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Gaussian Intrinsic Entanglement

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Gaussovská vnitřní kvantová provázanost

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Declaration
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Abstract	The thesis deals with a proposed entanglement measure called Gaussian intrinsic entanglement (GIE) [1]. It presents the theory of entanglement measures, the theory of Gaussian states and the theory of GIE itself. GIE had been analytically calculated for two-mode Gaussian states including all symmetric partial minimum uncertainty states, weakly mixed asymmetric squeezed thermal states with partial minimum uncertainty, and weakly mixed symmetric squeezed thermal states. We extend the theory of GIE by analytical formulas for the class of Gaussian states with minimum negativity for fixed global and local purities (GLEMS) [2] and we calculate the GIE for special examples of these states. Further, we compare our results with another entanglement measure called Gaussian Rényi-2 entanglement of formation (GR2EoF) [3] and strengthen the hypothesis of equivalence of these entanglement measures.
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Abstrakt	Práce se zabývá navrženou mírou kvantové provázanosti nazvanou gaussovská vnitřní kvantová provázanost [1]. Představuje teorii měr kvantové provázanosti, teorii gaussovských stavů a samotnou teorii gaussovské vnitřní kvantové provázanosti. Gaussovská vnitřní kvantová provázanost byla analyticky spočtena pro dvoumódové gaussovské stavy zahrnující všechny symetrické stavy s částečnou minimální neurčitostí, slabě smíšené asymetrické stlačené termální stavy s částečnou minimální neurčitostí a slabě smíšené symetrické stlačené termální stavy. Tuto teorii jsme rozšířili o analytické výpočty pro třídu gaussovských stavů s minimální negativitou pro fixované globální a lokální purity a spočetli gaussovskou vnitřní kvantovou provázanost pro konkrétní příklady těchto stavů. Dále jsme naše výsledky porovnali s jinou mírou kvantové provázanosti nazvanou Gaussian Rényi-2 entanglement of formation a tím posílili hypotézu ekvivalence těchto dvou měr.
Klíčová slova	kvantová fyzika, provázanost, gaussovská vnitřní kvantová provázanost, gaussovské stavy, míra provázanosti
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Contents

Introduction

'Not only does God play dice but...he sometimes throws them where they cannot be seen.' - Stephen Hawking

Even though, Stephen Hawking used this comparison talking about black holes, it sufficiently describes how helpless a brain of man can be when it comes to quantum physics. On the quantum level, objects act differently. We have no experience with this character from classical world, and so most of our intuition, that we are used to rely on, fails. Surely, it was necessary to find the best mathematical description of phenomena in quantum world and the linear vector and matrix algebra hit the jackpot.

As a consequence of the vector characteristics of quantum states, the principle of superposition occurs, which is the cause of a correlation between quantum objects, that has no analogy in classical world. For instance, let us imagine two photons that are prepared in a state where both of them have horizontal or vertical polarization. If the two states with well defined local polarization are superimposed, the individual properties become uncertain, yet the global properties are still well defined. This results in correlation between the quantum systems which is called quantum entanglement and there is no doubt that it is one of the most interesting things in quantum physics.

With the development of quantum theory and experiment, entanglement transformed from a theoretical phenomenon to the valuable physical concept that can be observed and used in protocols. Further, with the onset of quantum information theory, it began to play a key role in quantum communication and enabled for example to implement quantum teleportation, which still sounds rather as a science fiction than a serious physical experiment.

It is known about physicists, that we like to work with idealized objects and it is not different when it comes to quantum physics. Speech is about so-called pure states, which are states that contain the maximum attainable information about the state. However, just like in classical physics we never really work with the material points and ideal homogeneous spheres, we cannot investigate only pure states. In practice, we always have less information about the state. Then we talk about mixed states. Unfortunately, the definition of quantum entanglement via the principle of superposition applies only to pure states, so naturally some other more generic definition is needed. To get it, we use the fact, that entangled states can be, or cannot be created using some specifics types of operations. In the case of pure states, we say that the pure-state entanglement is a correlation between quantum systems that cannot be created from pure product states using local unitary operations. For the mixed states we generalize the operations and we define mixed-state entanglement as a correlation that cannot be created by local operations and classical communication (LOCC).

When the entanglement and its presence is clearly defined, we need to determine quality of prepared entangled states, in other words, we need to quantify entanglement. This is the reason, why we need entanglement measures. There are several ways that lead to entanglement quantification but each of them has its pitfalls. The first issue is the fact that we can talk about entanglement measure only if it satisfies several axioms. As we expect, entanglement measure should be non-negative and it should be zero if and only if the state is not entangled or in other words, if the state is separable. Further, the entanglement measure should not increase under LOCC. These three axioms are very intuitive main axioms of entanglement measure and above that they define so-called entanglement monotone [4]. However, entanglement measure should satisfy four other axioms. These are reduction to marginal von Neumann entropy on pure states, convexity, additivity on tensor product and asymptotic continuity. Beyond the condition of satisfying these seven axioms, there is another snag we meet in theory of entanglement measures. The proposed entanglement measures are either computable, or they are usable in protocols, but not both. Clearly, the quantum information theory needs a measure, that will satisfy all seven mentioned axioms and moreover it offers a compromise between these two extremes.

There is a candidate, which could solve the situation, that is called intrinsic entanglement (IE). It is proposed entanglement measure, which was born from an idea of changing the order of optimization in the definition of classical measure of entanglement proposed by Gisin and Wolf [5]. They came up with an idea of entanglement quantification based on classical secret key agreement [6]. If we consider two communicating parties, Alice and Bob, and an adversary Eve, in the protocol of classical secret key agreement, Alice and Bob try to generate a common string of bits called secret key, about which Eve has no information. The generation of secret key is possible only if there are the so-called secret correlations between Alice's and Bob's variables. Further, we can quantify the secret correlations using the quantity called intrinsic conditional information [7]. It is interesting that the condition to establish secret correlations is very similar to definition of quantum entanglement. Namely, secret correlations cannot be established by local operations and public communications. It is obvious, that replacing the word 'public' by 'classical', we get the definition of quantum entanglement. Naturally, if there is an analogy in some way between secret correlations and quantum entanglement, one wonders if they can be quantified using the same principle. Finally, this is what lead Gisin and Wolf to define above mentioned classical measure of entanglement. Unfortunately, same as the most of proposed entanglement measures, it has several drawbacks. Primarily, it has not been proved, that the classical measure of entanglement does not increase under LOCC, so we do not even know, if one of the most important conditional axioms is fulfilled. Besides, it is among the measures of entanglement that are hard to compute.

For the time being, IE have been studied for the Gaussian scenario. There all the states, channels and measurements are Gaussian. In quantum physics, Gaussian states are defined as states with Gaussian Wigner function. This is very significant class of states and it is convenient to work with it. Firstly, Gaussian states are fully characterized by vector of first moments and covariance matrix. Secondly, Gaussian scenario is also experimentally feasible. If we apply the theory of intrinsic entanglement to Gaussian scenario, then we talk about Gaussian IE (GIE). Here we overcome the obstacle given by Gisin and Wolf's classical measure of entanglement, because not only it has been proven that GIE is monotonic under LOCC but also it is easier to be computed. Moreover, it is faithful, i.e., it is zero if and only if the state is separable. GIE has been already calculated for the following classes of two-mode Gaussian states, namely, all pure states and symmetric states with a three-mode purification, weakly mixed asymmetric squeezed thermal states with a three-mode purification and symmetric squeezed thermal states. In this thesis, we will investigate GIE for the class of Gaussian states with minimum negativity for fixed global and local purities (GLEMS) [2].

Another interesting fact, which comes to the surface is that there is possible equivalence between GIE and another proposed entanglement measure known as Gaussian Rényi-2 entanglement of formation (GR2EoF) [3]. The existing calculated results are equal for GIE and GR2EoF, which leads to the assumption that these two proposed entanglement measures are equal on all bipartite Gaussian states. This idea is very significant because GR2EoF has many relevant properties. Namely, it does not increase under Gaussian LOCC, it is additive on two-mode symmetric states and it can be interpreted in the context of the sampling entropy for the Wigner quasiprobabilty distribution, it satisfies [8] monogamy inequality [9] and also the Gaussian Rényi-2 version of the Koashi-Winter monogamy relation [10]. Finally, it can be analytically calculated for all two-mode Gaussian states with a three-mode purification, all symmetric states, and two-mode squeezed thermal states and numerically for all two-mode Gaussian states. All these properties would transfer to GIE and vice versa, if the equivalence was proven in general.

Chapter 1

Methods

In this section we will introduce the basic principles of quantum theory. We will explain, how quantum mechanics differs from classical mechanics and we will show the mathematical tools used to the description of quantum entanglement.

1.1 Introduction to Quantum Mechanics

Firstly, let us introduce three postulates of quantum mechanics.
The first postulate says, that to each physical system, there is assigned a complex separable Hilbert space $\mathscr{H}$, so-called state space of the system. In mathematical terms, Hilbert space is a complete vector space with a scalar product. Separable Hilbert space is a Hilbert space with orthonormal base formed by countable set of vectors.
According to the second postulate, each state of the considered system corresponds to a vector ray from the given Hilbert space, i.e one-dimensional subspace of the Hilbert space. In this Thesis, as it is usual in quantum mechanics, we will use Dirac's bra-ket symbolic, where a column vector is denoted by $|\psi\rangle$.
The last postulate says that each measurable physical quantity, i.e observable of the given system with state space $\mathscr{H}$, corresponds to a self-joining operator A on $\mathscr{H}$, for which $A = A^\dagger$ occurs, where $A^\dagger$ is so-called Hermitian transposed operator, which is a transposed and complex conjectured operator $A^\dagger = \left(A^\mathsf{T}\right)^*$.
In addition, results of measurement of given physical quantity are eigenvalues of the operator, which describes the quantity. After measurement the state of the system collapses into an eigenstate of the quantity corresponding to the measured eigenvalue.

