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INSTITUTE OF MATHEMATICS
ÚSTAV MATEMATIKY

APPLICATIONS OF FRACTIONAL CALCULUS IN CONTROL THEORY
VYUŽITÍ ZLOMKOVÉHO KALKULU V TEORII ŘÍZENÍ

BACHELOR'S THESIS
BAKALÁŘSKÁ PRÁCE

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Bachelor's Thesis Assignment

Institut:	Institute of Mathematics
Student:	**Daniel Kiša**
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As provided for by the Act No. 111/98 Coll. on higher education institutions and the BUT Study and Examination Regulations, the director of the Institute hereby assigns the following topic of Bachelor's Thesis:

Applications of fractional calculus in control theory

Brief description:

Control theory is a key application of fractional calculus, since derivatives of non-integer order pose a way how to improve so-called PID controllers, or fractional operators can be used for modelling of complex controlled systems.

Bachelor's Thesis goals:

The goal of this thesis is to demonstrate the basic differences between integer-order and fractional-order PID controllers, to formulate the fundamental problems of control theory in the theory of fractional calculus, and to discuss some of the solutions.

Recommended bibliography:

PODLUBNÝ, I. Fractional Differential Equations. Academic Press, 1998.

CAPONETTO, R., G. DONGOLA, L. FORTUNA A I. PETRÁŠ. Fractional Order Systems: Modelling and Control Applications. World Scientific Publishing, 2010.

Students are required to submit the thesis within the deadlines stated in the schedule of the academic year 2017/18.

In Brno, 31. 10. 2017

L. S.

prof. RNDr. Josef Šlapal, CSc. Director of the Institute	doc. Ing. Jaroslav Katolický, Ph.D. FME dean

Summary

This bachelor's thesis deals with the mathematical theory of fractional calculus and its applications in the field of control theory. We lay out the basics of control of linear time-invariant systems and discuss three of the classical problems - determining stability, controllability, and observability. In the second part, we introduce the Riemann-Liouville and Caputo differintegrals and formulate the above mentioned problems for a fractional-order linear time-invariant system. We discuss the solutions to them and show how they are derived.

Abstrakt

Tato bakalářská práce se zabývá matematickou teorií zlomkového kalkulu a jeho aplikacemi v oblasti teorie řízení. V první části jsou uvedeny základy řízení lineárních časově invariantních systémů, a jsou dále diskutovány tři klasické úlohy, a to určení stability, řiditelnosti a pozorovatelnosti. V druhé části je zaveden Riemann-Liouvillův a Caputův diferintegrál a jsou formulovány výše zmíněné problémy pro lineární časově invariantní systém zlomkého řádu. Opět jsou diskutována řešení a jejich odvození.

Keywords

fractional calculus, fractional linear systems, control theory, stability, controllability, observability

Klíčová slova

zlomkový kalkulus, zlomkové lineární systémy, teorie řízení, stabilita, řiditelnost, pozorovatelnost

Rozšířený abstrakt
Teorie řízení je odvětví aplikované matematiky a inženýrství zabývající se chováním dynamických systémů. Začala se rozvíjet a široce uplatňovat ve 20. století, především v jeho druhé polovině díky rozvoji počítačových technologií. Zároveň s jejím rozmachem byly zkoumány možnosti aplikací matematické teorie neceločíselných derivací na inženýrské úlohy, a ukázalo se, že mnohé objekty a struktury v teorii řízení skrývají v tomto ohledu velký potenciál.
Problematika zlomkového kalkulu, neboli odvětví matematiky věnujícího se derivacím a integrálum neceločíselného řádu, je téměř tak stará, jako klasický integrální a diferenciální počet. První zmínka pochází z korespondence mezi l'Hospitalem a Leibnizem z roku 1695, kde spolu diskutovali možnost derivace o řádu $\frac{1}{2}$. Od té doby byla tato oblast studována mnoha významnými matematiky, můžeme jmenovat například Liouvilla, Riemanna, Eulera nebo Abela. Nebyly však dlouho známy aplikace na reálné problémy. To se však ve 20. století změnilo, a dnes je tato bohatá teorie využívána při modelování viskoelastických materiálů, proteinů, ve zpracovávání signálů, robotice a mnoha dalších odvětvích.
Tato bakalářská práce je zaměřena na aplikace této oblasti matematiky v teorii řízení. První část se skládá z úvodu do řízení lineárních časově invariantních systémů. Jsou zde diskutovány úlohy určení stability, řiditelnosti a pozorovatelnosti. Co se týče stability, rozlišujeme zde několik pojmů, a to asymptotická stabilitu, BIBO stabilitu a vnitřní stabilitu. Text obsahuje jejich definice, věty, které určují podmínky pro dosažení stability, a několik příkladů sloužících k ilustraci. Problémy řiditelnosti a pozorovatelnosti jsou rovněž zadefinovány, pro jejich analýzu jsou uvedeny dvě možná kritéria. Prvním z nich je regularita příslušné Gramovy matice, tedy existence její inverze. Toto kritérium platí pro obecnější případ lineárního systému, jehož matice koeficientů mohou být závislé na čase. Je však náročnější na výpočet. Druhé kritérium, čímž je plná hodnost příslušné matice řiditelnosti, respektive pozorovatelnosti, platí sice pouze pro systémy nezávislé na čase, ale je výpočetně jednodušší.
Druhá část je věnována zlomkovému kalkulu. Nejprve jsou zavedeny dvě speciální funkce, které jsou pro teorii neceločíselných derivací klíčové, a to gamma funkce, která zobecňuje faktoriál, a Mittag-Lefflerova funkce, kterou lze chápat jako generalizaci exponenciální funkce. Následně jsou zadefinovány oba diferintegrály, které jsou v práci použity, tedy Riemann-Liouvillův a Caputův. Uvedeny jsou také jejich Laplaceovy transformace, které jsou klíčové pro analýzu lineárních systémů zlomkového řádu. Tyto systémy jsou představeny v další kapitole. V té jsou nejprve pro Riemann-Liouvillův i Caputův systém odvozena vyjádření stavového vektoru na čase. Dále je prozkoumána asymptotická a BIBO stabilita, spolu s odvozením vět, které stanovují jejich kritéria. Je zde ukázáno, že pro volbu řádu $0 < \alpha < 1$ je dosaženo větší oblasti stability. Pro řiditelnost a pozorovatelnost jsou dále odvozena obě pozměněná kritéria z první části práce.

KIŠA, D.: *Applications of fractional calculus in control theory.* Brno: Vysoké učení technické v Brně, Fakulta strojního inženýrství, 2018. 37 s. Vedoucí bakalářské práce Ing. Tomáš Kisela, Ph.D.

I declare that I have written the bachelor's thesis *Applications of fractional calculus in control theory* on my own and under the guidance of my supervisor Ing. Tomáš Kisela, Ph.D., using the sources listed in the references.

Daniel Kiša

I would like to express my thanks to the supervisor of my thesis, Ing. Tomáš Kisela, Ph.D., for his patience, kindness, valuable advice, and the time that he spent helping me.

Daniel Kiša

Contents

1 Introduction

Control theory is a branch of mathematics and engineering that deals with the behaviour of dynamical systems. Although the fundamental idea of automatic control is centuries old, the discipline underwent a rapid rise in the 20th century. This was especially true of the '50s and the '60s, when many interesting problems that lacked the underlying theory emerged in the context of increasing availability of efficient digital computing and events such as the Space Race. In the last few decades, it has been demonstrated that many objects and phenomena can be with great success modeled with the use of fractional calculus.

The theory of fractional calculus - or of integrals and derivatives of non-integer order - is as old as classical calculus itself. It was first mentioned in 1695 in Leibniz's correspondence with l'Hospital, where they pondered on the idea of a derivative of order one half. Since then, the topic has been studied by many great mathematicians, including Liouville, Riemann, Abel, Euler or Grünwald. For a long time, it was considered to be exclusively a field of pure mathematics. However, in recent history, many applications have been found. Besides control theory, which has already been mentioned, we can name modeling of viscoelastic materials, polymers and proteins, electromagnetism, chaos, robotics, signal processing, and many more.

In this bachelor's thesis, we will look at three principal problems of control theory - stability, controllability, and observability of linear systems. First, we will analyse them for integer-order systems, and then look at what changes when we move on to the fractional case. The thesis is organised as follows.

In the second chapter, we will describe some of the mathematical apparatus that will be needed throughout the thesis. Next, we will do a brief introduction to the world of control theory. There, we will define the questions of stability, controllability, and observability and state the solutions to them for integer-order linear time-invariant systems. In the fourth chapter, we will delve into the theory of fractional calculus and define the operators - or differintegrals - that we will use in this thesis. And finally, in the last chapter, we will examine fractional-order linear time-invariant systems and fine the conditions for their stability, controllability, and observability.

