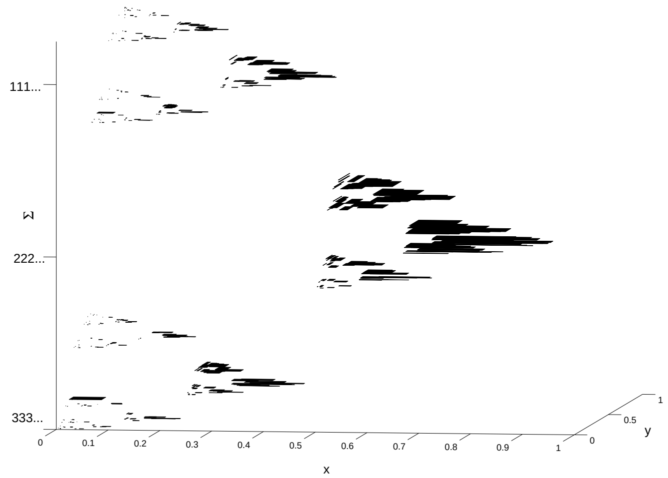

Figure 9: Lifted IMS

and $s_i(\sigma) := i\sigma := \omega$, with $\omega_1 = i$ and $\omega_{i+1} = \sigma_i$, for $i = 1, 2, \ldots$ We have already used

$$d_{X\times\Sigma}((x,\sigma),\,(y,\,\theta)) = d_X(x,\,y) + d_\Sigma(\sigma,\,\theta).$$

In order to show that $\hat{F}_i$ are contractions $i = 1, 2, \ldots, m$, we need not consider general compact subsets of $X \times \Sigma$. It suffices to treat $A \times \{\sigma\}$, $\sigma \in \Sigma$, $A \in K(X)$. Therefore, let us write $(A,\,\sigma)$ for $A \times \{\sigma\}$.

For the distance $d_{X\times\Sigma_H}((A,\,\sigma),\,(B,\,\theta))$, we have

$$d_{X\times\Sigma_H}((A,\sigma),\,(B,\,\theta)) = \max\{\sup_{(a,\,\sigma)\in(A,\sigma)}(\inf_{(b,\,\theta)\in(B,\,\theta)} d_{X\times\Sigma}((a,\,\sigma),(b,\,\theta))),$$

$$\sup_{(b,\,\theta)\in(B,\,\theta)}(\inf_{(a,\,\sigma)\in(A,\sigma)} d_{X\times\Sigma}((a,\,\sigma),(b,\,\theta))\}.$$

We will prove that $\hat{F}_i$ are contractions. For any $(x,\,\sigma)$ and $(y,\,\theta) \in X \times \Sigma$, we can write

$$d_{X\times\Sigma_H}(\hat{F}_i(x,\,\sigma),\,\hat{F}_i(y,\,\theta)) =$$

$$\max\{\sup_{(a,\,\eta)\in\hat{F}_i(x,\,\sigma)}\{\inf_{(b,\,\omega)\in\hat{F}_i(y,\,\theta)} d_{X\times\Sigma}((a,\,\eta),(b,\,\omega))\},$$

$$\sup_{(b,\,\omega)\in\hat{F}_i(y,\,\theta)}\{\inf_{(a,\,\eta)\in\hat{F}_i(x,\,\sigma)} d_{X\times\Sigma}(a,\,\eta),(b,\,\omega)\}\}.$$

Our calculation is easy because of (15). Thus,

$$d_{X\times\Sigma_H}(\hat{F}_i(x,\,\sigma),\,\hat{F}_i(y,\,\theta)) =$$

$$\max\{\sup_{(a,\,i\sigma)\in\hat{F}_i(x,\,\sigma)}(\inf_{(b,\,i\theta)\in\hat{F}_i(y,\,\theta)} d_{X\times\Sigma}((a,\,i\sigma),(b,\,i\theta)),$$

$$\sup_{(b,\,i\theta)\in\hat{F}_i(y,\,\theta)}(\inf_{(a,\,i\sigma)\in\hat{F}_i(x,\,\sigma)} d_{X\times\Sigma}(a,\,i\sigma),(b,\,i\theta))\} =$$

$$\max\{\sup_{(a,\,i\sigma)\in\hat{F}_i(x,\,\sigma)}(\inf_{(b,\,i\theta)\in\hat{F}_i(y,\,\theta)} d_X(a,\,b)+d_\Sigma(i\sigma,\,i\theta),$$

$$\sup_{(b,\,i\theta)\in\hat{F}_i(y,\,\theta)}(\inf_{(a,\,i\sigma)\in\hat{F}_i(x,\,\sigma)} d_X(a,\,b)+d_\Sigma(i\sigma,\,i\theta))\} =$$

$$d_\Sigma(i\sigma,\,i\theta)+\max\{\sup_{(a,\,i\sigma)\in F_i(x)}(\inf_{b\in F_i(y)} d_X(a,\,b)),$$

$$\sup_{(b,\,i\theta)\in F_i(y)}(\inf_{(a,\,i\sigma)\in F_i(x)} d_X(a,\,b))\} =$$

$$d_\Sigma(i\sigma,\,i\theta)+d_H(F_i(x),\,F_i(y)) \le$$

$$\frac{1}{2}d_\Sigma(\sigma,\,\theta)+r_i d(x,\,y) \le$$

$$\max\left\{\frac{1}{2},\,r_i\right\} d_{X\times\Sigma}((x,\sigma),\,(y,\,\theta)).$$

*THEOREM 25. *The system* $\hat{F}=\{(X\times\Sigma,\,d_{X\times\Sigma}),\,\hat{F}_1,\,\hat{F}_2,\,\ldots,\,\hat{F}_n\}$ *is an IMS consisting of contractions with respect to the metrics* $d_{X\times\Sigma}$ *and* $d_{X\times\Sigma_H}$. *The projections of* $\hat{A}$ *onto* X *and* Σ *are* A^* *and* Σ, *respectively.*

Proof. Since $\hat{F}_i : X\times\Sigma \to K(X\times\Sigma)$ are contractions in a complete metric space, we obtain the first part of the theorem from Theorem 18. The second part follows from $\hat{A}=\bigcup_{\sigma\in\Sigma}(A^*_\sigma,\,\sigma)$ and Corollary 4. □

EXAMPLE 4. The Fat Sierpiński triangle is an attractor of the IMS $F=\{[0,\,1]^2,\,F_i : [0,\,1]^2 \to K([0,\,1]^2),\,i=1,\,2,\,3\}$,

$$F_1\begin{pmatrix} x\\ y\end{pmatrix} := \left\{\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}\begin{pmatrix} x\\ y\end{pmatrix}+\begin{pmatrix}0\\ \frac{1}{2}\end{pmatrix}\right\},$$

$$F_2\begin{pmatrix} x\\ y\end{pmatrix} := \begin{pmatrix}[\frac{1}{3},\,\frac{1}{2}] & 0\\ 0 & [\frac{1}{3},\,\frac{1}{2}]\end{pmatrix}\begin{pmatrix} x\\ y\end{pmatrix}+\left\{\begin{pmatrix}\frac{1}{2}\\ 0\end{pmatrix}\right\},$$

$$F_3\begin{pmatrix} x\\ y\end{pmatrix} := \left\{\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}\begin{pmatrix} x\\ y\end{pmatrix}+\begin{pmatrix}0\\ 0\end{pmatrix}\right\},$$

where

$$\left(\left[\frac{1}{3},\,\frac{1}{2}\right]\cdot x,\,\left[\frac{1}{3},\,\frac{1}{2}\right]\cdot y\right) := \left(\left[\frac{x}{3},\,\frac{x}{2}\right],\,\left[\frac{y}{3},\,\frac{y}{2}\right]\right).$$

For the lifted Fat Sierpiński triangle, see Figure 9.

REMARK 50. In further examples, we will consider the contractions in a different order (see e.g. Example 6).

5.3. Ergodic approach

In this section, we will give basic theorems of the ergodic theory. They will enable us to draw fractal measures and estimate integrals of functions defined on fractals. First, we will introduce dynamical systems.

DEFINITION 31. [Ba1, Definition 3.1, p. 130] *A dynamical system* is a transformation $f : X \to X$ on a metric space (X, d). It is denoted by $\{X, f\}$. *The orbit* of a point $x \in X$ is the sequence $\{f^k(x)\}_{k=0}^{\infty}$.

DEFINITION 32. [Ba1, Definition 3.3, p. 132] Let $\{X, f\}$ be dynamical system and let x_f be a fixed point of f. The point x_f is called *an attractive fixed point of* f if there is a number $\epsilon > 0$ so that f maps the ball $O(x_f, \epsilon)$ into itself, and moreover f is a contraction mapping on $O(x_f, \epsilon)$. The point x_f is called *a repulsive fixed point of* f if there are numbers $\epsilon > 0$ and $C > 1$ such that

$$d(f(x_f),\, f(y)) \geq Cd(x_f,\, y), \text{ for all } y \in O(x_f,\, \epsilon).$$

DEFINITION 33. [Ba1, Definition 4.1, p. 140] Let $(X, f_1, f_2, \ldots, f_m)$ be a hyperbolic IFS with totally disconnected attractor A^*. *The associated shift transformation on* A^* is the transformation $S : A^* \to A^*$ defined by

$$S(a) := f_i^{-1}(a) \text{ for } a \in f_i(A^*),$$

where f_i is viewed as transformation on A^*. The dynamical system $\{A^*, S\}$ is called the shift dynamical system associated with the IFS.

DEFINITION 34. [Ba1, Definition 5.2, p. 146] Two dynamical systems $\{X_1, f_1\}$ and $\{X_2, f_2\}$ are said to be *equivalent,* or *topologically conjugate,* if there is a homeomorphism $\theta : X_1 \to X_2$ such that

$$f_1(x_1) = \theta^{-1} \circ f_2 \circ \theta(x_1), \text{ for all } x_1 \in X_1,$$

$$f_2(x_2) = \theta \circ f_1 \circ \theta^{-1}(x_2), \text{ for all } x_2 \in X_2.$$

In other words, the two dynamical systems are related by the commutative diagram (see Figure 10).

REMARK 51. The only dynamical system we meet here, is the shift dynamical system.

THEOREM 26. [Ba1, Theorem 5.1, p. 147] *Let* $\{X, f_1, f_2, \ldots f_m\}$ *be a totally disconnected hyperbolic IFS and let* $\{A^*, S\}$ *be the associated shift dynamical system. Let* Σ *be the associated code space of* m *symbols and let* $T : \Sigma \to \Sigma$ *be defined by*

$$T(\sigma_1\sigma_2\sigma_3\ldots) := \sigma_2\sigma_3\sigma_4\ldots, \text{ for all } \sigma = \sigma_1\sigma_2\sigma_3\cdots \in \Sigma.$$

Then the two dynamical systems $\{A^*, S\}$ *and* $\{\Sigma, T\}$ *are equivalent. The homeomorphism that provides this equivalence is* $\phi : \Sigma \to A^*$*, defined in Definition 34. Moreover,* ϕ *protects repulsive, attractive cycles and periodic points, too.*

$$\begin{tikzcd} X_1 \arrow[r, "\theta"] \arrow[d, "f_1"] & X_2 \arrow[d, "f_2"] \\ X_1 \arrow[r, "\theta"] & X_2 \end{tikzcd}$$

Figure 10: Commutative diagram

From now on, we will consider in addition a measure. We will estimate it on measurable subsets by means of special dynamical systems.

DEFINITION 35. [BKS, Definition 2.1 p. 22] Let $(X, \mathcal{A}, \mu)$ be a fixed measure space. Then *a measure-preserving transformation* T of the measure space $(X, \mathcal{A}, \mu)$ is a mapping

$$T : X \to \mathcal{A}$$

of the underlying set X of the measure space to itself, which satisfies the following properties:

1. T is measurable, i.e. if A is any element of the $\sigma-$algebra $\mathcal{A}$ of the measure space (that is A is a measurable subset of X) then the subset

   $$T^{-1}(A) := \{x \in X : T(x) \in A\}$$

   also belongs to the σ algebra $\mathcal{A}$ (that is $T^{-1}(A)$ is also a measurable subset of X).

2. T preserves the measure μ, i.e. for any $A \in \mathcal{A}$, not only is $T^{-1}(A) \in \mathcal{A}$ as in 1), but also

   $$\mu(T^{-1}(A)) = \mu(A),$$

   where $\mu(\cdot)$ denotes the measure of an element $\cdot$ of $\mathcal{A}$.

THEOREM 27. [BKS, Theorem 2.2, p. 41] *Let T be a measure-preserving transformation of the probability space $(X, \mathcal{A}, \mu)$, and let B be any element of $\mathcal{A}$. Set*

$$S_n(x) := card\{i : 0 \leq i < n, T^i(x) \in B\}$$

and

$$A_n(x) := \frac{1}{n} S_n(x) \quad (x \in X).$$

Then for $\mu-$almost every $x \in X$,

$$A(x) := \lim_{n \to \infty} A_n(x)$$

exists.

Let $\mu(X) = 1$. It seems logical that the time the orbit of a measure preserving transformation T spends in the set B relates to the measure of B. It is tempting to estimate the measure $\mu(B)$ by $A(x)$. However, our assumptions are not sufficient. Transformation $T : X \to X$, $T(x) = x$, is always measure-preserving and we can not obviously get measure $\mu(B)$ from $A(x)$. We need in addition ergodicity.

Definition 36. [BKS, Definition 2.3, p. 41] A measure preserving transformation T is *ergodic* if whenever $f : X \to \mathbb{R}$ is a measurable function such that

$$f(T(x)) = f(x)$$

for $\mu-$almost all $x \in X$, then f is $\mu-$almost everywhere equal to constant.

We will use that isomorphism of dynamical systems preserves ergodicity.

Definition 37. [BKS, Definition 2.7, p. 45] Let

$$S = (X, \mathcal{A}, \mu, T)$$

and let

$$S' = (X', \mathcal{A}', \mu', T')$$

be two dynamical systems (i.e. T and T' are measure preserving transformations of the respective measure spaces $(X, \mathcal{A}, \mu)$ and $(X', \mathcal{A}', \mu')$). Then S and S' are *isomorphic* if there exists a mapping

$$\phi : X \to X'$$

(an isomorphism) such that

1. ϕ is measurable,
2. for each $A' \in \mathcal{A}'$, $\mu(\phi^{-1}(A')) = \mu'(A')$,
3. for $\mu-$almost all $x \in X, \phi(T(x)) = T'(\phi(x))$,
4. ϕ is invertible, i.e. there exists a mapping

   $$\psi : X' \to X,$$

   measure preserving, such that $\psi(\phi(x)) = x$ for $\mu-$almost all $x \in X$ and $\phi(\psi(x')) = x$ for $\mu-$almost all $x \in X$ and $\phi(\psi(x')) = x'$ for $\mu-$almost all $x' \in X'$.

If only properties 1), 2) and 3) are required, ϕ is called a homomorphism and S' is said to be a factor of S.

The following theorem tells us implicitly that the shift dynamical system is ergodic. We will use it to find a measure of subsets of attractors of IFSs or to calculate integrals on these attractors.

THEOREM 28. *(*[BD, Theorem 6, p. 261]*) Let $\{X, f_i, p_i, p_i > 0, i = 1, 2, \ldots, n\}$ be a hyperbolic IFS with an attractor A^* and invariant measure μ such that f_i is one-to-one on A^* for $i = 1, 2, \ldots, n$ and $f_i(A^*) \cap f_j(A^*) = \emptyset$, $i \neq j$. Then a measurable function $T : A^* \to A^*$ is given by $T(x) = f_i^{-1}(x)$ for $x \in f_i(A^*)$. It is such that $(A^*, \mathbb{B}(A^*), \mu, T)$ is a measure-preserving system, in the sense of Billingsley (cf.* [Bi]*), isomorphic to $(\Sigma, \mathbb{B}(\Sigma), \rho, s)$, where $s : \Sigma \to \Sigma$ is the Bernoulli shift operator*

$$s(\sigma_1, \sigma_2, \sigma_3, \ldots) = \sigma_2, \sigma_3, \ldots$$

In particular, $(X, \mathbb{B}(X), \mu, T)$ is ergodic, mixing, and has entropy

$$h(T) = -\sum_{i=1}^{n} p_i \ln p_i.$$

The Birkhoff ergodic theorem gives us a prescription how to calculate and approximate integrals on fractals.

THEOREM 29. *(Birkhoff's ergodic theorem)*[Fa2, Theorem 6.1, p. 98] *Let $T : X \to X$, let μ be a finite measure on X that is invariant under T, and let $\phi \in L^1(\mu)$. Then the limit*

$$\Phi(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(T^j(x))$$

exists for $\mu-$almost all x. Moreover, if μ is ergodic then

$$\Phi(x) = \frac{1}{\mu(X)} \int \phi d\mu$$

for $\mu-$almost all x.

COROLLARY 5. [Fa2, Corollary 6.2, p. 100] *If μ is ergodic then*

$$\Phi(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(T^j(x))$$

is almost everywhere constant.

This means that we can estimate

$$\frac{1}{\mu(X)} \int \phi d\mu$$

by

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \phi(T^j(x)).$$

Corollary 6. [BKS, p. 41] *Let T be an ergodic measure-preserving transformation of the probability space $(X,\, \mathcal{A},\, \mu)$, and let B be any element of $\mathcal{A}$. Then $\mu(B) = A(x)$ for almost all $x \in X$. Particularly, it means*

$$\mu(B) = \lim_{n\to\infty} \frac{1}{n}\sum_{t=0}^{n-1} \chi_B(T^t(x)).$$

We will need theorems with a more general function $f : X \to X$ than χ_B. The following corollary of Birkhoff's ergodic theorem enables us to draw measures induced by IFS and their supports-fractals and estimate integrals.

Theorem 30. *(Elton's theorem)*[Ba1, Theorem 7.1, p. 364] *Let $(X,\, d)$ be a compact metric space. Let $(X,\, f_1,\, f_2,\, \dots,\, f_m;\, p_1,\, p_2,\, \dots,\, p_m)$ be a hyperbolic IFS with probabilities. Let $(X,\, d)$ be a compact metric space. Let $\{x_n\}_{n=0}^{\infty}$ denote an orbit of the IFS produced by the random iteration algorithm starting at x_0. That is*

$$x_n = f_{\sigma_n} \circ f_{\sigma_{n-1}} \circ \dots f_{\sigma_1}(x_0),$$

where the maps are chosen independently according to probabilities $p_1,\, p_2,\, \dots,\, p_m$, for $n = 1,\, 2,\, \dots$ Let μ be the unique invariant measure for the IFS. Then with probability one (that is, for all code sequences $\sigma_1,\, \sigma_2,\, \dots$ except for a set of sequences having probability zero),

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} f(x_k) = \int_X f(x)d\mu(x),$$

for all continuous functions $f : X \to \mathbb{R}$ and all x_0.

Remark 52. The theorem holds for more general cases than we treat (cf. [E]). The space can be locally compact, p_i's can be functions of x and f_i's can be contraction mappings "on average" ($\sum_i p_i r_i < 1$).

The following corollary gives a prescription how to draw fractals and fractal measures by means of Elton's theorem.

Corollary 7. [Ba1, Corollary 7.1, p. 365] *Let B be a Borel subset of X and let $\mu(\partial B) = 0$. Let $N(B,\, n) = card\{x_0,\, x_1,\, x_2,\, \dots,\, x_n\} \cap B$, for $n = 0,\, 1,\, 2,\, \dots$ Then with probability one,*

$$\mu(B) = \lim_{n\to\infty} \left\{ \frac{N(B,\, n)}{n+1} \right\},$$

for all starting points x_0. That is the "mass" of B is the proportion of iteration steps when running the random iteration algorithm, which produces points in B.

5.4. Chaos game

This subsection is devoted to the chaos game. It is the most popular way to image fractals and invariant measures. In order to understand how it works, we will introduce necessary notions from the theory of chaos. In addition, we will apply the chaos game to hyperIFSs to visualize attractors of underlying IMSs.

Fractals were developed together with computers and ability to draw them. The first pictures and the theory can be found in [Ma3], [Ba1], [BD] and [PJS]. For fractals generated by IFSs, we have two basic possibilities how to draw attractors. The first one follows from the Banach theorem. Let $F = \{(X, d), f_1, \ldots, f_m\}$ be an IFS, $F : K(X) \to K(X)$ induced operator and A^* the attractor (X is usually $\mathbb{R}^2$). We draw $n-$th iterate $F^n(A)$, where $A \in K(X)$ is arbitrary, instead of A^*. These sets are close for n great enough due to the Banach theorem, which implies

$$d_H(A, A^*) \leq \frac{d_H(F(A), A)}{1 - r}$$

and

$$d_H(F^n(A), A^*) \leq \frac{r^n d_H(F(A), A)}{1 - r}.$$

Since multivalued fractals are fixed point of contracting operator $F : K(X) \to K(X)$, we can image attractors of IMS in the same way. However, this approach may not be effective. We need to store and process complicated sets and count with errors (for multivalued case see [AFGL], [Fi]).

Barnsley (cf. [BD], [Ba1]) introduced the chaos game for IFS. It demands less memory and process less complicated objects. Therefore, the chaos game is the most popular way to draw fractals. Given an IFS with probabilities $\{X, f_i, p_i, i = 1, 2, \ldots, m\}$, we construct a sequence

$$\{x_i\}_{i=1}^n, \ x_i \in X,$$

where

$$x_{i+1} = f_{\sigma_i}(x_i), \ \sigma_i \in \{1, 2, \ldots, m\}.$$

Contractions f_j are taken with given probability $P(\sigma_i = j) = p_j$, $j \in \{1, 2, \ldots, m\}$. We divide the space X, usually $\mathbb{R}^2$, to small squares-pixels and calculate the ratio of points x_i which lie in each pixel. In this way we obtain measures of any pixel. The fractal is approximated by the pixels which have a positive ratio.

Barnsley describes extensively the chaos game and its relationship to the shift dynamical system in [Ba1, pp. 168–169].

Consider the hyperbolic IFS $\{\mathbb{R}^2, f_1, f_2\}$ with an attractor A^*. Let $a \in A^*$; suppose that the address of a is $\sigma \in \Sigma$ the associated code space. That is

$$a = \phi(\sigma).$$

With the aid of a random-number generator, a sequence of one million ones and twos is selected. For example, suppose that the actual sequence produced is the following one, which has been written from right to left,

$$21\ldots 12122211$$

By this we mean that the first number chosen is a 1, then a 1, then three 2's, and so on. Then the following sequence of points on the attractor is computed:

$$\begin{gathered}
a = \phi(\sigma)\\
f_1(a) = \phi(1\sigma)\\
f_1 \circ f_1(a) = \phi(11\sigma)\\
f_2 \circ f_1 \circ f_1(a) = \phi(112\sigma)\\
f_2 \circ f_2 \circ f_1 \circ f_1(a) = \phi(2211\sigma)\\
f_2 \circ f_2 \circ f_2 \circ f_1 \circ f_1(a) = \phi(22211\sigma)\\
f_1 \circ f_2 \circ f_2 \circ f_2 \circ f_1 \circ f_1(a) = \phi(122211\sigma)\\
f_2 \circ f_1 \circ f_2 \circ f_2 \circ f_2 \circ f_1 \circ f_1(a) = \phi(2122211\sigma)\\
f_1 \circ f_2 \circ f_1 \circ f_2 \circ f_2 \circ f_2 \circ f_1 \circ f_1(a) = \phi(12122211\sigma)\\
\vdots\\
f_2 \circ f_1 \circ \cdots f_1 \circ f_2 \circ f_1 \circ f_2 \circ f_2 \circ f_2 \circ f_1 \circ f_1(a) = \phi(21 \cdots 12122211\sigma)
\end{gathered}$$

We imagine that instead of plotting the points as they are computed, we keep a list of the one million computed points. This done, we plot the points in the reverse order from the order in which they were computed. That is, we begin by plotting the point $\phi(21 \cdots 12122211\sigma)$ and we finish by plotting the point $\phi(\sigma)$. What we will see? We will see one million points on the orbit of the shift dynamical system $\{A^*,\, S\}$, namely, $\{S^n(\phi(21 \cdots 12122211\sigma))\}_{n=0}^{1000000}$.

Remark 53. We have not mentioned probabilities yet. For example, the chaos game for the IFS $\{X,\, f_1,\, f_2,\, f_3,\, p_1 = 0.1,\, p_2 = 0.3,\, p_3 = 0.6\}$ with an attractor A^* produces points in A^*_1 with probability 0.1, in A^*_2 with probability 0.3, A^*_3 with probability 0.6 and A^*_{33} with probability 0.36. Then orbits of the shift dynamical system spend 0.1 of time in A^*_1 and 0.36 of time in A^*_{33} for almost all sequences generated by the chaos game.

We will show that the shift dynamical system is also chaotic, which means that almost all orbits of the chaos game are dense in the attractor.

Definition 38. ([Ba1, Definition 8.2, p. 167]) A dynamical system $\{X,\, f\}$ is *transitive* if, whenever U and V are open subsets of the metric space $(X,\, d)$, there exists a finite integer n such that

$$U \cap f^n(V) \neq \emptyset.$$

Definition 39. ([Ba1, Definition 8.3, p. 167]) A dynamical system $\{X,\, f\}$ is *sensitive to initial conditions* if there exists $\delta > 0$ such that, for any $x \in X$ and any neighbourhood $O(x,\, \epsilon)$, there is $y \in O(x,\, \epsilon)$ and integer $n \geq 0$ such that $d(f(x),\, f(y)) > \delta$.