A normalized state vector $|\psi\rangle$ contains the maximal attainable information about the state. This kind of states we call pure states. Due to the probability character of quantum mechanics, we perform the measurement on the sufficiently large set of particles. The state vector $|\psi\rangle$ describes so-called pure statistical set, in which each particle is prepared in the same state $|\psi\rangle$.
As a consequence of various imperfections of the real equipment preparing the particles, we always have less information about the state of prepared particles. In general, we know that the particle of the set is in the state $|\psi_j\rangle$ with probability p_j. This kind of states we call mixed states and they cannot be described with a single ket vector. Instead, it is described with a density matrix ρ. In general, it can be written as

$$\rho = \sum_j p_j \left|\psi_j\right\rangle \left\langle\psi_j\right|, \tag{1.1}$$

where $0 \leq p_j \leq 1$, $\forall j$ and $\sum_j p_j = 1$ occurs.
The density matrix is a Hermitian operator $\rho = \rho^\dagger$, it is positively semi-definitive $\langle\psi| \rho |\psi\rangle \geq 0$, $\forall |\psi\rangle$ of trace one $\operatorname{Tr} \rho = 1$.

1.2 Quantum Entanglement

In quantum mechanics we can generate composite quantum systems consisting of a pair subsystems labeled as A and B. Let $\mathscr{H}_A$ and $\mathscr{H}_B$ be the Hilbert spaces of the systems. If a state $|\Psi\rangle_{AB}$ cannot be written as a product state, i.e.

$$|\Psi\rangle_{AB} \neq |\psi\rangle_A |\phi\rangle_B , \tag{1.2}$$

where $|\psi\rangle_A$ and $|\phi\rangle_B$ are local states of subsystems A and B, then the state $|\Psi\rangle_{AB}$ is called entangled. Otherwise, non-entangled states we call separable states.
Now, we will take a look at some examples of separable and entangled pure states.
Let us take two systems A and B with Hiblert spaces $\mathscr{H}_A$ and $\mathscr{H}_B$ and the states are given by kets $|0\rangle_A$ and $|0\rangle_B$, which are the basis kets of the relevant Hilbert spaces. Then to the composite system we will assign the Hiblert space $\mathscr{H}_{AB} = \mathscr{H}_A \otimes \mathscr{H}_B$ and the state of the composite system will be given by ket $|\Psi\rangle_{AB} = |0\rangle_A \otimes |0\rangle_B$.
On the other hand, the example of entangled pure states can be written as $|\Psi_-\rangle = \frac{1}{\sqrt{2}} (|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B)$, where $\{|0\rangle_A , |1\rangle_A\}$ $(\{|0\rangle_B , |1\rangle_B\})$ are basis kets of the Hiblert space $\mathscr{H}_A$ $(\mathscr{H}_B)$.
Here the ket $|\Psi_-\rangle$ is one of four Bell states, which are maximally entangled pure states.

In the context of mixed states, a state ρ_{AB} is called entangled if it cannot be written as the following mixture of product states

$$\rho^{(AB)} \neq \sum_i p_i \rho_i^{(A)} \otimes \rho_i^{(B)}. \tag{1.3}$$

In physical terms, quantum entanglement is a synonym for a sort of correlations in quantum mechanics. The difference between classical and quantum correlation is based on their resources. Let us imagine two parties communicating via a quantum channel. In practice, every quantum channel is lossy which leads to the depreciation of transmitted information. To increase the amount of the reached information, the parties can use two options. Firstly, the quality of the communication can increase by improving the quality of local operations performed by individual parties. Secondly, they can use classical communication to coordinate the quantum operations of the opposite party. Local operations and classical communication (LOCC) are operations that can create classical but not quantum correlations. Hence, the quantum entanglement can be defined as a sort of correlations that cannot be created by LOCC [31].

Chapter 2

Entanglement measures

Having defined the concept of entanglement we now move to quantification of the amount of entanglement in a given quantum state. To do so, we use entanglement measures.

2.1 Examples of entanglement measures

Establishment and investigation of entanglement measures is highly motivated, since they play an important role in many cases. Namely, in the sphere of theoretical physics they provide bounds on several hardly computable quantities [32] or they are essential tool in proofs of impossibility [33] or limitation [34] in some quantum-information protocols. Utilization of entanglement measures does not end with theoretical quantum physics. In experiment, they are used to estimate quality of prepared entangled states [35] and entangling gates [36] and they are indispensable in verification of successful demonstration of some protocols in quantum communication, for instance entanglement distillation [37].

In general, entanglement measure is a mathematical quantity that should possess features associated with properties of entanglement. These are summarized in seven axioms that any good entanglement measure should satisfy. Firstly, entanglement measure should be non-negative function. Secondly, it should be zero on all separable states. Thirdly, it should not increase under LOCC. A function satisfying these three axioms is so-called entanglement monotone [4]. The other four axioms say that it should reduce to von Neumann entropy on pure states, it should be convex, additive on tensor product and asymptotically continuous function [31].

For now, most existing entanglement quantifiers either do not satisfy some of the mentioned axioms, or it has not been proven, yet. Another issue is that known entanglement measures are either physically meaningful or computable, but not both.
Let us introduce some of the measures, which quantify the entanglement in different ways.

We will begin with logarithmic negativity [11], which is defined as

$$E_N(\rho) = \log \|\rho^{\mathrm{T}_A}\|_1, \tag{2.1}$$

where ρ^{T_A} denotes the partial transpose of ρ with respect to party A and the trace norm $\|\rho^{\mathrm{T}_A}\|_1$ is defined as $\|\rho^{\mathrm{T}_A}\|_1 = tr|\rho^{\mathrm{T}_A}|$. It has been proven that logarithmic negativity is a monotonic function under LOCC and it satisfies two other axioms to belong to the class of entanglement monotones. Further, it has an operation interpretation as a cost of entanglement under positive partial transpose preserving operations (PPT-operations) [12]. In other words, it quantifies entanglement pursuant to how much a partial transpose of the given state deviates from a physical state. Moreover, one can see that it is easily computable. However, logarithmic negativity is not a convex function, so we cannot talk about a full-value entanglement measure.

On the other hand, there exist entanglement measures with very good operation meaning. These are distillable entanglement [13] and entanglement of formation [14].
Entanglement distillation is a process of using LOCC operations to transform a certain number of non-maximally entangled states into a smaller number of approximately maximally entangled states. Hereupon, distillable entanglement is a maximal number of maximally entangled states per copy that can be distilled from many copies of a given state (,i.e. in the asymptotic limit $n \to \infty$ of n identically prepared systems in the considered state), using LOCC. Unfortunately, even though this measure offers very good operational meaning, when it comes to its evaluation for general mixed states, it is extraordinarily difficult and for now, it has not been done, yet.
Entanglement of formation is closely related to distillable entanglement. Actually, it is its dual measure. It defines a number of maximally entangled states needed in order to prepare copies of a particular state [38]. What is more, it is a upper bound of distillable entanglement. Entanglement of formation provides a compromise between computability and physical meaning, however it is still in question whether it is additive or not.

Another option, how to quantify the entanglement are geometric measures [15]. This class of measures quantifies entanglement via distance of the investigated state from the set of separable states. However, the sophistication of their computation increases with increasing the number of included subsystems and also their physical meaning has yet to be unveiled.

On top of what have been said, none of the previously mentioned entanglement measures satisfies all seven axioms that a good entanglement measure should. Finally, there exists a measure that does and for the time being it is the only one known. The measure in question is the so-called squashed entanglement [16]. It is defined as an infimum of quantum conditional mutual information $I\left(A;B|E\right)$ of an extension of the investigated quantum state ρ_{AB} with respect to all the extensions ρ_{ABE}, i. e.

$$E_{sq}\left(\rho_{AB}\right) = \inf\left\{\frac{1}{2}I\left(A;B|E\right):\ \rho_{AB} = \mathrm{Tr}_E\rho_{ABE}\right\}. \tag{2.2}$$

Even though this way of entanglement quantification is most promising, its evaluation is extremely hard.

2.2 Entanglement quantification based on secret key agreement

The invention of the squashed entanglement the most promising measure discovered to date, has been in fact inspired by the so-called intrinsic information, being a quantity quantifying the amount of secret correlations in a given classical probability distribution. In fact, the intrinsic information [6] inspired invention of another measure called classical mutual information.

The secret key agreement protocol is a classical cryptographical protocol in which there are two honest parties Alice and Bob knowing correlated random variables A and B, and an eases dropper Eve, who knows a random variable E. These variables are distributed according to a probability distribution P_{ABE}. Alice and Bob can use noiseless but insecure communication channel to which Eve has a full access. The goal of the protocol is to generate a secret key and reduce the amount of information that Eve obtains about generated secret key to be negligible (Fig. 2.1). Generating such a secret key is conditioned by secret correlations between variables A and B, which are correlations that cannot be established by local operations and public communication. These secret correlations can be quantified and the quantity has been called intrinsic conditional information.

