MONTE CARLO SIMULATIONS APPLIED TO UNCERTAINTY IN MEASUREMENT

University of Hradec Králové
Faculty of Science

Tomáš Vodička

Faculty of Science

Course on Physical Measurements and Modelling

Department of Physics

University of Hradec Králové

January 2019

Univerzita Hradec Králové
Přírodovědecká fakulta

Zadání diplomové práce

Autor:	**Bc. Tomáš Vodička**
Studium:	S14FY002NP
Studijní program:	N1701 Fyzika
Studijní obor:	Fyzikální měření a modelování
Název diplomové práce:	**Monte Carlo Simulations Applied to Uncertainty in Measurement**
Název diplomové práce AJ:	Monte Carlo Simulations Applied to Uncertainty in Measurement

Cíl, metody, literatura, předpoklady:
Cílem práce je stanovit nejistoty pomocí metody Monte Carlo. V teoretické části Diplomové práce budou shrnuty typy nejistot měření, jejich stanovení, principy a vlastnosti. Další část práce bude obsahovat úvod do metody Monte Carlo a její aplikace. V praktické části bude zpracováno konkrétní měření a následně z daných výsledků stanoveny nejistoty měření nejprve analyticky, poté uvedenou metodou. Výsledky budou porovnány s výpočtem podle teoretických předpokladů. Závěrem budou výsledky porovnání diskutovány a zhodnoceny.

Garantující pracoviště:	Katedra fyziky, Přírodovědecká fakulta
Vedoucí práce:	Mgr. Josef Horálek, Ph.D.
Oponent:	RNDr. Jiří Lipovský, Ph.D.

Datum zadání závěrečné práce: 31.8.2015

DECLARATION

This Diploma Thesis is the result of my own work and does not include anything done in collaboration with others, except where explicitly stated in the text. It has not been previously submitted, in part or whole, to any university of the institution for any degree, or other qualification.

In Hradec Králové

(signature)

Annotation

This diploma thesis focuses on calculation of uncertainties by Monte Carlo method. The introduction provides an essential theoretical overview of measurement uncertainties. The diploma thesis also deals with the analysis of the Monte Carlo method and its application in solving the physical models of measurement from a different part of a physics. In conclusion are the calculated uncertainties compared to results that came from the Monte Carlo Simulation.

Keywords

Uncertainty of Measurement, Monte Carlo Simulation, Metrology.

Anotace

Tato diplomová práce je zaměřena na výpočet nejistot metodou Monte Carlo. V úvodu poskytuje základní teoretický přehled pro nejistoty měření. Dále pak diplomová práce uvadí rozbor metody Monte Carlo a její aplikaci při řešení fyzikálních modelů měření z různých odvětví fyziky. V závěru jsou vypočítané nejistoty porovnány s výsledky Monte Carlo Simulace

Klíčová slova

Nejistota měření, metoda Monte Carlo, Metrologie.

Vodička, T. *Monte Carlo Simulation Applied to Uncertainty in Measurement*. Hradec Králové, 2019. Diploma Thesis at Faculty of Science University of Hradec Králové. Thesis Supervisor Josef Horálek. 69 p.

ACKNOWLEDGMENTS

I would like to thank my supervisor for my diploma thesis Mgr. Josef Horálek, Ph.D. for willingness, time spent consultations and expert advice in its preparation.

CONTENTS

LIST OF TABLES

LIST OF FIGURES

List of Abbreviations and Acronyms

n	number of input quantity
M	number of Monte Carlo trials
Y	output quantity, the random output quantity for the Monte Carlo method
y	estimate output quantities, the mean value of the random output quantity
Q	input quantity
Q_i	i-th input quantity
q	Estimation of Q
q_i	Estimate of Q_i
$\bar{q}$	the arithmetic average of the measured values
$u_A(q)$	type A standard uncertainty
$u_B(q)$	type B standard uncertainty
$u_C(q)$	combined uncertainty
$U(q)$	expand uncertainty
k	coverage factor
s^2	sample variance
c_i	sensitivity coefficient
χ	coefficient probability distribution
A_{max}	the maximum deviation of the sources of uncertainty
$\tilde{y}$	output estimate for the Monte Carlo
$u(\tilde{y})$	the associated standard uncertainty for the Monte Carlo
η	the possible value of the random variable Y
$g(\xi_i)$	probability density values X_i
ξ_i	the possible value of the random variable X_i
G	the discrete distribution function
$r(q_i, q_k)$	correlation coefficient
$s(\bar{p}, \bar{q})$	covariance between the estimate of the input $\bar{p}, \bar{q}$.
$cov(\bar{p}, \bar{q})$	covariance between the estimate of the input $\bar{p}, \bar{q}$.
p	coverage probability
y_{low}	left hand endpoint of coverage interval
y_{high}	right hand endpoint of coverage interval
$\propto$	parameter of the probability
$[\gamma_1, \gamma_2]$	pair of random numbers
y_r	r-th model value
q_r	r-th Monte Carlo draw from the probability density functions (PDF)

1 INTRODUCTION

Metrology is a science that deals with the measurement of various technical and physical quantities. If the same physical quantity is repeated with the same conditions several times in succession, different values are usually given. However, the measured value is one correct value. Each deviation of the measured value of the correct value is generally called an error. Based on the demand for a more versatile approach to measuring accuracy, the International Commission has formulated a new approach that generalizes the concept of error and is called uncertainty. Measurement uncertainty, unlike an error, characterizes the range of the measured value around the measurement result, which as expected contains the actual value of the measured quantity. The basis for determining uncertainties is the statistical approach for random and systematic errors. A specific probability distribution, which describes how the measured value values, can deviate from the actual value is assumed. Rules for calculating uncertainties are described in Chapter 2 and are based on the familiar Guide to the expression of Uncertainty in Measure (GUM). Due to many limitations, the GUM method has been extended by the first supplement, which provides a calculation of uncertainties by the Monte Carlo method.

The Monte Carlo method is a numerical computational method based on the use of random variables and probability theory. The Monte Carlo method is a class of algorithms for system simulation. It is a method using random or pseudo-random numbers. It uses a repeated sampling of random variables to simulate random events. An introduction to the Monte Carlo method, including representative examples and history, is described in Chapter 3. Based on the first supplement of the GUM method, a calculation of the uncertainties by the Monte Carlo method was formulated in Chapter 4.

There are five case studies from different branches of physics. Each case study is specific to its approach to best demonstrate the difficulty of calculating GUM uncertainties. The calculations focus primarily on the determination of sensitivity coefficients, correlations, covariance and coverage intervals. The Monte Carlo method was calculated for individual case studies according to the algorithm presented in the appendix of the thesis. For individual variables according to the first GUM appendix, random M values were generated according to the division corresponding to the given measurement. The resulting uncertainty and coverage range were determined from the generated values. The results of both methods were compared and commented. In conclusion, depending on individual case studies, the advantages of the Monte Carlo method are described compared to the classical GUM method.

1.1 Terminology and basic concepts

To understand the problems in the field of metrology, it is essential to explain some basic concepts. The basic ideas with definitions can be found in the International Vocabulary of Metrology - Basic and general concepts and associated terms (1) Augmented definition for uncertainty can be found in Guide to the expression of uncertainty in measurement (2) where the rules of evaluating and expressing uncertainty in measurement (GUM) are described. The base of the GUM was used to prepare the essential concepts of this work. The Propagation of distributions, using the Monte Carlo method is clarified in Supplement 1 and the Guide to the expression of uncertainty in measurement (3).

Quantity. "*Property of a phenomenon, body or substance, where the property has a magnitude that can be expressed as a number and a reference*." It can be measured or counted. The value of a given quantity is given by comparison with a fixed value of the quantity of the same kind we choose for the measuring unit. For example, the length of an object can be determined by comparing it to an object of known length, such as a ruler.

True quantity value. "*Quantity value consistent with the definition of quantity*." The true quantity value is the value corresponding to the real dimension of the measured quantity value. It is a one-dimensional value for continuous quantities with a large number of decimal places, which we obtain with exact measurements at high demands on the technical maturity of the gauges, the competence of the staff and the measurement time.

Reference quantity value. “*Quantity value, used as a basis for comparison with values of quantities of the same kind*.” In other words, it is a specific value affecting the quantities to which we measure the result.

Measured quantity value. “*Quantity value is representing a measurement result*.” It is the quantity value that is measured in practice, being represented as a measurement result. Available measurement techniques can measure the length of an object.

Measurement result. “*Set of quantity values being attributed to a measurand together with any other available relevant information*.” The measured quantity value is affected by the systematic and random effects that together determine the measurement error. The measured value is the data subtracted from the gauge, which is given by the true quantity value of the measured dimension and the instantaneous magnitude of measurement errors. For example, the result of the length of an object is represented by a measurement result: 126 mm $\pm$ 0.2 mm.

Measurement accuracy. “*Closeness of agreement between a measured quantity value and the true quantity value of the measurand*.” Measurement accuracy is a metrological concept whose statistical measure is the measurement error. It tells us the difference between what we measured and what we should measure.

The sensitivity of measuring system. “*Quotient of the change in an indication of a measuring system and the corresponding change in a value of a quantity being measured*.” The sensitivity indicates the minimum change in the measured value the meter is capable of indicating.

Resolution. “*Smallest change in a quantity being measured that causes a perceptible change in the corresponding indication*.” For modern gauges, the resolution is given by the least significant bits of the imaging device

Measurement uncertainty. "*Non-negative parameter characterizing the dispersion of the quantity values being attributed to a measurand, based on the information used.*" In other words, is the designation for a parameter that is related to outcome measurements and characterizes the range of values that can be rationally assigned to the measurand. It is a new methodology for processing measurement results.

Coverage interval. "*Interval containing the set of true quantity values of a measurand with a stated probability, based on the information available*." This parameter provides limits within which the true quantity values may be found with a determined probability. So for the length of an object example, there could be 95% probability of finding the true value of the length within the interval of 125.8 mm to 126.2 mm.

2 UNCERTAINTY OF MEASUREMENT

The purpose of measurement is to determine the size of the measured variables, characterizing a particular specific property. Specifications of measured values can also require data on other variables such as time, temperature or force. Individual measurements are usually burdened by variations of noise, known as errors. Results are expressed using appropriate estimates of mean μ and the corresponding uncertainty associated with noise.

Classical statistics, based on the definition of probability as relative frequency limits, provides all equipment the expression of uncertainty as to the confidence interval of the parameter μ. Expressing uncertainty is philosophically closer to subjective definition of probability as the degree of trust or faith. This probability, however, has more to do with lack of knowledge than the result of the repeated experiment.

Until now, it was customary in the evaluation of measurement data to work with errors. New evaluation is done through the expression of uncertainty in measurement. Let us now speak briefly about the foundations of the theory of errors, so that they can better compare with the new concept of uncertainty, which replaces the concept of errors.

Measurement uncertainty characterizes the range of measured values about the result of the measurement, which can be assigned to the value of the measured quantity. Measurement uncertainty concerns the result of a measurement, measuring devices, the values used constants, corrections, and so on. To which the uncertainty of the measurement result depends. The basis of measurement uncertainty is the statistical approach to an assumed particular probability distribution which describes how the indicated value can deviate from the actual value. (4) (5)

2.1 Measurement Model

The measurement model is defined as the relationship between the input and output variables for a given measurement. It represents not only the principle, procedure, and method of measurement but also the influence of the environment in which the measurement takes place or the knowledge and experience of the worker carrying out the measurement.

The measurands are the particular quantities subject to measurement as the dependence of the output quantities Y on the input quantities Q_i. The measurands are considered those that are the target of measurement. Usually, one output quantity Y depends on a number of input quantities $Q_i (i = 1, 2, \dots, N)$ according to the function depends in (2.1).

$$Y = f(Q_1, Q_2, \dots, Q_N) \tag{2.1}$$

The function f can represent the measurement procedure and the measurement method and describes how the values of the output quantity Y and their estimation y are determined from the values of the input variables Q_i and their estimates q_i.

2.2 The Uncertainty of Direct Measurement

To determine the magnitude of the uncertainty principle two following methods are possible. Type A of standard uncertainty and type B of standard uncertainty.

Random errors mostly cause type A standard uncertainty and determine the static analysis of the measured values obtained under precisely defined measurement conditions. Here applies a mathematical and statistical approach.

