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FAKULTA STROJNÍHO INŽENÝRSTVÍ
ÚSTAV MATEMATIKY

FACULTY OF MECHANICAL ENGINEERING
INSTITUTE OF MATHEMATICS

STOCHASTIC ORDINARY DIFFERENTIAL EQUATIONS

STOCHASTICKÉ OBYČEJNÉ DIFERENCIÁLNI ROVNICE

DIPLOMOVÁ PRÁCE
MASTER'S THESIS

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ZADÁNÍ DIPLOMOVÉ PRÁCE

student(ka): Bc. Michal Bahník

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Ředitel ústavu Vám v souladu se zákonem č.111/1998 o vysokých školách a se Studijním a zkušebním řádem VUT v Brně určuje následující téma diplomové práce:

Stochastické obyčejné diferenciálni rovnice

v anglickém jazyce:

Stochastic ordinary differential equations

Stručná charakteristika problematiky úkolu:

Stochastické diferenciální rovnice popisují spojité jevy s náhodnými vlivy, koeficienty v rovnicích a následně i řešení jsou náhodné procesy. Náhodnost koeficientů je modelována přidáním násobku tzv. Brownova procesu.

Cíle diplomové práce:

Zavedení základních pojmů, přehled teorie stochastického integrálu a diferenciálu pro technické aplikace, formulace počátečních úloh, řešení lineárních rovnic a výpočet řešení konkrétních úloh.

Seznam odborné literatury:

L.C.Evans: An introduction to stochastic differential equations, AMS, 2013.
B.Oksendal: Stochastic differential equations, An introduction with applications, Springer, Berlin 2000.

Vedoucí diplomové práce:prof. RNDr. Jan Franců, CSc.

Termín odevzdání diplomové práce je stanoven časovým plánem akademického roku 2014/15.

V Brně, dne 20.11.2014

VYSOKÉ UČENÍ TECHNICKÉ V BRNĚ FAKULTA STROJNÍHO INŽENÝRSTVÍ
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11

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Děkan

Abstrakt

Diplomová práce se zabývá problematikou obyčejných stochastických diferenciálních rovnic. Po souhrnu teorie stochastických procesů, zejména tzv. Brownova pohybu je zaveden stochastický Itôův integrál, diferenciál a tzv. Itôova formule. Poté je definováno řešení počáteční úlohy stochastické diferenciální rovnice a uvedena věta o existenci a jednoznačnosti řešení. Pro případ lineární rovnice je odvozen tvar řešení a rovnice pro jeho střední hodnotu a rozptyl. Závěr tvoří rozbor vybraných rovnic.

Abstract

This thesis deals with the issue of stochastic ordinary differential equations. After the summary of the theory of stochastic processes, namely the Brownian motion, the stochastic Itô's integral, differential and so called Itô's formula are introduced. Thereafter the solution of the initial value problem for the stochastic equation is defined and the theorem of its existence and uniqueness is stated. For the case of the linear equation the explicit formula for the solution is derived as well as the equations for its expected value and variance. The last part is the analysis of selected equations.

Klíčová slova

Brownův pohyb, Itôova formule, Itôův integrál, Stochastické obyčejné diferenciální rovnice

Key-words

Brownian motion, Itô's formula, Itô's integral, Stochastic ordinary differential equations

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I declare that I wrote the master's thesis *Stochastic ordinary differential equations* independently under the leadership of prof. RNDr. Jan Franců, CSc. using the materials specified in the list of references.

Michal Bahník

I would like to thank my supervisor prof. RNDr. Jan Franců, CSc. for the numerous hints and comments in the conduct of my Master's thesis.

Michal Bahník

Contents

1 Introduction

Let us consider an initial value problem for ordinary differential equation

$$\begin{cases} \mathrm{d}x(t) = f(x(t), t)\ \mathrm{d}t, \\ x(0) = x_0\,. \end{cases}$$

the solution to this problem is a smooth function whose trajectory is similar to the one in the following figure

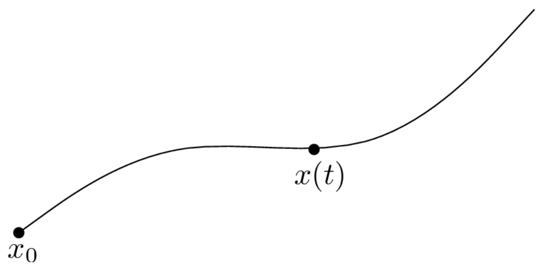

Figure 1.1: Solution trajectory for ODE

However, many times we have to model a phenomenont that is influenced by random "noise" and that is why it does not behave in the way that is predictable using the ordinary differential equations:

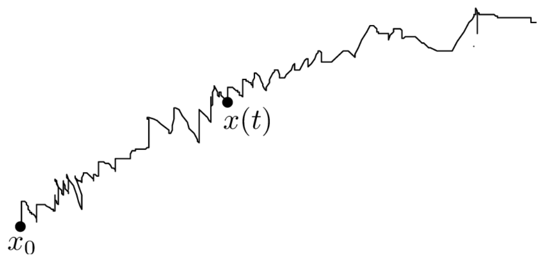

Figure 1.2: Sample path of the SDE

Therefore we need to add a "noise" term to the equation, that will model the random nature of such phenomenon.

$$\begin{cases} \mathrm{d}x(t) = f(x(t), t)\ \mathrm{d}t + \text{"}noise\text{"}, \\ x(0) = x_0\,. \end{cases}$$

The question that rises up now is how can we describe the "noise". The standard way to do it is to use a Brownian motion.

$$\begin{cases} \mathrm{d}x(t) = f(x(t), t)\ \mathrm{d}t + g(x(t), t)\ \mathrm{d}B_t, \\ x(0) = x_0, \end{cases}$$

where

(1) $x(t)$ is an unknown function,

(2) B_t is a Brownian motion.

In this thesis we will deal with the issue of Ordinary stochastic differential equations. They are applied in many brands of science such as physics, mathematical science, optimal control, etc. Before we can do it, we have to summarize the probability theory and then the theory of stochastic integral and differential. It will turn out that the stochastic integral can be defined in more than one way. The two basic definitions were given by Kiyoshi Itô and Ruslan Stratonovich. To make the text more simple and accessible, we will restrict ourselves to the case of Itô's integral. Another reason of doing it is that the Itô's definition is more suitable for treating the initial value problems. Moreover there exists a powerful tool that can be used in this case. After that we will state the general properties of the stochastic differential equations, derive the formula for the solution in the case of linear equations and finally we will solve the specific linear equations with and we will do the analysis of their solutions.

The basic sources for this thesis were [1] and [2] where authors present the stochastic differential equations in a very understandable way. Even simpler is the book [4] that was helpful to understood some more complex concepts since it gives simple examples of them. The preliminary part about probability theory and stochastic processes is based on [3], but this theory is included more or less in the first chapters of any book on stochastic differential equations. The additional sources were [5] and [6] as they both were used occasionally in order to confirm some ideas.

The thesis itself is divided as follows. In the sections 2 and 3 we summarize the basic probability theory and theory of stochastic processes that the whole thesis is based on. The most important here is the introduction of the *Brownian motion* with its properties that we are using throughout the whole thesis. In the section 4 we introduce the important concept of Itô's integral and the most important tool in stochastic analysis, the Itô's formula. The fifth section contains the basic facts about stochastic differential equations and it states also the existence and uniqueness theorem for stochastic equations. The sixth section is the most important section of this thesis where we occupy ourselves with the linear equations, we derive the explicit formula for the solution of general linear equation and also the ordinary differential equations for the expected value and variance of the solution. In the section 7 we analyse in details the solutions to specific linear equations, namely the *Geometric Brownian motion*, the *Brownian bridge*, oscillating process and the *Langevin equation*.

2 Basic probability theory

Before we can actually start dealing with stochastic differential equations, it is necessary to introduce the theory that we will use throughout the whole text. This section is based on [3], but the concepts are very basic, therefore they can be found in any textbook about probability.

2.1 Probability and probability space

In this subsection we will recall the important basic terms.

Definition 2.1 (The σ-algebra). Let Ω be a set (in our case the set of elementary events), the family $\mathscr{F}$ of subsets of the set Ω is called a *σ - algebra* if

(1) $\emptyset \in \mathscr{F}$,

(2) $A \in \mathscr{F} \Rightarrow A^C \in \mathcal{F} \qquad A^C = \Omega \setminus A$,

(3) $\{A_i\}_{i\geq 0} \subset \mathscr{F} \Rightarrow \bigcup\limits_{i=0}^{\infty} A_i \in \mathscr{F}$.

The couple $(\Omega, \mathscr{F})$ is called a *measurable space* and the elements of $\mathscr{F}$ are called events. Taking $\mathcal{C} \subset \Omega = \mathbb{R}$ and all the possible σ -algebras containing only open sets on Ω that contain also $\mathcal{C}$, there always exists the smallest one of them. We denote it by $\sigma(\mathcal{C})$ and call it the *Borel σ - algebra* and its elements are called *Borel sets*.

Now we will introduce a measure on $(\Omega, \mathscr{F})$ called *probability* or *probability measure*.

Definition 2.2 (The probability measure). A probability measure P on a measurable space $(\Omega, \mathscr{F})$ is a function $P : \mathscr{F} \to [0, 1]$ such that

(1) $P(\Omega) = 1$,

(2) $P\left(\bigcup\limits_{i=1}^{\infty} A_i\right) = \sum\limits_{i=1}^{\infty} P(A_i)$.

The triple $(\Omega, \mathscr{F}, P)$ is called a *probability space*. The following theorem shows some of the basic properties of the probability measure.

Theorem 2.1 (Properties of probability). *Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $A, B \in \Omega$. Then*

(1) $P(A^C) = 1 - P(A)$

(2) if $A \subseteq B$ *then* $P(B) = P(A) + P(B \setminus A) \geq P(A)$

(3) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(4) more generally, if A_i $(i = 1, \ldots, n)$ are events, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} P\left(\bigcap_{i=1}^{n} A_i\right)$$

The following definition shows the concept of conditional probability.

Definition 2.3 (**Conditional probability**)**.** Let $A, B \in \Omega$, $P(B) > 0$ then the *conditional probability* of A under condition B is given by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

2.2 Random variables

Let us Imagine that we are about to make a random experiment and we can evaluate its result with a number. Unfortunately, we cannot know the result in advance. Therefore the variable that assigns a certain value to the result of a random experiment is called a *random variable*.

Definition 2.4 (**Random variable**)**.** A real-valued function $X : \Omega \to \mathbb{R}$ is said to be $\mathscr{F}$-measurable if

$$\{\omega : X(\omega) \leq a\} \in \mathscr{F} \quad \text{for all } a \in \mathbb{R}.$$

The function X is called a (real-valued) *random variable*.

Any random variable is given by its *distribution function* which describes the likelihood that the random variable takes a certain value. There are two basic classes of random variables, *discrete* and *continuous*. The discrete random variables are far less important for our purposes so we will focus on the continuous ones. The definitions for the discrete case can be however found e.g. in [3].

Definition 2.5 (**Distribution function**)**.** Let X be a random variable. The *distribution function* of X is a function $F : \mathbb{R} \to [0, 1]$, such that

$$F(x) = P(X \leq x).$$

As is stated above, in this text we will use only the continuous random variables, therefore we will now define them.

Definition 2.6 (**Continuous random variable**)**.** A random variable X is said to be *continuous* if its distribution function F is continuous and if it can be expressed as

$$F(x) = \int_{-\infty}^{x} f(t)\ \mathrm{d}t$$

for some function $f : \mathbb{R} \to [0, \infty)$. The function f is called the *probability density* of the random variable X.

The important property that is studied in random variables is whether or not they are independent of each other.

Definition 2.7 (**Independence of random variables**)**.** The random variables $X_1, \ldots, X_n$ are independent if

$$F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \prod_{i=1}^{n} F_{X_i}(x_i),$$

where $F_{X_1,\dots,X_n}(x_1,\dots,x_n)$ is the cumulative distribution function of $X_1,\dots,X_n$ given by

$$F_{X_1,\dots,X_n}(x_1,\dots,x_n) = \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_n} f_{X_1,\dots,X_n}(s_1,\dots,s_n)\ \mathrm{d}s_1 \dots \mathrm{d}s_n.$$

The function $f_{X_1,\dots,X_n}(x_1,\dots,x_n)$ is called the cumulative density of $X_1,\dots,X_n$.

As a consequence of previous definition we have that

$$f_{X_1,\dots,X_n}(x_1,\dots,x_n) = \prod_{i=1}^{n} f_{X_i}(x_i),$$

The most important distribution is the *Normal distribution*. We will need this distribution to define the Brownian motion.

Definition 2.8 (Normal distribution). We say that random variable X has the *normal distribution* with parameters μ and σ^2 and we write $X \sim N(\mu, \sigma^2)$, if its probability density function is defined by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

The parameter μ is the *expected value* of X and σ^2 is its *variance*. The normal distribution is sometimes called *Gaussian*.

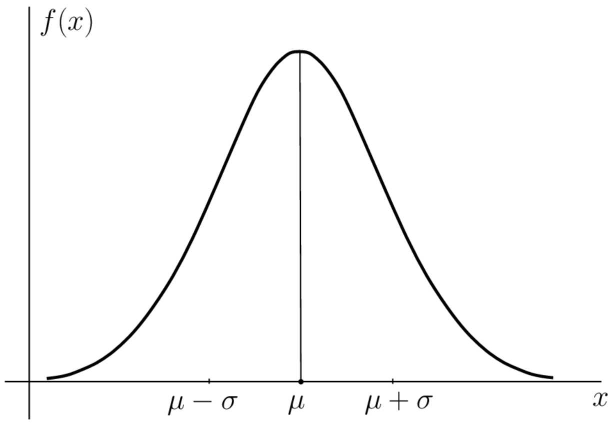

Figure 2.1: The density of the normal distribution

In the definition of the normal distribution we used the terms expected value and variance, so we should clarify their meaning.

Definition 2.9 (Expected value, variance and standard deviation). Let $(\Omega, \mathscr{F}, P)$ be a probability space and let X, a random variable, be an *integrable* function with respect to the measure P. Then if $\Omega = \mathbb{R}$, the number

$$\mathrm{E}X = \int_{\Omega} X(\omega)\ \mathrm{d}P(\omega) = \int_{-\infty}^{\infty} xf(x)\ \mathrm{d}x$$

is called the *expected value* or shortly the *expectation*, the number

$$\mathrm{V}X = \mathrm{E}(X - \mathrm{E}X)^2 = \mathrm{E}X^2 - (\mathrm{E}X)^2$$

is called the *variance* of random variable X and

$$\sigma_D = \sqrt{\mathrm{V}X},$$

is called its *standard deviation.*

More generally, if g is a real-valued measurable function of the random variable X, the $\mathrm{E}g(X)$ is given by the formula (2.1).

$$\mathrm{E}\left(g(X)\right) = \int\limits_{-\infty}^{\infty} g(x)f(x)\mathrm{d}x, \tag{2.1}$$

so there is no need to anyhow transform the probability density of X in order to compute the expectation of a function of X. The usefulness of (2.1) can be demonstrated on computing the variance of X, since

$$\mathrm{E}X^2 = \int\limits_{-\infty}^{\infty} x^2 f(x)\mathrm{d}x.$$

For $p \in (0, \infty)$ we denote L^p the family of random variables X such that $\mathrm{E}|X|^p < \infty$. The theorem 2.2 summarizes some properties of expected value and variance

Theorem 2.2. *Let $X, X_1, \dots\ X_n$ be random variables, then*

(1) $\mathrm{E}(aX + b) = a\mathrm{E}X + b$ *for all* $a, b \in \mathbb{R}$,

(2) $\mathrm{E}\left(\sum\limits_{i=1}^{n} X_i\right) = \sum\limits_{i=1}^{n} \mathrm{E}X_i$,

(3) $\mathrm{E}\left(\prod\limits_{i=1}^{n} X_i\right) = \prod\limits_{i=1}^{n} \mathrm{E}X_i$ *if X_i are independent,*

(4) $\mathrm{V}a = 0$ *for any* $a \in \mathbb{R}$,

(5) $\mathrm{V}(aX + b) = a^2\ \mathrm{V}X$ *for all* $a, b \in \mathbb{R}$,

(6) $\mathrm{V}\left(\sum\limits_{i=1}^{n} X_i\right) = \sum\limits_{i=1}^{n} \mathrm{V}X_i$ *if X_i are independent,*

(7) $\mathrm{V}X \geq 0$.

