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Fuzzy metody agregace v rozhodovacích úlohách

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Fuzzy methods of aggregation in decision making

A thesis submitted for the degree of Doctor of Philosophy

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Prohlášení

Prohlašuji, že na základě zadání jsem vytvořila tuto dizertační práci samostatně za vedení doc. RNDr. Jany Talašové, CSc., a že jsem v seznamu použité literatury uvedla všechny zdroje použité při zpracování práce.

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Abstract

In the decision making, which is a part of our everyday life, we often meet a situation in which we need to merge several points of view to reach a single decision. We usually have partial evaluations of the considered object and we need the overall one. The aggregation of these partial evaluations can be performed by various aggregation operators. The most widely applied one is the weighted average, but it cannot handle interacting criteria. Problems with some of these interacting criteria can be handled by the Choquet integral.

The data presented to us during a decision making problem are only rarely precise and unambiguous. More often we deal with vague descriptions, uncertain opinions and approximate values. The uncertainty burdening the input data can be processed properly with help of fuzzy sets theory. Since the input data can be transformed into fuzzy numbers, the aggregation operators, which are supposed to aggregate those data, need to be adjusted to accommodate this, i. e. they need to be fuzzified. In the thesis, the first and second level fuzzification of several aggregation operators, with extra attention devoted to the Choquet integral, is proposed. The first-level fuzzy Choquet integral handles partial fuzzy evaluations, while the second-level fuzzy Choquet integral is able to process even the uncertain weights of the sets of criteria modeled by fuzzy numbers. Together with the definitions of fuzzified aggregation operators, some theorems, which make the computation much easier, are presented.

The Choquet integral can be used only if the interactions among the criteria are of a specific type. In the cases of overly complicated interactions, the aggregation operators are recommended to be replaced by a base of fuzzy rules and a suitable approximate reasoning algorithm. In the thesis, bases of fuzzy rules are applied to a realistic issue arising in the field of clinical psychology. Two fuzzy expert systems are modeled to help the evaluator in quantitative interpretation of the results of MMPI-2 test.

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Notation and symbols

$\emptyset$ ... empty set
$\mathbb{N}$ ... set of natural numbers
$\mathbb{R}$ ... set of real numbers
$[a, b]$ ... closed interval with endpoints a and b
(a, b) ... open interval with endpoints a and b
$[a, b]^n$ ... n-th Cartesian power of interval $[a, b]$
$\wp(\Omega)$ ... power set of the set Ω
$C(x)$ ... membership degree of x in the fuzzy set C
$\operatorname{Ker} C$ ... kernel of the fuzzy set C
C_α ... α-cut of the fuzzy set C
$\operatorname{Supp} C$ ... support of the fuzzy set C
$\mathcal{F}(U)$ ... system of all fuzzy sets on the set U
$C \cap D$ ... intersection of fuzzy sets C and D
$C \cup D$ ... union of fuzzy sets C and D
$\mathcal{F}_N(\mathbb{R})$ ... system of all fuzzy numbers
$\mathcal{F}_N([a, b])$ ... system of all fuzzy numbers on the interval $[a, b]$
$f : U \to V$ ... mapping f from the set U to the set V
f^{-1} ... inverse function to function f
μ ... fuzzy measure
$\mu(U)$ ... fuzzy measure of the set U
m^μ ... Möbius transform of fuzzy measure μ
$\operatorname{Card}(U)$ ... cardinality of set U
$U \setminus V$ ... set difference
$\operatorname{Cl}(U)$ ... closure of the set U
$I^\mu(C_i, C_j)$ ... interaction index between i-th and j-th criterion
$I^\mu(U)$ ... interaction index for the set U
$\Phi(C_i)$ ... importance index of criterion C_i
$\widehat{\mu}$ ... FNV-fuzzy measure
F ... FNV-function

Chapter 1

Introduction

1.1 Decision making problem

Making the right decision is one of the most important skills a person may have. In critical situations, the right decisions can save lives and prevent disasters, but even in the everyday routine they mean the difference between making and not making a profit, or winning and loosing a game. Making the right choices is a skill, but it is a difficult skill to learn. It is therefore no surprise that a considerable attention was devoted to formalize the decision making process and apply mathematical methods to it in order to give powerful tools to the person making the decision and improve her ability to make the right one.

The decision making problem can be simply defined as a problem with more than one possible solution. Usually, the set of all these solutions is referred to as a set of alternatives or a set of actions and it is denoted by X. The set of alternatives may be finite and represented by $X = \{x_1, x_2, \ldots, x_n\}$, or it may be described analytically by systems of constrains which need to be fulfilled. In the latter case, the set of alternatives is usually infinite and the problem is solved by mathematical programming. In the rest of the thesis we will be assuming a finite set of alternatives.

Each alternative x_i, $i = 1, 2, \ldots, n$, is studied from one or, more often, from several points of view, which are called criteria, goals, or attributes. Given m criteria, the alternative x_i can be described by m values x_{ij} referred to as consequences, observations or performances of alternative x_i, $i = 1, 2, \ldots, n$, with respect to criterion C_j, $j = 1, 2, \ldots, m$. Usually it is described as $x_i =$

$(x_{i1}, x_{i2}, \ldots, x_{im})$. The consequences can be expressed in various ways. They can be real numbers, linguistic terms, or even fuzzy numbers, which will be in detail explained in chapter 3. The specific form of the consequences depends on the nature of criteria, but also on the decision making problem itself. Throughout the thesis we will focus on set of criteria, which is finite with more than one element.

In order to make a decision, the decision maker has to evaluate the alternatives. The evaluation can be of two main types. The first one is referred to as the ordinal type of the evaluation, which allows us to set a preference relation on the set of alternatives, but the numerical value, according to which the alternatives are ordered, bears no other significance. This kind of evaluation allows ordering the alternatives from the best to the worst, but it does not provide any information about the degree of difference between the consecutive alternatives. Given two alternatives x_p and x_q, the ordinal evaluation can help the decision maker to state that x_p is as good as, better, or worse than alternative x_q', but nothing else.

The second type of evaluation is called cardinal. Not only does it allow us to order the alternatives, but it also provides information about the intensity of the preference between them. The decision maker is therefore able to compare these intensities and claim, for example, that 'alternative x_p is much better than alternative x_q' or 'alternative x_p is slightly better than alternative x_q'. Moreover, if the cardinal evaluation is of the absolute character with respect to some previously established goal, then some boundaries are given (usually 0 and 1) and the decision maker is able to measure how good the alternative is with respect to the goal.

The process of evaluation is one of the main parts in the process of decision making. Its output helps the decision maker to make a decision. In the presence of more than one criterion or goal, we are talking about multiple criteria evaluation.

Decision making problem is a quite general notion covering wide range of problems with wildly differing sets of alternatives, criteria, or even decision makers. From now on we will deal with a specific combination of these, focusing on problems with a finite set of alternatives, a finite set of criteria and a single decision maker. This kind of problem will be, in short, referred to as MCDM (Multiple Criteria Decision Making) problem.

1.2 MCDM methods and techniques

There are two basic approaches to solving MCDM problems. The aim of the first one is to get the alternatives ordered and than pick the best one(s). The main idea is based on pairwise comparison of the alternatives with respect to each criterion separately. This provides the evaluator with m partial preference relations (one for each criterion), which are then aggregated in order to achieve the overall preference relation. Among the methods employing this approach, which are called the outranking methods, the best known are ELECTRE methods presented by Roy in 1968 [71] or younger PROMETHEE method presented in 1982 by Brans [11]. Both of the methods are still in active use and under development. For example, one of the big disadvantage of ELECTRE methods was overcome in 2009, when Figueira, Greco and Roy presented a modification of ELECTRE, which allowed tackling MCDM problems with interacting criteria [23]. For more detailed insight on outranking methods see, for example, [22], or [12].

The second approach to MCDM problems is based on one simple principle. Each alternative is assigned m different partial evaluations, one for each criterion, which are then consolidated into a single overall evaluation by a process called aggregation. Unlike the previous treatment, where we compared different alternatives among themselves in order to find out the ordering, i. e. the ordinal type of evaluation, here we to each alternative assign an evaluation, expressing not just if one alternative is better than another one, but also by how much. This is the, so called, cardinal type of evaluation.

There are many different and unconnected ways of actually assigning, aggregating, and interpreting the partial evaluations. One of the main approaches is the multiple attribute utility theory (MAUT) approach, the basic principle of which was originally put forth in 1947 by Von Neumann and Morgenstern in [88]. The first application of those principles to MCDM was presented in 1970 by Fishburn [24], but a concise theory came into wide acceptance only after it was presented in the work of Keeney and Raiffa [48]. We kindly invite any reader interested in more details on the MAUT approach to seek an exhaustive survey by Dyer in [21]. Other than MAUT, some methods providing the cardinal type of evaluation are, for example Saaty AHP [72] and its generalization, the ANP method [73] (which can also handle interactions between the criteria) or Partial

Goals Method (PGM) presented in [79].

In both the approaches usable to solve MCDM problems, the ordinal as well as the cardinal, there is a mutual and very important step to be taken. It is the aggregation of the partial elements. In the ordinal methods, the partial preferences are aggregated in order to obtain the overall preference. In cardinal methods, the overall evaluation of the alternative is a result of aggregating the partial ones. The most widely used aggregation operator, and certainly the best known, is weighted average. However, because of its limited applicability, some other aggregation operators have found their way into the MCDM methods: fuzzy integrals [45], [63], OWA operators [95], triangular norms [49], symmetric sums [75], uninorms [27] or null-norms [14]. Since then, the application of different aggregation operators in MCDM has been studied [15], [28], [30], [32], [38], [53], [55], [56], [57], [83].

In 1965, Zadeh introduced fuzzy sets [97], which have soon became a new and powerful tool for modeling vagueness and uncertainty. Soon after, the fuzzy sets were also applied to MCDM. In 1970, Bellman and Zadeh, [10], formed a first link between fuzzy sets and MCDM by proposing a, so called, fuzzy decision. The fuzzy decision is in the form of intersection of fuzzy sets, which are modeling the individual established goals on the set of alternatives. Others have employed a more direct route. Instead of devising an entirely novel method, they sought to adjust the existing aggregation operators in order to make them compatible with the newly devised fuzzy numbers. These new, fuzzified operators, could then be effectively applied even to problems exhibiting uncertainty.

The aggregation operator, which attracted the most attention, was the weighted average. The first fuzzy weighted average was presented in 1977 [3], but others soon followed [19], [40], [65]. The OWA operators were partially fuzzified later. In 2000 Mitchell and Schaefer [61] introduced OWA operator capable of aggregating fuzzy numbers. However, it worked only with crisp weights and the computation was quite complicated. Not even the fuzzy integrals managed to avoid the fuzzificaton, which lead to fuzzified fuzzy integrals. The fuzzified Sugeno integral was proposed by Wu et al. in 1998 [93] and [94]. Later, the Choquet integral was modified for fuzzy integrand by Meyer and Roubens (2006) [59], Yang et al. (2005) [96] or Wang (2006) [91].

Apart from the aggregation operators, whole MCDM methods were fuzzified as well. For example, in 1983, fuzzy extension of Saaty's AHP was introduced [87], in 2000, the PROMETHEE method was extended to accommodate uncertain input [29], and later, in 2007, fuzzy ELECTRE was presented [69].

Finally, there is another important tool brought to the MCDM field by fuzzy sets theory. A linguistically defined function in the form of base of fuzzy rules, together with approximate reasoning algorithms, allows to model any MCDM problem burdened with uncertainty or interacting criteria. Some interesting application of fuzzy rules can be found in [79] or [86].

1.3 Goals of the thesis

The theses aims to accomplish several goals.

We started, at the beginning of the introduction, by introducing the MCDM problem. One of the most important steps encountered in the process of solving this kind of a problem is the evaluation of alternatives performed by an aggregation operator. The first aim of the thesis is to study aggregation operators and their utilization in the models of MCDM.

In reality, most of the MCDM problems are burdened by uncertainty arising from imperfect data acquisition methods and vagueness of a human factor. In the formal treatment of this uncertainty, partial evaluations are modeled by fuzzy numbers and the aggregation operators should be able to handle them. Although some of the aggregation operators were already fuzzified, most of them passed only through the partial, so called, first-level fuzzification. The second aim of the thesis is to fully fuzzify selected aggregation operators, specifically the Choquet integral, and to propose a way of their effective calculation.

Finally, one of the amazing tools provided by fuzzy sets theory is a linguistically defined function. With help of several fairly simple "If-then" rules it is able to describe any relationship, even a fairly complicated one, among the input and output variables. Then, together with proper selection of an approximate reasoning algorithm, it can be used in the process of multiple criteria evaluation. The third aim of the thesis is to study the influence of various aggregation operators on the fuzzy rules base performance and to apply it on a real MCDM problem.

1.4 Structure of the thesis

The structure of the thesis corresponds to the established goals. After a short introduction, a main part of the thesis begins. In the chapter 2, aggregation operators and their properties are described. We focus specifically on the special forms of the Choquet integral and the problems with its application in multiple criteria decision making. At the end of the chapter we take the first step in generalization of the Partial Goals Method and present a new algorithm for constructing fuzzy measures.

Next chapter, chapter 3, is devoted to fuzzification of aggregation operators introduced in chapter 2. In the first part of the chapter we present basic notions and theorems of fuzzy sets theory. The second part of the chapter then treats the fuzzification of the aggregation operators. There are two levels of fuzzification. The first-level fuzzification deals with uncertain partial evaluations and crisp weights of criteria. Whilst in the second-level fuzzification both of them, partial evaluations as well as the weights of criteria, are uncertain and modeled by fuzzy numbers. The chapter ends with complete Generalized Partial Goals Method and with a proposal of FNV-fuzzy measure construction method, including the algorithmic description.

The final chapter 4 brings the application of the fuzzy sets approach to a real MCDM problem. Two bases of fuzzy rules are created to solve a problem arising in psychology - interpretation of the MMPI-2 tests.

Chapter 2

Aggregation operators and their application in MCDM

One of the most important steps in the process of multi-criteria decision making is the evaluation. During the evaluation, the n partial evaluations or n partial preferences are replaced by one value, which then helps the decision maker to arrive at the decision and thus solve the MCDM problem. This formidable task is the responsibility of aggregation operators, which are going to be the main focus of the following chapter.

At first we put forth the underlaying definitions and present some rudimentary properties. After this general introduction we move towards and discuss behavior of aggregation operators, which are seeing the most use in decision making problems - weighted average, OWA operators and the Choquet integral. Of these three, we are most interested in the Choquet integral. Apart from its usual application we show how it can be joined with the generalized Partial Goals Method (PGM). To complete the treatment, we finish the chapter by proposing a new technique of constructing a fuzzy measure, which is a crucial part of the Choquet integral approach.

2.1 Aggregation operators - definition and properties

There are several definitions of aggregation operator. They differ author from author [15], [20], [55], [70]. In the following text we will use the one below [70]. Since the partial evaluations are either from the interval $[0,1]$, or they can be easily mapped on it, we will, in the rest of the thesis, restrict ourselves to the interval $[0,1]$.

Definition 2.1 *Aggregation operator A is a sequence $\{A_n\}_{n=1}^{\infty}$ of mappings (called aggregating mappings) $A_n : [0,1]^n \to [0,1]$ such that*

- $A_1(h) = h$ *for each* $h \in [0,1]$*;*
- $A_n(0,0,\ldots,0) = 0$ *and* $A_n(1,1,\ldots,1) = 1$ *for every* $n = 2,3,\ldots$*;*
- $A_n(h_1,\ldots,h_n) \leq A_n(g_1,\ldots,g_n)$ *whenever* $h_i \leq g_i$*, for each* $i = 1,2,\ldots,n$ *and every* $n = 2,3,\ldots$*.*

The three conditions in Definition 2.1 represent three quite naturally arising requirements. The first condition is called identity unary operation and it corresponds to aggregation of a singleton. The second condition is known as boundary condition. Simply put, any aggregation operator should give very bad output, if all the input values are very bad, and, in complete analogy, if all the input values are very good, the aggregated output should be very good as well.

The third condition guaranties the monotonicity with respect to each argument. If any part of the input improves, while the remaining input parameters remain fixed, the output value produced by aggregation operator should not get worse.

There is another property sometimes required from a definition of an aggregation operator: continuity.

Definition 2.2 *[70] Let $A = \{A_n\}_{n=1}^{\infty}$ be an aggregation operator. The aggregation operator A is called continuous, if for each $n \geq 2$ the aggregating mapping A_n is a continuous function of n variables.*

We understand the continuity of n-ary aggregating mapping A_n as classical continuity of any multivariate function. Continuity of n-ary aggregating mappings then guarantees that small changes in the input lead again only to small changes in the output (and in the limit of zero change the output remains unchanged).

There are, of course, other important properties, which can be extremely useful in describing the behavior of particular aggregation operators. The two most prominent are idempotence and symmetry, which are defined as follows:

Definition 2.3 *[70] Let $A = \{A_n\}_{n=1}^{\infty}$ be an aggregation operator. The aggregation operator A is called idempotent, if for each $n \in \mathbb{N}$ the aggregating mapping A_n is idempotent, i. e.*

$$A_n(h, \ldots, h) = h \text{ for all } h \in [0, 1]. \tag{2.1}$$

Idempotence is a typical property of aggregation operators which represent averages of various kinds. That is why the idempotent aggregation operators are sometimes called averaging aggregation operators [15]. If an aggregation operator is idempotent, then for all $n \in \mathbb{N}$ and for all $h_1, \ldots, h_n \in [0, 1]$ the following holds

$$\min\{h_1, \ldots, h_n\} \leq A(h_1, \ldots, h_n) \leq \max\{h_1, \ldots, h_n\}. \tag{2.2}$$

The last of the properties, which will be mentioned during the text of the thesis, is symmetry.

Definition 2.4 *[15] Let $A = \{A_n\}_{n=1}^{\infty}$ be an aggregation operator. The aggregation operator A is called a symmetric aggregation operator, if for all $n \in \mathbb{N}$ and for all $h_1, \ldots, h_n \in [0, 1]$, the following holds*

$$A_n(h_1, \ldots, h_n) = A_n(h_{\alpha(1)}, \ldots, h_{\alpha(n)}), \tag{2.3}$$

for all permutations $\alpha = (\alpha(1), \ldots, \alpha(n))$ of $(1, \ldots, n)$

In multiple criteria decision making, the symmetric aggregation operators are usually used if all the criteria are of the same importance.

2.2 Weighted average

Weighted average is one of the best known aggregation operators and it is certainly the one most often applied.

Definition 2.5 *Weighted average of real numbers* $h_1, \ldots, h_n$, $h_i \in [0,1]$, $i = 1, 2, \ldots, n$ *with weights* $w_1, \ldots, w_n$, $w_i \geq 0$, $i = 1, 2, \ldots, n$, *is a real number* $y \in [0,1]$ *given by*

$$y = \frac{\sum_{i=1}^{n} w_i h_i}{\sum_{i=1}^{n} w_i}. \tag{2.4}$$

If the weights are normalized, i.e. $\sum_{i=1}^{n} w_i = 1$, *then the formula 2.4 is simplified to*

$$y = \sum_{i=1}^{n} w_i h_i. \tag{2.5}$$

Let us analyze the formula 2.5 a little bit. From MCDM point of view, the weights w_i, $i = 1, 2, \ldots, n$, enable us to express the various importance of the given criteria. To each criterion C_i there is a weight $w_i \geq 0$, $i = 1, 2, \ldots, n$ assigned according to one basic rule: the more important the criterion is, the bigger its weight should be. Values h_i, $i = 1, 2, \ldots, n$ now stand for partial evaluations of the object with respect to the particular criteria C_i, $i = 1, 2, \ldots, n$. Partial evaluations are supposed to be standardized: they have to refer to the same scale (percentages, points, degrees of fulfillment, etc.). Most often they come from the unit interval and we shall keep this custom.

Weighted average is the most popular aggregation operator used in many MCDM methods. As an example, we can name method of MAUT [48], [24], Saaty AHP [72] or Partial Goals Method (PGM) [79]. According to [15] it is a continuous and idempotent operator, which is not symmetric. Application of the weighted average operator is straightforward, there are some limitations, however, which must be observed.

First, it is safe to use weighted average only, when the criteria describing the MCDM problem do not interact. The application of weighted average on the problem with interacting criteria may lead to confusing and erroneous results.

Second, weighted average is suitable only for problems, where the weights of the criteria are related solely to the criteria (evaluations with respect to these criteria) themselves, and they do not depend on the actual values of the evaluations.

Finally, the weights of the criteria should differ. If they do not differ, then the evaluator cares more about the actual highs of the partial evaluations than about the particular criteria connected to them. In such the case, another aggregation operator, for example the OWA operator, should be used.

Example 2.1 *Let us consider two students A and B. Their teacher is reciting the alphabet and students are supposed to write the letters down on the paper. Let us suppose we want to evaluate a student's ability to write down individual letters. The alphabet consists of 26 letters, which represent 26 criteria in multi-criteria evaluation problem. All the letters are of the same importance, therefore the weights of the criteria are equal to* $\frac{1}{26}$*. The teacher evaluates each written letter with value from* $[0, 1]$*, where 0 represents the worst and 1 the best evaluation.*

Student A knows all the letters, but her handwriting is poor. The symbols are barely recognizable. Therefore, the partial evaluations of the student can be expressed as $(0.5, \ldots, 0.5)$*.*

Student B knows only half of the letters, but she can write down the letters she knows in perfect handwriting and is rewarded by evaluations of 1. The other letters, the missing ones, are evaluated by 0. The vector of partial evaluations can be then expressed as $(1, \ldots, 1, 0, \ldots, 0)$*.*

If we use weighted average to aggregate students' partial evaluations, both the students will be evaluated equally by number 0.5. And yet, there is a big difference between them. Student A can, even if imperfectly, write any text she needs, while student B can't!

This example demonstrates that weighted average is not always a good choice. In the following, we shall look at another operator, which is better suited for this kind of tasks. It is called the OWA operator.

2.3 OWA

Ordered weighted average operators have been introduced by Yager in 1988 [95]. They are usually referred to as OWA operators and we define them as follows:

Definition 2.6 *Ordered weighted average (OWA) of real numbers* $h_1, \ldots, h_n$, $h_i \in [0,1]$, $i = 1, 2, \ldots, n$, *with normalized weights* $w_1, \ldots, w_n$, $w_i \geq 0$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^{n} w_i = 1$, *is a real number* $y \in [0,1]$ *given by*

$$y = \sum_{i=1}^{n} w_i h^{(i)}, \tag{2.6}$$

where $(1), \ldots, (n)$ *is a permutation of indices, such that* $h^{(1)} \geq h^{(2)} \geq \cdots \geq h^{(n)}$.

Compared to the weighted average, there is one big difference. The aggregated values are reordered before they enter the calculation. It can be seen, in the formula 2.6, that while the values $h_i, i = 1, \ldots, n$, are reordered, the weights $w_i, i = 1, \ldots, n$, stay in the same order. Here, the weights are not tied to the particular criteria, but to the ordering of the partial evaluations. In other words, the output given by the OWA operator depends only on the actual values of the partial evaluations.

Because of the properties described in the previous paragraph, the OWA operators are suitable for situations in which the criteria are indifferent. If the criteria are of the same importance, we can focus solely on the structure of the actual partial evaluations, disregarding the connection between the partial evaluations and the criteria.

OWA operators are commutative, monotonic, continuous and idempotent [25]. They belong to the family of the averaging aggregation operators.

OWA operators can take many forms. By changing the vector of weights we can obtain a wide range of aggregation operators, some of which are well known even to people not interested in the theory of aggregation operators. Perhaps the best example of this are the operators minimum and maximum [28].

2.3.1 Maximum

Given a vector of normalized weights, such that $w_1 = 1$ and $w_i = 0$, $i = 2, 3 \ldots, n$, the OWA operator becomes maximum operator, i. e.

$$OWA(h_1, \ldots, h_n) = \max(h_1, \ldots, h_n). \tag{2.7}$$

In the formal logic, the maximum operator is used to model the logical operator "or". The truth degree of a logical disjunction corresponds to the maximum of the truthfulness degrees of all particular operands. In multiple criteria decision making, the maximum operator is applied when all of the criteria are interchangeable and it does not matter which one of them is evaluated highly. If we want to model the requirement "at least one criterion has to be fulfilled", then we look for the maximum.

When we apply the maximum operator, we should be aware that the alternatives $(1, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$ will be evaluated equally. As a consequence, the alternative $(1, 0, \ldots, 0)$ will be evaluated better than the alternative $(0.99, 0.99, \ldots, 0.99)$ and we should be prepared to accept that.

2.3.2 Minimum

If a vector of normalized weights is given as $w_n = 1$, $w_i = 0$, $i = 1, 2, \ldots, n-1$, then OWA operator is equal to minimum operator, i. e.

$$OWA(h_1, \ldots, h_n) = \min(h_1, \ldots, h_n). \tag{2.8}$$

In the theory of logic, the minimum operator is used to model logical operator "and". Logical conjunction is true if and only if all of its operands are true. Its truthfulness degree is equal to the minimum of truthfulness degrees of the particular operands. In multiple criteria decision making, the minimum operator serves well to model the requirement "All of the criteria should be fulfilled". Minimum expresses the following idea: An object is evaluated highly, only if all of its partial evaluations are high.

The minimum operator does, similarly to maximum, exhibit some peculiarities in it's behavior and the evaluator should be ready to take them into an account. When using minimum as an aggregation tool, we should expect the same evaluation for the alternative $(0, 1, \ldots, 1)$ as for the alternative $(0, 0, \ldots, 0)$. As another example, if we compare two alternatives $(0.01, 0.01, \ldots, 0.01)$ and $(0, 1, \ldots, 1)$ using minimum, the first option will be preferred to the alternative.

Maximum and minimum operators are only two extremal forms of the OWA operators. The particular forms depend on the weighting vector. According to [70], by appropriate choice of the weighting vector, OWA operators can also find and bring into the forefront any k-th smallest partial evaluation, i.e. it can, for example, also turn into median. The application of OWA operators in MCDM was studied thoroughly in [28].

The popularity of OWA operators can be demonstrated by an example of their practical application. Today, the overall evaluation of a figure skater is computed by OWA. At first, the performance of the figure skater is evaluated by nine experts. Each expert evaluates the performance with one mark. Then, two of the marks are eliminated and only seven marks are used to determine the overall evaluation. The marks, which are eliminated, are the smallest and the highest partial evaluations. The overall evaluation is calculated as an average of the rest of the marks. This procedure ensures that large variation of the evaluations is avoided.

Example 2.2 *Let us imagine a swimming course, in which children are taught four swimming styles: breaststroke, front crawl, butterfly and backstroke. At the end of the course, the children are evaluated with respect to each particular swimming style, i.e. four particular evaluations are assigned to each child. Two different questions can now arise:*

1. *Did the child learn to swim?*
2. *Is the child talented enough to participate in the swimming competition, where all four swimming styles are needed?*

As for the first question, if we want to find out if the child can swim or not, we usually do not care which swimming style is the child's preferred one. The child does not need to manage all four swimming styles to avoid drowning. In such the case we will be happy, if the child manages one style, not mattering which, and the right aggregation operator would then be maximum.

The situation changes with the second question. Now we are not satisfied with a child who can only swim in one style. Now we want more. We are looking for a child who can compete in all four swimming styles. Since the child must be good at all four styles, we choose minimum to evaluate child's performance.