Known or estimable causes, cause type B standard uncertainty. These establish the procedures that are not defined in the standard. For more complex installations, requiring increased accuracy it is necessary to perform a detailed analysis of errors and fix them accordingly uncertainty type B. The resulting standard uncertainty u_B is determined by their geometric sum. The sum of the squares of the standard uncertainty of type A and the resulting standard uncertainty of type B is obtained so. Combined standard uncertainty u_C. (6) (7)

2.2.1 Type A evaluation of standard uncertainty

Type A evaluation of standard uncertainty is determined from repeated measurements of the same measured value under the same conditions. The uncertainties are reduced with an increasing number of repeated measurements. Random errors are expected with a normal distribution. (6)

Normal distribution assumes the existence of a primary distribution of repeatability and random errors, illustrated in figure 2-1. Where samples are taken, the sample means and sample standard deviation are calculated, it is assumed these represent the mean and standard deviation of the population distribution. However, this equivalence is only approximate. It is responsible for Student distribution used instead of the normal distribution to calculate confidence limits around the sample mean. (5)

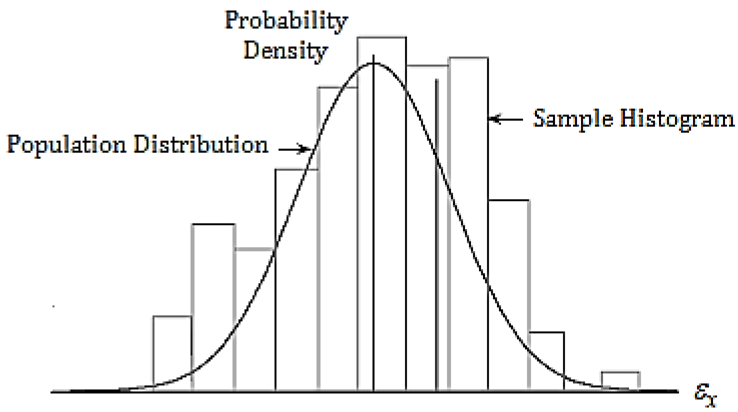


Figure 2-1. Repeatability Distribution

Estimation data measured variable y is given by sample meaning $\bar{q}$ from measured value q_i by relationship (2.2).

$$\bar{q} = \frac{1}{n}\sum_{i=1}^{n} q_i \tag{2.2}$$

Where $\bar{q}$ is selective arithmetic average, n is the number of trials and q_i is the individual measured values. The estimated variance of the measured values, referred to as sample variance $s^2(x)$ is given by (2.3).

$$s^2(q_i) = \frac{1}{n-1}\sum_{i=1}^{n}(q_i - \bar{q})^2 \tag{2.3}$$

where s^2 is the sample variance. Root of the sample variance is obtained from sample standard deviation $s(q_i)$, which characterizes the variance of the measured values around the sample mean q. Sample variance mean $s^2(\bar{q})$ is given by (2.4).

$$s^2(\bar{q}) = \frac{s^2(q_i)}{n} \tag{2.4}$$

When the number n of repeated measurements is lower than ten, the reliability of a Type A evaluation of standard uncertainty, as expressed by equation (2.5).

$$u_A(q) = s(\bar{q}) = \sqrt{\frac{1}{n(n-1)}\sum_{i=1}^{n}(q_i - \bar{q})^2} \tag{2.5}$$

Where$u(\bar{q})$ is associated with the input estimate $\bar{q}$ is the experimental standard deviation of the mean.

From equation (2.5) it follows that $u_A(q)$ will be the smaller the more repeated measurements n. Assuming not enough repetitive measurements, we can estimate $u_A(q)$ if the number of measurements is less than 10. If the data comes from the normal distribution, follow equation (2.6).

$$u_A(q) = k_{u_A}\sqrt{\frac{1}{n(n-1)}\sum_{i=1}^{n}(q_i - \bar{q})^2} \tag{2.6}$$

Where k_{u_A} is a safety factor, the size of which is given in the table 2-1.

Table 2-1. Safety factor table for an estimate $u_A(q)$.

n	2	3	4	5	6	7	8	9	10 +
k_{u_A}	7,0	2,3	1,7	1,4	1,3	1,3	1,2	1,2	1

From the methodological point of view, it is not recommended to perform repeat measurements of less than 10 for the determination of type A uncertainties. It is also not advisable to perform measurements in an environment where the measurement is not usually performed. (6) (8)

2.2.2 Type B evaluation of standard uncertainty

In addition to the influences described by the Type A method, the measurement system also has sources that can be traced to contextual cause, size of variability and static behavior without the need for repeated measurements. Non-statistical methods determine sources of uncertainty type B. Their amount depends on the operator's decision. Repeated measurements can not reduce Their influence.

Measurands are based on rational judgments and use all available information on the measuring chain, method, and other factors, which may affect measurement results. Properties of type B sources of uncertainty of are as follows:

- are determined by non-static methods,
- their amount depends on the operator's decision,
- repeated measurements cannot reduce their influence.

Sources of uncertainty type B can be divided into two groups: variable systematic effect and random sources with known variability.

Variable systematic effects are knowable, quantifiable, workable, but their effect on the system is not stable. They have a permanent and variable component. The permanent component of a systematic effect has been compensated. For the processing of type B uncertainties only variable component apply. The sources are the influence of weather conditions, the effect of calibration, the impact of the sensor storage and the effect of the measuring cables.

Random sources with known variability are determined by estimation based on static distribution and probability interval. The sources are the influence of the resolution of the meter and the influence of the constants in the calculation of the indirectly measured quantity. (9)

Standard uncertainty type B, can be calculated in several ways. Only the way of estimating the source variability and the static distribution that is used in the practical part can be discussed here.

Calculation of type B uncertainty, is based on the partial uncertainties individual sources, where the value A_{max} is the maximum deviation of the sources of uncertainty.

$$U_{BZ}(q) = \frac{A_{max}}{\chi} \tag{2.7}$$

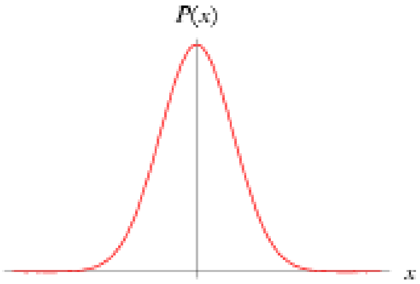
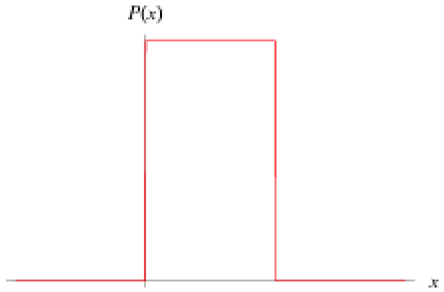
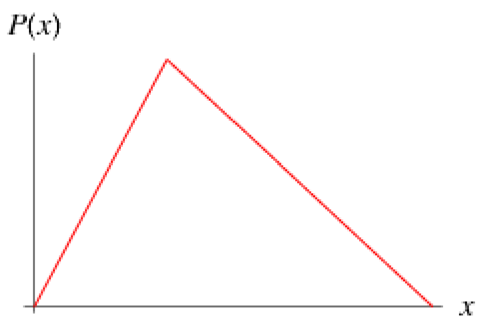
The equation (2.7) is an expression of standard calculation deviation another way, than the static method. As was said A_{max} describe the n spread of uncertainty sources and χ is the coefficient corresponding to the selected approximation of the probability distribution described in table 2-2. These probability distributions are the most known.

Numerically, the source of uncertainty can be determined by a separate measurement, such as the average value from repeated measurements, the technical conditions of measurement or an estimate.

The total uncertainty of type B is given by the geometric sum of the individual resources in the case that the individual sources are uncorrelated. (9)

$$u_B^2(q) = \sum_{i=1}^{n} u_{BZ}^2(q) \tag{2.8}$$

Table 2-2. Uncertainty Equations for Selected Distributions χ.

Distribution	Distribution Plot	Coefficient χ
Normal		2 $(P = 95\ \%)$
Uniform		$\sqrt{3} \approx 1.73$ $(P = 100\ \%)$
Triangular		$\sqrt{6} \approx 2.45$ $(P = 100\ \%)$

Uniform distribution is used in cases where any deviation from the nominal value may occur with equal probability. Uniform distribution is the most common approximation in practice. In the evaluation of standard uncertainty type B, it is often the only available information an interval $[a, b]$, then knowledge of the magnitude is characterized by the uniform distribution. Approximation normal or triangular distribution is used when there are more small deviations from the nominal value and the increasing size of deviation decreases the probability of their occurrence. (8) (11)

2.2.3 Combined standard uncertainty

The combined standard uncertainty $u_c(q)$ is a measure of uncertainty of the result obtained from a number of other variables. This is an estimate of the standard deviation associated with the result, which is equal to the positive square root of the combined variance obtained from all variance input variables and all potential covariates. Procedure for the determination of the combined standard uncertainty is different for uncorrelated and correlated to other variables.

For uncorrelated quantities, the combined standard uncertainty $u_c(q)$ is determined as the positive square root of the variance of the combined standard uncertainty, which is determined according to the relationship (2.9).

$$u_C(q) = \sqrt{u_A^2(q) + u_B^2(q)} \tag{2.9}$$

Where $u_A(q)$ is a type A of standard uncertainty, $u_B(q)$ is a type B of standard uncertainty. By obtaining equation (2.5) to (2.8) gets

$$u_C(q) = \sqrt{u_A^2(q) + \sum_{i=1}^{n} u_{BZ}^2(q)} \tag{2.10}$$

The equation (2.10) points to merging both uncertainties and comparing their magnitudes. In conclusion, if $u_A(q)$ will be significantly higher than $u_B(q)$, may be assumed that the measurement system dominated by random effects, and should be in the context of measures to improve the focus on these influences. If $u_B(q)$ will be significantly higher than $u_A(q)$ it may be assumed that either there is an improperly designed system of measurement in the system or a dominant source of the type B. This fact again gives instructions to improve the measurement system.

Numerically uncertainty is equal to the standard deviation of measurement variability investigated. (6) (8)

2.3 The Uncertainty of Indirect Measurement

Evaluation of the uncertainty of measurement of output estimates can be calculated on the base of the particular operations uncertainty measurement of output.

Quantity Y, representative output value, known as function of values $Q_1, Q_2, \dots, Q_m$. This quantity can be directly measured or whose estimates, uncertainties and covariance is known from other sources. Then we can write

$$Y = f(Q_1, Q_2, \dots, Q_m) \tag{2.11}$$

Estimate y output quantity of Y is given by equation (2.12)

$$y = f(q_1, q_2, \dots, q_m) \tag{2.12}$$

For uncorrelated input quantities, the square of the standard uncertainty associated with the output estimate y is given equation (2.13).

$$u^2(y) = \sum_{i=1}^{N} c_i^2 u^2(q_i) \tag{2.13}$$

Where $u(y)$ is an uncertainty estimate of output quantity, $u(x_i)$ is uncertainty estimate of input quantity and c_i is sensitivity coefficient. Furthermore equation (2.14) describe sensitivity coefficient.

$$c_i = \frac{\partial f}{\partial q_i} = \left.\frac{\partial f}{\partial Q_i}\right|_{Q_i = q_i \dots Q_N = q_N} \tag{2.14}$$

In case estimates are correlated, it is necessary to consider the covariance between the estimates, which are another component of the resulting uncertainty. Correlating input variables to output variables and uncertainty, is determined from the relation (2.15)

$$u^2(y) = \sum_{i=1}^{N} c_i^2 u^2(q_i) + 2 \sum_{i=2}^{N} \sum_{k<i}^{N-1} c_i c_k \, u(q_i, q_k) \tag{2.15}$$

Where $u(q_i, q_k)$ is the covariance between the mutually correlated estimates q_i and q_k, which can be as two interdependent different variables, as well as two values of the same magnitude, between which there is a certain correlation bond. (6) (11)

2.4 Correlated quantities

Correlated quantities are assumed that resources are infinitely independent (uncorrelated). It means that individual resources do not have a standard mathematical or physical basis, a standard organizational or technical base — covariance expected value of the product of the deviations of two random variables from their respective means. The covariance of two independent random variables is zero. (5)

If two input quantities Q_i and Q_k are known to be correlated. The covariance associated with the two estimates is q_i and q_k. In the case of available n measured values of two quantities P and Q the estimates are presented by arithmetic means $\bar{p}$ and $\bar{q}$ is given by (2.16).

$$s(\bar{p}, \bar{q}) = \frac{1}{n(n-1)} \sum_{j=1}^{n} (p_j - \bar{p})(q_j - \bar{q}) \qquad (2.16)$$

Where $s(\bar{p}, \bar{q})$ is covariance between the estimated input $\bar{p}, \bar{q}$, n is the number of measured value and p_j, q_j is the estimate of two input quantities.

In a case without the available n measured values of two quantities, for each source of each pair estimated to estimate the correlation coefficient $r(q_i, q_k)$, where $i \neq k$ and $|r| \leq 1$ and showing the degree of correlation characterized between the estimates. Values close to zero correspond to a weak dependence, values close to one strong a dependence. Relevant covariance value is determined from the relationship

$$u_B(q_i, q_k) = u(q_i)u(q_k)r(q_i, q_k) \qquad (2.17)$$

If the two input variables Q_1 and Q_2 with estimates by q_1 and q_2 are functions of the independent variables Q_i can be expressed by relations (2.18).

$$Q_1 = g_1(Q_1, Q_2, \dots, Q_L)$$
$$Q_2 = g_2(Q_1, Q_2, \dots, Q_L) \tag{2.18}$$

Furthermore, determine the covariance between the estimates q_1 and q_2 provided the estimates $q_i(l = 1,2, \dots, L)$ are uncorrelated. In this case the covariance is given by equation (2.19).

$$u(q_1, q_2) = \sum_{l=1}^{L} c_{1l} c_{2l} u^2(q_\mathrm{i}) \tag{2.19}$$

Where c_{1l} and $c_{2l}$are the sensitivity coefficients derived from the functions $g_1$and g_2.

If the two input variables Q_1 and Q_2 with estimates by q_1 and q_2 are functions of the dependent variables Q_i can be expressed by relations 2.20.

$$u^2(y) = \sum_{i=1}^{N} u_i^2(y) + 2 \sum_{i=1}^{N-1} \sum_{k=i+1}^{N} u_i(y)\, u_k\, (y) r(q_i, q_k) \tag{2.20}$$

Where $r(q_i, q_k)$ is the correlation coefficient, $u_i(y)$is the standard uncertainty of the output estimate y resulting from the standard uncertainty of the input estimate q_i given by

$$u_i(y) = c_i u(q_i) \tag{2.21}$$

If you cannot determine the correlation coefficient it is recommended to determine the maximum correlation effects on the resulting uncertainty means the upper limit of the estimate of the standard uncertainty of the measured quantity. It means that if there is not enough information for an accurate assessment of covariance, it is possible to specify an upper limit of uncertainty. (6) (7) (9)

2.5 Expand standard uncertainty

The measurement result in the form of $y \pm u(y)$ defines the actual measured value with a relatively small probability, approximately 65%. This probability is generally insufficient. Therefore, the effort to determine the interval at which the value occurs with a probability of close to 100%. In practice, therefore introduces Expanded uncertainty U. The term expanding is understood combined uncertainty multiplication constant (coefficient of expansion) so as to establish an agreed probability zone.