Moreover, the following lemma is very useful.

Lemma 2.1 (Hölder's inequality). *Let X, Y be two random variables such that $X \in \mathrm{L}^p$ and $Y \in \mathrm{L}^q$. if $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ Then*

$$|\mathrm{E}(XY)| \leq \left(\mathrm{E}|X|^p\right)^{\frac{1}{p}} \cdot \left(\mathrm{E}|Y|^q\right)^{\frac{1}{q}}.$$

2.3 Conditional expectation

Despite we have already defined the expectation and also the conditional probability, sometimes we encounter not just one, but a family of conditions. This is the reason why we need a more general concept of *conditional expectation.*

Let X be a random variable on a probability space $(\Omega, \mathscr{F}, P)$ such that $EX < \infty$. Let $\mathscr{G} \subset \mathscr{F}$ be a sub σ-algebra of $\mathscr{F}$ so that $(\Omega, \mathscr{G})$ is a measurable space. In general, X is not a random variable on $(\Omega, \mathscr{G}, P)$, i.e.

$$\{\omega : X(\omega) \leq a\} \notin \mathscr{G} \quad \text{for all } a \in \mathbb{R}.$$

What we are looking for is such Y that is a random variable on $(\Omega, \mathscr{G})$, that takes the same values as X in the sense that

$$\int\limits_G Y(\omega)\ \mathrm{d}P(\omega) = \int\limits_G X(\omega)\ \mathrm{d}P(\omega) \quad \text{for all } G \in \mathscr{G}.$$

It has been proven that such Y exists almost surely unique. We call it the *conditional expectation* of X under the condition $\mathscr{G}$ and we write

$$Y = \mathrm{E}\left(X|\mathscr{G}\right).$$

If $\mathscr{G}$ is a σ-algebra generated by Y we also write

$$\mathrm{E}\left(X|\mathscr{G}\right) = \mathrm{E}\left(X|Y\right).$$

. The following theorem states some properties of the conditional expectation.

Theorem 2.3. *The conditional expectation has these properties*

(1) $\mathrm{E}(\mathrm{E}(X|\mathscr{G})) = \mathrm{E}X$*,*

(2) $\mathscr{G} = \{\emptyset, \Omega\} \Rightarrow \mathrm{E}(X|\mathscr{G}) = \mathrm{E}X$*,*

(3) $X \geq 0 \Rightarrow \mathrm{E}(X|\mathscr{G}) \geq 0$*,*

(4) X *is* $\mathscr{G}$*-measurable* $\Rightarrow \mathrm{E}(X|\mathscr{G}) = X$*,*

(5) $a, b \in \mathbb{R} \Rightarrow \mathrm{E}(aX + bY|\mathscr{G}) = a\ \mathrm{E}(X|\mathscr{G}) + b\ \mathrm{E}(Y|\mathscr{G})$*,*

(6) $X \leq Y \Rightarrow \mathrm{E}(X|\mathscr{G}) \leq \mathrm{E}(Y|\mathscr{G})$*,*

(7) if $\sigma(X), \mathscr{G}$ *are independent* $\Rightarrow \mathrm{E}(X|\mathscr{G}) = \mathrm{E}X$*,*
particularly, if X, Y *are independent* $\Rightarrow \mathrm{E}(X|Y) = \mathrm{E}X$*.*

All of these properties hold almost surely.

3 Stochastic processses

In the previous section we defined what a random variable is. But sometimes we are interested in modelling something that cannot be interpreted as a single random variable, but rather as a sequence of random variables in time. We call this sequence a stochastic process. In general there are two basic classes of processes with respect to images, discrete-time and continuous-time and we can find everything about both of them in [3]. We shall deal with the processes with continuous time and continuous image.

The example in the subsection 3.1 shows such a process in an intuitive way. Despite the fact that we will focus on the continuous stochastic processes, for the sake of giving a simple example, we will show a discrete one. In the subsections after 3.1 we however introduce the stochastic processes in the precise mathematical sense.

3.1 Simple symmetric random walk

Take a sequence $\{X_n\}_{n\geq 1}$ of independent and identically distributed (i.i.d.) random variables, that can only take values from $\{-1, 1\}$ and such that

$$P(X_n = 1) = P(X_n = -1) = \frac{1}{2},$$

so they can be interpreted as flips of a coin. Let us say, that we are standing at position $S_0 = 0$ at time $n = 0$ and let

$$S_n = \sum_{i=1}^{n} X_i \tag{3.1}$$

be our position at time n, assuming that we go forth if $X_n = 1$, which corresponds to the tails of the coin, and we go back if $X_n = -1$ corresponding to the heads of the coin. Let us fix ω and let

$$T\ T\ H\ T\ H\ T\ T\ H\ T\ T\ H$$

be a random sequence of heads and tails for our ω. So using (3.1) we can compute $S_{11} = 3$. The figure 3.1 shows a sample path of this process

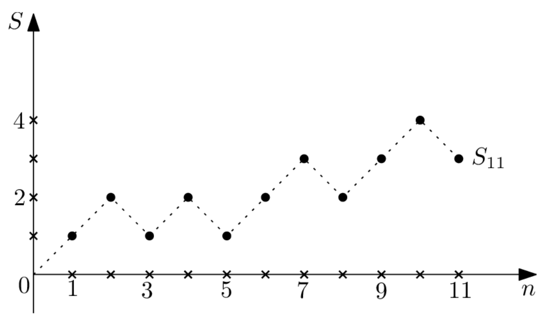

Figure 3.1: Simple symmetric random walk

3.2 Basic facts about stochastic processes

Definition 3.1 (Stochastic process). A stochastic process is a family of random variables

$$\{X_t(\omega)\}_{t \in T},$$

defined on $(\Omega, \mathcal{F}, P)$. The t is a parameter belonging to a parameter space T.

(1) For fixed $t = t_0$ we get $X_{t_0}(\omega) : \Omega \to \mathbb{R}$ a single random variable.

(2) For fixed $\omega = \omega_0$ we get a random function $X_t(\omega_0) : T \to \mathbb{R}$ which is called a *trajectory* or *sample path* of $X_t(\omega)$.

In the following text we will frequently use the identification $X_t = X_t(\omega)$ in order to make the text more simple. But we shall not forget, that $X_t(\omega) = X(t, \omega)$ is not just a function of t but a function of ω as well.

The parameter space can be discrete (we talk about discrete time stochastic process) or continuous. Since the main aim of this text is the studying of stochastic differential equations, we will omit the discrete case and focus more on the continuous one. For our needs through this text we will assume that the parameter space is either the interval the whole $\mathbb{R}_0^+$ or we will consider even a bounded interval $[0, t]$, $0 < t < \infty$.

3.3 Brownian motion

Browninan motion is the name that was originally given to the movement of grains suspended in water by the Scottish botanist Robert Brown. In mathematics, this process is often called the *Wiener process*, but we will remain with the term Brownian motion. For the mathematical description we will consider that it is a stochastic process $B_t(\omega)$, which we interpret as the position of the grain ω at given time t.

Let us show now the standard definition of Brownian motion.

Definition 3.2 (One-dimensional Brownian motion). A one-dimensional Brownian motion on a probability space $(\Omega, \mathscr{F}, P)$ is a real-valued continuous process with following properties

(1) $B_0 = 0$ almost surely;

(2) Increments $B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_k} - B_{t_{k-1}}$ are independent for all $0 \leq t_1 < t_2 < \ldots < t_k < \infty$,

(3) for $0 \leq s < t < \infty$ the increment $B_t - B_s$ has the Normal distribution with mean equal to zero and variance $t - s$.

Sometimes we have to talk about the Brownian motion $\{B\}_{0 \leq t < T}$ on some interval $[0, T]$, where $T > 0$. The definition for such process remains the same, but in 2. and 3. we have to replace the ∞ with T.

The previous definition is sometimes written in different way. Before we show the equivalent definition, we have to clarify what is the *past* and *future* of the Brownian motion, because we need this notions to define the term *Filtration* that is used in the other definition.

Definition 3.3 (**History and future of the Brownian motion**)**.** Let $\{B_t\}_{t\geq 0}$ be a one-dimensional Brownian motion defined on $(\Omega, \mathscr{F}, P)$.

(a) The σ-algebra $\mathscr{B}^-(t) := \mathscr{F}(B(s)|0 \leq s \leq t)$ is called the *history* of $\{B_t\}_{t\geq 0}$ up to the time t.

(b) The σ-algebra $\mathscr{B}^+(t) := \mathscr{F}(B(s) - B(t)|s \geq t)$ is called the *future* of $\{B_t\}_{t\geq 0}$ beyond the time t.

Definition 3.4 (**Filtration**)**.** A family $\mathcal{F}(t)$ of σ-algebras $\subseteq \mathscr{F}$ is called a *filtration* if

(1) $\mathcal{F}(t) \supseteq \mathcal{F}(s),\ \forall t \geq s \geq 0,$

(2) $\mathcal{F}(t) \supseteq \mathscr{B}^-(t),\ \forall t \geq 0,$

(3) $\mathcal{F}(t)$ is independent of $\mathscr{B}^+(t),\ \forall t \geq 0.$

A process $\{X_t\}_{t\geq 0}$ is said to be *adapted*, if for all t, X_t is $\mathcal{F}_t$ measurable.

Definition 3.5 (**One-dimensional Brownian motion 2**)**.** Let $(\Omega, \mathscr{F}, P)$ be a probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. A one-dimensional Brownian motion is a real-valued continuous process with following properties

(1) $B_0 = 0$ almost surely;

(2) for $0 \leq s < t < \infty$ the increment $B_t - B_s$ is independent of $\mathcal{F}_s$,

(3) for $0 \leq s < t < \infty$ the increment $B_t - B_s$ has the Normal distribution with mean equal to zero and variance $t - s$.

In what follows we will assume that we work on $(\Omega, \mathscr{F}, P)$ a complete probability space with the Brownian motion B_t defined on it. Now we will present some properties of Brownian motion.

(1) The trajectory of Brownian motion has almost surely infinite variation on every interval $[a, b]$, i.e.

$$\sup \sum_{i=1}^{k} |B_{t_i}(\omega) - B_{t_{i-1}}(\omega)| = \infty. \tag{3.2}$$

The supremum in (3.2) is taken over every partition $a = t_1 \leq t_2 \leq \cdots \leq t_k = b$ of the interval $[a, b]$.

(2) It has however a finite quadratic variation on every $[a, b]$.

$$\langle B, B\rangle_t = \sup \sum_{i=1}^{k} |B_{t_i}(\omega) - B_{t_{i-1}}(\omega)|^2 = b - a, \tag{3.3}$$

In particular, if $[a, b] = [0, t]$

$$\langle B, B\rangle_t = t.$$

The supremum in (3.3) is again taken over every partition $a = t_1 \leq t_2 \leq \cdots \leq t_k = b$ of the interval $[a, b]$.

(3) For almost every ω the sample path $B_t(\omega)$ is nowhere differentiable.

(4) $\{-B_t\}$ is a Brownian motion with respect to the same filtration $\mathcal{F}_t$.

(5) B_t is a continuous square-integrable Martingale

The stochastic process X_t is said to be square-integrable if $\mathrm{E}X_t^2 < \infty$ for all $t \geq 0$. We should also give the definition of the Martingale.

Definition 3.6 (Martingale). A real valued $\{\mathcal{F}_t\}$-adapted process M_t is called a martingale with respect to the filtration $\{\mathcal{F}_t\}$ or simply a *Martingale* if

$$\mathrm{E}(M_t|\mathcal{F}_s) = M_s \quad \text{almost surely for all} \quad 0 \leq s < t < \infty.$$

Before we state the next property, we need the theorem.

Theorem 3.1 (Strong law of large numbers). *Let $M_t = M$ be a real valued martingale vanishing at $t = 0$ and let $\langle M, M\rangle_t$ be its quadratic variation. Then*

$$\lim_{t\to\infty} \langle M, M\rangle_t = \infty \quad a.s. \quad \Rightarrow \quad \lim_{t\to\infty} \frac{M_t}{\langle M, M\rangle_t} = 0 \quad a.s.$$

and also

$$\limsup_{t\to\infty} \frac{\langle M, M\rangle_t}{t} < \infty \quad a.s. \quad \Rightarrow \quad \lim_{t\to\infty} \frac{M_t}{t} = 0 \quad a.s.$$

(6) According to the theorem 3.1,

$$\lim_{t\to\infty} \frac{B_t}{t} = 0 \quad a.s.$$

(7) The law of iterated logarithm holds for B_t

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t\ln(\ln t)}} = 1 \quad a.s. \qquad \liminf_{t\to\infty} \frac{B_t}{\sqrt{2t\ln(\ln t)}} = -1 \quad a.s.$$
$$\limsup_{t\to 0} \frac{B_t}{\sqrt{2t\ln(\ln \frac{1}{t})}} = 1 \quad a.s. \qquad \liminf_{t\to 0} \frac{B_t}{\sqrt{2t\ln(\ln \frac{1}{t})}} = -1 \quad a.s.$$

Our list of properties is a summary of most important ones of them from [1], [2], [3], [4],[8] and [7].

The existence of such a process is assured by the next theorem

Theorem 3.2. *Let $(\Omega, \mathscr{F}, P)$ be a probability space on which countably many random variables $\{X_n\}_{n=1}^{\infty} \sim N(0,1)$ are defined. Then there exists a one-dimensional Brownian motion $B(t,\omega)$ defined for all $\omega \in \Omega, t \geq 0$.*

The proof of theorem 3.2 is based on the construction of mentioned process. It can be found in [4].

4 Stochastic integrals and Itô's formula

If we look at an integral form of a stochastic differential equation

$$X_t = X_0 + \int_0^t f(X_s, s)\ \mathrm{d}s + \int_0^t g(X_s, s)\ \mathrm{d}B_s,$$

we know exactly the meaning of the first integral

$$\int_0^t f(X_s, s)\ \mathrm{d}s,$$

because it is a classical Lebesgue integral the definition of which can be found in any book on functional analysis (e.g in [9]). The second integral

$$\int_0^t g\left(X_s, s\right)\ \mathrm{d}B_s, \tag{4.1}$$

on the other hand, cannot be interpreted in that way. We could think of defining it in the Stieltjes sense as the integral of function f with respect to function g

$$\int_a^b f(t)\ \mathrm{d}g(t) = \lim_{|\pi|\to 0} \sum_{i=1}^{n} f(\tau_i)\left(g(t_{i+1}) - g(t_i)\right),$$

where τ_i is an arbitrary point of the interval $[t_i, t_{i+1}]$ and π is the partition of $[a, b]$. The existence of the limit requires g to have a finite variation. In our case, when we would like to define an integral with respect to a Brownian motion B_t, this property is not satisfied, as has been already said. Therefore we cannot define it like that. In this section we will show the precise procedure of defining the stochastic integral.