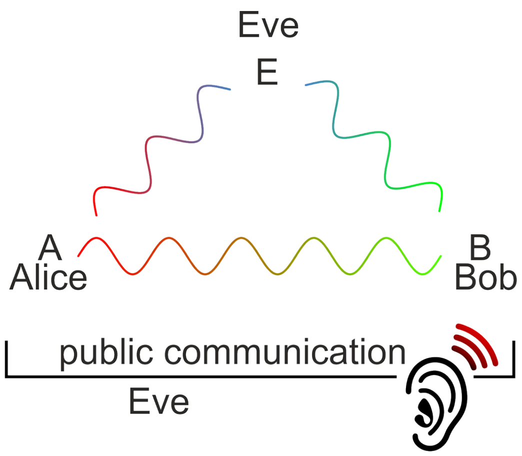


Figure 2.1: Secret key agreement protocol

If we pass to quantum key agreement, the probability distribution P_{ABE} is replaced by a quantum state vector $\Psi \in \mathscr{H}_A \otimes \mathscr{H}_B \otimes \mathscr{H}_E$, where $\mathscr{H}_A(\mathscr{H}_B,\mathscr{H}_E)$ is Hilbert space of Alice's (Bob's, Eve's) system. Additionally, Eve can carry out generalized measurements, which means that the set $\{|z\rangle\}$ is not generally an orthonormal basis but any set generating $\mathscr{H}_E$ and fulfilling the completeness condition $\sum_z |z\rangle\langle z| = \mathbb{1}_{\mathscr{H}_E}$. After all the parties carry out their measurements, they obtain the probability distribution P_{ABE}. Further, Alice and Bob's marginal distribution P_{AB} is analogical with the partial state ρ_{AB} that is obtained by tracing over $\mathscr{H}_E$, i.e., $\rho_{AB} = \mathrm{Tr}_{\mathscr{H}_E}(P_\Psi)$.

It should be add, that quantum entanglement and classical intrinsic information are not only analogies in notions but they can be swapped onto each other by a quantum measurement $P(A;B|E) = \mathrm{Tr}\left(|\Psi\rangle_{ABE}\langle\Psi|\,\Pi_A \otimes \Pi_B \otimes \Pi_E\right)$, i. e, that the probability distribution P_{ABE} can be obtained from Ψ by performing the measurements in certain basis and then the probability distribution P_{ABE} has strictly positive intrinsic information if and only if the state ρ_{AB} is entangled. Nevertheless, this does not pay for all cases. For instance, if Alice and Bob do not choose the measurement basis wisely, the intrinsic information can be zero, even though ρ_{AB} is entangled. Conversely, Eve can perform such a bad measurements that will make the intrinsic information positive even if ρ_{AB} is separable. This leads to conclusion, that intrinsic conditional information cannot generally quantify quantum entanglement and some optimization over all possible measurements on all sides must be involved. This lead Gisin and Wolf to propose an entanglement measure called classical measure of entanglement [5]

$$\mu\left(\rho_{AB}\right) := \min_{\{|z\rangle\}} \left(\max_{\{|x\rangle\},\{|y\rangle\}} \left(I\left(A; B \downarrow E\right)\right)\right), \tag{2.3}$$

where $I\left(A; B \downarrow E\right)$ is intrinsic conditional information between A and B given E, which is further maximized over all conditional purifications $|\Psi\rangle$ Alice's and Bob's measurements and then it is minimized over all Eve's measurements.

Classical measure of entanglement seems to be a good candidate for entanglement measure, as the max-min optimization guarantees, that the intrinsic information is strictly positive if and only if ρ_{AB} is entangled and so the first condition for good entanglement measure is fulfilled. Moreover, it reduces to von Neumann entropy on pure states and it is a convex function. Unfortunately, the monotonicity under LOCC remains a question. Secondly, it is hard to compute it for most of the mixed states.

By changing the order of optimization in definition of classical measure of entanglement (2.3), we get so-called intrinsic entanglement (IE), which is another proposed entanglement measure [1]. This measure will be more precisely discussed later.

Chapter 3

Gaussian states

In this section, we focus on a specific class of quantum states, so-called Gaussian states. These are quantum states for which the Wigner function is Gaussian. This class is very significant due to its convenient mathematical properties and moreover, these states can be easily prepared in experiments. In more detail, Gaussian states can be fully described by a vector of first moments and by the covariance matrix of second moments and henceforth they are characterized by a finite number of parameters, despite living in an infinite-dimensional Hilbert state space. Above that, first moments can be set to zero by displacement without any loss of generality for studying entanglement. Further, pure Gaussian states naturally saturate Heisenberg uncertainty relations and they are extremal on von Neumann entropy, mutual information, conditional entropy, secret key and some entanglement measures.
In laboratory, they can be realized by systems implying light, atomic ensembles, trapped ions, or optomechanical systems [17]. What is more, they can be easily transformed by simple linear optical elements such as beam splitters and squeezers as well as they can be conveniently measured by homodyne detection.

3.1 Characteristics of Gaussian states

In the beginning, let us focus on required mathematical apparatus and its application on Gaussian states.
Gaussian states naturally occur in continuous variable (CV) systems. These are systems with infinite dimensional Hilbert state space $\dim \mathscr{H} = \infty$, and they can be realized, e.g., by light modes. Each mode is characterized by canonically conjugate operators, most often by the position operator $\hat{x}$ and the momentum operator $\hat{p}$, or equivalently, by the annihilation operator $\hat{a}$ and the creation operator $\hat{a}^{\dagger}$. These operators are canonically conjugate variables and they satisfy canonical commutation rule, which in the case of position and momentum operators reads as

$$[\hat{x},\hat{p}] = i. \tag{3.1}$$

The most common example of CV system is quantum linear harmonic oscillator, which can be realized for instance by a mode of the electromagnetic field or a vibration mode of a trapped ion.

Previous description occurs for single mode systems. If we want to work with systems of N modes, the Hilbert space is given by the tensor product of Hilbert spaces of particular systems

$$\mathscr{H}_N = \bigotimes_{i=1}^{N} \mathscr{H}. \tag{3.2}$$

To desribe these systems we need $2N$ quadrature operators $\hat{x}_1,\hat{p}_1,\hat{x}_2,\hat{p}_2,\ldots \hat{x}_N,\hat{p}_N$, which fulfill canonical commutation rules

$$[\hat{x}_i,\hat{p}_j] = i\delta_{ij}, \quad [\hat{x}_i,\hat{x}_j] = [\hat{p}_i,\hat{p}_j] = 0, \tag{3.3}$$

where δ_{ij} is the Kronecker symbol. For our purposes is convenient to arrange the operators into a vector

$$\hat{\mathbf{r}} = (\hat{x}_A,\hat{p}_A,\ldots \hat{x}_N,\hat{p}_N)^{\mathrm{T}}. \tag{3.4}$$

Then, the canonical commutation rule given by Eq.(3.3) can be written compactly as

$$[\hat{r}_i,\hat{r}_j] = i\Omega_{Nij}, \tag{3.5}$$

where

$$\Omega_N = \bigoplus_{i=1}^{N} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.6}$$

is the so called symplectic matrix.

As it was mentioned above, Gaussian states are defined as quantum states with Gaussian-shaped Wigner quasiprobability distribution. This function is defined as

$$W(\mathbf{r}) = \frac{1}{(2\pi)^N} \int e^{i\mathbf{x}'^{\mathrm{T}}\cdot\mathbf{p}} \left\langle \mathbf{x} - \frac{\mathbf{x}'}{2} \right| \hat{\rho} \left| \mathbf{x} + \frac{\mathbf{x}'}{2} \right\rangle \mathrm{d}^N\mathbf{x}', \tag{3.7}$$

and it can be estimated from characteristic function of density matrix by using Fourier transformation. In the Eq.(3.7) $\mathbf{r} = (x_A,p_A,\ldots,x_N,p_N)^{\mathrm{T}}$, $\mathbf{x}'^{\mathrm{T}} \cdot \mathbf{p} = \sum_{i=1}^{N} x'_i p_i, \mathrm{d}^N\mathbf{x}' = \mathrm{d}x'_1\mathrm{d}x'_2\ldots\mathrm{d}x'_N$, and

$$\left| \mathrm{x} \pm \frac{\mathbf{x}'}{2} \right\rangle = \left| x_1 \pm \frac{x'_1}{2} \right\rangle \otimes \left| x_2 \pm \frac{x'_2}{2} \right\rangle \otimes \ldots \otimes \left| x_N \pm \frac{x'_N}{2} \right\rangle. \tag{3.8}$$