The basic definition for the expanded uncertainty of measurement is followed by equation (2.22)

$$U = k * u_C(y), \tag{2.22}$$

Where U is the expanded uncertainty, $u_C(y)$ is the combined standard uncertainty and k is the coverage factor.

The value of k depends on the probability distribution of the measurement result. In practice, the different expansion coefficients of the type of division and the desired probability value. In the case of a normal distribution of measurement results, where coverage factor $k = 2$, corresponds to a 95% probability.

Some other coverage factors (for a normal distribution) are $k = 1$ for a confidence level of approximately 68 percent, $k = 2.58$ for a confidence level of 99. (11) (4)

3 INTRODUCTION TO MONTE CARLO METHODS

The Monte Carlo Method is a numerical method of solving mathematical problems by the simulation of random variables. (12) The random variables were created for the solved example, and the mean value was identical with the solution of the original problem. Stochastic and deterministic problems can be solved, although the random variables had the same solution to the problem. It is not possible to achieve this solution because many random experiments can be done. The sample average was created based of these randomized trials which are approximations of the mean. (13)

The term "Monte Carlo" is dated around 1944. Since then there have been many other works. In 1945, J. Neumann suggested using the apparatus of probability using a computer used in the development of the atomic bomb. S.A Ulam, N. Metropolis, H. Kahn and E. Fermi also participated on the development of methods. The detailed history of the facility can be found at (14). Monte Carlo method thus uses the apparatus of probability theory and mathematical statistics, which are also included in this work. A deeper interpretation of the theory of probability and mathematical statistics can be found at (13).

The first recorded use of the Monte Carlo method was in 1777 by French scientist Georges de Buffon determining the number using the random throwing of needles on plain covered parallels. The random experiment is known as Buffon's needle.

We have a plane which is covered with parallel and whose distance is d. On this plane were accidentally thrown a needle of length $l, l \leq d$. See figure 3-1. The question arose: What is the probability that the parallels would be intersect by the needle. (15) (16)

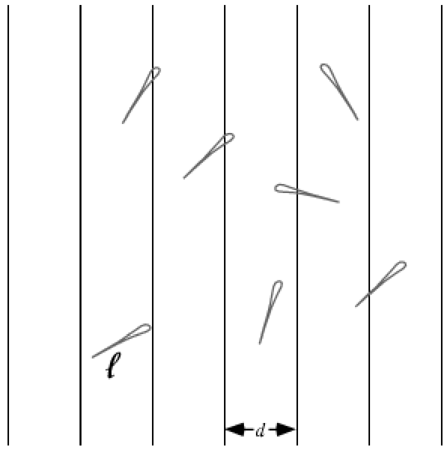


Figure 3-1. Buffon's needle experiment.

If the distance of the needle from the closest parallel were described as x and angle, which were formed between needle and parallels as φ, the needle position on the plane would be described by a pair (φ, x), where $0 \leq \varphi \leq \pi$ and $0 \leq x \leq d/2$. Needle crossed any of parallels if and only if this condition would be valid $x \leq (l/2)\ sin\varphi$. The coordinates display (φ, x), can be seen in figure 3-2.

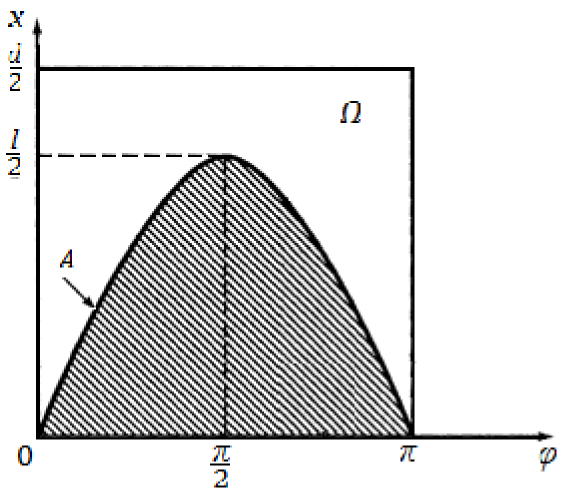


Figure 3-2. Coordinates the center of the needle.

Coordinates of the center of the needle (φ, x) can be any value of square Ω. The crossing of some parallels could happen if coordinates of the center of the needle would lie in the hatched area A. The probability P intersecting parallel lines would be equal to the ratio of surface area A to square area Ω. Mathematical expression can be seen on (3.1).

$$P = \int_0^{\pi} d\varphi \int_0^{\left(\frac{l}{2}\right) sin\varphi} d\varphi \frac{2}{\pi d} = \frac{l}{\pi d} \int_0^{\pi} d\varphi\, sin\varphi = \frac{2l}{\pi d} \qquad (3.1)$$

The probability of parallel needle transaction was expressed as the number π, in this experiment. If n throws were performed and the throws frequency m of needles that crossed the parallels were determined, relative frequency n/m estimate the probability P and calculate the number π by relation $2l/(\pi d) \doteq m/n$ (15) (16).

In 1901, Italian mathematician Mario Lazzarini throws needles3408 times, and the number π get the value of $355/113 = 3.14159292$, which was a surprisingly good result. It is clear that the described method of determining the number π is considerably lengthy (17). However, after the introduction of computers, there was an opportunity to attempt faster simulation. Thanks to computing speed and solving several tasks at the same time, the speed of this process got rapidly faster. Of course, the broader dissemination of the Monte Carlo was not possible until the computers became widespread. (18)

3.1 Random Numbers Generator

Solving tasks by Monte Carlo method was based on repeated random experiments. Useful calculation required vast quantities of these random experiments. A random experiment was realized by modeling which means operations with random numbers. Parameters inserted into the generator of random numbers have to fulfill at least one of these three conditions. The first condition is that parameters have to characterize where the random numbers start. The second condition characterizes a maximum span between the numbers, and the last condition characterize the value that cannot be exceeded. We have two different types of random numbers. Pseudorandom numbers and true random numbers

Pseudorandom numbers are deterministic algorithms, which create long strings of numbers seemingly having a proper random distribution. Later, these sequences are repeated, and the quality distribution is reduced. Among the simple hand-feasible method includes a method of secondary squares designed by John Von Neumann. Its implementation is very simple, and the results are poor statistical properties. Nowadays, there are programs for generating random numbers.

True random numbers are a measured physical phenomenon, which is assumed to be random, and subsequently, compensate for deviations. The first approach was throwing dice, flipping a coin, roulette, etc. For use in statistics or encryption, this is too slow and inefficient. (13) (19)

3.2 Monte Carlo Technique

The Monte Carlo method can be applied to numerous models. Monte Carlo method can be applied for many techniques.

- solving differential equations,
- solving definite integrals,
- modeling of stochastic systems,
- calculation of uncertainties.

3.2.1 Estimating Pi Using the Monte Carlo Method

For an illustrative example of the Monte Carlo method, the numbers π was estimated. Figure 3-3 is represents the square of size two, in which an inscribed circle of a maximum. If the sides of the square are parallel to the mathematical axes Cartesian coordinate system, the origin of the coordinate system lies at the center of the circle. Each point inside the square is defined by an ordered pair of numbers$[\gamma_1, \gamma_2]$, where is $\gamma_i \in< -1,1 >$ is a random number. Randomly selected point unit square lies in a circular sector.

On the basis of geometrical probability, equation (3.2) can be written.

$$\rho = \frac{S_{circle}}{S_{square}} = \frac{\pi r^2}{(2r^2)} = \frac{\pi}{4} \qquad (3.2)$$

If the sum of the generated numbers and the radius of the circle is known, it is possible to write equation (3.3) for number π. (20) (12) (13)

$$\pi = \frac{S_{square} N_{circle}}{r^2 N_{trials}} \tag{3.3}$$

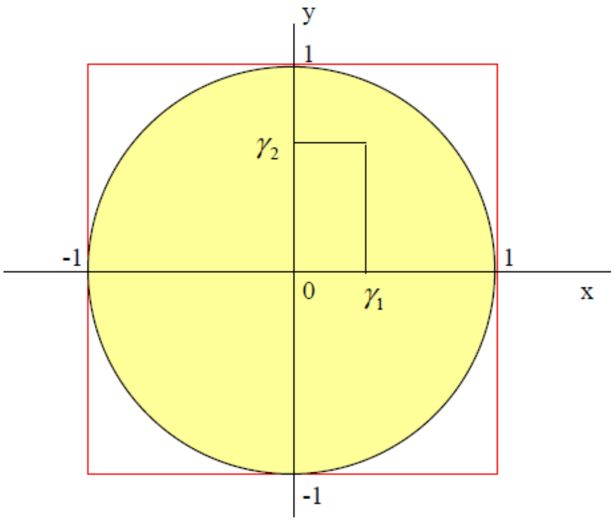


Figure 3-3. Estimate π number.

The numbers of trials were varied from 1,000 to 1,000,000 in Rstudio. The error in the estimate of π was also calculated. Results are illustrated in table 3-2.

Table 3-1. Estimating of number π by Rstudio.

$\mathbf{N}_{trials}$	$\boldsymbol{\pi}_{MC}$	**Error** [%]
1,000	3.012	4.13
10,000	3.175	1.05
100,000	3.143	0.04
1,000,000	3.141	0.02

3.2.2 Monte Carlo Integration

A lot of numerical methods can solve the definite integral. In any cases, the results of the definite integral can achieve high accuracy. However, definite calculation by the Monte Carlo method with the same accuracy requires much more computation.

For estimation, a one-dimensional integral by Monte Carlo method according to equation (3.4), can be used the knowledge of the previous example 3.2.1 –

Estimating π number. Ring in this case replaced the common geometric formation. It can be seen in figure 3-4.

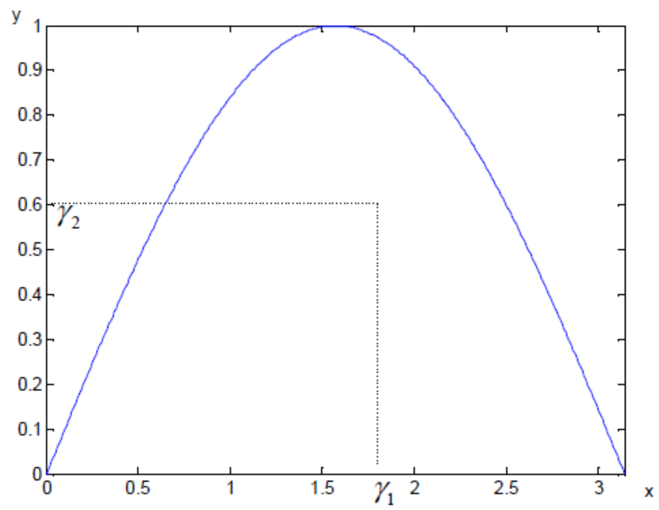


Figure 3-4. Calculating a one-dimensional integral.

$$\int_a^b f(q)dq \tag{3.4}$$

As a first step, the most integrated features should be determined, which can be used, for example, the function $max()$. Then the rectangle sides AB, BC must be defined. Random numbers $[\gamma_1, \gamma_2], \gamma_1 \in < a, b >, \gamma_2 \in < 0, \max(f(x)) >$ must be generated. and the ratio between the points that lie within the integrated area and all generated numbers must be calculated. The value of the integral can be determined according to the relationship (3.5). (20)

$$I = \frac{N_{accept}}{N_{trials}} S_{rectangle} \tag{3.5}$$

The demonstration is made on the calculation of the integration sine function over the interval $< 0, \pi >$, given by equation (3.6).

$$\int_0^\pi sin(x)dx = 2 \tag{3.6}$$

The analytic result is equal to 2. The value obtained by MCM is 2 as well. For the calculation, 10^6 random numbers are used. Since the program Rstudio generated random numbers with a uniform probability distribution over the interval $< 0,1 >$. Concerning equation (3.7), must be interval transformed to follow interval $< a, b >$

$$\gamma' = a + (b - a)\gamma. \tag{3.7}$$

Results are illustrated in table 3-2. The table shows the numbers of trials that were varied from 1,000 to 1,000,000 in Rstudio. The errors in integration were also calculated. It is evident that the result is more accurate the more the number of generated values are chosen. (21) (12) (14)

Table 3-2. Error in MCM Integration.

$\mathbf{N}_{trials}$	$\mathbf{I}_{MC}$	**Error** [%]
1,000	1.955	2.25
10,000	2.004	0.2
100,000	1.999	0.05
1,000,000	2	0

4 Monte Carlo simulation applied to metrology

As was indicated in Chapter 3, the principle of MCM is to generate the random numbers by the probability density function of input variables and their assignment in the measurement model and calculated probability function output variables. (22) As in GUM method, it is necessary before the calculation of uncertainties to specify the following:

a. definition of the measurand and input quantities;
b. modeling;
c. estimation of the probability density functions (PDFs) for the input quantities;
d. setup and run the Monte Carlo simulation;
e. summarizing and expression of the results.