It turns out that here, unlike the case of Stieltjes integral, it matters which point τ_i we choose. Essentially there are two approaches. The first one is called the *Itô's integral* and it is the first definition of stochastic integral ever. The choice here is the left side point of the interval $[t_i, t_{i+1}]$. It is usually denoted normally as (4.1).

Stratonovich gave an alternative definition and his integral is denoted adding $\circ$ to (4.1).

$$\int_0^t g\left(X_s, s\right) \circ \mathrm{d}B_s,$$

for the sake of making a clear distinction between the two cases. This time we let $\tau_{\frac{i}{2}}$ be the middle point of $[t_i, t_{i+1}]$. However, in this text we will reduce ourselves to the Itô's case, because it is more suitable for the initial value problems for stochastic differential equations. We will show not only the definition, but also an important rule of computing it which is a key tool in stochastic analysis, so-called *Itô's formula*.

4.1 Itô's integral

In this section we will define the Itô type stochastic integral

$$\int_0^t f(X_s, s)\,\mathrm{d}B_s, \tag{4.2}$$

with respect to a Brownian motion $\{B_t\}_{t\geq 0}$. The construction will be made according to [1] As we already said, the Brownian motion is nowhere differentiable, therefore (4.2) cannot be defined in the usual way and there is a lot of stochastic processes that are not integrable even in this way. But still we can define a stochastic integral for a large class of processes.

We will define (4.2) step by step. First we will define it for the simplest class of processes, so-called *simple processes* and then we will extend this definition to a larger class of processes.

Definition 4.1 (**Simple process**). A real valued stochastic process $g = \{g(t)\}_{a\leq t\leq b}$ is said to be a *simple process* if there exists a partition $a = t_0 < t_1 < \cdots < t_k = b$ of [a,b], and bounded random variables ξ_i, $0 \leq i \leq k-1$ such that ξ_i is $\mathcal{F}_{t_i}$-measurable and

$$g(t) = \xi_0 I_{[t_0,t_1]}(t) + \sum_{i=1}^{k-1} \xi_i I_{(t_i,t_{i+1}]}(t). \tag{4.3}$$

where $I_{(t_i,t_{i+1}]}(t)$ is the characteristic function of the interval $(t_i, t_{i+1}]$ defined as follows

$$I_{(t_i,t_{i+1}]}(t) = \begin{cases} 1 & \text{for} \quad t \in (t_i, t_{i+1}], \\ 0 & \text{otherwise} \ . \end{cases}$$

We denote $\mathcal{M}_0([a,b];\mathbb{R})$ the family of all such processes. For a simple process, the Itô's integral is built as follows.

Definition 4.2. For a simple process $g(t) \in \mathcal{M}_0([a,b];\mathbb{R})$ of the form (4.3) we define

$$\int_a^b g(t)\,\mathrm{d}B_t = \sum_{i=0}^{k-1} \xi_i (B_{t_{i+1}} - B_{t_i}) \tag{4.4}$$

and we call it the *Itô's integral* of g with respect to Brownian motion $\{B_t\}_{t\geq 0}$.

The next lemma states a useful property of the Itô's integral of a simple process.

Lemma 4.1 (**Itô's isometry**). *Let $\phi(t)$ be bounded simple process, then*

$$\mathrm{E}\left(\int_a^b \phi(t)\,\mathrm{d}B_t\right)^2 = \mathrm{E}\int_a^b \phi^2(t)\,\mathrm{d}t$$

We have shown the construction of the stochastic integral for a simple process. Our aim in this moment is to extend this definition to a larger class of processes that satisfy the definition 4.3.

Definition 4.3. Let $0 \le a < b < \infty$. Denote by $\mathcal{M}^2([a,b],\mathbb{R})$ the space of all real-valued measurable $\mathcal{F}_t$-adapted processes $f = \{f(t)\}_{a \le t \le b}$ such that

$$\|f\|_{a,b}^2 = \mathrm{E}\int_a^b |f(t)|^2 \mathrm{d}t < \infty.$$

We identify f and $\overline{f}$ in $\mathcal{M}^2([a,b],\mathbb{R})$ if $\|f - \overline{f}\|_{a,b}^2 = 0$. In this case we say that they are equivalent and write $f = \overline{f}$.

The result that will enable us to extend the definition is the following lemma whose proof can be found in [1].

Lemma 4.1. *for any $f \in \mathcal{M}^2([a,b],\mathbb{R})$ there exists a sequence $\{g_n\}$ of simple processes such that*

$$\lim_{n\to\infty} \mathrm{E}\int_a^b |f(t) - g_n(t)|^2 \mathrm{d}t = 0. \tag{4.5}$$

Using the previous lemma we can define the stochastic integral of the of the process $f(t)$ belonging to the $\mathcal{M}^2([a,b],\mathbb{R})$.

Definition 4.4. Let $f \in \mathcal{M}^2([a,b],\mathbb{R})$. The *Itô's integral* of f with respect to $\{B_t\}_{t\ge 0}$ is defined by

$$\int_a^b f(t)\ \mathrm{d}B_t = \lim_{n\to\infty}\int_a^b g_n(t)\ \mathrm{d}B_t,$$

where $\{g_n\}$ is a sequence of simple processes that have stochastic integral (4.4) and that satisfy (4.5).

The next theorem summarizes the properties of Itô's integral.

Theorem 4.1 (Properties of the Itô's integral). *Let $X_t, Y_t \in \mathcal{M}^2[0,T]$ and let $0 \le S < U < T$ and c a constant, then*

(1) $\int_S^T X_s\ \mathrm{d}B_s = \int_S^U X_s\ \mathrm{d}B_s + \int_U^T X_s\ \mathrm{d}B_s,$

(2) $\int_0^T (c\cdot X_s + Y_s)\ \mathrm{d}B_s = c\int_0^T X_s\ \mathrm{d}B_s + \int_0^T Y_s\ \mathrm{d}B_s,$

(3) $\mathrm{E}\int_0^T X_s\ \mathrm{d}B_s = 0,$

(4) $\int_0^T X_s\ \mathrm{d}B_s$ *is $\mathcal{F}_T$ measurable.*

(5) the extension of Itô's isometry to $\mathcal{M}^2[0,T]$:

$$\mathrm{E}\left(\int_0^t X_s\ \mathrm{d}B_s\right)^2 = \mathrm{E}\int_0^t X_s^2\ \mathrm{d}s$$

(6) the Itô's integral of X_t is normally distributed with expected value given by (3) and variance given by (5)

The Itô's integral that we defined is actually a definite integral. For the application to the stochastic differential equations we will need the indefinite Itô's integral.

Definition 4.5 (Indefinite Itô's integral). Let $f(t) \in \mathcal{M}^2([0,T];R)$. Define

$$I(t) = \int_0^t f(s)\ \mathrm{d}B_s \quad \text{for } 0 \le t \le T, \tag{4.6}$$

where $I(0) = \int_0^0 f(s)\ \mathrm{d}B_s = 0$. We call (4.6) the indefinite Itô's integral of $f(t)$.

Since the indefinite integral is defined using the definition of definite integral, it has the same properties given by theorem 4.1.

Now we will show a theorem which states that the indefinite integral can be chosen to be continuous.

Theorem 4.2. *Let $X_t \in \mathcal{M}^2[0,T]$ then there exist a t-continuous version of*

$$I(t) = \int X_t(\omega)\ \mathrm{d}B_s(\omega), \quad 0 \le t \le T,$$

it means that there exists a t-continuous stochastic process $J(t)$ on $(\Omega, \mathscr{F}, P)$ such that

$$P\left(I(t) = J(t)\right) = 1.$$

We could stop here the process of extension the definition and build all the following theory on the class of $\mathcal{M}^2(\Omega)$ processes. As a matter of fact, it is done in this way in [4]. But according to some other books such as [1] or [2] we can extend the integral into even larger class of functions.

Let $\{B_t\}_{t\ge 0}$ be a one-dimensional Brownian motion on a probability space $(\Omega, \mathscr{F}, P)$. We denote $\mathcal{L}^1(\mathbb{R}_+;\mathbb{R})$ resp. $\mathcal{L}^2(\mathbb{R}_+;\mathbb{R})$ the spaces of all real valued measurable $\mathcal{F}_t$-adapted processes $\{f(t)\}_{t\ge 0}$ such that

$$\int_0^T |f(t)|\mathrm{d}t < \infty, \quad \text{resp.} \quad \left(\int_0^T |f(t)|^2\mathrm{d}t\right)^{\frac{1}{2}} < \infty \quad \text{a.s. for every } T > 0.$$

The extension is similar to the one that has been carried out for the case of $\mathcal{M}^2$ processes therefore we will omit it and refer to [1], where it is shown in a detailed way. What is important for us is that the integral exists for all the $f(t) \in \mathcal{L}^2(\mathbb{R}_+;\mathbb{R})$. We should also mention the fact that some of the properties given by theorem 4.1 does not hold here.

4.2 Itô's formula

In previous section we defined the Itô's integral. However, the definition is not very useful to evaluate an actual integral. We will formulate an important rule that will enable us to compute the Itô's integral more easily and quickly. This rule is called the *Itô's formula.*

Definition 4.6 (Itô's process). A one-dimensional *Itô's process* is a continuous adapted process X_t on $t \geq 0$, that has the form

$$X_t = X_0 + \int_0^t f(s, X_s)\ \mathrm{d}s + \int_0^t g(s, X_s)\ \mathrm{d}B_s, \tag{4.7}$$

where $f \in \mathcal{L}^1(\mathbb{R}_+; \mathbb{R}^d)$ and $g \in \mathcal{L}^2(\mathbb{R}_+; \mathbb{R}^d)$. We can also say that X_t has the stochastic differential $\mathrm{d}X_t$ given by

$$\mathrm{d}X_t = f(t, X_t)\ \mathrm{d}t + g(t, X_t)\ \mathrm{d}B_t. \tag{4.8}$$

Now we can proceed to the actual Itô's formula.

Theorem 4.3 (Itô's formula in one dimension). *Let X_t be an Itô's process satisfying* (4.7) *and let it have a stochastic differential* (4.8). *Let $V \in C^2(\mathbb{R} \times \mathbb{R}_+)$ and let it also be in $C^1(\mathbb{R})$, then $V = V(X_t, t)$ is also an Itô's process with stochastic differential given by*

$$\mathrm{d}V = \left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\ f(t, X_t) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}\ g^2(t, X_t)\right] \mathrm{d}t + \frac{\partial V}{\partial x}\ g(t, X_t)\ \mathrm{d}B_t \ a.s., \tag{4.9}$$

The relation (4.9) is called the *One-dimensional Itô's formula.* The proof is rather technical and long, so we will show just the basic ideas of it (see [1] for details).

Ideas of the proof:

(1)(2) The first two ideas are to assume that X_t is bounded by some constant K. So we do not have to pay attention to the values of $V(X_t, t)$ for $X_t \notin [-K, K]$ and also that $V(X_t, t)$ is continuously twice differentiable in both X_t and t.

(3) The third step is to show the (4.9) for the case of simple processes.

(4) Now we fix $t > 0$ arbitrarily assuming that $V(X_t, t)$ and all its derivatives up to the order 2 are bounded and we take $f(t), g(t)$ two simple processes. It can be shown that all the V, $\frac{\partial V}{\partial t}$, $\frac{\partial V}{\partial x}$, $\frac{\partial^2 V}{\partial t^2}$, $\frac{\partial^2 V}{\partial xt}$, $\frac{\partial^2 V}{\partial x^2}$ can be approximated by simple processes.

The following rules are used when one has to use the Itô's formula

$$\mathrm{d}t\mathrm{d}t = 0, \quad \mathrm{d}t\mathrm{d}B_t = 0, \quad \mathrm{d}B_t\mathrm{d}B_t = \mathrm{d}t. \tag{4.10}$$

Theorem 4.4 (Itô's formula - alternative form). *Under the assumptions of theorem 4.3, the stochastic differential of $V = V(X_t, t)$ can be written as*

$$\mathrm{d}V = \frac{\partial V}{\partial t}\mathrm{d}t + \frac{\partial V}{\partial x}\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(\mathrm{d}X_t)^2 \tag{4.11}$$

The (4.9) can be obtained from (4.11) substituting (4.8) and using (4.10), so they are equivalent. Some authors, such as Oksendal [2], speak only about the form given by the theorem 4.4. The reason might be the greater straightforwardness of (4.11).

Let us also state the Itô's formula for the case of product of two stochastic processes.

Theorem 4.5 (**Product Itô's formula**). *Let X_t and $\hat{X}_t$ be two Ito processes satisfying* (4.7) *and let them have the Ito differential* (4.8). *Let $V \in C^2(\mathbb{R}^2 \times \mathbb{R}_+)$ and let it also be in $C^1(R)$. Then $V = V(X_t, \hat{X}_t, t) = X_t\hat{X}_t$ is also an Itô's process with stochastic differential given by*

$$\mathrm{d}(X_t\hat{X}_t) = \hat{X}_t\mathrm{d}X_t + X_t\mathrm{d}\hat{X}_t + g(X_t, t)\hat{g}(\hat{X}_t, t)\mathrm{d}t \tag{4.12}$$

We will show now an example of use of Itô's formula. Let B_t be a Brownian motion and we would like to compute the integral

$$\int_0^t B_s \,\mathrm{d}B_s.$$

Note that the Brownian motion has the simplest Itô's differential

$$\mathrm{d}B_t = 0 \cdot \mathrm{d}t + 1 \cdot \mathrm{d}B_t,$$

so in this case $f(t, X_t) = 0$ and $g(t, X_t) = 1$. In order to compute the given integral, we apply the Ito formula to $V(B_t, t) = B_t^2$ and get

$$\begin{aligned} \mathrm{d}V(B_t, t) &= \left[0 + 0 \cdot 2B_t + \frac{1}{2} \cdot 2 \cdot 1^2\right] \mathrm{d}t + 1 \cdot 2B_t\mathrm{d}B_t, \\ \mathrm{d}(B_t^2) &= 2B_t \,\mathrm{d}B_t + \mathrm{d}t, \end{aligned}$$

we integrate the both sides and end up with

$$B_t^2 = 2\int_0^t B_s \,\mathrm{d}B_s + t,$$

from which we deduce

$$\int_0^t B_s \,\mathrm{d}B_s = \frac{B_t^2 - t}{2}.$$

5 Stochastic differential equations

5.1 Basic notions

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space and let us consider an Itô type stochastic differential equation with initial condition $X_0 = \eta$:

$$\begin{cases} \mathrm{d}X_t = f(X_t, t)\mathrm{d}t + g(X_t, t)\mathrm{d}B_t, \\ X_0 = \eta. \end{cases} \tag{5.1}$$

By the definition of stochastic differential, the previous equation can be rewritten to its integral version

$$X_t = \eta + \int_0^t f(X_s, s)\ \mathrm{d}s + \int_0^t g(X_s, s)\ \mathrm{d}B_s. \tag{5.2}$$

Let us now define what is the solution to (5.1).