In the case of Gaussian states it reduces to

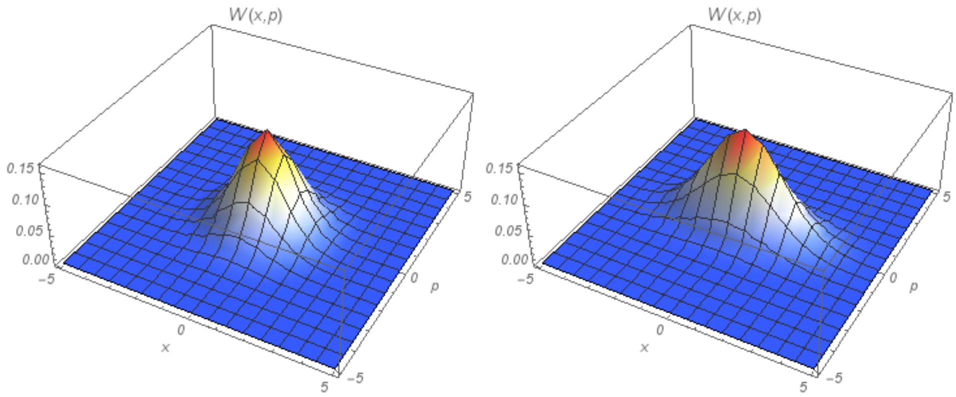
$$W(\mathbf{r}) = \frac{e^{-(\mathbf{r}-\mathbf{d})^{\mathrm{T}}\gamma^{-1}(\mathbf{r}-\mathbf{d})}}{\pi^N\sqrt{\det\gamma}}, \tag{3.9}$$

where $\mathbf{d}$ is a vector of first moments with elements $d_i = \langle \hat{r}_i \rangle = \mathrm{Tr}(\hat{\rho}\hat{r}_i)$ and γ is the so-called covariance matrix (CM) with elements

$$\gamma_{ij} = \langle \hat{r}_i\hat{r}_j + \hat{r}_j\hat{r}_i \rangle - 2\langle \hat{r}_i \rangle \langle \hat{r}_j \rangle. \tag{3.10}$$

The examples of Wigner function for vacuum state and squeezed state are illustrated in Fig. 3.1. Due to the Gaussianity of the Wigner function, all N-mode Gaussian states are fully characterized by $2N \times 2N$ CM (3.10) and by $2N \times 1$ vector of first moments $\hat{\mathbf{r}}$. The vector of first moments can be set to zero by local displacements. Since this operation has no influence on the entanglement of the state, it will be assumed to be zero in the rest of the Thesis.

Additionally, entanglement does not change under local unitaries and therefore we



(a) Vacuum state (b) Squeezed state

Figure 3.1: Wigner quasiprobability distribution for vacuum state (left) and state squeezed in quadrature x (right).

can work without lost of any generality only with the standard form [18] of CM for two-mode Gaussian states

$$\gamma_{AB} = \begin{pmatrix} a & 0 & c_x & 0 \\ 0 & a & 0 & c_p \\ c_x & 0 & b & 0 \\ 0 & c_p & 0 & b \end{pmatrix}, \tag{3.11}$$

where $c_x \geq |c_p| \geq 0$, to which any two-mode CM can be brought by local Gaussian unitaries. Further, we will work with CMs satisfying $c_x c_p < 0$, since all the other states are separable [18] and thus any faithful entanglement measure is zero for them.
It is convenient to work with other standard form of CM with new parameters k_x and k_p

$$\gamma_{AB} = \begin{pmatrix} a & 0 & k_x & 0 \\ 0 & a & 0 & -k_p \\ k_x & 0 & b & 0 \\ 0 & -k_p & 0 & b \end{pmatrix}, \tag{3.12}$$

where $k_x \equiv c_x$, $k_p \equiv |c_p| = -c_p$ and $k_x \geq k_p > 0$. Using the parameters of CM (3.12) we can determine conditions that will ensure, that we work with CM of a physical state, which is also entangled.
Firstly, the CM (3.12) of a physical quantum state must satisfy the Heisenberg uncertainty principle $\gamma_{AB} + i\Omega \geq 0$ [18], which in terms of parameters a, b, k_x and k_p reads as [19]

$$\begin{aligned} (ab - k_x^2)(ab - k_p^2) + 1 &\geq a^2 + b^2 - 2k_x k_p, \\ ab - k_x^2 &\geq 1. \end{aligned} \tag{3.13}$$

Secondly, CM (3.12) describes an entangled state if and only if

$$(ab - k_x^2)(ab - k_p^2) + 1 < a^2 + b^2 + 2k_x k_p. \tag{3.14}$$

3.2 Symplectic diagonalization

Additionally, we need to introduce unitary operations that preserve Gaussian character of states. These operations are called Gaussian unitary operations and they are represented by a real $2N \times 2N$ symplectic matrix S, which satisfies condition

$$S\Omega_N S^{\mathrm{T}} = \Omega_N. \tag{3.15}$$

On the CM level, the symplectic transformation acts as

$$\gamma' = S\gamma S^{\mathrm{T}}. \tag{3.16}$$

The symplectic transformations are related to Williamson's [40] theorem, which says, that for any N-mode CM γ there always exists a symplectic transformation S which transforms it to the Williamson's normal form

$$S\gamma S^{\mathrm{T}} = \mathrm{diag}\left(\nu_1,\nu_1,\ldots,\nu_N,\nu_N\right), \tag{3.17}$$

where $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_N$ are the so-called symplectic eigenvalues. For our further calculations it is also convenient to introduce the symplectic rank R of a CM, which is the number of its symplectic eigenvalues different from one.

Previous mathematical description of symplectic transformation can be interpreted to physical language as a global Gaussian unitary, which can bring any two-mode Gaussian state into a tensor product of two thermal states with CMs $\nu_j \mathbb{1}, j = 1,2$, where $\mathbb{1}$ is the 2×2 identity matrix.
For the derivation of GIE, which will be explained in Chapter 5, we must know not only the symplectic eigenvalues of CM (3.12) but also the respective symplectic matrix S, that brings the CM to the Williamson's normal form [26].
We can derive the symplectic matrix S for any CM (3.12). In particular, for the two-mode case which is relevant here, the Williamson's normal form is given by

$$\gamma_{AB}^{(0)} \equiv \left(\bigoplus_{i=1}^{R} \nu_i \mathbb{1}\right) \bigoplus \mathbb{1}_{2(2-R)}. \tag{3.18}$$

and the symplectic eigenvalues can be calculated from the eigenvalues of CM γ_{AB} using the method of Ref. [39]. Here, the matrix S is sought in the form of a product $S = \left(\bigotimes_{i=1}^{2} U^*\right) V^{\mathrm{T}}$, where

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \tag{3.19}$$

and V contains in its columns the eigenvectors $u_{\nu_1} = (ix_1,x_3,ix_2,x_4)^{\mathrm{T}}$ and $w_{\nu_2} = (ix_5,x_7,ix_6,x_8)^{\mathrm{T}}$ corresponding to the eigenvalues ν_1 and ν_2 of the matrix $i\Omega\gamma_{AB}$. The eigenvectors u_{ν_1} and w_{ν_2} are chosen to S be real. Using the aforementioned method, considering that S must satisfy symplectic condition $S\Omega_2 S^{\mathrm{T}} = \Omega_2$ (3.15) and it does not contain any x-p elements, we will get the set of equations for variables x_j, $j = 1,2,...,8$, which solution will allow us to express the symplectic matrix S in the form

$$S = \begin{pmatrix} x_1 & 0 & x_2 & 0 \\ 0 & x_3 & 0 & x_4 \\ x_5 & 0 & x_6 & 0 \\ 0 & x_7 & 0 & x_8 \end{pmatrix} \tag{3.20}$$

where the parameters $x_1,\ldots,x_8$ are some functions of parameters a,b,k_x and k_p of CM (3.12) [26].

Pursuant to the relations between parameters of CM (3.12) a,b,k_x and k_p we will sort our investigation into several cases. As it will be showed in Chapter 4, GIE has been already calculated for some of them. For instance, these are the symmetric states fulfilling the conditions $a = b$ and $k_x \geq k_p > 0$. In this case, symplectic matrix S describes a composition of a balanced beam splitter described by symplectic matrix

$$U_{BS} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix} \tag{3.21}$$

and local squeezing transformations of quadratures x_A and p_B described by symplectic matrices

$$S_A = \begin{pmatrix} z_A^{-1} & 0 \\ 0 & z_A \end{pmatrix}, \quad S_B = \begin{pmatrix} z_B & 0 \\ 0 & z_B^{-1} \end{pmatrix}, \tag{3.22}$$

with $z_A = \sqrt[4]{\frac{a+k_x}{a-k_p}} > 1$ and $z_B = \sqrt[4]{\frac{a+k_p}{a-k_x}} > 1$.
Thus, the symplectic matrix S reads as a product

$$S_1 = (S_A \oplus S_B) U_{BS}. \tag{3.23}$$

Our investigation will be focused on states with a symplectic matrix describing a composition of local squeezers described by symplectic matrices (3.22) with eigenvalues $z_A = \sqrt{\frac{a}{\nu_1}} > 1$ and $z_B = \sqrt{\frac{b}{\nu_2}} > 1$ and quantum non-demolition interaction (QND) with symplectic matrix given as

$$S_{QND} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & q \\ -q & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.24}$$

with interaction constant $q = \frac{k_x}{a} = \frac{k_p}{b}$.
Symplectic matrix S_{QND} itself can be further decomposed to the product of two beam splitters and local squeezers on modes A and B as

$$S_{QND} = U_{BS_1} (S_{Sq_A} \oplus S_{Sq_B}) U_{BS_2}, \tag{3.25}$$

where

$$U_{BS_j} = \begin{pmatrix} T_j \mathbb{1} & R_j \mathbb{1} \\ -R_j \mathbb{1} & T_j \mathbb{1} \end{pmatrix}, \quad j = 1,2 \tag{3.26}$$

are symplectic matrices describing the beam splitters, in general not balanced, with transmission parameters T_j and reflection parameters R_j, and local squeezers with symplectic matrices

$$S_{Sq_k} = \begin{pmatrix} e^{2r_k} & 0 \\ 0 & e^{-2r_k} \end{pmatrix}, \quad k = A,B \tag{3.27}$$

with corresponding squeezing parameter r_k. Then the symplectic matrix of intended states will read as

$$S_2 = (S_A \oplus S_B) S_{QND} \tag{3.28}$$

and it corresponds to the states with parameters of CM (3.12) fulfilling relations $a > b$ and $bk_x = ak_p$. In our work we will investigate also the states with aforementioned relations fulfilling the opposite inequality that can be achieved again by using the method

of Ref. [39] or by using the mode exchange symmetry, which is described in the last section of Chapter 5.
Above that we will also deal with the more generic asymmetric states, i. e., states for which $a \neq b$ and $bk_x \neq ak_p$ occurs.