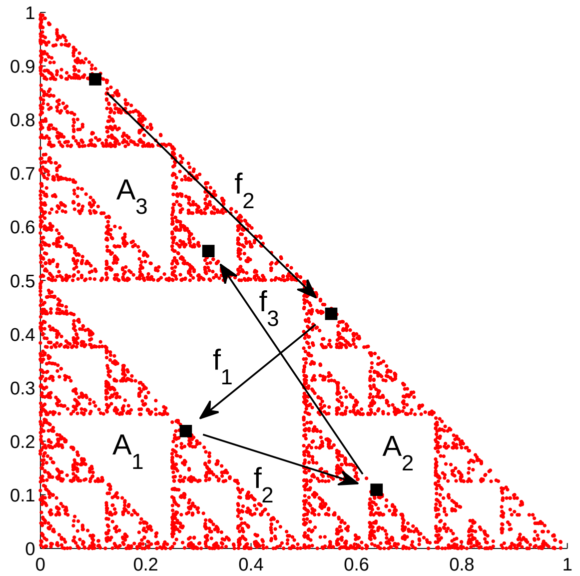

Figure 11: Chaos game for the Sierpiński triangle

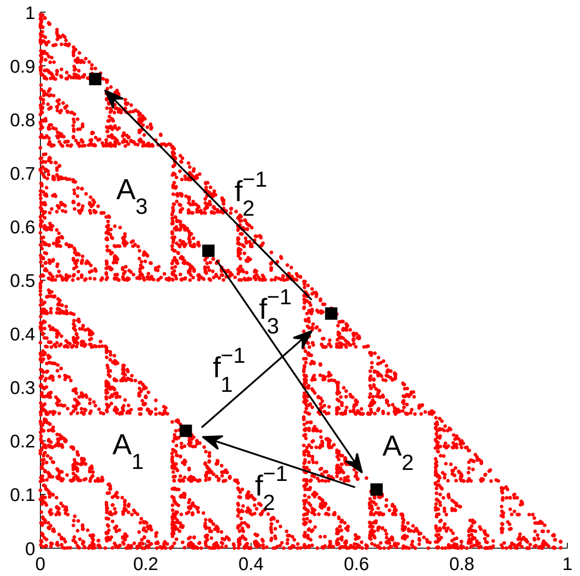

Figure 12: Shift dynamical system for the Sierpiński triangle

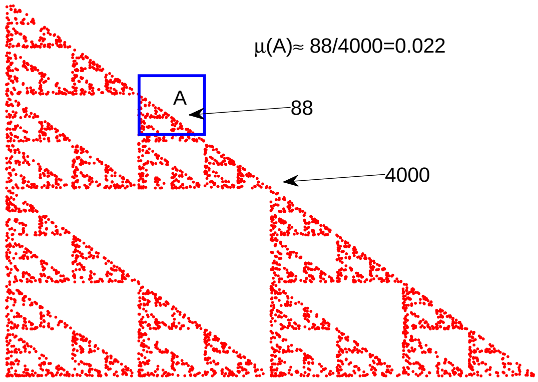

Figure 13: Approximation of a measure of a set using chaos game

We are ready to give a definition of chaos.

DEFINITION 40. ([Ba1, Definition 8.4, p. 167]) A dynamical system $\{X, f\}$ is *chaotic* if

1. it is transitive;

2. it is sensitive to initial conditions;

3. the set of periodic points is dense in X.

THEOREM 31. *(*[Ba1, Theorem 8.1, p. 167]*) The shift dynamical system associated with a totally disconnected hyperbolic IFS of two or more transformations is chaotic.*

Running the chaos game, we almost always produce small errors. Since the shift dynamical system is chaotic, the orbit with errors diverges from the exact one. However, the following theorem assures us that there is another orbit which is close to the one with errors.

THEOREM 32. [Ba1, Theorem 7.1, p. 159] *Let $\{X; f_1, f_2, \dots, f_n\}$ be a hyperbolic IFS of contractivity r, where $0 < r < 1$. Let A^* denote the attractor of the IFS and suppose that each of the transformations $f_i : A^* \to A^*$ is invertible. Let $\{A^*, S\}$ denote the associated shift dynamical system in the case that the IFS is totally disconnected. Let $\{\tilde{x}_i\}_{i=0}^{\infty} \in A^*$ be an approximate orbit of S, such that*

$$d(\tilde{x}_{i+1}, S(\tilde{x}_i)) \leq \theta, \textit{ for all } i = 1, 2, 3, \dots,$$

for some fixed constant θ with $0 \leq \theta \leq \mathrm{diam}(A^*)$. Then there is an exact orbit $\{x_i = S^i(x_0)\}_{i=0}^{\infty}$ for some $x_0 \in A^*$, such that

$$d(\tilde{x}_{i+1}, x_{i+1}) \leq \frac{r\theta}{(1-r)}, \textit{ for all } i = 1, 2, 3, \dots$$

Now, we proceed to hyperIFSs. Since hyperIFSs are IFSs, we can run the chaos game for any hyperIFS. Moreover, we will use the chaos game for hyperIFSs to draw attractors of underlying IMSs. Notice that the set of address sets of a multivalued fractal forms a hyperfractal.

*THEOREM 33. *Let us consider a hyperIFS generated by the IMS* $F = \{(X, d), F_1, F_2, \ldots, F_n\}$, *where* $F_i : X \to K(X)$, A^* *is an attractor of the IMS and* α^* *is an attractor of the associated hyperIFS. Let us also consider an orbit of the chaos game* $\hat{\alpha} = \{\hat{A}_i\}_{i=1}^k$, $\hat{A}_i \in K(X)$, *where* $k \in \mathbb{N} \cup \{\infty\}$, *such that* $d_{H_H}(\hat{\alpha}, \alpha^*) \leq \epsilon$. *Then* $d_H(\cup_i \hat{A}_i, A^*) \leq \epsilon$, *which can be written*

$$d_H\left(\bigcup_i \hat{A}_i, \bigcup_{\sigma\in\Sigma} A^*_\sigma\right) \leq \epsilon.$$

Proof. $d_{H_H}(\hat{\alpha}, \alpha^*) \leq \epsilon$ implies that, for each $A^*_\sigma \in \alpha^*$, there exists $\hat{A}_i \in \hat{\alpha}$ such that

$$d_H(A^*_\sigma, \hat{A}_i) \leq \epsilon.$$

It follows that, for each $x \in A^*_\sigma$, there exists $y \in \hat{A}_i$ such that $d(x, y) \leq \epsilon$. In the same way, for each $\hat{A}_i \in \hat{\alpha}$, there exists $A^*_\sigma \in \alpha^*$ such that

$$d_H(\hat{A}_i, A^*_\sigma) \leq \epsilon.$$

This implies that, for each $x \in \hat{A}_i$, there exists $y \in A^*_\sigma$ such that $d(x, y) \leq \epsilon$. We arrive to

$$d_H\left(\bigcup_i \hat{A}_i, \bigcup_{\sigma\in\Sigma} A^*_\sigma\right) =$$

$$\max\{\sup_{x\in\bigcup_i \hat{A}_i}\{\inf_{y\in A^*}\{d(x, y)\}\}, \sup_{x\in A^*}\{\inf_{y\in\bigcup_i \hat{A}_i}\{d(x, y)\}\}\} \leq \epsilon.$$

□

Hence, one can use the chaos game for hyperfractals to draw underlying multivalued fractals with the same accuracy.

REMARK 54. The preceding derivations also follow from the fact, that the metric d_{H_H} is "stronger" than the metric d_H, which is stated in the next theorem.

THEOREM 34. *(*[Ba2, Theorem 1.13.8]*) Let* (X, d) *be a metric space. Let* $\alpha, \beta \in K(K(X))$ *be such that*

$$\{a \in A : A \in \alpha\}, \{b \in B : B \in \beta\} \in K(X).$$

Then

$$d_H(\{a \in A : A \in \alpha\}, \{b \in B : B \in \beta\}) \leq d_{H_H}(\alpha, \beta).$$

REMARK 55. We will also use the chaos game for hyperfractals to image a measure on multivalued fractals, but we will need in addition the theory from the following section.

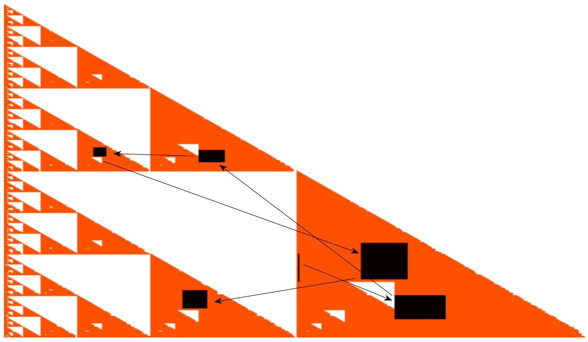

Figure 14: Chaos game for hyperIFS

6. Visualization and dimension of hyperfractals

Since hyperfractals are attractors of IFS, we can explore their self-similarity and dimension. We will also visualize their structure. The theory of convex sets provides us an effective tool.

6.1. Convex sets and support functions

Convex sets will play a key role in investigation of dimension and visualization of hyperfractals. Let us remind notions related to convex sets, convex hulls and support functions.

Let $(E, \|.\|)$ be a real Banach space and $A \subset E$, $B \subset E$ its subsets. Defining, as usual (cf. e.g. [AB], [AG2])

$$A + B := \{x \mid x = a + b,\, a \in A,\, b \in B\},$$

$$c \cdot A := \{x \mid x = c \cdot a,\, a \in A\},\; c \in \mathbb{R},$$

we can say the following. If A and B are convex subsets of E, then $A + B$, and $c \cdot A$ are convex (cf. e.g. [Be], [DS]). In the special case $E = \mathbb{R}^m$, we have also that (cf. e.g. [Be, Theorem 1.4.1]) $\mathcal{Q}A$ is convex for $A \in K_{Co}(\mathbb{R}^m)$, $\mathcal{Q} \in \mathbb{R}^{m\times m}$.

Defining still the *convex hull* $\mathrm{conv}(A)$ of $A \in K(E)$ as (see e.g. [DS, Chapter V.2]

$$\mathrm{conv}(A) := \{x \in E \mid x = \sum_{i=1}^{p} \alpha_i a_i,\, \sum_{i=1}^{n+1} \alpha_i = 1,\, \alpha_i \geq 0,\, a_i \in A,\, i = 1, \ldots, p\},$$

$p = 1, 2, \ldots,$ it is obviously the smallest convex set containg $A \subset E$. Let us note that, in $\mathbb{R}^m$, we can simply fix $p = m + 1$.

Lemma 9. *For any A, $B \subset E$, it holds (see e.g.* [DS, Lemma V.2.4]*):*

1. *conv*$(A+B) = $ *conv*$(A) +$ *conv*(B),
2. *conv*$(c \cdot A) = c \cdot$ *conv*(A), $c \in \mathbb{R}$.
   In the special case $E = \mathbb{R}^m$, *we have also that*
3. *conv*$(\mathscr{Q}A) = \mathscr{Q}$*conv*$(A)$, $\mathscr{Q} \in \mathbb{R}^{m\times m}$.

We can describe compact convex sets in $\mathbb{R}^m$ with closed halfspaces [MV]

$$F := \{x \in \mathbb{R}^m;\ x'a \le c, a \in \mathbb{R}^m,\ c \in \mathbb{R}\}.$$

The intersection of such halfspaces defines a compact convex subset

$$K \sim \bigcap_{\alpha\in A} F_\alpha.$$

For polygons, A is finite.

We need not consider all a's but it suffices to take $\{a \in \mathbb{R}^m,\ |a| = 1\}$ and we get c's as values of a support function (see [MV, p. 328], [Sch, p. 37]).

Definition 41. *A support function* $\mathrm{supp}_M(x)$ of a compact set $M \in \mathbb{R}^m$ is defined

$$\mathrm{supp}_M(x) := \max(m'x,\ m \in M,\ x \in \mathbb{R}^m,\ ||x|| = 1).$$

Let us remind basic properties of support functions (see [DK]).

Lemma 10. *Let* $M_1,\ M_2 \in K_{Co}(\mathbb{R}^m)$. *Then*

$$supp_{M_1+M_2} = supp_{M_1} + supp_{M_2},$$

$$supp_{\lambda M} = \lambda supp_M,\ \lambda \ge 0.$$

Lemma 11. *Let* $M_1,\ M_2 \in K_{Co}(\mathbb{R}^m)$. *Then*

$$d_{\mathrm{H}}(M_1,\ M_2) = \max_x |\mathrm{supp}_{M_1}(x) - \mathrm{supp}_{M_2}(x)|.$$

Let us denote by $S(K_{Co}(\mathbb{R}^m))$ the set of all support functions for sets in $K_{Co}(\mathbb{R}^m)$.

Remark 56. The correspondence between $K_{Co}(\mathbb{R}^m)$ and $S(K_{Co}(\mathbb{R}^m))$ is one to one.

The metric spaces $(K_{Co}(\mathbb{R}^m),\ d_H)$ and $(S(K_{Co}(\mathbb{R}^m)),\ d_H)$ are identical. Hence, $(S(K_{Co}(\mathbb{R}^m)),\ d_H)$ is complete.

Lemma 12. *(*[DK, p. 13]*) Let* $A,\ B,\ C \in (K_{Co}(\mathbb{R}^m),\ d_H)$, *then*

$$d_H(A+C,\ B+C) = d_H(A,\ B). \tag{16}$$

As a consequence of the previous lemma, we obtain the following lemma.

Lemma 13. *A map* $F : K_{Co}(\mathbb{R}^m) \to K_{Co}(\mathbb{R}^m)$,

$$F(A) = r\mathscr{Q}A + C,$$

where $r \in [0,\, 1)$, $\mathscr{Q} \in \mathbb{R}^{m\times m}$ *is orthonormal and* $C \in K_{Co}(\mathbb{R}^n)$, *is a similitude.*

Proof. Since $A,\, B \subset K_{Co}(\mathbb{R}^m)$ are convex sets, so must be, for each $r\mathscr{Q}A$ and $r\mathscr{Q}B$. Furthermore, since it is well-known (see e.g. [Hu]) that, in Euclidean spaces $(\mathbb{R}^m,\, d_{\mathrm{Eucl}})$, $\mathscr{Q}$ acts as an isometry, we obtain in view of (16) that

$$d_H(F(A),\, F(B)) = d_H(r\mathscr{Q}A + C,\, r\mathscr{Q}B + C) = rd_H(\mathscr{Q}A,\, \mathscr{Q}B) =$$
$$r\max\{\sup_{a\in\mathscr{Q}A}\inf_{b\in\mathscr{Q}B} d_{\mathrm{Eucl}}(a,\, b),\, \sup_{b\in\mathscr{Q}B}\inf_{a\in\mathscr{Q}A} d_{\mathrm{Eucl}}(a,\, b)\} =$$
$$r\max\{\sup_{a\in A}\inf_{b\in B} d_{\mathrm{Eucl}}(\mathscr{Q}a,\, \mathscr{Q}b),\, \sup_{b\in B}\inf_{a\in A} d_{\mathrm{Eucl}}(\mathscr{Q}a,\, \mathscr{Q}b)\} =$$
$$r\max\{\sup_{a\in A}\inf_{b\in B} d_{\mathrm{Eucl}}(a,\, b),\, \sup_{b\in B}\inf_{a\in A} d_{\mathrm{Eucl}}(a,\, b)\} =$$
$$rd_H(A,\, B),$$

i.e. F is similitude, as required. □

We will compare the Hausdorff distance of compact sets and their convex hulls. Hence, let us state three more lemmas.

Lemma 14. [DK, p. 13] *Let* $A,\, B \in K(\mathbb{R}^m)$. *Then* $A \subset B \Rightarrow \mathrm{supp}_A(x) \le \mathrm{supp}_B(x)$.

We discussed and proved the following lemma in [AR2].

Lemma 15. *For* $A,\, B,\, C \in K(E)$, *we have that*

$$d_H(A, B) \ge d_H(conv(A), conv(B)) \tag{17}$$

and

$$d_H(conv(A + C), conv(B + C)) = d_H(conv(A), conv(B)). \tag{18}$$

Lemma 16. *Let* $B \in \mathbb{R}^m$ *be a compact set. It has the same support function as its convex hull.*

Proof. We can prove two inequalities instead of equality $\mathrm{supp}_B(x) = \mathrm{supp}_{\mathrm{conv}(B)}(x)$.

First, it follows from Lemma 14 that $\mathrm{supp}_B(x) \le \mathrm{supp}_{\mathrm{conv}(B)}(x)$.

Second, let $a \in \mathbb{R}^m$ be such that $|a| = 1$. Let us assume that for some $y \in \mathrm{conv}(B)$, $y'a = c$. Then we can write

$$y = \alpha_1 y_1 + \alpha_2 y_2 + \cdots \alpha_{n+1} y_{n+1},$$

where $y_i \in B$, $i = 1,\, 2,\, \ldots,\, n+1$, $\sum_{i=1}^{n+1} \alpha_i = 1$. Using properties of scalar product, we have

$$y'a = \alpha_1 y_1' a + \alpha_2 y_2' a + \cdots \alpha_{n+1} y_{n+1}' a = c.$$

Observe that at least one $y_{\max} \in \{y_1,\, \ldots y_{n+1}\}$ fulfills $y_{\max} a \ge c$. It implies $\mathrm{supp}_B(x) \ge \mathrm{supp}_{\mathrm{conv}(B)}(x)$. □

We are ready to give a theorem which will help us to calculate the Hausdorff dimension of hyperfractals and visualize their particular class.

*Theorem 35. *Let us consider two IMS $F = \{\mathbb{R}^m,\, F_i,\, i = 1,\, 2,\, \ldots,\, n\}$ and $F^c = \{\mathbb{R}^m,\, F_i^c,\, i = 1,\, 2,\, \ldots,\, n\}$, where*

$$F_i : \mathbb{R}^m \to K(\mathbb{R}^m), F_i(x) = \{r_i \mathscr{Q}_i x\} + C_i,\, i = 1,\, 2,\, \ldots,\, n,$$

and

$$F_i^c : \mathbb{R}^m \to K(\mathbb{R}^m), F_i(x) = \{r_i \mathscr{Q}_i x\} + \mathrm{conv}(C_i),\, i = 1,\, 2,\, \ldots,\, n,$$

$C_i \in K(\mathbb{R}^m)$, $r_i \in [0,\, 1)$ are reals, $\mathscr{Q}_i$ are orthonormal $m \times m-$matrices. Let us also consider the associated hyperIFS $\phi = \{(K(\mathbb{R}^m),\, d_H),\, F_i,\, i = 1,\, 2,\, \ldots,\, n\}$ and $\phi^c = \{(K_{Co}(\mathbb{R}^m),\, d_H),\, F_i^c,\, i = 1,\, 2,\, \ldots,\, n\}$ with attractors α^* and α^c. Then address sets A_σ^* and A_σ^c of α^* and α^c have the same set of support functions. This means

$$\mathrm{supp}_{A_\sigma^*} = \mathrm{supp}_{A_\sigma^c},\, \forall \sigma \in \Sigma$$

and also

$$\mathrm{conv} A_\sigma^* = A_\sigma^c,\, \forall \sigma \in \Sigma.$$

Proof. We will prove that $\mathrm{supp}(A_\sigma) = \mathrm{supp}(A_\sigma^c)$, $\sigma \in \Sigma$. We can see from Lemma 9 that $\mathrm{conv}(A + B) = \mathrm{conv}(A) + \mathrm{conv}(B)$. Let $A \in K(\mathbb{R}^m)$ then

$$\mathrm{conv}(F_j(A)) = F_j^c(\mathrm{conv}(A)),\, \forall j \in \{1,\, 2,\, \ldots,\, n\},\, A \in K(\mathbb{R}^m),$$

follows from

$$\mathrm{conv}(F_j(A)) = \mathrm{conv}(r_j \mathscr{Q}_j(A) + C_j) = r_j \mathscr{Q}_j \mathrm{conv}(A) + \mathrm{conv}(C_j) = F_j^c(\mathrm{conv}(A)).$$

Thus, we have from mathematical induction

$$\mathrm{conv}(F_{i_1 i_2 \ldots i_n}(A)) = F^c_{i_1 i_2 \ldots i_n}(\mathrm{conv}(A)),\, \forall n \in \mathbb{N}.$$

Since the space $K_{Co}(\mathbb{R}^m)$ is complete, the sequences have the same limit,

$$\lim_{n \to \infty} \mathrm{conv}(F_{\sigma_1 \sigma_2 \ldots \sigma_n}(A)) = \lim_{n \to \infty} F^c_{\sigma_1 \sigma_2 \ldots \sigma_n}(\mathrm{conv}(A)).$$

It follows

$$\mathrm{conv}(A_\sigma^*) = A_\sigma^c$$

and from Lemma 16 also

$$\mathrm{supp}_{A_\sigma^*} = \mathrm{supp}_{A_\sigma^c},\, \forall \sigma \in \Sigma.$$

□

Remark 57. The theorem could be generalized for affine mappings.

6.2. Visualization of hyperfractals

Hyperfractals lie in a hyperspace which is complicated infinite-dimensional nonlinear space. However, the space $K_{Co}(\mathbb{R}^m)$ can be embedded in a linear space due to Rådström [Ra]. We will make at least projections of hyperfractals. We will use support functions of compact sets.

We know that

$$\mathrm{supp}_A(x) = \sup_u \{u'x,\, u \in A,\, |x| = 1\} \tag{19}$$

and (cf. Figure 24)

$$d_H(A,\, B) = \max_{x\in\mathbb{R}^n,\, |x|=1} |\mathrm{supp}_A(x) - \mathrm{supp}_B(x)|. \tag{20}$$

In order to draw, for instance, a three dimensional projection of sets $A, B \in K_{Co}(\mathbb{R}^m)$, we choose $x_i \in \mathbb{R}^m$, $|x_i| = 1$, $i = 1, 2, 3$. We draw a three dimensional graph, where each axis corresponds to one x_i. Thus, $\mathrm{supp}_A(x_i)(= \sup_u\{u'x_i,\, u \in A\})$ and $\mathrm{supp}_B(x_i)(= \sup_u\{u'x_i,\, u \in B\})$ will be coordinates of A and B. In other words, we create the map

$$V : (K_{Co}(\mathbb{R}^n),\, d_H) \to (\mathbb{R}^d,\, d_{\max}),$$

$$V(M) = \begin{pmatrix} \mathrm{supp}_M(x_1) \\ \mathrm{supp}_M(x_2) \\ \vdots \\ \mathrm{supp}_M(x_d) \end{pmatrix},$$

where $d = 3$. Thus, $A \in K(\mathbb{R}^n)$ is represented by coordinates $(\mathrm{supp}_A(x_1),\, \mathrm{supp}_A(x_2),\, \mathrm{supp}_A(x_3))$ in our graph. We consider the space $\mathbb{R}^d$ with the metric $d_{\max,}$ because the Hausdorff distance between A and B is greater or equal than the maximum of differences in coordinates in this coordinate system (cf. equation (20)). Usually, we get only projections of the metric structure of fractals in hyperspaces.

Remark 58. We can naturally generalize the map V to

$$V : (K(\mathbb{R}^n),\, d_H) \to (\mathbb{R}^d,\, d_{\max}),$$

since

$$\mathrm{supp}_A = \mathrm{supp}_{\mathrm{conv}(A)}$$

and

$$V(A) = V(\mathrm{conv}(A)).$$

Let us show the easiest cases. We will discuss visualizing of hyperfractals in $K_{Co}(\mathbb{R})$, $K_{Co}(\mathbb{R}^2)$ and the behaviour of the visualization of a fractal set of singletons.

There are only two vectors of length 1 in $\mathbb{R}^1$ ($x_1 = 1$, $x_2 = -1$). Therefore, we can draw hyperfractals from $K_{Co}(\mathbb{R}^1)$ in two-dimensional pictures. For $A = [a, b] \in K_{Co}(\mathbb{R}^1)$, we have $\mathrm{supp}_A(-1) = \sup_y\{-1y,\, y \in A\} = -a$, $\mathrm{supp}_A(1) = \sup_y\{y,\, y \in A\} = b$. We arrive to $V([a, b]) = (-a, b)$.

Example 5. Let us consider an attractor α^* of the hyperIFS $\phi = \{(K_{Co}([0, 1]), d_H), F_1, F_2, F_3\}$ induced by an IMS $\{([0, 1], d_{\mathrm{Eucl}}), F_1, F_2, F_3\}$, where $F_i : [0, 1] \to K_{Co}([0, 1])$, $i = 1, 2, 3$,

$$\begin{aligned} F_1(x) &= \{rx\}, \\ F_2(x) &= \{rx + 1 - r\}, \\ F_3(x) &= [rx,\, rx + 1 - r] = \{rx\} + [0, 1 - r], \end{aligned}$$

for $r = \frac{1}{2}$. Running the chaos game for the hyperIFS, we obtain a sequence of intervals $\{A_i\}_{i=1}^k$, which can be easily visualized by means of the map V (see Figure 15). Note that we get the same picture as for the attractor B^* of the IFS $F = \{([-1, 0] \times [0, 1],\, d_{\mathrm{Eucl}}),\, g_1,\, g_2,\, g_3\}$,

$$g_i(x) = \frac{x + c_i}{2},$$

where $c_1 = (0,\, 0)'$, $c_2 = (-1,\, 1)'$, $c_3 = (0,\, 1)'$. This follows from

$$g_i(x) = V F_i V^{-1}(x),\ x \in [-1, 0] \times [0, 1].$$

Although the structures of the attractors α^* and B^* differ in metrics, the Hausdorff dimension of the attractors is the same ($D = \frac{\log 3}{\log 2}$). Since the maximum metric and Euclidean metric are equivalent,

$$d_{\max}(x_1,\, x_2) \le d_{\mathrm{Eucl}}(x_1,\, x_2) \le \sqrt{2} d_{\max}(x_1,\, x_2),\ x_1,\, x_2 \in \mathbb{R}^2,$$

it follows from Proposition 9.