The thesis draws from the books and articles listed on the last page. Specifically, the book [2] was used as a foundation for the introduction to control theory and integer-order linear systems. The books [3] and [4] were the basis for defining the differintegrals, studying their properties and connections with the Mittag-Leffler function. The articles and works [1], [5], [6], [7] and [8] were used to state and prove the theorems about stability, controllability and observability.

2 Preliminaries

In this chapter, we will remind the reader of some of the essentials that will be needed through the whole thesis. Namely, we will define the Laplace transform, show that the exponential function can be extended to matrices and state the Cayley-Hamilton theorem and its corollaries.

2.1 The Laplace Transform

Definition 2.1 (The Laplace transform)**.** Let $f(t)$ be a real function defined on $\langle 0, \infty)$. Then the *Laplace transform* of $f(t)$ is the function $\mathcal{L}\{f(t)\}(s)$ defined by

$$\mathcal{L}\{f(t)\}(s) = F(s) = \int_0^{\infty} f(t)e^{-st}dt \tag{2.1}$$

for all complex numbers s such that the improper integral converges.

Definition 2.2 (The inverse Laplace transform)**.** Let $F(s)$ be a complex function such that $F(s) = \mathcal{L}\{f(t)\}(s)$. Then the *inverse Laplace transform* of $F(s)$ is a real function $\mathcal{L}^{-1}\{F(s)\}(t)$ which has the property

$$\mathcal{L}^{-1}\{F(s)\}(t) = f(t). \tag{2.2}$$

2.1.1 Properties of the Laplace Transform

The Laplace Transform has a number of important properties, some of which we will state here.

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\}(s) = \alpha\mathcal{L}\{f(t)\}(s) + \beta\mathcal{L}\{f(t)\}(s) \tag{2.3}$$

$$\mathcal{L}\{f^{'}(t)\}(s) = s\mathcal{L}\{f(t)\}(s) - f(0) \tag{2.4}$$

$$\mathcal{L}^{-1}\{F(s) \cdot G(s)\}(t) = \mathcal{L}^{-1}\{F(s)\}(t) * \mathcal{L}^{-1}\{G(s)\}(t) \tag{2.5}$$

2.2 The Matrix Exponential

The well known exponential function can be extended to accept a square matrix as its input.

Definition 2.3 (The matrix exponential)**.** Let $\mathbf{A}$ be an $n \times n$ real or complex matrix. Then the *matrix exponential* of $\mathbf{A}$ is defined by the power series

$$e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!} \tag{2.6}$$

where $\mathbf{A}^0$ is defined to be the identity matrix $\mathbf{I}$ of the same dimensions as $\mathbf{A}$.

Matrix exponentials are important in expressing the solution of systems of ordinary differential equations. Namely, the solution of the homogenous system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{2.7}$$

where A is a constant matrix, is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 \tag{2.8}$$

2.3 The Cayley-Hamilton Theorem

The Cayley-Hamilton theorem is a crucial result in linear algebra.

Theorem 2.4. *Let $p(\lambda) = det\,(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0$ be the characteristic equation of a square matrix $\boldsymbol{A}$. Then the matrix $\boldsymbol{A}$ satisfies the equation. Symbolically, we can express this as*

$$p(\boldsymbol{A}) = \boldsymbol{O}, \tag{2.9}$$

where $\boldsymbol{O}$ is the zero matrix.

Corollary 2.5. *The Cayley-Hamilton theorem yields a way to express the n^{th} power of a matrix. This is best explained on an example. Let*

$$\boldsymbol{A} = \begin{bmatrix} 0 & 3 \\ -2 & 5 \end{bmatrix}. \tag{2.10}$$

Then the characteristic equation is given by

$$p(\lambda) = \lambda^2 - 5\lambda + 6 = 0 \tag{2.11}$$

Since according to the theorem, $\boldsymbol{A}$ must also satisfy the equation, we get

$$\boldsymbol{A}^2 - 5\boldsymbol{A} + 6\boldsymbol{I} = 0 \tag{2.12}$$

$$\boldsymbol{A}^2 = 5\boldsymbol{A} - 6\boldsymbol{I} \tag{2.13}$$

Analogously, we can express higher powers of $\boldsymbol{A}$ as well.

Corollary 2.6. *Given an analytic function*

$$f(x) = \sum_{k=0}^{\infty} a_k x^k, \tag{2.14}$$

the analytic function of matrix $\boldsymbol{A}$

$$f(\boldsymbol{A}) = \sum_{k=0}^{\infty} a_k \boldsymbol{A}^k \tag{2.15}$$

can be expressed as a matrix polynomial of degree less than n.

$$f(\boldsymbol{A}) = \sum_{k=0}^{n-1} c_k \boldsymbol{A}^k \tag{2.16}$$

3 Basics of Control Theory

Many important physical systems can be expressed in the form of a state-space representation, which models the system as a set of input, output and state variables related by first-order differential equations. As the state variables are functions of time, the representation is also often called the "time-domain approach". Most of this chapter draws from [2].
We will concern ourselves mainly with the general linear system, which can be efficiently described in matrix form

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \\ \mathbf{y}(t) &= \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t) \end{aligned} \tag{3.1}$$

and its simpler form, the linear time-invariant system

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{aligned} \tag{3.2}$$

The first equation will be referred to as the *state equation*, the second as the *output equation.*

For an n^{th} order system (a system that can be represented by an n^{th} order differential equation) with r inputs and m outputs, the matrix dimensions are as follows:

$\mathbf{x}(t) \in \mathbb{R}^{n\times 1}$ is the state vector
$\mathbf{A}(t) \in \mathbb{R}^{n\times n}$ is the state matrix
$\mathbf{B}(t) \in \mathbb{R}^{n\times r}$ is the input matrix
$\mathbf{u}(t) \in \mathbb{R}^{r\times 1}$ is the input vector
$\mathbf{C}(t) \in \mathbb{R}^{m\times n}$ is the output matrix
$\mathbf{D}(t) \in \mathbb{R}^{m\times r}$ is the feedthrough matrix
$\mathbf{y}(t) \in \mathbb{R}^{m\times 1}$ is the output vector

Remark. We will use the abbreviation LTI for the qualifier *linear time-invariant* from now on.

3.1 Transfer Function

Transfer function representation is one of the most powerful tools in control system analysis. Given the system (3.2), the transfer function

$$\mathbf{T}(s) = \frac{\mathbf{Y}(s)}{\mathbf{U}(s)} \tag{3.3}$$

can be obtained by taking the Laplace transform of the system equations with zero initial conditions. The state equation gives

$$\left[s\mathbf{I} - \mathbf{A}\right]\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s), \tag{3.4}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & 1 \end{bmatrix}. \tag{3.5}$$

By solving for $\mathbf{X}(s)$, we get

$$\mathbf{X}(s) = \left[s\mathbf{I} - \mathbf{A}\right]^{-1} \mathbf{B}\mathbf{U}(s). \tag{3.6}$$

From the output equation, we know that the output and state vectors are related by

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) = \mathbf{C}\left[s\mathbf{I} - \mathbf{A}\right]^{-1} \mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s). \tag{3.7}$$

This gives us the final expression for the transfer function

$$\mathbf{T}(s) = \frac{\mathbf{Y}(s)}{\mathbf{U}(s)} = \mathbf{C}\left[s\mathbf{I} - \mathbf{A}\right]^{-1} \mathbf{B}(s) + \mathbf{D}. \tag{3.8}$$

3.2 State-Transition Matrix

The state-transition matrix $\Phi(t, t_0)$ is a matrix whose product with the state vector $\mathbf{x}$ at time t_0 gives the state vector $\mathbf{x}$ at a later time t (see [5]). For the linear system (3.1), the solution is given by

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \Phi(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)d\tau \tag{3.9}$$

In the general case of the system (3.1), the state-transition matrix can be obtained from the system's fundamental solution matrix.
The fundamental solution matrix $\mathbf{P}(t)$ is an $n \times n$ matrix, where each column represents one of the n linearly independent solutions to the zero-input state equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) \tag{3.10}$$

The matrix $\mathbf{P}(t)$ will then also satisfy the equation

$$\dot{\mathbf{P}}(t) = \mathbf{A}(t)\mathbf{P}(t) \tag{3.11}$$

The state-transition matrix is given by

$$\Phi(t, t_0) = \mathbf{P}(t)\mathbf{P}^{-1}(t_0) \tag{3.12}$$

and has the following properties.

$$\begin{aligned} \Phi(t, t) &= \mathbf{I} \\ \Phi(t_2, t_1)\Phi(t_1, t_0) &= \Phi(t_2, t_0) \\ \Phi^{-1}(t, t_0) &= \Phi(t_0, t) \end{aligned}$$

Remark. For the linear time-invariant system (3.2), the matrix is given by the matrix exponential

$$\Phi(t, t_0) = e^{\mathbf{A}(t - t_0)} \tag{3.13}$$

3.3 Stability

Determining the stability of a system is one of the central problems of control theory. In this section we will present some of the commonly used definitions of stability and explain the differences between them. The definitions are taken from [2].