3.3 Construction of CM of purification γ_π

We will use Williamson theorem to find a Gaussian purification of state given by CM (3.12), so we can construct a CM of the purification γ_π. The pure Gaussian state $|\Psi\rangle_{ABE}$ must satisfy condition $\mathrm{Tr}_E |\Psi\rangle_{ABE} \langle\Psi| = \rho_{AB}$. The construction of CM γ_π depends on the symplectic rank of the CM (3.12).
In the case of $R = 0$ and so $\gamma'_{AB} = \mathrm{diag}\,(1{,}1{,}1{,}1)$ describes a pure state $|\Psi\rangle_{AB}$, purifying subsystem E is independent of modes A and B. Thus, $|\Psi\rangle_{ABE} = |\psi\rangle_{AB} |\varphi\rangle_E$, where $|\varphi\rangle_E$ is the state of purifying system E, whence $\gamma_\pi = \gamma_{AB} \oplus \gamma_E$, where γ_E is a CM of $|\varphi\rangle_E$.
On the other hand, in the case of $R > 0$, the principle of purification lies in replacing modes with $\nu_i > 1$, $i = 1, \ldots, R$ in $\gamma_{AB}^{(0)}$ with modes of the two-mode squeezed vacuum state. Performing this purification we get (2+R)-mode CM

$$\gamma_\pi^{(0)} = \begin{pmatrix} \gamma_{AB}^{(0)} & \gamma_{ABE}^{(0)} \\ \left(\gamma_{ABE}^{(0)}\right)^{\mathrm{T}} & \gamma_E^{(0)} \end{pmatrix}, \tag{3.29}$$

where

$$\gamma_{ABE}^{(0)} = \begin{pmatrix} \bigoplus_{i=1}^{R} \sqrt{\nu_i^2 - 1}\sigma_z \\ \mathbb{O}_{2(2-R)} \times 2R \end{pmatrix}, \quad \gamma_E^{(0)} = \bigoplus_{i=1}^{R} \nu_i \mathbb{1}, \tag{3.30}$$

where $\sigma_z = \mathrm{diag}\,(1, -1)$ is the diagonal Pauli-z matrix and $\mathbb{O}_{I\times J}$ is the $I \times J$ zero matrix.
Applying a symplectic matrix $S \oplus \mathbb{1}_E$ to CM (3.29) gives us the final CM of purification

$$\gamma_\pi = \begin{pmatrix} \gamma_{AB} & \gamma_{ABE} \\ \gamma_{ABE}^{\mathrm{T}} & \gamma_E \end{pmatrix}, \tag{3.31}$$

with

$$\gamma_{ABE} = S^{-1}\gamma_{ABE}^{(0)}, \quad \gamma_E = \gamma_E^{(0)} \quad \text{and} \quad \gamma_{AB} = S^{-1}\gamma_{AB}^{(0)}\left(S^{\mathrm{T}}\right)^{-1}. \tag{3.32}$$

3.4 Gaussian states with minimum negativity for fixed global and local purities (GLEMS)

In this Thesis, we will focus on the specific class of Gaussian states called GLEMS which are Gaussian states with minimum negativity for fixed global and local purities [2],[20]. All GLEMS satisfy the condition $\nu_2 = 1$ and they are the least entangled Gaussian states. Further we will work only with the states fulfilling condition $a +$ $+ b - 1 > \sqrt{\det\gamma_{AB}}$ since they contain all entangled GLEMS. Besides, this subclass saturates Heisenberg uncertainty principle expressed by inequality (3.13) and so these states are states with partial minimum uncertainty.
At last, GLEMS naturally appear in cryptographical settings involving two-mode

squeezed vacuum with one mode transmitted through a purely lossy channel.

As we already mentioned, for all GLEMS $\nu_2 = 1$, whereas the other symplectic eigenvalue can be calculated as

$$\nu \equiv \nu_1 = \sqrt{\det \gamma_{AB}}. \tag{3.33}$$

The Thesis is focused on GLEMS with $\nu > 1$. We will further select this set into subsets according to the certain conditions expressed by parameters of symplectic matrix S (3.20).

We will focus our investigation on the four sets of GLEMS. Each set is characterized by the relations between parameters of S (3.20). In the Thesis we will use the following denotation:

$$\begin{aligned}
\rho_{AB}^{(4)} : &\quad a > b,\, bk_x = ak_p, \\
\rho_{AB}^{(5)} : &\quad a < b,\, ak_x = bk_p, \\
\rho_{AB}^{(6)} : &\quad a > b,\, bk_x \neq ak_p, \\
\rho_{AB}^{(7)} : &\quad a < b,\, ak_x \neq bk_p.
\end{aligned} \tag{3.34}$$

In addition, as we will show in Chapter 5, the transition from the state $\rho_{AB}^{(4)}$ to the state $\rho_{AB}^{(5)}$ and from the state $\rho_{AB}^{(6)}$ to the state $\rho_{AB}^{(7)}$ corresponds to the mode exchange between modes A and B. This fact brings some advantages to the calculations of Gaussian intrinsic entanglement, an entanglement measure introduced in the following chapter.

Chapter 4

Gaussian Intrinsic Entanglement

We already introduced theory of entanglement measures in Chapter 2, we mentioned some of existing entanglement measures and explained why prospecting new ways of quantifying entanglement is still an open topic. Once we also defined Gaussian states in Chapter 3, we have in our hands all tools to handle with a new proposed entanglement measure called Gaussian intrinsic entanglement (GIE) [1].

4.1 Definition and properties of GIE

Intrinsic entanglement (IE) is a new proposed entanglement measure. It is based on the idea of changing the order of optimization in the definition of classical measure of entanglement μ (2.3) and thus it is defined as

$$E_{\downarrow}(\rho_{AB}) = \sup_{\{|A\rangle,|B\rangle\}} \left\{ \inf_{\{|E\rangle,|\Psi\rangle\}} \left[I\left(A;B\downarrow E\right) \right] \right\}. \tag{4.1}$$

Because of the cryptographically based origin of IE, its definition results from the definition of intrinsic information in classical secret key agreement, therefore in the Eq.(4.1) we use notation as follows. A (Alice) and B (Bob) are two subsystems of entangled state ρ_{AB} and E (Eve) is the purifying subsystem. Further, $|\Psi\rangle$ is a purification of state ρ_{AB}, i.e. $\mathrm{Tr}_E \left|\Psi\right\rangle \left\langle\Psi\right| = \rho_{AB}$.
We already mentioned, that IE was born from the idea of changing the order of optimization in the definition of classical measure of entanglement (2.3). By comparing its definition to the definition of IE (4.1), one can see, that relation $E_{\downarrow} \leq \mu$, due to the max-min inequality [21].
If we restrict our investigation to the cases, in which all the states, measurements and channels are Gaussian, then we talk about Gaussian IE (GIE). In the case of two-mode state ρ_{AB} with purifying subsystem E it is defined as

$$E_{\downarrow}^{G}\left(\rho_{AB}\right) = \sup_{\Gamma_A,\Gamma_B} \inf_{\Gamma_E} \left[I\left(A;B|E\right) \right], \tag{4.2}$$

where

$$I\left(A;B|E\right) = \frac{1}{2} \ln \left(\frac{\det \sigma_A \det \sigma_B}{\det \sigma_{AB}} \right). \tag{4.3}$$

Further,

$$\sigma_{AB} = \gamma_{AB|E} + \Gamma_A \oplus \Gamma_B, \tag{4.4}$$

$\sigma_{A,B}$ are local submatrices of σ_{AB} and Γ_A (Γ_B) is a single-mode CM of pure-state Gaussian measurement on a mode A (B).
Next,

$$\gamma_{AB|E} = \gamma_{AB} - \gamma_{ABE} \left(\gamma_E + \Gamma_E\right)^{-1} \gamma_{ABE}^{\mathrm{T}} \tag{4.5}$$

is a CM of conditional mutual state $\rho_{AB|E}$ [22]. This state is obtained by a Gaussian measurement with CM Γ_E on purifying subsystem E of the purification with CM γ_π given by Eq. (3.29). Furthermore, applying symplectic transformation to the CM (4.5), we get

$$\gamma_{AB|E}^{(0)} = \gamma_{AB}^{(0)} - \gamma_{ABE}^{(0)} \left(\gamma_E^{(0)} + \Gamma_E\right)^{-1} \left(\gamma_{ABE}^{(0)}\right)^{\mathrm{T}}, \tag{4.6}$$

where CMs $\gamma_{AB}^{(0)}$,$\gamma_{ABE}^{(0)}$ and $\gamma_E^{(0)}$ are given by Eqs.(3.18) and (3.30).