2.4 Fuzzy integrals - Choquet and Sugeno integrals

The concept of fuzzy integrals was first introduced in 1953 by Gustave Choquet in his Theory of capacities [18]. The contemporary concept of the fuzzy integral, as integral with respect to fuzzy measure, was initially proposed by Höhle in [45] and later expanded upon by Murofushi and Sugeno [63].

Important role in definition of the fuzzy integral is played by fuzzy measure, which was introduced by Sugeno in 1974 [77] as a non-additive monotonous set function.

Definition 2.7 *A fuzzy measure on a finite nonempty set* Ω, $\Omega = \{C_1, \ldots, C_n\}$,*is a set function* $\mu : \wp(\Omega) \to [0, 1]$, *where* $\wp(\Omega)$ *is a power set of* Ω, *satisfying the following conditions:*

- $\mu(\emptyset) = 0$, $\mu(\Omega) = 1$
- $A \subseteq B$ *implies* $\mu(A) \leq \mu(B)$ *for any* $A, B \in \wp(\Omega)$.

Clearly, the fuzzy measure is a generalized measure, where the condition of additivity was replaced by weaker condition of monotonicity. This simple modification enabled us to define two of the so called fuzzy integrals, the Choquet integral and the Sugeno integral:

Definition 2.8 *Let* $\Omega = \{C_1, \ldots, C_n\}$ *be a finite nonempty set,* μ *be a fuzzy measure on* Ω, *and* $f : \Omega \to [0, 1]$, *the discrete Choquet integral of* f *is then defined as follows:*

$$(C)\int_{\Omega} f\, d\mu = \sum_{i=1}^{n} [f(C_(i)) - f(C_{(i-1)})]\mu(B_{(i)}), \tag{2.9}$$

where $(1), (2), \ldots, (n)$ *is a permutation of indices* $1, 2, \ldots, n$ *such that* $f(C_{(1)}) \leq f(C_{(2)}) \leq \cdots \leq f(C_{(n)})$, $B_{(i)} = \{C_{(i)}, C_{(i+1)}, \ldots, C_{(n)}\}$ *and for* $f(C_{(0)}) = 0$ *by convention.*

Definition 2.9 *Let* $\Omega = \{C_1, \ldots, C_n\}$ *be a finite nonempty set,* μ *be a fuzzy measure on* Ω, *and* $f : \Omega \to [0, 1]$, *the discrete Sugeno integral of* f *is then* *defined as follows:*

$$(S)\int_{\Omega} f\,d\mu = \max\{\min\{f(C_{(i)}),\mu(B_{(i)})\}|i=1,2,\ldots,n\}, \tag{2.10}$$

where $(1),(2),\ldots,(n)$ *is a permutation of indices* $1,2,\ldots,n$ *such that* $f(C_{(1)}) \leq f(C_{(2)}) \leq \cdots \leq f(C_{(n)})$, $B_{(i)} = \{C_{(i)}, C_{(i+1)}, \ldots, C_{(n)}\}$.

Both the definitions 2.8 and 2.9 work with function f, the values of which are real numbers from the interval $[0,1]$. The fuzzy integrals can be defined for any real function as in, for example, [58]. Nevertheless, because of the theme of the thesis, we restrict ourselves to partial evaluations with values from the interval $[0,1]$. The connection between the functional values and the partial evaluations will be explained later.

Remark 2.1 *Comparing the definitions of the Choquet and Sugeno integrals, we can see that while the Choquet integral is defined with help of linear operators addition and multiplication, the Sugeno integral is defined by non-linear operators maximum and minimum.*

The definition of the Choquet integral can be also rewritten in equivalent form [17], which might be suitable for certain applications. This alternative form arises from our ability to transform fuzzy measures into the form of multilinear polynomials:

Definition 2.10 *[35] Let* μ *be a fuzzy measure on* Ω, $\Omega = \{C_1,\ldots,C_n\}$. *The Möbius transform of* μ, *denoted by* m^{μ}, *is the unique solution of the equation*

$$\mu(A) = \sum_{B\subseteq A} m^{\mu}(B),\ \forall A \subseteq \Omega, \tag{2.11}$$

given by

$$m^{\mu}(A) = \sum_{B\subseteq A} (-1)^{\mathrm{Card}(A\setminus B)}\mu(B). \tag{2.12}$$

The alternative form of the Choquet integral then looks as:

Theorem 2.1 *Let* $\Omega = \{C_1,\ldots,C_n\}$ *be a finite nonempty set,* μ *be a fuzzy measure on* Ω, *and* $f:\Omega \to [0,1]$. *Any Choquet integral of* f *can be then written* *as*

$$(C)\int_\Omega f\,d\mu = C_\mu(f(C_1),\dots,f(C_i)) = \sum_{A\subseteq\Omega} m^\mu(A)\cdot\min\{f(C_i)\mid C_i\in A\}, \quad (2.13)$$

where m^μ *is the Möbius transform of* μ*.*

Proof: See [57]. ■

When the fuzzy measure is additive, the Choquet integral becomes Lebesgue integral. In [30] Grabisch has proved the property formulated in the following theorem 2.2. The theorem guarantees that Sugeno and Choqet integrals always satisfy the condition 2.2.

Theorem 2.2 *The Sugeno and Choquet integrals are idempotent, continuous, monotonically nondecreasing operators.*

Proof: See [30]. ■

Fuzzy integrals have become very popular aggregation operators. Among the applications of Sugeno integral we can find prediction of wood strength [47], computer vision [78], or human reliability analysis [92]. In [84] the connection between the Hirsh index and Sugeno integral has been noted.

The Choquet integral has also been applied to many various areas [52]. It has found its use in pattern recognition [31], speech recognition [16], hand writing recognition [68], classification [37], evaluation of color images [82], design of audio speakers [46], or root dispersal models [64]. Aside of that, the Choquet integral has been successfully applied in multiple criteria decision making and multiple criteria evaluation [38], which is the main application studied in this thesis.

Let us look over the Definition 2.8 again and describe the formula 2.9 with respect to MCDM. In decision making, Ω represents the set of all partial goals or criteria, $\mu(A)$, $A\in\wp(\Omega)$, is interpreted as the weight of the set of partial goals (criteria) A, the values $f(C_i)=h_i$ represent partial evaluations with respect to the i-th criterion, and the value of the integral is interpreted as the overall evaluation.

Remark 2.2 *Because of its wide use in MCDM as an aggregation operator for partial evaluations* $f(C_i)=h_i$, $i=1,2,\dots,n$, *the symbol* $(C)\int_\Omega f\,d\mu$ *is often replaced by* $C_\mu(h_1,\dots,h_n)$.

With help of the new notation, the Choquet integral C_μ can be rewritten in another, for MCDM applications more illustrative, form:

$$C_\mu(h_1, \ldots, h_n) = h_{(n)}\mu(B_{(n)}) + \sum_{i=1}^{n-1} h_{(i)} \left[\mu(B_{(i)}) - \mu(B_{(i+1)})\right], \tag{2.14}$$

where $(1), (2), \ldots, (n)$ *is a permutation of indices* $1, 2, \ldots, n$ *such that* $f(C_{(1)}) \leq f(C_{(2)}) \leq \cdots \leq f(C_{(n)})$, $B_{(i)} = \{C_{(i)}, \ldots, C_{(n)}\}$, $i = 1, \ldots, n$.

There are two main reasons for the popularity of the Choquet integral. First, Choquet integral is capable of handling interactions among the criteria. Since in the real world almost everything is, to a certain extent, connected to anything, complete independence is always a theoretical abstraction. An approach, which allows us to model at least some interactions and thus move towards realistic description is therefore imminently useful. The second important property of the Choquet integral is that all of the commonly used aggregation operators presented in the previous sections can be obtained from it by a specific choice of parameters. Let us now look at the two features in greater detail.

2.4.1 Interactions among criteria

The Choquet integral can help the evaluator model two main kinds of interactions among the criteria: redundancy and synergy. In case of redundancy, the partial goals overlap and the corresponding criteria act as substitutes for each one to some extent. When redundancy appears, achieving just few selected partial goals is often enough to evaluate the alternative quite highly. The interactions among the criteria are described by a fuzzy measure.

Given two criteria, C_i and C_j, and fuzzy measure μ, then relation $\mu(\{C_i, C_j\}) < \mu(C_i) + \mu(C_j)$ suggests the criteria are redundant.

The fuzzy measure with property

$$\mu(A \cup B) \leq \mu(A) + \mu(B) \quad \forall A, B \in \wp(\Omega), A \cap B = \emptyset. \tag{2.15}$$

is called a subadditive fuzzy measure. The subadditive fuzzy measure implies redundancy.

The second type of interactions we consider is synergy. Synergy occurs, when fulfillment of some combination of partial goals from the given set of partial goals brings some additional value to the overall evaluation. If any partial goal from the combination is not fulfilled, then the overall evaluation of the alternative is reduced by more than just by the weight of the unfulfilled partial goal.

The synergy between the criteria can also be identified with help of a fuzzy measure. When, for a pair of criteria C_i and C_j and fuzzy measure μ, the inequality $\mu(\{C_i, C_j\}) > \mu(C_i) + \mu(C_j)$ holds, then there is a synergy between these criteria.

The synergy is implied by a supperaditive fuzzy measure [32]. This is a fuzzy measure fulfilling the following condition:

$$\mu(A \cup B) \geq \mu(A) + \mu(B) \quad \forall A, B \in \wp(\Omega), A \cap B = \emptyset. \tag{2.16}$$

Definition 2.11 *[62] Let μ be a fuzzy measure on $\Omega = \{C_1, \ldots, C_n\}$. The interaction index between two criteria C_i and C_j is*

$$I^{\mu}(C_i, C_j) = \sum_{A \subseteq \Omega \setminus \{C_i, C_j\}} \frac{(\mathrm{Card}(A))!(n - \mathrm{Card}(A) - 2)!}{(n-1)!} \cdot (\Delta_{ij}\mu)(A), \tag{2.17}$$

where

$$(\Delta_{ij}\mu)(A) = (\mu(A \cup \{C_i, C_j\}) + \mu(A) - \mu(A \cup \{C_i\}) - \mu(A \cup \{C_j\})). \tag{2.18}$$

The interaction index describes not only the intensity of the interaction between two criteria, but also the type of the interaction. The values of the interaction index ranges in $[-1, 1]$. For a pair of redundant criteria the interaction index is negative. Analogically, for a couple of synergic criteria the interaction index is calculated as positive. If the interaction index is equal to zero, then there are no interactions between the particular pair of criteria. The interaction index between two criteria has been extended by Grabisch to the interaction index for any set of criteria [33].

Definition 2.12 *Let μ be a fuzzy measure on $\Omega = \{C_1, \ldots, C_n\}$. The interaction* *index for any $A \subset \Omega$ is given by*

$$I^{\mu}(A) = \sum_{B \subset \Omega \setminus A} \frac{(n - \mathrm{Card}(B) - \mathrm{Card}(A))!(\mathrm{Card}(B))!}{(n - \mathrm{Card}(A) + 1)!} \sum_{K \subset A} (-1)^{\mathrm{Card}(A \setminus K)} \mu(B \cup K) \tag{2.19}$$

Because of the interactions allowed among the criteria, the overall importance of particular criterion C_i can not be simply given by fuzzy measure of $\{C_i\}$ alone. The full information about the criterion C_i is hidden in the values of all $\mu(A)$ such that $C_i \subseteq A$.

Definition 2.13 *Let μ be a fuzzy measure on $\Omega = \{C_1, \ldots, C_n\}$. The importance index or Shapley value Φ of criterion C_i with respect to μ is defined by [74]*

$$\Phi(C_i) = \sum_{A \subseteq \Omega \setminus \{C_i\}} \frac{(\mathrm{Card}(A))!(n - \mathrm{Card}(A) - 1)!}{n!} \left[\mu(A \cup \{C_i\}) - \mu(A)\right]. \tag{2.20}$$

Moreover, the Choquet integral is able to model the veto and favor effects [32].

Definition 2.14 *Suppose C_μ is a Choquet integral being used for a multiple criteria decision making problem.*

- *A criterion C_i is a veto for C_μ, if for any $(h_1, \ldots, h_i, \ldots, h_n) \in [0,1]^n$ holds*

$$C_\mu(h_1, \ldots, h_n) \leq h_i. \tag{2.21}$$

- *A criterion C_i is a favor for C_μ, if for any $(h_1, \ldots, h_i, \ldots, h_n) \in [0,1]^n$ holds*

$$C_\mu(h_1, \ldots, h_n) \geq h_i. \tag{2.22}$$

In other words, if the partial evaluation with respect to the veto criterion is high, then it does not influence the overall evaluation. On the other hand, if the partial evaluation with respect to the veto criterion is low, then also the overall evaluation will be low regardless of the rest of the partial evaluations.

Analogically, the low partial evaluation with respect to favor criterion has no effect on the overall evaluation. The favor criterion becomes important only if the partial evaluation with respect to it is high. Then the overall evaluation should be high as well and the rest of partial evaluations are not considered.

Criterion which is veto and favor at the same time is called a dictator. If criterion C_i is a dictator for C_μ, then $C_\mu(h_1, \ldots, h_i, \ldots, h_n) = h_i$ for any n-tuple $(h_1, \ldots, h_i, \ldots, h_n) \in [0,1]^n$.

According to [32] veto and favor effect can be modelled by the Choquet integral with special choice of fuzzy measure μ. If C_i is supposed to be veto for the decision making problem, then we can take the fuzzy measure μ with the property $\mu(A) = 0$ for any A such that $C_i \notin A$. Similarly, if C_i is considered as favor criterion for given decision making problem, then it suffices to choose the fuzzy measure μ such that $\mu(A) = 1$ for all sets A containing criterion C_i.

In 1998 Marichal [55] has defined veto and favor indices. They allow us to measure how much any given criterion can act as veto or favor criterion.

Definition 2.15 *Let μ be a fuzzy measure on $\Omega = \{C_1, \ldots, C_n\}$.*

- *The veto index for a Choquet integral C_μ and criterion $C_i \in \Omega$ is given by*

$$\mathrm{veto}(C_\mu, C_i) = 1 - \frac{1}{n-1} \sum_{K \subset \Omega \setminus \{C_i\}} \frac{(n - \mathrm{Card}(K) - 1)!(\mathrm{Card}(K))!}{(n-1)!} \mu(K). \tag{2.23}$$

- *The favor index for a Choquet integral C_μ and criterion $C_i \in \Omega$ is given by*

$$\begin{aligned} &\mathrm{favor}(C_\mu, C_i) = \\ &= \frac{1}{n-1} \sum_{K \subset \Omega \setminus \{C_i\}} \frac{(n - \mathrm{Card}(K) - 1)!(\mathrm{Card}(K))!}{(n-1)!} \mu(K \cup \{C_i\}) - \frac{1}{n-1}. \end{aligned} \tag{2.24}$$

Example 2.3 *Let us suppose we want to evaluate high school graduates' aptitude for study of science. The evaluation can be based on their results of mathematics (M), physics (Ph) and Chemistry (Ch) tests. Although these three disciplines differ from each other, they have a lot in common. Because of this overlap the evaluation obtained by the weighted average can be misleading. Fortunately, we can use the aggregation by Choquet integral instead.*

The overlap of the subjects can be observed by looking at the respective weights assigned to particular tests and groups of tests: $w_M = 0.5$, $w_{Ph} = 0.45$, $w_{Ch} = 0.4$,

$w_{M,Ph} = 0.8$, $w_{Ph,Ch} = 0.7$, $w_{M,Ch} = 0.75$, $w_{M,Ph,Ch} = 1$ *(see Fig.3.8). The weight of each subject is determined as the evaluation of a student with perfect score from this subject and zeroes from the rest of subjects.*

The tests' results are given by percentage of successfully answered questions. Then the overall evaluation Y can be obtained with help of the Choquet integral. For example, a student who achieved partial evaluations of the tests in the form $h_M = 0.9$, $h_{Ph} = 0.5$ and $h_{Ch} = 0.2$, is then evaluated as follows

$$Y = h_{Ch}[1 - w_{M,Ph}] + h_{Ph}[w_{M,Ph} - w_M] + h_M w_M. \tag{2.25}$$

Considering the weights and the tests' results, we evaluate the student by number $Y = 0.64$.

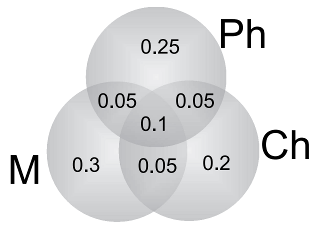


Figure 2.1: Fuzzy measure defined on the set of particular tests M, Ph, Ch.

Remark 2.3 *The Choquet integral can not be used, if the interactions among the criteria depend on particular values of the given criteria. As an example, we can present the problem of meal evaluation. A meal could be evaluated with respect to two criteria: "Meat" and "Side-dish". Note, that there are no global interactions among the criteria "Meat" and "Side-dish", but the evaluation of every meal depends on the combination of their values, e.g. we like fish with chips, but not fish with rice. Here, the type of the interaction between the two criteria has changed from synergy to redundancy.*

2.4.2 Transformations of the Choquet integral

In addition to its ability to handle the interactions among the criteria, the Choquet integral is known for its other property. The Choquet integral is a natural generalization of all the aggregation operators mentioned in the previous sections. By proper choice of fuzzy measure the Choquet integral can be reduced to minimum, maximum, OWA operators, or weighted average ([26], [70]).

In the case of the extreme synergy, the Choquet integral corresponds to the minimum operator. Here, the fulfillment of any partial goal does not affect the overall evaluation, unless all of the partial goals are achieved. The fuzzy measure, which turns the Choquet integral into minimum operator, has the property that $\mu(\Omega) = 1$, while $\mu(C) = 0$ for any other $C \in \wp(\Omega)$.

In the limit case of full redundancy, the Choquet integral corresponds to the maximum operator. In such the case, $\mu(\emptyset) = 0$ and $\mu(C) = 1$ for any $C \in \wp(\Omega)$.

The Choquet integral can also turn into weighted average. Whenever the fuzzy measure is additive, the Choquet integral takes the form of Lebesgue integral and, consequently, of the weighted average with $\mu(C_i)$ as the weights w_i, $i = 1, 2, \ldots, n$.

The Choquet integral becomes the OWA operator, if the fuzzy measure of any $A \in \wp(\Omega)$ depends only on its cardinality, i.e. the fuzzy measure of the sets with the same cardinality is the same. The weighting vector for such the OWA operator is then given by $w_i = \mu(A_i) - \mu(A_{i-1})$, where A_i are subsets of Ω with property $\mathrm{Card}(A_i) = i$, $i = 1, 2, \ldots, n$.

Remark 2.4 *The Sugeno integral can be transformed into some of the well known aggregation operators as well. The Sugeno integral can change into the maximum operator, if the fuzzy measure μ on Ω fulfills the conditions: $\mu(\emptyset) = 0$, $\mu(C) = 1$ for any $C \in \wp(\Omega) \setminus \{\emptyset\}$. Analogically, the Sugeno integral can transform into the minimum operator, the fuzzy measure μ on the set Ω just has to be as follows: $\mu(\Omega) = 1$ and $\mu(C) = 0$ for any $C \in \wp(\Omega) \setminus \{\Omega\}$.*

2.4.3 Generalized Partial Goals Method - Step 1

Because of its properties, the Choquet integral can be used to overcome some barriers, which do not allow the decision maker to use some of the MCDM methods based on a weighted average. In this section we will illustrate this approach on Partial Goals Method (PGM) [79].

The fundamental principle of the partial goals method is the following one: Let us suppose that the decision maker aims to achieve an overall goal G_0. Let the overall goal G_0 be covered by partial goals G_1, G_2, G_3. The partial evaluations $h_1, h_2, h_3 \in [0, 1]$, represent the degree of fulfilment of particular partial goals

G_1, G_2, G_3. Furthermore, $w_1, w_2, w_3 \in [0, 1]$, express the proportions of the three partial goals in the overall goal G_0. We are now looking for the overall evaluation of alternative x, which would express the degree of fulfillment of the overall goal G_0.

If $\{G_1, G_2, G_3\}$ form a partition of the overall goal (Fig. 2.2(a)), then the sum of the weights w_1, w_2, w_3 is equal to 1 (the weight of the overall goal) and also the weight of every subset of partial goals is equal to the sum of the weights of the particular goals in the subset, i.e. the measure on the power set of $\{G_1, G_2, G_3\}$ is additive. The overall evaluation of the alternative x is given by the weighted average of the partial evaluations, i.e.

$$h(x) = \sum_{j=1}^{3} h_j w_j. \tag{2.26}$$

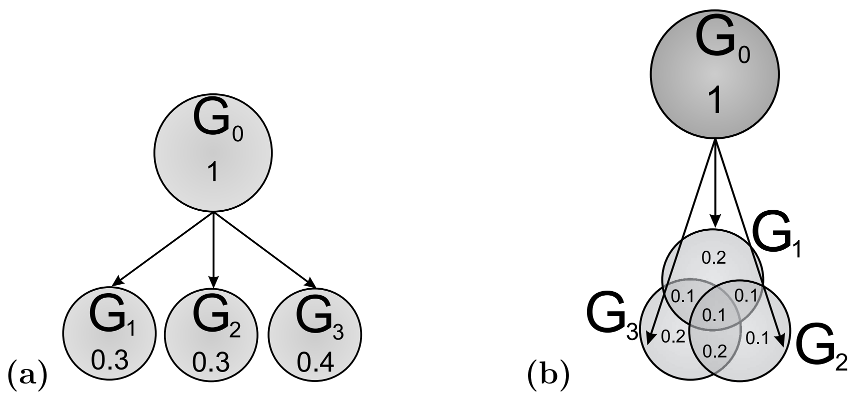


Figure 2.2: **(a)** Partial goals G_1, G_2, G_3 cover the overall goal G_0 and do not overlap - they form the partition of the overall goal G_0. **(b)** Partial goals G_1, G_2, G_3 cover the overall goal G_0, but they overlap - there is a redundance between criteria corresponding to the partial goals.

Nevertheless, the weighted average can not be used when partial goals G_1, G_2, G_3 overlap, as it is illustrated in Fig.2.2(b), where we can see that the proportions of the particular partial goals in the overall goal are $w_1 = 0.5$, $w_2 = 0.5$, $w_3 = 0.6$, and that their sum is not equal to 1. Moreover, the weight of any subset of partial goals $\{G_p, G_r\}$, $p, r \in \{1, 2, 3\}$, is not equal to the sum of the weights w_p and w_r. In other words, the measure defined on the power set of $\{G_1, G_2, G_3\}$ is subadditive. In this situation, the overall evaluation of the alternative x can be given by the value of the Choquet integral, i.e.

$$h(x) = h_{(1)} + (h_{(2)} - h_{(1)})\mu(\{G_{(2)}, G_{(3)}\}) + (h_{(3)} - h_{(2)})\mu(G_{(3)}), \tag{2.27}$$

or, in other form,

$$h(x) = h_{(1)}[1 - \mu(\{G_{(2)}, G_{(3)}\})] + h_{(2)}[\mu(\{G_{(2)}, G_{(3)}\}) - \mu(G_{(3)})] + h_{(3)}\mu(G_{(3)}), \tag{2.28}$$

where $(1), \ldots, (n)$ is a permutation of indices $1, \ldots, n$ such that $h_{(1)} \leq h_{(2)} \leq h_{(3)}$.

The replacement of weighted average with Choquet integral represents the first step in the process of generalization of PGM method. After the first step is taken, PGM is able to handle at least some of the interactions among the partial goals.

Unfortunately, in comparison with the weighted average the application of the Choquet integral requires much more information from the decision maker. Generally, the decision maker, who is modeling the interactions between the criteria (or overlaps of the goals), needs to set $2^n - 2$ values to describe the fuzzy measure properly. Note that determination of n weights in the case of the weighted average is already difficult enough.

2.4.4 Fuzzy measure construction

The application of the Choquet integral to multiple criteria decision making requires correct construction of the fuzzy measure. Given n criteria, the evaluator needs to set the weight of each subset of the set of criteria, $2^n - 2$ values in total (the weights of empty set and the whole set of criteria are given by the definition of fuzzy measure). Moreover, the fuzzy measure should maintain monotonicity arising due to inclusion and the evaluator should keep that in mind.

The construction of fuzzy measure can be much easier, if the sets of interacting criteria in the model do not contain more than k criteria. These types of fuzzy measures were presented in [33] as k-order or k-additive fuzzy measures.

Definition 2.16 *Let μ be a fuzzy measure on $\Omega = \{C_1, \ldots, C_n\}$. The fuzzy measure μ is k-additive if its Möbius transform satisfies $m^\mu = 0$ for all $A \subset \Omega$ such that* $\mathrm{Card}(A) > k$, *and there exists $A \subset \Omega$,* $\mathrm{Card}(A) = k$, *such that $m^\mu(A) \neq 0$.*

In the case of k-additive fuzzy measure, the construction of fuzzy measure requires to set only $\sum_{i=1}^{k}\binom{n}{i}$ values.

Various methods of constructing the fuzzy measure were already proposed. Some of them will be described in the following text.

Direct approaches are based on the knowledge of an expert. The expert is supposed to construct the fuzzy measure either by "assigning" the values of the fuzzy measure directly to the various combinations of criteria or by describing the relationship among the criteria with help of the Shapley values, the importance indices or veto and favor criteria.

Another approaches involve the training data. Let us suppose, we have a set of objects $x_1, \ldots, x_m$. Each object x_i, $i = 1, \ldots, m$, is characterized by n-tuple of partial evaluations $(h_{i1}, \ldots, h_{in})$ and one overall evaluation y_i. Then the fuzzy measure can be constructed with help of these objects, i.e. training data.

One of the methods based on training data is looking for fuzzy measure by minimizing the total square error E^2.

$$E^2 = \sum_{i=1}^{m} (C_\mu(h_{i1}, \ldots, h_{in}) - y_i)^2 \tag{2.29}$$

In this case, by considering the properties of fuzzy measure (monotonicity, boundary conditions) and some additional pieces of information from the expert (preferences between the criteria, synergy or redundancy effects, veto or favor criteria), the construction of fuzzy measure can be transformed to a problem of quadratic programming under constraints with $2^n - 2$ unknown values of fuzzy measure μ [82]. The process of looking for solution requires a lot of training data and a big amount of memory. Moreover, in general the problem does not have a unique solution [60].

To eliminate some of the above mentioned drawbacks of the quadratic program, several different methods of solution have been proposed: the Heuristic Least Mean Squares method [31], the applications of genetic algorithms [50], [90] or the neural networks [89].

In [58] the 2-order fuzzy measure is constructed with help of linear programming using the set of prototype alternatives and the ability of the decision maker to give some information about the ranking of the alternatives, the ranking of the criteria and the nature of the interaction index between the pairs of criteria.

The idea is to find the fuzzy measure, which maximizes the difference between two differently evaluated alternatives.

In [51] Labreuche and Grabisch have proposed to apply the MACBETH approach [4] to construction of the fuzzy measure. The idea is based on pairwise comparisons of criteria coalitions and an assumption that the decision maker is able to set the intensity of differences between the various pairs of those coalitions.

A new method for a fuzzy measure construction

In the following text, a new approach of the fuzzy measure construction, published in [5], will be presented.