Definition 3.1 (Asymptotic Stability)**.** Consider the linear time-invariant system (3.2). We will say that the system is *asymptotically stable* if all the states approach zero with time, that is when

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0} \tag{3.14}$$

The necessary and sufficient condition for asymptotic stability is quite simple.

Theorem 3.2. *The system* (3.2) *is asymptotically stable if and only if all the eigenvalues of its state matrix* $\boldsymbol{A}$ *are in the left half-plane.*

Another form of stability is BIBO stability (*bounded-input, bounded-output*).

Definition 3.3 (BIBO Stability)**.** We say that the system (3.2) is *BIBO stable* if the system's output is bounded for every input that is bounded.

$$|\mathbf{u}(t)| \leq N \quad \Rightarrow \quad |\mathbf{y}(t)| \leq M \tag{3.15}$$

where M and N are finite upper bounds of $\mathbf{u}(t)$ and $\mathbf{y}(t)$.

Here, the condition is very similar.

Theorem 3.4. *The system* (3.2) *is BIBO stable if and only if all the poles of the system's transfer function are in the left half-plane.*

Lemma 3.5. *If the system* (3.2) *is asymptotically stable, it is also BIBO stable.*

The converse is true only in the absence of pole-zero cancellations in the process of obtaining the transfer function.

3.3.1 Internal Stability

The notion of internal stability is similar to BIBO stability in that it is also concerned with the system transfer functions. We will say that a system is *internally stable* if all signals within the system are bounded for all bounded inputs.
Apparently, internal stability is a stronger claim than BIBO stability. Equivalently, the difference between internal stability and BIBO stability can be stated using transfer functions. While BIBO stability requires the transfer function of the whole system to be stable, internal stability requires all possible transfer functions between all inputs and outputs to be stable.

3.3.2 Examples

Example 3.6. Consider the linear time-invariant system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 2 & 3 \\ -4 & -5 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}(t)$$

$$\mathbf{y}(t) = \begin{bmatrix}1 & 1\end{bmatrix}\mathbf{x}(t).$$

The eigenvalues of the system can be obtained by computing the determinant

$$|\mathbf{A} - s\mathbf{I}| = \begin{vmatrix} 2-s & 3 \\ -4 & -5-s \end{vmatrix} = s^2 + 3s + 2 = (s+1)(s+2).$$

The real parts of both eigenvalues are negative, the system is therefore asymptotically stable and BIBO stable.

Example 3.7. Consider the control system depicted in the figure 1 with transfer functions

$$\frac{\mathbf{Y}(s)}{\mathbf{R}(s)} = \frac{1}{s+2} \qquad \frac{\mathbf{Y}(s)}{\mathbf{D}(s)} = \frac{s+1}{(s-1)(s+2)}.$$

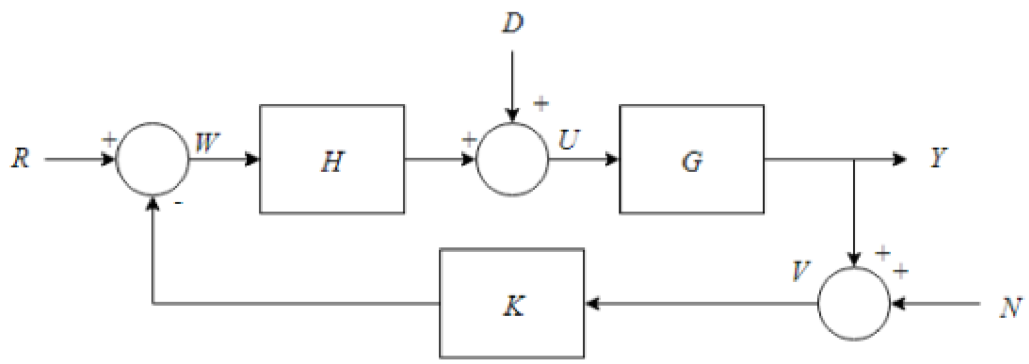

Figure 1: A system with an input **R**, a disturbance **D**, and an output **Y**.

The input-output transfer function is clearly stable, hence we can say that the system is BIBO stable. However, the transfer function between the disturbance and the output has a pole lying in the right half-plane and is therefore unstable. Any disturbance in the system will grow unbounded, the system is internally unstable.

3.4 Controllability

Another important part of control system analysis is determining its controllability. We say that a system is *completely controllable* if the system can be moved from any initial state $\mathbf{x}(t_0)$ to any final state $\mathbf{x}(t_f)$ by applying a control input $\mathbf{u}(t)$ over a finite time interval (t_0, t_f). Note that the definition does not say that a state can be maintained, but only that it can be reached.
We will also introduce the notion of controllability on an interval (see [5]).

Definition 3.8 (Controllability on an interval)**.** Let $t_0, t_f \in \mathbb{R}$ such that $t_0 < t_f$. We will say that the system (3.2) is *controllable on* (t_0, t_f) if and only if for all state vectors $\mathbf{x}_0, \mathbf{x}_f$ there exists an input vector $\mathbf{u}(t)$ such that $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\mathbf{x}(t_f) = \mathbf{x}_f$.

Remark. If a system is controllable on the interval (t_0, t_f) for some input $\mathbf{u}(t)$, we will say that the vector $\mathbf{u}(t)$ *steers* $\mathbf{x}_0$ to $\mathbf{x}_f$ on (t_0, t_f).

Remark. A system is completely controllable if it is controllable on every interval (t_0, t_f).

The next theorem draws from [5].

Theorem 3.9. *The system* (3.1) *is controllable on* (t_0, t_f) *if and only if the* $n \times n$ *controllability gramian matrix given by*

$$\boldsymbol{W}_c(t_0, t_f) = \int_{t_0}^{t_f} \Phi(t_f, \tau)\boldsymbol{B}(\tau)\boldsymbol{B}^T(\tau)\Phi^T(t_f, \tau)d\tau \tag{3.16}$$

is invertible, where $\Phi(t_f, t_0)$ *is the system's state transition matrix.*

For a linear time-invariant system , there is a simpler method of determining its controllability that relies on constructing the so-called controllability matrix, as it can be shown that the matrix has the same rank as the controllability gramian matrix (see [2]).

Definition 3.10 (Controllability matrix)**.** Consider the system (3.2). Then the *controllability matrix* of the system is given by

$$\mathbf{M}_c = [\mathbf{B} \mid \mathbf{AB} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}]. \tag{3.17}$$

Theorem 3.11. *The system* (3.2) *is completely controllable if and only if its controllability matrix is of full rank.*

3.4.1 Examples

Example 3.12. Consider the system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}(t).$$

The system has the controllability matrix

$$\mathbf{M}_c = [\mathbf{B} \mid \mathbf{AB}],$$

where

$$\mathbf{AB} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Thus,

$$\mathbf{M}_c = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix},$$

which is of rank 1. The system is not completely controllable.

Example 3.13. Consider another system, this time of order 3.

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mathbf{u}(t).$$

We will first compute the matrices **AB** and $\mathbf{A}^2\mathbf{B}$.

$$\mathbf{AB} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

$$\mathbf{A}^2\mathbf{B} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}.$$

Then the controllability matrix is given by

$$\mathbf{M}_c = [\mathbf{B} \mid \mathbf{AB} \mid \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and the determinant is

$$det(\mathbf{M}_c) = 1$$

The matrix is of full rank, the system is completely controllable.

3.5 Observability

Very similar to the previous concept is the notion of observability. A system is *completely observable* if any initial state $\mathbf{x}(t_0)$ can be reconstructed by examining the system output $\mathbf{y}(t)$ over a finite time interval (t_0, t_f).

Definition 3.14 (Observability on an interval)**.** Let $t_0, t_f \in \mathbb{R}$ such that $t_0 < t_f$. We will say that the system (3.2) is *observable on* (t_0, t_f) if and only if the initial state $\mathbf{x}(t_0)$ can be determined from the system output $\mathbf{y}(t)$ over a finite time interval (t_0, t_f) (see [5]).