Now, we have defined general analytical formula of GIE 4.2. Let us mention the properties, that GIE offers as an entanglement measure. Firstly, due to the reversed optimization in its definition, it is easier to be computed than the classical measure of entanglement. In addition, it has been proven [1] that GIE is zero if and only if the given state ρ_{AB} is separable. Likewise, it does not increase under Gaussian local trace-preserving operations and classical communication (GLTPOCC).
Since it was found that optimum in GIE is always reached by homodyne and heterodyne detection, it is experimentally feasible and thus it is physically meaningful.
Lastly, another questioned property belonging to a good entanglement measure is an operational meaning. There exists a conjecture, that GIE is an upper bound of speed of generating a secret key in a cryptographical scenario with so-called public Eve, i.e. a secret key agreement protocol in which Eve gives up some of the information and shares it with communicating parties Alice and Bob.

4.2 Existing results of GIE

Before we discuss our results of GIE in Chapter 5, let us show already acquired analytical formulas of GIE published in [1] for following three sets of Gaussian states.

Firstly, GIE was already calculated for symmetric GLEMS [2]. Surely, since this is a class of GLEMS, it fulfills all the properties of GLEMS mentioned in Chapter 3. Additionally, due to the symmetry, the condition $a = b$ occurs. All the states, which belong to the class of entangled symmetric GLEMS we denote as $\rho_{AB}^{(1)}$ and GIE is given by

$$E_\downarrow^G\left(\rho_{AB}^{(1)}\right) = \ln\left(\frac{a}{\sqrt{a^2 - k_p^2}}\right). \tag{4.7}$$

If also $k_x = k_p \equiv k$ is fulfilled, then the state reduces to the pure state $\left(\equiv \rho_{AB}^{(p)}\right)$ and also condition $a^2 - k^2 = 1$ occurs. Thus, one can see that GIE in this case is given by

$$E_\downarrow^G\left(\rho_{AB}^{(p)}\right) = \ln\left(a\right). \tag{4.8}$$

Another class of states are symmetric squeezed thermal states [23]. For all these states a condition $k_x = k_p \equiv k$ is fulfilled. Once more, these states are symmetric, hence the $a = b$ occurs. Additionally, they are entangled if and only if $a - k < 1$ is satisfied

[24],[25]. We denote them as $\rho_{AB}^{(2)}$ and GIE for all these states which also satisfy inequality $a \leq 2.41$ reads as

$$E_{\downarrow}^{G}\left(\rho_{AB}^{(2)}\right) = \ln\left[\frac{(a-k)^2+1}{2(a-k)}\right]. \tag{4.9}$$

The third states of the set are asymmetric squeezed thermal GLEMS. We denote them as $\rho_{AB}^{(3)}$ and they fulfill conditions $k_x = k_p \equiv k$, $\nu_2 = 1$ and $a \neq b$. For all these states satisfying $\sqrt{ab} \leq 2.41$ GIE is equal to

$$E_{\downarrow}^{G}\left(\rho_{AB}^{(3)}\right) = \ln\left(\frac{a+b}{|a-b|+2}\right). \tag{4.10}$$

These results were compared to the results of other entanglement measures for the given states. Their relations will be introduced in the following section.

4.3 Relation to other entanglement measures

One of the questioned issues of any entanglement measure is its relation to other entanglement measures. Firstly, let us investigate a relation between GIE and the most popular entanglement measure, logarithmic negativity, in general defined by Eq. (2.1). For a symmetric two-mode Gaussian state ρ_{AB}, the logarithmic negativity reads as

$$E_N(\rho_{AB}) = \max\left[0, -\ln\tilde{\nu}_-\right]. \tag{4.11}$$

where $\tilde{\nu}_- = \sqrt{(a-k_x)(a-k_p)}$ is the lower symplectic eigenvalue of the partial transpose $\rho_{AB}^{T_A}$ of the state ρ_{AB} with respect to mode A (where the symbol T_A stays for the partial transposition with respect to mode A). [1].
In the case of symmetric states, for which $a = b$, we can fuse GIE formulas (4.7) and (4.9) into

$$E_{\downarrow}^{G}(\rho_{AB}) = \begin{cases} \ln\left[\frac{\tilde{\nu}_- + (\tilde{\nu}_-)^{-1}}{2}\right], & \text{if } \tilde{\nu}_- < 1; \\ 0, & \text{if } \tilde{\nu}_- \geq 1, \end{cases} \tag{4.12}$$

One can see, that for the symmetric states, both, logarithmic negativity and GIE, are monotonically decreasing functions and by comparing Eqs. (4.12) and (4.11), one finds a relation $E_N(\rho_{AB}) \geq E_{\downarrow}^{G}(\rho_{AB})$.
Nevertheless, in the case of the Gaussian states with positive partial transpose, aforementioned relation does not apply anymore, since logarithmic negativity vanishes but GIE is strictly positive, due to its faithfulness property, and thus we can conclude that there is no fixed hierarchy between these two measures.

Anyway, during the investigation of GIE, it showed up, that on pure states it is equal to another Gaussian entanglement measure, so-called Gaussian-Rényi-2 entanglement of formations (GR2EoF) [3]. Since GR2EoF is computable for symmetric and also some classes of asymmetric Gaussian states, the results of GIE and GR2EoF were also compared for the states $\rho_{AB}^{(1)}$, $\rho_{AB}^{(2)}$ and $\rho_{AB}^{(3)}$ and surprisingly, they were equal. This lead to a hypothesis that GIE is also equivalent to GR2EoF on all bipartite Gaussian states.
GR2EoF is equipped with significant properties. Firstly, it is monotonic under all Gaussian LOCC. Secondly, it satisfies monogamy inequality [9] and Gaussian Rényi-2

version of Koashi-Winter monogamy relation [10]. Finally, it is additive on two-mode symmetric states and it offers an operational interpretation as sampling entropy for Wigner quasiprobability distribution.

If the equivalence between GIE and GR2EoF was proven, then all the properties of GR2EoF would transfer to GIE and vice versa. Hence, one of our goals in this work will not be only to calculate GIE for the given states but also to find the formulas of GR2EoF and compare our results, so we can strengthen or disprove the hypothesis.

Chapter 5

Results

We investigated GIE for three other types of GLEMS, namely the states $\rho_{AB}^{(4)}$, $\rho_{AB}^{(5)}$ and $\rho_{AB}^{(6)}$ which are defined at the end of Chapter 3. We showed that GIE is symmetric with respect to the exchange of modes A and B, which allows us to derive it also for an example of a state $\rho_{AB}^{(7)}$. Finally, we computed the results of logarithmic negativity and GR2EoF for the same states and compared them to GIE.

5.1 Derivation of GIE

Let us begin with brief explanation of the strategy we used to obtain new results of GIE. The first step is to calculate an upper bound [26] of GIE

$$U\left(\rho_{AB}\right) \equiv \inf_{\Gamma_E}\left[\mathcal{I}_c^G\left(\rho_{AB|E}\right)\right], \tag{5.1}$$

where

$$\mathcal{I}_c^G\left(\rho_{AB|E}\right) = \sup_{\Gamma_A,\Gamma_B}\left[I\left(A;B|E\right)\right] \tag{5.2}$$

is so-called Gaussian classical mutual information (GCMI) [27] of the conditional state $\rho_{AB|E}$ with CM (4.5) and inequality $E_{\downarrow}^{G}\left(\rho_{AB}\right) \leq U\left(\rho_{AB}\right)$ holds due to the max-min inequality [21]. In general, the GCMI can be calculated only numerically. However, if the CM parameters a, b and c_x satisfy condition

$$\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} + \frac{1}{\sqrt{ab}} - \sqrt{ab - c_x^2} \geq 0, \tag{5.3}$$

the optimization in Eq. (5.2) can be performed analytically and the GCMI reads as

$$\mathcal{I}_c^G\left(\rho_{AB|E}\right) = \frac{1}{2}\ln\left(\frac{\tilde{a}\tilde{b}}{\tilde{a}\tilde{b} - \tilde{c}_x^2}\right), \tag{5.4}$$

where $\tilde{a}$, $\tilde{b}$ and $\tilde{c}_x$ are parameters of the standard form of the CM of the conditional state $\rho_{AB|E}$

$$\gamma_{AB|E} = \begin{pmatrix} \tilde{a} & 0 & \tilde{c}_x & 0 \\ 0 & \tilde{a} & 0 & \tilde{c}_p \\ \tilde{c}_x & 0 & \tilde{b} & 0 \\ 0 & \tilde{c}_p & 0 & \tilde{b} \end{pmatrix}, \tag{5.5}$$

where $\tilde{c}_x \geq |\tilde{c}_p| \geq 0$. The GCMI in the form of Eq. (5.4) is reached by double homodyne detection of quadratures x_A and x_B on modes A and B.
In terms of the parameters of CM (5.5) condition (5.3) reads as

$$\sqrt{\frac{\tilde{a}}{\tilde{b}}} + \sqrt{\frac{\tilde{b}}{\tilde{a}}} + \frac{1}{\sqrt{\tilde{a}\tilde{b}}} - \sqrt{\tilde{a}\tilde{b} - \tilde{c}_x^2} \geq 0. \tag{5.6}$$

Further, it shows up, that for all the cases, a double homodyne detection of quadratures x_A and x_B is the optimal measurement on modes A and B for any CM Γ_E. We used this fact in our calculations and restricted our investigation to the states fulfilling a condition

$$\sqrt{\nu_1\nu_2} \leq 2 + \frac{1}{\sqrt{ab}}, \tag{5.7}$$

since for the states satisfying (5.7) the condition (5.3) is fulfilled and the double homodyning is an optimal measurement.