Definition 5.1 (Solution of SDE). A stochastic process $\{X_t\}_{0\leq t\leq T}$ is called a solution of (5.1) if

(1) it is continuous and $\mathcal{F}_t$-adapted,

(2) $f(X_t, t) \in \mathcal{L}^1([0, T]; \mathbb{R})$ and $g(X_t, t) \in \mathcal{L}^2([0, T]; \mathbb{R})$

(3) equation (5.2) holds for every $t \in [0, T]$

5.2 Existence and uniqueness of the solution

Now we defined what the solution of a stochastic differential equation is, we can turn to the important question of it's existence and uniqueness. The following theorem tells us, what conditions have to be satisfied to ensure both of them.

Theorem 5.1 (Existence and uniqueness). *Let $T > 0$ and $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $g : [0, T] \times \mathbb{R}^{d\times m} \to \mathbb{R}^{d\times m}$ be measurable functions satisfying*

$$|f(t, x)| + |g(t, x)| \leq C(1 + |x|)$$

for some constant C and

$$|f(t, x) - f(t, y)| + |g(t, x) - g(t, y)| \leq D|x - y|$$

for some constant D. Let η be a random variable independent of sigma algebra $\mathcal{F}_t$ generated by $B_s, s \geq 0$ and such that

$$\mathrm{E}|\eta|^2 < \infty.$$

Then the stochastic differential equation (5.1) has a unique t-continuos solution X_t which is adapted to the filtration $\mathcal{F}_t^Z$ generated by $B_s, s \leq t$ and η and has the property that

$$\mathrm{E}\left[\int_0^T |X_t|^2 \mathrm{d}t\right] < \infty.$$

In other words, the solution X_t is unique in $\mathcal{M}^2[0, T]$.

The theorem 5.1 is the formulation that can be found in [2] also with its proof. Other formulations, which are equivalent to this one, can be found in [1] or [4]. This theorem is very important, so it feels natural to state also its proof. But first we will state a lemma that will enable us to prove it.

Lemma 5.1 (**Gronnwall's inequality**). *Let $T > 0$ and $c \geq 0$. Let $u(\cdot)$ be a Borel measurable bounded non-negative function on $[0, T]$, and let $v(\cdot)$ be a non-negative integrable function on $[0, T]$. If*

$$u(t) \leq c + \int_0^t v(s)u(s)\mathrm{d}s \text{ for all } 0 \leq t \leq T,$$

then

$$u(t) \leq c \cdot \exp\left(\int_0^t v(s)\mathrm{d}s\right) \text{ for all } 0 \leq t \leq T.$$

Now we can proceed to the proof of the existence and uniqueness theorem, that has two steps. First we will prove the uniqueness. Let $X_1(t,\omega) = X_t(\omega)$ and $X_2(t,\omega) = \widehat{X}_t(\omega)$ be solutions with initial values Z and $\widehat{Z}$.

Put $a(s,\omega) = f(s, X_s) - f(s, \widehat{X_s})$ and $\gamma(s,\omega) = g(s, X_s) - g(s, \widehat{X_s})$. Then using the elementary inequality

$$|a + b + c|^2 \leq 3\left(|a|^2 + |b|^2 + |c|^2\right)$$

we obtain

$$\begin{aligned}
\mathrm{E}|X_t - \widehat{X_t}|^2 &= \mathrm{E}\left(Z - \widehat{Z} + \int_0^t a \; \mathrm{d}s + \int_0^t \gamma \; \mathrm{d}B_s\right)^2 \\
&\leq 3\mathrm{E}|Z - \widehat{Z}|^2 + 3\mathrm{E}\left(\int_0^t a \; \mathrm{d}s\right)^2 + 3\mathrm{E}\left(\int_0^t \gamma \; \mathrm{d}B_s\right)^2 \\
&\leq 3\mathrm{E}|Z - \widehat{Z}|^2 + 3t\mathrm{E}\left(\int_0^t a^2 \; \mathrm{d}s\right) + 3\mathrm{E}\left(\int_0^t \gamma^2 \; \mathrm{d}s\right) \\
&\leq 3\mathrm{E}|Z - \widehat{Z}|^2 + 3(1+t)D^2 \int_0^t \mathrm{E}|X_s - \widehat{X_s}|^2 \; \mathrm{d}s. \qquad (5.3)
\end{aligned}$$

So the function $v(t) = \mathrm{E}|X_t - \widehat{X_t}|^2$ satisfies

$$v(t) \leq F + A\int_0^t v(s) \; \mathrm{d}s,$$

where $F = \mathrm{E}|Z - \widehat{Z}|^2$ and $A = 3(1+T)D^2$. By the Gronnwall's inequality we deduce that

$$v(t) \leq F \; \exp(At).$$

Now since $Z = \widehat{Z}$ and $F = 0$ then $v(t) = 0$ for all $t \geq 0$. Hence

$$P\left[|X_t - \widehat{X}_t| = 0 \text{ for all } t \in \mathbb{Q} \cap [0, T]\right] = 1,$$

where $\mathbb{Q}$ denotes the rational numbers. By continuity of $t \to |X_t - \widehat{X}_t|$ it follows that

$$P\left[|X_1(t,\omega) - X_2(t,\omega)| = 0 \text{ for all } t \in [0, T]\right] = 1$$

and the uniqueness is proved.

The second step of the proof is showing that the solution to (5.1) exists. The proof is similar to the proof of existence of solution to the IVP for an ordinary differential equation. The idea is to define the Picard iterations and show that they converge to our solution. First we define the $Y_t^{(0)} = X_0$ and $Y_t^{(k)}$ in this way:

$$\mathrm{Y}_t^{(k+1)} = X_0 + \int_0^t f(Y_s^{(k)}, s)\, \mathrm{d}s + \int_0^t g(Y_s^{(k)}, s)\, \mathrm{d}B_s$$

Then we carry out a computation similar to (5.3) and we end up with

$$\mathrm{E}|\mathrm{Y}_t^{(k+1)} - Y_t^{(k)}|^2 \leq 3(1+\mathrm{T})\mathrm{D}^2 \int_0^t \mathrm{E}|\mathrm{Y}_s^{(k)} - Y_s^{(k-1)}|^2 \mathrm{d}s,$$

for $k \geq 1$ and $t \leq \mathrm{T}$ and

$$\mathrm{E}|Y_t^{(1)} - Y_t^{(0)}|^2 \leq 2t^2\mathrm{C}^2\left(1 + \mathrm{E}|X_0|^2\right) + 2t\mathrm{C}^2\left(1 + \mathrm{E}|X_0|^2\right) \leq \mathrm{A}_1 t,$$

Where A_1 is a constant, that depends only on C, $E|X_0|^2$ and T. Therefore by induction we can get

$$\mathrm{E}|\mathrm{Y}_t^{(k+1)} - \mathrm{Y}_t^{(k)}|^2 \leq \frac{\mathrm{A}_2^{k+1} t^{k+1}}{(k+1)!}$$

for $t \in [0, \mathrm{T}]$, $k \geq 0$ and a constant A_2, which depends on C, D, $E|X_0|^2$ and T.

Let us now denote λ a Lebesgue measure on $[0, \mathrm{T}]$ and we take $m > n \geq 0$ we get

$$\|\mathrm{Y}_t^{(m)} - Y_t^{(n)}\|_{L_2(\lambda\times P)} = \|\sum_{k=n}^{m-1}\left(\mathrm{Y}_t^{(k+1)} - Y_t^{(k)}\right)\|_{L_2(\lambda\times P)} \leq \sum_{k=n}^{m-1}\|\mathrm{Y}_t^{(k+1)} - \mathrm{Y}_t^{(k)}\|_{L_2(\lambda\times P)}$$

$$= \sum_{k=n}^{m-1}\left(\mathrm{E}\left[\int_0^{\mathrm{T}} |\mathrm{Y}_t^{(k+1)}\mathrm{Y}_t^{(k)}|^2 \mathrm{d}t\right]\right)^{\frac{1}{2}} \leq \sum_{k=n}^{m-1}\left(\int_0^{\mathrm{T}} \frac{\mathrm{A}_2^{k+1}t^{k+1}}{(k+1)!}\right)^{\frac{1}{2}} = \sum_{k=n}^{m-1}\left(\frac{\mathrm{A}_2^{k+1}\mathrm{T}^{k+2}}{(k+2)!}\right)^{\frac{1}{2}}.$$

Now we take the last term and we compute the following limit

$$\lim_{n,m\to\infty} \sum_{k=n}^{m-1}\left(\frac{\mathrm{A}_2^{k+1}\mathrm{T}^{k+2}}{(k+2)!}\right)^{\frac{1}{2}} = 0,$$

so $\{Y_t^{(n)}\}_{n=0}^{\infty}$ is a Cauchy sequence in $L^2(\lambda \times P)$ and since it is a complete space, this sequence is also convergent. Let us define

$$X_t := \lim_{n\to\infty} Y_t^{(n)}.$$

Then X_t is $\mathscr{F}_t^Z$-measurable, because $Y_t^{(n)}$ is for all n. Now we have to prove that X_t satisfies (5.1).

For all n and all $t \in [0, \mathrm{T}]$ we have

$$Y_t^{(n)} = X_0 + \int_0^t f(s, Y_s^{(n)})\ \mathrm{d}s + \int_0^t g(s, Y_s^{(n)})\ \mathrm{d}B_s$$

If we let $n \to \infty$ and use the Hölder inequality in the first case and the Ito isometry in the second, we show that

$$\int_0^t f(s, Y_s^{(n)})\ \mathrm{d}s \to \int_0^t f(s, X_s)\ \mathrm{d}s,$$
$$\int_0^t g(s, Y_s^{(n)})\ \mathrm{d}B_s \to \int_0^t g(s, X_s)\ \mathrm{d}B_s.$$

Using the expressions above, we can write

$$X_t = X_0 + \int_0^t f(s, X_s)\ \mathrm{d}s + \int_0^t g(s, X_s)\ \mathrm{d}B_s \quad \text{for all t} \in [0, \mathrm{T}].$$

We proved that X_t satisfies the (5.1). The final step is to show that the integral can be chosen continuously, but that is a direct consequence of theorem 4.2. The proof is complete.

6 Linear stochastic differential equations - Theoretical results

We have already cleared up what the solution to a stochastic differential equation is and under which conditions it exists and it is unique. Now we would like to find a way to obtain the solution. For the most of the stochastic equations the explicit formula for the solution does not exist. In such cases we have to use the approximate solutions such as *Euler-Maruyama* numerical scheme that is described in details in [1], but we will not discuss them in this text. We will concentrate on the case of the linear stochastic differential equations for which the explicit solution can be found.

In addition to finding the explicit formula for the solution, we will derive the equations to obtain its expected value and variance. These equations will be ordinary differential equations, so they will be much more easy to solve.

Let us now state how we classify the linear equations.

(1) The general form of *linear Stochastic differential equation* is the following one

$$\begin{cases} \mathrm{d}X_t = [a(t)X_t + b(t)]\,\mathrm{d}t + [c(t)X_t + d(t)]\,\mathrm{d}B_t, \\ X_0 = \eta, \end{cases} \tag{6.1}$$

where $a(t)$, $b(t)$, $c(t)$, $d(t)$ are real-valued functions of time t and to satisfy the assumptions of the existence and uniqueness theorem, we assume that they are bounded. We also assume, for the same reason, $\mathrm{E}\eta^2 < \infty$. X_t is the unknown stochastic process.

(2) The linear equation is said to be *homogeneous* if $b(t) = d(t) = 0$, so (6.1) yields

$$\begin{cases} \mathrm{d}X_t = a(t)X_t\ \mathrm{d}t + c(t)X_t\ \mathrm{d}B_t, \\ X_0 = \eta. \end{cases} \tag{6.2}$$

(3) The linear equation is said to be *Stochastic differential equation in narrow sense* if $c(t) = 0$, so (6.1) transforms into

$$\begin{cases} \mathrm{d}X_t = [a(t)X_t + b(t)]\,\mathrm{d}t + d(t)\ \mathrm{d}B_t, \\ X_0 = \eta. \end{cases} \tag{6.3}$$

6.1 First example

We will show now an easy example of such an equation. Let us consider following initial value problem for the stochastic differential equation with constant coefficients λ and σ.

$$\begin{cases} \mathrm{d}X_t = \lambda X_t\ \mathrm{d}t + \sigma X_t\ \mathrm{d}B_t, \\ X_0 = \eta. \end{cases} \tag{6.4}$$

We can rewrite the equation from (6.4) as follows

$$\frac{\mathrm{d}X_t}{X_t} = \lambda\ \mathrm{d}t + \sigma\ \mathrm{d}B_t,$$

and integrate both sides over $[0, t]$ and get the expression

$$\int_0^t \frac{\mathrm{d}X_t}{X_t} = \lambda t + \sigma B_t. \tag{6.5}$$

We would like to evaluate the left-hand side. In order to do that, we have to use the Ito formula of the function $V(X_t, t) = \ln X_t$.

$$\begin{aligned}
\mathrm{d}\left(\ln X_t\right) &= \left[0 + \frac{1}{X_t}\lambda X_t + \frac{1}{2}\frac{-1}{X_t^2}\sigma^2 X_t^2\right] \mathrm{d}t + \frac{1}{X_t}\sigma X_t \; \mathrm{d}B_t, \\
\mathrm{d}\left(\ln X_t\right) &= \left[\lambda - \frac{1}{2}\sigma^2\right] \mathrm{d}t + \sigma \; \mathrm{d}B_t, \\
\mathrm{d}\left(\ln X_t\right) &= \frac{\mathrm{d}X_t}{X_t} - \frac{1}{2}\sigma^2 \; \mathrm{d}t.
\end{aligned}$$

Now using (6.5) we compute

$$\lambda t + \sigma B_t = \ln \frac{X_t}{X_0} + \frac{1}{2}\sigma^2 t,$$

therefore

$$X_t = \eta \mathrm{e}^{\left(\lambda - \frac{1}{2}\sigma^2\right)t + \sigma B_t} \tag{6.6}$$

The expression (6.6) is the explicit solution to the IVP (6.4). We can go further now and we can compute its expected value.

$$\begin{aligned}
\mathrm{E}X_t = \mathrm{E}\left[\eta \mathrm{e}^{\left(\lambda - \frac{1}{2}\sigma^2\right)t + \sigma B_t}\right] &= \eta \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} \; \mathrm{e}^{-\frac{x_t^2}{2t}} \mathrm{e}^{\left(\lambda - \frac{1}{2}\sigma^2\right)t + \sigma x_t} \; \mathrm{d}x_t, \\
&= \frac{\eta}{\sqrt{2\pi t}} \; \mathrm{e}^{\left(\lambda - \frac{1}{2}\sigma^2\right)t} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{x_t^2}{2t}} \mathrm{e}^{\sigma x_t} \; \mathrm{d}x_t = \frac{\eta}{\sqrt{2\pi t}} \; \mathrm{e}^{\left(\lambda - \frac{1}{2}\sigma^2\right)t} \sqrt{2\pi t} \; \mathrm{e}^{\frac{\sigma^2}{2}t}, \\
\mathrm{E}X_t = \eta \mathrm{e}^{\lambda t}.
\end{aligned}$$

Now we can compute also its variance. First we evaluate the second moment $\mathrm{E}X_t^2$

$$\begin{aligned}
\mathrm{E}X_t^2 = \mathrm{E}\left[\eta^2 \mathrm{e}^{2\left(\lambda - \frac{1}{2}\sigma^2\right)t + 2\sigma B_t}\right] &= \eta^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} \; \mathrm{e}^{-\frac{x_t^2}{2t}} \mathrm{e}^{2\left(\lambda - \frac{1}{2}\sigma^2\right)t + 2\sigma x_t} \; \mathrm{d}x_t, \\
&= \frac{\eta^2}{\sqrt{2\pi t}} \; \mathrm{e}^{2\left(\lambda - \frac{1}{2}\sigma^2\right)t} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{x_t^2}{2t}} \mathrm{e}^{2\sigma x_t} \; \mathrm{d}x_t = \frac{\eta^2}{\sqrt{2\pi t}} \; \mathrm{e}^{2\left(\lambda - \frac{1}{2}\sigma^2\right)t} \sqrt{2\pi t} \; \mathrm{e}^{2\sigma^2 t}, \\
\mathrm{E}X_t^2 = \eta^2 \mathrm{e}^{\left(2\lambda + \sigma^2\right)t}.
\end{aligned}$$

Now we use both quantities that we have reached and substitute them into the definition formula of the variance

$$\begin{aligned}
\mathrm{V}X_t &= \mathrm{E}\left(X_t - \mathrm{E}X_t\right)^2 = \mathrm{E}X_t^2 - \mathrm{E}^2 X_t, \\
\mathrm{V}X_t &= \eta^2 \mathrm{e}^{2\lambda t}\left(e^{\sigma^2 t} - 1\right).
\end{aligned}$$

We computed the expected value and variance directly from the solution. There is however another way to do it. In the following text we will derive a method, how to obtain both these characteristics without knowing the explicit solution.