Theorem 3.15. *The system* (3.1) *is observable on* (t_0, t_f) *if and only if the* $n \times n$ *observability gramian matrix given by*

$$\boldsymbol{W}_o(t_0, t_f) = \int_{t_0}^{t_f} \Phi(t_f, \tau)\boldsymbol{C}^T(\tau)\boldsymbol{C}(\tau)\Phi^T(t_f, \tau)d\tau \tag{3.18}$$

is invertible, where $\Phi(t_f, t_0)$ *is the system's state transition matrix.*

Similarly to controllability, for the time-invariant system, we can define the observability matrix.

Definition 3.16 (Observability matrix)**.** Consider again the linear time-invariant system (3.2). Then its *observability matrix* is given by

$$\mathbf{M}_o = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}. \tag{3.19}$$

Theorem 3.17. *The system* (3.2) *is completely observable if and only if its observability matrix is of full rank (see* [2]*).*

Theorem 3.18. *If a system is uncontrollable or unobservable, it will have a pole-zero cancellation in its transfer function. Conversely, any pole-zero cancellation in the system's transfer function implies either uncontrollabity or unobservability.*

Remark. There is a strong connection between the observability and controllability matrices. Namely, the observability test on the system (3.2) is equivalent to the controllability test on a dual system with the state equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{C}^T\mathbf{u}(t) \tag{3.20}$$

3.5.1 Examples

Example 3.19. Consider the system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} -2 \\ 3 \end{bmatrix} \mathbf{u}(t)$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t).$$

Its observability matrix can be calculated.

$$\mathbf{M}_o = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}.$$

The determinant can be shown to be

$$det(\mathbf{M}_o) = 2.$$

Thus, the system is completely observable.

4 Basics of Fractional Calculus

Fractional calculus is a branch of mathematical analysis that deals with differentiation operators and integration operators of arbitrary order. The idea of generalizing integer-order derivatives and repeated integrals to real number orders is more than 300 years old, and there are several possible ways of defining them that are in use (see [3] and [4]). These definitions are usually named after their author and have distinct advantages and disadvantages.
In this text, we will only consider the Riemann-Liouville differintegral and the Caputo differintegral. However, before defining them, we will examine a couple of special functions.

4.1 The Gamma Function

In calculus of integer order, the factorial function naturally arises in expressions for the n-th derivative of a polynomial and the n-th integral.

$$\frac{d^n}{dt^n}(t^n) = n! \tag{4.1}$$

$$I_a^n = \int_a^t \int_a^{\tau_{n-1}} \cdots \int_a^{\tau_1} f(\tau) d\tau d\tau_1 \dots d\tau_{n-1} = \frac{1}{(n-1)!} \int_a^t (t-\tau)^{n-1} f(\tau) d\tau \tag{4.2}$$

Remark. The expression for the n-th integral above is called the *Cauchy formula for repeated integration.*

The gamma function is an extension of the factorial function to complex numbers. For a natural number n, the following holds.

$$\Gamma(n) = (n-1)! \tag{4.3}$$

The next definition is taken from [3].

Definition 4.1 (The gamma function)**.** Let z be a complex number with $\Re(z) > 0$. We will call the convergent improper integral

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt \tag{4.4}$$

the gamma function of z.

The gamma function also retains the recurrence relation

$$z\Gamma(z) = \Gamma(z+1). \tag{4.5}$$

This, in addition to $\Gamma(1) = 1$ gives us the connection (4.3) to the factorial function.

4.2 The Mittag - Leffler Function

The exponential function e^z plays a key role in the theory of integer-order differential equations. For fractional differential equations, we will use its two-parameter generalization (for more information, see [3]).

Definition 4.2 (The two-parameter Mittag-Leffler function)**.**

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \quad \beta > 0 \tag{4.6}$$

Remark. If $\beta = 1$, we talk of the one-parameter Mittag-Leffler function denoted by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \tag{4.7}$$

From the definition (4.2), we can derive many useful relationships, such as

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z \tag{4.8}$$

$$E_{1,2}(z) = \frac{e^z - 1}{z} \tag{4.9}$$

$$E_{2,1}(z^2) = cosh(z) \tag{4.10}$$

$$E_{2,2}(z^2) = \frac{sinh(z)}{z} \tag{4.11}$$

Remark. We can also - analogously to the matrix exponential mentioned in (2.6) - define the matrix extension of the Mittag-Leffler function

$$E_{\alpha,\beta}(\mathbf{A}) = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{\Gamma(\alpha k + \beta)}. \tag{4.12}$$

This family of functions will see extensive use in the analysis of fractional order linear systems.

4.3 The Riemann-Liouville Differintegral

This approach uses Cauchy's formula for repeated integration (4.2) and generalizes it to an arbitrary order by defining the Riemann-Liouville integral (definition taken from [4]).

Definition 4.3 (The Riemann-Liouville integral)**.** Let a, b be real numbers, such that $a < b$. Let α be a positive real number. Let $f(t)$ be integrable on $\langle a, b\rangle$ and $t \in \langle a, b\rangle$. We will call the expression

$$I_a^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t f(\tau)(t-\tau)^{\alpha-1} d\tau \tag{4.13}$$

the Riemann-Liouville integral of order α.

We will then define the Riemann-Liouville fractional derivative by using the above defined Riemann-Liouville integral followingly.

Definition 4.4 (The Riemann-Liouville fractional derivative)**.** Let a, b be real numbers, such that $a < b$. Let α be a positive real number. Let $f(t)$ be integrable on $\langle a, b\rangle$ and $t \in \langle a, b\rangle$. Let also $f(t)$ be at least $\lceil\alpha\rceil$-times differentiable on $\langle a, b\rangle$, where the symbol $\lceil\alpha\rceil$ denotes the ceiling function

$$\lceil\alpha\rceil = min\{z \in \mathbb{Z} | z \geq \alpha\}. \tag{4.14}$$

We will call the expression

$$D_a^\alpha f(t) := \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}} \left[I_a^{\lceil\alpha\rceil-\alpha} f(t)\right] = \frac{1}{\Gamma(n-\alpha)} \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}} \int_a^t f(\tau)(t-\tau)^{\lceil\alpha\rceil-\alpha-1} d\tau \tag{4.15}$$

the Riemann-Liouville fractional derivative of order α.

We can then combine these definitions to obtain the Riemann-Liouville differintegral.

Definition 4.5 (The Riemann-Liouville differintegral)**.** Let α, a, b be real numbers, such that $a < b$. Let $f(t)$ be integrable on $\langle a, b\rangle$ and $t \in \langle a, b\rangle$. If $\alpha > 0$, let also $f(t)$ be at least $\lceil\alpha\rceil$-times differentiable on $\langle a, b\rangle$.

$$\mathbf{D}_a^\alpha f(t) := \begin{cases} I_a^\alpha f(t) & \text{for } \alpha < 0 \\ f(t) & \text{for } \alpha = 0 \\ \frac{1}{\Gamma(n-\alpha)} \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}} \int_a^t f(\tau)(t-\tau)^{\lceil\alpha\rceil-\alpha-1} d\tau & \text{for } \alpha > 0 \end{cases} \tag{4.16}$$

Remark. The symbol $\mathbf{D}_a^\alpha$ will stand for the Riemann-Liouville differintegral throughout this thesis.

The properties of the Riemann-Liouville differintegral will be discussed later.

4.4 The Caputo Differintegral

An alternative definition of the fractional derivative was proposed by Caputo. Caputo's derivative has a clear advantage over the Riemann-Liouville derivative when it comes to modelling fractional physical systems, in that the initial conditions of the system are of integer orders and as such have known physical interpretations.

Definition 4.6 (The Caputo differintegral)**.** Let α, a, b be real numbers, such that $a < b$. Let $f(t)$ be integrable on $\langle a, b\rangle$ and $t \in \langle a, b\rangle$. If $\alpha > 0$, let also $f(t)$ be at least $\lceil\alpha\rceil$-times differentiable on $\langle a, b\rangle$.

$${}^C\mathbf{D}_a^\alpha f(t) := \begin{cases} \mathbf{D}_a^\alpha f(t) & \text{for } \alpha \leq 0 \\ \mathbf{D}_a^{-(\lceil\alpha\rceil-\alpha)} \left[\frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}} f(t)\right] & \text{for } \alpha > 0 \end{cases} \tag{4.17}$$

We can see that the two differintegrals are equal for $\alpha <= 0$, i.e. the fractional integrals are the same. However, the fractional derivatives differ. In Caputo's approach, the classical integer-order derivative is applied first, before the fractional integral, whereas in the Riemann-Liouville approach, it is the other way around.