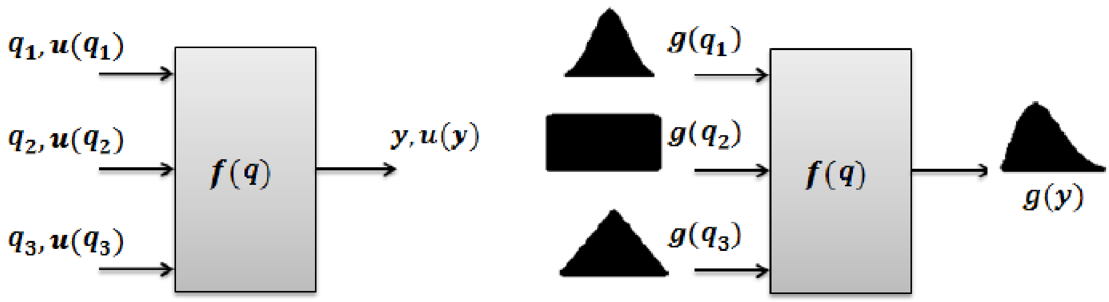


Figure 4-1. Illustrations of the methodologies. Propagation of uncertainties on the left and propagation of distribution on the right.

Figure 4-1 represents the propagation of uncertainties, where q_1, q_2 and q_3 are input quantities. Then $u(q_1), u(q_2)$ and $u(q_3)$ are their uncertainties and y and $u(y)$ are the measurand and its uncertainty. However, propagation of distribution, where $g(q_1), g(q_2)$ and $g(q_3)$ are distribution functions of the input quantities and $g(y)$ is the distribution function of the measurand.

4.1 Basic Principles

Firstly, measurands and input quantities must be defined. It has to be clear witch quantity could be the final object of measurement. Further, we have to identify all the variables, known as the input source, which directly or indirectly influence the determination of the measured. The example we can see on equation (4.1).

$$y = f(q_1, q_2, q_3, q_4) \tag{4.1}$$

where y is a function of four different input sources $q_i \geq 1$. The next step is modeling measurement procedure. It should be modeled in order to have the measured as a result of all the input sources. Example we can see on equation (4.2)

$$y = \frac{q_1(q_2 + q_3)}{q_4^2} \tag{4.2}$$

Construction of a flowchart helps with visualizing modeling process of experiments. Problems may arise if we want to define what impact they could have on the measurement of input sources.

This step is also significant. All the details about the estimation of the uncertainties of input sources are described in chapter 2. Unlike GUM the most appropriate density functions for each of the input quantities must be found. (22) (23) There are many other probability distributions used for input variables that are outside the scope of this project.

After all, the input PDFs have been defined, some Monte Carlo trials following in step (d). Because uncertainty provides a maximum of two valid points, most cases, merely choose $M > 10^4$. For complicated uncertainty calculations $M = 10^6$ is sufficient. For complicated computations, it may be advisable to reduce M to a minimum. In that case, we must use an adaptive methodology for determining M.

Let's suppose that our model calculations could consist of a high enough number of trials. Therefore, this document does not consist of an adaptive methodology. This methodology is described in the 1 Supplement of GUM. However, the principle is to check after each trial for stabilization of the results of interest Representation of the expected result we can see in equation (4.3).

$$M > \frac{10^4}{1-p} \tag{4.3}$$

Where $100p\%$ is the selected coverage probability. For example, if we have coverage probability 95%, so $p = 0{,}95$ and M should be at least 200,000.

After setting M trials an algorithm for estimating measurement uncertainty needs to be chosen. Requirements for a reliable simulation is a pseudorandom number generator. Of course, this depend on our software.

4.2 Implementation of a Monte Carlo Method

The Monte Carlo method is implemented using an algorithm that can be summarized in six steps base on Supplement 1 to the Guide to the Expression of Uncertainty in measurement (3).

4.2.1 Numbers if Monte Carlo trials

First of all, you need to create a mathematical model to measure $Y = f\ (q)$, where Y is the scalar output quantity and, q represents N input variables. Each variable q_i is considered as a random variable with possible value ξ_i, with medium vala ue q_i and probability the of density $g(\xi_i)$. Y is a random variable with possible value η, the mean value y and the probability of density $g(\eta)$. Furthermore, it is necessary to choose the number of trials M of Monte Carlo method and coverage of probability p.

4.2.2 Sampling from a probability distribution

Each input quantity q_i generates M random vectors q_r,$r = 1, \ldots, M$ according to the density of distribution of uncertainties. Thus generated $M \cdot N$ numbers.

4.2.3 Evaluation of the model

Generated numbers are substituted into the measurement model $y_r = f(q_r), r = 1,\ldots,M$. The measurement model r-th element contains $q_{1,r},\ldots,q_{N,r},q_{i,r}$ random number according to the density of distribution of uncertainties.

4.2.4 Representation of the distribution function for the output quantity

The values $y_r, r = 1,\ldots,M$ must be sorted into a non-declining order. This ordered model is designated as $y_{(r)}, r = 1,\ldots,M$. After that, the discrete distribution function G is determined from the values $y_{(r)}$.

4.2.5 Estimate of the output quantity

The average of the quantity $\overline{y}$ must be calculated by (4.4) and standard deviation by (4.5).

$$\overline{y} = \frac{1}{M}\sum_{r=1}^{M} y_r \tag{4.4}$$

$$u(\overline{y}) = \sqrt{\frac{1}{M-1}\sum_{r=1}^{M}(y_r - \tilde{y})} \tag{4.5}$$

4.2.6 Coverage interval for the output.

The coverage interval for Y can be estimated from the discrete form G. It can be calculated as $q = pM$. Then $[y_{low}, y_{high}]$ is a $100p\%$ coverage interval for Y, where $y_{low} = y_{(r)}$ and $y_{high} = y_{(r+q)}$ for $r = 1,\ldots,M-q$. $100p\%$ probabilistically symmetric coverage interval is calculated by $r = (M-q)/2$. The shortest $100p\%$ coverage interval r^* can be determined as $y_{(r^*+q)} - y_{r^*} \leq y_{(r+q)} - y_{(r)}$ for $r = 1,\ldots,M-q$.

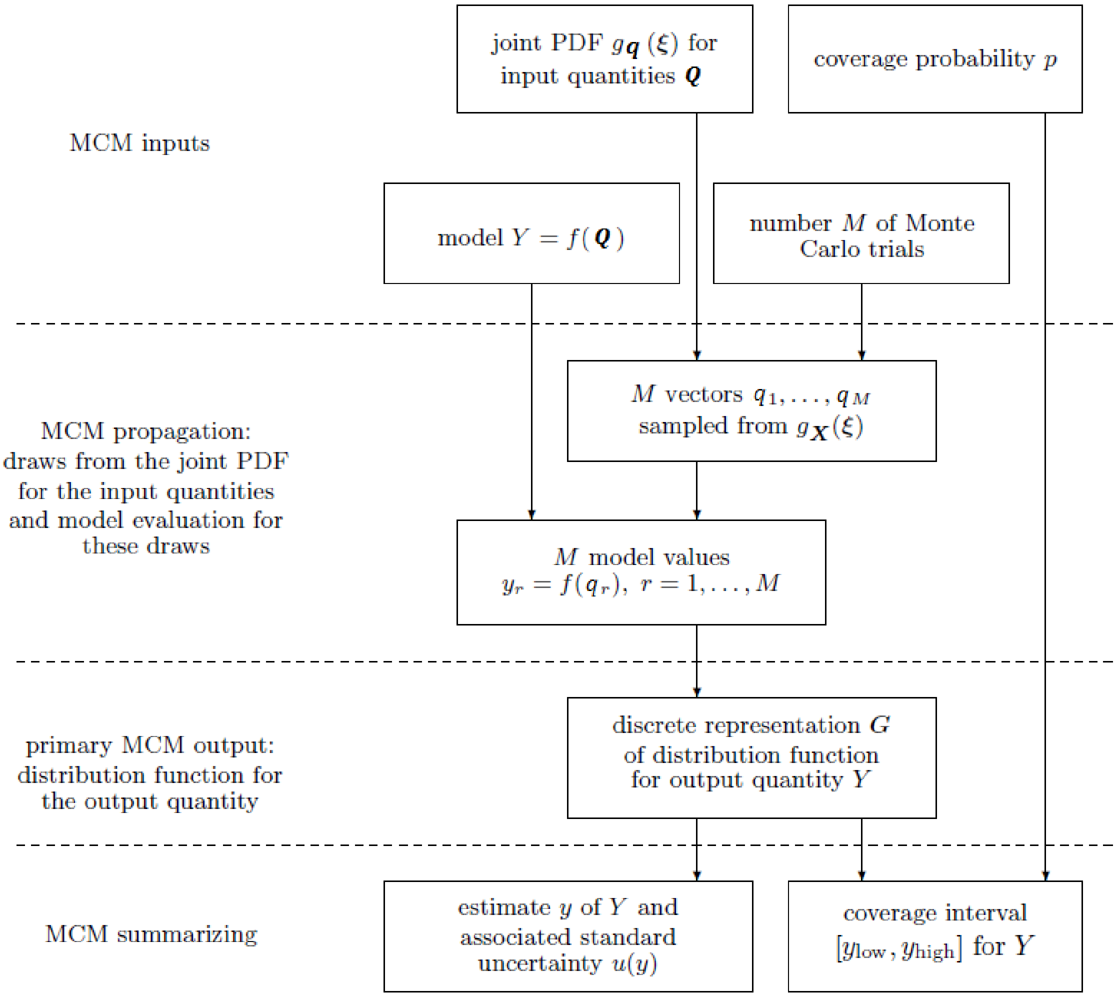


Figure 4-2. The propagation and summarizing stages of uncertainty evaluation using MCM to implement the propagation of distributions.

Figure 4-2 is represented implementation of the procedure in the flow diagram. All of the steps showed at the start of chapter 4 are stacked to form flow chart.

5 Case studies: Volume of a Solid Object

The cuboid was chosen as the measured object. Length, Width, and Height were measured by caliper with tolerance 0.02 mm for all three sides. For this reason the correlation must be considered. The volume of the cuboid can be expressed by equation (5.1).

$$V = abc \tag{5.1}$$

Length, Width, and Height were repeatedly measured ten times. Measured values were recorded in table 5-1.

Table 5-1. Measured value for case studies 5: Volume of a Solid Object.

i	1	2	3	4	5	6	7	8	9	10
$\boldsymbol{a}\,[\boldsymbol{cm}]$	6.141	6.135	6.139	6.146	6.140	6.138	6.144	6.139	6.142	6.144
$\boldsymbol{b}[\boldsymbol{cm}]$	4.222	4.219	4.215	4.225	4.224	4.218	4.217	4.221	4.219	4.214
$\boldsymbol{c}\,[\boldsymbol{cm}]$	2.322	2.319	2.323	2.327	2.318	2.319	2.323	2.321	2.318	2.324

Length (a). The average length was calculated to 6.141 cm, with uncertainty type A of $1.041*10^{-3}$ cm. Uncertainty type B was estimated as caliper tolerance divided by expected PDF. The best PDF is Uniform distribution, and the results of uncertainty type B is $1.155*10^{-2}$ cm.

Width (b). The average width is 4.219 cm. Uncertainty type A was calculated to $1.147*10^{-3}$ cm, and uncertainty type B is $1.155*10^{-2}$ cm as well. PDF was also estimated as uniform distribution.

Height (c). Average height is 2.321 cm. Uncertainty type A is $9.333*10^{-4}$ cm, and uncertainty type B is $1.155*10^{-2}$ cm. The best PDF is a uniform distribution.

Every single uncertainty with their PDFs was summarized in table 5-2 and was calculated according to chapter 2. As was mentioned at the beginning, all measurements were finished with the same caliper for all three sides. In this case, there is a powerful correlation between measurements. Theoretical background of correlation can be found in chapter 2.3.

Final Uncertainty of this case studies can be calculated base on equation (5.2).

$$u^2(V) = c_a^2 u^2(a) + c_b^2 u^2(b) + c_c^2 u^2(c) + 2c_a c_b u(a,b) + 2c_a c_c u(a,c) + 2c_b c_c u(b,c) \tag{5.2}$$

For each measured value, the following applies.

$$u^2(a) = u_A^2(a) + u_B^2(a), u^2(b) = u_A^2(b) + u_B^2(b), u^2(c) = u_A^2(c) + u_B^2(c) \tag{5.3}$$

The covariance method A is determined according to the relationship (5.4) because there is a real pair of measurements.

$$u_A(a,b) = cov(a,b);\ u_A(a,c) = cov(a,c);\ u_A(b,c) = cov(b,c) \tag{5.4}$$

For a covariance determined by method B, a correlation factor equals one, which was used because it is a strong correlation.

$$u_B(a,b) = u_B(a)u_B(b)r(a,b) = u_B^2(V);\ u_B(a,c) = u_B(a)u_B(c)r(a,c) = u_B^2(V);\ u_B(b,c) = u_B(b)u_B(c)r(b,c) = u_B^2(V) \tag{5.5}$$

Every single partial derivation has the meaning of sensitivity coefficients, and they were estimated according to the equations below.

$$c_a = \frac{\partial V}{\partial a} = bc, c_b = \frac{\partial V}{\partial b} = ac, c_c = \frac{\partial V}{\partial c} = bc \tag{5.6}$$

Concerning to equation (5.2) and chapter 2.5 it is necessary to expand the final uncertainty by two. It means that the final results should be in the interval with probability 95%. The results can be seen in table 5-3.

Table 5-2. Input sources and associated PDFs with their parameters for the estimation of uncertainty for the measurement for the volume of a solid object.