Let us suppose that object x has to achieve n partial goals $G_1, \ldots, G_n$ describing the overall goal G_0. Let the object x fully achieve partial goal G_1 and totally fail with respect to the rest of the partial goals. Object x then can be denoted as $x = (1, 0, \ldots, 0)$ and for its overall evaluation $h_{(1,0,\ldots,0)}$ obtained by Choquet integral it holds

$$h_{(1,0,\ldots,0)} = 0 \cdot \mu(\{G_1, \ldots, G_n\}) + \cdots + 0 \cdot \mu(\{G_1, G_2\}) + 1 \cdot \mu(G_1) = \mu(G_1). \quad (2.30)$$

Analogically, we can see that the evaluations of objects $(1, 1, 0, \ldots, 0)$ and $(1, 1, 1, 0, \ldots, 0)$ are equal to $\mu(\{G_1, G_2\})$ and $\mu(\{G_1, G_2, G_3\})$, respectively, i.e.

$$h_{(1,1,0,\ldots,0)} = 0 \cdot \mu(\{G_1, \ldots, G_n\}) + \cdots + 1 \cdot \mu(\{G_1, G_2\}) + 0 \cdot \mu(G_1), \quad (2.31)$$

$$h_{(1,1,1,0,\ldots,0)} = 0 \cdot \mu(\{G_1, \ldots, G_n\}) + \cdots + 0 \cdot \mu(\{G_1, G_2, G_3\}) + \cdots + 0 \cdot \mu(G_1). \quad (2.32)$$

All in all, the values of fuzzy measure of the sets of partial goals are equal to the evaluations of the corresponding objects. As a consequence, the construction of the fuzzy measure can be simplified by evaluation of 2^n imaginary objects $x^1 = (0, 0, \ldots, 0)$, $x^2 = (1, 0, \ldots, 0)$, $x^3 = (0, 1, \ldots, 0)$, …, $x^{2^n} = (1, 1, \ldots, 1)$.

The evaluation can be done in two steps:

1. We use the Pairwise Comparison Method and order all the imaginary objects.

2. We describe the intensity of the preferences between the subsequent classes of indifferent objects with help of linguistic descriptors.

In the first step, we order the objects decreasingly by comparing all pairs of the given objects. During the process, we create a matrix $A = \{a_{ij}\}_{i,j=1}^{2^n}$, where $a_{ij} = 1$ if the object from the i-th row achieves the overall goal better than the object from the j-th column, $a_{ij} = 0.5$ if the object from the i-th row is as good as the object from the j-th column, and $a_{ij} = 0$ otherwise. To each object x^i there is assigned value $\sum_{j=1}^{2^n} a_{ij}$, $i = 1, 2, \ldots, 2^n$, and all the imaginary objects are arranged into several groups $\Pi_1 \succ \Pi_2 \succ \cdots \succ \Pi_k$, $k \leq 2^n$, such that objects in each group are assigned the same value and the objects in group Π_j are assigned higher value than objects in group Π_{j+1}, $j = 1, 2, \ldots, k-1$.

Using this approach, we need to set only the upper triangle of the matrix, because $a_{ij} = 1 - a_{ji}$ for any $i, j \in \{1, 2, \ldots, 2^n\}$. Moreover, some values are not optional (they are fixed from the definition of fuzzy measure), therefore the evaluator needs to set at most $2^{2n-1} - 2^{n-1} - 1$ values. During the process of creating the matrix A, we can also employ the inclusion and transitivity properties to reduce the number of needed elements even further. For example, for $n = 3$ the 27 required parameters can be reduced up to only 9. The real number of reduced parameters will depend on the actual choice of values and the nature of the problem and without that knowledge it can be only bounded from above and from below.

In the second step, we use the ordering of the groups $\Pi_1 \succ \Pi_2 \succ \cdots \succ \Pi_k$, and add information about the intensity of the relation between the groups Π_j, Π_{j+1}, $j = 1, 2, \ldots, k-2$. The intensity of the relation r_j, $j = 1, 2, \ldots, k-2$, can be described linguistically, e.g. by terms "as good as","slightly better", "quite better", "strongly better" or "extremely better", with numbers $1, t_1, t_2, t_3, t_4$ quantifying the relation. For example we may consider numbers $1, 2, 3, 4, 5$. If $r_j = t_j$, it means that objects in the group Π_j are t_j-times better than the objects in the group Π_{j+1}.

The evaluation h^j of the objects from Π_j is then given as follows, $h^1 = 1$, $h^j = \frac{1}{r_1 \cdots r_{j-1}}$, $j = 2, 3, \ldots, k-1$, $h^k = 0$. For example, let us suppose $(1, 1, \ldots, 1) \in \Pi_1 \succ (1, 1, \ldots, 1, 0) \in \Pi_2$. If we describe the relation between the first and the second object as "slightly better", then the evaluation of the object $(1, 1, \ldots, 1, 0)$ is $\frac{1}{t_1}$ and $\mu(\{G_1, G_2, \ldots, G_{n-1}\}) = \frac{1}{t_1}$.

It is also possible to have the matrix A populated only by ones and zeros, when $a_{ij} = 1$ if object from the i-th row is better or as good as the object from the j-th column, and $a_{ij} = 0$ otherwise. However, creating this matrix requires more parameters to be set by the evaluator, for n criteria $2^{2n} - 3^n$ parameters are needed.

For ease of use to the evaluator it is very convenient to present the construction of the fuzzy measure in a form of an algorithm:

Algorithm 2.1

Step 1: *Generate imaginary objects x^i, $i = 1, 2, \ldots, 2^n$, such that each object corresponds to a different subset of the set of partial goals $G_1, G_2, \ldots, G_n$, and it absolutely satisfies the goals in the subset and totally fails in achieving all the others.*

Step 2: *Create matrix $A = \{a_{ij}\}_{i,j=1}^{2^n}$: For each $j = 1, 2, \ldots, 2^n$ check if a_{ij} has already been set. If yes, then $a_{ji} = 1 - a_{ij}$. Otherwise check following:*

a. *Inclusion: if the set of partial goals corresponding to the object x^i is a proper superset (equal to, proper subset) of the set of partial goals corresponding to the object x^j, then $a_{ij} = 1$ ($a_{ij} = 0.5$, $a_{ij} = 0$). Proceed to the next element of matrix A.*

b. *Transitivity: if there exists an index z such that $a_{iz} = a_{zj} = 1$ ($a_{iz} = a_{zj} = 0$, $a_{iz} = a_{zj} = 0.5$), then $a_{ij} = 1$ ($a_{ij} = 0$, $a_{ij} = 0.5$). Proceed to the next element of matrix A.*

c. *Goal fulfilment: compare object x^i with object x^j. If the object x^i fulfils the overall goal G_0 better than the object x^j, then $a_{ij} = 1$; if both objects achieve the overall goal equally, then $a_{ij} = 0.5$; otherwise $a_{ij} = 0$. Proceed to the next element of matrix A.*

Step 3: *For each $i = 1, 2, \ldots, 2^n$ calculate preference index $\sum_{j=1}^{2^n} a_{ij}$.*

Step 4: *Order the objects decreasingly according to their preference indexes and bunch them into groups Π_q, $q = 1, 2, \ldots, k$, $k \leq 2^n$, such that all the members of the group gave the same preference index.*

Step 5: *For $q = 1, 2, \ldots, k-1$ compare groups Π_q, Π_{q+1} and quantify their relationship by number r_q, which can attain one of the five values $\{1, t_1, t_2, t_3, t_4\}$, where each value corresponds to one linguistic term of "as good as", "slightly better", "quite better", "strongly better" or "extremely better", respectively.*

Step 6: For $q = 2, 3, \ldots, k-1$ *calculate the evaluation* h^q *of the objects from group* Π_q*:* $h^q = \frac{1}{r_1 \cdots r_{q-1}}$*. The remaining two evaluations are* $h^1 = 1$ *and* $h^k = 0$*.*

Step 7: Fuzzy measure of each subset of the set of partial goals is then equal to the evaluation h^q *of the corresponding object* $x^i \in \Pi_q$*.*

Chapter 3

Fuzzification of aggregation operators

In multiple criteria decision making we rarely make the decision while having a complete knowledge of the parameters of the situation. The uncertainty is part of the world and we can meet it in every element of a decision making problem. There are two kinds of uncertainty. The first one comes from not knowing the future. Any given scenario may or may not occur and we are not sure what the future will be. This kind of uncertainty can be called stochastic uncertainty and it is handled by probability theory. Another uncertainty comes from the vagueness of human language and thinking. The future is not problem here. We are certain about the occurrence of the event, but the words, which are describing the event, are vague and therefore they represent the source of uncertainty - we lack the knowledge what did the words exactly mean. This type of uncertainty is referred to as lexical uncertainty [86] and that is what the fuzzy set theory is working with.

Let us stay with the lexical uncertainty. In MCDM, the lexical uncertainty can influence several inputs of the decision making problem. At first, in an MCDM problem, we distinguish between quantitative and qualitative criteria. The consequences of the former are measured and then evaluated with help of expertly set evaluative functions. On the other hand, consequences of the latter are, for each alternative, evaluated directly by an expert. Both the evaluations can be uncertain. The measurements can be inaccurate or distorted by round off errors. The expertly set evaluations are based on the expert's opinion and experience, and therefore they can be uncertain as well. Sometimes some of the data can be even missing. The measurement could not be made or the expert was not able to evaluate the alternative. In such the scenario we should take into the consideration that the evaluation can come from the whole evaluation scale, i.e. we work with a very uncertain piece of information.

Second, most of the MCDM methods work with criteria of different importance. The importance of each criterion is expressed by its weight. Sometimes the weights are calculated with help of previous data or a training set. But very often the weights are set expertly. Both the processes produce the weights burdened with uncertainty.

All the uncertainty and missing data can be well modeled by fuzzy numbers [20]. Fuzzy numbers allow us to consider the uncertainty in the MCDM process and to transfer it from the inputs to the output. Then, knowing the value of the output is uncertain, the decision makers can adjust their verdict.

Working with fuzzy numbers requires modification of the classical aggregation operators in order to allow them to handle fuzzy inputs. The process of modification is called fuzzification. In this chapter, the fuzzification of the main aggregation operators described in the previous chapter will be presented.

3.1 Introduction to fuzzy sets

One can meet several different definitions of a set. The most widespread one can be found [54] in the following form: *A set is a well defined collection of objects.* Here, the words "well defined" are of great importance. They are saying that for any given object it can be unambiguously determined whether the object does or does not belong to the set. It means that the property, which is common to all the objects from the set, is well defined and there are no doubts whether the object has the property or not. Each two sets can be united, intersected or subtracted. The sets and the whole set theory are one of main building stones of mathematics.

However, sometimes the property, which is supposed to define a set, is not described properly. Its meaning may differ from person to person and for some objects it is difficult to decide whether they possess the property or not. This kind of property is called vague, uncertain or fuzzy.

The concept of fuzzy sets was introduced in 1965 by L. A. Zadeh, professor of Systems Theory at the University of California, Berkeley. In his publication "Fuzzy sets" [97] he generalized the classical set and laid the foundations of fuzzy set theory.

Definition 3.1 *Let U be a nonempty set. A fuzzy set C on U is defined by the mapping $C : U \to [0, 1]$.*

For each $x \in U$ the value $C(x)$ is called a membership degree of the element x in the fuzzy set C and $C(\cdot)$ is a membership function of the fuzzy set C.

The system of all fuzzy sets on the set U is denoted by $\mathcal{F}(U)$.

Remark 3.1 *Note that the membership function of a fuzzy set is really a generalization of the characteristic function of a classical set.*

The characteristic function of a set assigns to any object one of two numbers, 0 or 1, depending upon the object having the defining property or not. On the other hand, the membership function of a fuzzy set assigns to any object the truth degree of the statement: "The object has the defining property." The truth degree ranges continuously from zero to one and expresses to what extent the object possesses the property.

Every fuzzy set can be described by its support, kernel, height and so called α-cuts:

Definition 3.2 *Let C be a fuzzy set on a nonempty set U and $\alpha \in [0, 1]$.*

- *The kernel of fuzzy set C is a set*

$$\operatorname{Ker} C = \{x \in U \mid C(x) = 1\}. \tag{3.1}$$

- *The α-cut of fuzzy set C is a set*

$$C_\alpha = \{x \in U \mid C(x) \geq \alpha\}. \tag{3.2}$$

- *The support of fuzzy set C is a set*

$$\operatorname{Supp} C = \{x \in U \mid C(x) > 0\}. \tag{3.3}$$

- *The height* $\mathrm{hgt}(C)$ *of fuzzy set* C *is a number*

$$\mathrm{hgt}(C) = \sup_{x \in U} C(x). \tag{3.4}$$

Fuzzy sets can be also intersected and united.

Definition 3.3 *Let* C *and* D *be fuzzy sets on* U*.*

- *The intersection of fuzzy sets* C *and* D *is a fuzzy set* $C \cap D$ *on* U *with membership function given for any* $x \in U$ *by*

$$(C \cap D)(x) = \min\{C(x), D(x)\}. \tag{3.5}$$

- *The union of fuzzy sets* C *and* D *is a fuzzy set* $C \cup D$ *on* U *with membership function given for any* $x \in U$ *by*

$$(C \cup D)(x) = \max\{C(x), D(x)\}. \tag{3.6}$$

Fuzzy sets are defined on some universal set U, which can be practically any set of arbitrary objects: trees, people, cities, functions, etc. Nevertheless, we usually work with a special type of fuzzy sets, which are defined on the set of real numbers and fulfill some additional conditions. They are called fuzzy numbers and their definition differs from author to author. The fuzzy numbers appearing in the rest of the text are defined as follows:

Definition 3.4 *A fuzzy number* C *is a fuzzy set on the set of real numbers* $\mathbb{R}$ *with the following properties:*

- *The kernel of* C *is nonempty,*
- *for all* $\alpha \in (0, 1]$ *the* α*-cuts of* C *are closed intervals,*
- *the support of* C *is bounded.*

The family of all fuzzy numbers will be denoted by $\mathcal{F}_N(\mathbb{R})$*.*

Remark 3.2 *If* C *is a fuzzy number and* $\mathrm{Supp}\, C \subseteq [a, b]$*, then* C *is referred to as a fuzzy number on* $[a, b]$ *and the set of all fuzzy numbers on* $[a, b]$ *is denoted by* $\mathcal{F}_N([a, b])$*.*

Remark 3.3 *Real numbers are often called "crisp" numbers and they can be modeled by fuzzy numbers. A real number c can be described by fuzzy number C on $\mathbb{R}$ with membership function defined for any $x \in \mathbb{R}$ as*

$$C(x) = \begin{cases} 1, & \text{for } x = c, \\ 0, & \text{otherwise.} \end{cases} \tag{3.7}$$

Analogically, closed intervals can also be represented by a special type of fuzzy numbers. Interval $[a, b]$ can be modeled as fuzzy number D on $\mathbb{R}$ such that:

$$\forall x \in \mathbb{R} : D(x) = \begin{cases} 1, & \text{for } x \in [a, b], \\ 0, & \text{otherwise.} \end{cases} \tag{3.8}$$

Remark 3.4 *According to [20, 67] a fuzzy number C with a membership function $C(.)$ can be alternatively described by a couple of functions $\underline{c} : [0, 1] \to \mathbb{R}$, $\overline{c} : [0, 1] \to \mathbb{R}$, such that $[\underline{c}(\alpha), \overline{c}(\alpha)] = C_\alpha, \alpha \in (0, 1]$, $[\underline{c}(0), \overline{c}(0)] = \mathrm{Cl}(\mathrm{Supp}(C))$, where $\mathrm{Cl}(\cdot)$ stands for the closure of a set. Therefore each fuzzy number C can be denoted by $C = \{[\underline{c}(\alpha), \overline{c}(\alpha)], \alpha \in [0, 1]\}$.*

In general, fuzzy numbers can take many forms. As the most simple fuzzy number is considered a linear fuzzy number (see Definition 3.5), which can be fully determined by only three or four points lying in its support.

Definition 3.5 *A linear fuzzy number on the interval $[a, b]$ that is determined by four points $(x_1, 0), (x_2, 1), (x_3, 1), (x_4, 0)$, $a \le x_1 \le x_2 \le x_3 \le x_4 \le b$, is a fuzzy number C with the membership function depending on parameters x_1, x_2, x_3, x_4, as follows*

$$\forall x \in [a, b] : C(x, x_1, x_2, x_3, x_4) = \begin{cases} 0, & \text{for } x < x_1; \\ \frac{x - x_1}{x_2 - x_1}, & \text{for } x_1 \le x < x_2; \\ 1, & \text{for } x_2 \le x \le x_3; \\ \frac{x_4 - x}{x_4 - x_3}, & \text{for } x_3 < x \le x_4; \\ 0, & \text{for } x_4 < x. \end{cases}$$

Remark 3.5 *Linear fuzzy number C determined by four points $(x_1, 0)$, $(x_2, 1)$, $(x_3, 1)$, $(x_4, 0)$ will be called a trapezoidal fuzzy number and denoted by $C \sim (x_1, x_2, x_3, x_4)$. If $x_2 = x_3$, then the linear fuzzy number C will be called a triangular fuzzy number and denoted by $C \sim (x_1, x_2, x_4)$.*

Similarly to crisp numbers, fuzzy numbers can be ordered, although there are several ways how to order and compare them. In the rest of the text we will employ the ordering described in Definition 3.6. It imposes only partial order, because some pairs of fuzzy numbers are incomparable this way, but for the purpose of the present work it is sufficient.

Definition 3.6 *We say that a fuzzy number* $C = \{[\underline{c}(\alpha), \overline{c}(\alpha)], \alpha \in [0,1]\}$ *is less than or equal to a fuzzy number* $D = \{[\underline{d}(\alpha), \overline{d}(\alpha)], \alpha \in [0,1]\}$*, denoted by* $C \leq D$*, if* $\underline{c}(\alpha) \leq \underline{d}(\alpha)$ *and* $\overline{c}(\alpha) \leq \overline{d}(\alpha)$ *for any* $\alpha \in [0,1]$*.*

In the following text, we will quite often use the term: FNV-function. It is the kind of function which to any element of a given set Ω assigns a fuzzy number. In MCDM it is the function which describes the process of evaluation quite well. To any evaluated object it assigns a fuzzy number expressing the uncertain (partial or overall) evaluation of the object.

Definition 3.7 *Let* Ω *be a nonempty set and* $\mathcal{F}_N(\mathbb{R})$ *be a system of all fuzzy numbers. A mapping* F*,* $F : \Omega \to \mathcal{F}_N(\mathbb{R})$*, is called a fuzzy number-valued function (FNV-function).*

If partial evaluations of an alternative are modeled by fuzzy numbers they can not be aggregated with help of a classical aggregation operator. Instead a fuzzified aggregation operator, aggregating an n-tuple of fuzzy numbers into a single one, has to be used.

The process of fuzzifying the aggregation operator takes advantage of the following theorems based on the extension principle.

Definition 3.8 *[Extension principle] Let* $U_1, \ldots, U_n$*,* V *be nonempty sets. The fuzzification of a mapping* $f : U_1 \times \cdots \times U_n \to V$ *is defined as a mapping* $f_F : \mathcal{F}(U_1) \times \cdots \times \mathcal{F}(U_n) \to \mathcal{F}(V)$*, which to any* n*-tuple of fuzzy sets* $C_i \in \mathcal{F}(U_i)$*,* $i = 1, \ldots, n$*, assigns a fuzzy set* $f_F(C_1, \ldots, C_n) \in \mathcal{F}(V)$ *with the membership function given for any* $d \in V$ *by*

$f_F(C_1, \ldots, C_n)(d) =$

$$= \begin{cases} \sup\{\min\{C_1(c_1), \ldots, C_n(c_n)\} | d = f(c_1, \ldots, c_n), c_i \in U_i, i = 1, \ldots, n\} \\ \hfill \text{if } f^{-1}(d) \neq \emptyset; \\ 0 \hfill \text{otherwise.} \end{cases} \tag{3.9}$$

Definition 3.8 is too general, it works with any mapping and any n-tupple of fuzzy sets. In practice, however, we usually work with fuzzy numbers and continuous functions. Under these circumstances the fuzzification can be done more easily according to Theorem 3.1.

Theorem 3.1 *Let $f : [0,1]^n \to [0,1]$ be a real continuous function of n real variables. Let f_F be a fuzzification of the function f according to the extension principle. Then for any n-tuple of fuzzy numbers $C_1, C_2, \ldots, C_n$ on $[0,1]$, $D = f_F(C_1, C_2, \ldots, C_n)$ is a fuzzy number on $[0,1]$, such that for any $d \in [0,1]$ the following holds:*

$$D(d) = \begin{cases} \max\{\min\{C_1(c_1), C_2(c_2), \ldots, C_n(c_n)\} \\ \qquad |d = f(c_1, c_2, \ldots, c_n), c_i \in [0,1], i = 1, 2, \ldots, n\} & \text{if } f^{-1}(d) \neq \emptyset; \\ 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

Moreover, for any $\alpha \in (0,1]$ holds:

$$D_\alpha = f(C_{1\alpha}, \ldots, C_{n\alpha}) = \left[\min_{\substack{c_i \in C_{i\alpha} \\ i = 1, \ldots, n}} f(c_1, \ldots, c_n), \quad \max_{\substack{c_i \in C_{i\alpha} \\ i = 1, \ldots, n}} f(c_1, \ldots, c_n) \right]. \tag{3.11}$$

Proof: See [85]. ■

In practice, we usually describe a fuzzy number D with help of two functions $\underline{d}$ and $\overline{d}$. If the fuzzy number D is a result of fuzzified function applied on $n-$tuple of fuzzy numbers $C_1, \ldots, C_n$, then the shape of the functions $\underline{d}$ and $\overline{d}$ depends on the look of the functions $\underline{c}_i$ and $\overline{c}_i$, $i = 1, \ldots, n$, and the function which has to be fuzzified. The rule, which allows us to effectively find the shape of functions $\underline{d}$ and $\overline{d}$, is described in the theorem 3.2 proven in [67].

Theorem 3.2 *Let a real continuous function $f : [0,1]^n \to [0,1]$ be non-decreasing in all of its variables. Let $C_i = \{[\underline{c}_i(\alpha), \overline{c}_i(\alpha)], \alpha \in [0,1]\}$, $i = 1, \ldots, n$, be* *fuzzy numbers on* $[0,1]$. *Then for a fuzzy number* $D = f_F(C_1, \ldots, C_n)$, $D = \{[\underline{d}(\alpha), \overline{d}(\alpha)], \alpha \in [0,1]\}$, *the following holds for any* $\alpha \in [0,1]$:

$$\underline{d}(\alpha) = f(\underline{c}_1(\alpha), \ldots, \underline{c}_n(\alpha)), \tag{3.12}$$

$$\overline{d}(\alpha) = f(\overline{c}_1(\alpha), \ldots, \overline{c}_n(\alpha)). \tag{3.13}$$

Proof: See [67]. ■

A powerful property of fuzzy sets is their ability to capture a word with vague meaning and describe it with help of mathematical language. Most of the people prefer to describe an object or a principle by words rather than by mathematics. The ability to transform the words into mathematical objects is therefore very useful. The main role in the, so called, linguistical modeling is played by linguistical variables (variables with words as values), which was created by Zadeh [98].

Definition 3.9 *A linguistic variable is a quintuple* $(X, T(X), U, G, M)$, *where* X *is the name of the variable,* $T(X)$ *is the set of its linguistic values (linguistic terms),* U *is the universe, which the mathematical meanings of the linguistic terms are modeled on,* G *is the syntactical rule for generating the linguistic terms, and* M *is the semantic rule, which to every linguistic term* $\mathcal{C}$ *assigns its meaning* $C = M(\mathcal{C})$ *as a fuzzy set on* U.

Remark 3.6 *If the set of linguistic terms is given explicitly, then the linguistic variable is denoted by* $(X, T(X), U, M)$.

The introduction of linguistic variable allows us to present a linguistically defined function (see Definition 3.10), which is able to take the linguistically described relationship, rule, or dependence among various values of linguistic variables, and express it with help of mathematical tools.

Definition 3.10 *Let* $(X_j, T(X_j), U_j, M_j)$, $j = 1, 2, \ldots, m$, *and* $(Y, T(Y), V, M)$ *be linguistic variables and* U_j, $j = 1, \ldots, m$, V *be closed intervals. Let* $\mathcal{C}_{ij} \in T(X_j)$ *and* $C_{ij} = M_j(\mathcal{C}_{ij}) \in F_N(U_j)$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. *Let* $\mathcal{D}_i \in T(Y)$ *and* $D_i = M(\mathcal{D}_i) \in F_N(V)$, $i = 1, 2, \ldots, n$. *Then the following scheme* F

$$\begin{array}{l} \textit{If } X_1 \textit{ is } \mathcal{C}_{11} \textit{ and } \dots \textit{ and } X_m \textit{ is } \mathcal{C}_{1m}, \textit{ then } Y \textit{ is } \mathcal{D}_1 \\ \textit{If } X_1 \textit{ is } \mathcal{C}_{21} \textit{ and } \dots \textit{ and } X_m \textit{ is } \mathcal{C}_{2m}, \textit{ then } Y \textit{ is } \mathcal{D}_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \textit{If } X_1 \textit{ is } \mathcal{C}_{n1} \textit{ and } \dots \textit{ and } X_m \textit{ is } \mathcal{C}_{nm}, \textit{ then } Y \textit{ is } \mathcal{D}_n \end{array} \tag{3.14}$$

is called a linguistically defined function (base of fuzzy rules).

The base of rules is a valuable tool for multiple criteria evaluation. It allows us to describe the relationship between the partial evaluations of an alternative, or even between its actual performances and overall evaluation, in any kind of MCDM problem. It is therefore a suitable tool for tackling situations, in which the interactions among the partial goals or the criteria are so complicated that the application of any aggregation operators is difficult or even impossible.

The application of a base of rules follows a simple principle: to specific combinations of input values we assign a specific output value. The combinations need to be chosen carefully, so each input vector of values satisfies at least one of the rules. Of course, in the process of multiple criteria evaluation the partial evaluations at our disposal only rarely match any of the rules perfectly. This would be a problem if we used crisp numbers, but for a vector of fuzzy inputs a fuzzy output can be always obtained by using one of the several algorithms of an approximate reasoning. The most popular and the most widely used one is the Assilian and Mamdani approach [2], which can be described as follows:

Let F be the base of rules looking as:

$$\begin{array}{l} \text{If } X_1 \text{ is } \mathcal{C}_{11} \text{ and } \dots \text{ and } X_m \text{ is } \mathcal{C}_{1m}, \text{ then } Y \text{ is } \mathcal{D}_1 \\ \text{If } X_1 \text{ is } \mathcal{C}_{21} \text{ and } \dots \text{ and } X_m \text{ is } \mathcal{C}_{2m}, \text{ then } Y \text{ is } \mathcal{D}_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \text{If } X_1 \text{ is } \mathcal{C}_{n1} \text{ and } \dots \text{ and } X_m \text{ is } \mathcal{C}_{nm}, \text{ then } Y \text{ is } \mathcal{D}_n \end{array} \tag{3.15}$$

and let us assume the observed values to be

$$X_1 \text{ is } \mathcal{C}'_1 \text{ and } X_2 \text{ is } \mathcal{C}'_2 \text{ and } \dots \text{ and } X_m \text{ is } \mathcal{C}'_m. \tag{3.16}$$

Then by entering the observed values into the base of rules F, according to the Assilian-Mamdani algorithm, we obtain the output value

$$Y = \mathcal{D}', \tag{3.17}$$

where the $\mathcal{D}'$ is the linguistic approximation of a fuzzy set D^M. The membership function of the fuzzy set D^M is defined for all $y \in V$ as follows:

$$D^M(y) = \max\{D_1^M(y), \ldots, D_n^M(y)\}, \tag{3.18}$$

where

$$D_i^M(y) = \min\{h_i, D_i(y)\}, \tag{3.19}$$

$$h_i = \min\{\text{hgt}(C_{i1} \cap C_1'), \ \ldots, \text{hgt}(C_{im} \cap C_m')\}, \text{ for } i = 1, \ldots, n. \tag{3.20}$$

In the description of Assilian-Mamdani algorithm we have mentioned the expression "linguistic approximation". As it has been already said, fuzzy numbers are capable of processing vague linguistic expressions. They allow us to model the meaning of *tall man, long waiting* or *big lunch.* Considering the linguistical variable *age*, we are now capable of modeling the meanings of *young age, middle age* or *old age* by fuzzy numbers Y, M and O. However, what is now the meaning of fuzzy number F, which lies somewhere in between the other values? According to linguistic approximation, the linguistical term we are looking for is the term corresponding to that of the fuzzy numbers Y, M, O, which is the most similar to F. The exact meaning of linguistic approximation is given in Definition 3.11.