If we assume stricter conditions for the order of the derivative $\alpha > 0, \alpha \notin \mathbb{N}$, we can write the fractional derivative in the form

$$ {}^{C}\mathbf{D}_a^{\alpha} f(t) = \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha} \int_a^t (t-\tau)^{\lceil\alpha\rceil - \alpha - 1} f^{(\lceil\alpha\rceil)}(\tau) d\tau. \tag{4.18} $$

4.5 Properties of the Differintegrals

The differintegrals keep some of the properties of classical derivatives and integrals, or generally have direct analogies to them. We can, for example, mention linearity, which is satisfied by both of our differintegrals.

$$ \mathbf{D}_a^{\alpha}(\mu f(t) + \nu g(t)) = \mu \mathbf{D}_a^{\alpha} f(t) + \nu \mathbf{D}_a^{\alpha} g(t) \tag{4.19} $$

$$ {}^{C}\mathbf{D}_a^{\alpha}(\mu f(t) + \nu g(t)) = \mu {}^{C}\mathbf{D}_a^{\alpha} f(t) + \nu {}^{C}\mathbf{D}_a^{\alpha} g(t) \tag{4.20} $$

However, we will be more interested in the Laplace transforms of the differintegrals, because we will need those to prove many of our theorems in the next chapter. This time, the transform of our operators will produce different results (see [3] for more information and proofs).

$$ \mathcal{L}\{\mathbf{D}_a^{\alpha} f(t)\}(s) = s^{\alpha} F(s) - \sum_{k=0}^{\lceil\alpha\rceil - 1} s^k \mathbf{D}_a^{\alpha - k - 1} f(a) \tag{4.21} $$

$$ \mathcal{L}\{{}^{C}\mathbf{D}_a^{\alpha} f(t)\}(s) = s^{\alpha} F(s) - \sum_{k=0}^{\lceil\alpha\rceil - 1} s^{\alpha - k - 1} f^{(k)}(a) \tag{4.22} $$

5 Fractional Order Linear Time-Invariant Systems

Many physical and engineering phenomena can be efficiently described by fractional-order dynamical systems. In this chapter, we will look at the fractional equivalents of the linear time-invariant system (3.2) discussed previously. Such systems see use in the field of control theory.

Definition 5.1 (The Caputo LTI system)**.** Let $\alpha \in (0, 1\rangle$. We will call the system

$$\begin{aligned} {}^{C}\mathbf{D}_0^{\alpha}\mathbf{x}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t), \end{aligned} \tag{5.1}$$

where the vectors **x, u, y** and the matrices **A, B, C, D** have the same dimensions as in the system (3.2) the Caputo LTI system.

We can express $\mathbf{x}(t)$ similarly to (3.9) (see [6]). First, we will state two results which we will not prove here.

Lemma 5.2.

$$[s^{\alpha}\boldsymbol{I} - \boldsymbol{A}]^{-1} = \sum_{k=1}^{\infty} \boldsymbol{A}^{k-1}s^{-k\alpha} \tag{5.2}$$

Lemma 5.3.

$$\mathcal{L}^{-1}\{s^{-\alpha-1}\}(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)} \quad \textit{for } \alpha > -1 \tag{5.3}$$

Theorem 5.4. *Consider the system* (5.1)*. Then the state vector* $\boldsymbol{x}(t)$ *at an arbitrary time* $t > 0$ *is given by the equation*

$$\boldsymbol{x}(t) = E_{\alpha}(\boldsymbol{A}t^{\alpha})\boldsymbol{x}_0 + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\boldsymbol{A}(t-\tau)^{\alpha})\boldsymbol{B}\boldsymbol{u}(\tau)d\tau. \tag{5.4}$$

Proof. Let us take the Laplace transform of the state equation.

$$\mathcal{L}\{{}^{C}\mathbf{D}_0^{\alpha}\mathbf{x}(t)\}(s) = \mathcal{L}\{\mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\}(s) \tag{5.5}$$

Then by (4.22), we get

$$\mathbf{I}\left(s^{\alpha}\mathbf{X}(s) - s^{\alpha-1}\mathbf{x}_0\right) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \tag{5.6}$$

$$\mathbf{X}(s) = [s^{\alpha}\mathbf{I} - \mathbf{A}]^{-1} \cdot \left[\mathbf{B}\mathbf{U}(s) + s^{\alpha-1}\mathbf{x}_0\right]. \tag{5.7}$$

By using the lemma (5.2), we can rewrite this as

$$\mathbf{X}(s) = \left[\sum_{k=1}^{\infty} \mathbf{A}^{k-1}s^{-k\alpha}\right] \cdot \mathbf{B}\mathbf{U}(s) + \mathbf{x}_0 \sum_{k=0}^{\infty} \mathbf{A}^{k}s^{-k\alpha-1}. \tag{5.8}$$

If we now take the inverse Laplace transform of the equation (5.8) above and use the properties (2.3) and (2.5), we obtain

$$\mathbf{x}(t) = \mathcal{L}^{-1}\{\left[\sum_{k=1}^{\infty} \mathbf{A}^{k-1}s^{-k\alpha}\right]\}(t) * \mathcal{L}^{-1}\{\mathbf{B}\mathbf{U}(s)\}(t) + \mathcal{L}^{-1}\{\mathbf{x}_0 \sum_{k=0}^{\infty} \mathbf{A}^{k}s^{-k\alpha-1}\}(t). \tag{5.9}$$

Now we use the lemma (5.3) to get

$$\mathbf{x}(t) = \left[\sum_{k=1}^{\infty} \frac{A^{k-1}t^{\alpha k-1}}{\Gamma(\alpha k)}\right] * [\mathbf{Bu}(t)] + \mathbf{x}_0 \sum_{k=0}^{\infty} \frac{A^k t^{\alpha k}}{\Gamma(\alpha k+1)}. \tag{5.10}$$

Finally, if we recall the definition of the Mittag-Leffler function, we can write this as

$$\mathbf{x}(t) = \left[t^{\alpha-1}E_{\alpha,\alpha}(\mathbf{A}t^{\alpha})\right] * [\mathbf{Bu}(t)] + \mathbf{x}_0 E_{\alpha}(\mathbf{A}t^{\alpha}). \tag{5.11}$$

By rearranging, we get the final formula.

$$\mathbf{x}(t) = E_{\alpha}(\mathbf{A}t^{\alpha})\mathbf{x}_0 + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\mathbf{A}(t-\tau)^{\alpha})\mathbf{Bu}(\tau)d\tau \tag{5.12}$$

□

Similarly, we can define the linear-time invariant system with the Riemann-Liouville differintegral. Note that the initial condition is given by a fractional derivative.

Definition 5.5 (The Riemann-Liouville LTI system)**.** Let $\alpha \in (0,1\rangle$. We will call the system

$$\begin{aligned} &\mathbf{D}_0^{\alpha}\mathbf{x}(t) = \mathbf{Ax}(t) + \mathbf{Bu}(t), \quad \mathbf{D}_0^{\alpha-1}\mathbf{x}(0) = \mathbf{x}_0 \\ &\mathbf{y}(t) = \mathbf{Cx}(t) + \mathbf{Du}(t), \end{aligned} \tag{5.13}$$

Theorem 5.6. *Consider the system* (5.5)*. Then the state vector* $\boldsymbol{x}(t)$ *at an arbitrary time* $t > 0$ *is given by the equation*

$$\boldsymbol{x}(t) = t^{\alpha-1}E_{\alpha,\alpha}(\boldsymbol{A}t^{\alpha})\boldsymbol{x}_0 + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\boldsymbol{A}(t-\tau)^{\alpha})\boldsymbol{Bu}(\tau)d\tau \tag{5.14}$$

Remark. The proof of this theorem is virtually identical to the theorem (5.4) for the Caputo LTI system.

Now we can move on towards examining the notions of stability, controllability and observability of fractional-order systems.

5.1 Stability

In this section, we will establish the conditions for asymptotic stability and BIBO stability for fractional-order linear time-invariant systems.

5.1.1 Asymptotic Stability

First, we will state an important lemma that will help us prove the theorem explaining the conditions for stability. The lemma is stated and proved in [3].

Lemma 5.7. *Let* $\alpha \in (0,1\rangle$ *and* β *an arbitrary real number. Let* z *be a complex number such that* $\frac{\alpha\pi}{2} < |arg(z)| \leq \pi$*. Then for an arbitrary integer* $p \geq 1$*, the following asymptotic expansion is valid as* $|z| \to \infty$*.*

$$E_{\alpha,\alpha}(z) = -\sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta-\alpha k)} + O\left(|z|^{-1-p}\right) \tag{5.15}$$

Now we can move forward and state the theorem for the Riemann-Liouville LTI system. The theorem is taken from [8]

Theorem 5.8. *The system* (5.5) *is asymptotically stable if and only if*

$$|arg(\lambda_i(\boldsymbol{A}))| > \frac{\alpha\pi}{2} \quad \forall i = 1, \ldots, n, \tag{5.16}$$

where λ_i is the i^{th} eigenvalue of $\boldsymbol{A}$.