Remark 59. Note that the shadow of α^* is [0, 1].

Now, we turn our attention to the case of hyperIFS in $K(\mathbb{R}^2)$. Let us remind the definition of a support function

$$\mathrm{supp}_M(x) = \sup_m\{m'x,\, m \in M,\, |x| = 1\}. \tag{21}$$

It is much more comfortable to consider one angle θ instead of two coordinates $x \in \mathbb{R}^2$, $|x| = 1$ in $\mathrm{supp}(x)$. Hence, we write, for $x \in \mathbb{R}^2$, $|x| = 1$, and $m \in M$, $M \in K(\mathbb{R}^2)$,

$$x = (\cos\theta,\, \sin\theta),\ m = r_m(\cos\phi,\, \sin\phi).$$

Then

$$\mathrm{supp}_M(\theta) = \sup_{m \in M}\{r_m(\cos\theta\cos\phi + \sin\theta\sin\phi)\}$$

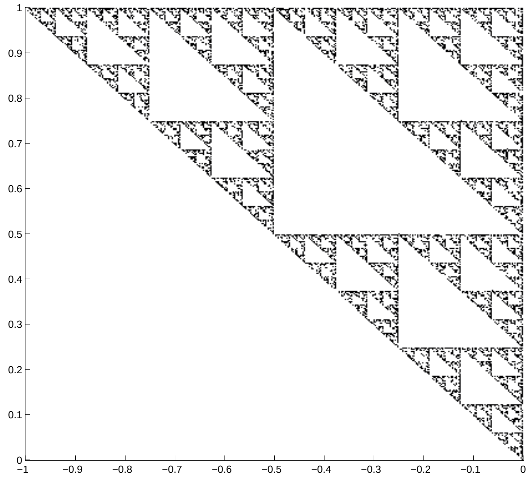

Figure 15: Sierpiński hypertriangle

and it follows

$$\mathrm{supp}_M(\theta) = \sup_{m \in M} \{r_m(\cos(\theta - \phi)\}.$$

Example 6. We will consider a hyperIFS associated with an IMS $F = \{[0,\,1]^2,\, F_i : [0,\,1]^2 \to K([0,\,1]^2),\, i = 1,\,2,\,3\}$,

$$F_1\begin{pmatrix} x \\ y \end{pmatrix} := \left\{\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\},$$

$$F_2\begin{pmatrix} x \\ y \end{pmatrix} := \left\{\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}\right\},$$

$$F_3\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} [\frac{1}{3},\,\frac{1}{2}] & 0 \\ 0 & [\frac{1}{3},\,\frac{1}{2}] \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \left\{\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}\right\},$$

where

$$\left(\left[\frac{1}{3},\,\frac{1}{2}\right] \cdot x,\ \left[\frac{1}{3},\,\frac{1}{2}\right] \cdot y\right) := \left(\left[\frac{x}{3},\,\frac{x}{2}\right],\ \left[\frac{y}{3},\,\frac{y}{2}\right]\right).$$

The attractor of the underlying IMS is called the fat Sierpiński triangle.

Let $x_1 = (\frac{1}{2},\,\frac{\sqrt{3}}{2})$, $x_2 = (-\frac{1}{2},\,\frac{\sqrt{3}}{2})$ and $x_3 = (-\frac{1}{2},\,-\frac{\sqrt{3}}{2})$ which corresponds to $\theta_1 = \frac{\pi}{3}$, $\theta_2 = \frac{2\pi}{3}$ and $\theta_3 = \frac{4\pi}{3}$, respectively. We run the chaos game for the hyperIFS and obtain a sequence of compact convex sets $\{A_i\}_{i=1}^k$. We show, for $k = 10$, sets A_i, $i = 1,\,\ldots,\,k$, their support functions and projections to $\mathbb{R}^3$ in Figure 21. For $k = 500$, the structure of the hyperfractal can be seen in Figure

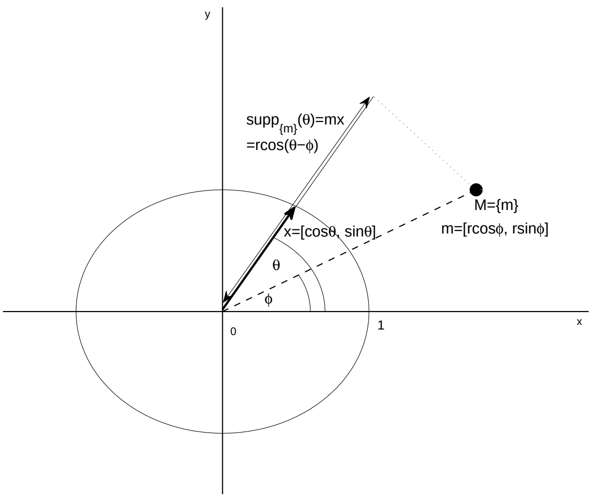

Figure 16: Support function of a one-point set

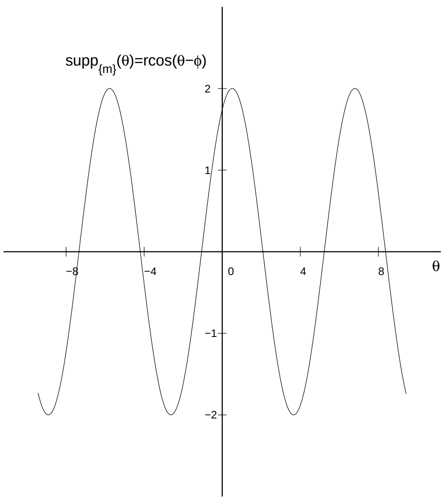

Figure 17: Support function of a one-point set

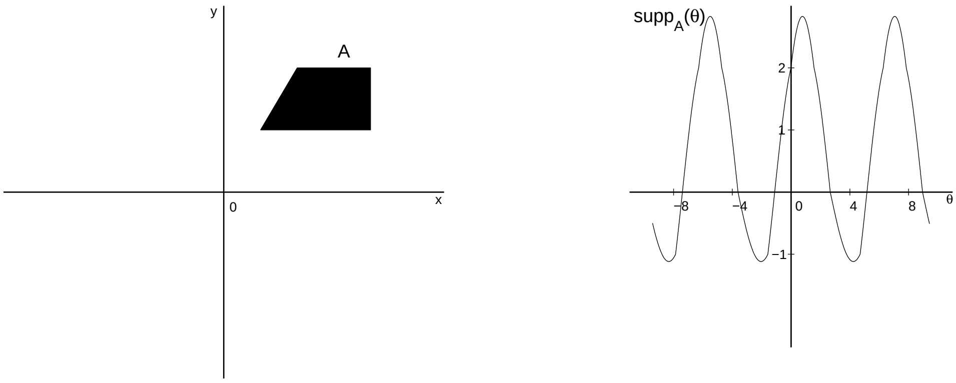

Figure 18: Set and its support function

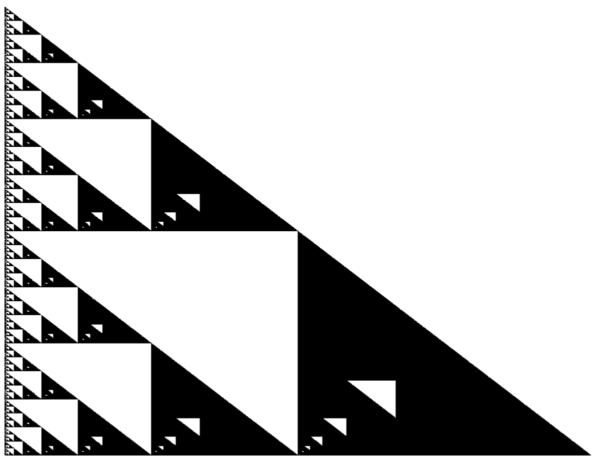

Figure 19: Fat Sierpiński triangle

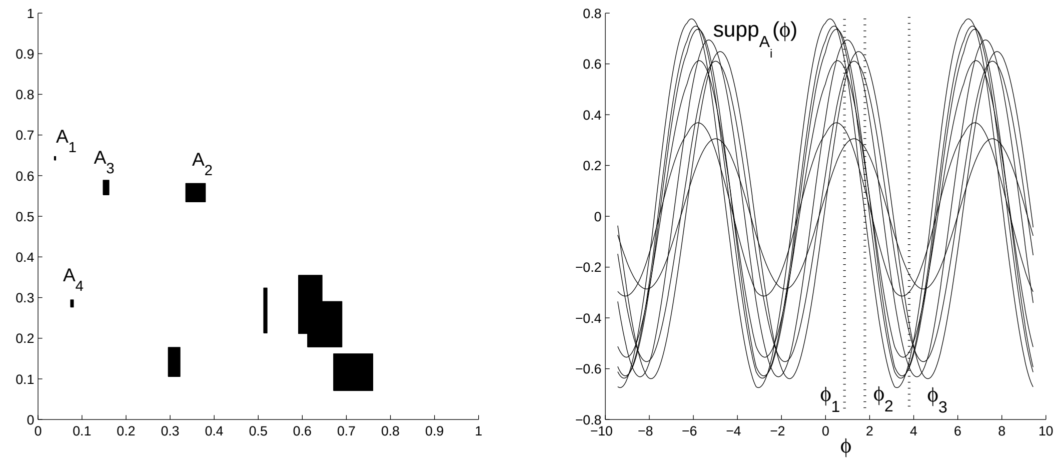

Figure 20: Support functions of 10 sets from Example 6

22. Finally, we visualize the structure of the fat Sierpiński hypertriangle in Figure 23.

We will find out that a hyperIFS in $K_{Co}(\mathbb{R}^2)$ consisting of similitudes has an elegant interpretation in the space of support functions. Let us discuss how support functions of images of convex sets in similitudes look like. Similitudes in $\mathbb{R}^2$ are compositions of homotheties, translations, and orthonormal transformations (reflections and rotations). Similitudes in $K_{Co}(\mathbb{R}^2)$ are generalization of similitudes in $\mathbb{R}^2$. Furthermore, addition of convex sets is involved. Moreover, we can regard translation as addition of a one-point set (see [DK, p. 14]). We reviewed the behaviour of support functions of homotheties and translations in Lemma 10. It remains to show the support functions of reflected and rotated sets.

For $\mathscr{Q} \in \mathbb{R}^{n\times n}$, $M \in K(\mathbb{R}^n)$, we can write

$$\operatorname{supp}_{\mathscr{Q}M}(x) = \max(n'x,\, n \in \mathscr{Q}M) =$$

$$= \max(m'\mathscr{Q}'x,\, m \in M) = \operatorname{supp}_M(\mathscr{Q}'x).$$

In the particular case of $n = 2$, $x = (\cos\theta, \sin\theta)'$ and matrix of rotation $\mathscr{Q}$,

$$\mathscr{Q} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix},$$

we obtain

$$\mathscr{Q}'x = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\phi\cos\theta + \sin\phi\sin\theta \\ -\sin\phi\cos\theta + \cos\phi\sin\theta \end{pmatrix} = \begin{pmatrix} \cos(\theta-\phi) \\ \sin(\theta-\phi) \end{pmatrix}.$$

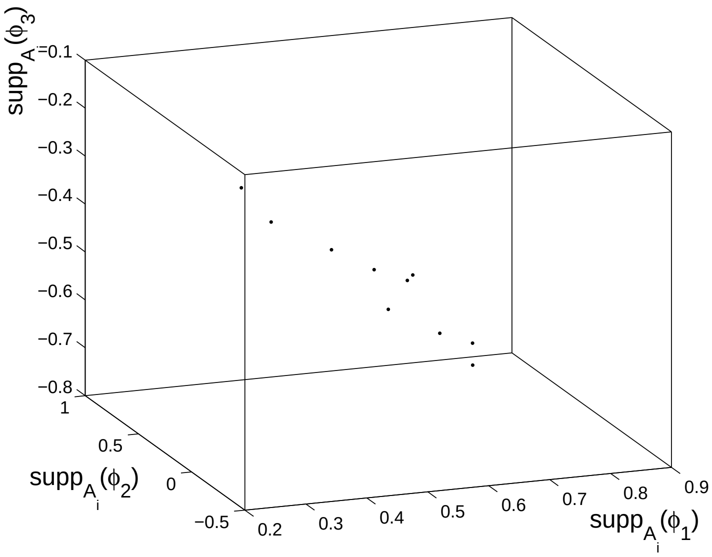

Figure 21: Visualization of sets

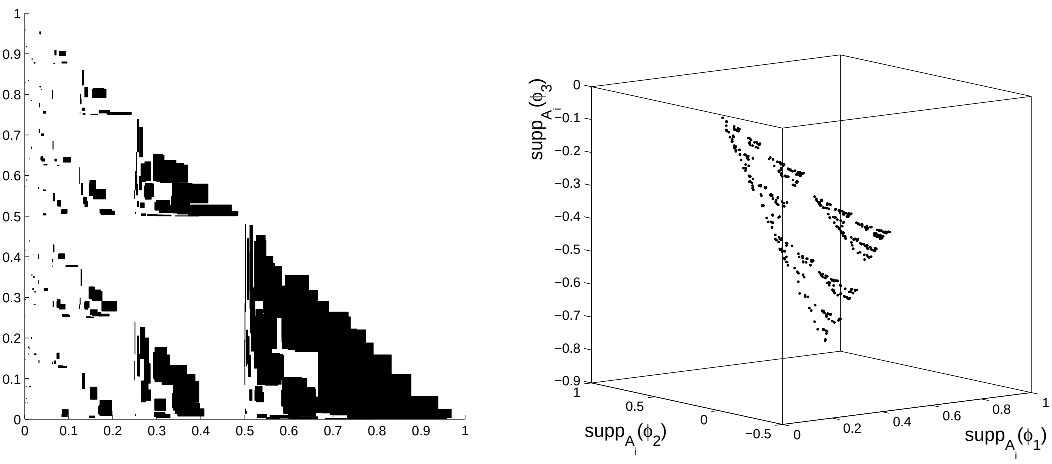

Figure 22: Visualization of sets of the fat Sierpiński triangle

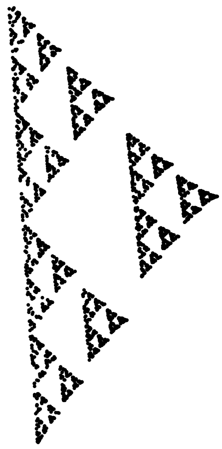

Figure 23: Structure of the hyperfractal associated to the fat Sierpiński triangle

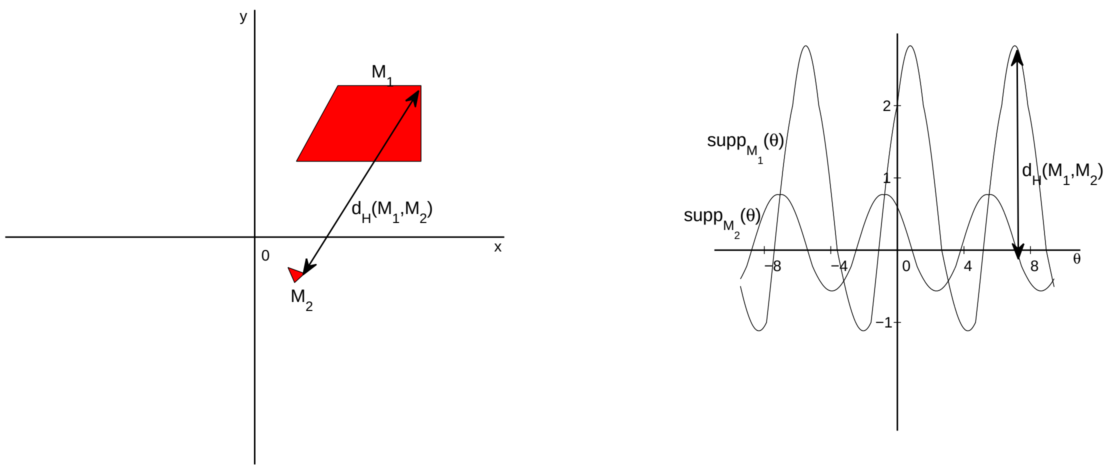

Figure 24: Support function and the Hausdorff distance

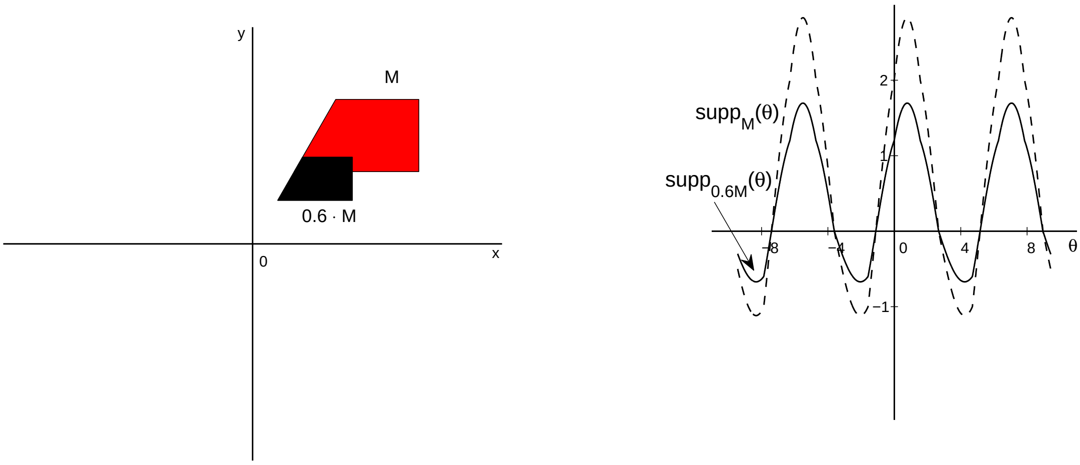

Figure 25: Support function and homothety

Thus,

$$\mathrm{supp}_{\mathscr{Q}M}(\theta) = \mathrm{supp}_M(\theta - \phi)$$

means only translation of support function.

In the same way, we obtain, for the matrix of reflection

$$\mathscr{Q} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

that

$$\mathrm{supp}_{\mathscr{Q}M}(\theta) = \mathrm{supp}_M(-\theta).$$

This means reflection of support function. Hence, similitudes in $K_{Co}(\mathbb{R}^2)$ have a natural explanation in the set of support functions $S(K_{Co}(\mathbb{R}^2))$.

Let us consider the hyperIFS

$$\{K_{Co}(\mathbb{R}^2),\ F_i,\ i = 1,\ 2,\ \ldots,\ n\}, \tag{22}$$

$$F_i : K_{Co}(\mathbb{R}^2) \to K_{Co}(\mathbb{R}^2), F_i(x) = r_i \mathscr{Q}_i A + C_i,\ i = 1,\ 2,\ \ldots,\ n,$$

$C_i \in K_{Co}(\mathbb{R}^2)$, r_i are reals, $\mathscr{Q}_i$ are orthonormal $2 \times 2-$matrices. Thus,

$$\mathscr{Q}_i = \mathscr{R}_i \cdot \mathscr{O}_i(\phi),$$

where $\mathscr{R}_i$ are matrices of reflection or identity and $\mathscr{O}_i(\phi)$, $\det(\mathscr{O}_i(\phi)) = 1$, matrices of rotation for $i = 1,\ 2,\ \ldots,\ n$. Let us consider the operators

$$T_i :\ S(K_{Co}(\mathbb{R}^2)) \to S(K_{Co}(\mathbb{R}^2)),$$

$$T_i(f)(\theta) = r_i \cdot f(\det(\mathscr{R}_i)(\theta - \phi)) + \mathrm{supp}_{C_i}(\theta).$$

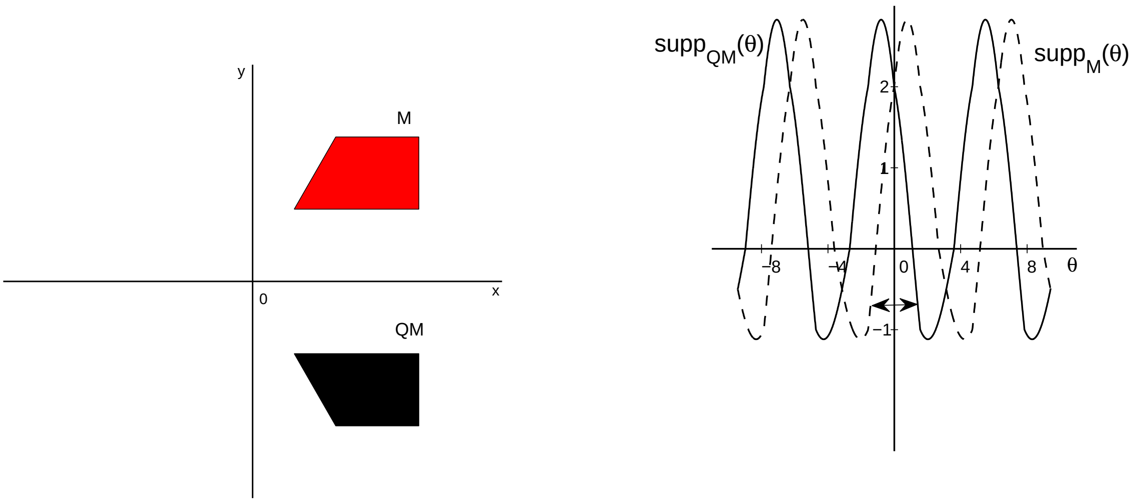

Figure 26: Support function of reflected set

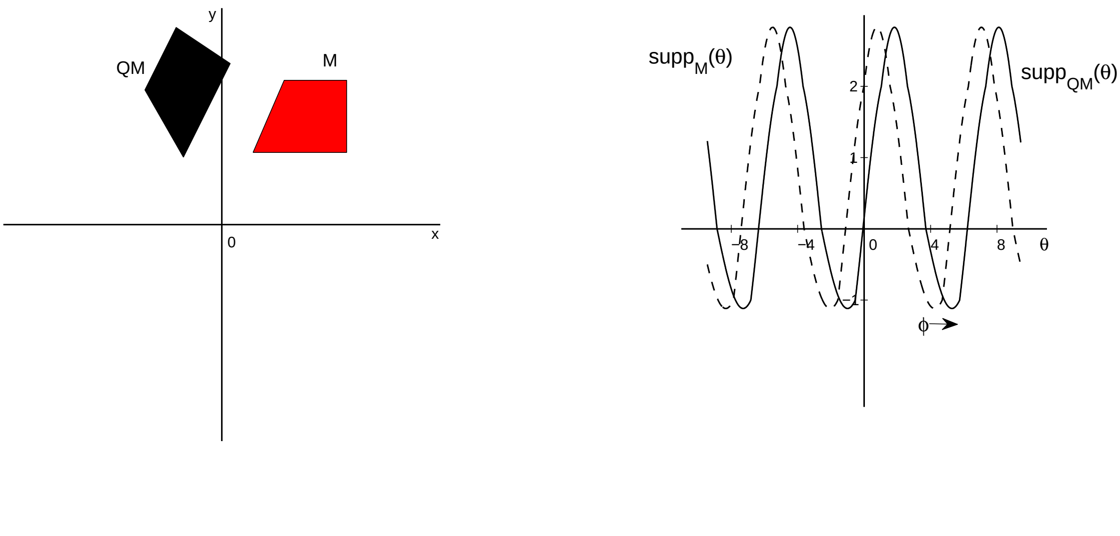

Figure 27: Support function of rotated set

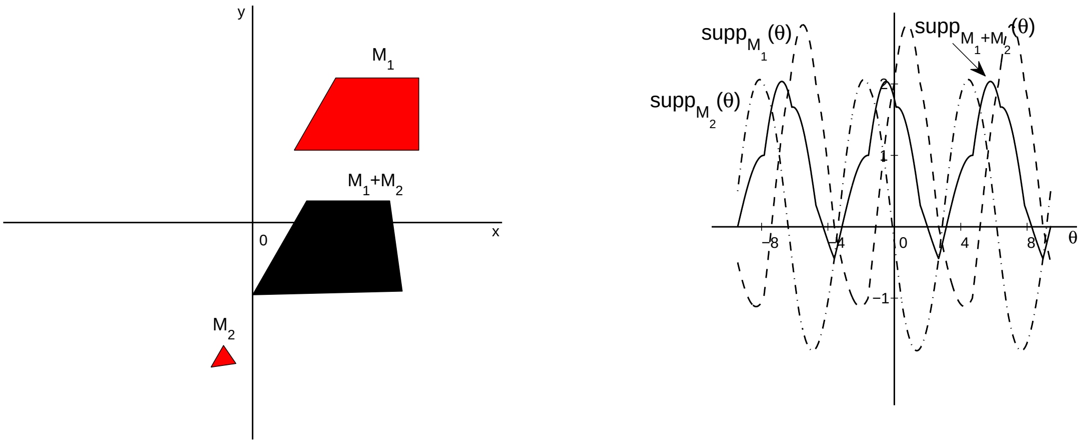

Figure 28: Support function and addition of sets

*THEOREM 36. *The IFS (22) and*

$$\{S(K_{Co}(\mathbb{R}^2)),\, d_{\mathrm{H}}),\, T_i,\, i = 1,\, 2,\, \ldots,\, n\}$$

are equivalent.