Definition 3.11 *[79] Let $(X, T(X), [a, b], M)$ be a linguistic variable, $T(X) = \{\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s\}$ and $M(\mathcal{T}_i) = T_i$, $i = 1, 2, \ldots, s$, are fuzzy numbers on the interval $[a, b]$. Let C be a fuzzy set on interval $[a, b]$ with Borel-measurable membership function. Let for given C a fuzzy set P_C be defined on set $\{\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s\}$ by following formula*

$$P_C(T_i) = 1 - \frac{\int_a^b |C(x) - T_i(x)| dx}{\int_a^b (C(x) + T_i(x)) dx}, \tag{3.21}$$

where $i = 1, 2, \ldots, s$. Then a linguistic approximation of fuzzy set C by linguistic variable X is a linguistic term T_{i_0}, $i_0 \in \{1, 2, \ldots, s\}$ such that

$$P_C(T_{i_0}) = \max_{i=1,\ldots,s} P_C(T_i). \tag{3.22}$$

3.2 First level fuzzification of aggregation operators

In the previous section, the basic notions of fuzzy set theory and their role in MCDM were explained. Because of the uncertainty of the input data, the usually applied MCDM methods have to be modified. It is usually convenient to start this alteration by modification of the aggregation operators, namely their fuzzification.

Complete fuzzification of the aggregation operators will be done in two steps. In the first one, the partial evaluations of alternatives are modeled by fuzzy numbers, while weights of the sets of criteria remain crisp. In the second step we consider also the weights of the sets of criteria to be uncertain and therefore we describe them by fuzzy numbers as well.

From now on we are going to focus on a single alternative x. This can be done without loss of generality, as the aggregation operator treats all alternatives equally, and it simplifies the notation and improves clarity. The evaluations of the alternative with respect to a set of criteria $\Omega = \{C_1, C_2, \dots, C_n\}$ are modelled by fuzzy numbers $H_1, H_2, \dots, H_n$, $H_j = \{[\underline{h}_j(\alpha), \overline{h}_j(\alpha)], \alpha \in [0,1]\}$, $j = 1, 2, \dots, n$.

3.2.1 First-level fuzzy weighted average

Weighted average of n real numbers is a linear function of n variables with parameters w_i, $i = 1, 2, \dots, n$, fulfilling the condition $\sum_{i=1}^{n} w_i = 1$. It is a continuous function, therefore its fuzzification can be done with help of Definition 3.8 and Theorem 3.1.

Definition 3.12 *Let H_i, $i = 1, 2, \dots, n$ be fuzzy numbers on $[0,1]$ and v_i, $i = 1, 2, \dots, n$ be normalized weights. First-level fuzzy weighted average of $H_1, H_2, \dots, H_n$ with normalized weights $v_1, v_2, \dots, v_n$ is a fuzzy number Y with a membership function given for any $y \in [0,1]$ by*

$$\begin{aligned} Y(y) \quad = \quad & \max\Big\{ \min\big\{H_1(h_1), \dots, H_n(h_n), \big\} \mid \\ & h_i \in [0,1], i = 1, \dots, n, y = \sum_{i=1}^{n} v_i h_i \Big\}. \end{aligned} \tag{3.23}$$

According to the definition 2.1, each n-ary aggregation operator is a non-decreasing function in all of its n real variables. Moreover, weighted average is continuous [15]. Therefore all the assumptions of Theorem 3.2 are fulfilled and the first-level fuzzy weighted average of n fuzzy numbers from Definition 3.12 can be expressed as the fuzzy number described in Theorem 3.3.

Theorem 3.3 *Let* $H_i = \{[\underline{h_i}(\alpha), \overline{h_i}(\alpha)], \alpha \in [0,1]\}$, $i = 1, 2, \ldots, n$ *be fuzzy numbers on* $[0,1]$ *and* v_i, $i = 1, 2, \ldots, n$ *be normalized weights. The first-level fuzzy weighted average of* $H_1, H_2, \ldots, H_n$ *with normalized weights* $v_1, v_2, \ldots, v_n$ *is a fuzzy number* Y, $Y = \{[\underline{y}(\alpha), \overline{y}(\alpha)], \alpha \in [0,1]\}$, *given for any* $\alpha \in [0,1]$ *by*

$$\underline{y}(\alpha) = \sum_{i=1}^{n} v_i \underline{h}_i(\alpha), \tag{3.24}$$

$$\overline{y}(\alpha) = \sum_{i=1}^{n} v_i \overline{h}_i(\alpha) \tag{3.25}$$

Proof: Theorem is derived directly from Definition 3.8 and Theorem 3.1. ∎

Remark 3.7 *Although Definition 3.12 describes the first-level fuzzy weighted average properly, it is Theorem 3.3, which makes the calculation much more simple. Moreover, according to Theorem 3.3, first-level fuzzy weighted average keeps the linearity of fuzzy numbers, i.e. if the input fuzzy numbers are linear, the output fuzzy number is going to be linear as well, because the functions* $\underline{y}$ *and* $\overline{y}$ *will be calculated as a linear combination of* n *linear functions (See Figure 3.1).*

Remark 3.8 *The weighed average of the uncertain evaluations with crisp weights is described in [79] in detail.*

3.2.2 First-level fuzzy OWA

Because of the continuity of OWA operators, all the theorems and definitions applied to fuzzification of the weighted average from the previous section can be employed also in the fuzzification of the OWA operators.

Definition 3.13 *Let* H_i, $i = 1, 2, \ldots, n$ *be fuzzy numbers on* $[0,1]$ *and* v_i, $i = 1, 2, \ldots, n$ *be normalized weights. First-level fuzzy ordered weighted average of*

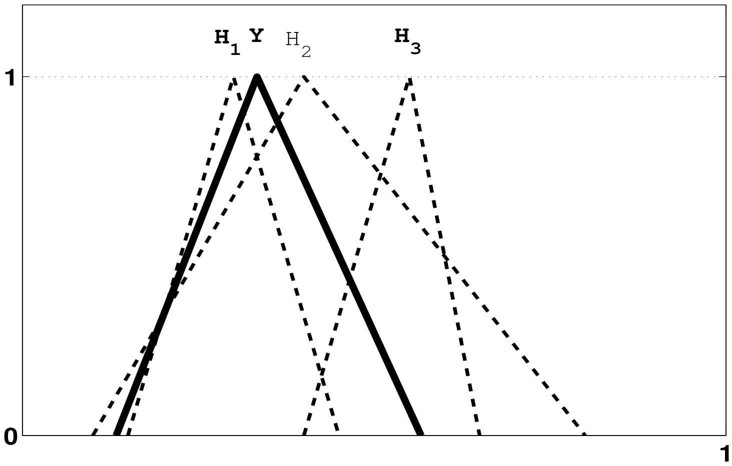


Figure 3.1: The linear fuzzy number Y as a first-level fuzzy weighted average of fuzzy numbers H_1, H_2 and H_3, where $H_1 \sim (0.15, 0.3, 0.45)$, $H_2 \sim (0.1, 0.4, 0.8)$, $H_3 \sim (0.4, 0.55, 0.65)$, $v_1 = 2/3$, $v_2 = 1/3$, $v_3 = 0$.

$H_1, H_2, \dots, H_n$ *with normalized weights* $v_1, v_2, \dots, v_n$ *is a fuzzy number* Y *with a membership function given for any* $y \in [0,1]$ *by*

$$\begin{aligned} Y(y) \;&=\; \max\Big\{ \min\{H_1(h_1), \dots, H_n(h_n), \} \mid \\ & \qquad h_i \in [0,1], i = 1, \dots, n, y = \sum_{i=1}^{n} v_i h^{(i)} \Big\} \end{aligned} \tag{3.26}$$

where $(1), \dots, (n)$ *denotes a permutation of indices* $1, \dots, n$ *such that* $h^{(1)} \geq h^{(2)} \geq \cdots \geq h^{(n)}$.

Theorem 3.4 *Let* $H_i = \{[\underline{h_i}(\alpha), \overline{h_i}(\alpha)], \alpha \in [0,1]\}$, $i = 1, 2, \dots, n$ *be fuzzy numbers on* $[0,1]$ *and* v_i, $i = 1, 2, \dots, n$ *be normalized weights. The first-level fuzzy ordered weighted average of* $H_1, H_2, \dots, H_n$ *with normalized weights* $v_1, v_2, \dots, v_n$ *is a fuzzy number* Y, $Y = \{[\underline{y}(\alpha), \overline{y}(\alpha)], \alpha \in [0,1]\}$, *given for any* $\alpha \in [0,1]$ *by*

$$\underline{y}(\alpha) = \sum_{i=1}^{n} v_i \underline{h}^{(i)}(\alpha), \tag{3.27}$$

where $\underline{h}^{(1)}(\alpha) \geq \underline{h}^{(2)}(\alpha) \geq \cdots \geq \underline{h}^{(n)}(\alpha)$,

$$\overline{y}(\alpha) = \sum_{i=1}^{n} v_i \overline{h}^{(i)}(\alpha) \tag{3.28}$$

where $\overline{h}^{(1)}(\alpha) \geq \overline{h}^{(2)}(\alpha) \geq \cdots \geq \overline{h}^{(n)}(\alpha)$,

Proof: Theorem is derived directly from Definition 3.8 and Theorem 3.1. ■

Remark 3.9 *Analogically to the first-level fuzzy weighted average, the first-level fuzzy ordered weighted average of n fuzzy numbers is easier to calculate with help of Theorem 3.4. Contrary to the first-level weighted average, the first-level OWA operator does not preserve the linearity of the input fuzzy numbers. The Figure 3.2 illustrates that the output can be in the form of a piecewise linear fuzzy number, even if the input fuzzy numbers were linear.*

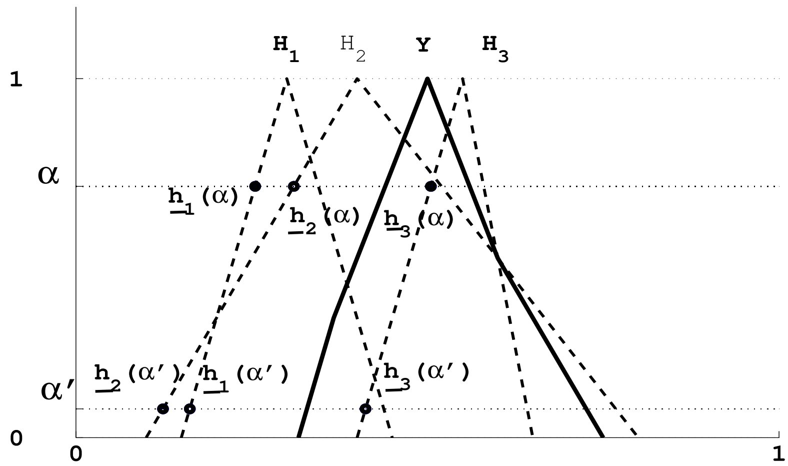


Figure 3.2: The piecewise linear fuzzy number Y as a first-level fuzzy ordered weighted average of fuzzy numbers H_1, H_2 and H_3, where $H_1 \sim (0.15, 0.3, 0.45)$, $H_2 \sim (0.1, 0.4, 0.8)$, $H_3 \sim (0.4, 0.55, 0.65)$, $v_1 = 2/3$, $v_2 = 1/3$, $v_3 = 0$.

The fuzzification of the special cases of OWA operators, maximum and minimum operator, follows the same principle as the previous operators.

Definition 3.14 *Let* $H_i = \{[\underline{h_i}(\alpha), \overline{h_i}(\alpha)], \alpha \in [0,1]\}$, $i = 1, 2, \ldots, n$ *be fuzzy numbers on* $[0,1]$. *First-level fuzzy maximum of* $H_1, H_2, \ldots, H_n$ *is a fuzzy number*

Y with a membership function given for any $y \in [0,1]$ by

$$Y(y) = \max\Big\{ \min\{H_1(h_1), \ldots, H_n(h_n),\} \mid h_i \in [0,1], i = 1, \ldots, n, y = \max\{h_1, \ldots, h_n\}\Big\} \tag{3.29}$$

Definition 3.15 *Let $H_i = \{[\underline{h_i}(\alpha), \overline{h_i}(\alpha)], \alpha \in [0,1]\}$, $i = 1, 2, \ldots, n$ be fuzzy numbers on $[0,1]$. First-level fuzzy minimum of $H_1, H_2, \ldots, H_n$ is a fuzzy number Y with a membership function given for any $y \in [0,1]$ by*

$$Y(y) = \max\Big\{ \min\{H_1(h_1), \ldots, H_n(h_n),\} \mid h_i \in [0,1], i = 1, \ldots, n, y = \min\{h_1, \ldots, h_n\}\Big\} \tag{3.30}$$

Theorem 3.5 *Let $H_i = \{[\underline{h_i}(\alpha), \overline{h_i}(\alpha)], \alpha \in [0,1]\}$, $i = 1, 2, \ldots, n$ be fuzzy numbers on $[0,1]$. The first-level fuzzy maximum of $H_1, H_2, \ldots, H_n$ is a fuzzy number Y, $Y = \{[\underline{y}(\alpha), \overline{y}(\alpha)], \alpha \in [0,1]\}$, given for any $\alpha \in [0,1]$ by*

$$\underline{y}(\alpha) = \max\{\underline{h}_1(\alpha), \ldots, \underline{h}_n(\alpha)\} \tag{3.31}$$

$$\overline{y}(\alpha) = \max\{\overline{h}_1(\alpha), \ldots, \overline{h}_n(\alpha)\} \tag{3.32}$$

Proof: As a one of the OWA operators, maximum is a continuous aggregation operator. The theorem can be therefore derived directly from Definition 3.8 and Theorem 3.1. ■

Theorem 3.6 *Let $H_i = \{[\underline{h_i}(\alpha), \overline{h_i}(\alpha)], \alpha \in [0,1]\}$, $i = 1, 2, \ldots, n$ be fuzzy numbers on $[0,1]$. The first-level fuzzy minimum of $H_1, H_2, \ldots, H_n$ is a fuzzy number Y, $Y = \{[\underline{y}(\alpha), \overline{y}(\alpha)], \alpha \in [0,1]\}$, given for any $\alpha \in [0,1]$ by*

$$\underline{y}(\alpha) = \min\{\underline{h}_1(\alpha), \ldots, \underline{h}_n(\alpha)\} \tag{3.33}$$

$$\overline{y}(\alpha) = \min\{\overline{h}_1(\alpha), \ldots, \overline{h}_n(\alpha)\} \tag{3.34}$$

Proof: The proof is analogical to the proof of Theorem 3.5. ■

Again, the first-level fuzzy minimum and the first-level fuzzy maximum of n linear fuzzy numbers do not have to be always linear fuzzy numbers. Figure 3.3 presents the result of the aggregation of three triangular fuzzy numbers as a piecewise linear fuzzy number.

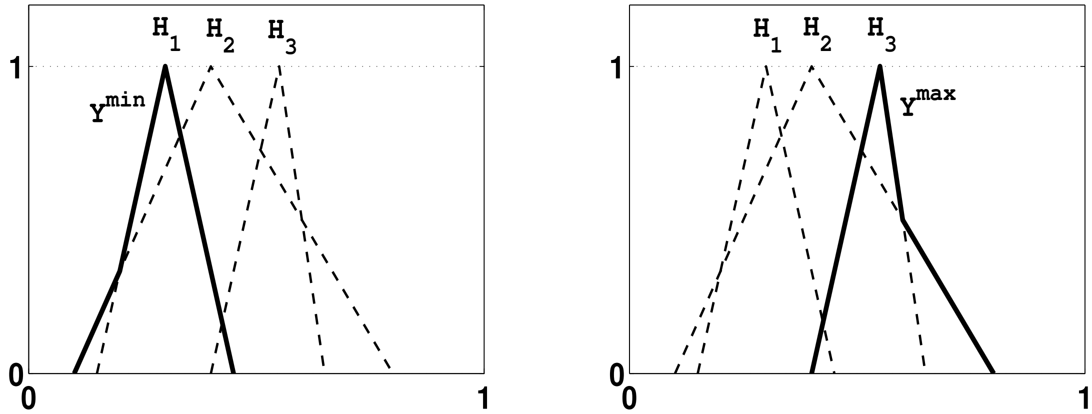


Figure 3.3: The piecewise linear fuzzy numbers Y^{min} and Y^{max} as a first-level fuzzy minimum and fuzzy maximum of fuzzy numbers H_1, H_2 and H_3, where $H_1 \sim (0.15, 0.3, 0.45)$, $H_2 \sim (0.1, 0.4, 0.8)$, $H_3 \sim (0.4, 0.55, 0.65)$.

Remark 3.10 *Note that similar to the crisp scenario, first-level fuzzy maximum and first-level fuzzy minimum are special cases of first-level fuzzy OWA. The weighting vectors which change first-level fuzzy OWA to first-level fuzzy maximum and first-level fuzzy minimum are* $(1, 0, \ldots, 0)$ *and* $(0, \ldots, 0, 1)$ *respectively.*

Remark 3.11 *Another approach to fuzzification of the OWA operators based on the extension principle can be seen, for example, in [61].*

3.2.3 First-level fuzzy Choquet integral

Here, the weights of the sets of criteria are described by a fuzzy measure $\mu : \wp(\Omega) \to [0, 1]$, i.e. by crisp values. According to the extension principle in combination with Theorem 3.1 the first-level fuzzy Choquet integral required for this type of aggregation is defined as follows:

Definition 3.16 *Let* $\Omega = \{C_1, \ldots, C_n\}$ *be a nonempty finite set,* μ *be a fuzzy measure on* Ω*, and* $F : \Omega \to \mathcal{F}_N([0, 1])$*,* $F(C_i) = H_i$*,* $i = 1, \ldots, n$*, be a FNV-function. The discrete Choquet integral of* F *with respect to the fuzzy measure* μ *is defined as a fuzzy number* Y *with a membership function given for any* $y \in [0, 1]$

by

$$\begin{aligned} Y(y) &= \max\Big\{ \min\{H_1(h_1),\ldots,H_n(h_n)\} \mid h_i \in [0,1], i=1,\ldots,n, \\ & y = h_{(n)}\mu(B_{(n)}) + \sum_{i=1}^{n-1} h_{(i)}\left[\mu(B_{(i)}) - \mu(B_{(i+1)})\right], \\ & B_{(i)} = \{C_{(i)},\ldots,C_{(n)}\}, i=1,\ldots,n\Big\}, \end{aligned} \tag{3.35}$$

where $(1),\ldots,(n)$ *denotes a permutation of indices* $1,\ldots,n$ *such that* $h_{(1)} \leq h_{(2)} \leq \cdots \leq h_{(n)}$. *The first-level fuzzy Choquet integral will be denoted by* $Y = (C)\int_\Omega F\mathrm{d}\mu$.

Remark 3.12 *In reaction to [7], where Definition 3.16 was presented, there has been proposed another form of definition of first-level fuzzy Choquet integral in [101]:*

$$\begin{aligned} Y(y) &= \max\Big\{ \min\{H_1(h_1),\ldots,H_n(h_n)\} \mid y = (C)\int_\Omega f d\mu, \\ & \text{where } f:\Omega \to [0,1] \text{ such that } f(C_i) = h_i,\ i=1,\ldots,n\Big\}. \end{aligned} \tag{3.36}$$

As the Choquet integral is continuous and monotonic [36], a simple way of calculating the first-level fuzzy Choquet integral is given by the following theorem.

Theorem 3.7 *Let* $H_i = \{[\underline{h_i}(\alpha), \overline{h_i}(\alpha)], \alpha \in [0,1]\}$, $i = 1,2,\ldots,n$ *be fuzzy numbers on* $[0,1]$. *The Choquet integral of FNV-function* F *from Definition 3.16 with respect to the fuzzy measure* μ, $Y = (C)\int_\Omega F\mathrm{d}\mu$, $Y = \{[\underline{y}(\alpha), \overline{y}(\alpha)], \alpha \in [0,1]\}$, *is given for any* $\alpha \in [0,1]$ *by*

$$\underline{y}(\alpha) = \sum_{i=1}^{n-1} \underline{h}_{(i)}(\alpha)\left[\mu(B_{(i)}) - \mu(B_{(i+1)})\right] + \underline{h}_{(n)}(\alpha)\mu(B_{(n)}), \tag{3.37}$$

where $(1),\ldots,(n)$ *is a permutation of indices* $1,\ldots,n$ *such that* $\underline{h}_{(1)}(\alpha) \leq \cdots \leq \underline{h}_{(n)}(\alpha)$, $B_{(i)} = \{C_{(i)},\ldots,C_{(n)}\}$, $i=1,\ldots,n$, *and*

$$\overline{y}(\alpha) = \sum_{i=1}^{n-1} \overline{h}_{(i)}(\alpha)\left[\mu(B_{(i)}) - \mu(B_{(i+1)})\right] + \overline{h}_{(n)}(\alpha)\mu(B_{(n)}) \tag{3.38}$$

where $(1),\ldots,(n)$ *is a permutation of indices* $1,\ldots,n$ *such that* $\overline{h}_{(1)}(\alpha) \leq \cdots \leq$

$\overline{h}_{(n)}(\alpha)$, $B_{(i)} = \{C_{(i)}, \ldots, C_{(n)}\}$, $i = 1, \ldots, n$.

Proof: The theorem can be straightforwardly proven using Theorem 3.2. ∎

Remark 3.13 *Similarly to the notation* $(C)\int_\Omega f\,d\mu = C_\mu(h_1, \ldots, h_n)$, $f(C_i) = h_i$, $i = 1, \ldots, n$, *the first-level fuzzy Choquet integral* $Y = (C)\int_\Omega F \mathrm{d}\mu$ *can be denoted by* $C_\mu(H_1, \ldots, H_n)$, *where* $F(C_i) = H_i$, $i = 1, \ldots, n$. *If* $H_i = \{[\underline{h_i}(\alpha), \overline{h_i}(\alpha)], \alpha \in [0,1]\}$, $i = 1, 2, \ldots, n$, *then the equations 3.37 and 3.38 from Theorem 3.7 can be also rewritten in another form:*

$$\underline{y}(\alpha) = C_\mu(\underline{h}_1(\alpha), \underline{h}_2(\alpha), \ldots, \underline{h}_n(\alpha)) \tag{3.39}$$

$$\overline{y}(\alpha) = C_\mu(\overline{h}_1(\alpha), \overline{h}_2(\alpha), \ldots, \overline{h}_n(\alpha)) \tag{3.40}$$

Remark 3.14 *The first-level fuzzification of the Choquet integral can be also done in another way. In [59], Meyer and Roubens used an alternative definition of the Choquet integral, described by Theorem 2.1, and applied the extension principle on operations of addition, scalar multiplication and minimum. After this they defined the fuzzy extension of the Choquet integral of FNV-function* $F : \Omega \to \mathcal{F}_N([0,1])$ *as*

$$\widetilde{C}_\mu(F(C_1), \ldots, F(C_n)) = \widetilde{\sum}_{A\subseteq\Omega} m^\mu(A) \,\tilde{\cdot}\, \widetilde{\min}\{F(C_i) \mid C_i \in A\}, \tag{3.41}$$

$$\left(\begin{array}{c} \text{or by using the previously presented notation} \\ \widetilde{C}_\mu(H_1, \ldots, H_n) = \widetilde{\sum}_{A\subseteq\{1,\ldots,n\}} m^\mu(A) \,\tilde{\cdot}\, \widetilde{\min}\{H_i \mid i \in A\}, \end{array} \right)$$

where m^μ *is the Möbius transform of fuzzy measure* μ *and operators* $\widetilde{+}, \tilde{\cdot}, \widetilde{\min}$ *are extensions of the operators* $+, \cdot, \min$ *such that for given two fuzzy numbers* C, D *and crisp number* $p \in \mathbb{R}$ *the following holds:*

$$C \widetilde{+} D(y) = \sup\left\{\min\{C(c), D(d)\} \mid c + d = y\right\}, \tag{3.42}$$

$$(p \,\tilde{\cdot}\, C)_\alpha = p \cdot C_\alpha, \forall\alpha \in [0,1], \tag{3.43}$$

$$\widetilde{\min}\{C, D\}(y) = \sup\left\{\min\{C(c), D(d)\} \mid \min\{c, d\} = y\right\}. \tag{3.44}$$

By using the other form of the Choquet integral they avoided the tricky reordering step in the process of calculation. Nevertheless, both the definitions are equivalent.

Moreover, in [96], Yang et al. proposed the fuzzification of the Choquet integral with respect to the, so called, signed fuzzy measure. That is a generalized fuzzy measure, which can attain arbitrary real numbered values, even negative ones. Finally, Wang. et al. [91] studied a way of using the Choquet integral to obtain crisp output from fuzzy valued input.

3.2.4 Generalized Partial Goals Method - Step 2

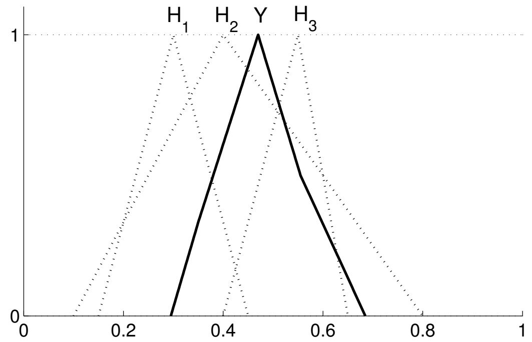
The application of first-level fuzzy Choquet integral in MCDM will be demonstrated on the next step of PGM generalization.

Let H_1, H_2, H_3 be fuzzy numbers on $[0, 1]$ modeling the partial evaluations of the alternative x with respect to three partial goals G_1, G_2, G_3. Each partial fuzzy evaluation H_i is a triangular fuzzy number, i.e. it is described by a triplet of real numbers - the first and the third determine the support of H_i, while the second determines the kernel. In this particular example $H_1 \sim (0.15, 0.3, 0.45)$, $H_2 \sim (0.1, 0.4, 0.8)$, and $H_3 \sim (0.4, 0.55, .65)$, and they represent the fuzzy degrees of fulfillment of particular partial goals. The partial goals G_1, G_2, G_3 cover the overall goal and the proportion of each one in the overall goal is set expertly as a crisp number (see Fig. 3.4(b)): $\mu(G_1) = 0.5$, $\mu(G_2) = 0.5$ and $\mu(G_3) = 0.6$. Moreover, the expert also sets the proportion of each couple of the partial goals in the overall goal: $\mu(\{G_1, G_2\}) = 0.8$, $\mu(\{G_2, G_3\}) = 0.8$ and $\mu(\{G_1, G_3\}) = 0.9$. Finally, $\mu(\{G_1, G_2, G_3\}) = 1$.