6.2 General form of solution

Now that we have seen that the equation from previous example could be solved analytically, we would like to find a way of solving all the linear equations.

Solution to homogeneous linear equation

First we will derive the explicit formula of solution to the homogeneous case (6.2). We will perform a similar procedure as we did in the first example, but this time the coefficients are not constant. The initial step is to divide both sides of equation by X_t

$$\frac{\mathrm{d}X_t}{X_t} = a(t)\mathrm{d}t + c(t)\mathrm{d}B_t.$$

Integrating both sides over the interval $[0, t]$ we yield

$$\int_0^t \frac{\mathrm{d}X_t}{X_t} = \int_0^t a(s)\ \mathrm{d}s + \int_0^t c(s)\ \mathrm{d}B_s. \tag{6.7}$$

Now to evaluate the integral on the left-hand side, we use the Ito formula of the function $V(X_t, t) = \ln X_t$.

$$\begin{aligned}
\mathrm{d}\left(\ln X_t\right) &= \left[\frac{1}{X_t}a(t)X_t - \frac{1}{2}\frac{1}{X_t^2}c^2(t)X_t^2\right]\mathrm{d}t + \frac{1}{X_t}c(t)X_t\mathrm{d}B_t,\\
\mathrm{d}\left(\ln X_t\right) &= \left[a(t) - \frac{1}{2}c^2(t)\right]\mathrm{d}t + c(t)\mathrm{d}B_t,\\
\mathrm{d}\left(\ln X_t\right) &= \frac{\mathrm{d}X_t}{X_t} - \frac{1}{2}c^2(t)\mathrm{d}t.
\end{aligned}$$

Now using (6.7) we compute

$$\begin{aligned}
\ln\frac{X_t}{X_0} &= \int_0^t a(s)\ \mathrm{d}s + \int_0^t c(s)\ \mathrm{d}B_s - \int_0^t \frac{1}{2}c^2(s)\mathrm{d}s,\\
\mathrm{e}^{\ln\frac{X_t}{X_0}} &= \mathrm{e}^{\int_0^t \left(a(s) - \frac{1}{2}c^2(s)\right)\ \mathrm{d}s + \int_0^t c(s)\ \mathrm{d}B_s}.
\end{aligned}$$

Which we can rewrite and obtain the explicit solution to the initial value problem (6.2).

$$X_t = \eta\cdot\mathrm{e}^{\int_0^t \left(a(s) - \frac{1}{2}c^2(s)\right)\ \mathrm{d}s + \int_0^t c(s)\ \mathrm{d}B_s}. \tag{6.8}$$

Solution to non-homogeneous linear equation

Now that we know how the solution to homogeneous equation looks like, we can turn to the non-homogeneous one. To solve the initial value problem (6.1), we can use the solution to the corresponding homogeneous problem

$$\begin{cases} \mathrm{d}Y_t = a(t)Y_t\mathrm{d}t + c(t)Y_t\mathrm{d}B_t, \\ Y_0 = 1, \end{cases}$$

which is

$$Y_t = \mathrm{e}^{\int\limits_0^t \left(a(s)-\frac{1}{2}c^2(s)\right)\ \mathrm{d}s + \int\limits_0^t c(s)\ \mathrm{d}B_s}, \tag{6.9}$$

according to the (6.8). Let us now compute $\mathrm{d}Y_t^{-1}$ using the Ito formula

$$\begin{aligned} \mathrm{d}Y_t^{-1} &= -\frac{1}{Y_t^2}\ \mathrm{d}Y_t + \frac{1}{2}\frac{2}{Y_t^3}\left(c(t)X_t\right)^2\mathrm{d}t, \\ &= -Y_t^{-2}\left(a(t)Y_t\ \mathrm{d}t + c(t)Y_t\ \mathrm{d}B_t\right) + Y_t^{-1}c^2(t)\ \mathrm{d}t, \\ &= -\left(a(t)Y_t^{-1}\mathrm{d}t + c(t)Y_t^{-1}\mathrm{d}B_t\right) + Y_t^{-1}c^2(t)\ \mathrm{d}t. \end{aligned}$$

Now we apply the product version of Ito formula (4.12) to $X_tY_t^{-1}$.

$$\mathrm{d}\left(X_tY_t^{-1}\right) = \left(\mathrm{d}X_t\right)Y_t^{-1} + \left(\mathrm{d}Y_t^{-1}\right)X_t - \left(c(t)X_t + d(t)\right)c(t)Y_t^{-1}\mathrm{d}t$$

Substituting (6.1) and (6.2) into it we obtain

$$\begin{aligned} \mathrm{d}\left(X_tY_t^{-1}\right) &= \left[\left(a(t)X_t + b(t)\right)\mathrm{d}t + \left(c(t)X_t + d(t)\right)\mathrm{d}B_t\right]Y_t^{-1} \\ &\quad + \left[-\left(a(t)Y_t^{-1}\mathrm{d}t + c(t)Y_t^{-1}\mathrm{d}B_t\right) + Y_t^{-1}c^2(t)\ \mathrm{d}t\right]X_t \\ &\quad - \left(c(t)X_t + d(t)\right)c(t)Y_t^{-1}\mathrm{d}t, \\ \mathrm{d}\left(X_tY_t^{-1}\right) &= \left(b(t) - c(t)d(t)\right)Y_t^{-1}\ \mathrm{d}t + d(t)Y_t^{-1}\mathrm{d}B_t. \end{aligned}$$

Taking integral form of the expression above we get

$$X_tY_t^{-1} = X_0Y_0^{-1} + \int\limits_0^t \left(b(s) - c(s)d(s)\right)Y_s^{-1}\ \mathrm{d}s + \int\limits_0^t d(s)Y_s^{-1}\mathrm{d}B_s.$$

Reminding that $Y_0 = 1$ we can drop Y_0^{-1}

$$X_tY_t^{-1} = X_0 + \int\limits_0^t \left(b(s) - c(s)d(s)\right)Y_s^{-1}\ \mathrm{d}s + \int\limits_0^t d(s)Y_s^{-1}\mathrm{d}B_s,$$

and multiplying both sides by Y_t we yield

$$X_t = X_0Y_t + Y_t\int\limits_0^t \left(b(s) - c(s)d(s)\right)Y_s^{-1}\ \mathrm{d}s + Y_t\int\limits_0^t d(s)Y_s^{-1}\mathrm{d}B_s. \tag{6.10}$$

The only thing that is left to do is to substitute (6.9) into (6.10).

In the following text we will deal with the question of properties and characteristics, such as expected value and variance, of the solution that we just obtained.

6.3 Integral theorems

Before we can start deriving the formulas for the expected value and variance, we have to state some theorems that we will need in order to do some important steps. These are the well known *Fubini* and *Tonelli* theorems that both plays key role in integration on L^p spaces in the functional analysis. First we will introduce the general versions of both theorems and then we will show how can we interpret them for our purposes.

Theorem 6.1 (**Tonelli**). *Let $F(x,y) : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a measurable function satisfying*

(1) $\int\limits_{\Omega_2} |F(x,y)| \mathrm{d}\mu_2 < \infty$ *for almost every $x \in \Omega_1$ and*

(2) $\int\limits_{\Omega_1} \left(\int\limits_{\Omega_2} |F(x,y)| \mathrm{d}\mu_2 \right) \mathrm{d}\mu_1 < \infty$,

then

$$\iint\limits_{\Omega_1 \times \Omega_2} |F(x,y)| \, \mathrm{d}\mu_1 \mathrm{d}\mu_2 < \infty. \tag{6.11}$$

Theorem 6.2 (**Fubini**). *Assume that* (6.11) *holds. Then*

$$\iint\limits_{\Omega_1 \times \Omega_2} F(x,y) \, \mathrm{d}\mu_1 \mathrm{d}\mu_2 = \int\limits_{\Omega_1} \left(\int\limits_{\Omega_2} F(x,y) \mathrm{d}\mu_2 \right) \mathrm{d}\mu_1 = \int\limits_{\Omega_2} \left(\int\limits_{\Omega_1} F(x,y) \mathrm{d}\mu_1 \right) \mathrm{d}\mu_2.$$

Now we will show how to obtain the stochastic versions of these theorems. If we identify

$$F(x,y) = X(t,\omega), \quad \Omega_1 = \Omega, \quad \Omega_2 = [0,t],$$

then the Tonelli theorem yields the following form

Theorem 6.3 (**Tonelli - Stochastic version**). *Let $F(x,y) : \Omega \times [0,t] \to \mathbb{R}$ be a measurable function satisfying*

(1) $\int\limits_0^t |X(s,\omega)| \mathrm{d}s < \infty$ *for almost every $\omega \in \Omega$ and*

(2) $\mathrm{E} \int\limits_0^t |X(s,\omega)| \mathrm{d}s < \infty$,

then

$$\iint\limits_{\Omega \times [0,t]} f(x_s) |X(s,\omega)| \, \mathrm{d}s \, \mathrm{d}x < \infty, \tag{6.12}$$

where $f(x_s)$ is the probability density function of X_t for fixed $t = s$.

And we also transform the Fubini's theorem

Theorem 6.4 (**Fubini - Stochastic version**). *Assume that* (6.12) *holds. Then*

$$\iint\limits_{\Omega \times [0,t]} f(x_s) X(s,\omega) \, \mathrm{d}s \, \mathrm{d}x = \int\limits_0^t \mathrm{E} X_s \, \mathrm{d}s = \mathrm{E} \int\limits_0^t X_s \, \mathrm{d}s.$$

6.4 Observation

In this section we will make an observation about the linear stochastic differential equations. We will show that both $f(X_t, t) = a(t)X_t + b(t)$ and $g(X_t, t) = c(t)X_t + d(t)$ not only belongs to $\mathcal{L}^2[0, t]$ but also to $\mathcal{M}^2[0, t]$. Namely we want to show that

$$\mathrm{E}\int_0^t |a(s)X_s + b(s)|^2 \mathrm{d}s < \infty \qquad \text{and} \qquad \mathrm{E}\int_0^t |c(s)X_s + d(s)|^2 \mathrm{d}s < \infty.$$

Let us begin with simple evaluation of the square of our function

$$\mathrm{E}\int_0^t \left[c^2(s)X_s^2 + 2c(s)d(s)X_s + d^2(s)\right] \mathrm{d}s =$$

$$= \mathrm{E}\int_0^t [c^2(s)X_s^2]\mathrm{d}s + \mathrm{E}\int_0^t [2d(s)c(s)X_s]\mathrm{d}s + \mathrm{E}\int_0^t d^2(s)\mathrm{d}s.$$

We know from the existence and uniqueness theorem that $X_t \in \mathcal{M}^2[0, t]$. By the assumptions on coefficients of the linear equation we also now that $c(t)$ and $d(t)$ are bounded with some constants $\hat{c}$ and $\hat{d}$.

Therefore

$$\mathrm{E}\int_0^t [c^2(s)X_s^2]\mathrm{d}s = \int_0^t [c^2(s)\mathrm{E}X_s^2] \leq \int_0^t [\hat{c}^2\mathrm{E}X_s^2]\mathrm{d}s \leq \hat{c}^2\mathrm{E}\int_0^t X_s^2\mathrm{d}s < \infty$$

and

$$\mathrm{E}\int_0^t d^2(s)\mathrm{d}s = \int_0^t d^2(s)\mathrm{d}s \leq \int_0^t \hat{d}^2\mathrm{d}s < \infty.$$

In the case of the middle term, the following computation is required

$$\mathrm{E}\int_0^t [2d(s)c(s)X_s]\mathrm{d}s = \int_0^t [2d(s)c(s)\mathrm{E}X_s]\mathrm{d}s \leq \int_0^t [2\hat{d}\hat{c}\mathrm{E}X_s]\mathrm{d}s = 2\hat{d}\hat{c}\int_0^t \mathrm{E}X_s\mathrm{d}s.$$

Now using the Hölder inequality it follows that

$$\mathrm{E}X_s \leq \left(\mathrm{E}X_s^2\right)^{\frac{1}{2}}.$$

Using the previous expression and the monotonicity property of the Lebesgue integral we can proceed (without losing of generality we put $2\hat{d}\hat{c} = 1$)

$$\int_0^t \mathrm{E}X_s\mathrm{d}s \leq \int_0^t \left(\mathrm{E}X_s^2\right)^{\frac{1}{2}} \mathrm{d}s.$$

If $\mathrm{E}X_s^2 < 1$, then

$$\int\limits_0^t \left(\mathrm{E}X_s^2\right)^{\frac{1}{2}} \mathrm{d}s < \int\limits_0^t \mathrm{d}s = t < \infty.$$

If $\mathrm{E}X_s^2 \geq 1$, then

$$\int\limits_0^t \left(\mathrm{E}X_s^2\right)^{\frac{1}{2}} \mathrm{d}s \leq \int\limits_0^t \left(\mathrm{E}X_s^2\right) \mathrm{d}s = \mathrm{E}\int\limits_0^t X_s^2 \mathrm{d}s < \infty.$$

We have proved that $g(X_t, t) = [c(t)X_t + d(t)]$ belongs to $\mathcal{M}^2[0,t]$ Since $f(X_t, t)$ has the same form, the computation would be exactly the same, so $f(X_t, t)$ belongs to $\mathcal{M}^2[0,t]$ as well.

6.5 Expected value

We have already shown the way to obtain the general solution to a linear stochastic differential equation, but the solution itself does not tell us everything about the process. In this subsection we would like to derive an equation, that will enable us to compute the expected value of a process that solves the given SDE, without solving the equation and computing it directly from the solution as we did in the introductory example.