We will prove that our approach behaves well to hyperfractals with a known structure. Therefore, we will study the behaviour of the operator V to hyperfractals consisting of singletons. Let us consider the IFS $\{\mathbb{R}^n,\, f_i,\, i = 1,\, 2,\, \ldots,\, m\}$ and the IMS $\{\mathbb{R}^n,\, F_i,\, i = 1,\, 2,\, \ldots,\, m\}$, where $F_i(x) = \{f_i(x)\}$. Let us consider also an attractor α^* of an associated hyperIFS $\{(K(\mathbb{R}^n),\, F_i),\, i = 1,\, 2,\, \ldots,\, m\}$. Calculation of support functions of address sets (singletons) and visualization of α^* is easy. If $A = \{a\}$, $a \in \mathbb{R}^n$, then

$$\mathrm{supp}_A(x) = a'x.$$

We obtain

$$V(A) = \begin{pmatrix} \mathrm{supp}_A(x_1) \\ \mathrm{supp}_A(x_2) \\ \vdots \\ \mathrm{supp}_A(x_d) \end{pmatrix} = \begin{pmatrix} a'x_1 \\ a'x_2 \\ \vdots \\ a'x_d \end{pmatrix} = \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_d' \end{pmatrix} a =: \mathscr{V} a,$$

which is a linear mapping.

It is worth using n points x_i in our case. We obtain $V : K(\mathbb{R}^n) \to \mathbb{R}^n$. Since we consider only singletons, let us simplify our notation and write $V(a)$ instead of $V(\{a\})$. Thus, we consider $V : \mathbb{R}^n \to \mathbb{R}^n$ with $\mathscr{V}^{(n \times n)}$. If x_i, $i = 1,\, 2,\, \ldots,\, n$, are linearly independent, $\det(\mathscr{V}) \neq 0$ and it holds

$$|\lambda_i| > 0$$

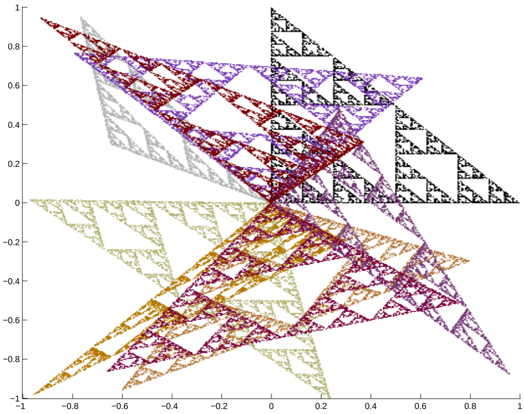

Figure 29: Possible visualizations of the Sierpiński triangle

for eigenvalues of $\mathscr{V}$.

We will show that $V : (\mathbb{R}^n, d_{\text{Eucl}}) \to (\mathbb{R}^n, d_{\max})$ is a bi-Lipschitz transformation. If we had

$$V : (\mathbb{R}^n, d_{\text{Eucl}}) \to (\mathbb{R}^n, d_{\text{Eucl}}),$$

transformation V would be bi-Lipschitz with constants $\lambda_{\min}$ and $\lambda_{\max}$,

$$|\lambda_{\min}| d_{\text{Eucl}}(a_1, a_2) \le d_{\text{Eucl}}(V(a_1), V(a_2)) \le |\lambda_{\max}| d_{\text{Eucl}}(a_1, a_2).$$

Metrics d_{Eucl} and $d_{\max}$ are equivalent, i.e.

$$d_{\max}(a_1, a_2) \le d_{\text{Eucl}}(a_1, a_2) \le \sqrt{n} d_{\max}(a_1, a_2) \,\forall a_1, a_2 \in \mathbb{R}^n.$$

We arrive to

$$\frac{|\lambda_{\min}|}{\sqrt{n}} d_{\text{Eucl}}(a_1, a_2) \le d_{\max}(V(a_1), V(a_2)) \le |\lambda_{\max}| d_{\text{Eucl}}(a_1, a_2).$$

Thus, V is a bi-Lipschitz transformation (see also Figure 29). Moreover, it follows from Proposition 9 that

$$\dim_H V(\alpha^*) = \dim_H(\alpha^*).$$

Remark 60. In this case, we probably choose

$$x_i = e_i = (\underbrace{0, 0, \ldots, 0}_{i-1}, 1, 0, \ldots, 0), \, i = 1, 2, \ldots, n.$$

However, for every $n-$tuple of linearly independent x_i, we obtain $\det \mathscr{V} \neq 0$.

Remark 61. This kind of visualization is suitable for fractals of lower dimension (for class of hyperfractals from the following theorem) not for hyperfractals from [AR1] (see Example 8).

6.3. Dimension and self-similarity of hyperfractals

It is not easy to calculate the Hausdorff dimension of hyperfractals, but we can succeed in the case of almost similitudes and fulfilling a separation condition. We will also discuss relationship between self-similar fractals and hyperfractals.

First, let us supply a theorem which is a stronger version of Theorem 7 in [AR2] and a few examples on the calculation of the Hausdorff dimension of hyperfractals.

*THEOREM 37. *Consider the hyperIFS* $\phi = \{(K(\mathbb{R}^m),\, d_H),\, F_i,\, i = 1,\, 2,\, \ldots,\, n\}$,

$$F_i : K(\mathbb{R}^m) \to K(\mathbb{R}^m), F_i(A) = r_i \mathscr{Q}_i A + C_i,\ i = 1,\, 2,\, \ldots,\, n,$$

$C_i \in K(\mathbb{R}^m)$, $r_i \in [0,\, 1)$ *are reals,* $\mathscr{Q}_i$ *are orthonormal* $m\times m-$*matrices. Consider also the hyperIFS* $\phi^c = \{(K_{Co}(\mathbb{R}^m)),\, d_H),\, F_i^c,\, i = 1,\, 2,\, \ldots,\, n\}$,

$$F_i^c : K_{Co}(\mathbb{R}^m) \to K_{Co}(\mathbb{R}^m), F_i^c(A) = r_i \mathscr{Q}_i A + \operatorname{conv}(C_i),\ i = 1,\, 2,\, \ldots,\, n.$$

Assume that the attractor α^c *of* ϕ^c *is totally disconnected. Then the Hausdorff dimension of the attractor* α^* *of* ϕ *can be calculated by means of the Moran formula and* $\dim_H(\alpha^*) = \dim_H(\alpha^c)$.

Proof. Let us consider the hyperIFS $\phi = \{(K_{Co}(\mathbb{R}^m),\, d_H),\, F_i,\, i = 1,\, 2,\, \ldots,\, n\}$ with attractor α^* and $\phi^c = \{(K_{Co}(\mathbb{R}^m),\, d_H),\, F_i^c,\, i = 1,\, 2,\, \ldots,\, n\}$, where $F_i^c(A) = r_i \mathscr{Q}_i A + \operatorname{conv}(C_i),\ A \in K_{Co}(\mathbb{R}^m)$ with a totally disconnected attractor α^c.

Firstly, ϕ^c is a hyperIFS consisting of similitudes and its attractor is totally disconnected. Therefore, its Hausdorff dimension D can be calculated by means of the Moran formula. We will prove that α^* has a similar metric structure.

We know from Theorem 35 that

$$\operatorname{conv}(\alpha^*_\sigma) = \alpha^c_\sigma,\ \forall \sigma \in \Sigma.$$

Since α^c is totally disconnected, there exists $d_H^{\min} > 0$ such that

$$d_H(F_i(A^c_\sigma),\, F_j(A^c_{\sigma'})) \geq d_H^{\min},\ i \neq j,\ \sigma,\, \sigma' \in \Sigma.$$

Lemma 15 implies

$$d_H(F_i(A^*_\sigma),\, F_j(A^*_{\sigma'})) \geq d_H(F_i(A^c_\sigma),\, F_j(A^c_{\sigma'})) \geq d_H^{\min}.$$

In order to calculate the Hausdorff dimension of α^*, we will find a bi-Lipschitz mapping of α^c onto α^*.

For any $j \in \mathbb{N}$, we can write:

$$A^c_{i_1 i_2 \ldots i_j \ldots} = F^c_{i_1 \ldots i_{j-1}}(A_{i_j i_{j+1} \ldots})$$

and

$$A^c_{i_1 i_2 \ldots i'_j \ldots} = F^c_{i_1 \ldots i_{j-1}}(A_{i'_j i'_{j+1} \ldots}).$$

Let us estimate the distance $d_H(A^c_{i_1i_2\dots i_j\dots},\ A^c_{i_1i_2\dots i'_j\dots})$. Since F_i are similitudes,

$$d_H(A^c_{i_1i_2\dots i_j\dots},\ A^c_{i_1i_2\dots i'_j\dots}) =$$

$$\begin{aligned}&= d_H(F^c_{i_1\dots i_{j-1}}(A^c_{i_ji_{j+1}\dots}),\ F^c_{i_1\dots i_{j-1}}(A^c_{i'_ji'_{j+1}\dots}))\\ &= r_{i_1i_2\dots i_{j-1}}\cdot d_H(A^c_{i_ji_{j+1}\dots},\ A^c_{i'_ji'_{j+1}\dots})\\ &\geq r_{i_1i_2\dots i_{j-1}}\cdot d_H^{\min}.\end{aligned}$$

Moreover, Lemma 15 and Lemma 35 imply

$$d_H(A^*_{i_ji_{j+1}\dots},\ A^*_{i'_ji'_{j+1}\dots}) \geq d_H(A^c_{i_ji_{j+1}\dots},\ A^c_{i'_ji'_{j+1}\dots}).$$

On the other hand, since F_i are contractions with factors r_i, $i = 1, 2, \dots, n$,

$$d_H(A^*_{i_1i_2\dots i_j\dots},\ A^*_{i_1i_2\dots i'_j\dots}) \leq r_{i_1i_2\dots i_{j-1}}\mathrm{diam}(\alpha^*).$$

Observe that $\alpha^* \in K(K(\mathbb{R}^m))$ implies $\mathrm{diam}(\alpha^*) < \infty$.

We obtain from these inequalities

$$d_H(A^c_{i_1i_2\dots i_{j-1}i_j\dots}, A^c_{i_1i_2\dots i_{j-1}i'_j\dots}) \leq$$

$$d_H(A^*_{i_1i_2\dots i_{j-1}i_j\dots}, A^*_{i_1i_2\dots i_{j-1}i'_j\dots}) \leq \frac{\mathrm{diam}(\alpha^*)}{d_H^{\min}} d_H(A^c_{i_1i_2\dots i_{j-1}i_j\dots}, A^c_{i_1i_2\dots i_{j-1}i'_j\dots}).$$

Applying Proposition 9, for $f : \alpha^c \to \alpha$, $f(A^c_{i_1i_2\dots i_{j-1}i_j\dots}) = A_{i_1i_2\dots i_{j-1}i_j\dots}$, the Hausdorff dimension of α^* is really D.

□

*Corollary 8. *If the assumptions of Theorem 37 are fulfilled by the hyperIFS $\phi = \{(K(\mathbb{R}^m), d_H), F_i, i = 1, 2, \dots, n\}$, then they are fulfilled by any other hyperIFS $\phi' = \{(K(\mathbb{R}^m), d_H), F'_i, i = 1, 2, \dots, n\}$, where $\mathscr{Q}_i = \mathscr{Q}'_i$, $r_i = r'_i$ and conv(C_i) =conv(C'_i).*

Proof. There exists one hyperIFS ϕ^c for both the hyperIFSs ϕ and ϕ'. □

Example 7. Let us consider the IMS $F = \{([0, 1]^2, d_{\mathrm{Eucl}}),\ F_i, i = 1, 2, \dots, 5\}$, where $F_i : [0, 1]^2 \to K([0, 1]^2)$,

$$F_i(x) = \{f_i(x)\},\ i = 1, 2, 3, 4,$$

$$f_1\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$f_2\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{2}{3} \end{pmatrix},$$

$$f_3\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix},$$

$$f_4\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix},$$

$$F_5\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \left[\frac{17}{48}, \frac{19}{48}\right] \\ \left[\frac{17}{48}, \frac{19}{48}\right] \end{pmatrix}.$$

Let us also consider a hyperIFS $\phi = \{(K_{Co}([0,\ 1]^2), d_H),\ F_i,\ i = 1,\ 2,\ \ldots,\ 5\}$ induced by the IMS F. Since its attractor α^* is a set of convex sets and it fulfills the assumptions of Theorem 37, we can calculate its Hausdorff dimension in $K_{Co}([0,\ 1]^2)$.

Let us consider the IMS $F' = \{([0,\ 1]^2,\ d),\ F'_i,\ i = 1,\ 2,\ \ldots,\ 5\}$, where $F'_i = F_i$, $i = 1,\ 2,\ 3,\ 4$,

$$F'_5\begin{pmatrix} x \\ y \end{pmatrix} := \left\{\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right\} + C'_5,$$

$$C'_5 = \left\{\left(\frac{17}{48}, \frac{17}{48}\right)', \left(\frac{17}{48}, \frac{19}{48}\right)', \left(\frac{19}{48}, \frac{17}{48}\right)', \left(\frac{19}{48}, \frac{17}{49}\right)'\right\}.$$

Let us denote by α' an attractor of the associated hyperIFS $\phi' = \{(K_{Co}([0,\ 1]^2), d_H),$ $F'_i,\ i = 1,\ 2,\ \ldots,\ 5\}$. Since the assumptions of Theorem 37 are fulfilled by the hyperIFS ϕ and $\mathrm{conv}(C'_i) = C_i$, the Hausdorff dimension of attractors α^* and α' is the same. The support functions of sets with the same addresses are the same, too. It follows from Theorem 35 that visualizations of these hyperfractals are the same.

REMARK 62. The hyperIFS ϕ' is equal to a superIFS $\{(K([0,\ 1]^2), d_H),\ G_i,\ i = 1,\ 2,\ 3,\ 4,\ 5\}$, where

$$G_i(A) = \bigcup_j \bigcup_{x \in A} \{g^i_j(x)\},$$

$$g^i_1 = f_i,\ i = 1,\ 2,\ 3,\ 4,$$

$$g^5_i\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + a_i.$$

$$a_1 = \left(\frac{17}{48}, \frac{17}{48}\right)',\ a_2 = \left(\frac{17}{48}, \frac{19}{48}\right)',\ a_3 = \left(\frac{19}{48}, \frac{17}{48}\right)',\ a_4 = \left(\frac{19}{48}, \frac{19}{48}\right)'.$$

We call α' a hyperfractal here because we are interested in the dimension of $\alpha^* \subset K(\mathbb{R}^2)$, not in the dimension of $A^*_\sigma \subset \mathbb{R}^2$ as Barnsley (see [Ba2]).

EXAMPLE 8. Stimulated from quantitative linguistics, we gave another example of a fractal on the boundary between superfractals and hyperfractals in [AR1]. We called it Cantor-like hyperset. It consists of Cantor sets and it is self-similar in $K([0,\, 1])$. Denoting

$$C^m := \{x \in [0,\, 1] |\, x = \frac{l}{m},\, l = 0,\, 1,\, \ldots,\, m-1,\, m \in \mathbb{N},\, m \geq 2\},$$

let us consider the system $\{K([0,\, 1]),\, F_j = 1,\, 2,\, \ldots,\, 2^m - 1\}$ of contractions $F_j : K([0,\, 1]) \to K([0,\, 1])$,

$$F_j(B) := r \cdot B + C_j,\ j = 1,\, 2,\, \ldots,\, 2^m - 1, \tag{23}$$

where $r < \frac{1}{m}$, $C_j \subset C^m$, $C_j \neq \emptyset$, $C_i \neq C_j$, $i \neq j$. Observe that although the hyperspace $K([0,\, 1], d_H)$ has not a linear structure, the sum as well as the product in (23) is well (point-wise) induced from $[0,\, 1]$.

We will prove that, for each $j = 1,\, 2,\, \ldots,\, 2^m - 1$, the contractions F_j are similitudes. Let us denote

$$j_{\min} := \min(C_j),\ j_{\max} := \max(C_j)$$

and

$$D_j := \{i \in \mathbb{N} |\, \frac{i}{m} \in C_j\},\ j = 1,\, 2,\, \ldots,\, 2^m - 1.$$

Let us, furthermore, denote

$$I_i := \left[\frac{i-1}{m}, \frac{i-1}{m} + r\right],\ i = 1,\, 2,\, \ldots,\, m,$$

and, for an arbitrary $A \in K([0,\, 1])$, ${}_jA_k$ such that

$${}_jA_k := F_j(A) \cap I_k,\ k = 1,\, 2,\, \ldots,\, m.$$

We can see that

$$F_j(A) = \bigcup_{k=1}^{m} {}_jA_k \tag{24}$$

and

$$d_H({}_jA_k,\, {}_jB_k) = r \cdot d_H(A,\, B),\ \text{for all } k \in D_j. \tag{25}$$

We must guarantee that, for all $A,\, B \in K([0,\, 1])$, we have

$$d_H(F_j(A),\, F_j(B)) = r \cdot d_H(A,\, B),\ j = 1,\, 2,\, \ldots,\, 2^m - 1. \tag{26}$$

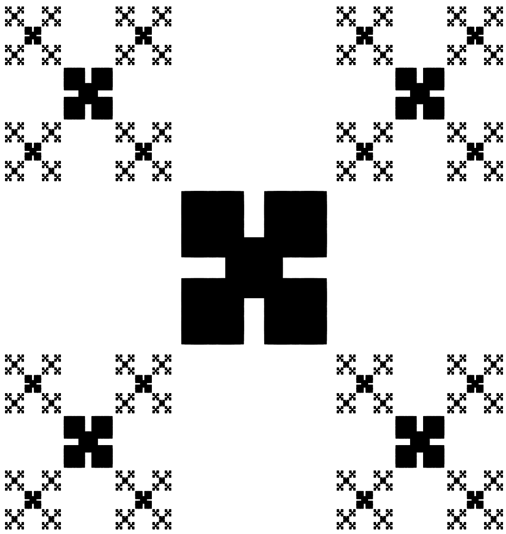

Figure 30: Attractor of underlying IMS for hyperfractal

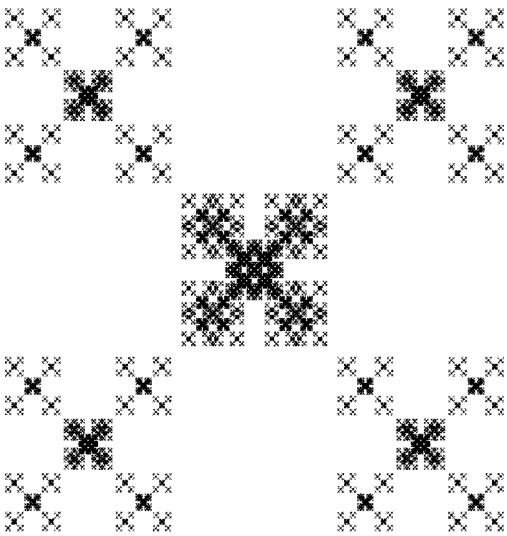

Figure 31: Attractor of underlying IMS for hyperfractal

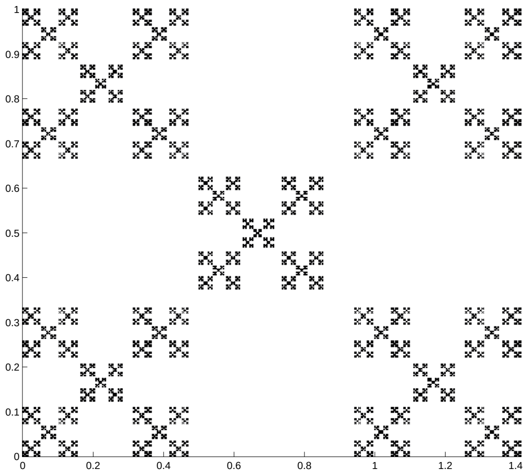

Figure 32: Structure of the hyperfractal in a hyperspace

If the set C_j is a singleton, the proof is trivial. Otherwise, we will prove two reverse inequalities implying equality (26). It follows from (24), (25) and Lemma 1 that

$$d_H(F_j(A), F_j(B)) \leq r \cdot d_H(A, B), \text{ for all } A,\, B \in K([0,\, 1]). \tag{27}$$

We will also prove

$$d_H(F_j(A), F_j(B)) \geq r \cdot d_H(A,\, B), \text{ for all } A,\, B \in K([0,\, 1]). \tag{28}$$

Since the sets A, B are compact, we can find ${}_j\overline{x} \in A$ such that

$$d({}_j\overline{x},\, B) = d_H(A,\, B); \tag{29}$$

otherwise, we can interchange the sets. Denoting ${}_j\overline{x}_k := F_j({}_j\overline{x}) \cap A_k$, it satisfies

$$d({}_j\overline{x}_k,\, {}_jB_k) = d_H({}_jA_k,\, {}_jB_k), \text{ for all } k \in D_j.$$

In order (28) to be satisfied, there must exist at least one $l \in \mathbb{N}$ such that

$$d({}_j\overline{x}_l,\, {}_jB_l) \leq d({}_j\overline{x}_l,\, {}_jB_i), \text{ for every } i \in D_j.$$

Let us show that

$$l = j_{\min} \text{ or } l = \ j_{\max}.$$

If it is not so, ${}_j\overline{x}_{j_{\min}}$ must be located behind the centre of $I_{j_{\min}}$ (to be closer to $B_{j_{\min}+1}$ than $B_{j_{\min}}$) and ${}_j\overline{x}_{j_{\max}}$ must be located at the same time in front of the centre of $I_{j_{\max}}$ which is impossible. Thus,

$$d_H(F_j(A),\, F_j(B)) \geq d({}_j\overline{x}_l,\, F_j(B)) = d({}_j\overline{x}_l,\, {}_jB_l) = d_H({}_jA_l,\, {}_jB_l),$$

where (cf. (29))

$$d_H({}_jA_l,\ {}_jB_l) = r \cdot d_H(A,\ B),$$

i.e. F_j are similitudes, for every $j = 1,\ 2,\ \ldots,\ 2^m - 1$, as claimed.

Since

$$F_i(K([0,\ 1])) \cap F_j(K([0,\ 1])) = \emptyset,\ \text{for all } i \neq j;\ i, j = 1,\ 2,\ \ldots,\ 2^m - 1,$$

it follows from Lemma 8 that the dimension of the associated hyperattractor (obtained as a unique fixed point in $K(K([0,\ 1])))$ can be calculated by means of the Moran-Hutchinson formula as

$$D = \frac{\log\,(2^m - 1)}{\log \frac{1}{r}},\ \text{where } \frac{1}{r} > m. \tag{30}$$

Let us return our attention to the motivation examples from Introduction. We will show that their visual self-similarity follows from the fact that all of them are shadows of self-similar fractals.

Let us consider the picture with embedded squares. Let $F = \{[-0.5,\ 0.5]^2,\ F_1,\ F_2\}$, be an IMS such that

$$\begin{aligned} &F_1(x) = \Box, \\ &F_2(x) = \{r\mathscr{Q}(x)\}\,, \end{aligned}$$

where $\Box = \partial([-0.5,\ 0.5]^2)$, $r = 0.75$,

$$\mathscr{Q} = \begin{pmatrix} \cos(\frac{\pi}{12}) & \sin(\frac{\pi}{12}) \\ -\sin(\frac{\pi}{12}) & \cos(\frac{\pi}{12}) \end{pmatrix}.$$

Instead of the IMS F, we can study the associated hyperIFS $\phi = \{K([-0.5,\ 0.5]^2),\ F_1,\ F_2\}$. Mappings F_1 and F_2 are similitudes and they satisfy the strong open set condition. The embedded squares are a shadow of the self-similar attractor α^* of ϕ (cf. Figure 33). Applying our method of visualization, we get a projection of a structure of the hyperfractal α^* in the hyperspace in Figure 34. We obtain the Hausdorff dimension of the hyperfractal α^* by means of the Moran formula,

$$\dim_H(\alpha^*) = 0.$$

Let us look at the picture of a tree similarly. It suffices to define a hyperIFS $\phi = \{(K_{Co}([0,\ 1]^2),\ d_H),\ F_1,\ F_2,\ F_3\}$, containing three mappings. The hyperIFS is induced by the IMS $F = \{[0,\ 1]^2,\ F_1,\ F_2,\ F_3\}$,

$$F_1\begin{pmatrix} x \\ y \end{pmatrix} = \left\{ \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{2} - \frac{\sqrt{2}r}{2} \\ \frac{1}{2} - \frac{\sqrt{2}r}{2} \end{pmatrix} \right\},$$

$$F_2\begin{pmatrix} x \\ y \end{pmatrix} = \left\{ \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{2} - \frac{\sqrt{2}r}{2} \\ \frac{1}{2} + \frac{\sqrt{2}r}{2} \end{pmatrix} \right\},$$

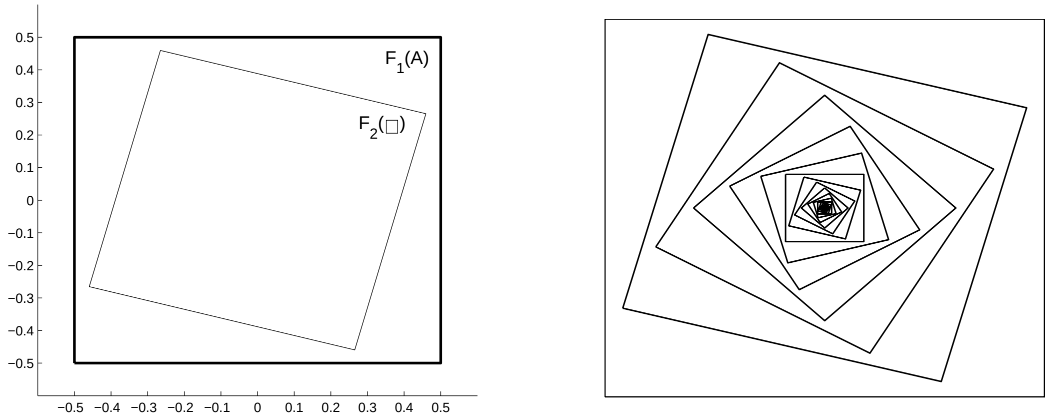

Figure 33: Embedded squares

$$F_3\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \{0.5\} \\ [0,\, 0.5] \end{pmatrix},$$

where $r = 0.45$. All the induced maps F_1, F_2, F_3 are similitudes and the SOSC is satisfied. The tree is again a shadow of a self-similar hyperfractal. For the projection of the structure of the hyperfractal, see Figure 36. We obtain the non-trivial Hausdorff dimension D of the hyperfractal from the Moran formula,

$$D = \frac{\log 2}{\log \frac{1}{0.45}}.$$

The Sierpiński triangle is a shadow of a hyperfractal, too. Instead of the IFS $F = \{I^2,\, f_1,\, f_2,\, f_3\}$, it suffices to consider the hyperIFS $\phi = \{K(I^2),\, F_1,\, F_2,\, F_3\}$, where $F_i(A) = \cup_{x\in A}\{f_i(x)\}$. The shadow of the hyperfractal is the Sierpiński triangle.