Proof. As the stability of the system does not depend on the input vector $\mathbf{u}(t)$, we can, without loss of generality, set $\mathbf{u}(t) = \mathbf{0}$. By (5.6), we can then express the state vector as

$$\mathbf{x}(t) = \mathbf{x}_0 t^{\alpha-1} E_{\alpha,\alpha}(\mathbf{A}t^{\alpha}). \tag{5.17}$$

Let us suppose that the matrix $\mathbf{A}$ is diagonalizable. This implies that it can be written as

$$\mathbf{A} = \mathbf{T}\Lambda\mathbf{T}^{-1}, \tag{5.18}$$

where Λ is the diagonal matrix of eigenvalues of $\mathbf{A}$

$$\Lambda = diag\left[\lambda_1, \ldots, \lambda_n\right]. \tag{5.19}$$

Remark. This is not necessarily true, but the idea of the proof is the same as in the general case, where we instead decompose $\mathbf{A}$ into $\mathbf{TJT}^{-1}$ with $\mathbf{J}$ being the unique Jordan canonical form of $\mathbf{A}$. The general proof involves more tedious calculations and will be omitted.

Then

$$E_{\alpha,\alpha}(\mathbf{A}t^{\alpha}) = \mathbf{T}E_{\alpha,\alpha}(\Lambda t^{\alpha})\mathbf{T}^{-1} = \mathbf{T}diag\left[E_{\alpha,\alpha}(\lambda_1 t^{\alpha}), \ldots, E_{\alpha,\alpha}(\lambda_n t^{\alpha})\right]\mathbf{T}^{-1}. \tag{5.20}$$

Now if we consider our assumption

$$|arg(\lambda_i(\mathbf{A}))| > \frac{\alpha\pi}{2} \quad \forall i = 1, \ldots, n, \tag{5.21}$$

we see that we can use the lemma (5.7).

$$E_{\alpha,\alpha}(\lambda_i t^{\alpha}) = -\sum_{k=1}^{p} \frac{(\lambda_i t^{\alpha})^{-k}}{\Gamma(\alpha - \alpha k)} + O\left(|\lambda_i t^{\alpha}|^{-1-p}\right) \tag{5.22}$$

This expression approaches 0 as $t \to \infty$ for all $i = 1, \ldots, n$. Because the matrix norm induced by the L_2 vector norm is given by the largest singular value of the matrix and singular values are square roots of eigenvalues, we can deduce that

$$0 = \lim_{t\to\infty} E_{\alpha,\alpha}(\lambda_i t^{\alpha}) = \lim_{t\to\infty} \|E_{\alpha,\alpha}(\Lambda t^{\alpha})\| = \lim_{t\to\infty} \|E_{\alpha,\alpha}(\mathbf{A}t^{\alpha})\| = \lim_{t\to\infty} E_{\alpha,\alpha}(\mathbf{A}t^{\alpha}). \tag{5.23}$$

Now going back to our expression for $\mathbf{x}(t)$ (5.17), we can see that

$$\lim_{t\to\infty} \mathbf{x}(t) = \lim_{t\to\infty} \mathbf{x}_0 t^{\alpha-1} E_{\alpha,\alpha}(\mathbf{A}t^{\alpha}) = \mathbf{0} \tag{5.24}$$

and the proof is done. □

Remark. Observe that by choosing $\alpha = 1$, this also proves the theorem (3.2) for integer-order systems.

Remark. Unlike in integer-order systems where the state $\mathbf{x}(t)$ decays to $\mathbf{0}$ at an exponential rate, in the fractional case, it decays algebraically at the rate of $t^{-\alpha}$.

For the Caputo system (5.1), the condition for asymptotic stability is the same. The expression for $\mathbf{x}(t)$

$$\mathbf{x}(t) = E_\alpha(\mathbf{A}t^\alpha)\mathbf{x}_0 \tag{5.25}$$

used in the proof is different, but it has no bearing on the validity of the proof, as the lemma (5.7) still holds.

5.1.2 BIBO Stability

We will keep our usual definition of BIBO stability (3.15). Before we articulate the condition for BIBO stability, we will derive the transfer function of the Caputo system (5.1). However, because the transfer function of a system is obtained by taking the Laplace transform of its equations with zero initial conditions and the Laplace transforms of the Riemann-Liouville differintegral (4.21) and the Caputo differintegral (4.22) differ only in their treatment of the initial condition, the transfer function of the Riemann-Liouville system (5.5) is the same.

Now, when we were proving the theorem (5.4), we took the Laplace transform of the state equation and arrived at the equation (5.7)

$$\mathbf{X}(s) = \left[s^\alpha \mathbf{I} - \mathbf{A}\right]^{-1} \cdot \left[\mathbf{B}\mathbf{U}(s) + s^{\alpha-1}\mathbf{x}_0\right]. \tag{5.26}$$

To reiterate, the transfer function is derived for the case $\mathbf{x}_0 = 0$. The equation (5.26) then simplifies into

$$\mathbf{X}(s) = \left[s^\alpha \mathbf{I} - \mathbf{A}\right]^{-1} \cdot \left[\mathbf{B}\mathbf{U}(s)\right]. \tag{5.27}$$

The Laplace transform of the output equation is

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s). \tag{5.28}$$

When we substitute (5.27) into the output equation, we obtain

$$\mathbf{Y}(s) = \mathbf{C}\left[s^\alpha \mathbf{I} - \mathbf{A}\right]^{-1} \cdot \left[\mathbf{B}\mathbf{U}(s)\right] + \mathbf{D}\mathbf{U}(s). \tag{5.29}$$

The transfer function is defined as

$$\mathbf{T}(s) = \frac{\mathbf{Y}(s)}{\mathbf{U}(s)}, \tag{5.30}$$

and so we reach the conclusion

$$\mathbf{T}(s) = \mathbf{C}\left[s^\alpha \mathbf{I} - \mathbf{A}\right]^{-1} \mathbf{B} + \mathbf{D}. \tag{5.31}$$

Theorem 5.9. *Let λ_i be the i^{th} pole of the transfer function $\boldsymbol{T}(s)$. Then the systems* (5.1) *and* (5.5) *are stable in the BIBO sense if and only if*

$$|arg(\lambda_i(\boldsymbol{A}))| > \frac{\alpha\pi}{2} \quad \forall i = 1, \ldots, n. \tag{5.32}$$

The theorem is taken from [1].

Remark. Just like in the integer-order case, asymptotic stability implies BIBO stability, but the reverse implication is only true if no pole-zero cancellations occured in obtaining the transfer function.

5.2 Controllability

In studying controllability and observability, we will only do so for the case of the Caputo differintegral. The reason for this is that in the Riemann-Liouville case, we cannot keep our usual definitions of controllability and observability, because the initial conditions are given for a fractional derivative of order $\alpha - 1$ which do not have a known physical interpretation.
To reiterate, for the Caputo system (5.1), we can keep the definition of controllability introduced previously.
First, we will show that for a system to be controllable on (t_0, t_f), we only have to demonstrate that for every state $\mathbf{x}_f$, some input $\mathbf{u}(t)$ steers $\mathbf{0}$ to $\mathbf{x}_f$ in the time interval (t_0, t_f).

Lemma 5.10. *The system* (5.1) *is controllable on* $(t_0, t_f) \Leftrightarrow \forall \boldsymbol{x}_f \in \mathbb{R}^n \ \exists \boldsymbol{u}(t) \in L^2(\langle t_0, t_f\rangle, \mathbb{R}^m)$: $\boldsymbol{u}(t)$ *steers* $\boldsymbol{0}$ *to* $\boldsymbol{x}_f$ *on* (t_0, t_f).