It is advantageous to rewrite Eq.(5.1) as

$$U(\rho_{AB}) = -\ln\sqrt{1 - h_{\min}}, \tag{5.8}$$

with

$$h_{\min} \equiv \inf_{\Gamma_E}\left(\frac{\tilde{c}_x^2}{\tilde{a}\tilde{b}}\right). \tag{5.9}$$

Hence, minimization of RHS of Eq.(5.1) boils down to minimization of RHS in Eq.(5.9).

The next step is to saturate the upper bound, since the saturated bound is equal to the requested GIE $U(\rho_{AB}) = E_{\downarrow}^{G}(\rho_{AB})$. To perform the saturation, we find convenient expression of conditional mutual information [26], which simplifies its minimization over all CMs Γ_E. Additionally, using the double homodyne detection on quadratures x_A and x_B, which is characterized by CMs $\Gamma_A^t \equiv \mathrm{diag}\left(e^{-2t}, e^{2t}\right)$ and $\Gamma_B^t \equiv \mathrm{diag}\left(e^{-2t}, e^{2t}\right)$ in the limit $t \to +\infty$, the minimization of conditional mutual information reduces to finding the quantity

$$L(\rho_{AB}) \equiv \inf_{\Gamma_E}\left[I_h(A;B|E)\right] = \frac{1}{2}\ln\left(\frac{ab}{ab - k_x^2}\right) + \frac{1}{2}\ln\mathscr{K}_{\min}, \tag{5.10}$$

where

$$\mathscr{K}_{\min} \equiv \inf_{\Gamma_E}\mathscr{K}_h \tag{5.11}$$

and $\mathscr{K}_h$ is obtained from the quantity

$$\mathscr{K} = \frac{\det\left(\Gamma_E + X_A\right)\det\left(\Gamma_E + X_B\right)}{\det\left(\Gamma_E + X_{AB}\right)\det\left(\Gamma_E + \gamma_E\right)}, \tag{5.12}$$

where

$$X_j = \gamma_E - \gamma_{jE}^{\mathrm{T}}\left(\Gamma_j + \gamma_j\right)^{-1}\gamma_{jE} \tag{5.13}$$

with CMs Γ_A and Γ_B replaced by the CMs characterizing homodyne detection $\Gamma_A = \Gamma_A^t$ and $\Gamma_B = \Gamma_B^t$ and taking the limit $t \to +\infty$.

Finally, if we find some states ρ_{AB}, for which the quantity $L(\rho_{AB})$ (5.10) is equal to the upper bound $U(\rho_{AB})$ (5.1), then we have found at fixed measurements on modes A and B the minimal conditional mutual information with respect to all CMs Γ_E, which saturates the upper bound (5.1) and thus it coincidents with GIE.

5.2 GIE for $\rho_{AB}^{(4)}$, $\rho_{AB}^{(5)}$ and $\rho_{AB}^{(6)}$

The procedure described in the previous chapter can be used to evaluation of GIE for four sets of states defined in Eqs.(3.34). We have reached the following results.

Firstly, we performed the derivation of GIE for the state $\rho_{AB}^{(4)}$, which is characterized by conditions $a > b$ and $bk_x = ak_p$. Besides, for this state Eq. (3.33) boils down to

$$\nu \equiv \nu_1 = \sqrt{a^2 - k_x k_p}. \tag{5.14}$$

In the calculation of upper bound (5.1) we utilized an inequality [26]

$$\frac{\tilde{c}_x^2}{\tilde{a}\tilde{b}} \geq \frac{k_x k_p}{a^2}. \tag{5.15}$$

This lower bound is tight, since it is attained for CM Γ_E with parameters and conditions corresponding to the homodyne detection of quadrature x_E on mode E. Thus, $h_{\min} = k_x k_p / a^2$ and then applying this result to Eq.(5.1) one gets

$$U\left(\rho_{AB}^{(4)}\right) = \ln\left(\frac{a}{\nu}\right). \tag{5.16}$$

Next, to evaluate the quantity (5.10), we need to express the matrix (5.13) in Eq.(5.12) for the case of homodyne detection in the limit $t \to +\infty$, i.e.

$$X_k = \nu \mathbb{1} - \alpha_k \left|0\right\rangle \left\langle 0\right|, \tag{5.17}$$

$k = A$,B,AB and

$$\alpha_A = \left(\frac{\nu^2 - 1}{a}\right) x_3^2, \quad \alpha_B = \left(\frac{\nu^2 - 1}{b}\right) x_4^2,$$
$$\alpha_{AB} = \left(\frac{\nu^2 - 1}{ab - k_x^2}\right)\left(a x_4^2 + b x_3^2 - 2k_x x_3 x_4\right) \tag{5.18}$$

and $\left|0\right\rangle = (1{,}0)^{\mathrm{T}}$.
Using the formula [28]

$$\det\left(\chi + \left|c\right\rangle \left\langle r\right|\right) = \left(1 + \left\langle r\right| \chi^{-1} \left|c\right\rangle\right) \det \chi, \tag{5.19}$$

which holds for any invertible matrix χ, and we can express the Eq.(5.12) as

$$\mathscr{K}_h = \frac{(1 - \alpha_A Q)(1 - \alpha_B Q)}{1 - \alpha_{AB} Q}, \tag{5.20}$$

with the variable

$$Q \equiv \left\langle 0\right| \left(\Gamma_E + \nu \mathbb{1}\right)^{-1} \left|0\right\rangle = \frac{\tau\left[\cosh(2t)\cos(2\varphi)\right] + \nu}{\tau^2 + 2\tau\cosh(2t)\nu + \nu^2}. \tag{5.21}$$

Hence, evaluating the quantity corresponds to minimizing the function (5.20) with the single variable Q (5.21) over $\varphi \in [0{,}\pi)$, $\tau \geq 1$ and $t \geq 0$. Using the conditions $a > b$ and $bk_x = ak_p$, Eqs. (5.18) simplify to

$$\alpha_A = \alpha_{AB} = \frac{\nu^2 - 1}{\nu}, \quad \alpha_B = \left(\frac{\nu^2 - 1}{\nu}\right)\frac{k_x^2}{ab} \tag{5.22}$$

and thus

$$\mathscr{K}_h = 1 - \alpha_B Q. \tag{5.23}$$

Since $\alpha_B > 0$, we get the following chain of equations:

$$\mathscr{K}_{\min} \equiv \inf_{\Gamma_E} \mathscr{K}_h = \inf_{Q \in \left(0, \frac{1}{\nu}\right)} \mathscr{K}_h = 1 - \left(\frac{\nu^2 - 1}{\nu^2}\right) \frac{k_x^2}{ab}, \tag{5.24}$$

where the infimum lies at the boundary point $1/\nu$, which can be reached by homodyne detection of quadrature x_E on the mode E characterized by parameters $\varphi = \pi/2$, $\tau = 1$ in the limit $t \to +\infty$.
If we substitute the final expression in Eq.(5.24) to Eq.(5.10), one finds

$$L\left(\rho_{AB}^{(4)}\right) = \ln\left(\frac{a}{\nu}\right), \tag{5.25}$$

which is equal to the upper bound (5.16) and thus GIE for the state $\rho_{AB}^{(4)}$ is given by

$$E_{\downarrow}^{G}\left(\rho_{AB}^{(4)}\right) = \ln\left(\frac{a}{\nu}\right). \tag{5.26}$$

Now, let us calculate GIE for a particular test state $\rho_{AB}^{\prime(4)}$. This state will be given by CM (3.12) with parameters $a = 2\sqrt{2}$, $b = k_x = \sqrt{2}$ and $k_p = 1/\sqrt{2}$, which satisfies all the conditions for the class of states $\rho_{AB}^{(4)}$. Also, we have to verify that our test state satisfies conditions (3.13),(3.14), (5.3) and (5.7), so we know that in our hands we have physical entangled state for which the homodyne detection is an optimal measurement and the upper bound can be analytically optimized. One can easily find out that our state fulfills all the conditions. Now, one can use the simple analytical formula (5.26) and reach the result

$$E_{\downarrow}^{G}\left(\rho_{AB}^{\prime(4)}\right) = \ln\left(2\sqrt{\frac{2}{7}}\right) \doteq 0.068. \tag{5.27}$$

If we move to the state $\rho_{AB}^{(5)}$ with conditions $a < b$, $ak_x = bk_p$ and

$$\tilde{\nu} \equiv \nu_1 = \sqrt{b^2 - k_x k_p}, \tag{5.28}$$

the procedure of derivation GIE will be analogous with the case of $\rho_{AB}^{(4)}$ and we will get the result

$$U\left(\rho_{AB}^{(5)}\right) = L\left(\rho_{AB}^{(5)}\right) = \ln\left(\frac{b}{\tilde{\nu}}\right) \tag{5.29}$$

and thus

$$E_{\downarrow}^{G}\left(\rho_{AB}^{(5)}\right) = \ln\left(\frac{b}{\tilde{\nu}}\right). \tag{5.30}$$

According to the symmetry with the respect to the exchange of parameters a and b, which occurs in all four conditions (3.13),(3.14), (5.3) and (5.7), as our test state $\rho_{AB}^{\prime(5)}$ we chose CM (3.12) with parameters $b = 2\sqrt{2}$, $a = k_x = \sqrt{2}$ and $k_p = 1/\sqrt{2}$. Once more, using the final formula (5.30), we get the result

$$E_{\downarrow}^{G}\left(\rho_{AB}^{\prime(5)}\right) = \ln\left(2\sqrt{\frac{2}{7}}\right) \doteq 0.068. \tag{5.31}$$