Because sums of proportions of the partial goals are not equal to proportions of the pairs (they are bigger), we can conclude there are some interactions between the criteria corresponding to those goals (criteria are partially redundant). In accordance with section 2.4.3 these interactions can be handled by the Choquet integral. Nevertheless, the Choquet integral presented in Definition 2.8 is not able to work with uncertain values and the aggregation needs to be performed by the first-level fuzzy Choquet integral. The result calculated with the help of the Theorem 3.7 can be seen in Fig. 3.4(a).

Remark 3.15 *As Figure 3.4 shows, the result of aggregation of fuzzy numbers is a fuzzy number of different type. While the fuzzy numbers on the input are linear, the output fuzzy number is only piecewise linear fuzzy number.*

The application of first-level fuzzy Choquet integral in PGM method repre-

(a) 

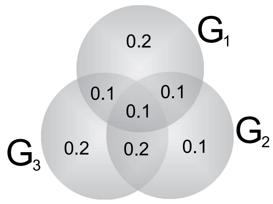
(b) 

Figure 3.4: **(a)** Fuzzy numbers $H_1 \sim (0.15, 0.3, 0.45)$, $H_2 \sim (0.1, 0.4, 0.8)$ and $H_3 \sim (0.4, 0.55, .65)$, modelling the partial evaluations and fuzzy number Y modelling the overall evaluation achieved by the first-level fuzzy Choquet integral. **(b)** Fuzzy measure defined on the set of partial goals G_1, G_2, G_3.

sents the second step in the process of PGM generalization. Now, the evaluator may use PGM to solve the multiple criteria evaluation problem with interacting criteria and uncertainty affecting the partial evaluations.

3.3 Second-level fuzzification of aggregation operators

So far the weights of the sets of criteria were considered to be crisp numbers. However, as the weights are usually set expertly they are burdened by uncertainty, and they should be modeled by fuzzy numbers as well.

During the second-level fuzzification, the aggregation operators are considered functions of $2n$ variables: n partial evaluations and n weights describing the importance of the criteria. This is different for the Choquet integral where, because of nonadditivity of the fuzzy measure, the number of variables is equal to $n + 2^n - 2$. The definitions of second-level fuzzy aggregation operators adhere to the extension principle with constraints given by the properties of classical additive or generalized monotonous fuzzy measure. The theorems explaining the calculation of the result of second-level fuzzy aggregation are derived from Theorems 3.1 and 3.2.

In a crisp scenario, the weighted average and OWA operators take advantage of normalized weights, i.e. the weights whose sum is equal to one. If the uncertainty occurs and the weights are replaced by fuzzy numbers, the condition of normality should still hold, at least in some adjusted form. In [65], [66] a special structure of normalized fuzzy weights was proposed.

Definition 3.17 (Normalized fuzzy weights) *Fuzzy numbers V_1, V_2, ..., V_n defined on $[0,1]$ form an $n-$tuple of normalized fuzzy weights, if for all $\alpha \in (0,1]$ and for any $i = \{1,2,\dots,n\}$ the following holds: for any $v_i \in V_{i\alpha}$ there exist $v_j \in V_{j\alpha}$, $j = 1,2,\dots,n$, $j \neq i$, such that*

$$v_i + \sum_{j=1, j\neq i}^{n} v_j = 1. \tag{3.45}$$

Remark 3.16 *Normalized fuzzy weights and a more general tool for modeling uncertain weights - fuzzy vector of normalized weights, were studied in detail in [67].*

3.3.1 Second-level fuzzy weighted average

Weighted average is the most popular aggregation operator in MCDM and it is therefore hardly surprising it was among the first aggregation operators to be fuzzified. The first attempt, made by [3], was later followed by [19], [40] and [65]. In the following we shall use the formalism and results originally presented in [65].

Definition 3.18 *Let H_i, $i = 1,2,\dots,n$, be fuzzy numbers on $[0,1]$ and V_i, $i = 1,2,\dots,n$, be normalized fuzzy weights. Second-level fuzzy weighted average of $H_1, H_2, \dots, H_n$ with normalized fuzzy weights $V_1, V_2, \dots, V_n$ is defined as fuzzy number Y with a membership function given for any $y \in [0,1]$ by*

$$\begin{aligned} Y(y) \;=\; \max\Big\{ & \min\big\{H_1(h_1),\dots,H_n(h_n),V_1(v_1),\dots,V_n(v_n)\big\} \mid \\ & h_i \in [0,1], i = 1,\dots,n, v_j \geq 0, j = 1,\dots,n, \sum_{j=1}^{n} v_j = 1 \\ & y = \sum_{i=1}^{n} v_i h_i \Big\}. \end{aligned} \tag{3.46}$$

Theorem 3.8 *Let $H_i = \{[\underline{h}_i(\alpha), \overline{h}_i(\alpha)], \alpha \in [0,1]\}$, $i = 1, 2, \dots, n$, be fuzzy numbers on $[0,1]$ and $V_i = \{[\underline{v}_i(\alpha), \overline{v}_i(\alpha)], \alpha \in [0,1]\}$, $i = 1, 2, \dots, n$, be normalized fuzzy weights. Let Y, $Y = \{[\underline{y}(\alpha), \overline{y}(\alpha)], \alpha \in [0,1]\}$, be second-level fuzzy weighted average of $H_1, H_2, \dots, H_n$ with normalized fuzzy weights $V_1, V_2, \dots, V_n$. Then for any $\alpha \in [0,1]$ the following holds:*

$$\underline{y}(\alpha) = \min\{\sum_{i=1}^{n} v_i \underline{h}_i(\alpha) | v_i \in [\underline{v}_i(\alpha), \overline{v}_i(\alpha)], \sum_{i=1}^{n} v_i = 1, i = 1, \dots, n\}, \tag{3.47}$$

and

$$\overline{y}(\alpha) = \max\{\sum_{i=1}^{n} v_i \overline{h}_i(\alpha) | v_i \in [\underline{v}_i(\alpha), \overline{v}_i(\alpha)], \sum_{i=1}^{n} v_i = 1, i = 1, \dots, n\}, \tag{3.48}$$

Proof: See [67] ■

Remark 3.17 *According to the previous theorem, the functions $\underline{y}(\alpha), \overline{y}(\alpha)$ can be obtained by solving two linear programming problems for each $\alpha \in [0,1]$. Nevertheless, in [65] and [66] another way for computing $\underline{y}(\alpha), \overline{y}(\alpha)$ was proposed. Importantly, the newly proposed algorithm can find the functions $\underline{y}(\alpha), \overline{y}(\alpha)$ without solving any mathematical programming problems.*

Remark 3.18 *Even if the input fuzzy numbers and fuzzy weights are linear fuzzy numbers, the result of the aggregation by second-level fuzzy weighted average generally does not have to be a linear fuzzy number at all. Figure 3.5 shows an example, when the output fuzzy number is piecewise quadratic.*

3.3.2 Second-level fuzzy OWA

The second level fuzzification of the OWA operators can be performed using the same principle as in the case of the weighted average.

Definition 3.19 *Let H_i, $i = 1, 2, \dots, n$, be fuzzy numbers on $[0,1]$ and V_i, $i = 1, 2, \dots, n$, be normalized fuzzy weights. Second-level fuzzy ordered weighted average of $H_1, H_2, \dots, H_n$ with normalized fuzzy weights $V_1, V_2, \dots, V_n$ is defined*

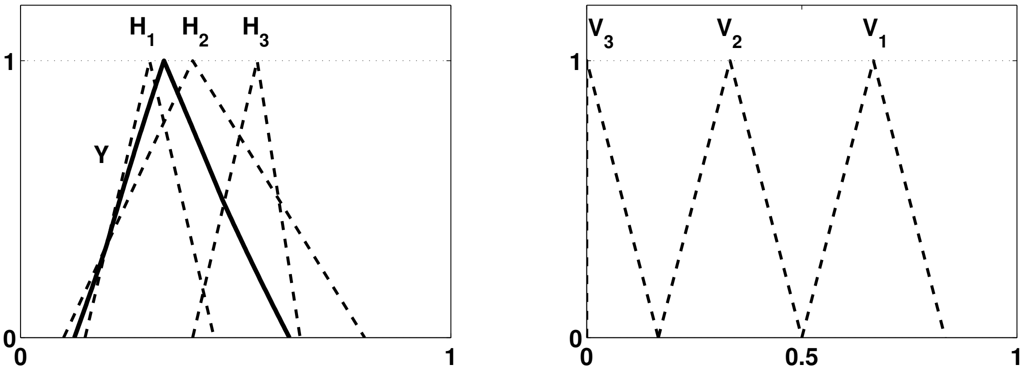


Figure 3.5: Fuzzy numbers $H_1 \sim (0.15, 0.3, 0.45)$, $H_2 \sim (0.1, 0.4, 0.8)$ and $H_3 \sim (0.4, 0.55, 0.65)$ modelling the partial evaluations and fuzzy number Y modelling the overall evaluation achieved by the second-level fuzzy weighted average of fuzzy numbers H_1, H_2, H_3 with normalized fuzzy weights V_1, V_2, V_3. $V_1 \sim (1/2, 2/3, 5/6)$, $V_2 \sim (1/6, 1/3, 1/2)$ and $V_3 \sim (0, 0, 1/6)$.

as a fuzzy number Y with a membership function given for any $y \in [0,1]$ by

$$\begin{aligned} Y(y) \quad = \quad & \max\Big\{ \min\big\{H_1(h_1), \ldots, H_n(h_n), V_1(v_1), \ldots, V_n(v_n)\big\} \mid \\ & h_i \in [0,1], i = 1, \ldots, n, v_j \geq 0, j = 1, \ldots, n, \sum_{j=1}^{n} v_j = 1 \\ & y = \sum_{i=1}^{n} v_i h^{(i)} \Big\} \end{aligned} \tag{3.49}$$

where $(1), \ldots, (n)$ denotes a permutation of indices $1, \ldots, n$ such that $h^{(1)} \geq h^{(2)} \geq \cdots \geq h^{(n)}$.

Theorem 3.9 *Let $H_i = \{[\underline{h}_i(\alpha), \overline{h}_i(\alpha)], \alpha \in [0,1]\}$, $i = 1, 2, \ldots, n$, be fuzzy numbers on $[0,1]$ and $V_i = \{[\underline{v}_i(\alpha), \overline{v}_i(\alpha)], \alpha \in [0,1]\}$, $i = 1, 2, \ldots, n$, be normalized fuzzy weights. Let Y, $Y = \{[\underline{y}(\alpha), \overline{y}(\alpha)], \alpha \in [0,1]\}$, be second-level fuzzy ordered weighted average of $H_1, H_2, \ldots, H_n$ with normalized fuzzy weights $V_1, V_2, \ldots, V_n$. Then for any α the following holds:*

$$\underline{y}(\alpha) = \min\{\sum_{i=1}^{n} v_i \underline{h}^{(i)}(\alpha) | v_i \in [\underline{v}_i(\alpha), \overline{v}_i(\alpha)], \sum_{i=1}^{n} v_i = 1, i = 1, \ldots, n\}, \tag{3.50}$$

and

$$\overline{y}(\alpha) = \max\{\sum_{i=1}^{n} v_i \overline{h}^{(i)}(\alpha) | v_i \in [\underline{v}_i(\alpha), \overline{v}_i(\alpha)], \sum_{i=1}^{n} v_i = 1, i = 1, \dots, n\}, \quad (3.51)$$

Proof: The proof follows naturally from the continuity and monotonicity of OWA operator and from the Theorem 3.1. ∎

According to Theorem 3.9, for each $\alpha \in [0,1]$ the values $\underline{y}(\alpha)$, $\overline{y}(\alpha)$ are calculated as minimum/maximum of ordered weighted averages of values $\underline{h}_i(\alpha)$, $\overline{h}_i(\alpha)$ and some values $v_i \in [\underline{v}_i(\alpha), \overline{v}_i(\alpha)]$, $i = 1, \dots, n$. Nevertheless, once the values $\underline{h}_i(\alpha)$, $\overline{h}_i(\alpha)$ are ordered for given $\alpha \in [0,1]$, the ordering is fixed and the minimum/maximum is calculated from the weighted averages. To simplify the practical computation, the following algorithm, as a modification of algorithm for calculation of fuzzy weighted average published in [67], was proposed:

Algorithm 3.1

Step 1: Let for any $\alpha \in [0,1]$, $\underline{h}_{(1)}(\alpha), \underline{h}_{(2)}(\alpha), \dots, \underline{h}_{(n)}(\alpha)$ *and* $\overline{h}^{(1)}(\alpha), \overline{h}^{(2)}(\alpha), \dots, \overline{h}^{(n)}(\alpha)$ *be such permutations of values* $\underline{h}_1(\alpha), \underline{h}_2(\alpha), \dots, \underline{h}_n(\alpha)$ *and* $\overline{h}_1(\alpha), \overline{h}_2(\alpha), \dots, \overline{h}_n(\alpha)$ *that following holds:* $\underline{h}_{(1)}(\alpha) \le \underline{h}_{(2)}(\alpha) \le \dots, \le \underline{h}_{(n)}(\alpha)$ *and* $\overline{h}^{(1)}(\alpha) \ge \overline{h}^{(2)}(\alpha) \ge \dots, \ge \overline{h}^{(n)}(\alpha)$.

Step 2: Let for $k \in \{1, 2, \dots, n\}$ *values* $v_k^L(\alpha)$ *and* $v_k^P(\alpha)$ *be given by formulas*

$$v_k^L(\alpha) = 1 - \sum_{i=k+1}^{n} \overline{v_i}(\alpha) - \sum_{i=1}^{k-1} \underline{v_i}(\alpha) \quad (3.52)$$

$$v_k^P(\alpha) = 1 - \sum_{i=1}^{k-1} \overline{v_i}(\alpha) - \sum_{i=k+1}^{n} \underline{v_i}(\alpha) \quad (3.53)$$

Step 3: Let k^* *and* k^{**} *denote such indices that following holds:*

$$\underline{v}_{k^*}(\alpha) \le v_{k^*}^L \le \overline{v}_{k^*} \quad (3.54)$$

$$\underline{v}_{k^{**}}(\alpha) \le v_{k^{**}}^P \le \overline{v}_{k^{**}} \quad (3.55)$$

Step 4: Then

$$\underline{y}(\alpha) = \sum_{i=1}^{n-k^*} \underline{h}_{(i)}(\alpha)\overline{v}_{n-i+1}(\alpha) + \underline{h}_{n-k^*+1}(\alpha)v_{k^*}^L(\alpha) + \sum_{i=n-k^*+2}^{n} \underline{h}_{(i)}(\alpha)\underline{v}_{n-i+1}(\alpha) \tag{3.56}$$

$$\overline{y}(\alpha) = \sum_{i=1}^{k^{**}-1} \overline{h}^{(i)}(\alpha)\overline{v}_i(\alpha) + \overline{h}^{k^{**}}(\alpha)v_{k^{**}}^P(\alpha) + \sum_{i=k^{**}+1}^{n} \overline{h}^{(i)}(\alpha)\underline{v}_i(\alpha) \tag{3.57}$$

Remark 3.19 *Similarly to the second-level fuzzy weighted average, application of the second-level fuzzy OWA with linear fuzzy weights on linear fuzzy numbers does not have to produce a linear fuzzy number. The Figure 3.6 illustrates the example, where the result of aggregation is a piecewise quadratic fuzzy number.*

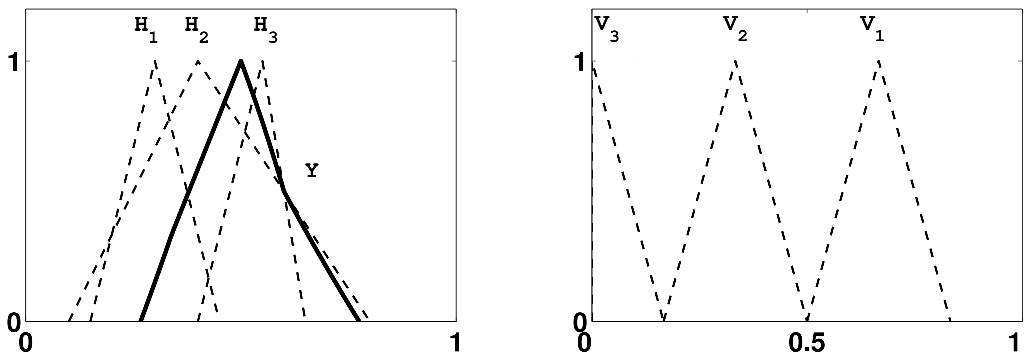


Figure 3.6: Fuzzy numbers H_1, H_2, H_3 modeling the partial evaluations and fuzzy number Y modeling the overall evaluation achieved by the second-level fuzzy ordered weighted average of fuzzy numbers H_1, H_2, H_3 with normalized fuzzy weights V_1, V_2, V_3. $H_1 \sim (0.15, 0.3, 0.45)$, $H_2 \sim (0.1, 0.4, 0.8)$, $H_3 \sim (0.4, 0.55, 0.65)$, $V_1 \sim (1/2, 2/3, 5/6)$, $V_2 \sim (1/6, 1/3, 1/2)$ and $V_3 \sim (0, 0, 1/6)$.

Remark 3.20 *Unlike in the case of the first-level fuzzification, there is no need to define second-level fuzzy maximum and minimum. This is a consequence of the nature of these two aggregation operators. Usually, if maximum or minimum operator is applied, then the evaluator intends to find the biggest or the smallest value, respectively. In such the scenario, the evaluator is sure about the weights and looking for approximately biggest or smallest value does not make sense.*

Remark 3.21 *Simultaneously with Definition 3.19 and Theorem 3.9 being published in [80], another approach to fuzzification of OWA appeared in [99]. The* *definition is equivalent, but our method of calculation is more transparent and easier to use.*

3.3.3 Second-level fuzzy Choquet integral

In the second level of the fuzzification process, the weights of the criteria or the partial evaluations are considered to be uncertain and modeled by special system of fuzzy numbers. However, the Choquet integral does not work with simple weights. The original Choquet integral is defined as an integral with respect to fuzzy measure, which describes the importance of the various combinations of criteria or partial goals.

Despite its name, a fuzzy measure does not work with fuzzy numbers. Fuzzy measure of any given set is a crisp number. Therefore, in order to proceed in the fuzzification, the fuzzy measure, defined in the previous section, has to be modified. The new kind of set function, the values of which are fuzzy numbers, will be called a FNV-fuzzy measure.

Definition 3.20 *A FNV-fuzzy measure on a finite set Ω, $\Omega = \{C_1, C_2, \ldots, C_n\}$, is a set function $\widehat{\mu} : \wp(\Omega) \to \mathcal{F}_N([0,1])$ satisfying the following axioms:*

- $\widehat{\mu}(\emptyset) = 0$, $\widehat{\mu}(\Omega) = 1$,
- $C \subseteq D$ *implies* $\widehat{\mu}(C) \leq \widehat{\mu}(D)$ *for any* $C, D \in \wp(\Omega)$.

Remark 3.22 *The notation $\widehat{\mu}(C) \leq \widehat{\mu}(D)$ in the second condition of Definition 3.20 expresses the ordering of fuzzy numbers defined in Definition 3.6.*

Following this definition, the extension principle, and Theorem 3.1, the second-level fuzzy Choquet integral can be defined as:

Definition 3.21 *Let $\Omega = \{C_1, \ldots, C_n\}$ be a nonempty finite set, $B_1, \ldots, B_{2^n-1}$ be all its nonempty subsets, $\widehat{\mu}$ be a FNV-fuzzy measure on Ω, and $F : \Omega \to \mathcal{F}_N([0,1])$, $F(C_i) = H_i$, $i = 1, \ldots, n$, be a FNV-function. The discrete Choquet integral of F with respect to FNV-fuzzy measure $\widehat{\mu}$ is defined as a fuzzy number*

Y with a membership function given for any $y \in [0,1]$ by

$$\begin{aligned} Y(y) \;=\; & \max\Big\{ \min\big\{H_1(h_1),\ldots,H_n(h_n),\widehat{\mu}(B_1)(\mu_1),\ldots,\widehat{\mu}(B_{2^n-1})(\mu_{2^n-1})\big\} \mid \\ & h_i \in [0,1], i=1,\ldots,n, \mu_j \in [0,1], j=1,\ldots,2^n-1, \\ & y = h_{(n)}\beta_n + \sum_{i=1}^{n-1} h_{(i)}\left[\beta_i - \beta_{i+1}\right], \beta_k - \beta_{k+1} \geq 0, k=1,\ldots,n-1, \\ & \textit{and for } i=1,\ldots,n, \textit{ it holds that } \beta_i = \mu_j, \\ & \textit{where } j \in \{1,\ldots,2^n-1\} \textit{ such that } B_j = \{C_{(i)},\ldots,C_{(n)}\}\Big\} \end{aligned} \tag{3.58}$$

where $(1),\ldots,(n)$ denotes a permutation of indices $1,\ldots,n$ such that $h_{(1)} \leq h_{(2)} \leq \cdots \leq h_{(n)}$. The second-level fuzzy Choquet integral will be denoted by $Y = (C)\int_\Omega F \mathrm{d}\widehat{\mu}$.

Note that in Definition 3.21 we are looking for the minimum among $(n + 2^n - 1)$ membership degrees, even though there are only $2n$ variables (n partial evaluations and n weights) in the formula for Choquet integral. The seemingly large number $(n + 2^n - 1)$ is a consequence of the need to consider all possible subsets of the set of partial goals, because for each n-tuple $h_1,\ldots,h_n$ a different n-tuple of subsets B_i of the set Ω is relevant. The relevant n-tuple of subsets B_i is given by the condition in Definition 3.21. The remaining membership degrees, among which we are looking for the minimum but which are irrelevant for given n-tuple $h_1,\ldots,h_n$, can be considered equal to 1, and therefore do not influence the minimum value.

Remark 3.23 *As with Definition 3.16, Definition 3.21 can be rewritten in the alternative form [101]:*

$$\begin{aligned} Y(y) \;=\; & \max\Big\{ \min\{H_1(h_1),\ldots,H_n(h_n),\mu_F(B_{(1)})(\mu_1),\ldots,\mu_F(B_{(n)})(\mu_n)\} \mid \\ & y = (C)\int_\Omega f d\mu, \textit{ where } f:\Omega \to [0,1] \textit{ such that } f(C_i) = h_i, \\ & i = 1,\ldots,n, \textit{ and } \mu \textit{ is a fuzzy measure on } \Omega \textit{ such that } \mu(B_{(i)}) = \mu_i, \\ & i = 1,\ldots,n, \textit{ where } (i) \textit{ is a permutation such that } h_{(1)} \leq \ldots \leq h_{(n)}, \\ & \textit{and } B_{(i)} = \{C_{(i)},\ldots,C_{(n)}\}\Big\}. \end{aligned} \tag{3.59}$$

During the review process of [6], one of the unknown reviewers suggested yet another form of the definition 3.21:

Definition 3.22 *Let $\Omega = \{C_1, \dots, C_n\}$ be a nonempty finite set, $\widehat{\mu}$ be a FNV-fuzzy measure on Ω, and $F : \Omega \to \mathcal{F}_N([0,1])$ be FNV-function. Let $\mathcal{M}$ and $\mathcal{F}$ be the sets of all fuzzy measures $\mu : \wp(\Omega) \to [0,1]$ and functions $f : \Omega \to [0,1]$, respectively. Let for any $y \in [0,1]$ there be a relation $R_y \subseteq \mathcal{M} \times \mathcal{F}$ such that $(\mu, f) \in R_y$ if and only if $y = (C)\int_\Omega f \; d\mu$. Then Choquet integral of FNV-function F with respect to FNV-fuzzy measure $\widehat{\mu}$ is defined as a fuzzy number Y with a membership function given for any $y \in [0,1]$ by*

$$Y(y) = \max_{(\mu,f)\in R_y} \min_{i=1,\dots,n} \min(F(C_i)(f(C_i)), \widehat{\mu}(B_{(i)})(\mu(B_{(i)}))),$$

where $(1), \dots, (n)$ denotes a permutation of indices $1, \dots, n$ such that $f(C_{(1)}) \leq f(C_{(2)}) \leq \dots \leq f(C_{(n)})$ and $B_{(i)} = \{C_{(i)}, C_{(i+1)}, \dots, C_{(n)}\}$.

Because the Choquet integral is continuous and monotonic with respect to its argument, we can use Theorem 3.1 and Theorem 3.2 again and lay down Theorem 3.10, which can be used to calculate the integral.

Theorem 3.10 *Let $H_i = \{[\underline{h}_i(\alpha), \overline{h}_i(\alpha)], \alpha \in [0,1]\}$, $i = 1, 2, \dots, n$, be fuzzy numbers on $[0,1]$. The Choquet integral of FNV-function F from Definition 3.21 with respect to the FNV-fuzzy measure $\widehat{\mu}$, $Y = (C)\int_\Omega F \mathrm{d}\widehat{\mu}$, $Y = \{[\underline{y}(\alpha), \overline{y}(\alpha)], \alpha \in [0,1]\}$, is given for any $\alpha \in [0,1]$ by*

$$\begin{aligned} \underline{y}(\alpha) &= \min\Big\{ \sum_{i=1}^{n-1} \underline{h}_{(i)}(\alpha) \left[\mu_{(i)} - \mu_{(i+1)}\right] + \underline{h}_{(n)}(\alpha)\mu_{(n)} \mid \\ &\quad \mu_{(i)} \in [\underline{\mu}_{(i)}(\alpha), \overline{\mu}_{(i)}(\alpha)], B_{(i)} = \{C_{(i)}, \dots, C_{(n)}\}, i = 1, \dots, n, \\ &\quad \mu_{(j)} - \mu_{(j+1)} \geq 0, j = 1, \dots, n-1 \Big\}, \end{aligned} \tag{3.60}$$

where $(1), \dots, (n)$ denotes a permutation of indices $1, \dots, n$ such that $\underline{h}_{(1)}(\alpha) \leq \dots \leq \underline{h}_{(n)}(\alpha)$, and $\underline{\mu}_i(\alpha)$, $\overline{\mu}_i(\alpha)$ are functions such that $(\widehat{\mu}(B_i))_\alpha = [\underline{\mu}_i(\alpha), \overline{\mu}_i(\alpha)]$; and

$$\begin{aligned} \overline{y}(\alpha) &= \max\Big\{ \sum_{i=1}^{n-1} \overline{h}_{(i)}(\alpha) \left[\mu_{(i)} - \mu_{(i+1)}\right] + \overline{h}_{(n)}(\alpha)\mu_{(n)} \mid \\ &\quad \mu_{(i)} \in [\underline{\mu}_{(i)}(\alpha), \overline{\mu}_{(i)}(\alpha)], B_{(i)} = \{C_{(i)}, \dots, C_{(n)}\}, i = 1, \dots, n, \\ &\quad \mu_{(j)} - \mu_{(j+1)} \geq 0, j = 1, \dots, n-1 \Big\} \end{aligned} \tag{3.61}$$

where $(1), \ldots, (n)$ *is a permutation of indices* $1, \ldots, n$ *such that* $\overline{h}_{(1)}(\alpha) \leq \cdots \leq \overline{h}_{(n)}(\alpha)$ *and* $\underline{\mu}_i(\alpha)$, $\overline{\mu}_i(\alpha)$ *are again functions such that* $(\widehat{\mu}(B_i))_\alpha = [\underline{\mu}_i(\alpha), \overline{\mu}_i(\alpha)]$.