Input source	**Type**	**PDF**	**PDF parameters**
Length [cm]			
- due to repeatability	A	Gaussian	Mean: 6.141 cm; u_A: 1.041*10^{-3} cm
- due to certificate	B	Uniform	u_B: 1.155*10^{-2} cm
Width [cm]			
- due to repeatability	A	Gaussian	Mean: 4.219 cm; u_A: 1.147*10^{-3} cm
- due to certificate	B	Uniform	u_B: 1.155*10^{-2} cm
Height [cm]			
- due to repeatability	A	Gaussian	Mean: 2.321 cm; u_A: 9.333*10^{-4} cm
- due to certificate	B	Uniform	u_B: 1.155*10^{-2} cm

Monte Carlo Simulation was set to run M=10^7 trials. Random values in Rstudio were generated for each side of a cuboid with uniform distribution. Model of measurement was chosen according to equation (5.6) where Q_1 represent length, Q_2 represent width and Q_3 is a height.

$$Y = f(Q) = Q_1 * Q_2 * Q_3 \tag{5.6}$$

By assigning the quantities, the relationship (5.1) is obtained. Then the average final volume of a the cuboid was evaluated. Measurement model $y_r = f(q_r), r = 1, \ldots, M$ contained M values. By shorting the values of the model $y_r = f(q_r)$ to a non-decreasing order the discrete distribution function was obtained. Then the average value was calculated with regard to equation (4.4). Then the shortest coverage interval must be found with alpha factor equal to 0.05. It means the shortest coverage interval contain 95% generated pseudorandom numbers. This interval respond standard uncertainty of Y for probability 95%.

The picture 5-1 represent the histogram of density probability — vertical lines are labeling 95% coverage interval obtained by the Monte Carlo Method.

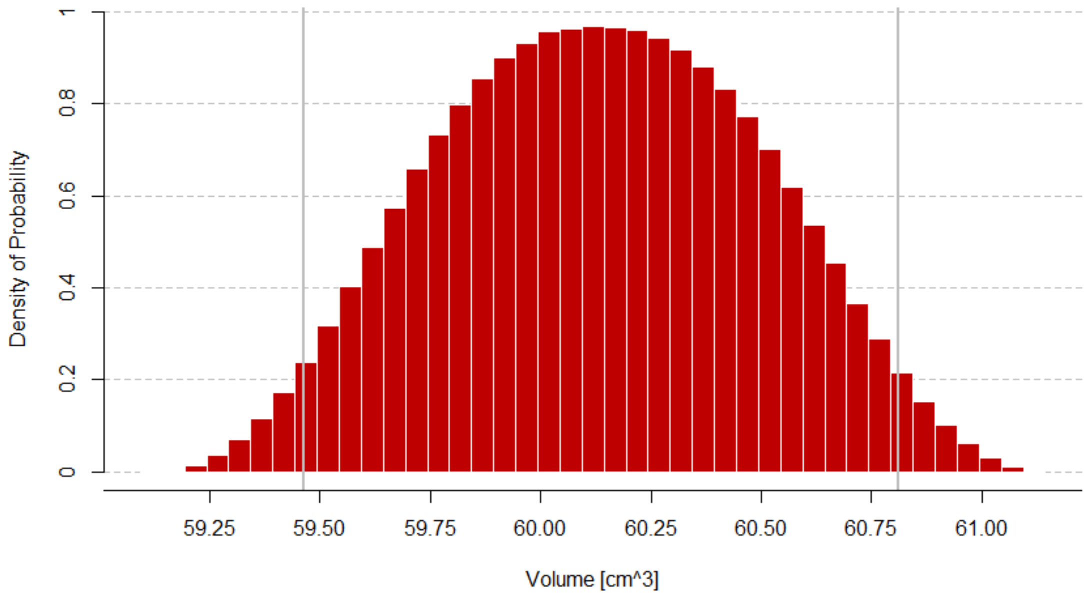


Figure 5-1. Histogram representing the resulting PDF for Volume of a solid object.

The significant results are represented in table 5-2. The averages obtained from both method is matching. Uncertainty belongs to GUM is bigger than uncertainty from MCM. That is a consequence of correlation because MCM does not count with it. It means that uncertainty calculated according to GUM method is the most likely. However for quick estimate it is better to use MCM because correlation calculation is demanding.

Table 5-3. Summarization of significant results for GUM and MCM at case studies 5.

Parameter	**GUM**	**MCM**
Mean	60.148	60.135
Uncertainty	1.162	0.673
Low endpoint for 95%	58.987	59.465
High endpoint for 95%	61.31	60.81

6 CASE STUDIES: DENSITY MEASUREMENT

In determining the density of the solid object, it was assumed that the temperature of the solid was assayed at room temperature. The volume of the solid object measured is based on the previous example of volume measurement with uncertainty. The weighing of the solid object was done on the electronic laboratory scales with tolerance 0.01 mg for ten times, and the measured values are shown in table 6-1 marked as z. A correction was made for the air lift according to the relationship (6.1) where z is the average calculated from the ten measured values was set. Density of calibration weights $\rho_z = 8400\ kg/m^3$ and air density $\rho_v = 1.2 kg/m^3$. Density of the solid object was calculated base on equation (6.2).

Table 6-1. Measured values for case studies 6: Density measurement.

$\boldsymbol{i}$	$\boldsymbol{z}\ [\boldsymbol{g}]$	$\boldsymbol{m}\ [\boldsymbol{g}]$
1	68.7123	69.4243
2	68.7121	69.4241
3	68.7118	69.4238
4	68.7125	69.4245
5	68.7121	69.4241
6	68.7119	69.4239
7	68.712	69.4240
8	68.7127	69.4247
9	68.7126	69.4246
10	68.7122	69.4242

$$m = \sum_{i=1}^{n} z_i + \left(V - \frac{z_i}{\rho_z}\right)\rho_v \tag{6.1}$$

$$\rho = \frac{m}{V} \tag{6.2}$$

Final Uncertainty can be calculated base on equation (6.3).

$$u(\rho) = \sqrt{\left(\frac{\partial\rho}{\partial V}u(V)\right)^2 + \left(\frac{\partial\rho}{\partial m}u(m)\right)^2} \tag{6.3}$$

The uncertainty of volume of a solid object $u(\rho)$ was established as a results came from case studies 5 and it is equal to 0.011 m^3. Uncertainty of the mass $u(m)$ was calculated regards to equation (2.9) where uncertainty type A was calculated from equation (2.5) and type B from the equation (2.7). The best PDF if uniform. Final uncertainty of the the mass is $5.77*10^{-6}$ kg.

The partials derivation as sensitivity coefficients were appointed to equation (6.3). The theoretical background can be seen in chapter 2.3. Due to this chapter, sensitivity coefficients were set according to the equations below.

$$\frac{\partial\rho}{\partial V} = -\frac{m}{V^2} \tag{6.4}$$

$$\frac{\partial\rho}{\partial m} = \frac{1}{V} \tag{6.5}$$

Ultimate results for the density of the object is 0.12 kg/m^3 calculated from equation (6.2). Concerning the equation, (6.3) the final uncertainty was equal to 0.0022 kg/m^3 with probability 68.2%. However, a request from GUM has represented the results with probability of 95%. In this case, results extend by two and uncertainty is 0.44 kg/m^3. Shortest coverage interval was estimated for low endpoint as 0.1109 kg/m^3 and for a hight endpoint 0.1154 kg/m^3. All of the significant results were recorded in table 6-3.

Table 6-2. Input sources and associated PDFs with their parameters for the estimation of uncertainty for the measurement for the Density measurement.

Input source	Type	PDF	PDF parameters
Mass [kg]			
- due to repeatability	A	Gaussian	Mean: 0.069 kg; u_A: 9.52*10^{-8} kg
- due to certificate	B	Uniform	u_B: 5.77*10^{-6} kg
Volume [m^3]			
- due to certificate	B	Uniform	Mean: 0.60 m^3; u: 0.012 m^3

Monte Carlo Simulation was set to run M=10^7 trials. Input quantities were selected according to table 6-2. Random values for volume and mass with uniform PDF were generated in Rstudio. Generated values were provided to model measurement (6.6) concerning equation (6.2).

$$Y = f(Q) = \frac{Q_1}{Q_2} \tag{6.6}$$

With regarding chapter, 4.2.4 results came from measurement model shorted to non-declining sequence, and the distribution function G was obtained. Then the average of density was calculated as 0.16 kg/m^3 with uncertainty 0.016 kg/m^3. Coverage interval 95% was estimated base on chapter 4.2.6. The results are recorded in table 6-3.

Shown in the picture 6-1, is a histogram of density probability for the density of a solid object. Vertical lines are represented 95% coverage interval calculated MMC.

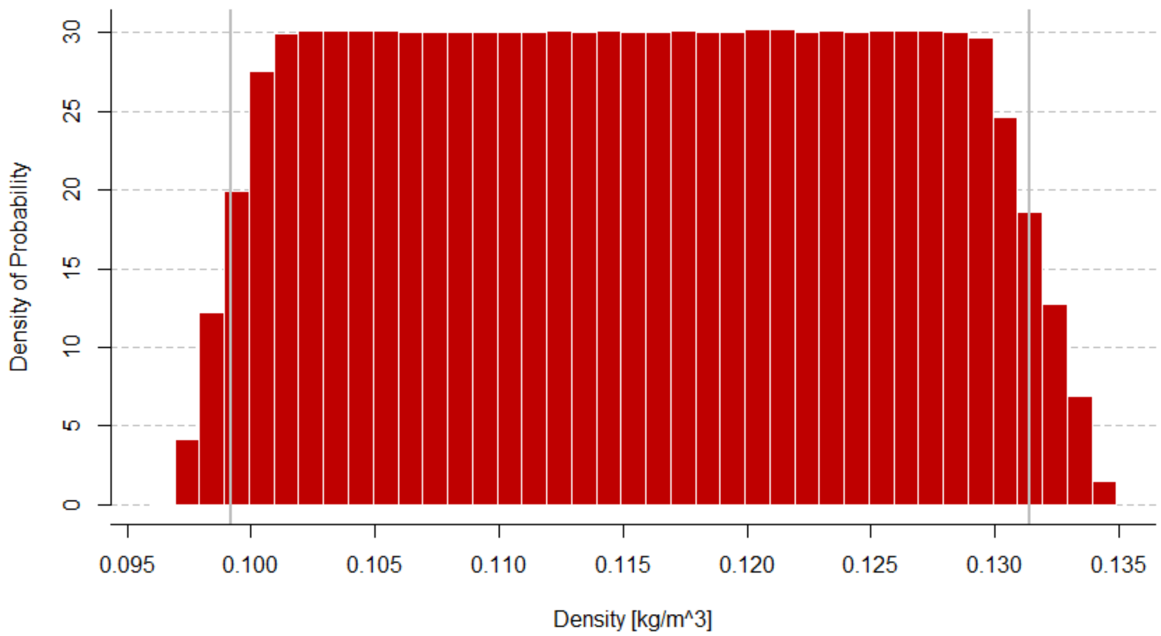


Figure 6-1. Histogram representing the resulting PDF for Density of a solid object.

Also from table 6-3 can be seen the comparison between GUM and MCM. At first view is evident that the standard uncertainty belongs to GUM method is bigger than the uncertainty get from MCM. Nevertheless, the algorithm of MCM did not include a sensitivity coefficient so it might be bigger. However, means are matching and endpoints for coverage interval were overlapped. It follows that the results are correct and methods are compatible.

Table 6-3. Summarization of significant results for GUM and MCM at case studies 6.

Parameter	**GUM**	**MCM**
Mean	0.1154	0.1154
Uncertainty	0.0197	0.0161
Low endpoint for 95%	0.0957	0.0992
High endpoint for 95%	0.1351	0.1314

7 CASE STUDIES: VOLTAGE DIVIDER UNLOADED

The voltage divider allows redistributing input voltage to required output voltage. The unloaded voltage divider has no load on the output voltage from which no current is drawn. The current passing through both resistors is the same. The total voltage is divided between resistors R1 and R2. For the output voltage, the unloaded divider applies to the Ohm law following equations (7.1). Diagram of the unloaded voltage divider is represented in image 7-1.

$$V_{out} = V_{in}\left(\frac{R_2}{R_1 + R_2}\right) \tag{7.1}$$

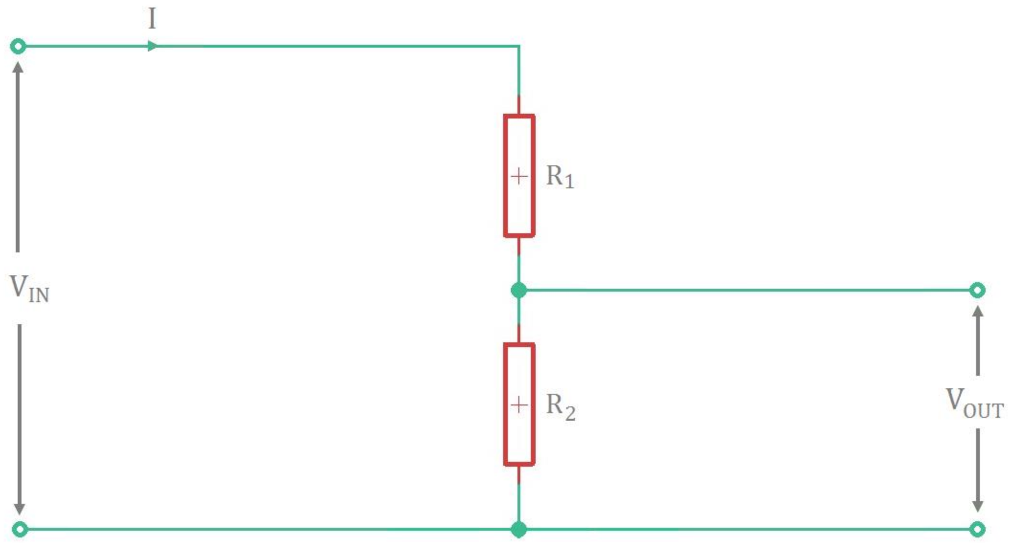


Figure 7-1. Diagram of an unloaded voltage divider.