Let us consider an initial value problem for a linear stochastic differential equation,

$$\begin{cases} \mathrm{d}X_t = [a(t)X_t + b(t)]\,\mathrm{d}t + [c(t)X_t + d(t)]\,\mathrm{d}B_t, \\ X_0 = \eta, \end{cases}$$

which can be rewritten to its integral form

$$X_t = \eta + \int\limits_0^t [a(s)X_s + b(s)]\ \mathrm{d}s + \int\limits_0^t [c(s)X_s + d(s)]\ \mathrm{d}B_s.$$

Taking the expected value of both sides we obtain

$$\mathrm{E}X_t = \mathrm{E}\eta + \mathrm{E}\int\limits_0^t [a(s)X_s + b(s)]\ \mathrm{d}s + \mathrm{E}\int\limits_0^t [c(s)X_s + d(s)]\ \mathrm{d}B_s.$$

Now we can apply the Fubini theorem and continue in treating the expression for expectation.

$$\mathrm{E}X_t = \mathrm{E}\eta + \int\limits_0^t [a(s)\mathrm{E}X_s + b(s)]\ \mathrm{d}s + \mathrm{E}\int\limits_0^t [c(s)X_s + d(s)]\ \mathrm{d}B_s.$$

As we have already proved, $[c(s)X_s + d(s)] \in \mathcal{M}^2[0,t]$. Therefore by the theorem 4.1

$$\mathrm{E}\int_0^t [c(s)X_s + d(s)]\ \mathrm{d}B_s = 0.$$

Denoting $m(t) = \mathrm{E}X_t$ we obtain

$$m(t) = \mathrm{E}\eta + \int_0^t [a(s)m(s) + b(s)]\ \mathrm{d}s,$$

which has a differential form

$$\begin{cases} \dot{m}(t) = a(t)m(t) + b(t), \\ m(0) = \mathrm{E}\eta. \end{cases} \tag{6.13}$$

The solution to the initial value problem (6.13) is the expected value of the solution X_t of the original problem. Realizing that (6.13) only contains the coefficients from the deterministic part of (6.1), we can deduce that the expected value behaves in the same way as the solution to the deterministic analogue of (6.1), namely

$$\begin{cases} \dot{x}(t) = a(t)x(t) + b(t)\mathrm{d}t, \\ x(0) = \eta. \end{cases}$$

6.6 Variance

In the previous subsection we derived the formula for the expected value of the solution of given stochastic differential equation. In this section we will derive a formula for its variance.

First we recall that the variance of a random variable is defined as follows

$$\mathrm{V}X_t = \mathrm{E}\left(X_t - \mathrm{E}X_t\right)^2 = \mathrm{E}X_t^2 - \left(\mathrm{E}X_t\right)^2. \tag{6.14}$$

In order to derive the variance formula, we can use the fact that we have already computed the expected value, so we know that

$$\left(\mathrm{E}X_t\right)^2 = m^2(t),$$

where $m(t)$ solves (6.13). In the remainder of this subsection we will derive an ordinary differential equation in order to evaluate the second term of the variance

$$\mathrm{E}X_t^2 = \int_\Omega x^2(t)\ f(x(t))\ \mathrm{d}x,$$

which is called the second moment of X_t. Let us apply the Itô's formula to the X_t^2

$$\begin{aligned} \mathrm{d}X_t^2 &= 2X_t\ \mathrm{d}X_t + g(X_t,t)^2\mathrm{d}t, \\ \mathrm{d}X_t^2 &= 2X_t\ \mathrm{d}X_t + \left(c(t)X_t + d(t)\right)^2\mathrm{d}t. \end{aligned}$$

Substituting (5.1) into it we get

$$\begin{aligned}
\mathrm{d}X_t^2 &= 2X_t\left[(a(t)X_t + b(t))\,\mathrm{d}t + (c(t)X_t + d(t))\,\mathrm{d}B_t\right] \\
&\quad + \left(c^2(t)X_t^2 + 2c(t)d(t)X_t + d^2(t)\right)\mathrm{d}t, \\
\mathrm{d}X_t^2 &= \left[\left(2a(t) + c^2(t)\right)X_t^2 + (2b(t) + 2c(t)d(t))\,X_t + d^2(t)\right]\mathrm{d}t \\
&\quad + \left(2c(t)X_t^2 + 2d(t)X_t\right)\mathrm{d}B_t.
\end{aligned}$$

The integral form of expression above is

$$\begin{aligned}
X_t^2 = \eta^2 &+ \int_0^t \left[\left(2a(s) + c^2(s)\right)X_s^2 + (2b(s) + 2c(s)d(s))\,X_s + d^2(s)\right]\mathrm{d}s \\
&+ \int_0^t \left(2c(s)X_s^2 + 2d(s)X_s\right)\mathrm{d}B_s
\end{aligned} \tag{6.15}$$

Taking the expected value of (6.15) and denoting $P(t) = \mathrm{E}X_t^2$ we obtain

$$P(t) = \mathrm{E}\eta^2 + \int_0^t \left[\left(2a(s) + c^2(s)\right)P(s) + (2b(s) + 2c(s)d(s))\,m(s) + d^2(s)\right]\mathrm{d}s,$$

which we rewrite to the differential form and get the final equation for the second moment of X_t

$$\begin{cases} \dot{P}(t) = \left(2a(t) + c^2(t)\right)P(t) + (2b(t) + 2c(t)d(t))\,m(t) + d^2(t), \\ P(0) = \mathrm{E}\eta^2, \end{cases} \tag{6.16}$$

where $a(t), b(t), c(t), d(t)$ are the coefficients from (5.1) and $m(t)$ is a solution to (6.13). The final formula for the variance is then

$$\mathrm{V}X = P(t) - m^2(t). \tag{6.17}$$

In this section we reached the final expressions for the solution, expected value and variance. We will use all these results in the next section, where we will apply them to the concrete examples.

7 Linear stochastic differential equations - Examples

In this section we will give some examples of linear stochastic differential with a detailed analysis of their solutions together with the visualization of their sample paths.

To visualize the sample path of the solution, we have to make a partition of the time interval and display only values in the given points, because otherwise it would be impossible to display it, since it is a nowhere differentiable function. Unless otherwise specified, we will display the trajectory over the interval [0,3] and we will take 500 points from that interval and display the values in them. The visualization will be performed by the software *MATLAB*.

If possible, we will also make the comparison with corresponding non-stochastic analogues.

7.1 The Langevin equation

As our first example we take the historically oldest stochastic differential equation. Langevin wrote down the equation of motion for a particle according to the Newton's law of motion. He assumed, that two forces affect the particle

(1) A systematic force $-\zeta\dot{x}(t)$, which represents a dynamical friction experienced by the particle

(2) A rapidly fluctuating force $F(t) = \sigma\dot{B}_t$, which is caused by the impacts of the molecules of the liquid on the particle.

If we let $S(t) = \dot{x}(t)$ be the velocity of the particle and $S(0) = \eta$ be the initial value of velocity, the initial value problem for the motion is then

$$\begin{cases} m\dot{S}(t) = -\zeta S(t) + \sigma\dot{B}_t, \\ S(0) = \eta, \end{cases}$$

where ζ is the constant coefficient of friction. Without losing of generality (since all the m, ζ and σ are constants) we put $m = 1$. The equation then yields the following form

$$\dot{S}(t) = -\zeta S(t) + \sigma\dot{B}_t$$

and the corresponding Itô's problem is

$$\begin{cases} \mathrm{d}S_t = -\zeta S_t \ \mathrm{d}t + \sigma \ \mathrm{d}B_t, \\ S_0 = \eta. \end{cases} \tag{7.1}$$

The initial value problem (7.1) is a stochastic differential equation in narrow sense (6.3). According to the theory from the subsection 6.2, in order to obtain the solution, we first have to solve the corresponding homogeneous problem

$$\begin{cases} \mathrm{d}W_t = -\zeta W_t \ \mathrm{d}t, \\ W_0 = 1. \end{cases}$$

This problem is easy to solve because it is an ordinary differential equation and its solution is

$$W_t = \mathrm{e}^{-\zeta t}. \tag{7.2}$$

And now we will use the formula (6.10)

$$S_t = \eta W_t + W_t \int\limits_0^t \sigma W_s^{-1} \mathrm{d}B_s. \tag{7.3}$$

Substituting (7.2) into (7.3) we get the solution

$$S_t = \eta \mathrm{e}^{-\zeta t} + \int\limits_0^t \sigma \mathrm{e}^{-\zeta(t-s)} \ \mathrm{d}B_s, \tag{7.4}$$

where $\int\limits_0^t \sigma \mathrm{e}^{-\zeta(t-s)} \ \mathrm{d}B_s$ is according to the theorem 4.1 normally distributed with expectation equal to 0 and variance equal to

$$\int\limits_0^t \sigma^2 \mathrm{e}^{-2\zeta(t-s)} \ \mathrm{d}s = \sigma^2 \frac{1 - \mathrm{e}^{-2\zeta t}}{2\zeta}. \tag{7.5}$$

The solution (7.4) is called the *Ornstein-Uhlenbeck process*. We can use the theorem 4.1 because the function $\sigma \mathrm{e}^{-\zeta(t-s)}$ is deterministic and does not explode in final time, therefore it belongs to $\mathcal{M}^2[0, t]$. In the figure 7.1 we can see two sample paths of (7.4) for the following values

$$t = 3, \quad \eta = 3, \quad \sigma = 0.5, \quad \zeta = \pm 1.$$

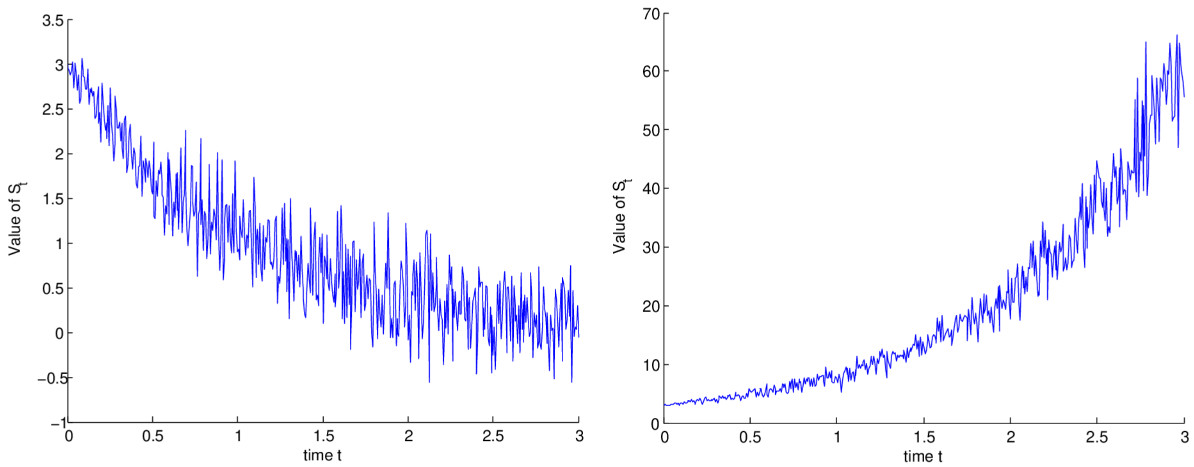

Figure 7.1: The sample path of (7.4) for $\zeta = 1$ on the left and for $\zeta = -1$ on the right

As we can see from the figures, the parameter ζ plays the key role with respect to the property of the solution to tend to zero or infinity as times approaches ∞. So we can summarize as follows

$$\zeta > 0 \Rightarrow \lim_{t \to \infty} S_t = S \quad \text{and} \quad \zeta < 0 \Rightarrow \lim_{t \to \infty} S_t = \infty \quad ,$$

where S is normally distributed, as we will verify at the end of this example, with expectation 0 and variance $\frac{\sigma^2}{2\zeta}$. The final question is how the solution would behave when

$\zeta = 0$. Our first guess shall be that it would tend to the value of the initial condition, but looking at (7.5) we see that the stochastic part of the solution would have infinite variance. Therefore our conclusion is that for $\zeta = 0$ the S_T will fluctuate between arbitrary values as $t \to \infty$.

However, realizing the physical meaning of ζ, i.e. that it is a friction coefficient, it does not make a physical sense taking $\zeta \leq 0$.

Let us focus now on the parameter σ. From (7.5) we can deduce that it affects the variance of the stochastic part and taking the values

$$\sigma_1 = 0.1 \quad \sigma_2 = 0.8$$

we can confirm it in the figure 7.2.

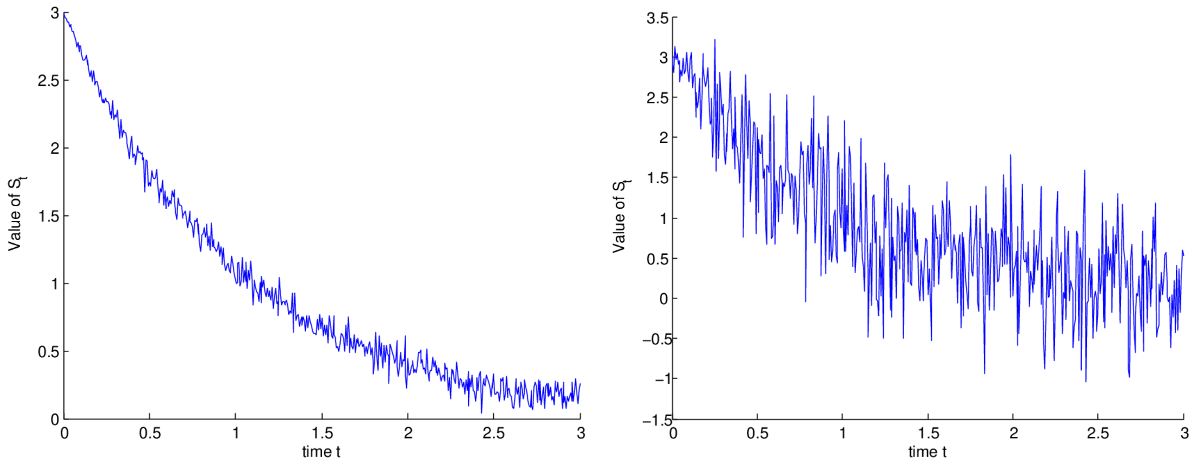

Figure 7.2: The sample path of (7.4) for $\sigma = 0.1$ on the left and for $\sigma = 0.8$ on the right

Taking negative values of σ does not make any difference, because in (7.5) we have σ^2 and if we let $\sigma = 0$ we would lose the stochastic behaviour of the solution, since it would have a zero variance.

We will go forward now and compute the expectation of S_t. As for this example the equation (6.13) reaches

$$\begin{cases} \dot{m}(t) = -\zeta m(t), \\ m(0) = \eta, \end{cases}$$

which has the explicit solution

$$m(t) = \eta e^{-\zeta t}. \tag{7.6}$$

We can easily deduce, that (7.4) reaches (7.6) as σ tends to zero. The figure 7.4 displays the (7.6) for the values $\eta = 3$ and $\zeta = 1$.

The variance of S_t should be given by (7.5). Let us now verify it using the method derived in the subsection 6.6.

$$\begin{cases} \dot{P}(t) = -2\zeta P(t) + \sigma^2, \\ P(0) = \eta^2. \end{cases} \tag{7.7}$$

The solution to (7.7) is

$$P(t) = \left(\eta^2 - \frac{\sigma^2}{2\zeta}\right) \mathrm{e}^{-2\zeta t} + \frac{\sigma^2}{2\zeta}.$$

Taking the (6.17)

$$\mathrm{V}S_t = P(t) - m^2(t) = \left(\eta^2 - \frac{\sigma^2}{2\zeta}\right) \mathrm{e}^{-2\zeta t} + \frac{\sigma^2}{2\zeta} - \eta^2 \mathrm{e}^{-2\zeta t},$$
$$\mathrm{V}S_t = \sigma^2 \frac{1 - \mathrm{e}^{-2\zeta t}}{2\zeta},$$

which is exactly the same as (7.5). The figures 7.3 and 7.4 shows the expected value and variance of S_t.