Proof: The proof follows from the monotonicity and continuity of the Choquet integral in combination with Theorems 3.1 and 3.2. ∎

However, arriving at the desired result with help of this theorem is unwieldy as it requires multiple minimizations over varying sets of parameters. Fortunately the calculation can be significantly simplified by employing the following theorem published in [7]:

Theorem 3.11 *Let* $H_i = \{[\underline{h}_i(\alpha), \overline{h}_i(\alpha)], \alpha \in [0,1]\}$, $i = 1, 2, \ldots, n$, *be fuzzy numbers on* $[0,1]$. *The Choquet integral of FNV-function F from definition 3.21 with respect to the FNV-fuzzy measure* $\widehat{\mu}$, $Y = (C) \int_\Omega F \mathrm{d}\widehat{\mu}$, $Y = \{[\underline{y}(\alpha), \overline{y}(\alpha)], \alpha \in [0,1]\}$, *is given for any* $\alpha \in [0,1]$ *by*

$$\underline{y}(\alpha) = \sum_{i=1}^{n-1} \underline{h}_{(i)}(\alpha) \left[\underline{\mu}_{(i)}(\alpha) - \underline{\mu}_{(i+1)}(\alpha)\right] + \underline{h}_{(n)}(\alpha)\underline{\mu}_{(n)}(\alpha), \tag{3.62}$$

where $(1), \ldots, (n)$ *is a permutation of indices* $1, \ldots, n$ *such that* $\underline{h}_{(1)}(\alpha) \leq \cdots \leq \underline{h}_{(n)}(\alpha)$, $\widehat{\mu}(B_{(i)}) = \{[\underline{\mu}_{(i)}(\alpha), \overline{\mu}_{(i)}(\alpha)], \alpha \in [0,1]\}$, $B_{(i)} = \{C_{(i)}, \ldots, C_{(n)}\}$, $i = 1, \ldots, n$; *and*

$$\overline{y}(\alpha) = \sum_{i=1}^{n-1} \overline{h}_{(i)}(\alpha) \left[\overline{\mu}_{(i)}(\alpha) - \overline{\mu}_{(i+1)}(\alpha)\right] + \overline{h}_{(n)}(\alpha)\overline{\mu}_{(n)}(\alpha), \tag{3.63}$$

with $(1), \ldots, (n)$ *as a permutation of indices* $1, \ldots, n$ *such that* $\overline{h}_{(1)}(\alpha) \leq \cdots \leq \overline{h}_{(n)}(\alpha)$, *and* $\widehat{\mu}(B_{(i)}) = \{[\underline{\mu}_{(i)}(\alpha), \overline{\mu}_{(i)}(\alpha)], \alpha \in [0,1]\}$, $B_{(i)} = \{C_{(i)}, \ldots, C_{(n)}\}$, $i = 1, \ldots, n$.

Proof:

According to Theorem 3.10, for any $\alpha \in [0,1]$ the values $\underline{y}(\alpha)$ are determined by the solution of the following linear programming problem:

Minimize

$$f(\mu_{(1)}, \ldots, \mu_{(n)}) = \sum_{i=1}^{n-1} \underline{h}_{(i)}(\alpha) \left[\mu_{(i)} - \mu_{(i+1)}\right] + \underline{h}_{(n)}(\alpha)\mu_{(n)} \tag{3.64}$$

under constraints

$$\begin{aligned}&\mu_{(j)} - \mu_{(j+1)} \geq 0, \quad j = 1, \dots, n-1,\\&\mu_{(i)} \in [\underline{\mu}_{(i)}(\alpha), \overline{\mu}_{(i)}(\alpha)], B_{(i)} = \{C_{(i)}, \dots, C_{(n)}\}, \quad i = 1, \dots, n.\end{aligned} \tag{3.65}$$

Let us define $\underline{h}_0(\alpha) = 0$. The problem can be then rewritten into the following form:

Minimize

$$f(\mu_{(1)}, \dots, \mu_{(n)}) = \sum_{i=1}^{n} \left[\underline{h}_{(i)}(\alpha) - \underline{h}_{(i-1)}(\alpha)\right] \mu_{(i)} \tag{3.66}$$

under constraints

$$\begin{aligned}&\mu_{(j)} - \mu_{(j+1)} \geq 0, \quad j = 1, \dots, n-1,\\&\underline{\mu}_{(i)}(\alpha) \leq \mu_{(i)} \leq \overline{\mu}_{(i)}(\alpha), B_{(i)} = \{C_{(i)}, \dots, C_{(n)}\}, \quad i = 1, \dots, n.\end{aligned} \tag{3.67}$$

Fuzzy numbers $\widehat{\mu}(B_i) = \{[\underline{\mu}_i(\alpha), \overline{\mu}_i(\alpha)], \alpha \in [0,1]\}$ are values of a FNV-fuzzy measure and are therefore defined on the interval $[0,1]$. As a consequence, the convex polyhedron defined by the constraints is bounded and the solution exists. Because the objective function $f(\mu_{(1)}, \dots, \mu_{(n)})$ is linear, the solution will be at a vertex of the polyhedron. Furthermore, the values $(\underline{h}_{(i)}(\alpha) - \underline{h}_{(i-1)}(\alpha))$ are non-negative for any $\alpha \in [0,1]$, $i = 1, 2, \dots, n$, thus the function $f(\mu_{(1)}, \dots, \mu_{(n)})$ is non-decreasing. It is apparent then that the minimum is attained at the point $(\underline{\mu}_{(1)}(\alpha), \dots, \underline{\mu}_{(n)}(\alpha))$, which is one of the vertices of the polyhedron, where the condition (3.67) is satisfied as per Definition 3.20. The minimum can be now expressed as

$$\underline{y}(\alpha) = \sum_{i=1}^{n} \left[\underline{h}_{(i)}(\alpha) - \underline{h}_{(i-1)}(\alpha)\right] \underline{\mu}_{(i)}(\alpha), \tag{3.68}$$

which is equivalent to

$$\underline{y}(\alpha) = \sum_{i=1}^{n-1} \underline{h}_{(i)}(\alpha) \left[\underline{\mu}_{(i)}(\alpha) - \underline{\mu}_{(i+1)}(\alpha)\right] + \underline{h}_{(n)}(\alpha)\underline{\mu}_{(n)}(\alpha). \tag{3.69}$$

The second statement of the theorem can be proven analogically, searching for maximum instead of minimum. ■

Remark 3.24 *Similarly to the first-level fuzzy Choquet integral, the second-level fuzzy Choquet integral* $Y = (C)\int_{\Omega} F\mathrm{d}\widehat{\mu}$ *can be denoted also by* $C_{\widehat{\mu}}(H_1, \ldots, H_n)$*. Moreover, if* F *is a FNV-function from definition 3.21,* $H_i = \{[\underline{h}_i(\alpha), \overline{h}_i(\alpha)], \alpha \in [0,1]\}$*,* $i = 1 \ldots, n$*, and* $\widehat{\mu}$ *is a FNV-fuzzy measure on* Ω *such that* $\widehat{\mu}(C) = \{[\underline{\mu}_C(\alpha), \overline{\mu}_C(\alpha)], \alpha \in [0,1]\}$ *for any* $C \in \wp(\Omega)$*, then the equations 3.62 and 3.63 from Theorem 3.11 can be rewritten in another form:*

$$\underline{y}(\alpha) = C_{\underline{\mu}_\alpha}(\underline{h}_1(\alpha), \ldots, \underline{h}_n(\alpha)), \tag{3.70}$$

where $\underline{\mu}_\alpha$ *is a fuzzy measure on* Ω *such that* $\underline{\mu}_\alpha(C) = \underline{\mu}_C(\alpha)$ *for all* $C \in \wp(\Omega)$*;*

$$\overline{y}(\alpha) = C_{\overline{\mu}_\alpha}(\overline{h}_1(\alpha), \ldots, \overline{h}_n(\alpha)), \tag{3.71}$$

where $\overline{\mu}_\alpha$ *is a fuzzy measure on* Ω *such that* $\overline{\mu}_\alpha(C) = \overline{\mu}_C(\alpha)$ *for all* $C \in \wp(\Omega)$*.*

Remark 3.25 *The Theorems 3.11 and 3.9 are being used for computation of second-level fuzzy OWA and second-level fuzzy Choquet integral by software FuzzME, which was developed to model multiple criteria evaluation problems [81].*

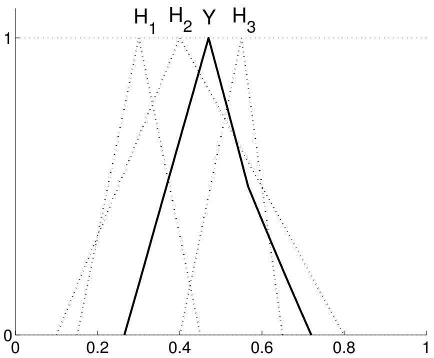
Remark 3.26 *The Sugeno integral can be fuzzified as well. In [93] and [94] Wu et. al extended the Sugeno integral to accommodate fuzzy-valued functions and fuzzy number fuzzy measures. In order to fuzzify the Sugeno integral, an interval number fuzzy measure and an interval number function were defined and applied to representation theorem of fuzzy sets. In [41], a similar approach was applied by Guo, Zhang and Wu to fuzzification of a generalized fuzzy integral. Later, in [42], this general approach was used to define the fuzzified Choquet integral of fuzzy-valued function with respect to fuzzy number fuzzy measures. The definition employed the interval number fuzzy measures and interval number functions.*

3.3.4 Generalized Partial Goals Method - Step 3

The incorporation of the second-level fuzzy Choquet integral into Partial Goals Method is the third and final step of PGM generalization.

Let the decision making problem be the same as in Section 3.2.4, but with some complexity added. Here, the proportions of the partial goals as well as the proportions of all the subsets of the partial goals in the overall goal are now uncertain. An expert described them by fuzzy numbers illustrated in Fig.3.7(b) and by doing that the FNV-fuzzy measure on set of partial goals was constructed. In such the case, the aggregation of the partial evaluations H_1, H_2, H_3 can not be done by first-level fuzzy Choquet integral, but its second-level version has to be employed instead. The overall evaluation of an alternative is then given by the second-level fuzzy Choquet integral. The figure Fig.3.7(a) shows the partial evaluations H_1, H_2, H_3 and the overall evaluation Y computed with help of Theorem 3.11.

The Generalized Partial Goals Method - Step 3 (GPGM) is a powerful tool. Above the capability to handle multiple criteria evaluation problem with interactions, it also deals with uncertainty. After the third step of generalization, it is able to work with uncertain partial evaluations as well as with uncertain significance of the partial goals and the subsets of partial goals.

(a) 

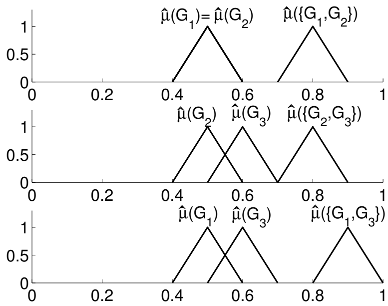
(b) 

Figure 3.7: **(a)** Fuzzy numbers H_1, H_2, H_3 describing the partial evaluations and the overall evaluation Y. **(b)** Fuzzy numbers modeling the contribution of the partial goals to the overall goal, i.e. FNV-fuzzy measure.

The comparison of the results of all three steps of PGM generalization is demonstrated by the following example.

Example 3.1 *Let us go back to the Example 2.3 from section 2.1. The graduates of high schools are being evaluated with respect to their ability to study a science. According to their tests' results they are evaluated with respect to three criteria:*

- *mathematics (M),*
- *physics (Ph),*

- *chemistry (Ch).*

Those criteria corresponds to three partial goals

- *A student of science needs to be able to apply the basic principals of mathematics (G_1)*
- *A student of science needs to be able to apply the basic principals of physics (G_2)*
- *A student of science needs to be able to apply the basic principals of chemistry (G_3)*

In the Example 2.3 the partial evaluations were presented by percentages of successfully answered questions. Given the partial evaluations $h_M = 0.9$*,* $h_{Ph} = 0.5$ *and* $h_{Ch} = 0.2$ *and fuzzy measure defined on the set of partial goals as described by Figure 3.8(b), the crisp overall evaluation of a graduate was calculated with help of Choquet integral as* $Y = 0.64$*.*

(a)

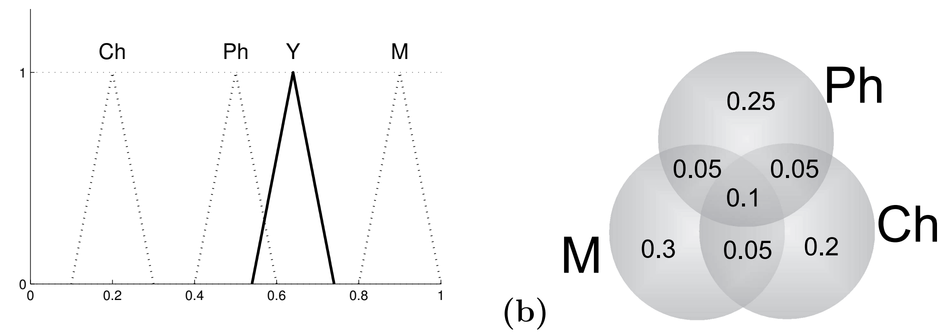


Figure 3.8: **(a)** Fuzzy numbers M, Ph, Ch describing uncertain partial evaluations of a student and the overall evaluation Y. **(b)** Fuzzy measure defined on the set of particular tests M, Ph, Ch.

However, the problem may become more complicated. The particular tests are usually evaluated expertly. For example, in mathematics the evaluator must consider what part of the problem did the student solve. It is therefore more natural to express the test results by fuzzy numbers, which can, unlike the crisp numbers, take into account the uncertainty of the evaluation. In such the case, the overall evaluation Y *can be computed with help of the first-level fuzzy Choquet integral. For the partial evaluations* $H_M \sim (0.8, 0.9, 1)$*,* $H_{Ph} \sim (0.4, 0.5, 0.6)$ *and*

$H_{Ch} \sim (0.1, 0.2, 0.3)$ *and fuzzy measure described with help of Fig. 3.8(b) the overall evaluation of a graduate is modeled by fuzzy number* Y *in Fig.3.8(a).*

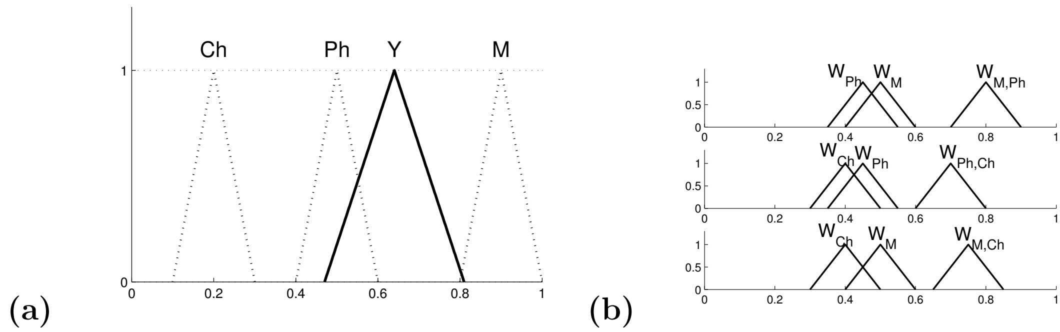


Figure 3.9: **(a)** Fuzzy numbers M, Ph, Ch describing uncertain partial evaluations of a student and the overall evaluation Y. **(b)** Fuzzy numbers modelling the proportions of the particular tests and their groups in the overall evaluation, i.e. FNV-fuzzy measure.

Moreover, the complexity of the problem may grow again. The weights of the particular tests (their proportions in the overall evaluation) are also set expertly and therefore uncertain. Then, it is convenient to model the uncertainty of the weights by fuzzy numbers W_M*,* W_{Ph}*,* W_{Ch} *(see Fig.3.9(b)) and compute the overall evaluation* Y *by the second-level fuzzy Choquet integral (Fig.3.9(a)).*

Let us discuss the difference between the crisp and the fuzzy evaluation. The first overall evaluation of the student, $Y = 0.64$*, computed with crisp inputs, is high enough to classify the student as 'above-average'. But if we admit the uncertainty of the partial evaluations, because they are 'estimated' expertly, the overall evaluation stops looking so good. Moreover, if also the uncertainty of the weights is taken into the consideration, the uncertainty of the output fuzzy number grows and the original overall evaluation as 'above-average' is not so convincing anymore. The result, which can be seen in Fig.3.9, reveals a distinct possibility that the student is not reaching even the average evaluation, therefore his/her aptitude for science may not be sufficient.*

3.4 FNV-fuzzy measure construction

In the last section, we have defined a FNV-fuzzy measure. Similarly to the original fuzzy measure, even this one is a set function, but its values are fuzzy numbers expressing the contributions of the particular partial criteria and their groupings to the overall evaluation. It is able to model the interactions among the criteria together with the uncertainty which accompanies their weights.

To apply the FNV-fuzzy measure to multiple criteria evaluation methods, namely the GPGM, the FNV-fuzzy measure has to be constructed properly. Similarly to the fuzzy measure, the FNV-fuzzy measure can be also set directly, nevertheless it is more difficult to check the monotonicity. A direct approach is convenient only for a small number of criteria besides the evaluator needs some experience with solving the given problems to be able to construct the FNV-fuzzy measure directly.

Analogically to the crisp scenario, there is also the indirect way of constructing the FNV-fuzzy measure published in [5]:

Similarly as in section 2.4.4, we can construct a set of imaginary alternatives and group them into decreasingly ordered classes with help of Pairwise Comparison Method. Then we describe the intensity of the preferences between each successive classes of objects. This time the linguistic terms 'as good as', 'slightly better', 'quite better', 'strongly better' or 'extremely better' are associated with fuzzy numbers 1, T_1, T_2, T_3, T_4. After assigning the value $R_j \in \{1, T_1, T_2, T_3, T_4\}$, $R_j = \{[\underline{r}_j(\alpha), \overline{r}_j(\alpha)], \alpha \in [0,1]\}$ to any pair of successive groups Π_j, Π_{j+1}, $j = 1, \ldots, k-2$, we can evaluate the objects from Π_j, $j = 2, \ldots, k-1$, by fuzzy number $H_j = \{[\underline{h}^j(\alpha), \overline{h}^j(\alpha)], \alpha \in [0,1]\}$, where for all $\alpha \in [0,1]$ holds

$$\underline{h}^j(\alpha) = \frac{1}{\overline{r}_1(\alpha) \cdots \overline{r}_{j-1}(\alpha)}, \tag{3.72}$$

$$\overline{h}^j(\alpha) = \frac{1}{\underline{r}_1(\alpha) \cdots \underline{r}_{j-1}(\alpha)}. \tag{3.73}$$

The objects from the group Π_1 and Π_k are evaluated by $H^1 = 1$, $H^k = 0$, respectively. The evaluations of the objects are equal to the weights of the corresponding sets of the partial goals and thus the FNV-fuzzy measure is set.

The construction of the FNV-fuzzy measure can be described also in the form of an algorithm:

Algorithm 3.2

Step 1: *Generate imaginary objects x^i, $i = 1, 2, \ldots, 2^n$, such that each object corresponds to a different subset of the set of partial goals $G_1, G_2, \ldots, G_n$, and it absolutely satisfies the goals in the subset and totally fails in achieving all the others.*

Step 2: *Create matrix $A = \{a_{ij}\}_{i,j=1}^{2^n}$: For each $j = 1, 2, \ldots, 2^n$ check if a_{ij} has already been set. If yes, then $a_{ji} = 1 - a_{ij}$. Otherwise check the following:*

a. *Inclusion: if the set of partial goals corresponding to the object x^i is a proper superset (equal to, proper subset) of the set of partial goals corresponding to the object x^j, then $a_{ij} = 1$ ($a_{ij} = 0.5$, $a_{ij} = 0$). Proceed to the next element of matrix A.*

b. *Transitivity: if there exists an index z such that $a_{iz} = a_{zj} = 1$ ($a_{iz} = a_{zj} = 0$, $a_{iz} = a_{zj} = 0.5$), then $a_{ij} = 1$ ($a_{ij} = 0$, $a_{ij} = 0.5$). Proceed to the next element of matrix A.*

c. *Goal fulfilment: compare object x^i with object x^j. If the object x^i fulfils the overall goal G_0 better than the object x^j, then $a_{ij} = 1$; if both objects achieve the overall goal equally, then $a_{ij} = 0.5$; otherwise $a_{ij} = 0$. Proceed to the next element of matrix A.*

Step 3: *For each $i = 1, 2, \ldots, 2^n$ calculate preference index $\sum_{j=1}^{2^n} a_{ij}$.*

Step 4: *Order the objects decreasingly according to their preference indexes and bunch them into groups Π_q, $q = 1, 2, \ldots, k$, $k \leq 2^n$, such that all the members of the group gave the same preference index.*

Step 5: *For $q = 1, 2, \ldots, k-1$ compare groups Π_q, Π_{q+1} and quantify their relationship by fuzzy number $R_q = \{\left[\underline{r_q}(\alpha), \overline{r_q}(\alpha)\right], \alpha \in [0,1]\}$, which can attain one of the five values $\{1, T_1, T_2, T_3, T_4\}$, where each value corresponds to one linguistic term of "as good as", "slightly better", "quite better", "strongly better" or "extremely better", respectively.*

Step 6: *For $q = 2, 3, \ldots, k-1$ calculate the evaluation $H_q = \{\left[\underline{h^q}(\alpha), \overline{h^q}(\alpha)\right], \alpha \in [0,1]\}$ of the objects from group Π_q in the following way: For all $\alpha \in [0,1]$*

calculate $\underline{h^q}(\alpha) = \frac{1}{\overline{r_1}(\alpha)\cdots\overline{r_{q-1}}(\alpha)}$ *and* $\overline{h^q}(\alpha) = \frac{1}{\underline{r_1}(\alpha)\cdots\underline{r_{q-1}}(\alpha)}$*. The remaining two evaluations are* $H^1 = 1$ *and* $H^k = 0$*.*

Step 7: FNV-fuzzy measure of each subset of the set of partial goals is then equal to the evaluation h^q *of the corresponding object* $x^i \in \Pi_q$*.*

Because of the demanding construction of FNV- fuzzy measure, it may seem that the implementation of fuzzified Choquet integral to multiple criteria evaluation problem is too complicated for practical applications. But let us not step ahead of ourselves and compare it to other methods. Any multiple criteria evaluation problem can be described by fuzzy rules base [2], which can deal with any kind of interactions among the criteria, even if the partial evaluations are uncertain and modeled by fuzzy numbers. However, the formulation of fuzzy rules requires an experienced and patient expert. Considering n criteria, each of which can attain m different values, the expert has to formulate m^n fuzzy rules to describe the problem properly. If we compare it to the number of parameters the expert has to set during the direct construction of FNV-fuzzy measure, i.e. 2^n fuzzy numbers, we can see that for $m > 2$ the construction of fuzzy rule base is more demanding than direct construction of FNV-fuzzy measure. As it was mentioned before, our technique for FNV-fuzzy measure construction requires setting at most $2^{2n-1} - 2^{n-1} - 1$ parameters in the first and $2^n - 2$ parameters in the second step, the exact number depending on the problem we are solving. In total, that is at most $2^{2n-1} + 2^{n-1} - 3$ parameters, where $2^{2n-1} - 2^{n-1} - 1$ of them are only zeroes or ones and, consequently, fairly easy to set. It can be seen that for $m > 3$ the application of fuzzified Choquet integral with our technique for FNV-fuzzy measure construction requires less parameters than construction of the complete fuzzy rules base does. Of course, this can be expected, since the base of rules, in comparison with the Choquet integral, can be applied to more general problems. The same conclusion can be made for application of fuzzy measure and original Choquet integral.

The following example demonstrates how the FNV-fuzzy measure can be constructed with help of the approach described by the algorithm above.

Example 3.2 *Let us discuss the problem from the Example 3.1 once again. Last time we have proposed how to solve the problem when the fuzzy measure and FNV-fuzzy measure on the set of partial goals are set directly. In this example,* *we are assuming the same conditions, moreover we are focusing on two additional problems:*

- *indirect fuzzy measure construction,*
- *indirect FNV-fuzzy measure construction.*

At first, we construct fuzzy measure with help of the technique described in section 2.4.4. We create the imaginary objects $x^1 = (0,0,0)$, $x^2 = (1,0,0)$, $x^3 = (0,1,0)$, $x^4 = (0,0,1)$, $x^5 = (1,1,0)$, $x^6 = (0,1,1)$, $x^7 = (1,0,1)$, $x^8 = (1,1,1)$ *and order them decreasingly with help of pairwise comparison method and matrix A. During the process we ask ourselves the following kinds of questions: "Is the aptitude of a student, who is excellent in mathematics and physics and completely fails in chemistry* (x^5) *better than the aptitude of a student who excels only in chemistry* (x^4)*?" With regards to the school's focus on mathematics, the answer is YES and the element* a_{54} *of the matrix A is equal to 1. Analogically, we analyze each pair of the imaginary objects and finally, following the algorithm described above, we arrive to the ordering:*

$$x^8 \succ x^5 \succ x^7 \succ x^6 \succ x^2 \succ x^3 \succ x^4 \succ x^1 \tag{3.74}$$

Afterwards, we choose numbers $t_1 = 1.1$, $t_2 = 1.2$, $t_3 = 1.3$ *and* $t_4 = 1.4$ *to describe the intensity of the relationship between each two consecutive objects starting with* x^8 *and* x^5. *The alternative* $x^8 = (1,1,1)$ *with all three partial goals fully achieved is for us* quite better *than the alternative* $x^5 = (1,1,0)$ *with excellent knowledge of mathematics and physics but total failure in chemistry. The linguistic term 'quite better' corresponds to the number* t_2, *therefore* $r_1 = t_2 = 1.2$ *and the overall evaluation* h^1 *of object* x^5 *is given by*

$$h^1 = \frac{1}{1.2} = 0.83. \tag{3.75}$$

Hence

$$\mu(M,P) = 0.83. \tag{3.76}$$

Analogically we compare all remaining imaginary objects and we construct the fuzzy measure: $\mu(M,Ph,Ch) = 1$, $\mu(M,Ph) = 0.83$, $\mu(M,Ch) = 0.76$, $\mu(Ph,Ch) = 0.69$, $\mu(M) = 0.49$, $\mu(Ph) = 0.45$, $\mu(Ch) = 0.41$.

Then, given the partial evaluations $H_M \sim (0.8, 0.9, 1)$*,* $H_{Ph} \sim (0.4, 0.5, 0.6)$ *and* $H_{Ch} \sim (0.1, 0.2, 0.3)$*, the overall evaluation of the student, as calculated by first-level fuzzy Choquet integral with respect to our fuzzy measure, is equal to triangular fuzzy number* $Y \sim (0.545, 0.645, 0.745)$*.*

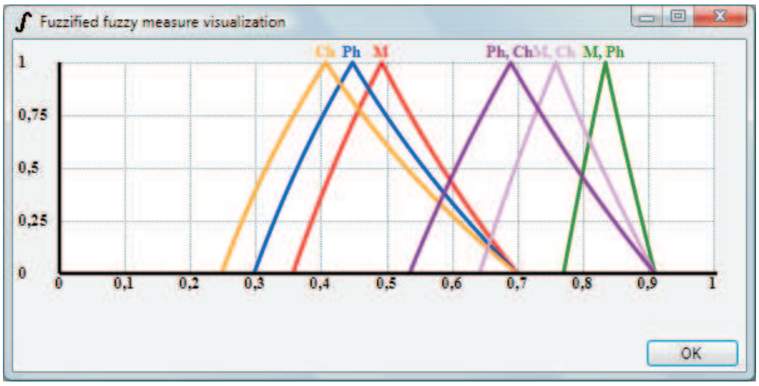


Figure 3.10: FNV-fuzzy measure.