Measurements were made in the laboratory where the temperature was set to 22°C, humidity 31% and pressure was 1006 hPa.

Input Voltage (V_{in}). HPM2000 programmable power supply with a tolerance of 1% with respect to a data sheet was used for measurement. Input voltage was set to 10V.

Absolute uncertainty type B is eaqual to 0.0058 V and it was estimated based on equation (2.7).

Resistance (R_1). The value for resistance is state in a certification 5% and nominal value is equal to 220Ω. Then the absolute uncertainty type B was calculated as 6.35 Ω. Uncertainty type A is not needed in this case due to datasheet characteristics.

Resistance (R_2). The nominal value for a second resistance is 1 kΩ with tolerance 5% as specified on datasheet. Absolute uncertainty was calculated to 28.87 Ω Detail of the calculation can be seen in attached excel sheet.

Final Uncertainty can be calculated base on equation (7.2).

$$u_{V_{out}} = \sqrt{\left(u_{V_{in}}\frac{\partial V_{out}}{\partial V_{in}}\right)^2 + \left(u_{R_1}\frac{\partial V_{out}}{\partial R_1}\right)^2 + \left(u_{R_2}\frac{\partial V_{out}}{\partial R_2}\right)^2} \qquad (7.2)$$

Based on the chapter 2.2. the sensitivity coeficients should be calculated as partial derivation according to the equations below.

$$\frac{\partial V_{out}}{\partial V_{in}} = \frac{R_2}{R_1 + R_2} \qquad (7.3)$$

$$\frac{\partial V_{out}}{\partial R_1} = -v_{in}\frac{R_2}{(R_1 + R_2)^2} \qquad (7.4)$$

$$\frac{\partial V_{out}}{\partial R_2} = -v_{in}\left(\frac{R_2}{R_1 + R_2} - \frac{R_2}{(R_1 + R_2)^2}\right) \qquad (7.5)$$

Quantities $u_{V_{in}}, u_{R_1}$ and u_{R_2} were determined as a tolerance of the products transfered to absolute value devided by PDF. The best PDF that best represents this case is uniform distribuitons because the tolerances has been read from the datasheets.

With respect to the equation (7.2) is a final uncertainty equal to 0.061 Ω with probability 68.2%, however, request from GUM represent the results with probability 95%. In this case, result extend by two and uncertainty is 0.12 Ω.

Table 7-1. Input sources and associated PDFs with their parameters for the estimation of uncertainty for the measurement for the Voltage divider.

Input source	Type	PDF	PDF parameters
Input Voltage[V]			
- due to certificate	B	Uniform	u_B: 0.00577 V
Resistance [Ω]			
- due to certificate	B	Uniform	$u_B(R_1)$: 6.3508 Ω
			$u_B(R_2)$: 28.8675 Ω

Monte Carlo Simulation was set to run M=10^7 trials in software Rstudio. Input quantities were selected according to table 7-1. Random values with uniform distribution for all three input quantities were generated. Based on equation (7.1) the measurement model was created and can be represented by equation (7.6) where the generated values were appointed.

$$Y = f(Q) = Q_3\left(\frac{Q_2}{Q_1 + Q_2}\right) \qquad (7.6)$$

Measurement model $y_r = f(q_r), r = 1, \ldots, M$ contains M generated values. By assigning the values of the measurement model to non-decreasing order, a discrete distribution function G was obtained, average was calculated using equation (4.4). Then the uncertainty and 95% coverage interval was calculated and the results were recorded to table 7-2.

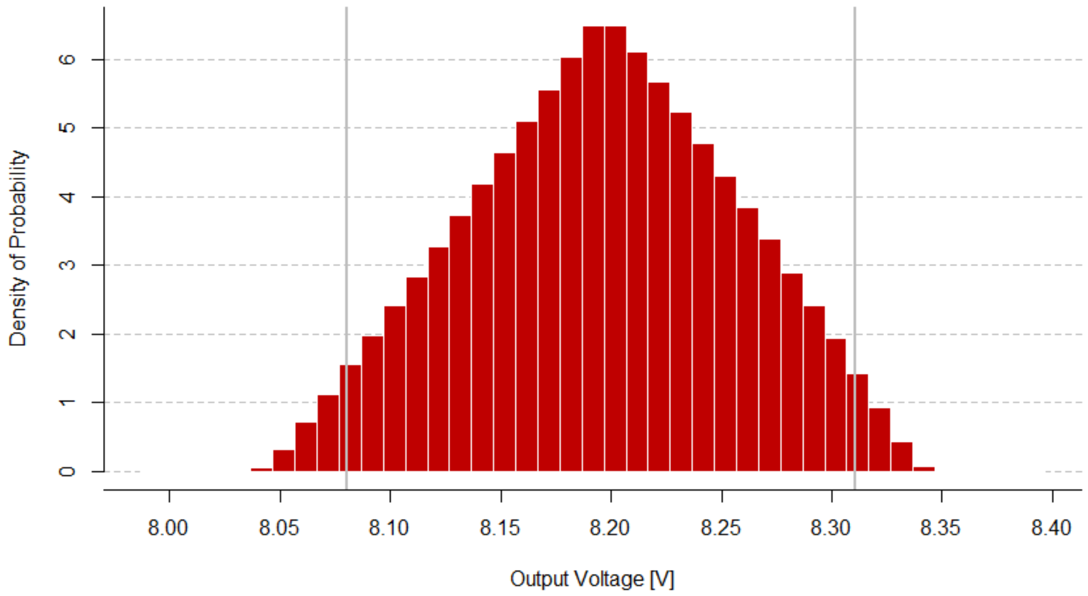


Figure 7-2. Histogram representing the resulting PDF for Output Voltage.

Figure 7-2 representing the density of probability for output voltage. Vertical lines showed coverage interval where the uncertainty should be in 95% probability.

Table 7-2. Summarization of significant results for GUM and MCM at case studies 7.

Parameter	GUM	MCM
Mean	8.1967	8.1959
Uncertainty	0.1211	0.1151
Low endpoint for 95%	8.0756	8.0805
High endpoint for 95%	8.3177	8.3107

As can be seen from table 7-2 both methods are compatible. Means they are exactly matched for both methods. Uncertainties are with 5% related difference. Endpoints for low and high position are leaner for MCM method. Calculation in this case study did not include statistical approach by reason of value used from the datasheet.

8 CASE STUDIES: AMMETER VOLTMETER METHOD

The Ammeter-Voltmeter method is the most popular and simple method for the measurement of resistance. It uses one ammeter and one voltmeter. The technique was used for the measurement of low resistance according to circuit shown in figure 8-1. The voltmeter measured the actual voltage at the measured resistance. However, the ammeter shows the sum of the currents passing through the measured resistance and the voltmeter. This method is suitable where the current passing through the measured resistance is considerably higher than the current passing through the voltmeter. With the known resistance of voltmeter, the correction can be made and the unknown resistance calculated according to equation (8.1).

$$R_m = \frac{V}{I - \frac{V}{R_V}} \tag{8.1}$$

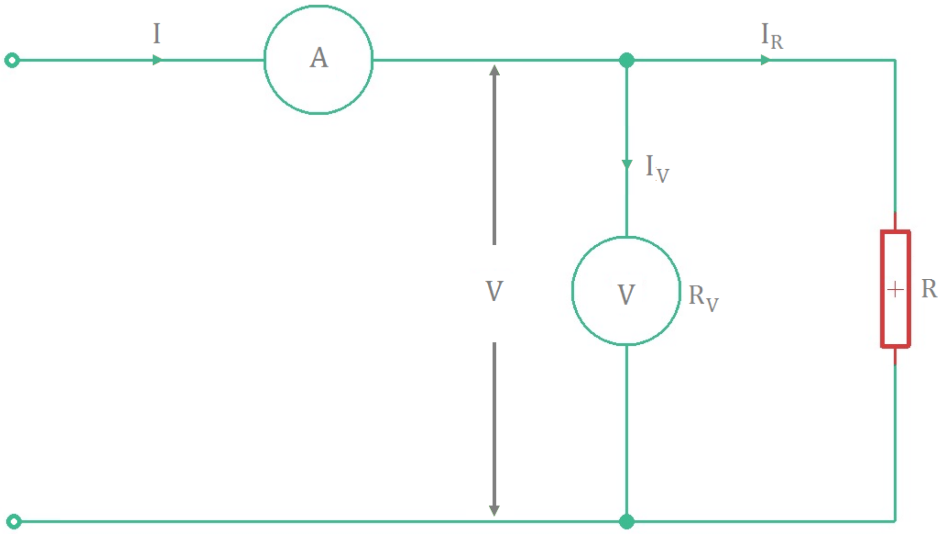


Figure 8-1. Diagram of low resistance measurement.

Measurements were made in the laboratory under normal conditions with the followed specification. Temperature was 22°C, humidity 31% and pressure 1006 hPa. For voltage measurement, a voltmeter Fluke 117 with resistance of $10*10^5 \Omega$ was used. For current measurement an Agilent 34401A ammeter and Power supply HMP2020 was used. Measured values were recorded into table 8-1.

Table 8-1. Measured values for case studies 8: Ammeter Voltmeter Method

i	1	2	3	4	5	6	7	8	9	10
$V\ [V]$	4.339	4.339	4.340	4.340	4.341	4.341	4.342	4.340	4.339	4.338
$I\ [mA]$	95.94	95.99	95.99	95.98	95.98	95.99	95.97	95.97	95.98	95.91

Voltage [V]. Voltage was repeatedly measured ten times. The average was given as 4.340V with uncertainty type A equal to 0.379 mV calculated based on chapter 2.2.1. Source of uncertainty was classified as type B equal to instrument error A_{max} divided by PDF exactly as was described in chapter 2.2.2. The error of digital voltmeter was established according to datasheet from the error range δ_1 and error reading δ_2. Together with the average of measured value Q and used range M from the following equation $\delta = \delta_1 Q + \delta_2 M$ calculated the final instrument error. As it was expected that values in the datasheet come from correct metrology procedures, the best estimation of PDF was a uniform distribution and it is equal to 0.0195V

Current [A]. The current was repeatedly measured ten times. The average was estimated the same way as the voltage calculation, and it is equal to 95.969 mA. Uncertainty type A is 7.968 μA. Regards to the datasheet of the Ammeter and the determination process of instrument error described above the uncertainty type B was established as 1.131 mA.

Probability density functions with every single input source and theoretical results were recorded at table 8-2.

Table 8-2. Input sources and associated PDFs with their parameters for the estimation of uncertainty for the measurement for the Ammeter Voltmeter Method.

Input source	Type	PDF	PDF parameters
Voltage [V]			
- due to repeatability	A	Gaussian	Mean: 4.340 mV; u_A: 0.38 mV
- due to certificate	B	Uniform	u_B: 0.019 V
Ampere [A]			
- due to repeatability	A	Gaussian	Mean: 95.969 mA; u_A: 7.97 μA
- due to certificate	B	Uniform	u_B: 1.13 mA

Final uncertainty can be calculated according to equation (8.2).

$$u(R) = \sqrt{\left(\frac{\partial R_m}{\partial V} u(V)\right)^2 + \left(\frac{\partial R_m}{\partial I} u(I)\right)^2} \tag{8.2}$$

Partial derivatives have the meaning of the sensitivity coefficients, and it was calculated with equation (2.14). The results of the partial derivatives with respect to voltage and current are represented by equation (8.3) and (8.4).

$$\frac{\partial R_m}{\partial V} = \frac{R_v^2 * I}{(IR_V - V)^2} \tag{8.3}$$

$$\frac{\partial R_m}{\partial I} = \frac{R_v^2 * V}{(IR_V - V)^2} \tag{8.4}$$

Parts of the final uncertainty for voltage and current was calculated from equation (2.9) where the uncertainty type A and type B was calculated in the paragraph above.

Resistance was calculated with respect to equation (8.1), and the value is equal to 45.22 Ω. Uncertainty for the resistance was from equation (8.2) calculated to 0.57 Ω for 68% coverage interval. Then the expanded uncertainty for 95% coverage interval is equal to 1.14 Ω. In this case, its low endpoint is 44.08 Ω and the High endpoint is 46.39 Ω.

In Rstudio the Monte Carlo Simulation set to run M=10^7 trials. Input quantities were set according to table 8-2, both input quantities were generated by random values with uniform distribution. Generated values were substituted in measurement model (8.1) where R_V is voltmeter resistivity.

$$Y = f(Q) = \frac{Q_1}{Q_2 - \frac{Q_1}{R_V}} \tag{8.1}$$

Generated values came from the measurement model were shorted to non-declining order, and the distribution function G was obtained. The average was estimated in regards to equation (4.4). Uncertainty and endpoints estimated by MCM were recorded in table 8-3.