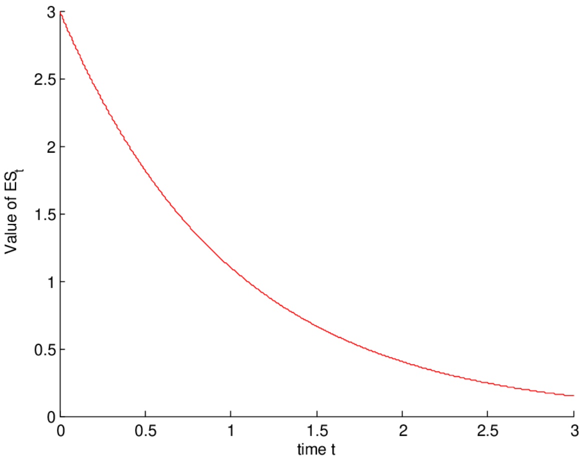

Figure 7.3: The expectation of S_t

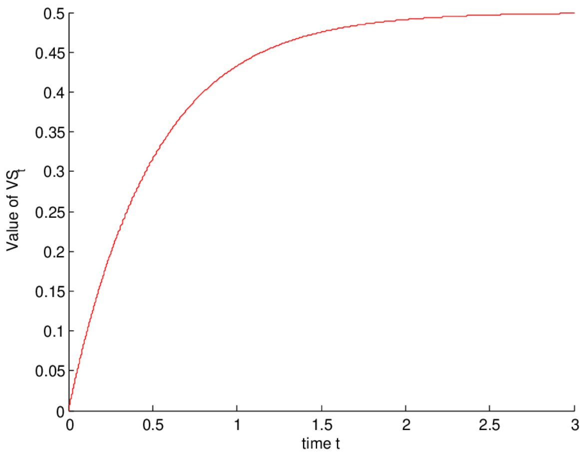

Figure 7.4: The variance of S_t

Before we turn ourselves to another example, we will make a final observation. Letting the $t \to \infty$ we can deduce

$$\lim_{t\to\infty} \mathrm{V}S_t = \frac{\sigma^2}{2\zeta}, \quad \text{and} \quad \lim_{t\to\infty} m(t) = 0,$$

therefore

$$\lim_{t\to\infty} S_t = S \sim \mathrm{N}(0, \frac{\sigma^2}{2\zeta}),$$

as we have already expected. There are a lot of books dealing with the issue of this example, such as [10], which is dedicated entirely to the Langevin equation and its applications.

7.2 Geometric Brownian motion

The second example that we will deal with is the *Geometric Brownian motion.* It is a stochastic process that follows the homogeneous linear equation with constant coefficients discussed in the subsection 6.1 in an introductory way. Now that we are equipped with the formulas from the further part of section 6, we can analyse this problem in a more sophisticated way.

First we will show how the specific terms of the equation can be interpreted and then we will do the analysis of the solution in the mathematical point of view.

The *Geometric Brownian motion* is a stochastic process that satisfies this Itô's equation

$$\begin{cases} \mathrm{d}S_t = \lambda S_t \ \mathrm{d}t + \sigma S_t \ \mathrm{d}B_t, \\ S_0 = \eta. \end{cases} \tag{7.8}$$

This model was frequently used in economics to model the asset prices. The S_t represents the price of an asset at time t and both λ and σ are both positive constants. We define as $\frac{\mathrm{d}S_t}{S_t}$ the return of the asset price at time t. The question that rises up is how can we model the returns. The classical model decomposes the returns into two parts. The first one is predictable and deterministic. It gives contribution

$$\lambda \ \mathrm{d}t,$$

where λ measures the average rate of growth of an asset price. Usually we call it the *drift* of $\frac{\mathrm{d}S_t}{S_t}$. The second contribution models the stochastic change in asset price due to the unexpected external effects and we are taking it into account by adding the term

$$\sigma \ \mathrm{d}B_t,$$

where the number σ measures the standard deviation of the returns and we call it the *volatility* of $\frac{\mathrm{d}S_t}{S_t}$. By putting all this together we obtain the stochastic differential equation (7.8). More about the economical meaning of this problem can be found in [1], we will turn to the question of its mathematical properties.

As has been said in the beginning, (7.8) is homogeneous, therefore we can use directly the formula (6.8) to solve it.

$$S_t = \eta \ \mathrm{e}^{\int_0^t (\lambda - \frac{\sigma^2}{2}) \ \mathrm{d}s + \int_0^t \sigma \ \mathrm{d}B_s} = \eta \ \mathrm{e}^{(\lambda - \frac{\sigma^2}{2})t + \sigma B_t}.$$

We obtained exactly the same solution as in the example in section 6. Let us now turn to the sample properties of S_t. By the consequence of the strong law of large numbers (theorem 3.1, property of the Brownian motion (6)),

$$\lim_{t\to\infty} \frac{1}{t} \ln S_t = \lim_{t\to\infty} \frac{1}{t} \left(\left(\lambda - \frac{\sigma^2}{2} \right) t + \sigma B_t \right) = \lambda - \frac{\sigma^2}{2} \quad a.s. \text{ if } \lambda \neq \frac{\sigma^2}{2}.$$

And if $\lambda = \frac{\sigma^2}{2}$ we show by the law of iterated logarithm (property (7) of the Brownian motion) that

$$\limsup_{t\to\infty} \frac{\ln S_t}{\sqrt{2t \ln(\ln t)}} = \sigma \quad a.s. \qquad \liminf_{t\to\infty} \frac{\ln S_t}{\sqrt{2t \ln(\ln t)}} = -\sigma \quad a.s.$$

Hence we can make the conclusion, that as $t \to \infty$

(1) $S_t \to \infty$ almost surely if $\lambda > \frac{\sigma^2}{2}$,

(2) $S_t \to 0$ almost surely if $\lambda < \frac{\sigma^2}{2}$,

(3) S_t can take arbitrary values from $[0, \infty)$ for every t if $\lambda = \frac{\sigma^2}{2}$.

We will demonstrate our conclusion on a concrete values of σ and λ. We set $\eta = 3$, $\sigma = 0.1$ and $\lambda = 1$ to obtain the solution in the figure 7.5.

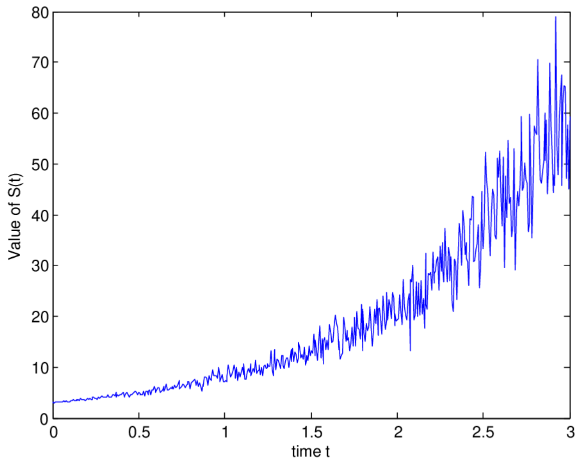

Figure 7.5: The solution trajectory for $\lambda = 1$ and $\sigma = 0.1$

In order to show the exponential decrease to 0 of S_t in the second case we had to extend the time interval t into $[0, 10^6]$, because the decrease is really slow. The solution for $\eta = 3$, $\sigma = 0.001$ and $\lambda = 10^{-8}$ is shown in the figure 7.6.

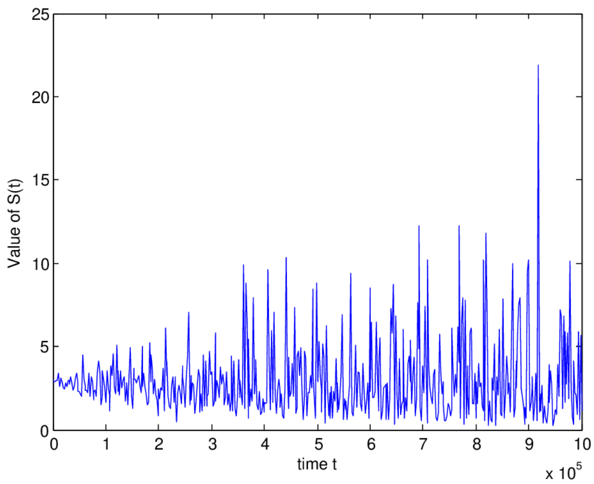

Figure 7.6: The solution trajectory for $\lambda = 10^{-8}$ and $\sigma = 0.001$

To be complete, we add the solution for the last case, i.e. $\lambda = \frac{\sigma^2}{2}$, as well. We set $\lambda = 0.005$ and $\sigma = 0.1$ and display the solution over the interval $[0, 100]$ in the figure 7.7.

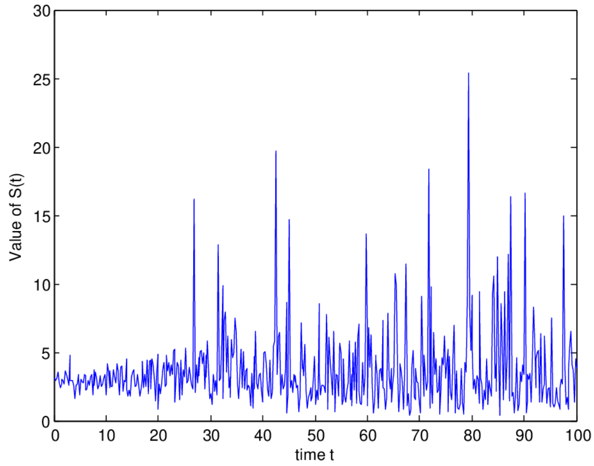

Figure 7.7: The solution trajectory for $\lambda = 0.005$ and $\sigma = 0.1$

We can advance now to the question of the expected value and variance of S_t. Let us take the equation (6.13) and use it to compute the expected value of S_t. Considering the original problem (7.8), the (6.13) gets the following form

$$\begin{cases} \dot{m}(t) = \lambda m(t) \\ m(0) = \eta, \end{cases}$$

that is easy enough to solve and we obtain

$$m(t) = \eta \mathrm{e}^{\lambda t},$$

which is the expression for the expectation of S_t and it is exactly the same result that we computed in **6.1** directly from the solution.

The figures 7.8 and 7.9 show the expected value of S_t for the values of σ and λ written in their descriptions.

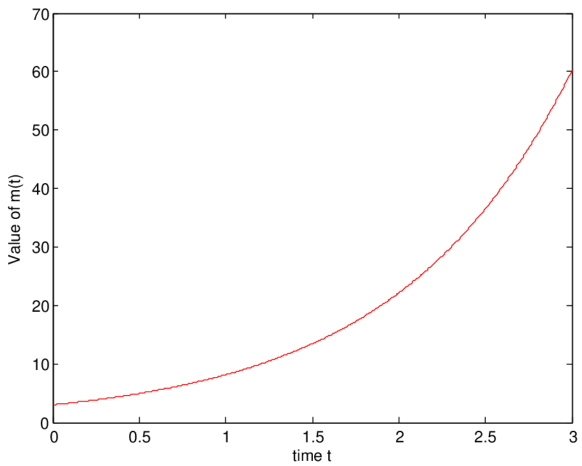

Figure 7.8: Expected value of S_t for $\lambda = 1$ and $\sigma = 0.1$

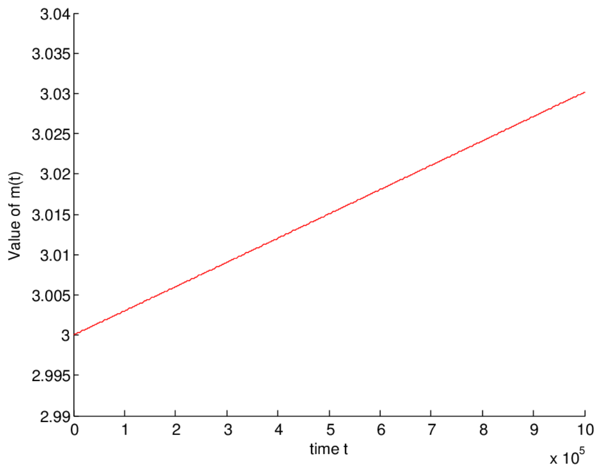

Figure 7.9: Expected value of S_t for $\lambda = 10^{-8}$ and $\sigma = 0.001$

Interesting fact about the behaviour of the expectation is that even if the solution tends to 0 (second case), its expectation still approaches infinity as $t \to \infty$.

Now we will focus on the variance of S_t. The equation for the second moment (6.16) can be modified to

$$\begin{cases} \dot{P}(t) = (2\lambda + \sigma^2)\, P(t), \\ P(0) = \eta^2, \end{cases}$$

which has explicit solution

$$P(t) = \eta^2 \mathrm{e}^{\left(2\lambda + \sigma^2\right)t}.$$

The variance is then by (6.17)

$$\begin{aligned} \mathrm{V}S_t &= P(t) - m^2(t) = \eta^2 \mathrm{e}^{\left(2\lambda+\sigma^2\right)t} - \eta^2 \mathrm{e}^{2\lambda t}, \\ \mathrm{V}S_t &= \eta^2 \mathrm{e}^{2\lambda t}\left(\mathrm{e}^{\sigma^2 t} - 1\right). \end{aligned}$$

and the standard deviation is given by

$$\sigma_d = \eta\ \mathrm{e}^{\lambda t}\sqrt{\mathrm{e}^{\sigma^2 t} - 1}.$$

The $\mathrm{V}S_t$ is always increasing, because both λ and σ are greater than zero. The figure 7.10 shows the variance of S_t for the values specified in its description. For completeness we are also adding the figure 7.11 where we display a sample path of the solution together with its expectation $\mathrm{E}S_t$ and $\mathrm{E}S_t \pm \sigma_d$.

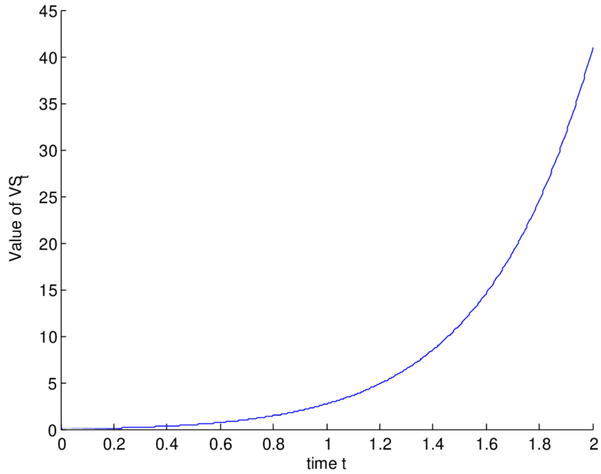

Figure 7.10: Variance of S_t for $\lambda = 1$ and $\sigma = 0.2$

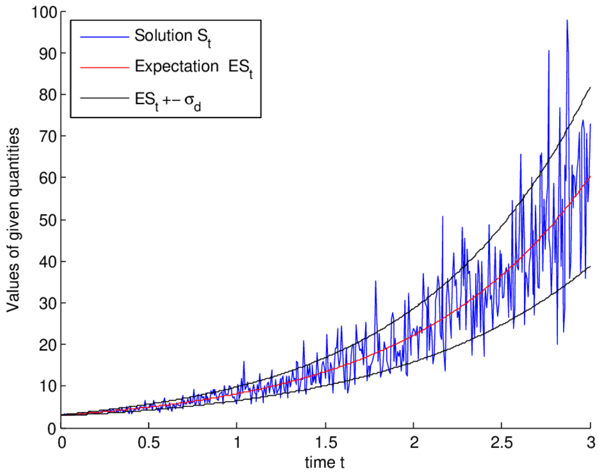

Figure 7.11: The solution S_t together with its characteristics for $\lambda = 1$ and $\sigma = 0.2$

We have seen two examples of equations with constant coefficients when the solution basically either went to infinity or to zero, depending on the values of the coefficients. The next example will present some sort of oscillation in the solution.