Proof. The left implication is obviously true from the definition, because if the system is controllable, then every $\mathbf{x}_0$ is steered to every $\mathbf{x}_f$ by some $\mathbf{u}(t)$. We can then choose $\mathbf{x}_0 = \mathbf{0}$ and the first part of the proof is done.
Now let us prove the right implication. Let $\mathbf{x}_0$, $\mathbf{x}_f \in \mathbb{R}^n$. Next, we choose an arbitrary $\mathbf{x}_0$ and define $\tilde{\mathbf{x}}_f$

$$\tilde{\mathbf{x}}_f = \mathbf{x}_f - E_\alpha(\mathbf{A}(t_f - t_0)^\alpha)\mathbf{x}_0. \tag{5.33}$$

We assume that there exists $\mathbf{u}(t)$ that steers $\mathbf{0}$ to $\tilde{\mathbf{x}}_f$ on (t_0, t_f). Then by (5.4), we can express $\tilde{\mathbf{x}}_f$ as

$$\tilde{\mathbf{x}}_f = \int_{t_0}^{t_f} (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\mathbf{A}(t-\tau)^\alpha)\mathbf{B}\mathbf{u}(\tau)d\tau \tag{5.34}$$

Next, we substitute for $\tilde{\mathbf{x}}_f$.

$$\mathbf{x}_f = E_\alpha(\mathbf{A}(t_f - t_0)^\alpha)\mathbf{x}_0 + \int_{t_0}^{t_f} (t_f-\tau)^{\alpha-1} E_{\alpha,\alpha}(\mathbf{A}(t_f-\tau)^\alpha)\mathbf{B}\mathbf{u}(\tau)d\tau \tag{5.35}$$

And again, from (5.4) we know that this is the expression for $\mathbf{x}(t_f)$ if $\mathbf{x}(t_0) = \mathbf{x}_0$.

$$\mathbf{x}_f = \mathbf{x}(t_f). \tag{5.36}$$

The system is controllable on (t_0, t_f). □

Next, we will state the necessary condition for the system (5.1) to be controllable. The theorem and the main ideas of the proof are taken from [6].

Theorem 5.11. *The system* (5.1) *is controllable on* (t_0, t_f) *if and only if the* $n \times n$ *controllability gramian matrix given by*

$$\boldsymbol{W}_c(t_0, t_f) = \int_{t_0}^{t_f} (t_f-\tau)^{\alpha-1} E_{\alpha,\alpha}(\boldsymbol{A}(t_f-\tau)^\alpha)\boldsymbol{B}\boldsymbol{B}^T E_{\alpha,\alpha}(\boldsymbol{A}^T(t_f-\tau)^\alpha)d\tau \tag{5.37}$$

is invertible.

Proof. Let us start with the right implication. Because the gramian matrix is invertible, we can choose the input vector to be

$$\mathbf{u}(t) = \mathbf{B}^T E_{\alpha,\alpha}(A^T(t_f - t)^\alpha)\mathbf{W}_c^{-1}(t_0, t_f)\left[\mathbf{x}_f - E_\alpha(A(t_f - t_0)^\alpha)\mathbf{x}_0\right]. \tag{5.38}$$

By (5.4), we can express $\mathbf{x}(t_f)$ as

$$\mathbf{x}(t_f) = E_\alpha(\mathbf{A}(t_f - t_0)^\alpha)\mathbf{x}_0 + \int_{t_0}^{t_f} (t_f - \tau)^{\alpha-1} E_{\alpha,\alpha}(\mathbf{A}(t_f - \tau)^\alpha)\mathbf{B}\mathbf{B}^T$$

$$\times E_{\alpha,\alpha}(A^T(t_f - t)^\alpha)\mathbf{W}_c^{-1}(t_0, t_f)\left[\mathbf{x}_f - E_\alpha(A(t_f - t_0)^\alpha)\mathbf{x}_0\right] d\tau \tag{5.39}$$

The term $\mathbf{W}_c^{-1}(t_0, t_f)\left[\mathbf{x}_f - E_\alpha(A(t_f - t_0)^\alpha)\mathbf{x}_0\right]$ inside the integral does not depend on τ, which means that we can pull it out and obtain

$$\mathbf{x}(t_f) = E_\alpha(\mathbf{A}(t_f - t_0)^\alpha)\mathbf{x}_0 + \mathbf{W}_c(t_0, t_f)\mathbf{W}_c^{-1}(t_0, t_f)\left[\mathbf{x}_f - E_\alpha(A(t_f - t_0)^\alpha)\mathbf{x}_0\right] \tag{5.40}$$

Here we see that $\mathbf{W}_c(t_0, t_f)\mathbf{W}_c^{-1}(t_0, t_f)$ is by definition equal to $\mathbf{I}$ and the terms $E_\alpha(\mathbf{A}(t_f - t_0)^\alpha)\mathbf{x}_0$ cancel each other out. We finally arrive to the conclusion

$$\mathbf{x}(t_f) = \mathbf{x}_f. \tag{5.41}$$

Hence, the system is controllable.
We will prove the left implication by contradiction. We will assume that the system is controllable, but the gramian matrix is not invertible.
If the gramian matrix is not invertible, then there exists a nonzero $\tilde{\mathbf{x}}$ with the property

$$\tilde{\mathbf{x}}^T \mathbf{W}_c \tilde{\mathbf{x}} = 0. \tag{5.42}$$

Now, if we substitute the matrix into this equation, we obtain

$$\tilde{\mathbf{x}}^T \left[\int_{t_0}^{t_f} (t_f - \tau)^{\alpha-1} E_{\alpha,\alpha}(\mathbf{A}(t_f - \tau)^\alpha)\mathbf{B}\mathbf{B}^T E_{\alpha,\alpha}(\mathbf{A}^T(t_f - \tau)^\alpha) d\tau\right] \tilde{\mathbf{x}} = 0, \tag{5.43}$$

which is the same as

$$\int_{t_0}^{t_f} (t_f - \tau)^{\alpha-1} \tilde{\mathbf{x}}^T E_{\alpha,\alpha}(\mathbf{A}(t_f - \tau)^\alpha)\mathbf{B}\mathbf{B}^T E_{\alpha,\alpha}(\mathbf{A}^T(t_f - \tau)^\alpha)\tilde{\mathbf{x}} d\tau = 0. \tag{5.44}$$

Now we use the fact that when taking a transpose of multiplied matrices, their order reverses and we can rewrite the equation (5.44) in the form

$$\int_{t_0}^{t_f} (t_f - \tau)^{\alpha-1} \left[\tilde{\mathbf{x}}^T E_{\alpha,\alpha}(\mathbf{A}(t_f - \tau)^\alpha)\mathbf{B}\right] \left[\tilde{\mathbf{x}}^T E_{\alpha,\alpha}(\mathbf{A}(t_f - \tau)^\alpha)\mathbf{B}\right]^T d\tau = 0. \tag{5.45}$$

Let us define

$$\Psi = \tilde{\mathbf{x}}^T E_{\alpha,\alpha}(\mathbf{A}(t_f - \tau)^\alpha)\mathbf{B}. \tag{5.46}$$

The equation (5.45) becomes

$$\int_{t_0}^{t_f} (t_f - \tau)^{\alpha-1} \Psi \Psi^T d\tau = 0. \tag{5.47}$$

Since Ψ is a matrix of dimensions $1 \times r$, the factor $\Psi\Psi^T$ is simply the squared L_2-norm of Ψ.

$$\int_{t_0}^{t_f} (t_f - \tau)^{\alpha-1} \|\Psi\|^2 \, d\tau = 0. \tag{5.48}$$

The integrand is greater or equal to zero for all $\tau \in (t_0, t_f)$, which implies that

$$\Psi = \tilde{\mathbf{x}}^T E_{\alpha,\alpha}(\mathbf{A}(t_f - \tau)^\alpha)\mathbf{B} = \mathbf{0} \quad \forall \tau \in (t_0, t_f)\,. \tag{5.49}$$

Now we can choose

$$\mathbf{x}_0 = [E_\alpha(\mathbf{A}(t_f - t_0)^\alpha)]^{-1} \tilde{\mathbf{x}}. \tag{5.50}$$

Our assumption is that the system is controllable, which means that there exists a control $\mathbf{u}(t)$ that steers $\mathbf{x}_0$ to $\mathbf{0}$.

$$\mathbf{x}(t_f) = \mathbf{0} = E_\alpha(\mathbf{A}(t_f - t_0)^\alpha)\mathbf{x}_0 + \int_{t_0}^{t_f} (t_f - \tau)^{\alpha-1} E_{\alpha,\alpha}(\mathbf{A}(t_f - \tau)^\alpha)\mathbf{B}\mathbf{u}(\tau) d\tau \tag{5.51}$$

Left-multiplying the above equation (5.51) by $\tilde{\mathbf{x}}^T$, we obtain

$$\tilde{\mathbf{x}}^T \tilde{\mathbf{x}} + \int_{t_0}^{t_f} (t_f - \tau)^{\alpha-1} \tilde{\mathbf{x}}^T E_{\alpha,\alpha}(\mathbf{A}(t_f - \tau)^\alpha)\mathbf{B}\mathbf{u}(\tau) d\tau = \mathbf{0}. \tag{5.52}$$

However, the term $\tilde{\mathbf{x}}^T E_{\alpha,\alpha}(\mathbf{A}(t_f - \tau)^\alpha)\mathbf{B}$ inside the integral is known to be zero, and thus

$$\tilde{\mathbf{x}}^T \tilde{\mathbf{x}} = \mathbf{0}. \tag{5.53}$$

This is only true for $\tilde{\mathbf{x}} = 0$, which is a contradiction with our assumption that $\tilde{\mathbf{x}}$ is nonzero. The negation of the implication is false and so the proof is done. □

However, as is the case in integer-order systems, there exists an easier method of determining controllability (see [7]).