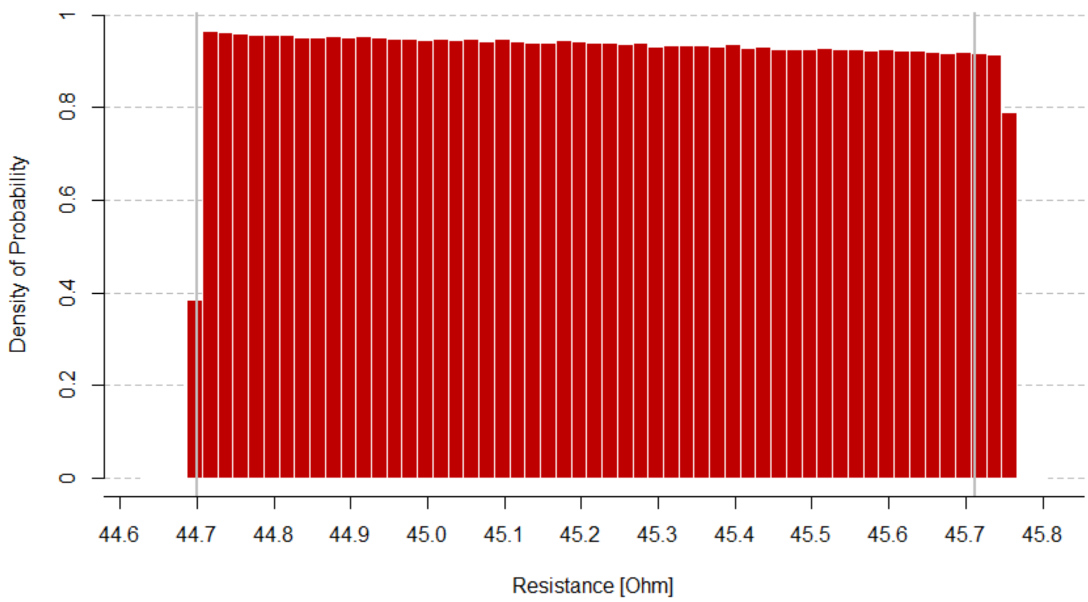


Figure 8-2. Histogram representing the resulting PDF for Resistance.

Figure 8-2 representing the density of probability resistance R. Vertical lines showed 95% coverage interval calculated by MCM. Uncertainty obtained by MCM is smaller and endpoints for low and high position are inside the interval calculated based on GUM method.

Table 8-3. Summarization of significant results for GUM and MCM at case studies 8.

Parameter	GUM	MCM
Mean	45.222	45.226
Uncertainty	1.1409	0.5054
Low endpoint for 95%	44.081	44.7005
High endpoint for 95%	46.363	45.7113

9 CASE STUDIES: POWER MEASUREMENT

An indirect measurement of DC power is realized by transferring power to other electrical quantities. The output of DC current at load has defined the product voltage and the current flowing, mathematically expressed by equation (9.1).

$$P = V * I \tag{9.1}$$

Diagram of the circuit connection can be seen from picture 8-1. Ammetr measures the actual current passing thru the resitance of the ammetr and voltmeter. Curent which passes through the voltmeter is very low, so can be skiped, however if the voltmeter resistance is high – thousnads ohms, the correction must be done accoring to equation (9.2).

$$P = V * I - \Delta P_v \tag{9.2}$$

where $\Delta P_v = V^2/R_V$ Than the final power can be estimated.

Power calculation used the data collected in previous chapter – case studies 8. Voltage and current were repeatedly measured ten times. According to table 8-1, the average for voltage is given at 4.340V with uncertainty 0.379 mV for type A and 0.0195 for type B. The average current was estimated at 95.969 mA. Uncertainty type A is 7.968 μA and type B is 1.131 mA. Results can be found at table 8-2.

Final uncertainty calculated based on the GUM method can be calculated from equation (9.3).

$$u(P) = \sqrt{\left(\frac{\partial P}{\partial V}u(V)\right)^2 + \left(\frac{\partial P}{\partial I}u(I)\right)^2} \tag{9.3}$$

Where partial derivatives $\partial P/\partial V$ and $\partial P/\partial I$ have the meaning of the sensitivity coefficients. The results of the partial derivatives are represented at equation (9.4) and (9.5).

$$\frac{\partial P}{\partial V} = I - \frac{2V}{R_V} \tag{9.4}$$

$$\frac{\partial P}{\partial I} = V \tag{9.5}$$

Uncertainties for voltage $u(V)$ and current $u(I)$ was calculated as combined uncertainty for single quantities according to equation (2.9).

Result of the power as stated in equation (9.2) is 0.416 W with uncertainty 0.0052 W for 68% coverage interval. 95% coverage interval for expand uncertainty calculated from equation (2.5) is 0.010 W. Low endpoint is 0.406 W and high endpoint is 0.427 W for 95% coverage interval.

Monte Carlo Simulation was set to run M=10^7 trials in Rstudio. For input quantities, random values were generated with uniform PDF. According to equation (9.2) the measurement model (9.6) was established. Generated values were substituted in the following equation (9.6) where R_V is voltmeter resistivity.

$$Y = f(Q) = Q_1 * Q_2 - \frac{Q_1^2}{R_V} \tag{9.6}$$

Generated values were shorted to non-declining order, and the distribution function G was obtained. The average was estimated as per equation 4.4. PDF of the output quantity is shown in Figure 9-1. That figure represented the density of probability wattage W. 95% coverage interval where the significant uncertainty is represented by vertical grey lines. Results obtained from the MCM are smaller in comparison to the results from the GUM method. High and low endpoints are shortest for MCM however, the means for both method are overlapped. In this case the uncertainty could be calculated properly. Results can be seen in table 9-1.

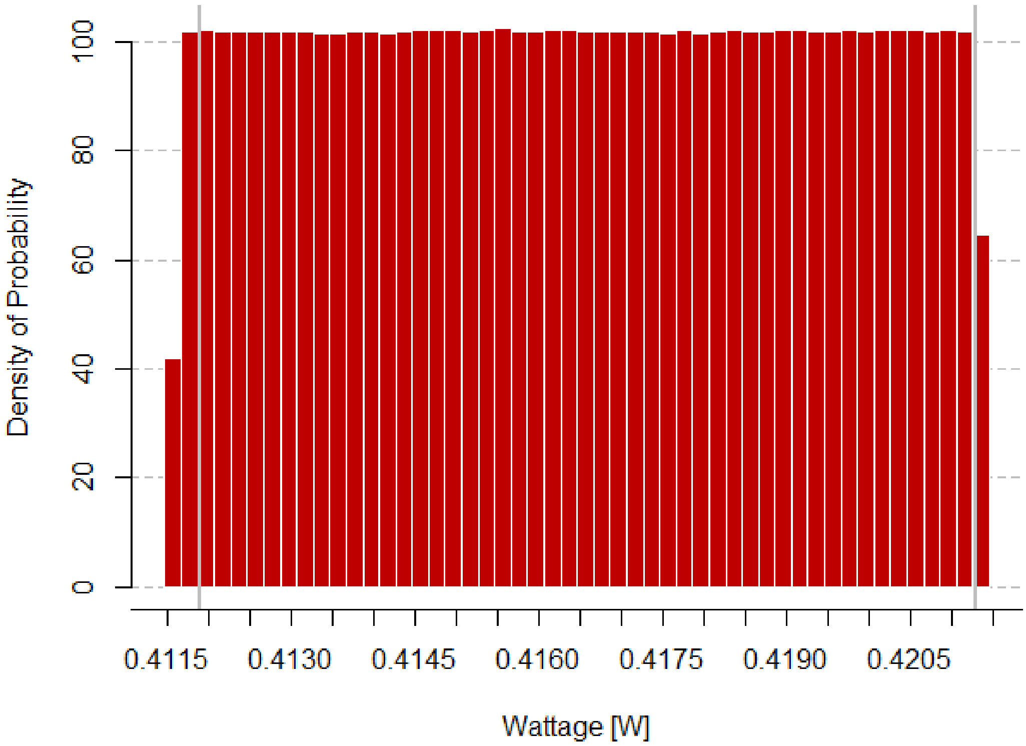


Figure 9-1. Histogram representing the resulting PDF for Wattage.

Regarding case study 8 – Ammeter Voltmeter method, the measurement model has been changed. Comparison output PDF shown in figure 8-2 with figure 9-1 showing the density of probability is higher, however, the PDF is still uniform.

Table 9-1. Summarization of significant results for GUM and MCM at case studies 9.

Parameter	**GUM**	**MCM**
Mean	0.416	0.416
Uncertainty	0.01	0.0047
Low endpoint for 95%	0.406	0.4119
High endpoint for 95%	0.427	0.4213

10 CONCLUSION

The thesis aimed to demonstrate the advantages of the Monte Carlo method against the GUM method in five case studies.

The theoretical part of the diploma thesis was divided into four chapters, including the introduction, which contains the basic concepts of measurement, which are presented in work. The second chapter explained the calculations of uncertainties for direct and indirect measurements with the most important relationships. The third chapter introduced the Monte Carlo method with representative examples for estimating the π number and for calculating a definite integral. The results demonstrated the accuracy of the method based on the count of generated numbers. The most important fourth chapter described the determination of uncertainties by the Monte Carlo method with all the basic assumptions.

Five case studies follow as a practical part of the diploma thesis. Each study was designed to demonstrate the complexity of the GUM method is best. Moreover, it is proven that the calculations by GUM method are complicated in comparison to Monte Carlo. These are indirect measurements focused on the different branches of physics. Case studies for GUM measurement include a theoretical analysis of every single task. Uncertainty type A and minor calculations are only commented for simplicity. These calculations can be viewed in the attached medium. The calculations are primarily focused on determining more demanding calculations. Analytical uncertainties calculations were performed in MS Excel based on the selected measurement model. Each case study includes, among other things, a brief description of the process, the measurement model from which the simulation was based, a histogram, and a comparison of the results of both methods with a concise conclusion.

In the first case study, the volume of a solid object is determined by the same gauge, which included calculation of covariance and correlation. The seemingly most straightforward case contains the most complicated calculations. The resulting uncertainty by 95% GUM is 1.16 cm. The resulting uncertainty of the Monte Carlo method was estimated at 0.63 cm. The output probability density function has corresponded to the normal distribution. The final uncertainty calculated by the Monte Carlo method is smaller by 42%

The second case study deals with density measurement. The calculation of uncertainties is based on a simple model of measurement. The uncertainty calculated by the GUM method for 95% probability was 0.020 kg/m^3. The uncertainty obtained by the Monte Carlo method is equal to 0.016 kg/m^3. It means that the resulting uncertainty is 20% smaller. The output probability density function corresponds to the trapezoid distribution.

In case study three, on the contrary, all input parameters are based on specifications. For the probability of 95%, the uncertainties obtained by both methods after rounding to two digits are equal to 0.12 V. From the result without rounding the MMC, is smaller by 5%. The probability density function of the resulting uncertainty corresponds to the triangle distribution.

In case studies 4 and 5, the uncertainty of the measuring instruments has been calculated. In the case of the fourth case study, the resulting uncertainty for the probability of 95% by the GUM method was 1.14 Ω. The uncertainty of the Monte Carlo method was 0.51 Ω. It means that MCM is smaller by 55%. For case study five uncertainty was estimated at 0.0047 W by MCM and it is smaller by 45% than uncertainty from GUM method, which is equal to 0.01 W. Both cases used the same input quantities, however measurement model was different. Output PDF changed as was recorded in previous chapter – case study 5 however, density probability still corresponds with uniform distribution.

The Monte Carlo calculated coverage interval, in which the calculated 95% probability variable is found, is narrower than the GUM method. It means that the resulting uncertainty obtained by the Monte Carlo method is much smaller. In the paragraph above, the Method is smaller by 30% of the case studies. Since intervals overlap and the mean values agree, we can assume that the calculations have been made correctly.

The advantage of MCM is the speed and simplicity of the algorithm. It is easy to get the right result, even without calculating partial derivations, sensitivity coefficients, or much else between calculations. Requirements are only placed on a high-quality random number generator that is already available in all paid and unpaid statistical programs such as Matlab, Octave, and Rstudio that were used in the calculations in this paper. The software then depends on the selected number of generated numbers.

10.1 Discussion

Results obtained from the MCM is smaller in average by 30% however, it cannot be said that the MCM is more precise. It depend for what the final uncertainty could be used. MCM is a great instrument for a quick estimate of the uncertainty nevertheless; in quality point of view the MCM method could not be more precise. It can be seen from case studies 3, there was not used any advance statistical calculation, so estimated uncertainties from both methods had a difference of just 5%. Otherwise the difference was much bigger for other case studies, by cause of covariance, correlation and sensitivity coefficient estimation.

11 REFERENCES

1. Joint Committee for Guides in Metrology. JCGM 200: International vocabulary of metrology - Basic and general concepts and associated terms (VIM). 2012.

2. Joint Committee for Guides in Metrology. JCGM 100: Evaluation of measurement data — Guide to the expression of uncertainty in measurement. 2008.

3. Joint Committee for Guides in Metrology. JCGM 101: Evaluation of measurement data — Supplement 1 to the "Guide to the expression of uncertainty in measurement" — Propagation of distributions using a Monte Carlo method. 2008.

4. Bell, Stephanie. World Meteorological Organization. A Beginner's Guide to Uncertainty of Measurement. [Online] August 1999. https://www.dit.ie/media/physics/documents/GPG11.pdf.

5. Measurement Uncertainty Analysis Principles and Methods. Washington DC 20546 : National Aeronautics and Space Administration, 2010. NASA-HDBK-8739.19-3.

6. EA Laboratory Committee. Evaluation of the Uncertainty of Measurement In Calibration. European Accreditation. [Online] 18 September 2013. http://www.european-accreditation.org/publication/ea-4-02-m-rev01--september-2013.

7. Evaluation of measurement data - Guide to the expression of uncertainty in measurement. . s.l. : JCGM, 1995. 92-67-10188-9.