7.3 Oscillating process

Let us consider the following initial value problem for stochastic differential equation

$$\begin{cases} \mathrm{d}X_t = \lambda \cos(t) X_t \ \mathrm{d}t + \sigma X_t \mathrm{d}B_t, \\ X_0 = \eta. \end{cases} \tag{7.9}$$

The assumptions of existence and uniqueness are satisfied since $\cos(t)$ is a bounded function and λ, σ are bounded constants. (7.9) is a homogeneous equation so we can use directly (6.8) to obtain the solution. It is

$$X_t = \eta \mathrm{e}^{\lambda \sin(t) - \frac{\sigma^2}{2} t + \sigma B_t}. \tag{7.10}$$

The figure 7.12 visualizes a sample paths of the solution (7.10) over the time interval $[0, 10]$ for the $\eta = 3$, $\sigma = 0.1$ and for three values of λ given in the legend.

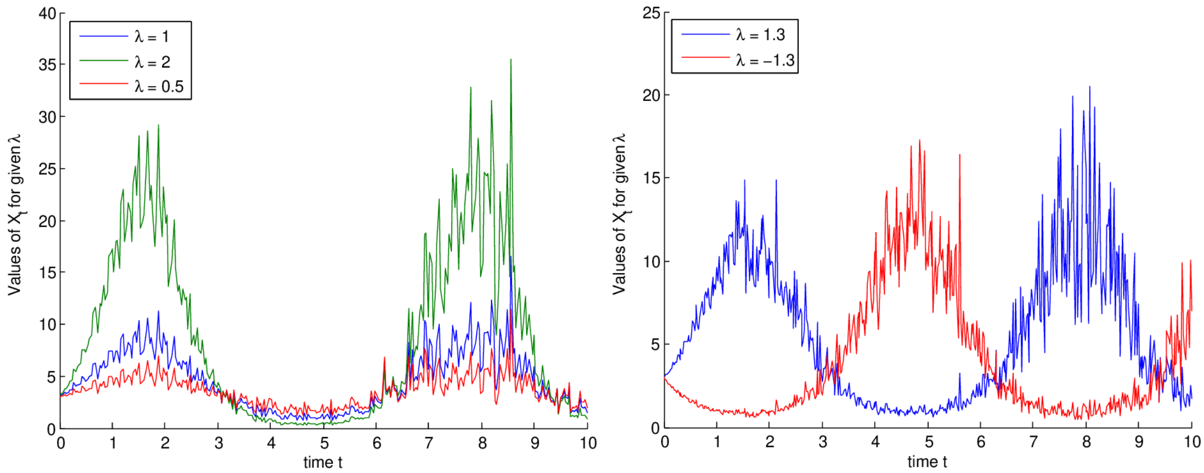

Figure 7.12: Oscillating process

We can guess now the influence of parameter λ. It drives the deterministic part of (7.10) and therefore the higher is the $|\lambda|$ the higher are the peaks in (7.10). We can also observe that the fluctuations due to the stochastic part are the highest at those peaks. Taking the negative value of λ generates the phase shift of size π.

Let us now focus on the parameter σ, since it drives the stochastic part of (7.10), it affects the size of fluctuations, as we can see in the figure 7.13.

We can advance and compute the expected value of (7.10). The equation (6.13) reaches

$$\begin{cases} \dot{m}(t) = \lambda \cos(t) m(t), \\ m(0) = \eta, \end{cases}$$

which has solution

$$m(t) = \eta \mathrm{e}^{\lambda \cos(t)}. \tag{7.11}$$

We can see, that X_t approaches $m(t)$ as $\sigma \to 0$. The expectation for $\lambda = 1$ is displayed in the figure 7.14.

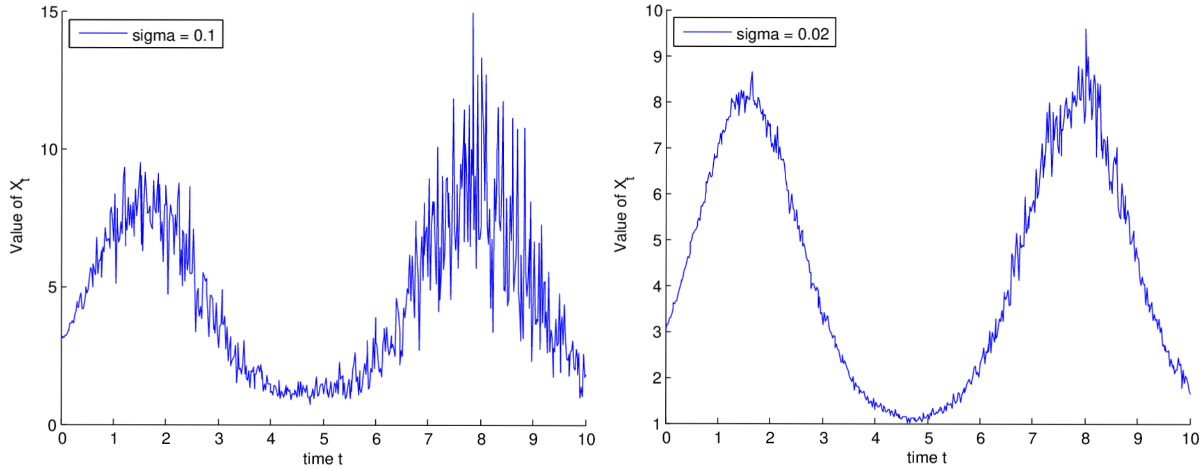

Figure 7.13: The sample paths of the solution X_t for $\sigma = 0.1$ and $\sigma = 0.02$

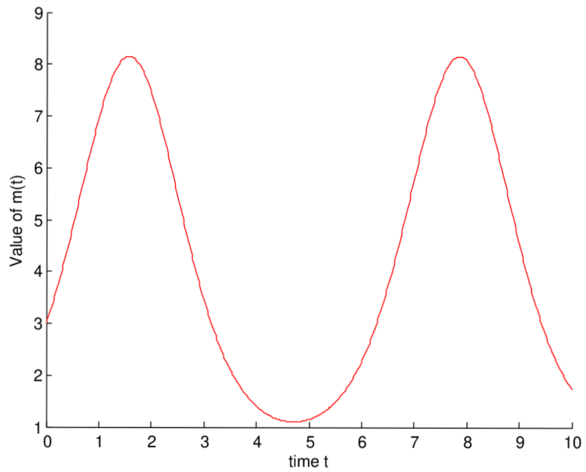

Figure 7.14: The expectation of X_t

Now we will compute the variance. The equation for the second moment is

$$\begin{cases} \dot{P}(t) = (2\lambda\cos(t) + \sigma^2)\, P(t), \\ P(0) = \eta^2, \end{cases}$$

so that its solution is

$$P(t) = \eta^2 e^{2\lambda \sin(t) + \sigma^2 t}. \tag{7.12}$$

Taking (7.12) and (7.11) we compute the variance of X_t

$$\mathrm{V}X_t = P(t) - m^2(t) = \eta^2 e^{2\lambda\sin(t)} \left(e^{\sigma^2 t} - 1\right).$$

The behaviour of $\mathrm{V}X_t$ is shown in the figure 7.15.

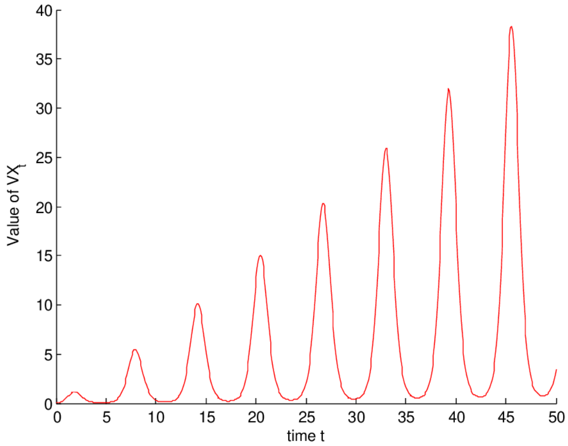

Figure 7.15: The variance of the oscillating process for $\lambda = 1$ and $\sigma = 0.1$

The standard deviation is

$$\sigma_d = \sqrt{\mathrm{V}X_t} = \eta \mathrm{e}^{\lambda \sin(t)} \sqrt{e^{\sigma^2 t} - 1}.$$

The last figure 7.16 puts together the solution, the expected value and $\mathrm{E}X_t \pm \sigma_d$ for $\lambda = 1$, $\sigma = 0.1$ and $\eta = 3$.

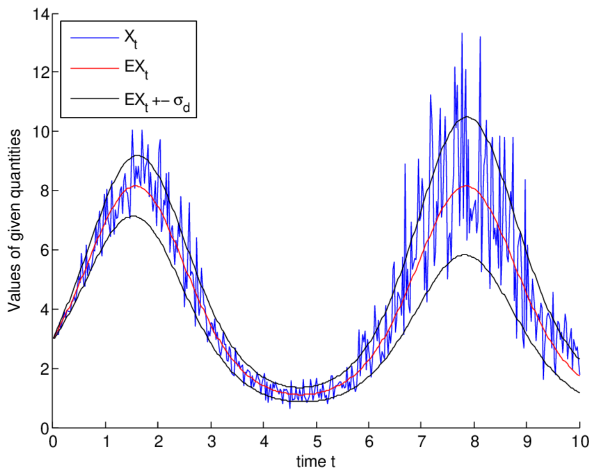

Figure 7.16: The sample path of the solution X_t fitted with its expectation and $\mathrm{E}X_t \pm \sigma_d$

7.4 Brownian bridge

The last example that we are giving is the so called *Brownian bridge*. It is the stochastic process that satisfies the initial value problem (7.13) on the interval $[0,1)$.

$$\begin{cases} \mathrm{d}X_t = \frac{b-X_t}{1-t}\,\mathrm{d}t + \mathrm{d}B_t, \\ X_0 = a. \end{cases} \tag{7.13}$$

We have to work only on the interval $[0,1)$ because for $t=1$ the $f(X_t,t)=\frac{b-X_t}{1-t}$ increases over every bound, which violates the existence and uniqueness theorem. Since (7.13) is not homogeneous, we first have to solve the corresponding homogeneous problem.

$$\begin{cases} \mathrm{d}Y_t = \frac{-Y_t}{1-t}\,\mathrm{d}t, \\ Y_0 = 1, \end{cases}$$

which is an initial value problem for an ordinary differential equation and its solution is

$$Y_t = 1-t.$$

Taking the formula (6.10) we obtain

$$X_t = (1-t)\left(a + \int_0^t \frac{b}{(1-s)^2}\,\mathrm{d}s + \int_0^t \frac{1}{1-s}\,\mathrm{d}B_s\right),$$

$$X_t = a(1-t) + b(1-t)\left[\frac{1}{1-s}\right]_0^t + (1-t)\int_0^t \frac{1}{1-s}\,\mathrm{d}B_s,$$

so the final expression for the solution of 7.13 is

$$X_t = a(1-t) + bt + I(t), \tag{7.14}$$

where

$$I(t) = (1-t)\int_0^t \frac{1}{1-s}\,\mathrm{d}B_s$$

is normally distributed with expectation

$$\mathrm{E}I(t) = 0 \tag{7.15}$$

and variance given by

$$\mathrm{V}I(t) = \mathrm{E}(1-t)^2\left(\int_0^t \frac{1}{1-s}\,\mathrm{d}B_s\right)^2 = (1-t)^2\int_0^t \frac{1}{(1-s)^2}\,\mathrm{d}s,$$

$$\mathrm{V}I(t) = t(1-t). \tag{7.16}$$

Since the variances of $a(1-t)$ and bt are equal to zero because they are deterministic functions, the (7.16) is also the variance of X_t. and the standard deviation of X_t is

$$\sigma_d = \sqrt{t(1-t)};$$

When we look at (7.14), we can easily deduce that in this case we don't need the equation (6.13) to obtain the expectation of X_t. It is straightforward that

$$\mathrm{E}X_t = a(1-t) + bt + \mathrm{E}\left((1-t)\int_0^t \frac{1}{1-s}\,\mathrm{d}B_s\right).$$

and since (7.15) holds,

$$\mathrm{E}X_t = a(1-t) + bt.$$

In the figure 7.17 we can see a trajectory of the Brownian bridge from $a = 1$ to $b = 2$ together with its expected value.

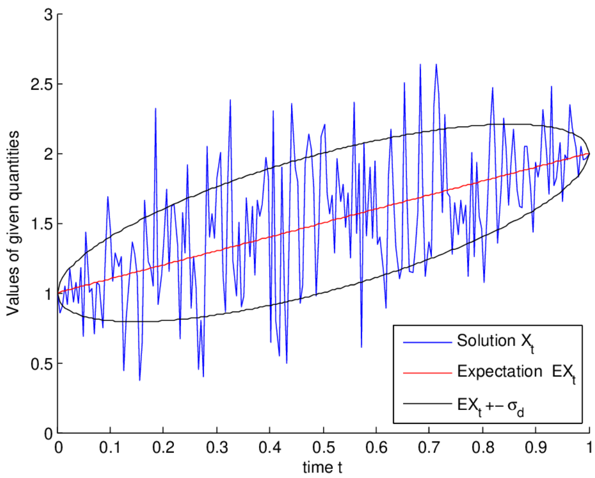

Figure 7.17: Brownian bridge

At the end we will make a remark about the name of this process. It is obvious that the value of X_t at $t = 0$ is a and its value at $t = 1$ is b and the process in between creates some sort of bridge from a to b. Since this "bridge" is normally distributed, it got the name Brownian bridge.

8 Conclusion

The goal of the thesis was to deal with the issue of stochastic differential equations. We devoted sections 2 and 3 to summarizing the probability theory and the theory of stochastic processes. Then we showed the construction of Itô's Integral and we stated its useful properties. We also established the most important formula in the stochastic analysis, so called *Itô's formula*, that we used to obtain important results. After that we actually started to deal with the stochastic differential equations. We defined what the solution is and we stated the theorem of existence and uniqueness of it. Then we focused on the special case of linear equations. We derived the general formula for the solution of linear stochastic differential equation. We also obtained the equations for the first and second moment of the solution as well. In order to do that, we had to transform the integral theorems of Fubini and Tonelli into their stochastic versions and use them in the process of deriving these equations. These results enabled us to compute the expected value and variance of the solution without the need of computing them directly from the it.

In the last section we used the results from the previous text and treated the specific problems. We also visualized our results in order to make them more clear. The first of them was the *Langevin* equation. It turned out that its solution tends to normally distributed random variable. The second example was the *Geometric brownian motion.* Also in this case we did a detailed analysis of its solution and we found out an interesting property of it, i.e. that for specific values of the coefficients of the equation, the solution tends to 0 as time tends to infinity while its expected value grows to infinity. We showed also an example with an oscillation in the solution. The last example was the *Brownian bridge* between a and b.

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