In the same way, we can regard all the attractors of classical IFSs as shadows of hyperfractals.

*Theorem 38. *Self-similar subsets of $\mathbb{R}^m$ form a subset of shadows of self-similar hyperfractals.*

Proof. Let us consider the IFS $F = \{\mathbb{R}^m,\, f_1,\, f_2,\, \dots,\, f_n\}$, the IMS $F = \{\mathbb{R}^m,\, F_1,\, F_2,\, \dots,\, F_n\}$ and an associated hyperIFS $\phi = \{(K(\mathbb{R}^m),\, d_H),\, f_1,\, f_2,\, \dots,\, f_n\}$, where $F_i(x) = \{f_i(x)\}$. Let us denote by A^* and α^* the attractors of F and ϕ, respectively. Note that

$$F(A) = \bigcup_{i=1}^{n}\bigcup_{x\in A}\{f_i(A)\}$$

is the same operator for the IFS and IMS. It implies that A^* is the attractor of the IFS and IMS. Moreover, it is a shadow of α^*.

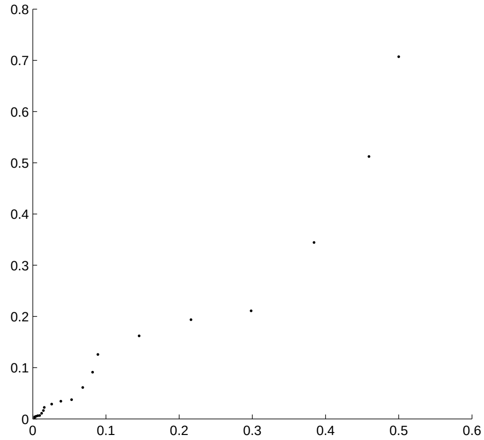

Figure 34: Structure in a hyperspace

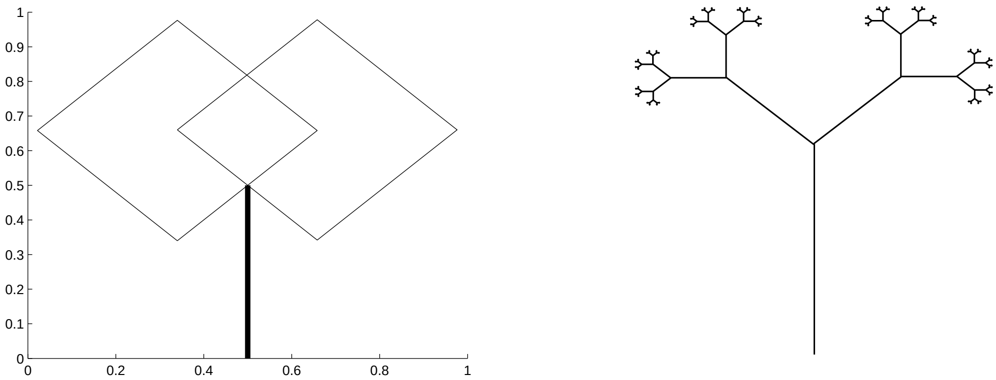

Figure 35: Tree

Figure 36: Structure in a hyperspace

Let us show that if A^* is self-similar then α^* is self-similar, too. Suppose that A^* is self-similar. Let us remind that the OSC and SOSC are equivalent in $\mathbb{R}^m$. Thus, self-similarity of A^* implies that there exists open $V \subset \mathbb{R}^m$ such that

1. $f_i(V) \subset V$,
2. $f_i(V) \cap f_j(V) = \emptyset$, $1 \le i,\, j \le n$, $i \ne j$,
3. $V \cap A^* \ne \emptyset$.

We will prove that ϕ fulfills SOSC, too. Let us consider the set of sets $K(V)$. We need to prove

1. $K(V)$ is open,
2. $F_i(K(V)) \subset K(V)$,
3. $F_i(K(V)) \cap F_j(K(V)) = \emptyset$, $1 \le i,\, j \le n$, $i \ne j$,
4. $K(V) \cap \alpha^* \ne \emptyset$.

1. Consider $A \in K(V)$. Since V is open, we have

   $$\forall x \in A\, \exists \epsilon > 0 :\; O(x,\, \epsilon) \subset V.$$

   Compactness of A implies

   $$\exists \epsilon_{\min} = \min_{x \in A} \{\sup_{\epsilon} \{\epsilon,\, O(x,\, \epsilon) \subset V\}\}.$$

   It follows that

   $$\{A' \in K(\mathbb{R}^m) :\; d_H(A',\, A) < \epsilon_{\min}\} \subset K(V).$$

   Hence, $K(V)$ is open.

2. Note that $F_i(A) = f_i(A)$, $\forall A \in K(\mathbb{R}^m)$. Consider $B \in F_i(K(V))$. It follows that

   $$\exists C \in K(V) :\; F_i(C) = B$$

   and

   $$B \subset F_i(V).$$

   Since $F_i(V) \subset V$, we have $B \in K(V)$ and $F_i(K(V)) \subset K(V)$.

3. Assume that $B \in F_i(K(V))$ and $B' \in F_j(K(V))$, $1 \le i,\, j \ge n$, $i \ne j$. We have shown that $B \subset F_i(V)$ and $B' \subset F_j(V)$. From

   $$F_i(V) \cap F_j(V) = \emptyset,\; 1 \le i,\, j \le n,\; i \ne j,$$

   we have $B \cap B' = \emptyset$ and also $F_i(K(V)) \cap F_j(K(V)) = \emptyset$.

4. Denote $\sigma \in \Sigma$, where $a^*_\sigma \in V \cap A^*$. Observe that $\{a^*_\sigma\} \in \alpha^*$ follows from

$$F_\sigma(\{x\}) = \{f_\sigma(x)\}.$$

We have also $\{a^*_\sigma\} \in K(V)$.

We proved that the attractor of the hyperIFS ϕ is also self-similar. □

Moreover, the metric structure (and also the Hausdorff dimension) of such hyperattractors is naturally the same as the structure of attractors of original IFSs. This means that the space of fractals generated by IFS is isometrically embedded in the space of hyperfractals.

7. Measure on multivalued fractals

We will use ergodic theorems to construct a measure on multivalued fractals. Particularly, we will find a shadow of an invariant measure on a hyperfractal, which will be supported on an underlying multivalued fractal.

Let us denote by

$$F = \{\mathbb{R}^n,\, F_i : \mathbb{R}^n \to K(\mathbb{R}^n),\, p_i > 0,\, i = 1, 2, \ldots, m,\, \sum_i p_i = 1\}$$

an IMS with probabilities. It is an underlying IMS for the hyperIFS with probabilities

$$\phi = \{K(\mathbb{R}^n),\, F_i : K(\mathbb{R}^n) \to K(\mathbb{R}^n),\, p_i > 0,\, i = 1, 2, \ldots, m\}.$$

We will treat a special case

$$\phi = \{K_{Co}(\mathbb{R}^n),\, F_i : K_{Co}(\mathbb{R}^n) \to K_{Co}(\mathbb{R}^n)\},$$

where $F_i(A) = r_i \mathcal{Q}_i(A) + C_i$, $A \in K_{Co}(\mathbb{R}^n)$. We can easily construct the attractor α^* and the invariant measure μ for the hyperIFS ϕ. We have

$$A^* = \bigcup_{\sigma \in \Sigma} A^*_\sigma$$

for the attractor A^* (a shadow of α^*) of the underlying IMS. In a similar way, we would like to find an underlying measure (i.e. a shadow of a measure μ) μ_S for μ. Hence, its support will be A^*.

We will calculate and visualize how often a Borel subset of the embedding space of A^* is visited during the chaos game for ϕ. Therefore, we need to evaluate how significant part of sets from an orbit of the chaos game visited the Borel set. We use a characteristic function $\chi_B(x)$ to distinguish whether a point $x \in \mathbb{R}^n$

belongs to a set $B \subset \mathbb{R}^n$. In order to evaluate how significant part of a set $A \subset \mathbb{R}^n$ is contained in $B \subset \mathbb{R}^n$, we apply the formula

$$\chi_B(A) := \frac{\mathrm{vol}_A(B \cap A)}{\mathrm{vol}_A(A)},\ \forall A,\ B \in K_{Co}(\mathbb{R}^n).$$

In Theorem 39, we will consider a hyperfractal $\alpha^* \subset K_{Co}(\mathbb{R}^n)$ such that $\dim_H(A^*_\sigma) = n$ for almost all $\sigma \in \Sigma$ (w.r.t an invariant measure). The following lemma concerns the kind of hyperIFS which we often treat.

*Lemma 17. *Let a hyperIFS* $\phi = \{K_{Co}(\mathbb{R}^n),\ F_i,\ p_i > 0\}$, $F_i(A) = r_i \mathscr{Q}_i A + C_i$ *be such that* $r_i > 0$, $\det(\mathscr{Q}_i) \neq 0$. *Let* $A^*_\omega \in \alpha^*$, $\omega \in \Sigma$ *be such that* $\dim A^*_\omega = n$. *Then* $\dim A^*_\sigma = n$ *for almost all* $\sigma \in \Sigma$.

Proof. It is obvious, for $r_i > 0$, $\det(\mathscr{Q}_i) \neq 0$, that

$$\dim F_i(A) \geq \dim A.$$

It follows

$$\dim A = n \Rightarrow F_{\mathbf{i}}(A) = n,\ \forall A \in K_{Co}(\mathbb{R}^n),\ \mathbf{i} = i_1 i_2 \ldots i_p.$$

Let us consider a singleton $\{x\}$, $x \in \mathbb{R}^n$, and $\omega \in \Sigma$. It follows from Theorem 3 that $F_\omega(\{x\}) = A^*_\omega$ and we assume that $\dim(A^*_\omega) = n$. Let us show that there exists an integer k such that $\dim(F_{\omega|k}(\{x\})) = n$. If it did not, we would have a sequence of sets $\{F_{\omega|k}(\{x\})\}_{k=0}^{\infty}$ with

$$\dim(A^*_{\omega|k}) < n,$$

which converge to a set A^*_ω of dimension n. Note that the integer k (if we take the least one) does not depend on x, since

$$F_{\omega|k}(\{x\}) = r_{\omega|k} \mathscr{Q}_{\omega|k}\{x\} + D,$$

where $D \in K_{Co}(\mathbb{R}^n)$. Thus, we have found an integer k and a mapping $F_{\omega|k} : K_{Co}(\mathbb{R}^n) \to K_{Co}(\mathbb{R}^n)$ which maps every element of $K_{Co}(\mathbb{R}^n)$ to a compact convex set of dimension n.

In order to keep our notation simple, let $\mathbf{i} = \sigma|k$ and $p = p_{i_1} p_{i_2} \cdots p_{i_k}$. Note that $p = \mu(\alpha^*_{\mathbf{i}})$. We will calculate $\mu(\cup_{j \in \Sigma'} \alpha^*_{\mathbf{ji}})$, where $|\mathbf{j}| < \infty$, a measure of a subset of sets α^* of dimension n. Let us denote

$$\alpha^{*l} = \bigcup \alpha^*_{\mathbf{j}_1 \mathbf{j}_2 \ldots \mathbf{j}_{l-1} \mathbf{i}},$$

where $|\mathbf{j}_q| = k$, $\mathbf{j}_q \neq \mathbf{i}$, $q \in \{1,\ 2,\ \ldots,\ l-1\}$. It holds

$$\mu(\alpha^{*l}) = (1-p)^{l-1} p.$$

Note that α^{*l_1}, α^{*l_2} are disjoint for $l_1 \neq l_2$. Finally,

$$\mu(\cup_{l \in \mathbb{N}} \alpha^{*l}) = \sum_{l \in \mathbb{N}} \mu(\alpha^{*l}) = \sum_{l \in \mathbb{N}} (1-p)^l p = 1.$$

$\square$

Remark 63. Let $F = \{(K_{Co}(\mathbb{R}^n), d_H), f_i, p_i, i = 1, 2, \ldots, m\}$ be a hyperIFS with an attractor α^*. Let us remind here a corresponding IFS on a code space $S = \{(\Sigma, d_\Sigma), s_i, p_i, i = 1, 2, \ldots, m\}$, where $s_i(\sigma) = i\sigma$, with an attractor Σ and invariant measure ρ. Note that $\mu(\beta) = \rho(\{\sigma \in \Sigma, A^*_\sigma \in \beta\})$, where $\beta \in \mathbb{B}(K_{Co}(\mathbb{R}^n))$.

Lemma 18. [Gr, Theorem 7.5] *Volume of a compact convex set in $\mathbb{R}^n$ is continuous with respect to the Hausdorff distance.*

Remark 64. Let us consider $A, B, A_1, A_2, \dots \in K_{Co}(\mathbb{R}^n)$, $A \cap B \neq \emptyset$, $A_i \cap B \neq \emptyset$, such that

$$\lim_{i\to\infty} d_H(A_i, A) = 0.$$

Observe that

$$\lim_{i\to\infty} d_H(A_i \cap B, A \cap B) = 0$$

and also

$$\lim_{i\to\infty} |\mathcal{L}^n(A_i \cap B) - \mathcal{L}^n(A \cap B)| = 0.$$

Hence, $\mathcal{L}^n(A \cap B)$ is also continuous in A on the set $\{A \in K_{Co}(\mathbb{R}^n), \mathcal{L}^n(A \cap B) \neq \emptyset\}$ w.r.t. the Hausdorff metric.

In the next step, we will define a set function

$$f(B) := \int_{\sigma\in\Sigma} \frac{\mathcal{L}^{d(\sigma)}(B \cap A^*_\sigma)}{\mathcal{L}^{d(\sigma)}(A^*_\sigma)} d\rho(\sigma),$$

for blocks $B \in K_{Co}(\mathbb{R}^n)$, $B = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$, where $d(\sigma) = \dim_H(A^*_\sigma)$.

Since $\mu(\{A^*_\sigma \in \alpha^*, \dim_H(A^*_\sigma) = n\}) = 1$, we have $\rho(\{\sigma \in \Sigma, \dim_H(A^*_\sigma) = n\}) = 1$. Thus, it suffices to consider n instead of $d(\sigma)$.

In order to apply the ergodic theorem to the shift dynamical system, we will prove integrability of $\frac{\mathcal{L}^n(B\cap A^*_\sigma)}{\mathcal{L}^n(A^*_\sigma)}$. Let us denote

$$\Sigma_B = \{\sigma \in \Sigma,\ \mathcal{L}^n(A^*_\sigma \cap B) > 0\}.$$

Lemma 18 and Remark 24 imply that $\mathcal{L}^n(A^*_\sigma)$ and $\mathcal{L}^n(A^*_\sigma \cap B)$ are continuous w.r.t. the Hausdorff metric. Since the address function

$$\phi(\sigma) = A^*_\sigma$$

is continuous w.r.t. the metrics d_Σ and d_H, $\mathcal{L}^n(A^*_\sigma)$ and $\mathcal{L}^n(A^*_\sigma \cap B)$ are continuous w.r.t. d_Σ. Moreover, Σ_B is open, which follows from continuity of $\mathcal{L}^n(A^*_\sigma \cap B)$. Hence, we integrate a continuous function on an open set,

$$\int_{\sigma\in\Sigma} \frac{\mathcal{L}^n(A^*_\sigma \cap B)}{\mathcal{L}^n(A^*_\sigma)} d\rho(\sigma) = \int_{\sigma\in\Sigma_B} \frac{\mathcal{L}^n(A^*_\sigma \cap B)}{\mathcal{L}^n(A^*_\sigma)} d\rho(\sigma).$$

Note that
$$\frac{\mathcal{L}^n(A^*_\sigma \cap B)}{\mathcal{L}^n(A^*_\sigma)}$$
is continuous (and positive) on Σ_B since $\mathcal{L}^n(A^*_\sigma) > 0$ for all $\sigma \in \Sigma_B$.

This integral exists and can be approximated by means of the ergodic theorem (Theorem 29). The dynamical system is $(\Sigma,\, \mathbb{B}(\Sigma),\, T,\, \rho)$, where T is the shift operator (ergodic, ρ preserving) on (totally disconnected) Σ.

It remains to define an outer measure for all Borel subsets and show that it is a Borel measure. Let us define, for all $B \subset \mathbb{R}^n$,

$$\mu_S(B) := \inf\{\sum_i f(B_i),\ B \in \cup B_i,\ B_i \text{ is a block}\}. \tag{31}$$

It is easy to prove that

1. $\mu_S(\emptyset) = 0$,
2. $\mu_S(A) \le \mu_S(B)$ for $A \subset B$,
3. $\mu_S(\cup_i B_i) \le \sum_i \mu_S(B_i)$.

In order to prove that μ_S is a Borel measure, we use the Carathéodory criterion. We will show that μ_S is a metric measure, that is

$$\mu_S(A \cup B) = \mu_S(A) + \mu_S(B),\ \forall A,\, B \in \mathbb{B}(\mathbb{R}^n),$$

such that $\operatorname{dist}(A,\, B) > 0$. First, let us note that, for all $\delta > 0$, block B can be divided into finite number of subblocks $\{B_i\}$ with diameter less than δ. Next, we will show that
$$f(B) = \sum_i f(B_i).$$
Since $\mathcal{L}^n$ is a Borel measure, it holds

$$\frac{\mathcal{L}^n(A^*_\sigma \cap B)}{\mathcal{L}^n(A^*_\sigma)} = \frac{\mathcal{L}^n(\cup_i B_i \cap A^*_\sigma)}{\mathcal{L}^n(A^*_\sigma)} = \sum_i \frac{\mathcal{L}^n(B_i \cap A^*_\sigma)}{\mathcal{L}^n(A^*_\sigma)},$$

for all $\sigma \in \Sigma$. Hence,

$$f(B) = \int_\Sigma \frac{\mathcal{L}^n(A^*_\sigma \cap B)}{\mathcal{L}^n(A^*_\sigma)} d\rho = \int_\Sigma \sum_i \frac{\mathcal{L}^n(B_i \cap A^*_\sigma)}{\mathcal{L}^n(A^*_\sigma)} d\rho =$$

$$= \sum_i \int_\Sigma \frac{\mathcal{L}^n(B_i \cap A^*_\sigma)}{\mathcal{L}^n(A^*_\sigma)} d\rho = \sum_i f(B_i).$$

Finally, from the previous, it follows that a decomposition of blocks B_i into subblocks leaves the sum in (31) unaltered. It suffices to consider only $\delta-$coverings in (31). Since the sets A, B are $\delta-$separated, no set from $\delta-$covering of $A \cup B$ can intersect both A and B. Hence,

$$\mu_S(A \cup B) = \mu_S(A) + \mu_S(B).$$

We arrive to the following theorem.

*THEOREM 39. *Let us consider the hyperIFS $\phi = \{K_{Co}(\mathbb{R}^n),\, F_i,\, p_i > 0\}$ such that $\dim \alpha^*_\sigma = n$, for almost all $\sigma \in \Sigma$. The set function μ_S defined in (31) is a Borel measure.*

REMARK 65. Notice that there exist more IFSs with the same attractor [Hu, 4.1 Motivation, p. 16]. If we know an invariant measure, we can distinguish the related IFS. In the same way, a shadow of a measure can help us to find the related IMS for a multivalued fractal.

EXAMPLE 9. Let us consider the hyperIFS $\phi = \{(K_{Co}([0,\, 1]), d_H),\, F_i,\, p_i,\, i = 1,\, 2,\, 3\}$ induced by an IMS $\{([0,\, 1], d_H),\, F_i,\, i = 1,\, 2,\, 3\}$, where $F_i : [0,\, 1] \to K_{Co}([0,\, 1])$, $i = 1,\, 2,\, 3$,

$$\begin{aligned} F_1(x) &= \{rx\}, \\ F_2(x) &= \{rx + 1 - r\}, \\ F_3(x) &= [rx,\, rx + 1 - r] = \{rx\} + [0, 1 - r]\}, \end{aligned}$$

for $r = \frac{1}{2.1}$ and $p_1 = 0.43$, $p_2 = 0.43$, $p_3 = 0.14$. We find a measure (Figure 41) on the multivalued fractal (Figure 37) constructing the associated hyperfractal and the invariant measure on it (Figure 38).

Figure 37: Multivalued fractal from Example 9

Let us also visualize a shadow of a measure for two IMS we have met.

EXAMPLE 10. We will consider the hyperIFS from Example 7 $\phi = \{K_{Co}([0,\, 1]^2),\, F_i,\, i = 1,\, 2,\, \ldots,\, 5\}$ with probabilities $p_1 = 0.21$, $p_2 = 0.19$, $p_3 = 0.19$, $p_4 = 0.21$, $p_5 = 0.2$.

EXAMPLE 11. Let us also image a shadow of a measure for the hyperIFS from Example 6 $\phi = \{K_{Co}([0,\, 1]^2),\, F_i,\, i = 1,\, 2,\, 3\}$ with probabilities $p_1 = 0.3$, $p_2 = 0.4$, $p_3 = 0.3$.

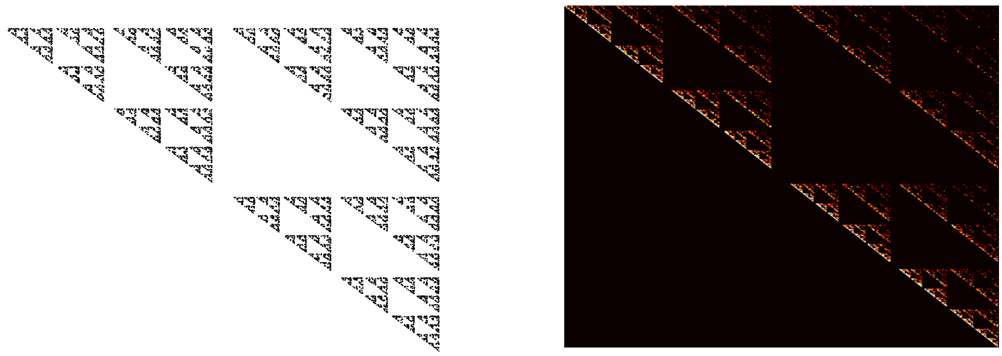

Figure 38: Hyperfractal and invariant measure from Example 9

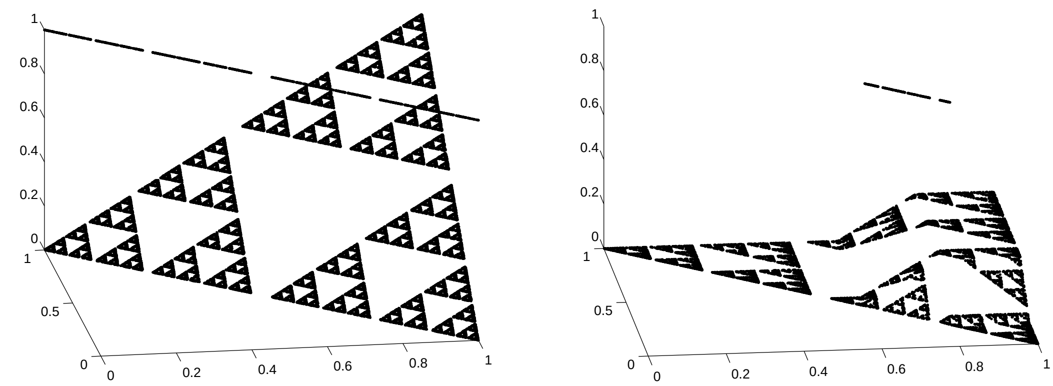

Figure 39: Graphs of functions $\mathcal{L}^{d(\sigma)}(A_\sigma^*)$ and $\mathcal{L}^{d(\sigma)}(A_\sigma^* \cap B)$ for $B = [0.2, 0.4]$

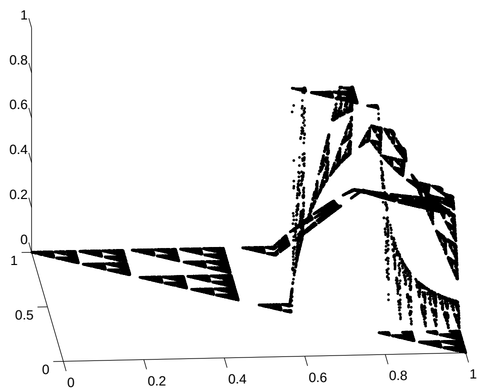

Figure 40: Function $\chi_B(A_\sigma^*) = \frac{\mathcal{L}^{d(\sigma)}(A_\sigma^* \cap B)}{\mathcal{L}^{d(\sigma)}(A_\sigma^*)}$

Figure 41: The "shadow" of the measure from Example 9

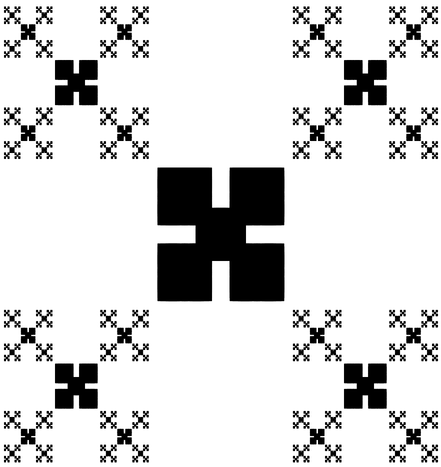

Figure 42: Attractor of the IMS from Example 10.