8. Vdoleček, František, Palenčař, Rudolf and Halaj, Martin. Nejistoty v měření II: nejistoty přímých měření. s.l. : Automa, 2001.

9. Němeček, Pavel. Nejistoty měření. Praha : Česká společnost pro jakost, 2008. 978-80-02-02089-9.

10. Castrup, Howard. CiteSeerX. Distributions for Uncertainty Analysis. [Online] 11 April 2007. http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.493.5377&rep=rep1&type=pdf.

11. Vdoleček, František, Palenčař, Rudolf and Halaj, Martin. Nejistoty v měření III: nejistoty nepřímých měření. s.l. : Automa, 2001.

12. Sobol', I. M. A primer for the Monte Carlo method. s.l. : CRC Press, 1994. 0-8493-8673-x.

13. Stefan, Weinzierl. Introduction to Monte Carlo methods. Cornell University Library. [Online] 23 June 2000. http://arxiv.org/abs/hep-ph/0006269v1. NIKHEF-00-012.

14. Metropolis, N. The Beginning of the Monte Carlo Method. [Online] 1987. http://jackman.stanford.edu/mcmc/metropolis1.pdf.

15. Weisstein, E. Buffon's Needle Problem. MathWorld. [Online] 2015. http://mathworld.wolfram.com/BuffonsNeedleProblem.html.

16. Larson, R. and Odoni, A. Buffon's Needle Experiment. Urban Operation research. [Online] 1999. http://web.mit.edu/urban_or_book/www/book/chapter3/3.3.1.html.

17. Eric, Weisstein. Wolfram MathWorld. MathWorld. [Online] Wolfram Research, 13 5 2015. http://mathworld.wolfram.com/.

18. Buffon's needle. Wikipedia. [Online] Wikimedia Foundation, 2001-. http://en.wikipedia.org/wiki/Buffon's_needle.

19. Random number generation. Wikipedia. [Online] Wikimedia Foundation, 2001- https://en.wikipedia.org/wiki/Random_number_generation#.22True.22_random_numbers_vs._pseudo-random_numbers.

20. John, Mathews and Kurtis, Fink. Numerical Methods Using Mathematica for Numerical Analysis. Module for Monte Carlo Pi. [Online] 2005. http://mathfaculty.fullerton.edu/mathews/n2003/MonteCarloPiMod.html.

21. Mohamed, Osman. Distributed Parameter Systems. Washington State University. [Online] 4 April 2006. http://www.eecs.wsu.edu/~osman/EE351_S06/montecarlo.pdf.

22. Couto, Paulo Roberto Guimarães, Damasceno, Jailton Carreteiro and Oliveira, Sérgio Pinheiro de. Monte Carlo Simulations Applied to Uncertainty in Measurement. [book auth.] Victor (Wai Kin) Chan. Theory and Applications of Monte Carlo Simulations. s.l. : InTech, 2013, p. 284.

23. M., Šíra. Elektrorevue. How to calculate uncertainties by means of Monte Carlo method in an easy manner and without any expensive software. [Online] 1 April 2014. http://www.elektrorevue.cz/file.php?id=200001286-8b7638c702.

12 APPENDICES

APPENDIX A – VOLUME OF A SOLID OBJECT

```
#trials numbers
nsim = 10^7

#input quantities
meanA = 6.141
meanB = 4.219
meanC = 2.321
deltaABC = .02

#generate of random numbers
randA <- runif(nsim,meanA - deltaABC, meanA + deltaABC)
randB <- runif(nsim,meanB - deltaABC, meanB + deltaABC)
randC <- runif(nsim,meanC - deltaABC, meanC + deltaABC)

#model of measurement
randV = randA*randB*randC
meanV = mean(randV)

#function for calculating the shortest coverage interval
sci <- function (values, alpha){
  sortedSim <- sort(values)
  nsim <- length(values)
  covInt <- sapply(1:(nsim-round((1-alpha)*nsim)), function(i) {
    sortedSim[1+round((1-alpha)*nsim)+(i-1)]-sortedSim[1+(i-1)]})
  lcl <- sortedSim[which(covInt==min(covInt))]
  ucl <- sortedSim[1+round((1-alpha)*nsim)+(which(covInt==min(covInt))-1)]
  c(lcl, ucl)
}

#shortest 95% coverage interval
(simInt <- round(sci(randV, alpha=.05), 4))

#plot the histogram
hist(randV, breaks = seq(min(randV)-0.05, max(randV)+0.05 ,0.05), probability =
1, axes = F, border = "white", col = rgb(0.75,0,0),xlab = "Volume [cm^3]",ylab
= "Density of Probability",main = "")
vgrid <- seq(0, 1, 0.2)
abline(h=vgrid, col="grey", lty=2)
hist(randV, add=TRUE, breaks = seq(min(randV)-0.05, max(randV)+0.05 ,0.05),
probability = 1, axes = F, border = "white", col = rgb(0.75,0,0),xlab = "Volume
[cm^3]",ylab = "Density of Probability",main = "")
abline(v=simInt, col="gray", lty=1, lwd=2)
axis(side=1, at=seq(0,1000,0.25))
axis(side=2, at=seq(0,1, 0.2), labels = seq(0,1, 0.2))

#output quantity
meanV
(simInt[2]-simInt[1])/2
```

APPENDIX B – DENSITY MEASUREMENT

```
#trials numbers
nsim = 10^7

#input quantities
meanm = .069424187
meanV = .6014861501
deltam = .01
deltaV = .0116534395

#generate of random numbers
randm <- runif(nsim,meanm - deltam, meanm + deltam)
randV <- runif(nsim,meanV - deltaV, meanV + deltaV)


#model of measurement
randRo = randm/randV
meanRo = mean(randRo)

#function for calculating the shortest coverage interval
sci <- function (values, alpha){
  sortedSim <- sort(values)
  nsim <- length(values)
  covInt <- sapply(1:(nsim-round((1-alpha)*nsim)), function(i) {
    sortedSim[1+round((1-alpha)*nsim)+(i-1)]-sortedSim[1+(i-1)]})
  lcl <- sortedSim[which(covInt==min(covInt))]
  ucl <- sortedSim[1+round((1-alpha)*nsim)+(which(covInt==min(covInt))-1)]
  c(lcl, ucl)
}

#shortest 95% coverage interval
(simInt <- round(sci(randRo, alpha=.05), 4))

#plot the histogram
hist(randRo,   breaks   =   seq(min(randRo)-0.001,   max(randRo)+0.001   ,0.001),
probability = 1, axes = F, border = "white", col = rgb(0.75,0,0),xlab = "Density
[kg/m^3]",ylab = "Density of Probability",main = "")
vgrid <- seq(0, 100, 5)
abline(h=vgrid, col="grey", lty=2)
hist(randRo,  add=TRUE,  breaks  =  seq(min(randRo)-0.001,  max(randRo)+0.001
,0.001), probability = 1, axes = F, border = "white", col = rgb(0.75,0,0),xlab
= "Density [kg/m^3]",ylab = "Density of Probability",main = "")
abline(v=simInt, col="gray", lty=1, lwd=2)
axis(side=1, at=seq(0,0.5,0.005))
axis(side=2, at=seq(0,100, 5), labels = seq(0,100, 5))

#output quantity
meanRo
(simInt[2]-simInt[1])/2
```

APPENDIX C – UNLOADED VOLTAGE DIVIDER

```
#trials numbers
nsim = 10^7

#input quantities
Vin = 10
R1 = 220
R2 = 1000
deltaVin = .01
deltaR1 = 11
deltaR2 = 50

#generate of rand numbers
randVin <- runif(nsim,Vin - deltaVin, Vin + deltaVin)
randR1 <- runif(nsim,R1 - deltaR1, R1 + deltaR1)
randR2 <- runif(nsim,R2 - deltaR2, R2 + deltaR2)

#model of measurement
randVout = randVin*(randR2/(randR1+randR2))
meanVout = mean(randVout)

#function for calculating the shortest coverage interval
sci <- function (values, alpha){
  sortedSim <- sort(values)
  nsim <- length(values)
  covInt <- sapply(1:(nsim-round((1-alpha)*nsim)), function(i) {
    sortedSim[1+round((1-alpha)*nsim)+(i-1)]-sortedSim[1+(i-1)]})
  lcl <- sortedSim[which(covInt==min(covInt))]
  ucl <- sortedSim[1+round((1-alpha)*nsim)+(which(covInt==min(covInt))-1)]
  c(lcl, ucl)
}

#shortest 95% coverage interval
(simInt <- round(sci(randVout, alpha=.05), 4))

#plot the histogram
hist(randVout,  breaks  =  seq(min(randVout)-0.05,  max(randVout)+0.05  ,0.01),
probability = TRUE, axes = F, border = "white", col = rgb(0.75,0,0),xlab =
"Output Voltage [V]",ylab = "Density of Probability",main = "")
vgrid <- seq(0, 6, 1)
abline(h=vgrid, col="grey", lty=2)
hist(randVout,add=TRUE,  breaks  =  seq(min(randVout)-0.05,  max(randVout)+0.05
,0.01), probability = TRUE, axes = F, border = "white", col = rgb(0.75,0,0),xlab
= "Output Voltage [V]",ylab = "Density of Probability",main = "")
abline(v=simInt, col="grey", lty=1, lwd=2)
axis(side=1, at=seq(0,10,0.05))
axis(side=2, at=seq(0,1000, 1), labels = seq(0,1000,1))

#output quantity
meanVout
(simInt[2]-simInt[1])/2
```

APPENDIX D – AMMETER VOLTMETER METHOD

```
#trials numbers
nsim = 10^7

#input quantities
meanU = 4.340
meanI = 95.969E-3
deltaU = .195E-3
deltaI= 1.13E-3

#generate of random numbers
randV <- runif(nsim,meanU - deltaU, meanU + deltaU)
randI <- runif(nsim,meanI - deltaI, meanI + deltaI)

#model of measurement
randR = randV/(randI-(randV/10E5))
meanR = mean(randR)

#function for calculating the shortest coverage interval
sci <- function (values, alpha){
  sortedSim <- sort(values)
  nsim <- length(values)
  covInt <- sapply(1:(nsim-round((1-alpha)*nsim)), function(i) {
    sortedSim[1+round((1-alpha)*nsim)+(i-1)]-sortedSim[1+(i-1)]})
  lcl <- sortedSim[which(covInt==min(covInt))]
  ucl <- sortedSim[1+round((1-alpha)*nsim)+(which(covInt==min(covInt))-1)]
  c(lcl, ucl)
}

#shortest 95% coverage interval
(simInt <- round(sci(randR, alpha=.05), 4))

#plot the histogram
hist(randR, breaks = seq(min(randR)-0.07, max(randR)+0.05 ,0.02), probability =
1, axes = F, border = "white", col = rgb(0.75,0,0),xlab = "Resistance [Ohm]",ylab
= "Density of Probability",main = "")
vgrid <- seq(0, 1, 0.2)
abline(h=vgrid, col="grey", lty=2)
hist(randR, add=TRUE, breaks = seq(min(randR)-0.07, max(randR)+0.05 ,0.02),
probability = 1, axes = F, border = "white", col = rgb(0.75,0,0),xlab =
"Resistance [Ohm]",ylab = "Density of Probability",main = "")
abline(v=simInt, col="gray", lty=1, lwd=2)
axis(side=1, at=seq(0,100,0.1))
axis(side=2, at=seq(0,1, 0.2), labels = seq(0,1, 0.2))

#output quantity
meanR
(simInt[2]-simInt[1])/2
```

APPENDIX E – POWER MEASUREMENT

```
#trials numbers
nsim = 10^7

#input quantities
meanU = 4.340
meanI = 95.969E-3
deltaU = .195E-3
deltaI= 1.13E-3

#generate of random numbers
randV <- runif(nsim,meanU - deltaU, meanU + deltaU)
randI <- runif(nsim,meanI - deltaI, meanI + deltaI)

#model of measurement
randP = randI-(2*randV/10e5)
meanP = mean(randP)

#function for calculating the shortest coverage interval
sci <- function (values, alpha){
  sortedSim <- sort(values)
  nsim <- length(values)
  covInt <- sapply(1:(nsim-round((1-alpha)*nsim)), function(i) {
    sortedSim[1+round((1-alpha)*nsim)+(i-1)]-sortedSim[1+(i-1)]})
  lcl <- sortedSim[which(covInt==min(covInt))]
  ucl <- sortedSim[1+round((1-alpha)*nsim)+(which(covInt==min(covInt))-1)]
  c(lcl, ucl)
}

#shortest 95% coverage interval
(simInt <- round(sci(randP, alpha=.05), 4))

#plot the histogram
hist(randP)
hist(randP,  breaks  =  seq(min(randP)-0.0001,  max(randP)+0.0001  ,0.0001),
probability  =  1,  axes  =  F,  border  =  "white",  col  =  rgb(0.75,0,0),xlab  =
"Resistance [Ohm]",ylab = "Density of Probability",main = "")
vgrid <- seq(0, 1, 0.2)
abline(h=vgrid, col="grey", lty=2)
hist(randP,  add=TRUE,  breaks  =  seq(min(randP)-0.07,  max(randP)+0.05  ,0.02),
probability  =  1,  axes  =  F,  border  =  "white",  col  =  rgb(0.75,0,0),xlab  =
"Resistance [Ohm]",ylab = "Density of Probability",main = "")
abline(v=simInt, col="gray", lty=1, lwd=2)
axis(side=1, at=seq(0,100,0.1))
axis(side=2, at=seq(0,1, 0.2), labels = seq(0,1, 0.2))

#output quantity
meanR
(simInt[2]-simInt[1])/2
```