Theorem 5.12. *The system* (5.1) *is completely controllable if and only if its controllability matrix*

$$\boldsymbol{M}_c = [\boldsymbol{B} \mid \boldsymbol{AB} \mid \cdots \mid \boldsymbol{A}^{n-1}\boldsymbol{B}] \tag{5.54}$$

is of full rank.

Proof. By (5.4), we can express the state vector $\mathbf{x}(t)$ as

$$\mathbf{x}(t) = E_\alpha(\mathbf{A}t^\alpha)\mathbf{x}_0 + \int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha}(\mathbf{A}(t - \tau)^\alpha)\mathbf{B}\mathbf{u}(\tau) d\tau. \tag{5.55}$$

Thanks to the corollary (2.6) of the Cayley-Hamilton theorem, we know that

$$t^{\alpha-1} E_{\alpha,\alpha}(\mathbf{A}t^\alpha) = \sum_{k=0}^{\infty} \frac{t^{k\alpha+\alpha-1}}{\Gamma(k\alpha + \alpha)} \mathbf{A}^k = \sum_{k=0}^{n-1} c_k(t)\mathbf{A}^k. \tag{5.56}$$

If we substite this into the expression for $\mathbf{x}(t)$ (5.55), we obtain

$$\mathbf{x}(t) - E_\alpha(\mathbf{A}t^\alpha)\mathbf{x}_0 = \sum_{k=0}^{n-1} \mathbf{A}^k\mathbf{B} \int_0^t c_k(t - \tau)u(\tau) d\tau. \tag{5.57}$$

This can be interpreted as meaning that the state vector $\mathbf{x}(t)$ is a linear combination of the vectors $\mathbf{A}^k\mathbf{B}$. We can rewrite the sum as the multiplication

$$\left[\mathbf{B} \mid \mathbf{AB} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}\right] \begin{bmatrix} d_0(t) \\ d_1(t) \\ \vdots \\ d_{n-1}(t) \end{bmatrix}, \tag{5.58}$$

where $d_k(t) = \int_0^t c_k(t-\tau)u(\tau)d\tau$. If the rank of the matrix $\left[\mathbf{B} \mid \mathbf{AB} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}\right]$ is less than n, then the column space of the matrix - which is the same as all linear combinations of $\mathbf{A}^k\mathbf{B}$ - does not include all possible states $\mathbf{x}(t)$ and the system is uncontrollable. If the rank is equal to n, the column space is $\mathbb{R}^{n\times 1}$ and the system is controllable.

□

5.3 Observability

In the case of the Caputo system (5.1), we can use the definition of observability mentioned in the third chapter. The necessary condition for observability is very similar to that of controllability (see [6]).

Theorem 5.13. *The system* (5.1) *is observable on* (t_0, t_f) *if and only if the* $n \times n$ *observability gramian matrix given by*

$$\boldsymbol{W}_o = \int_{t_0}^{t_f} E_\alpha(\boldsymbol{A}^T(\tau - t_0)^\alpha)\boldsymbol{C}^T\boldsymbol{C}E_\alpha(\boldsymbol{A}(\tau - t_0)^\alpha)d\tau \tag{5.59}$$

is invertible.

Proof. Since observability does not in any way depend on the input $\mathbf{u}(t)$, we can without loss of generality set $\mathbf{u}(t) = \mathbf{0}$. Then by (5.4), we can express the state vector at time t as

$$\mathbf{x}(t) = E_\alpha(\mathbf{A}(t - t_0)^\alpha)\mathbf{x}_0. \tag{5.60}$$

From the output equation, we then obtain

$$\mathbf{y}(t) = \mathbf{C}E_\alpha(\mathbf{A}(t - t_0)^\alpha)\mathbf{x}_0. \tag{5.61}$$

Now if we left-multiply (5.61) by $E_\alpha(\mathbf{A}^T(t - t_0)^\alpha)\mathbf{C}^T$ and integrate from t_0 to t_f, we get

$$\int_{t_0}^{t_f} E_\alpha(\mathbf{A}^T(\tau - t_0)^\alpha)\mathbf{C}^T\mathbf{y}(\tau)d\tau = \int_{t_0}^{t_f} E_\alpha(\mathbf{A}^T(\tau - t_0)^\alpha)\mathbf{C}^T\mathbf{C}E_\alpha(\mathbf{A}(\tau - t_0)^\alpha)\mathbf{x}_0 d\tau = \mathbf{W}_o\mathbf{x}_0. \tag{5.62}$$

$\mathbf{W}_o$ is invertible and $\mathbf{x}_0$ is uniquely determined to be

$$\mathbf{x}_0 = \mathbf{W}_o^{-1}\int_{t_0}^{t_f} E_\alpha(\mathbf{A}^T(\tau - t_0)^\alpha)\mathbf{C}^T\mathbf{y}(\tau)d\tau \tag{5.63}$$

If on the other hand the gramian matrix is not invertible, then there exists a nonzero $\tilde{\mathbf{x}}$ such that $\mathbf{W}_o\tilde{\mathbf{x}} = \mathbf{0}$. However, if we then choose $\hat{\mathbf{x}} = \mathbf{x}_0 + \tilde{\mathbf{x}}$, we can see that it also satisfies the equation

$$\int_{t_0}^{t_f} E_\alpha(\mathbf{A}^T(\tau - t_0)^\alpha)\mathbf{C}^T\mathbf{y}(\tau)d\tau = \mathbf{W}_o\hat{\mathbf{x}} = \mathbf{W}_o\left[\mathbf{x}_0 + \tilde{\mathbf{x}}\right] = \mathbf{W}_o\mathbf{x}_0. \tag{5.64}$$

Two different vectors satisfy the equation (5.64) and thus the initial state cannot be uniquely retrieved from the output. □

Nonetheless, there again exists a simpler condition (see [7]).

Theorem 5.14. *The system* (5.1) *is completely observable if and only if its observability matrix*

$$\boldsymbol{M}_o = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \\ \vdots \\ \boldsymbol{CA}^{n-1} \end{bmatrix} \tag{5.65}$$

is of full rank.

Proof. Without loss of generality, we set $\mathbf{u}(t) = \mathbf{0}$. By (5.4), we can write

$$\mathbf{y}(t) = \mathbf{C}E_\alpha(\mathbf{A}(t - t_0)^\alpha)\mathbf{x}_0. \tag{5.66}$$

We use the corollary (2.6) of the Cayley-Hamilton theorem and rewrite this in the form

$$\mathbf{y}(t) = \mathbf{C}\sum_{k=0}^{\infty} \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)}\mathbf{A}^k\mathbf{x}_0 = \mathbf{C}\sum_{k=0}^{n-1} \mathbf{A}^k d_k(t)\mathbf{x}_0. \tag{5.67}$$

This is the same as

$$\mathbf{y}(t) = [d_0(t), \ldots, d_{n-1}(t)] \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix} \mathbf{x}_0. \tag{5.68}$$

If the observability matrix is of full rank, it is invertible and we can express $\mathbf{x}_0$ as a function of $\mathbf{y}(t)$. The system is observable. If the condition is not met, then the system is not observable. □

We can see that the Caputo system (5.1)'s conditions for controllability are identical to those of the integer-order system (3.2).

6 Conclusions

In the first part of the thesis, we defined the problems of determining stability, controllability, and observability of integer-order linear time-invariant systems. We then stated the theorems that assert the conditions that need to be met for a system to be stable, controllable, or observable, and listed a couple of examples.
In the second part, we proceeded to define the Riemann-Liouville and Caputo differintegrals and examined some of their properties and differences. After that, we defined the corresponding fractional-order linear time-invariant systems of order $0 < \alpha \leq 1$ and analysed their stability, controllability, and observability. Here, most of the theorems are accompanied with detailed proofs.
The thesis should mainly serve as an introduction to fractional-order control and a summary of the most important results. It could be followed upon by looking at different kinds of systems, e.g. linear systems with time delay or nonlinear systems. Another interesting direction of development could be trying to find reasonable alternative definitions of controllability and observability for the Riemann-Liouville system.

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