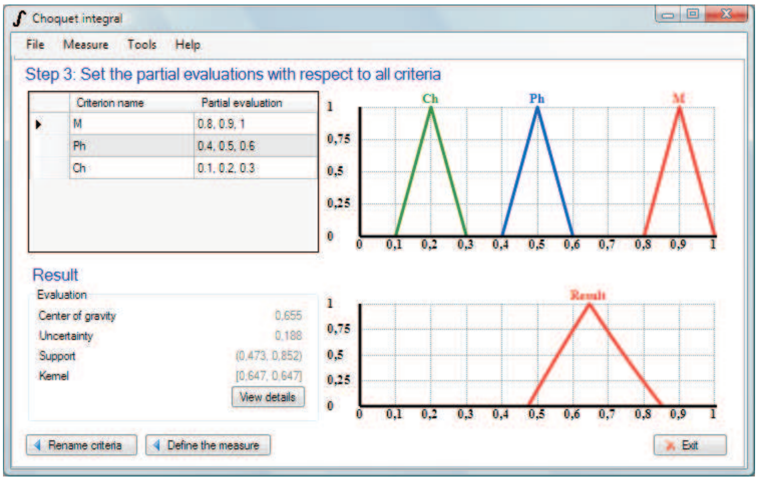


Figure 3.11: Partial evaluations and the overall evaluation calculated by second-level fuzzy Choquet integral.

Let us now calculate the overall evaluation using FNV-fuzzy measure. The construction of FNV-fuzzy measure is analogical to the construction of fuzzy measure in the previous paragraph with a single difference. To describe the intensity of the relationship between each two successive objects we use triangular fuzzy numbers $T_1 = (1, 1.1, 1.2)$*,* $T_2 = (1.1, 1.2, 1.3)$*,* $T_3 = (1.2, 1.3, 1.4)$ *and*

$T_4 = (1.3, 1.4, 1.5)$. *Graphical representation of the resulting FNV-fuzzy measure can be seen in Fig. 3.10.*

The overall evaluation of the student together with the partial evaluations is depicted in Fig. 3.11. As we can see, the evaluation is in the form of a fuzzy number. Its center of gravity 0.65 *corresponds the result from the previous paragraph.*

The calculations used in the example were implemented in a new software that was created by Holeček [100] for this purpose. The software is capable of modeling both the fuzzy measures and the partial evaluations by fuzzy numbers, and employs the method presented in [7] for the actual calculations. Demo version of the software can be found on [100].

Chapter 4

Fuzzy approach to quantitative interpretation of MMPI-2

One of the most powerful tools applied to MCDM problems is a base of fuzzy rules. It is a powerful tool with an extremely wide range of applicability. It can be used for problems with unknown interactions, complicated relationships between the possible values of all the criteria, or uncertain input data presented in linguistic form. Base of fuzzy rules allows us to create a linguistically described multiple criteria evaluation function, which in the process of multiple criteria evaluation plays the same role as aggregation operator usually does - it consolidates the partial evaluations into a single overal one.

This chapter is devoted to the example of applying a base of fuzzy rules to an important problem arising in psychology - the interpretation of the MMPI-2 tests. The results presented in this chapter were published in [8] and [9].

4.1 Introduction to MMPI-2

MMPI-2 (Minnesota Multiphasic Personality Inventory) is one of the most frequently used tests for characterization of personality features and psychic disorders. The first version of the test, MMPI, was developed by psychologist S. R. Hathaway and psychiatrist J. C. McKinley [43] of the Minnesota University. Their goal was to develop an instrument to describe patient's personality more effectively than what was allowed by the psychiatric interview with the patient, [1]. At the same time it was desirable to replace a great number of tests, focusing on single features, by a single test capable of full characterization. The fruit of their labor was an extensive testing method with applications far beyond the clinical practice. Today, a revised version of the test, MMPI-2 [13], is used. MMPI-2 is an important screening method for detecting pathological personality features, which is used in clinical practice, as well as in entrance interviews for universities, military, police, or leading positions [44].

Use of the MMPI-2 is very demanding. The examiner needs to possess knowledge of theory and application of psychological tests; he/she should have a Master degree in personal psychology and psychopathology, [44]. Furthermore, correct interpretation of the test requires experience with the MMPI-2 and a special training. For this reason, a software with transparent results providing solid basis for the clinic deliberation would be an enormous asset.

4.1.1 Quantitative interpretation of MMPI-2

An important part of the testing process is quantitative interpretation of the tests' results, [39]. Answers to 567 questionnaire questions are used to saturate a large number of scales (over 130). Their rough point values are then transformed into linear T-scores. Based on values of these, a codetype of the patient is determined.

The basis for the MMPI-2 interpretation is a determination of codetype, if possible. Each codetype is defined by T-scores of ten clinical scales (1-Hypochondriasis, 2-Depression, 3-Hysteria, 4-Psychopathic deviate, 5-Masculinity-Feminity, 6-Paranoia, 7-Psychasthenia, 8-Schizophrenia, 9-Hypomania, 0-Social introversion). Value of each T-score comes from the interval $[0, 120]$. Values higher than 65 are considered significantly elevated. According to number and type of increased clinical scales we define 55 different codetypes. Codetypes with one significantly elevated clinical scale are designated "Spike" (ten possible types), while two significantly elevated scales represent a "Two Point" (45 possible types). For a codetype to be well defined, there has to be at least five point difference between the T-scores of the highest scales and remaining T-scores. If this is not satisfied, there is a possibility of triad, for example, and it is not possible to use codetypes.

After finding the codetype, the agreement between patient's data and the respective prototypic profile is checked. In this testing, T-scores of all scales need to be considered. Each of 55 prototypic profiles is defined by specific values of all scales. To have a perfect match between the patient and a given prototypic profile, according to the instructions from the official MMPI manual [44], T-scores of patient's scales should not differ from T-scores of the profile by more than ten points.

For finding the T-scores and determining the codetype, the MMPI-2 software was developed [44]. This software finds the codetype only from the two highest T-scores and rest of the data is not involved in the process. This leads to loss of information and it is wasteful of the full MMPI-2 potential. Furthermore, the software does not strictly adhere to the five-point-difference condition and therefore may return an erroneous result.

In the following text, we present a mathematical model published in [8] and [9], which can help to find several codetypes best fitting the patient. The codetypes are determined in two steps. In the first step, the model searches for codetypes using the MMPI methodology with fuzzified conditions. In the second step, the additional suitable codetypes are found by comparing the patient's data to the prototypic profiles.

The Czech version of the MMPI-2 does not work with all of the scales. It uses and saves values of only 79 of them. To keep with the Czech localization, our mathematical model will consider this simplified version of the MMPI-2.

4.2 Designed mathematical model

The codetype determination requiring full satisfaction of all 79 conditions of a prototypic profile is problematic. Classification based on such a crisp mathematical model may not work, because only rarely a patient satisfies a prototypic profile fully. It will be shown that in a situation like this, as well as in many areas of social sciences and psychology, it is effective to use fuzzy approach described in section 3.1.

The quantitative interpretation of MMPI-2 is performed in two steps. First, based on the values of clinical scales, a patient's codetype is determined. This is followed by the verification, where the relevant prototypic profile is compared with the patient's data.

The proposed mathematical model respects this structure of MMPI-2. In the first step, the model finds the three clinical scales with the highest T-scores, and with help of the linguistically described function decides on a codetype. In the second step, the model works with values of all 79 scales and calculates the overlap between the linear T-scores of the patient and the prototypic profile of the codetype found in the previous step. Simultaneously the model searches for other prototypic profiles, which agree well with the patient's data.

4.2.1 Codetype determination

Two conditions are important for correct determination of the codetype. First, T-scores of significantly elevated scales must be higher than 65. Second, values of the highest scales must be at least five points higher than values of all the remaining scales. In practice, it is often difficult to strictly fulfill this conditions. It has shown to be more effective to use the fuzzy approach and define these conditions linguistically. Furthermore, use of the fuzzy set theory was instrumental in finding more variants of the codetype, which can be presented to the evaluator.

Prior to further processing, the scales need to be ordered from the highest T-score to the lowest. Based on the above mentioned requirements, we then define linguistic variables as:

1. (The First Scale Elevation,
   {Insignificant, Significant}, $[0, 120], M_1$),

2. (The Second Scale Elevation,
   {Insignificant, Significant}, $[0, 120], M_1$),

3. (The Third Scale Elevation,
   {Insignificant, Significant}, $[0, 120], M_1$),

4. (The Difference between the First Two Scales,
   {Small, Big Enough}, $[0, 120], M_2$),

5. (The Difference between the 2^{nd} and the 3^{rd} Scale,
   {Small, Big Enough}, $[0, 120], M_2$),

6. (Codetype Shape,
   {Spike, Two Point, Potential Triad, Within-Normal-Limits},
   $\{1, 2, 3, 4\}, N$),

where
M_1(Insignificant) $= IE \sim (0,0,63,65)$, M_1(Significant) $= SE \sim (63,65,120,$ $120)$, M_2(Small) $= SM \sim (0,0,0,5)$, M_2(Big Enough) $= BE \sim (0,5,120,120)$, N(Spike) $= S \sim (1,1,1,1)$, N(Two Point) $= 2P \sim (2,2,2,2)$, N(Potential Triad)$= PT \sim (3,3,3,3)$, N(Within-Normal-Limits) $= WNL \sim (4,4,4,4)$. Some of defined variables are illustrated in Fig. 4.1 and 4.2.

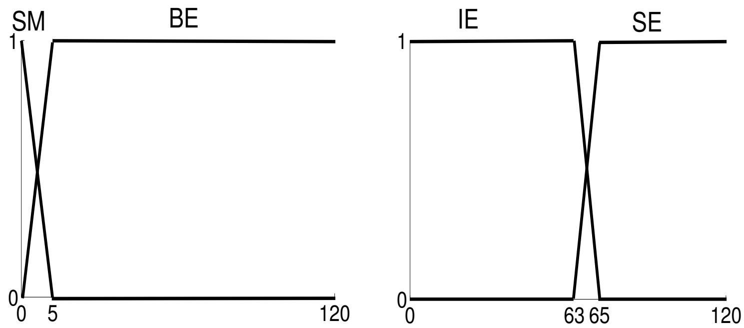


Figure 4.1: Input linguistic variables
Left: *The Difference between the First Two Scales* and its two linguistic values *Small* and *Big Enough* modelled by fuzzy numbers SM and BE.
Right: *The First Scale Elevation* and its two linguistic values *Insignificant* and *Significant* modelled by fuzzy numbers IE and SE.

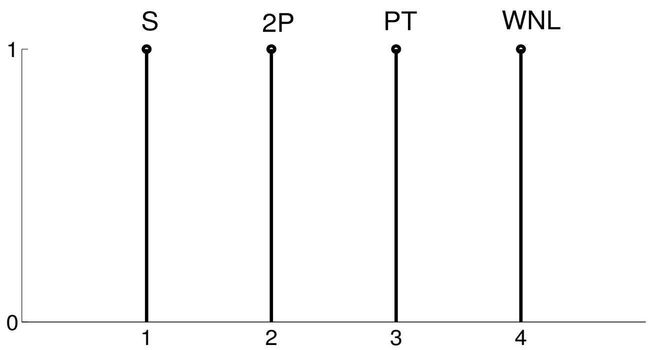


Figure 4.2: Output linguistic variable *Codetype Shape* and its four linguistic values *Spike, Two Point, Potential Triad* and *Within-normal-limits* modelled by fuzzy numbers *S,2P,PT* and *WNL.*

With help of these six linguistic variables and four rules we construct a base of rules F:

rule 1 If *The First Scale Elevation* is *Significant* and *The Second Scale Elevation* is *Insignificant* and *The Difference between the First Two Scales* is *Big Enough*, then the *Codetype Shape* is a *Spike.*

rule 2 If *The First Scale Elevation* is *Significant* and *The Second Scale Elevation* is *Significant* and *The Difference between the 2^{nd} and the 3^{rd} Scale* is *Big Enough*, then the *Codetype Shape* is *Two Point.*

rule 3 If *The First Scale Elevation* is *Significant* and *The Second Scale Elevation* is *Significant* and *The Third Scale Elevation* is *Significant* and *The Difference between the 2^{nd} and the 3^{rd} Scale* is *Small*, then the *Codetype Shape* is *Potential Triad.*

rule 4 If *The First Scale Elevation* is *Insignificant*, then the *Codetype Shape* is *Within-Normal-Limits.*

The base of rules F has five input linguistic variables - the three highest T-scores of clinical scales and the two differences between them - and one output linguistic variable, which determines the shape of the codetype.

Together with the Assilian-Mamdani approximate reasoning algorithm (see section 3.1), the linguistic function F forms an expert system for determination of the codetype shape. With values of clinical scales as an input, the model produces a fuzzy set D^M that helps the evaluator to determine possible codetype shapes. The membership degree of an element of the set $\{1, 2, 3, 4\}$ in fuzzy set D^M, representing a particular codetype shape, equals to the degree of satisfaction of the respective rule. See, for example, Fig. 4.3. To determine the complete codetype of the patient, we need to combine the information about the codetype shape with knowledge of the initial ordering of clinical scales. For example, if the codetype shape is Spike and the designation of the highest scale is 8-Schizophrenia, then the codetype is Spike 8.

4.2.2 Codetype verification

Each of all 55 codetypes is described in detail by a so called prototypic profile. Codetype verification is based on the calculation of the degree of agreement

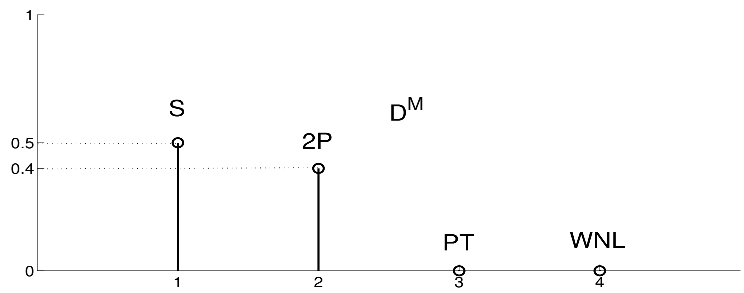


Figure 4.3: The fuzzy set D^M as obtained by entering input values $(67, 64, 62, 3, 2)$ into the designed expert system. The degrees of satisfaction express the possibility that the corresponding codetype shape is a Spike (possibility 50%) or a Two Point (possibility 40%).

between the patient's data and the prototypic profiles corresponding to the codetypes, which were determined in the first part of the model. Besides the verification, the model also searches for other prototypic profiles with a good overlap. Each profile is described by a vector of 79 real numbers representing values of the 79 scales with the T-scores ranging from 0 to 120. For a patient's profile to match a prototypic profile, all the patient's T-scores should be within 10 point distance from the prototypic values.

In the second part of the mathematical model we replaced all crisp numbers t_{ij} describing the prototypic profiles by linear fuzzy numbers $T_{ij} \sim (t_{ij} - 20, t_{ij} - 10, t_{ij} + 10, t_{ij} + 20)$, $i = 1, 2, \ldots, 55$, $j = 1, 2, \ldots, 79$. The example is illustrated in the Fig. 4.4. The kernel of the designed fuzzy number corresponds to the requirements of the methodic, i.e. if the patient's T-score is within 10 point distance from the prescribed value, there is a perfect match and the membership degree is equal to 1. The support of the fuzzy number was set at twice the length of the kernel, i.e. if the distance of the patient's T-score and the prototypic value is bigger than 20 points, then there is no match at all and the membership degree is zero.

Each i-th, $i = 1, 2, \ldots, 55$, prototypic profile is then described by 79 of these fuzzy numbers. Entering the patient's T-scores $t'_1, t'_2, \ldots, t'_{79}$ into the designed fuzzy numbers, we obtain 79 membership degrees $T_{i1}(t'_1), T_{i2}(t'_2), \ldots, T_{i,79}(t'_{79})$.

At this point we encountered a complication. According to Assilian-Mamdani

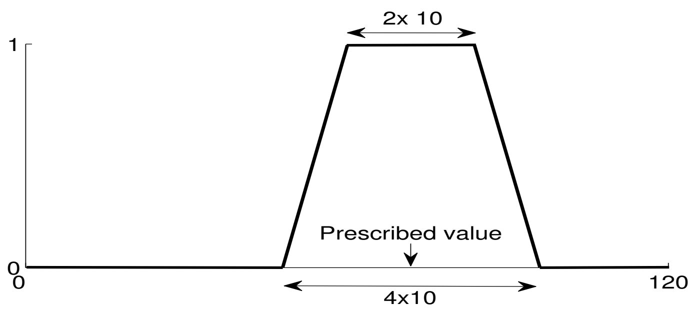


Figure 4.4: Fuzzy number replacing the crisp prototypic value of a scale. The membership degree corresponds to the degree of agreement between the patient's T-score and the prescribed value.

approach, the degree of agreement between the patient's data and the i-th prototypic profile should be calculated as minimum of these membership degrees, i. e.

$$h_i = \min(T_{i1}(t'_1), T_{i2}(t'_2)), \ldots, T_{i79}(t'_{79}),\ i = 1, 2, \ldots, 55. \tag{4.1}$$

Minimum is a common aggregation operator used for modeling the operation of logical conjunction, which should perfectly model the requirement:

"For a patient's profile to match a prototypic profile, all the patient's T-scores must be within 10 point distance from the prototypic values."

And yet, after several iterations of testing on real data, using the Matlab fuzzy toolbox, the minimum operator proved unfeasible. The problem is, a real patient rarely satisfies the full range of conditions. As a consequence, the overlap between the patient's and the prototypic profile was often determined as zero, even if these profiles were very similar.

To circumvent this issue, we started looking for another aggregation operator more suitable for this kind of problem. Finally, the arithmetic mean proved itself to be the most convenient in this case and the calculation of the degree of agreement was proposed as follows:

$$h_i = \frac{1}{79} \sum_{j=1}^{79} T_{ij}(t'_j),\ i = 1, 2, \ldots, 55. \tag{4.2}$$

The degree of agreement between the given data and the prototypic profile here represents the average satisfaction of all the 79 prescribed conditions. Compared to minimum, the arithmetic mean provides better information about the satisfaction of given conditions. This is illustrated in Figs. 4.5 and 4.6.

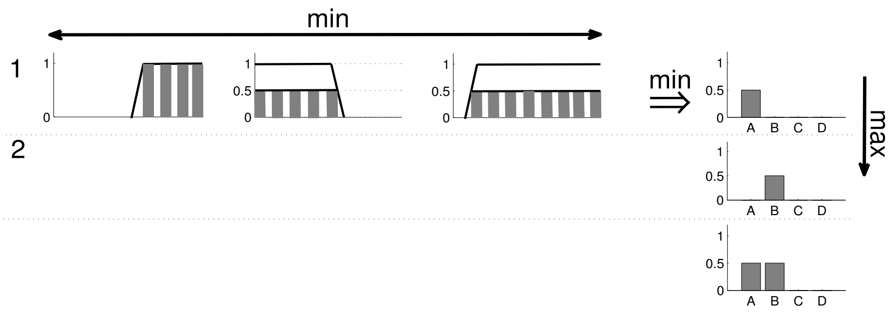


Figure 4.5: The Assilian-Mamdani approximate reasoning mechanism applying minimum.

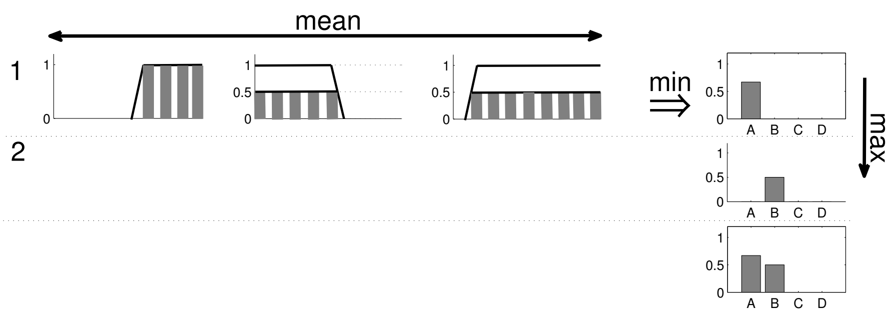


Figure 4.6: The modified Assilian-Mamdani approximate reasoning mechanism applying arithmetic average.

Remark 4.1 *In the future, the aggregation operator can be readjusted to the requirements of the examiner and the arithmetic mean can be replaced by an other aggregation operator: weighted mean, OWA, or, assuming the scales are interacting, even the Choquet integral.*

By applying the aforementioned approach we are able to test all the 55 prototypic profiles. The result can be modeled by a fuzzy set H on the set U, $U = \{1, 2, \ldots, 55\}$, where each integer between 1 and 55 corresponds to one prototypic profile and the membership degrees $H(i)$, $i = 1, 2, \ldots, 55$, represent the overlap of the respective prototypic profiles with the profile of the patient. The example is illustrated in Fig. 4.7.

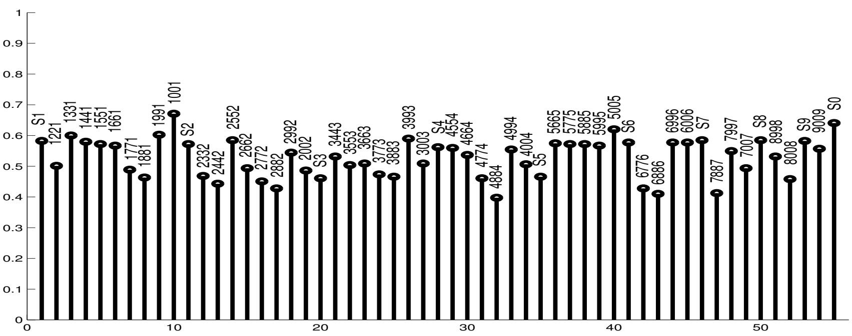


Figure 4.7: The fuzzy set H as obtained by comparing 79 given values with the prescribed prototypic profiles. The degrees of satisfaction represent the overlap between the prototypic profiles and the patient's profile.

4.3 The implementation of the mathematical model in MATLAB

Both parts of the proposed mathematical model were realized in MATLAB. At first, we used the Fuzzy Logic Toolbox to create the base of rules and to set the proper approximate reasoning algorithm. Then to each one of the 55 prototypic profiles we assigned a 79-tuple of fuzzy numbers, as was described in the previous section.

An example of the output can be seen in Fig. 4.8. The output of the utility is in the form of three figures and a linguistic description of the situation. The first figure presents values of clinical scales as obtained from the patient - the patient's profile. The second figure presents possible codetypes, together with their respective degrees of satisfaction. The third figure shows all the prototypic profiles and their overlap with the patient's profile. The evaluator can therefore decide, whether the found codetypes are in good agreement with the patient's available data. The linguistic output presents possible codetypes and three prototypic profiles with the best agreement. In addition it comments on a possibility of a triad or scales within normal limits.

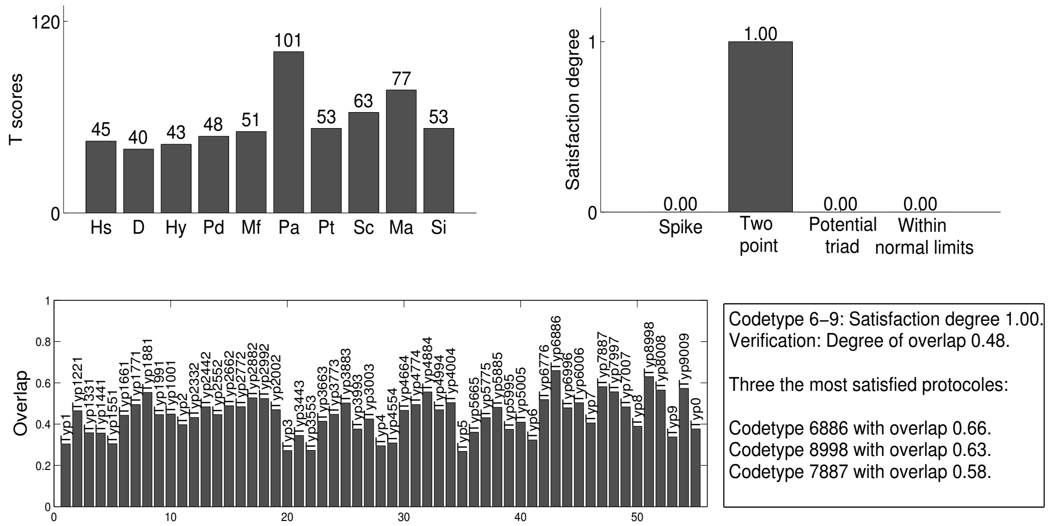


Figure 4.8: Three figures and linguistic description as returned by the MATLAB implementation of the model.

In Fig. 4.8 we demonstrate performance of the implementation. According to clinical scales values, codetype 6-9 was determined. The result is in agreement with the original software. However, during the prototypic profile analysis, the codetype 6-9 didn't show sufficient agreement, as the degree of overlap was only 0.48. The three most faithful profiles were those of codetypes 6-8/8-6, 8-9/9-8, and 7-8/8-7, with 6-8/8-6 showing the best overlap. This suggests that for further deliberation, codetypes 6-8/8-6 should be considered in addition to 6-9.

Chapter 5

Thesis results

In the chapter 2, the applications of aggregation operators, especially of the Choquet integral, to multiple criteria decision making are studied. Based on the principles explored in the chapter, the Partial Goals Method (PGM), originally presented in [79], is generalized for multiple criteria evaluation problem with overlapping or complementary partial goals. This generalization, forming a content of section 2.4.3 of this thesis, was published by as [5].

Chapter 3 is devoted to the fuzzification of OWA operators and the Choquet integral. The first and second-level fuzzy OWA operators were defined and published in 2008 [80] together with algorithm for their computation. Later, in 2010, the fuzzified Choquet integral was defined and published in [7]. The definition of the second-level fuzzy Choquet integral, based on the extension principle, was accompanied by Theorem 3.10 and Theorem 3.11, which significiantly ease its computation. Definitions 3.13, 3.19, 3.16, and 3.21, together with Theorems 3.4, 3.9, 3.7, 3.10, 3.11, and the proofs, represent one of the main results of the thesis.

The first and second-level fuzzy Choquet integrals were employed in order to fully generalize the Partial Goals Method (PGM). This new Generalized PGM (GPGM), published in 2011 in [5], is able to handle multiple criteria evaluation problems with uncertain partial evaluations and significance on the set of interacting partial goals described by fuzzy numbers. The PGM generalization for fuzzy input was treated in sections 2.4.3, 3.2.4 and 3.3.4 and is another result of the thesis.

In order to make the application of the second-level fuzzy Choquet integral more easier, a new technique for fuzzy measure and FNV-fuzzy measure construction was developed and published in [5] and [6]. Both the techniques were described also in the algorithmic way in sections 2.4.4, 3.4 and belong also among the results of the thesis.

The list of thesis results ends with two fuzzy expert systems described in Chapter 4. A real problem encountered in the field of the clinical psychology was modeled by bases of fuzzy rules in order to help the examiner to process and interpret the MMPI-2 test's results. The systems published in [8] and [9] are still being investigated, tuned and expanded upon with a possibility of obtaining a realistic application.

Chapter 6

Conclusion

There are lots of different problems, which can be met in multiple criteria decision making. In this thesis we focused on a single specific kind. It is a multiple criteria decision making problem with a finite set of possible solutions (alternatives) and a finite set of criteria, which is assessed by a single decision maker.