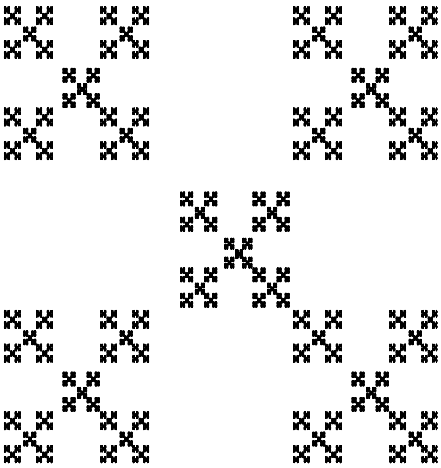

Figure 43: Structure of the hyperattractor from Example 10.

Figure 44: Invariant measure from Example 10 (its support is light).

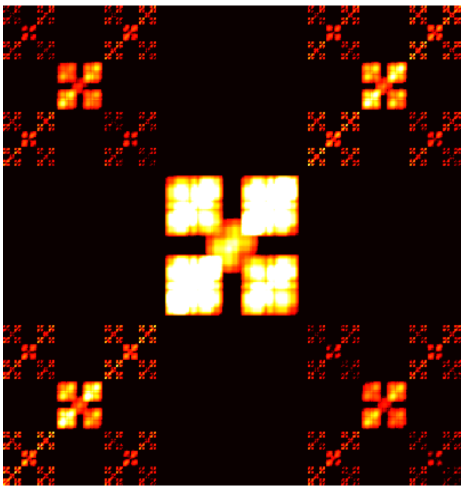

Figure 45: The “shadow” of the invariant measure from Example 10 (its support is light).

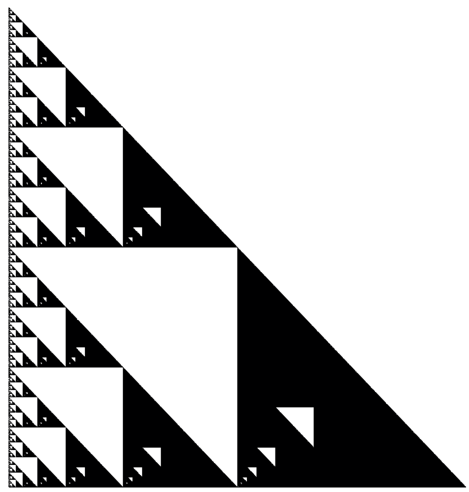

Figure 46: Fat Sierpiński's triangle.

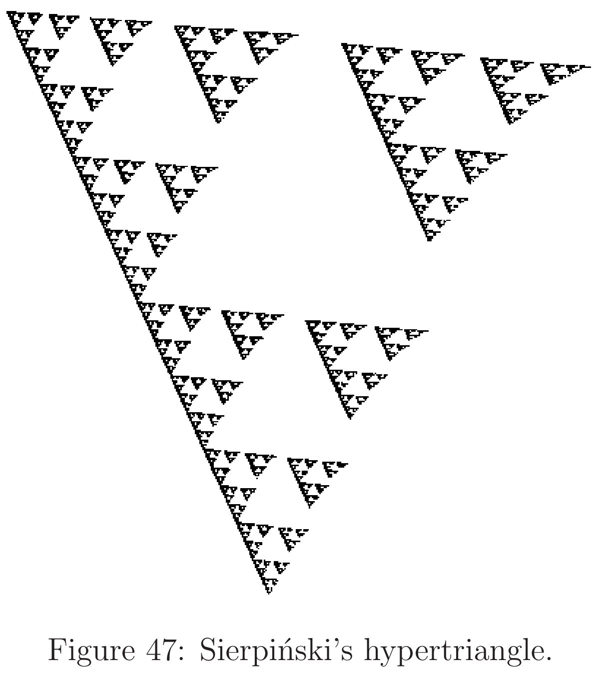

Figure 47: Sierpiński's hypertriangle.

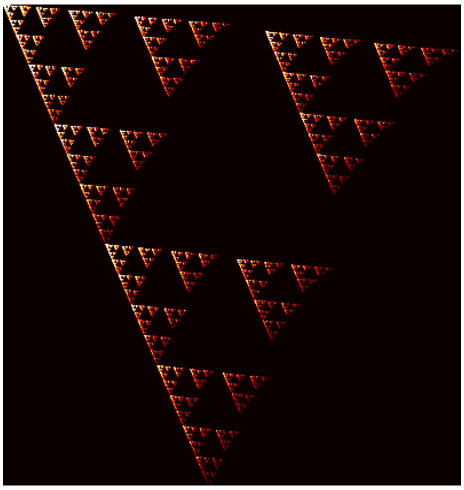

Figure 48: Invariant measure from Example 11 (its support is light).

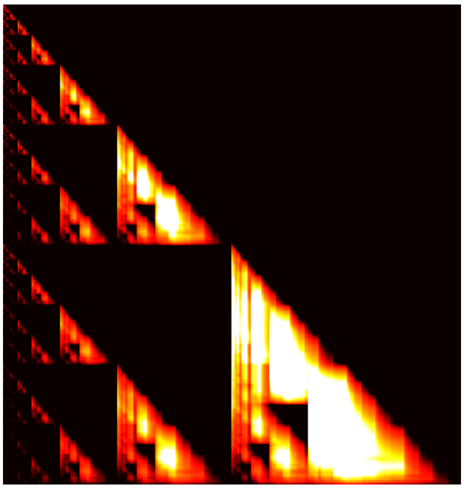

Figure 49: The “shadow” of the invariant measure from Example 11 (its support is light).

8. Fuzzy approach

We can extend our results for multivalued fractals and hyperfractals to particular spaces of fuzzy sets. First, we will remind the basic theory of fuzzy sets. Secondly, we will construct fuzzy fractals and associated fuzzy hyperfractals. Last, we will discuss visualization of these fractals and measures supported on them.

8.1. Fuzzy sets

During introduction to fuzzy sets, we will use notation and results from [DK]. See also [PR]. Fuzzy sets originated with Zadeh's 1965 paper [Z]. Fuzzy sets are considered with respect to a nonempty base set X of elements of interest. The essential idea is that each element $x \in X$ is assigned a membership grade $u(x)$ taking values in $[0,\, 1]$, with $u(x) = 0$ corresponding to non-membership, $0 < u(x) < 1$ to partial membership, and $u(x) = 1$ to full membership. According to Zadeh a *fuzzy subset* of X is a nonempty subset $\{(x,\, u(x)) : x \in X\}$ of $X \times [0,\, 1]$ for some function $u : X \to [0, 1]$. The function u itself is often used synonymously for the fuzzy set.

The only membership possibilities for an ordinary or *crisp* subset A of X are non-membership and full membership. Such a set can thus be identified with the fuzzy set on X given by its characteristic function $\chi_A : X \to [0,\, 1]$, that is with

$$\chi_A(x) \;\; = \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \notin A. \end{cases}$$

The $\alpha-$*level set* $[u]^\alpha$ of a fuzzy set u on X is defined as

$$[u]^\alpha := \{x \in X : u(x) \geq \alpha\}, \text{ for each } \alpha \in (0,\, 1],$$

while its support $[u]^0$ is the closure in the topology of X of the union of all of the level sets, that is

$$[u]^0 = \overline{\bigcup_{\alpha \in (0,\,1]} [u]^\alpha}.$$

An inclusion property follows immediately from the above definitions.

Proposition 16. *For all* $0 \leq \alpha \leq 1$

$$[u]^\beta \subseteq [u]^\alpha \subseteq [u]^0.$$

The *union, intersection and complement* of fuzzy sets can be defined pointwise in terms of their membership grades without using the extension principle (cf. Lemma 19). Consider a function $u : X \to [0,\, 1]$ as a fuzzy subset of a nonempty base space X and denote the totality of all such functions or fuzzy sets by $\mathcal{F}(X)$.

The union $u \vee v$ and the intersection $u \wedge v$ of $u,\, v \in \mathcal{F}(X)$ are defined, respectively, as

$$\begin{aligned} u \vee v(x) &= u(x) \vee v(x) := \max\{u(x),\, v(x)\}, \\ u \wedge v(x) &= u(x) \wedge v(x) := \min\{u(x),\, v(x)\}, \end{aligned}$$

for each $x \in X$. Clearly, $u \vee v$ and $u \wedge v \in \mathcal{F}(X)$.

Let us extend the definition of union for more than two fuzzy sets u_i

$$\bigvee_i u_i(x) \;:= \sup_i\{u_i(x)\},$$

for each $x \in X$. It is obvious that $\sup_i\{u_i(x)\} \in [0,\, 1]$. Hence, $\bigvee_i u_i \in \mathcal{F}(X)$.

The *Zadeh extension principle* allows a crisp mapping $f : X_1 \times X_2 \to Y$, where X_1, X_2 and Y are nonempty sets, to be extended to a mapping on fuzzy sets $\widetilde{f} : \mathcal{F}(X_1) \times \mathcal{F}(X_2) \to \mathcal{F}(Y)$, where

$$\widetilde{f}(u_1,\, u_2)(y) \;:= \begin{cases} \sup_{(x_1,\, x_2) \in f^{-1}(y)}(u_1(x_1),\, u_2(x_2)) & \text{if } \; f^{-1}(y) \neq 0 \\ 0 & \text{if } \; f^{-1}(y) = 0 \end{cases}$$

for $y \in Y$. Here, $f^{-1}(y) = \{(x_1,\, x_2) \in X_1 \times x_2 :\; f(x_1,\, x_2) = y\}$ may be empty or contain one or more points. The obvious generalization holds for mappings defined on an $N-$tuple $X_1 \times X_2 \times \cdots \times X_N$ where $N \geq 1$, with the $\wedge$ operator being superfluous when $N = 1$.

The definitions of addition and scalar multiplication of fuzzy sets in $\mathcal{F}(X)$ involve the extension principle and require the base set X to be a linear space. For the addition of two fuzzy sets $u,\, v \in \mathcal{F}(X)$ the Zadeh extension principle is applied to the function $f : X \times X \to X$ defined by $f(x_1,\, x_2) = x_1 + x_2$, to give

$$\widetilde{(u+v)}(x) = \sup_{x_1 + x_2 = x} (u(x_1) + v(x_2)),$$

for all $x \in X$, while for *scalar multiplication* of $u \in \mathcal{F}(X)$ by a nonzero scalar c the function $f : X \to X$ defined by $f(x) = cx$ is extended to

$$\widetilde{cu}(x) = u(x/c),$$

for all $x \in X$. Obviously both $\widetilde{u+v}$ and $\widetilde{cu}$ belong to $\mathcal{F}(X)$.

The totality of fuzzy sets $\mathcal{F}(X)$ on a base space X is often too broad and general to allow strong or specific enough results to be established, so various restrictions are often imposed on the fuzzy sets. In particular, a fuzzy set $u \in \mathcal{F}(X)$ is called *normal fuzzy set* if there exists at least one point $x_0 \in X$ for which $u(x_0) = 1$, so the 1-level set $[u]^1$ and hence every other level set $[u]^\alpha$ for $0 < \alpha < 1$ and the support $[u]^0$ are all nonempty subsets of X. For technical reasons, the level sets are often assumed to be compact and, when X is a linear space, also convex. In fact, the convexity of the level sets of a fuzzy set u is equivalent to its being a fuzzy convex fuzzy set, that is satisfying

$$u(\lambda x_1 + (1-\lambda)x_2) \geq u(x_1) \wedge u(x_2) \geq u(x_1) \wedge u(x_2),$$

for all $x_1,\, x_2 \in X,\, \lambda \in [0,\, 1]$.

In latter likewise in [DK], we will consider fuzzy subsets of the $n-$dimensional Euclidean space $\mathbb{R}^n$ for which the level sets are all nonempty, compact, and often also convex, subsets of $\mathbb{R}^n$.

Let us denote by $\mathcal{F}^n$ the set of all fuzzy sets of $\mathbb{R}^n$. Their membership function is

$$u : \mathbb{R}^n \to [0,\, 1].$$

In general, some level sets of a fuzzy set can be empty. Indeed, in the trivial case of $u(x) \equiv 0$, for all $x \in \mathbb{R}^n$, even the support is empty, than u here is called empty fuzzy set. We shall restrict our attention to the normal fuzzy sets which satisfy

1. u maps $\mathbb{R}^n$ onto I=$[0, 1]$. Obviously then $[u]^1 \neq \emptyset$, which is often as an alternative definition of a normal fuzzy set. It follows

$$[u]^\alpha \neq \emptyset, \text{ for all } \alpha \in I.$$

2. $[u]^0$ is a bounded subset of $\mathbb{R}^n$.

3. u is upper semicontinuous.

Hence, each level set $[u]^\alpha$, and also $[u]^0$ by definition, is a closed subset of $\mathbb{R}^n$. Moreover, they are all bounded since they are subsets of $[u]^0$, which is bounded, and so

Proposition 17. *$[u]^\alpha$ is a compact subset of $\mathbb{R}^n$, for all $\alpha \in I$.*

Proposition 18. *For any non-decreasing sequence $\alpha_i \to \alpha$ in I*

$$[u]^\alpha = \bigcap_{i \geq 1} [u]^{\alpha_i}.$$

The totality of fuzzy sets satisfying three assumptions above will be denoted by $\mathcal{D}^n$.

Remark 66. For the sake of completeness, if we use instead of $\mathbb{R}^n$ in the definition of $\mathcal{D}^n (= \mathcal{D}(\mathbb{R}^n))$ a metric space X it will be denoted by $\mathcal{D}(X)$.

Remark 67. [RF, Remark 2.2, p. 14]

(i) $u = v \Leftrightarrow [u]^\alpha = [v]^\alpha$, for all $\alpha \in I$.

(ii) We can define a partial order $\subseteq$ on D^n by setting $u \subseteq v \Leftrightarrow u(x) \leq v(x),\, \forall x \in \mathbb{R}^n$ ($\Leftrightarrow [u]^\alpha \subseteq [v]^\alpha$).

However, we will often restrict our attention to fuzzy convex sets.

4. u is fuzzy convex.

Hence, $[u]^\alpha$ is convex subset of $\mathbb{R}^n$ for any $\alpha \in [0, 1]$.

We denote by $\mathcal{E}^n$ the space of all fuzzy subsets u of $\mathbb{R}^n$ which satisfy all the four assumptions above, that is, normal, fuzzy convex upper semicontinuous fuzzy sets with bounded supports.

We can state

Proposition 19. *Let $u \in \mathcal{E}^n$ and write $C_\alpha = [u]^\alpha$ for $\alpha \in I$. Then*

1. *C_α is a nonempty compact convex subset of $\mathbb{R}^n$ for each $\alpha \in I$;*
2. *$C_\beta \subseteq C_\alpha$ for $0 \le \alpha \le \beta \le 1$;*
3. *$C_\alpha = \cap_{i=1}^\infty C_{\alpha_i}$ for any nondecreasing $\alpha_i \to \alpha$ in I. Or, equivalently, $d_H(C_{\alpha_i}, C_\alpha) \to 0$ as $\alpha_i \nearrow \alpha$.*

The converse of Proposition 19 holds.

Proposition 20. *Let $C = \{C_\alpha, \alpha \in I\}$ be a family of subsets of $\mathbb{R}^n$ satisfying 1., 2. and 3. of Proposition 19, and define $u : \mathbb{R}^n \to I$ by*

$$u(x) \ := \begin{cases} 0 & \text{if } \ x \notin C_0, \\ \sup_{(\alpha \in I :\, x \in C_\alpha)} & \text{if } \ x \in C_0. \end{cases}$$

Then $u \in \mathcal{E}^n$ with $[u]^\alpha = C_\alpha \in (0, 1]$ and

$$[u]^0 = \overline{\bigcup_{\alpha \in (0,\, 1]} C_\alpha} \subseteq C_0.$$

Compact (convex) sets in $\mathbb{R}^n$ belong to $\mathcal{D}^n$ ($\mathcal{E}^n$) due to the upper semicontinuity in assumption (3).

Proposition 21. *If $A \in K_{Co}(\mathbb{R}^n)$ then $\chi_A \in \mathcal{E}^n$.*

Definition 42. The *endograph*

$$\mathrm{end}(u) = \{(x, \alpha) \in \mathbb{R}^n \times I : u(x) \le \alpha\}.$$

It is a nonempty closed subset of $\mathbb{R}^n \times I$. Restricting to those points that lie above the support set, we obtain *supported endograph,* or *sendograph* for short, of u

$$\mathrm{send}(u) = \mathrm{end}(u) \cap ([u]^0 \times I),$$

which is a nonempty compact subset of $\mathbb{R}^n \times I$. In fact,

$$\mathrm{send}(u) = \bigcup \{[u]^\alpha \times \{\alpha\}, \, \alpha \in I\}.$$

We shall define addition and scalar multiplication of fuzzy sets in $\mathcal{E}^n$ levelsetwise, that is, for $u, v \in \mathcal{E}^n$ and $c \in \mathbb{R} - \{0\}$,

$$[u + v]^\alpha := [u]^\alpha + [v]^\alpha \tag{32}$$

and

$$[cu]^\alpha := c[u]^\alpha \tag{33}$$

for each $\alpha \in I$.

PROPOSITION 22. *$\mathcal{E}^n$ is closed under addition (32) and scalar multiplication (33).*

Proof. We use Proposition 19 to families of subsets $\{[u+v]^\alpha,\, \alpha \in I\}$ and $\{[cu]^\alpha,\, \alpha \in I\}$. Properties (1) and (2) follow from these properties for $\{[u]^\alpha\}$, $\{[v]^\alpha\}$, definitions (32) and (33) and the closedness of $K_{Co}(\mathbb{R}^n)$ under addition and scalar multiplication. In order to prove (3), let $\{\alpha_i\}$ be a nondecreasing sequence in I with $\alpha_i \to \alpha^- \in I$. Then by (32) and (33), by property (3) for $\{[u]^\alpha,\, \alpha \in I\}$ and $\{[v]^\alpha,\, \alpha \in I\}$.

$$d_H([u+v]^{\alpha_i},\, [u+v]^\alpha) \leq d_H([u]^{\alpha_i},\, [u]^\alpha) + d_H([v]^{\alpha_i},\, [v]^\alpha) \to 0.$$

$$d_H([cu]^{\alpha_i},\, [cu]^\alpha) = |c| d_H([u]^{\alpha_i},\, [u]^\alpha) \to 0.$$

Hence, property (3) of Proposition 19 is satisfied by both families. □

Here, we shall also define multiplication of fuzzy sets in $\mathcal{E}^n$ by orthonormal matrix levelsetwise, that is, for $u \in \mathcal{E}^n$ and $c \in \mathbb{R} - \{0\}$

$$[Qu]^\alpha = Q[u]^\alpha. \tag{34}$$

PROPOSITION 23. *$\mathcal{E}^n$ is closed under orthonormal matrix multiplication (34).*

Proof. The proof is the same as for $[cu]^\alpha$. We use Proposition 19 to family of subsets $\{[Qu]^\alpha,\, \alpha \in I\}$. Properties (1) and (2) follow from these properties for $\{[u]^\alpha\}$ and the closedness of $K_{Co}(\mathbb{R}^n)$ under matrix multiplication. In order to prove (3), let $\{\alpha_i\}$ be a nondecreasing sequence in I with $\alpha_i \to \alpha^- \in I$. Then by Proposition 13

$$d_H([Qu]^{\alpha_i},\, [Qu]^\alpha) = d_H([u]^{\alpha_i},\, [u]^\alpha) \to 0.$$

Hence, property (3) is satisfied. □

REMARK 68. Note that we can state the previous two propositions for $\mathcal{D}^n$ instead of for $\mathcal{E}^n$.

We used Zadeh's extension principle to define the addition and scalar multiplication of fuzzy sets. That is,

$$\widetilde{[u+v]}(z) = \sup_{z=x+y}\, (u(x),\, v(y)) \tag{35}$$

and

$$\widetilde{[cu]}(x) = u(x/c). \tag{36}$$

LEMMA 19. *In $\mathcal{E}^n$, definitions in equations (35) and (36) are equivalent to the level set definitions (32) and (33), respectively.*

Proof. Let $\alpha \in (0, 1]$. Then

$$\{x \in \mathbb{R}^n,\ \widetilde{cu}(x) \geq \alpha\} = \{x \in \mathbb{R}^n,\ u(x/c) \geq \alpha\} =$$

$$\{c\bar{x} \in \mathbb{R}^n,\ u(\bar{x}) \geq \alpha\} = c\{\bar{x} \in \mathbb{R}^n,\ u(\bar{x}) \geq \alpha\} = c[u]^\alpha = [cu]^\alpha$$

and so definitions (33) and (36) coincide. Now, suppose that

$$\widetilde{cu}(z) \geq \alpha.$$

By the definition of the supremum, there exist $x_k \in [u]^{\alpha(1-1/k)}$, $y_k \in [v]^{\alpha(1-1/k)}$ for $k = 1, 2, \ldots$ such that $x_k + y_k = z$ and so

$$\tilde{u} + \tilde{v}(z) \geq \min\{u(x_k),\ v(y_k)\} \geq \alpha(1 - 1/k).$$

Since $[u]^{\alpha(1-1/k)} \to [u]^\alpha$, $[v]^{\alpha(1-1/k)} \to [v]^\alpha$ with respect to the Hausdorff metric d_H, by the compactness of all these sets, there exist $x_{k_j} \in [u]^{\alpha(1-1/k)}$, $y_{k_j} \in [v]^{\alpha(1-1/k)}$, $\bar{x} \in [u]^\alpha$ and $\bar{y} \in [v]^\alpha$ such that $x_{k_j} \to \bar{x}$ and $y_{k_j} \to \bar{y}$. Hence, $x_{k_j} + y_{k_j} \to \bar{x} + \bar{y} \in [u]^\alpha + [v]^\alpha$ and $\{z \in \mathbb{R}^n,\ \widetilde{(u+v)}(z) \geq \alpha\} \in [u]^\alpha + [v]^\alpha$.

Conversely, if $\bar{x} \in [u]^\alpha$ and $\bar{y} \in [v]^\alpha$, so that $u(\bar{x}) \geq \alpha$ and $v(\bar{y}) \geq \alpha$, then with $z = \bar{x} + \bar{y}$

$$\widetilde{(u+v)}(z) \geq \min\{u(\bar{x}),\ v(\bar{y})\} \geq \alpha$$

and so $[u]^\alpha + [v]^\alpha \subseteq \{z \in \mathbb{R}^n,\ \widetilde{(u+v)}(z) \geq \alpha\} = [u]^\alpha + [v]^\alpha$. Thus, we have shown that

$$\{z \in \mathbb{R}^n,\ \widetilde{(u+v)}(z) \geq \alpha\} = [u]^\alpha + [v]^\alpha = [u+v]^\alpha,$$

so definitions (32) and (35) coincide. $\square$

We shall define, for an orthonormal matrix $\mathscr{Q}$ and $u \in \mathcal{E}^n$,

$$(\widetilde{\mathscr{Q}u})(z) := u(\mathscr{Q}^{-1}(x)).$$

In $\mathcal{E}^n$, this definition coincide with level set definition.

Remark 69. Since there is no need of convexity, this definition coincides also with level set definition in $\mathcal{D}^n$.

Lemma 20. *Let $\alpha \in I$. Then*

$$[\widetilde{\mathscr{Q}u}]^\alpha = \mathscr{Q}[u]^\alpha.$$

Proof. We can derive as in the case of $\widetilde{[cu]^\alpha}(x)$

$$[\widetilde{\mathscr{Q}u}]^\alpha = \{x,\ \mathscr{Q}u(x) \geq \alpha\} = \{x,\ u(\mathscr{Q}^{-1}(x)) \geq \alpha\}$$

$$= \{\mathscr{Q}y,\ u(y) \geq \alpha\} = Q\{y,\ u(y) \geq \alpha\} = \mathscr{Q}[u]^\alpha.$$

$\square$

The concept of support function of a nonempty compact (convex) subset of $\mathbb{R}^n$ can be usefully generalized to the fuzzy sets in $\mathcal{E}^n$. Let $u \in \mathcal{E}$ and define

$$\mathrm{supp}_u(\alpha,\, p) := \mathrm{supp}_{[u]^\alpha}(p) = \sup\{p'a,\, a \in [u]^\alpha\} \tag{37}$$

Proposition 24. *Let $u \in \mathcal{E}^n$. Then the support function s_u is*

1. *uniformly bounded on $I \times S^{n-1}$,*
2. *Lipschitz in $p \in S^{n-1}$ uniformly on I,*
3. *for each $\alpha \in I$*

$$d_H([u]^\alpha,\, [v]^\alpha) = \mathrm{supp}\{|s_u(\alpha,\, p) - s_v(\alpha,\, p)|,\, p \in S^{n-1}\}.$$

Proposition 25. *Let $u \in \mathcal{E}^n$. Then $s_u(\alpha,\, p)$ is nonincreasing in $\alpha \in I$ for each $p \in S^{n-1}$.*

The subset $\mathcal{G}^n$ of convex-sendograph fuzzy sets consists of those fuzzy sets $u \in \mathcal{E}^n$ for which the sendograph send(u) is a convex subset of $\mathbb{R}^n \times I$. Hence, $u \in \mathcal{G}$ if and only if $u : \mathbb{R}^n \to I$ is a concave function over its support $[u]^0$, that is if and only if

$$u(\lambda x + (1-\lambda)y) \geq \lambda u(x) + (1-\lambda)u(y),$$

for all $x,\, y \in [u]^0$ and $\lambda \in I$. Note that a fuzzy convex fuzzy set is not necessarily a convex-sendograph fuzzy set.

We will use the supremum metric d_∞ on $\mathcal{D}^n$ defined [DK] by

$$d_\infty(u,\, v) := \sup\{d_H([u]^\alpha,\, [v]^\alpha),\, \alpha \in I\},$$

for all $u,\, v \in \mathcal{D}^n$.

Let us supply basic properties of the metric d_∞.

Lemma 21. *Let $u,\, u_n \in \mathcal{D}^n$, $\forall n \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} d_\infty(u_n,\, u) = 0$$

if and only if

$$\lim_{n\to\infty} d_H([u_n]^\alpha,\, [u]^\alpha) = 0\, \forall \alpha \in I.$$

Proof. The claim follows from the definition of d_∞. □

Remark 70. [RF, Remark 2.4, p. 14] It is easy to see that

$$(X,\, d) \to (K(X),\, d_H) \to (\mathcal{D}(X),\, d_\infty)$$

are isometric embeddings (by mean $x \to \{x\}$ and $A \to \chi_A$, respectively).

In this work, the completeness of spaces turns out to be crucial. The following propositions enable us to construct fractals in particular spaces and hyperspaces of fuzzy sets.

PROPOSITION 26. [RF, Theorem 3.3, p. 15] *$(\mathcal{D}(X), d_\infty)$ is a complete metric space if and only if X is a complete metric space.*

PROPOSITION 27. [DK, Prop. 7.2.3] *$(\mathcal{E}^n, d_\infty)$ is a complete metric space.*

LEMMA 22. *Let $u, v \in \mathcal{G}^n$. Then $d_\infty(u, v) \geq d_H(\mathrm{send}(u), \mathrm{send}(v))$.*

Proof. Let $u, v \in \mathcal{G}^n$. We know that $\mathrm{send}(u) = \cup_{\alpha \in I} [u]^\alpha \times \alpha$ for any $u \in \mathcal{G}^n$. From

$$d_H(\cup_\alpha A_\alpha, \cup_\alpha B_\alpha) \leq \max_\alpha d_H(A_\alpha, B_\alpha),$$

we obtain

$$\begin{aligned} d_H(\mathrm{send}(u), \mathrm{send}(v)) &\leq \max_{\alpha \in I} d_H([u]^\alpha \times \alpha, [v]^\alpha \times \alpha) \\ &= \max_{\alpha \in I} d_H([u]^\alpha, [v]^\alpha) = d_\infty(u, v). \end{aligned}$$

□

PROPOSITION 28. *$(\mathcal{G}^n, d_\infty)$ is a complete metric space.*

Proof. Let $u_n \in \mathcal{G}^n$ be a Cauchy sequence. From

$$d_\infty(u_n, u_m) \to 0$$

and the previous lemma, we obtain

$$d_H(\mathrm{send}(u_n), \mathrm{send}(u_m)) \to 0.$$

Sets $\mathrm{send}(u_n)$, $\mathrm{send}(u_m)$ are nonempty compact convex. Since $\mathcal{E}^n$ is complete, the sequence u_n converge to $u \in \mathcal{E}^n$. Space $K_{Co}(\mathbb{R}^n)$ is complete, too, therefore $\mathrm{send}(u)$ is convex. We obtain $u \in \mathcal{G}^n$. □

We will often deal with compact sets of fuzzy sets. Let us finish the subsection with a lemma concerning their level sets.

*LEMMA 23. *Let $\mathcal{U}$ be a compact set of fuzzy sets $u_i \in \mathcal{D}^n$. Then*

$$\left[\bigvee_{u_i \in \mathcal{U}} u_i \right]^\alpha = \bigcup_{u_i \in \mathcal{U}} [u_i]^\alpha$$

for any $\alpha \in I$.

Proof. For a given $\alpha \in I$, we have

$$\left[\bigvee_{u_i \in \mathcal{U}} u_i\right]^\alpha = \left\{x \in \mathbb{R}^n,\ \sup_{u_i \in \mathcal{U}}\{u_i(x)\} \geq \alpha\right\}.$$

Let $x \in \mathbb{R}$ be such that $\sup_{u_i \in \mathcal{U}}\{u(x)\} \geq \alpha$, then there exists $u_k \in \mathcal{U}$, $k \in \mathbb{N}$ such that for any $\beta < \alpha$, there exists k_0, $u_k(x) \geq \beta$, for any $k > k_0$. Due to compactness of $\mathcal{U} \in K(\mathcal{D}^n)$, we can find subsequence, for the sake of simplicity u_k, which converges to $u \in \mathcal{U}$. Since $x \in [u_k]^\beta$, for all $\beta \leq \alpha$ for some $k > k_0$ (where $[u_k]^\beta$ are compact), it follows $x \in [u]^\beta$, for all $\beta > \alpha$. From the upper semicontinuity of u, we obtain $x \in [u]^\alpha$. Hence, we can write

$$\left[\bigvee_{u_i \in \mathcal{U}} u_i\right]^\alpha = \left\{x \in \mathbb{R}^n,\ \max_{u_i \in \mathcal{U}}\{u_i(x)\} \geq \alpha\right\} =$$

$$\{x \in \mathbb{R}^n,\ u_i(x) \geq \alpha,\ u_i \in \mathcal{U}\} = \bigcup_{u_i \in \mathcal{U}} [u_i]^\alpha.$$

□

8.2. Fuzzy fractals

Single-valued contractions are extended to construct fuzzy fractals, for example, in [CFMV], [FLV], [FMV] and [DK]. These fuzzy fractals are supported on skinny fractals. We will generalize our results on multivalued fractals and hyperfractals in a fuzzy way. Fuzzy fractals are shadows of associated fuzzy hyperfractals. It is not surprising that their address structures correspond.

Remark 71. In order to find the address structure of multivalued fractals, we need contractions F_i to be induced from multivalued mappings. The contractions F_i satisfy

$$F(A \cup B) = F(A) \cup F(B),\ \forall A,\ B \in K(X).$$

Then we can write

$$F_i(\cup_j F_j(A)) = \cup_j F_i(F_j(A)).$$

Similarly, when looking for the address structure of fuzzy fractals, we need contractions f_i to satisfy

$$f_i(u \vee v) = f_i(u) \vee f_i(v),\ \forall u,\ v \in \mathcal{D}^n.$$

Then we can write

$$f_i(\bigvee_j f_j(u)) = \bigvee_j f_i(f_j(u)),\ u \in \mathcal{D}^n.$$

Thus, we will consider only the contractions $f : \mathcal{D}^n \to \mathcal{D}^n$

$$f(u \vee v) = f(u) \vee f(v),\ \forall u,\, v \in \mathcal{D}^n.$$

in the following text.

*Lemma 24. *Let $f_i : \mathcal{D}^n \to \mathcal{D}^n$, $i = 1,\, 2,\, \ldots,\, m$, be contractions with factors of contractions r_i. Then $f : \mathcal{D}^n \to \mathcal{D}^n$,*

$$f(u) = \bigvee_i f_i(u)$$

is a contraction with factor $r = \max_i\{r_i\}$.

Proof. Let $u \in \mathcal{D}^n$. First, it follows from Lemma 23 that

$$[f(u)]^\alpha = \bigcup_{i=1}^{m} [f_i(u)]^\alpha.$$

Secondly, we will prove that $f : \mathcal{D}^n \to \mathcal{D}^n$ is a contraction. Note that the following holds for $[f(u)]^\alpha$:

1. it is a compact nonempty set, for each $\alpha \in I$,
2. $[f(u)]^\beta \subset [f(u)]^\alpha$, for $0 \le \alpha \le \beta \le 1$,
3. $[f(u)]^\alpha = \lim_{i\to\infty} [f(u)]^{\alpha_i}$, for any nondecreasing $\alpha_i \to \alpha \in I$.

Since these properties are satisfied by each $[f_i(u)]^\alpha$, they are satisfied by $[f(u)]^\alpha = \cup_i [f_i(u)]^\alpha$. Using $\alpha-$level sets, it is easy to show that f is a contraction,

$$\begin{aligned}
d_\infty(f(u),\, f(v)) &= \max_{\alpha\in I}\{d_H([f(u)]^\alpha,\, [f(v)]^\alpha)\} = \\
&= \max_{\alpha\in I}\{d_H(\bigcup_{i=1}^{m}[f_i(u)]^\alpha,\, \bigcup_{i=1}^{m}[f_i(v)]^\alpha)\} \\
&\le \max_{\alpha\in I}\{\max_{i=1,\,2,\,\ldots,\,m}\{d_H([f_i(u)]^\alpha,\, [f_i(v)]^\alpha)\}\} \\
&= \max_{i=1,\,2,\,\ldots,\,m}\{\max_{\alpha\in I}\{d_H([f_i(u)]^\alpha,\, [f_i(v)]^\alpha)\}\} \\
&= \max_{i=1,\,2,\,\ldots,\,m}\{d_\infty(f_i(u),\, f_i(v))\} \\
&= \max_{i=1,\,2,\,\ldots,\,m}\{r_i d_\infty(u,\, v)\} \\
&\le r d_\infty(u,\, v).
\end{aligned}$$

□

*THEOREM 40. *Let $f_i : \mathcal{D}^n \to \mathcal{D}^n$ be contractions for $i = 1, 2, \ldots, m$. There exists a unique $u^* \in \mathcal{D}^n$ such that*

$$f(u^*) = u^*.$$

Proof. It follows directly, since f is a contraction in a complete metric space $\mathcal{D}^n$. $\square$

DEFINITION 43. The above attractor u^* is called *a fuzzy fractal.* The system $f = \{(\mathcal{D}^n, d_\infty), f_i, i = 1, 2, \ldots, m\}$ is called *a fuzzy IFS.*

*THEOREM 41. *Let $f_i : \mathcal{D}^n \to \mathcal{D}^n$ be contractions for $i = 1, 2, \ldots, m$. Let $F = \{(\mathcal{D}^n, d_\infty), f_i\}$ be an IFS. There exists a unique $\mathcal{U}^* \in K(\mathcal{D}^n)$ such that*

$$F(\mathcal{U}^*) = \mathcal{U}^*,$$

where $F : K(\mathcal{D}^n) \to K(\mathcal{D}^n)$,

$$F(\mathcal{U}) = \bigcup_{i=1}^{m} f_i(\mathcal{U}),$$

$$f_i(\mathcal{U}) = \bigcup_{u \in \mathcal{U}} \{f_i(u)\}.$$

Proof. It follows directly since F is an IFS in a complete metric space $\mathcal{D}^n$. $\square$

DEFINITION 44. The above attractor $\mathcal{U}^*$ is called *a fuzzy hyperfractal.* The system $F = \{(\mathcal{D}^n, d_\infty), f_i, i = 1, 2, \ldots, m\}$ is called *a fuzzy hyperIFS.*

Let us proceed to the address structures of fuzzy fractals and fuzzy hyperfractals. We will treat them in the same way as the ones of multivalued fractals and hyperfractals. We can state the following proposition (see [Ba1, Theorem 2.1, p. 123]).

*PROPOSITION 29. *Let $\{\mathcal{D}^n, f_1, f_2, \ldots, f_m\}$ be a fuzzy hyperIFS. Let $\mathcal{U}^*$ denote the attractor of the fuzzy hyperIFS. Let (Σ, d_Σ) denote the code space associated with the fuzzy hyperIFS. For each $\sigma \in \Sigma$, $n \in \mathbb{N}$, and $u \in \mathcal{D}^n$, let*

$$\phi(\sigma, n, u) = f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_n}(u).$$

Then

$$\phi(\sigma) = \lim_{n \to \infty} \phi(\sigma, n, u)$$

exists, belongs to $\mathcal{U}^*$ and is independent of $u \in \mathcal{D}^n$. The function $\phi : \Sigma \to \mathcal{U}^*$ is continuous and onto.

*Theorem 42. *Let $\{\mathcal{D}^n,\, f_i,\, i = 1, 2, \ldots, m\}$ be a fuzzy IFS and u^* its attractor. Let*

$$\phi(\sigma) = \lim_{n\to\infty} \phi(\sigma,\, n,\, u),$$

where

$$\phi(\sigma,\, n,\, u) = f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_n}(u),\ \sigma \in \Sigma,\ n \in \mathbb{N},\ u \in \mathcal{D}^n.$$

Then $\phi(\sigma) \le u^*$. Let $a \in \mathbb{R}^n$, $u^*(a) = \alpha$. Then there exists $\omega \in \Sigma$ such that $\phi(\omega)(a) = \alpha$.

Proof. We know that

$$u^* = \lim_{n\to\infty} f^n(u),$$

for any $u \in \mathcal{D}^n$, where $f(u) = \bigvee_{i=1}^m f_i(u)$. Notice that

$$\phi(\sigma,\, n,\, u) \le f^n(u).$$

Since limits

$$\lim_{n\to\infty} \phi(\sigma,\, n,\, u) = \phi(\sigma)$$

and

$$\lim_{n\to\infty} f^n(u) = u^*$$

exist, it follows that

$$\phi(\sigma) \le u^*.$$

On the other hand, let $a \in \mathbb{R}^n$ be such that $u^*(a) = \alpha$. Consider $u \in \mathcal{D}^n$ and a sequence $\{u_n\}$, $u_n = f^n(u)$. Notice that

$$\lim_{n\to\infty} u_n = u^*.$$

Since $[u_n]^\alpha \to [u^*]^\alpha$ follows from Lemma 21, we can find a sequence $\{a_n\}$ such that $a_n \in [u_n]^\alpha$

$$\lim_{n\to\infty} a_n = a.$$

Moreover, for each a_n, there exists $\{\omega^{(n)}\}$ such that

$$a_n \in [\phi(\omega^{(n)},\, n,\, u)]^\alpha.$$

Notice that

$$a_1 \in \left[\bigvee_{i=1}^m f_i(u)\right]^\alpha \Rightarrow \exists i_1 :\, a_1 \in \left[f_{i_1}(u)\right]^\alpha,$$

$$a_2 \in \left[\bigvee_{i,\, j=1}^m f_{ij}(u)\right]^\alpha \Rightarrow \exists i_1,\, i_2 :\, a_2 \in \left[f_{i_1 i_2}(u)\right]^\alpha,$$

etc.

Since $(\Sigma,\, d_\Sigma)$ is compact, it follows that $\{\omega^{(n)}\}$ possesses a convergent subsequence with a limit $\omega \in \Sigma$. Without loss of generality, assume that $\lim_{n\to\infty} \omega^{(n)} = \omega$. Then the number of successive initial agreements between components of $\omega^{(n)}$ and ω increases without limit. That is, if

$$\alpha(n) = \operatorname{card}\{j \in \mathbb{N} : \omega_k^{(n)} = \omega_k \text{ for } 1 \le k \le j\},$$

then $\alpha(n) \to \infty$ as $n \to \infty$. It follows that

$$d(\phi(\omega,\, n,\, u),\, \phi(\omega^{(n)},\, n,\, u) \to 0.$$

From

$$a \in \lim_{n\to\infty} \left[\phi(\omega^{(n)},\, n,\, u)\right]^\alpha$$

it follows that

$$a \in \lim_{n\to\infty} \left[\phi(\omega,\, n,\, u)\right]^\alpha .$$

Hence, $\phi(\omega) \le u^*$ implies $\phi(\omega)(a) = \alpha$. $\square$

We obtain the following corollary.

*Corollary 9. *Let us consider the system $\{\mathcal{D}^n,\, f_i,\, i = 1,\, 2,\, \ldots,\, m\}$. Attractors u^* and $\mathcal{U}^*$ of the fuzzy IFS and fuzzy hyperIFS, respectively, have the same address structure,*

$$u^* = \bigvee_{\sigma\in\Sigma} f_\sigma(u),$$

$$\mathcal{U}^* = \bigcup_{\sigma\in\Sigma} \{f_\sigma(u)\},$$

for any $u \in \mathcal{D}^n$.

Definition 45. Let us call u^* *an underlying fuzzy fractal* to U^*.

Next, we will calculate the Hausdorff dimension of fuzzy hyperfractals. Let us give some notation and supporting lemmas. From now on, we will consider $f_i : \mathcal{D}^n \to \mathcal{D}^n$, $i = 1,\, 2,\, \ldots,\, m$, such that

$$f_i(u) = r_i \mathcal{Q}_i u + v_i, \tag{38}$$

where $r_i \in [0,\, 1)$, $\mathcal{Q}_i$ is an orthonormal matrix and $v_i \in \mathcal{D}^n$. These mappings can be described levelsetwise. Let us denote for any $\alpha \in I$

$$f_i^\alpha(C) := r_i \mathcal{Q}_i C + [v_i]^\alpha,$$

where $C \in K_{Co}(\mathbb{R}^n)$.

*Lemma 25. *Let f_i be as in (38), then $[f_i(u)]^\alpha = f_i^\alpha([u]^\alpha)$ for any $\alpha \in I$.*

Proof. It follows from equations (32), (33) and (34). $\square$

This implies the following lemma, which enables us to treat effectively level sets of fuzzy hyperfractals as hyperfractals.

*LEMMA 26. *Let* $F = \{(\mathcal{D}^n,\, d_\infty),\, f_i,\, i = 1,\, 2,\, \ldots,\, m\}$,

$$f_i(u) = r_i \mathscr{Q}_i u + v_i,$$

where $r_i \in [0,\, 1)$, $\mathscr{Q}_i$ *is an orthonormal matrix and* $v_i \in \mathcal{D}^n$ *be a fuzzy hyper-IFS with an attractor* $\mathcal{U}^*$. *Let* $F^\alpha = \{(K(\mathbb{R}^n),\, d_H),\, f_i^\alpha,\, i = 1,\, 2,\, \ldots,\, m\}$ *be a hyperIFS with an attractor* β^*. *Then*

$$[u_\sigma]^\alpha = B_\sigma^*.$$

Proof. We know that

$$[f_i(u)]^\alpha = f_i^\alpha([u]^\alpha),$$

for any $u \in \mathcal{D}^n$ and $\alpha \in I$. Note that $f_{\sigma_1\sigma_2\ldots\sigma_p}(u) = r_{\sigma_1\sigma_2\ldots\sigma_p}\mathscr{Q}_{\sigma_1\sigma_2\ldots\sigma_p}(u) + D$, $D \in \mathcal{D}^n$. This implies

$$[f_{\sigma_1\sigma_2\ldots\sigma_p}(u)]^\alpha = f^\alpha_{\sigma_1\sigma_2\ldots\sigma_p}([u]^\alpha),$$

for $p \in \mathbb{N}$. For $p \to \infty$, we obtain convergent sequences in $K(\mathbb{R}^n)$ on both sides of the equation. We arrive to $[u_\sigma^*]^\alpha = B_\sigma^*$. □

*LEMMA 27. *Let* $f_i : \mathcal{E}^n \to \mathcal{E}^n$ *be such that*

$$f_i(u) := r_i \mathscr{Q}_i u + v_i,$$

where $v_i \in \mathcal{E}^n$, $r \in [0,\, 1)$. *Then* f_i *is a similitude.*

Proof. The claim follows from the previous lemma and the convexity of level sets. For any $u,\, w \in \mathcal{E}$,

$$d_\infty(f_i(u),\, f_i(w)) = \max_{\alpha \in I} d_H([f_i(u)]^\alpha,\, [f_i(w)]^\alpha)$$

$$= \max_{\alpha \in I} r_i d_H([u]^\alpha,\, [w]^\alpha) = r_i d_\infty(u,\, w).$$

□

*THEOREM 43. *Let* $F = \{(\mathcal{E}^n,\, d_\infty),\, f_i,\, i = 1,\, 2,\, \ldots,\, m\}$ *be a fuzzy hyperIFS such that* $f_i : \mathcal{E}^n \to \mathcal{E}^n$,

$$f_i(u) := r_i \mathscr{Q}_i u + v_i,$$

where $v_i \in \mathcal{E}^n$, $r \in [0,\, 1)$, $\mathscr{Q}_i$ *are orthonormal. If its attractor is totally disconnected, its Hausdorff dimension can be calculated by means of the Moran formula.*

Proof. This follows from the previous lemma and Proposition 8. □

We will need a generalization of a convex hull for fuzzy sets. Let us write $v^c \in \mathcal{E}^n$ for $v \in \mathcal{D}^n$, where $[v^c]^\alpha := \mathrm{conv}[v]^\alpha$.

*LEMMA 28. *For any $v \in \mathcal{D}^n$, it holds that $v^c \in \mathcal{E}^n$.*

Proof. It follows directly that $[v^c]^\alpha$ are nonempty compact convex sets for any $\alpha \in I$. Observe also that $[v]^{\alpha_i} \to [v]^\alpha$ implies $[v^c]^{\alpha_i} \to [v^c]^\alpha$ for $\alpha_i \to \alpha$. $\square$

Let $F = \{(\mathcal{D}^n, d_\infty), f_i, i = 1, 2, \dots, m\}$ be a fuzzy hyperIFS such that $f_i : \mathcal{D}^n \to \mathcal{D}$,

$$f_i(u) := r_i \mathcal{Q}_i u + v_i,$$

where $v_i \in \mathcal{D}^n$, $r \in [0, 1)$. Let us define a fuzzy hyperIFS $F^c = \{(\mathcal{E}^n, d_\infty), f_i^c, i = 1, 2, \dots, m\}$ such that $f_i^c : \mathcal{E}^n \to \mathcal{E}^n$,

$$f_i^c(u) := r_i \mathcal{Q}_i u + v_i^c,$$

where $v_i^c \in \mathcal{E}^n$, $r \in [0, 1)$.

We will calculate the Hausdorff dimension of the attractor $\mathcal{U}^*$ of the fuzzy hyperIFS F with the help of the attractor $\mathcal{U}^c$ of F^c. We will state a few lemmas before.

*LEMMA 29. *Let $u \in \mathcal{D}^n$ then*

$$\mathrm{conv}[f_i(u)]^\alpha = [f_i^c(u^c)]^\alpha.$$

Proof. We obtain the claim directly from Lemma 25. $\square$

*LEMMA 30. *Let $\sigma \in \Sigma$ then*

$$\mathrm{conv}[u_\sigma^*]^\alpha = [u_\sigma^c]^\alpha.$$

Proof. Observe that

$$u_\sigma^* = f_\sigma(u) \forall u \in \mathcal{D}^n.$$

Lemma 26 implies that

$$[f_\sigma(u)]^\alpha = [f_\sigma^\alpha([u]^\alpha).$$

From Theorem 35, we obtain

$$\mathrm{conv}(f_\sigma^\alpha([u]^\alpha)) = f_\sigma^{\alpha c}([u^c]^\alpha) = [f_\sigma^c(u^c)]^\alpha.$$

$\square$

*LEMMA 31. *Let $u, v \in \mathcal{D}^n$, then*

$$d_\infty(u, v) \geq d_\infty(u^c, v^c).$$

Proof. Note that

$$d_H([u]^\alpha, [v]^\alpha) \geq d_H(\mathrm{conv}[u]^\alpha, \mathrm{conv}[v]^\alpha),$$

for each $\alpha \in I$. $\square$

*Theorem 44. *Let $F = \{(\mathcal{D}^n, d_\infty), f_i, i = 1, 2, \ldots, m\}$ be a fuzzy hyperIFS such that $f_i : \mathcal{D}^n \to \mathcal{D}$,*

$$f_i(u) := r_i \mathscr{Q}_i u + v_i,$$

where $r_i \in [0, 1)$, $\mathscr{Q}_i \in \mathbb{R}^{n\times n}$ is orthonormal $v_i \in \mathcal{D}^n$, $i = 1, 2, \ldots, m$. If the attractor $\mathcal{U}^c$ of F^c is totally disconnected, then the Hausdorff dimension of the attractor $\mathcal{U}^*$ of F can be calculated by means of the Moran formula and $\dim_H(\mathcal{U}^*) = \dim_H(\mathcal{U}^c)$.