One of the main problems arising during the process of decision making is to evaluate the given alternatives in order to decide if they fulfill our requirements, or, at lest, to determine which ones suit us the best. The process of multiple criteria evaluation fully employs aggregation operators. There are many families of aggregation operators studied with eye on their application to MCDM. In chapter 2, the most popular aggregation operators, with special emphasis on the Choquet integral, were treated. The unique properties of the Choquet integral allowed us to generalize PGM method for MCDM problems, in which the interactions among the criteria made use of weighted average impossible. One of the main drawbacks to proper application of the Choquet integral is its dire need for huge amount of information in the form of a fuzzy measure for a given set of criteria. This important problem has gathered a lot of attention in the community and several approaches to fuzzy measure construction, based on, for example, training data and linear or quadratic programming, were proposed. In the thesis we proposed a new method of constructing a fuzzy measure, a method based solely on the evaluator's ability to compare two well defined objects given to her.

For one reason or another, decision making problems are often burdened by uncertainty, which is integrated into the decision making models by employing the fuzzy sets theory. As a result, the aggregation operators are being adjusted to accommodate evaluations by fuzzy numbers in the process called 'fuzzification'. At the time the results of the thesis were formulated, fuzzified versions of several aggregation operators were already available, but their calculation was quite involved.

In chapter 3 we presented full fuzzification, up to the second level, of the Choquet integral and its several different forms. In addition to fuzzified aggregation operators we also proposed algorithms meant to ease their calculation. Those algorithms were afterwards integrated into FuzzME software, a tool developed for creating fuzzy models of multiple criteria evaluation and decision making.

The generalization of the Choquet integral to the second-level fuzzy Choquet integral led to the GPGM - Generalized Partial Goals Method. GPGM is able to help the evaluator with multiple criteria evaluation problem with interacting criteria and uncertain partial evaluations even if the partitions of the partial goals in the overall goal are set only vaguely.

For the second level fuzzification of the Choquet integral, the fuzzy measure, which is used by the integral to steer the aggregation, needed to be replaced by its generalization, the FNV-fuzzy measure. Unfortunately, the FNV fuzzy measure contains more information than the fuzzy measure and its construction is therefore much more demanding. To alleviate this issue and help the evaluator, we devoted the last part of the chapter 3 to a technique for constructing a FNV-fuzzy measure presented in a form of algorithm.

The applications of the particular aggregation operators are limited. In the case of strange relationships among the partial and overall evaluations, the application of fuzzy rules base and linguistically oriented modeling is recommended. In the chapter 4, the base of fuzzy rules was applied to a real problem occurring in the field of psychology. Two fuzzy expert systems were created to model the problem of the quantitative interpretation of results given by the MMPI-2 test. Moreover, the utilization of two different aggregation operators in the approximate reasoning algorithm was discussed.

Chapter 7

Resumé v českém jazyce

Dizertační práce se zabývá úlohou vícekriteriálního rozhodování s konečnou množinou variant a jedním rozhodovatelem. Při řešení tohoto problému se připouštějí různé typy interakcí mezi kritérii a neurčitost informací týkající se jak hodnot kritérií a dílčích hodnocení vzhledem k těmto kritériím, tak rozhodovatelových preferencí v množině kritérií.

Základním krokem v práci analyzovaného rozhodovacího procesu je vícekriteriální hodnocení variant, při němž je využíváno agregačních operátorů. Kapitola 2 se zabývá agregačními operátory, které jsou v úlohách vícekriteriálního hodnocení používány nejčastěji. Jsou to především vážený průměr, OWA operátory a Choquetův integrál. Důraz je kladen na Choquetův integrál, který umožňuje modelovat vícekriteriální hodnocení i v případech, kdy existují interakce mezi kritérii, ať už ve smyslu synergie nebo redundance. Fakt, že Choquetův integrál je vhodný nástroj pro zobecnění některých rozhodovacích metod pro případ vazeb mezi kritérii, byl demonstrován v sekci 2.4.3 při zobecnění metody dílčích cílů (PGM). Při využití Choquetova integrálu musí rozhodovatel poskytnout mnohem více informací o svých preferencích na množině kritérií, než v případě jiných běžně využívaných agregačních operátorů, například váženého průměru. Tyto jeho preference jsou reprezentovány fuzzy mírou, kterou je potřeba definovat na množině dílčích cílů, či kritérií. Problematikou zadávání fuzzy míry se zabývali již mnozí autoři. Bylo navrženo několik různých způsobů její konstrukce, často se například využívá množina trénovacích dat a matematické programování. V této práci je navržena nová metoda pro expertní zadávání fuzzy míry, která je založena na metodě párového porovnávání a schopnosti rozhodovatele vyjádřit intenzitu preference mezi dvěma uvažovanými objekty.

V praxi řešené rozhodovací problémy jsou často zatíženy neurčitosti, kterou je možno matematicky popsat pomocí nástrojů teorie fuzzy množin. Neurčitá dílčí hodnocení, která mohou být modelována fuzzy čísly, je potřeba agregovat pomocí tomu přizpůsobených, fuzzifikovaných, agregačních operátorů. V práci jsou uvedeny odkazy na práce autorů zabývajících se problematikou fuzzifikace agregačních operátorů a především pak prezentovány výsledky publikované v této oblasti autorkou dizertační práce. V kapitole 3 jsou představeny dva stupně fuzzifikace Choquetova integrálu a jeho speciálních případů. První úroveň fuzzifikace v kontextu modelů vícekriteriálního hodnocení a rozhodování znamená, že dílčí hodnocení jsou modelována fuzzy čísly, zatímco váhy skupin kritérií jsou modelovány fuzzy mírou popsanou na množině kritérií ostrými čísly z intervalu $[0, 1]$. Při druhé úrovni fuzzifikace uvažujeme kromě neurčitých hodnocení i neurčité váhy skupin kritérií - tyto váhy jsou modelovány FNV-fuzzy mírou. V práci byla dokázána důležitá tvrzení týkající se vlastností fuzzifikovaných agregačních operátorů a byly navrženy efektivní výpočetní algoritmy pro jejich výpočet. Plně fuzzifikovaný Choquetův integrál byl využit k dalšímu zobecnění metody dílčích cílů (GPGM) pro případ, kdy významnosti dílčích cílů jsou popsány FNV fuzzy mírou. Zadávání FNV-fuzzy míry je pochopitelně ještě složitější záležitostí než v případě fuzzy míry s hodnotami zadanými reálnými čísly. Poslední část kapitoly 3 obsahuje původní algoritmus pro zadávání FNV-fuzzy míry na základě expertních znalostí.

Choquetův integrál je možno použít, pokud vztahy mezi kritérii jsou typu synergie nebo redundance, či pokud chceme modelovat kritéria typu Veto, či Favor. V případě složitějších vztahů mezi dílčími a celkovými hodnoceními je vhodné použít bázi fuzzy pravidel a jazykově orientovaného programování. V kapitole 4 je představena aplikace báze fuzzy pravidel na reálný problém z oblasti klinické psychologie. Pomocí dvou fuzzy expertních systémů je zde modelován problém kvantitativní interpretace výsledků dotazníku MMPI-2. Na výsledky práce publikované v [8] and [9] navázal psychologický výzkum [76] a další výzkum v oblasti využití bází fuzzy pravidel v oblasti interpretace výsledků dotazníku MMPI pokračuje.

Výsledky předložené práce byly publikovány v odborných časopisech a předneseny na řadě mezinárodních konferencí. Algoritmy pro výpočet fuzzifikovaných

OWA operátorů a Choquetova integrálu byly implementovány do softwaru FuzzME, nástroje vhodného pro modelování problémů vícekriteriálního hodnocení a rozhodování.

Bibliography

[1] Archer, R. P.:*MMPI-A: Assessing adolescent psychopathology (Third edition).* Lawrence Erlbaum Associates Publishers, 2005.

[2] Assilian, P., Mamdani, E. H.:*An experiment in linguistic synthesis with a fuzzy logic controller.* International Journal of Man-Machine Studies 7, 1975, pp. 1-13.

[3] Baas, S., Kwakernaak, H.:*Rating and ranking of multiple-aspect alternatives using fuzzy sets.* Automatica 13, 1977, pp. 47–58.

[4] Bana e Costa, C. A., Vansnick, J.-C.:*A theoretical framework for measuring attractiveness by a categorical based evaluation technique (MACBETH).* Proceedings of the XIth international conference on multicriteria decision making, Coimbra, Portugal, 1994, pp. 15-24.

[5] Bebčáková, I., Holeček, P., Talašová, J.:*On the application of the fuzzified Choquet integral to multiple criteria evaluation.* Acta Polytechnica Hungarica, Vol.8 , No.3, 2011, pp. 65-78.

[6] Bebčáková, I., Talašová, J.:*Fuzzified Choquet integral: A way to model interacting criteria in decision making.* Proceedings of 28th International Conference on Mathematical Methods in Economics 2010, České Budějovice, 2010, p. 30-35.

[7] Bebčáková, I., Talašová, J., Pavlačka, O.:*Fuzzification of Choquet Integral and its application in multiple criteria decision making.* Neural network world, Vol. 20, 2010, pp. 125-137.

[8] Bebčáková, I., Talašová, J., Škobrtal, P.:*Fuzzy Approach to Quantitative Interpretation of MMPI-2.* Journal of Applied Mathematics, Vol. 2, Number 1, 2009, pp. 65-75.

[9] Bebčáková, I., Talašová, J., Škobrtal, P.:*Interpretation of the MMPI-2 test based on fuzzy set techniques.* Acta Univertsitatis Mathiae Belii, ser. Mathematics 16, 2010, pp. 5-16.

[10] Bellman, R. E., Zadeh, L. A.:*Decision-making in fuzzy environment.* Management Science 17 (4), 1970, pp. 141-164.

[11] Brans, J. P.:*L'Ingéniérie de la décision: Elaboration d'instruments d'aide à la décision. La méthode PROMETHEE.* In R. Nadeau and M. Landry (Eds): L'aide à la décision: Nature, Instruments et Perspectives d'Avenir, Presses de l'Université Laval, Québec, Canada, 1982, pp. 183-213.

[12] Brans, J. P., Mareschal, B.:*PROMETHEE methods.* In: Figueira, J., Greco, S., Ehrgott, M. (Eds.), Multiple Criteria Decision Analysis: The State of the Art Surveys. Springer Science, Business Media Inc., New York, 2005, pp. 163 – 196.

[13] Butcher, J. N., Dahlstrom, W. G., Graham, J. R., Tellegen, A., & Kaemmer, B.:*MMPI-2: Manual for administration and scoring*, University of Minnesota Press, Minneapolis, 1989.

[14] Calvo, T., De Baets, B., Fodor, J.:*The functional equations of Alsina and Frank for uninorms and null-norms.* Fuzzy Sets and Systems 120, 2001, pp. 15-24.

[15] Calvo, T., Kolesárová, A., Komrníková, M., Mesiar, R.:*Aggregation Operators: Properties, Classes and Construction Methods.* In: Calvo, T., Mayor, G., Mesiar, R. (Eds): Aggregation Operators: New trends and applications, Physica-Verlag, Heidelberg New York, 2002, pp. 3-104.

[16] Chang, S., Syllable, A.:*Articulatory-feature, and stress-accent model of speech recognition.* PhD. thesis, Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, 2002.

[17] Chateauneuf, A., Jaffray, J.-Y.:*Some characterizations of lower probabilitites and other monotone capacities through the use of Möbius inversion.* Math. Social Sci. 17 (3), 1989, pp. 263-283.

[18] Choquet G.*: Theory of capacities.* Ann. Inst. Fourier 5, 1953, pp. 131-295.

[19] Dong, W. M., Wong, F. S.:*Fuzzy weighted averages and implementation of the extension principle.* Fuzzy Sets and Systems 21, 1987, pp. 183-199.

[20] Dubois, D., Prade, H.(Eds.): *Fundamentals of fuzzy sets*. Kluwer Academic Publishers, Boston/London/Dordrecht, 2000.

[21] Dyer, J. S.:*MAUT - Multiattribute Utility Theory.* In: Figueira, J., Greco, S., Ehrgott, M. (Eds.), Multiple Criteria Decision Analysis: The State of the Art Surveys. Springer Science, Business Media Inc., New York, 2005, pp. 265 – 296.

[22] Figueira, J., Mousseau, V., Roy, B.:*ELECTRE methods.* In: Figueira, J., Greco, S., Ehrgott, M. (Eds.), Multiple Criteria Decision Analysis: The State of the Art Surveys. Springer Science, Business Media Inc., New York, 2005, pp. 133 – 162.

[23] Figueira J.R., Greco S., Roy B.:*ELEKTRE methods with interaction between criteria: An extension of the concordance index.* European journal of operational research, 2009, pp. 478-495.

[24] Fishburn, P. C.:*Utility theory for decision making.* John Wiley & Sons, New York, 1970.

[25] Fodor, J. C. , Keresztfalvi, T. :*Nonstandard conjunctions and implications in fuzzy logic*. International Journal of Approximate Reasoning 12, 1995, pp. 69-84.

[26] Fodor, J. C., Roubens, M.:*Fuzzy preference modeling and multi-criteria decision support.* Kluwer Acad. Publ., Dordrecht-Boston-London, 1994.

[27] Fodor, J., Yager, R., Rybalov, A.:*Structure of uninorms.* Internat. Journal of Uncertainty, Fuzziness and Knowledge-based systems 5, 1997, pp. 411-427.

[28] Fullér, R.:*OWA operators in Decision Making.* In: C. Carlsson (Ed): Exploring the limits of support systems, TUCS General Publications, No. 3, Turku Centre for Computer Science, Ȧbo, 1996, pp. 85-104.

[29] Goumas, M., Lygerou, V.:*An extension of the PROMETHEE method for decision making in fuzzy environment: Ranking of alternative energy exploitation projects.* European Journal of Operational Research 123, 2000, pp. 606-613.

[30] Grabisch, M.:*Fuzzy integral in multicriteria decision making.* Fuzzy sets and systems 69, 1995, pp. 279-298.

[31] Grabisch, M:*A new algorithm for identifying fuzzy measures and its application to pattern recognition.* Internat. Joint Conf. Fourth IEEE Internat. Conf. on Fuzzy Systems and the Second Internat. Fuzzy Engineering Symp., Yokohama, Japan, 1995, pp. 145-150.

[32] Grabisch, M.:*The application of fuzzy integrals in multicriteria decision making.* European Journal of Operational Research 89, 1996, pp. 445-456.

[33] Grabisch, M.:*k-order additive discrete fuzzy measures and their representation.* Fuzzy Sets and Systems 92, 1997, pp. 167-189.

[34] Grabisch, M., Kojadinovic, I., Meyer, P.:*A review of methods for capacity identification in Choquet integral based multi-attribute utility theory Applications of the Kappalab R package.* European Journal of Operational Research 186, 2008, pp. 766-785.

[35] Grabisch, M., Labreuche, C.:*A decade of application of the Choquet and Sugeno integrals in multi-criteria decision making.* Annals of Operations Research 175, 2010, pp. 247-286.

[36] Grabisch, M., Murofushi, T., Sugeno, M.(Eds.):*Fuzzy measures and integrals-theory and applications.* Physica Verlag, Heidelberg, New York, 2000.

[37] Grabisch, M., Nicolas, J.:*Classification by fuzzy integral: performace and tests.* Fuzzy Sets and Systems 65, 1994, pp. 225-271.

[38] Grabisch, M., Roubens, M.:*Application of the Choquet integral in multicriteria decision making.*In Grabisch, M., Murofushi, T., Sugeno, M.(Eds.): Fuzzy measures and integrals-theory and applications, Physica Verlag, Göttingen, 2000, pp. 348-374.

[39] Greene, Roger L.:*The MMPI-2 An interpretive manual (Second edition).* Allyn & Bacon, 1999.

[40] Guh, Y.-Y. et al.:*Fuzzy weighted average: The linear programming approach via Charles and Cooper's rule.* Fuzzy Sets and Systems 117, 2001, pp. 157-160.

[41] Guo, C., Zhang, D., Wu C.:*Fuzzy-valued fuzzy measures and generalized fuzzy integrals.* Fuzzy sets and systems 97, 1998, pp. 255-260.

[42] Guo C., Zhang, D.:*On Choquet integrals of fuzzy-valued functions with respect to fuzzy-valued fuzzy measures.* Proceedings of the Sixth International Conference on Machine Learning and Cybernetics, Hong Kong, 2007, pp. 3180-3183.

[43] Hathaway, S. R.,& McKinley, J. C.:*A multiphasic personality schedule (Minnesota):I. Construction of the schedule.* Journal of psychology 10, 249-254, 1940.

[44] Hathaway, S. R., McKinley, J. C., Netík, K.:*Minnesota Multiphasic Personality Inventory - 2 (First czech edition)*, Testcentrum, Praha, 2002.

[45] Höhle, U.:*Integration with respect to fuzzy measures.* Proceedings of IFAC Symposium on Theory and Applications of Digital Control, New Delhi, 1982, pp. 35-37.

[46] Inoue, K., Anzai, T.:*A study on the industrial design evalation based upon non-additive measures.* 7th Fuzzy System SYmposium, Nagoya, Japan, 1991, pp. 521-524 (in Japanese)

[47] Ishii, K., Sugeno, M.:*A model of human evaluation process using fuzzy measure.* Internat. J. Man-Mach. Stud. 22 (1), 1985, pp. 19-38.

[48] Keeney, R., Raiffa, H.:*Decisions with Multiple Objectives: Preferences and Value Tradeoffs.* Cambridge university press, 1993.

[49] Klement, E. P., Mesiar, R., Pap, E.:*Triangular norms.* Kluwer Academic Publishers, Boston, 2000.

[50] Klir, G. J., Wang, Z., Harmanec, D.:*Constructing fuzzy measures in expert systems.* Fuzzy Sets and Systems 92, 1997, pp. 251-264.

[51] Labreuche, Ch., Grabisch, M.:*The Choquet integral for the aggregation of interval scales in multicriteria decision making.* Fuzzy Sets and Systems 137, 2003, pp. 11-26.

[52] Liginlal, D., Ow, T. T.:*Modeling attitude to risk in human decision processes: An application of fuzzy measures.* Fuzzy Sets and Systems 157, 2006, pp. 3040-3054.

[53] Luo, X., Jennings, N.:*A spectrum of compromise aggregation operators for multi-attribute decision making.* Artificial Intellingence 171, 2007, pp. 161-184.

[54] Malik, S. C., Arora, S.:*Mathematical analysis.* New Age International (P) Ltd., New Delhi, 1992.

[55] Marichal, J.-L.:*Aggregation operators for multicriteria decision aid.* PhD. thesis, University of Liège, 1998.

[56] Marichal, J.-L.:*An axiomatic approach of the discrete Choquet integral as a tool to aggregate interacting criteria.* IEEE transactions on fuzzy systems, Vol. 8, No. 6, 2000, pp. 800-807.

[57] Marichal, J.-L.:*Aggregation of interacting criteria by means of the discrete Choquet integral.* In: Calvo, T., Mayor, G., Mesiar, R. (Eds): Aggregation Operators: New trends and applications, Physica-Verlag, Heidelberg New York, 2002, pp. 224-244.

[58] Marichal, J.-L., Roubens, M.:*Determination of weights of interacting criteria from a reference set.* European Journal of Operational Research 124, 2000, pp. 641-650.

[59] Meyer P., Roubens M.:*On the use of the Choquet integral with fuzzy numbers in multiple criteria decision support.* Fuzzy Sets and Systems 157, 2006, pp. 927-938.

[60] Miranda, P., Graisch, M.:*Optimization issues for fuzzy measures.* International Journal of Uncertainty, Fuzziness, and Knowledge-based Systems 7 (6), 1999, pp. 545-560.

[61] Mitchell, H. B., Schaefer, P. A.:*On ordering fuzzy numbers.* International Journal of Intelligent Systems 15, 2000, pp. 981-993.

[62] Murofushi, T., Soneda, S:*Techniques for reading fuzzy measures (III): interaction index.* Proceedings of the Ninth Fuzzy System Symp., Sapporo, Japan, 1993, pp. 693-696.

[63] Murofushi, T., Sugeno, M:*An interpretation of fuzzy measure and the Choquet integral as an integral with respect to a fuzzy measure.* Fuzzy Sets and Systems 29, 1989, pp. 201-227.

[64] Näther, W., Wälder, K.:*Applying fuzzy measures for considering interaction effects in root dispresal models.* Fuzzy Sets and Systems 158, 2007, pp. 572-582.

[65] Pavlačka O., Talašová J.:*Application of the fuzzy weighted average of fuzzy numbers in decision making models.* New dimensions in fuzzy logic and related technologies. Vol II. Proceedings of the 5th EUSFLAT Conferece, Ostrava, 2007, pp. 455-462.

[66] Pavlačka O., Talašová J.:*The fuzzy weighted average operation in decision making models.* Proceedings of the 24th International Conference Mathematical Methods in Economics, Typos, Plzeň, 2006, pp. 419-426.

[67] Pavlačka O.: *Fuzzy methods of decision making* (In Czech). PhD. thesis. Faculty of Science, Palacký University, Olomouc 2007.

[68] Pham, T. D.:*Combination of multiple cassifiers using adaptive fuzzy integral.* Proceedings of the IEEE Internat. Conf on Artificial Intelligence Systems, Taganrog, Russia, 2002, pp. 50-55.

[69] Qahri Saremi, & H., Montazer, G. A.: *Ranking the various structures of a website: Utilizing extended ELECTRE III Method based on fuzzy notions.* In WSEAS 8th international conference on fuzzy systems, Vancouver, CANADA, 2007.

[70] Ramík, J., Vlach, M.:*Generalized concavity in fuzzy omptimization and decision analysis.* Kluwer Academic Publisher, Boston/Dordrecht/London, 2002.

[71] Roy, B.:*Classement et choix en présence de point de vue multiples: Le méthode ELECTRE.* Revue Francaise d'Informatique et de Recherche Opérationelle 8, 1968, pp. 57-75.

[72] Saaty, T. L.:*The Analytic Hierarchy Process.* McGraw-Hill, New York, 1980.

[73] Saaty, T.:*The analytic hierarchy and analytic network processes for the measurement of intangible criteria and for decision-making.* In: Figueira, J., Greco, S., Ehrgott, M. (Eds.), Multiple Criteria Decision Analysis: The State of the Art Surveys. Springer Science, Business Media Inc., New York, 2005, pp. 345–407.

[74] Shapley, L. S.:*A value for n-person games.* In: Kuhn, H. W., Tucker, A. W. (Eds): Contribution to the theory of games, Vol. II, Annals of Mathematics Studies 28, Princeton University Press, Princeton, 1953, pp. 307-317.

[75] Silvert, W.:*Symmetric summation: A class of operations on fuzzy sets.* IEEE Trans. Systems, Man Cybernet. 9 (10), 1979, pp. 657-659.

[76] Škobrtal, P.: *Possibility of fuzzy sets application in psychology.* (In Czech). PhD. thesis. Faculty of Philosophy, Palacký University, Olomouc 2009.

[77] Sugeno, M.:*Theory of fuzzy integrals and its applications.* PhD. thesis, Tokyo Institute of Technology, 1974.

[78] Tahani, H., Keller, J.:*Information fusion in computer vision using the fuzzy integral.* IEEE Trans. Systems Man Cybernet 20 (3), 1990, pp. 733-741.

[79] Talašová, J.: *Fuzzy metody vícektriteriálního hodnocení a rozhodování.* Univerzita Palackého v Olomouci, Olomouc, 2003.

[80] Talašová J., Bebčáková, I.:*Fuzzification of Aggregation Operators Based on Choquet Integral.* Aplimat - Journal of Applied Mathematics, Vol. 1, 2008, pp. 463-474.

[81] Talašová, J., Holeček, P.: Multiple-Criteria Fuzzy Evaluation: *The FuzzME Software Package.* In Proceedings of the Joint 2009 International Fuzzy Systems Association World Congress and 2009 European Society of Fuzzy Logic and Technology Conference, Lisbon, Portugal, July 20-24, 2009, pp. 681-686.

[82] Tanaka, K., Sugeno, M.: *A study on subjective evaluation of color printing images.* Internat. J. Approx. Reason 5, 1991, pp. 213-222.

[83] Torra, V., Narukawa, Y.: *Modeling decisions: Information fusion and aggregation operators.* Springer-Verlag, Berlin Heidelberg, 2007.

[84] Torra, V., Narukawa, Y.: *The h-index and the number of citations: two fuzzy integrals.* IEEE Transactions on Fuzzy Systems, 2008, pp. 795-797.

[85] Viertl R.: *Statistical methods for non-precise data.* CRC Press, Boca Raton, Florida, 1996.

[86] von Altrock, C.: *Fuzzy logic and neurofuzzy applications explained.* Prentice Hall PTR, New Jersey, 1995.

[87] van Laarhoven, P.J.M., Pedrycs, W.: *A fuzzy extension of Saaty's priority theory.* Fuzzy Sets and Systems 11, 1983, 229-241.

[88] von Neumann, J., Morgenstern, O.: *Theory of Games and Economic Behavior.* Princeton University Press, Princeton, NJ, 1947.

[89] Wang, J.: *Determining fuzzy measures by using statistics and neural networks.* Proc. IFSA'95, Sao Paulo, 1995, pp. 519–521.

[90] Wang, Z., Leung, K. S., Wang, J.: *A genetic algorithm for determining nonadditive set functions in information fusion.* Fuzzy Sets and Systems 102, 1999, pp. 462-469.

[91] Wang, Z. et al.: *Real-valued Choquet integrals with fuzzy-valued integrand.* Fuzzy Sets and Systems 157, 2006, pp. 256-269.

[92] Washio, T., Takahashi, H., Kitamura, M.: *A method for supporting decision making on plant operation based on human reliability analysis with fuzzy integral.* Proc. of the IIZUKA'92: the Second Internat. Conf. on Fuzzy Logic and Neural Neworks, vol. 2, 1992, pp. 841-845.

[93] Wu, C. et al.:*Fuzzy number fuzzy measures and fuzzy integrals. (I) Fuzzy integrals of functions with respect to fuzzy number fuzzy measures.* Fuzzy Sets and Systems 98, 1998, pp. 355-360.

[94] Wu, C. et al.:*Fuzzy number fuzzy measures and fuzzy integrals. (II) Fuzzy integrals of fuzzy-valued functions with respect to fuzzy number fuzzy measures on fuzzy sets.* Fuzzy Sets and Systems 101, 1999, pp. 137-141.

[95] Yager, R. R.:*On ordered weighted averaging aggregation operators in multicriteria decision making.* IEEE Trans. Systems Man Cybernet. 18, 1988, pp. 183-190.

[96] Yang R., Wang Z., Heng P., Leung K.:*Fuzzy numbers and fuzzification of the Choquet integral.* Fuzzy sets and systems 153, 2005, pp. 95-113.

[97] Zadeh, L. A.:*Fuzzy sets.* Inform. Control 8, 1965, pp. 338-353.

[98] Zadeh, L. A.:*The concept of a linguistic variable and its application to approximate reasoning-I.* Information Sciences 8, 1975, pp. 199-249.

[99] Zhou, S.-M.:*Type-1 OWA operators for aggregating uncertain information with uncertain weights induced by type-2 linguistic quantifiers.* Fuzzy Sets and Systems 159, 2008, pp. 3281-3296.

[100] Holeček, P.:*FuzzME.* [online]. [cit. 2012-06-30]. Dostupné z: `fuzzme.wz.cz`

[101] *Fuzzy Choquet.* [online]. [cit. 2012-06-30]. Dostupné z: `http://aix-slx.upol.cz/~pavlacka/fuzzy_choquet.pdf`