Proof. We will proceed in the same way as in the proof of Theorem 37. Since $\mathcal{U}^c$ is totally disconnected and f_i^c are similitudes, we can obtain the Hausdorff dimension D of $\mathcal{U}^c$ (see Theorem 43). In order to calculate the Hausdorff dimension of $\mathcal{U}^*$, we will find a bi-Lipschitz mapping of $\mathcal{U}^c$ onto $\mathcal{U}^*$. Thus, we need to compare $d_\infty(u^*_\sigma, u^*_{\sigma'})$ and $d_\infty(u^c_\sigma, u^c_{\sigma'})$.

We know from Lemma 30 that

$$\operatorname{conv}([u^*_\sigma]^\alpha) = [u^c_\sigma]^\alpha, \ \forall \alpha \in I, \ \sigma \in \Sigma.$$

Since $\mathcal{U}^c$ is totally disconnected, there exists $d^{\min}_\infty > 0$ such that

$$d_\infty(f_i(u^c_\sigma), f_j(u^c_{\sigma'})) \geq d^{\min}_\infty, \ i \neq j, \ \sigma, \sigma' \in \Sigma.$$

Lemma 15 implies

$$d_\infty(f_i(u^*_\sigma), f_j(u^*_{\sigma'})) \geq d_\infty(f_i(A^c_\sigma), f_j(A^c_{\sigma'})) \geq d^{\min}_\infty.$$

Let us find a bi-Lipschitz mapping of $\mathcal{U}^c$ onto $\mathcal{U}^*$. For any $j \in \mathbb{N}$, we can write:

$$u^c_{i_1 i_2 \ldots i_j \ldots} = f^c_{i_1 \ldots i_{j-1}}(u_{i_j i_{j+1} \ldots})$$

and

$$u^c_{i_1 i_2 \ldots i'_j \ldots} = f^c_{i_1 \ldots i_{j-1}}(u_{i'_j i'_{j+1} \ldots}).$$

Let us estimate the distance $d_\infty(u^c_{i_1 i_2 \ldots i_j \ldots}, u^c_{i_1 i_2 \ldots i'_j \ldots})$. Since f_i^c are similitudes,

$$\begin{aligned} &d_\infty(u^c_{i_1 i_2 \ldots i_j \ldots}, u^c_{i_1 i_2 \ldots i'_j \ldots}) = \\ &= d_\infty(f^c_{i_1 \ldots i_{j-1}}(u^c_{i_j i_{j+1} \ldots}), f^c_{i_1 \ldots i_{j-1}}(u^c_{i'_j i'_{j+1} \ldots})) \\ &= r_{i_1 i_2 \ldots i_{j-1}} \cdot d_\infty(u^c_{i_j i_{j+1} \ldots}, u^c_{i'_j i'_{j+1} \ldots}) \\ &\geq r_{i_1 i_2 \ldots i_{j-1}} \cdot d^{\min}_\infty. \end{aligned}$$

Moreover, Lemma 31 implies

$$d_\infty(u^*_{i_j i_{j+1} \ldots}, u^*_{i'_j i'_{j+1} \ldots}) \geq d_\infty(u^c_{i_j i_{j+1} \ldots}, u^c_{i'_j i'_{j+1} \ldots}).$$

On the other hand, since f_i are contractions with a factor r_i, $i = 1, 2, \ldots, n$,

$$d_\infty(u^*_{i_1 i_2 \ldots i_j \ldots}, u^c_{i_1 i_2 \ldots i'_j \ldots}) \leq r_{i_1 i_2 \ldots i_j \ldots} \mathrm{diam}(\mathcal{U}^*).$$

Note that $\mathcal{U}^* \in K(\mathcal{E}^n)$ implies $\mathrm{diam}(\mathcal{U}^*) < \infty$.

We obtain from these inequalities

$$d_\infty(u^c_{i_1 i_2 \ldots i_{j-1} i_j \ldots}, u^c_{i_1 i_2 \ldots i_{j-1} i'_j \ldots}) \leq$$

$$d_\infty(u^*_{i_1 i_2 \ldots i_{j-1} i_j \ldots}, u^*_{i_1 i_2 \ldots i_{j-1} i'_j \ldots}) \leq \frac{\mathrm{diam}(\mathcal{U}^*)}{d_\infty^{\min}} d_\infty(u^c_{i_1 i_2 \ldots i_{j-1} i_j \ldots}, u^c_{i_1 i_2 \ldots i_{j-1} i'_j \ldots}).$$

Applying Proposition 9 for $f : \mathcal{U}^c \to \mathcal{U}^*$, $f(u^c_{i_1 i_2 \ldots}) = u^*_{i_1 i_2 \ldots}$, we obtain that the Hausdorff dimension of $\mathcal{U}^*$ is really D (cf. Proposition 9). □

*Corollary 10. *If the assumptions of Theorem 44 are fulfilled by the fuzzy hyperIFS* $F = \{(\mathcal{D}^n, d_\infty), f_i, i = 1, 2, \ldots, m\}$, $f_i(u) = r_i \mathcal{Q}_i u + v_i$, *then they are fulfilled by any other fuzzy hyperIFS* $F' = \{(\mathcal{D}^n, d_\infty), f'_i, i = 1, 2, \ldots, m\}$, *where* $f'_i(u) = r_i \mathcal{Q}_i u + v'_i$ $\mathcal{Q}_i = \mathcal{Q}'_i$, $r_i = r'_i$ *and* $v^c_i = v'^c_i$.

Proof. There exists one fuzzy hyperIFS F^c for both the fuzzy hyperIFSs F and F'. □

8.3. Visualization of fuzzy fractals and measures

Let us generalize our approach of visualization of multivalued fractals and hyperfractals to fuzzy fractals and fuzzy hyperfractals. We will also construct a measure on fuzzy fractals. We will proceed similarly as in the case of hyperfractals.

When drawing fuzzy sets here, we express levels of membership by levels of a grey colour. It is certainly more complicated to draw fuzzy fractals than multivalued fractals by means of the Banach theorem. Let $f = \{(\mathcal{D}^n, d_\infty), f_i, i = 1, 2, \ldots, m\}$ be a fuzzy IFS with an attractor u^*. Let us denote the attractor of the associated fuzzy hyperIFS $\mathcal{U}^*$. In order to avoid storing and processing $f^n(u)$, we use the chaos game.

We know that fuzzy fractals and fuzzy hyperfractals are related in a similar way as multivalued fractals and hyperfractals. Therefore, we can use the chaos game for fuzzy hyperIFS to draw attractors of fuzzy IFSs. Almost all orbits of the chaos game are dense in attractors of fuzzy hyperIFS. Let us give an analogy of Theorem 33.

*Theorem 45. *Let f be a fuzzy IFS and F an associated fuzzy hyperIFS with attractors u^* and $\mathcal{U}^*$, respectively. Let $\hat{\mathcal{U}} = \{\hat{u}_i, i \in \iota, \hat{u}_i \in \mathcal{D}^n\}$ be such that $d_{\infty_H}(\hat{\mathcal{U}}, \mathcal{U}^*) \leq \epsilon$. Then*

$$d_\infty\left(\bigvee_{i\in\iota} \hat{u}_i, \bigvee_{\sigma\in\Sigma} u^*_\sigma\right) \leq \epsilon.$$

Proof. Observe that, $d_{\infty_H}(\hat{\mathcal{U}}, \mathcal{U}^*) \le \epsilon$ implies that, for each $u^*_\sigma \in \mathcal{U}^*$, there exists $\hat{u}_i \in \hat{\mathcal{U}}$ such that

$$d_\infty(u^*_\sigma, \hat{u}_i) \le \epsilon.$$

Let $\alpha \in I$ be given. It follows that, for each $x \in [u^*_\sigma]^\alpha$, there exists $y \in [\hat{u}_i]^\alpha$ such that $d(x, y) \le \epsilon$. In the same way, for each $\hat{u}_i \in \hat{\mathcal{U}}$, there exists $u^*_\sigma \in \mathcal{U}^*$ such that

$$d_\infty(\hat{u}_i, u^*_\sigma) \le \epsilon.$$

It follows that, for each $x \in [\hat{u}_i]^\alpha$, there exists $y \in [u^*_\sigma]^\alpha$ such that $d(x, y) \le \epsilon$. We arrive to

$$d_\infty\left(\bigvee_{i\in\iota}\hat{u}_i, \bigvee_{\sigma\in\Sigma}u^*_\sigma\right) =$$

$$\max_\alpha\{\max\{\sup_{x\in\bigcup_{i\in\iota}[\hat{u}_i]^\alpha}\{\inf_{y\in[u^*]^\alpha}\{d(x, y)\}\}, \sup_{x\in[u^*]^\alpha}\{\inf_{y\in\bigcup_{i\in\iota}[\hat{u}_i]^\alpha}\{d(x, y)\}\}\}\} \le \epsilon.$$

□

Thus, we can use the chaos game for fuzzy hyperfractals to draw underlying fuzzy fractals with the same accuracy.

Let us start with a visualization of fuzzy hyperfractals. We define a mapping V_∞,

$$V_\infty : (\mathcal{D}^n, d_\infty) \to (\mathbb{R}^d, d_{\max}),$$

$$V_\infty(u) := \begin{pmatrix} \operatorname{supp}_{[u]^{\alpha_1}}(x_1) \\ \operatorname{supp}_{[u]^{\alpha_2}}(x_2) \\ \vdots \\ \operatorname{supp}_{[u]^{\alpha_d}}(x_d) \end{pmatrix},$$

where $x_i \in \mathbb{R}^n$, $|x_i| = 1$, and $\alpha_i \in I$ for $i = 1, \ldots, d$.

It is not difficult to prove that $d_{\max}(V_\infty(u), V_\infty(v)) \le d_\infty(u, v)$. This follows from

$$d_\infty(u, v) \ge \max_{\alpha\in I}\{\max_{i\in\{1,\ldots,d\}} |\operatorname{supp}_{[u]^{\alpha_i}}(x_i) - \operatorname{supp}_{[v]^{\alpha_i}}(x_i)|\}.$$

Let us proceed to measure on fuzzy fractals. Notice that fuzzy hyperfractals are attractors of ordinary IFSs and we can easily construct an invariant measure for a fuzzy hyperIFS $\{\mathcal{G}^n, f_i, p_i\}$.

*Lemma 32. *Let $r \in [0, 1)$, $\mathcal{Q} \in \mathbb{R}^{n\times n}$ is orthonormal and $v \in \mathcal{G}^n$, then*

$$f(u) = r\mathcal{Q}u + v \tag{39}$$

is a contraction in $\mathcal{G}^n$.

Proof. Let $u, v \in \mathcal{G}^n$, $r \in [0, 1)$, $\mathcal{Q} \in \mathbb{R}^{n\times n}$. We will prove that ru, $\mathcal{Q}u$, $u + v \in \mathcal{G}^n$. Let us remind that, for $u \in \mathcal{G}^n$, we have

$$u(\lambda x + (1-\lambda)y) \ge \lambda u(x) + (1-\lambda)u(y).$$

1. Let $r \in \mathbb{R}$, then

$$ru(\lambda x + (1-\lambda)y) = u(r^{-1}(\lambda x + (1-\lambda)y))$$
$$= u(\lambda r^{-1}x + (1-\lambda)r^{-1}y) \geq \lambda u(r^{-1}x) + (1-\lambda)u(r^{-1}y)$$
$$= \lambda ru(x) + (1-\lambda)ru(y).$$

   We arrive to $ru \in \mathcal{G}^n$.

2. Let $\mathcal{Q} \in \mathbb{R}^{n\times n}$, then we can proceed in the same way,

$$\mathcal{Q}u(\lambda x + (1-\lambda)y) = u(\mathcal{Q}^{-1}(\lambda x + (1-\lambda)y))$$
$$= u(\lambda \mathcal{Q}^{-1}x + (1-\lambda)\mathcal{Q}^{-1}y) \geq \lambda u(\mathcal{Q}^{-1}x) + (1-\lambda)u(\mathcal{Q}^{-1}y)$$
$$= \lambda \mathcal{Q}u(x) + (1-\lambda)\mathcal{Q}u(y).$$

   We arrive to $\mathcal{Q}u \in \mathcal{G}^n$.

3. Note that

$$u(\lambda x + (1-\lambda)y) \geq \lambda u(x) + (1-\lambda)u(y)$$

   is equivalent with

$$\lambda x + (1-\lambda)y \in [u]^{\lambda\alpha_1 + (1-\lambda)\alpha_2},$$

   where $u(x) = \alpha_1$, $u(y) = \alpha_2$. Hence, we can write

$$\lambda[u]^{\alpha_1} + (1-\lambda)[u]^{\alpha_2} \subset [u]^{\lambda\alpha_1+(1-\lambda)\alpha_2}.$$

   Let us consider $u+v$:

$$\lambda[u+v]^{\alpha_1} + (1-\lambda)[u+v]^{\alpha_2} = \lambda[u]^{\alpha_1} + \lambda[v]^{\alpha_1} + (1-\lambda)[u]^{\alpha_2} + (1-\lambda)[v]^{\alpha_2}$$
$$= \lambda[u]^{\alpha_1} + (1-\lambda)[u]^{\alpha_2} + \lambda[v]^{\alpha_1} + (1-\lambda)[v]^{\alpha_2} \subset [u]^{\lambda\alpha_1+(1-\lambda)\alpha_2} + [v]^{\lambda\alpha_1+(1-\lambda)\alpha_2}$$
$$= [u+v]^{\lambda\alpha_1+(1-\lambda)\alpha_2}.$$

   This is equivalent with

$$(u+v)(\lambda x + (1-\lambda)y) \geq \lambda(u+v)(x) + (1-\lambda)(u+v)(y).$$

□

In order to construct a shadow of an invariant measure on fuzzy hyperfractals, we use a similar approach as for a measure on multivalued fractals. (It will be a measure on a multivalued fractal.) We will calculate and visualize how often a Borel subset of embedding space $\mathbb{R}^n$ is visited during the chaos game for the IFS F. Moreover, we need to evaluate how significant part of fuzzy sets from an orbit of the chaos game visited the Borel set. The easiest way is to measure a volume of intersections of sendographs. We will consider a fuzzy hyperIFS $\{\mathcal{G}^n,\, f_i,\, p_i,\, p_i > 0,\, i = 1, 2, \ldots, m\}$ such that $\dim[u^*_\sigma]^0 = n$ for almost all $\sigma \in \Sigma$. Lemma 17 implies

*LEMMA 33. *Let* $\{\mathcal{G}^n,\ f_i,\ p_i,\ p_i > 0,\ i = 1,\ 2,\ \ldots,\ m\}$ *be a fuzzy hyperIFS, where* f_i, $i = 1,\ 2,\ \ldots m$ *are as in (39). Let* $\omega \in \Sigma$ *be such that* $\dim[u_\omega]^0 = n$, *then* $\dim[u^*_\sigma]^0 = n$ *for almost all* $\sigma \in \Sigma$.

*LEMMA 34. *Let* $u \in \mathcal{G}^n$ *be such that* $\dim([u]^0) = n$ *then* $\dim(\text{send}(u) = n + 1$.

Proof. The claim follows from the convexity of the sendograph $\text{send}(u)$ and $[u]^1 \neq \emptyset$. □

Thus, let us consider such a fuzzy hyperIFS that $\dim(\text{send}(u^*_\sigma)) = n + 1$ for almost all $\sigma \in \Sigma$. We define a set function f on the set of blocks $[a_1,\ b_1] \times [a_2,\ b_2] \times \cdots \times [a_n,\ b_n]$,

$$f(B) := \int_{\sigma\in\Sigma} \frac{\mathcal{L}^{n+1}(\text{send}(u^*_\sigma) \cap \text{send}(B))}{\mathcal{L}^{n+1}(\text{send}(u^*_\sigma))} d\rho(\sigma).$$

Observe that we can treat a set $B \in K_{Co}(\mathbb{R}^n)$ as a fuzzy set in $\mathcal{G}^n$.

In order to use the ergodic theorem, we prove integrability of $\frac{\mathcal{L}^{n+1}(\text{send}(u^*_\sigma)\cap\text{send}(B))}{\mathcal{L}^{n+1}(\text{send}(u^*_\sigma))}$. Let us denote

$$\Sigma_B = \{\sigma \in \Sigma,\ \mathcal{L}^n(\text{send}(u^*_\sigma) \cap \text{send}(B)) > 0\}.$$

Let us remind three facts. Volume of convex sets is continuous w.r.t. the Hausdorff metric (see Lemma 18), but we deal with d_∞ and d_Σ. Note that

$$d_H(\text{send}(u),\ \text{send}(v)) \leq \max(d_H([u]^\alpha,\ [v]^\alpha)) = d_\infty(u,\ v),\ \forall u,\ v \in \mathcal{G}^n,$$

follows from Lemma 22. The address function

$$\phi(\sigma) = u^*_\sigma,\ \sigma \in \Sigma,$$

is continuous w.r.t. the metrics d_∞ and d_Σ. These imply, for any $\sigma \in \Sigma$,

$$\forall\epsilon > 0\exists\delta_1 > 0 : d_H(\text{send}(u^*_\sigma),\ \text{send}(u^*_{\sigma'}) < \delta_1 \Rightarrow$$

$$\Rightarrow |\text{vol}(\text{send}(u^*_\sigma)) - \text{vol}(\text{send}(u^*_{\sigma'}))| < \epsilon,$$

$$\forall\delta_1 > 0\exists\delta_2 > 0 : d_\infty(u^*_\sigma,\ u^*_{\sigma'}) < \delta_2 \Rightarrow d_H(\text{send}(u^*_\sigma),\ \text{send}(u^*_{\sigma'})) < \delta_1,$$

$$\forall\delta_2 > 0\exists\delta > 0 : d_\infty(u^*_\sigma,\ u^*_{\sigma'}) < \delta \Rightarrow d_\Sigma(\sigma,\ \sigma') < \delta_2.$$

We obtain continuous dependence of $\mathcal{L}^n(\text{send}(u^*_\sigma))$ w.r.t. d_Σ. From Remark 64, we have also that $\mathcal{L}^{n+1}(\text{send}(u^*_\sigma) \cap \text{send}(B))$ is continuous in u^*_σ w.r.t. and d_Σ. Since Σ_B is a preimage (in $\mathcal{L}^{n+1}(\text{send}(u^*_\sigma)\cap\text{send}(B))$) of the open interval $(0, \infty)$, it is open.

Hence, we integrate a continuous function on the open set Σ_B. It follows that

$$f(B) = \int_{\sigma\in\Sigma} \frac{\mathcal{L}^{n+1}(\text{send}(u^*_\sigma) \cap \text{send}(B))}{\mathcal{L}^{n+1}(\text{send}(u^*_\sigma))} d\rho(\sigma)$$

$$= \int_{\sigma \in \Sigma_B} \frac{\mathcal{L}^{n+1}(\mathrm{send}(u^*_\sigma) \cap \mathrm{send}(B))}{\mathcal{L}^{n+1}(\mathrm{send}(u^*_\sigma))} d\rho(\sigma)$$

exists and it can be approximated by means of the ergodic theorem. It remains to define an outer measure, for all Borel sets of $\mathbb{R}^n$,

$$\mu_S(B) := \inf\{\sum f(B_i),\ B \subset \cup_i B_i,\ B_i \text{ are blocks}\}. \tag{40}$$

μ_S satisfies three conditions for an outer measure

1. $\mu_S(\emptyset) = 0$,
2. $\mu_S(A) \leq \mu_S(B)$ for $A \subset B$,
3. $\mu_S(\cup_i B_i) \leq \sum_i \mu_S(B_i)$.

In the same way as in the case of a measure on multivalued fractals, we can prove that it is a Borel measure by means of the Carathéodory criterion.

We obtain the following theorem.

*Theorem 46. *Let us consider the fuzzy hyperIFS $F = \{\mathcal{G}^n,\ f_i,\ p_i > 0\}$, such that $\dim([u^*_\sigma]^0) = n$ for almost all $\sigma \in \Sigma$. The set function μ_S defined in (40) is a Borel measure.*

Example 12. Consider the fuzzy IFS $f = \{(\mathcal{G}^2, d_\infty),\ f_i,\ p_i,\ i = 1, 2, 3\}$ and corresponding fuzzy hyperIFS $F = \{(\mathcal{G}^2, d_\infty),\ f_i,\ p_i,\ i = 1, 2, 3\}$, where $p_1 = p_2 = p_3 = \frac{1}{3}$,

$$[f_1(u)]^\alpha = \frac{1}{2}[u]^\alpha,$$

$$[f_2(u)]^\alpha = \frac{1}{2}[u]^\alpha + \left\{\left(0, \frac{1}{2}\right)'\right\},$$

$$[f_3(u)]^\alpha = \left\{\left(\frac{1}{2} + ax, by\right)',\ (x, y)' \in [u]^\alpha,\ a, b \in \left[\frac{1}{3} + \frac{1}{6}\alpha,\ \frac{1}{2}\right]\right\}.$$

For the images of the fuzzy fractal and measure on it, see Figure 50. Images of the structure of fuzzy hyperfractal and the invariant measure are in Figure 51. Notice that the 0-level set of the fuzzy attractor corresponds to the fat Sierpiński triangle and the 1-level set to the Sierpiński triangle.

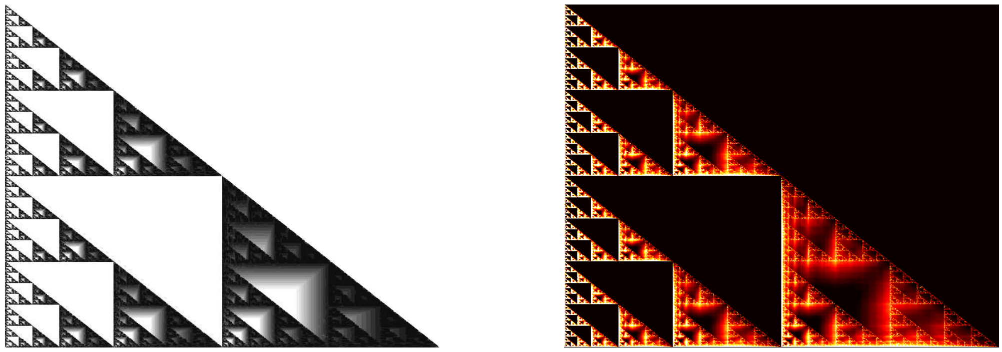

Figure 50: Fuzzy fat Sierpiński triangle and a measure from Example 12

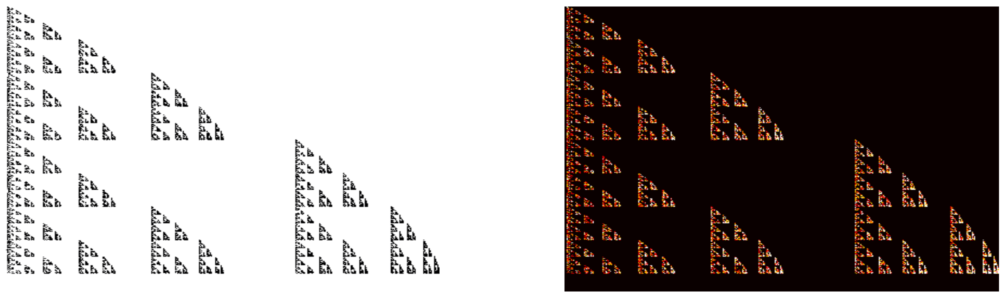

Figure 51: Projection of metric structure of the fuzzy fat Sierpiński triangle and invariant measure from Example 12

9. Summary

Let us briefly sum up our contribution to the theory of fractals and discuss open problems. The thesis connects the theories of multivalued fractals, fractals generated by iterated function systems and superfractals. We come up with the term hyperfractal. It answers the question set in [BHS1] about a multivalued analogy of attractors of super iterated function systems. Hyperfractals are attractors of iterated function systems and multivalued fractals are their shadows. Hyperfractals enable us to discuss the address structure of multivalued fractals, which is to our knowledge only implicitly present in [KLV2] and [KLMV]. We explain visual self-similarity and complexity of multivalued fractals by means of hyperfractals. We show that the set of self-similar attractors of IFSs is a subclass of shadows of self-similar hyperfractals. We also visualize structure of hyperfractals by means of support functions. Since we treat the hyperfractals as the first, their application for drawing multivalued fractals and measures supported by them applying the chaos game seems to be new. We generalized the theory to spaces of fuzzy sets. Fuzzy fractals were studied rarely. However, these fractals were not a direct generalization of multivalued fractals or hyperfractals like ours.

Thousands of articles were written about fractals but only few about this field. Hence, there is a lot of problems to investigate. Let us suggest a few.

- We found the address structure of multivalued fractals. It arises a question, which conditions are necessary to recognize the address structure of topological multivalued fractals.

- A degree of self-similarity of trees is usually found in a topological way (see e.g. [FGP]). It is worth considering also metric ways. We can regard a tree as a shadow of a hyperfractal and look for its degree of self-similarity.

- In order to apply our results to image compression, the inverse problem should be solved. It means efficient searching for an IMS or hyperIFS whose attractors are close to an original image.

- Our approach to fuzzy fractals has a disadvantage that all address fuzzy sets have nonempty levels. Therefore, generalizations should be investigated. Considering fuzzy sets as functions, the metric d_∞ does not seem natural. There are other metrics which can be applied (see [DK]).

- In the last years, hyperchaos was extensively studied. We have also treated the simplest case of hyperchaos, particularly shift dynamical system for hyperfractals. Our approach can be generalized to estimate dimension and visualize chaotic orbits in hyperspace.

List of Figures

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