Further, two other type of states can be investigated, namely $\rho_{AB}^{(6)}$ and $\rho_{AB}^{(7)}$. In accordance with the conditions (3.34), these are the most generic GLEMS, as the only fulfilled condition is $\nu_2 = 1$ in these cases. For this reason, our derivation will not simplify as it did in previous cases and we have to perform the general optimization described in the first section. Since this is very sophisticated process we decided to reach the numerical result for some test state at first. As the test state for the case of $\rho_{AB}^{(6)}$, we took the CM (3.12) with parameters $a = 2\sqrt{2}$, $b = \sqrt{2}$ and $k_{x,p} = (\sqrt{97} \pm 1)/8$. Again, we certified that all the conditions (3.13),(3.14), (5.3) and (5.7) are fulfilled and the final result of GIE is

$$E_{\downarrow}^{G}\left(\rho_{AB}^{\prime(6)}\right) = \ln\left(\frac{6}{5}\right) = 0.182322. \tag{5.32}$$

In the case of $\rho_{AB}^{(6)}$, the analytical formula of GIE have not been derived so far, however, the upper bound (5.1) can be expressed as [26]

$$U\left(\rho_{AB}^{(6)}\right) = \ln\left(\frac{a^2 - b^2}{\sqrt{D}}\right), \tag{5.33}$$

where

$$D = \left(a^2 - b^2\right)^2 + 4M\tilde{M} \tag{5.34}$$

with

$$M \equiv ak_x - bk_p, \quad \tilde{M} \equiv bk_x - ak_p. \tag{5.35}$$

5.3 Symmetry with respect to the mode exchange

Now, we will show that we can transfer from the case $a < b$ to the case $a > b$ by exchanging modes A and B. We will perform this via a simple orthogonal transformation carried out by orthonormal transformation symplectic matrix

$$T = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{5.36}$$

Applying this transformation to our CM (3.12) γ_{AB} we get a CM in standard form with exchanged modes

$$\gamma_{BA} = T\gamma_{AB}T^{\mathrm{T}}. \tag{5.37}$$

One can easily infer, that if we have a CM γ_{AB} with $a < b$, then the transformed CM γ_{BA} will correspond to the case when we have CM γ_{AB} but with $b < a$.

The transformation of the symplectic matrix S (3.20) is performed as follows

$$S = S'T, \tag{5.38}$$

where S' is the symplectic matrix that brings γ_{BA} to the Williamson normal form. This transformation will lead to the change of parameters of the symplectic matrix (3.20), whereas the symplectic eigenvalues will be unchanged.

Finally, we can say that the mode exchange will not change our GIE results, because GIE is invariant under the transformation $S \to (O_A \oplus O_B)\, S$, where O_A and O_B are local orthogonal symplectic matrices [26].

By exchanging the parameter a in the GIE result for the state $\rho_{AB}^{(4)}$ (5.26) with the parameter b, one gets the the analytical formula of GIE for the state $\rho_{AB}^{(5)}$ (5.30) and vice versa. This corresponds to the aforementioned mode exchange between modes A and B and obviously.
Using the symmetry in the case of the state $\rho_{AB}^{(6)}$ (5.33), we get the upper bound for the state $\rho_{AB}^{(7)}$ in the form

$$U\left(\rho_{AB}^{(7)}\right) = \ln\left(\frac{b^2 - a^2}{\sqrt{D}}\right). \tag{5.39}$$

Further, we can calculate the numerical value of the test state $\rho_{AB}^{\prime(7)}$ with parameters $a = \sqrt{2}$, $b = 2\sqrt{2}$ and $k_{x,p} = \left(\sqrt{97} \pm 1\right)/8$, which will be

$$E_{\downarrow}^{G}\left(\rho_{AB}^{\prime(7)}\right) = \ln\left(\frac{6}{5}\right) = 0.182322. \tag{5.40}$$

As we expected, we got the same result as for the test state $\rho_{AB}^{\prime(6)}$.

The symmetry with respect to the mode exchange provides another way of reaching some of the GIE formulas or it can be used to verify them.

5.4 Comparing GIE to logarithmic negativity and GR2EoF

As it had been done before for the the states $\rho_{AB}^{(1)}$, $\rho_{AB}^{(2)}$ and $\rho_{AB}^{(3)}$, we compared our results with two other entanglement measures, i.e. logarithmic negativity and GR2EoF.

Although, there is not a hierarchy between logarithmic negativity and GIE for Gaussian states in general, it may still hold for two-mode Gaussian states. This is because there are no two-mode entangled Gaussian states with positive partial transposition and thus both, logarithmic negativity and GIE vanish on the two-mode PPT states. To support the conjecture, we therefore calculated also the logarithmic negativity for our examples.
To calculate the logarithmic negativity, we used Eq. (4.11). Since for our classes of GLEMS the symmetry $a = b$ does not hold anymore, the symplectic eigenvalue $\tilde{\nu}_-$ is the lower eigenvalue of the matrix $i\Omega\gamma_{AB}^{(\mathrm{T})}$, where

$$\gamma_{AB}^{(\mathrm{T})} = \begin{pmatrix} a & 0 & k_x & 0 \\ 0 & a & 0 & k_p \\ k_x & 0 & b & 0 \\ 0 & k_p & 0 & b \end{pmatrix} \tag{5.41}$$

is the CM of the partial transpose with respect to the mode A $\rho_{AB}^{\mathrm{T}_A}$.
By inserting the parameters of our test state $\rho_{AB}^{\prime(4)}$, one gets the result

$$E_N\left(\rho_{AB}^{\prime(4)}\right) = 0.243201. \tag{5.42}$$

Comparing it to the result of GIE for this state in Eq. (5.27), one can see that the inequality $E_N\left(\rho_{AB}^{\prime(4)}\right) = 0.243201 > E_{\downarrow}^{G}\left(\rho_{AB}^{\prime(4)}\right) = \ln\left(2\sqrt{\frac{2}{7}}\right) \doteq 0.068$ holds.

For the state $\rho'^{(5)}_{AB}$ we get the result

$$E_N\left(\rho'^{(5)}_{AB}\right) = 0.243201 \tag{5.43}$$

and by comparing it to the result given by Eq.(5.31) we get the relation $E_N\left(\rho'^{(5)}_{AB}\right) = 0.243201 > E^G_{\downarrow}\left(\rho'^{(5)}_{AB}\right) = \ln\left(2\sqrt{\frac{2}{7}}\right) \doteq 0.068$.
In the case of state $\rho'^{(6)}_{AB}$, the logarithmic negativity is

$$E_N\left(\rho'^{(6)}_{AB}\right) = 0.367816 \tag{5.44}$$

and once again, in relation to GIE, the inequality $E_N\left(\rho'^{(6)}_{AB}\right) = 0.367816 > E^G_{\downarrow}\left(\rho'^{(6)}_{AB}\right) = \ln\left(\frac{6}{5}\right) = 0.182322$ holds.
For the state $\rho'^{(7)}_{AB}$, we will get the same numerical values as for the state $\rho'^{(6)}_{AB}$, i.e., $E_N\left(\rho'^{(7)}_{AB}\right) = 0.367816 > E^G_{\downarrow}\left(\rho'^{(7)}_{AB}\right) = \ln\left(\frac{6}{5}\right) = 0.182322$.

One can see that for all the cases, we got the relation

$$E_N\left(\rho_{AB}\right) > E^G_{\downarrow}\left(\rho_{AB}\right), \tag{5.45}$$

which supports our conjecture, that this hierarchy may hold for two-mode Gaussian states.

Nevertheless, it showed up that there may be even more interesting relation to another entanglement measure.
After the introduction of GIE, the subsequent investigation led to a remarkable finding. All calculated formulas were equal to another entanglement measure called Gaussian-Rényi-2 entanglement of formation. This fact spurred the thought of a possible equivalence between these two entanglement measures. We computed GR2EoF for our states $\rho^{(4)}_{AB}$, $\rho^{(5)}_{AB}$, $\rho'^{(6)}_{AB}$ and $\rho'^{(7)}_{AB}$ to strengthen or disprove the hypothesis of equivalence.

Firstly, let us consider a two-mode state reduction of a pure state of three modes A_1, A_2 and A_3 with CM in standard form [29]

$$\gamma_{A_1,A_2,A_3} = \begin{pmatrix} a_1 & 0 & c_3^+ & 0 & c_2^+ & 0 \\ 0 & a_1 & 0 & c_3^- & 0 & c_2^- \\ c_3^+ & 0 & a_2 & 0 & c_1^+ & 0 \\ 0 & c_3^- & 0 & a_2 & 0 & c_1^- \\ c_2^+ & 0 & c_1^+ & 0 & a_3 & 0 \\ 0 & c_2^- & 0 & c_1^- & 0 & a_3 \end{pmatrix}, \tag{5.46}$$

where

$$c_i^{\pm} = \frac{\sqrt{a_{--}a_{+-}} \pm \sqrt{a_{-+}a_{++}}}{4\sqrt{a_j a_k}} \tag{5.47}$$

with

$$\begin{aligned} a_{\mp-} &= \left(a_i \mp 1\right)^2 - \left(a_j - a_k\right)^2, \\ a_{\mp+} &= \left(a_i \mp 1\right)^2 - \left(a_j + a_k\right)^2. \end{aligned} \tag{5.48}$$

Here, $|a_j - a_k| + 1 \leq a_i \leq a_j + a_k - 1$, where $\{i,j,k\}$ run all over all possible permutations $\{1,2,3\}$. Hereupon, the reduction to the state $\rho_{A_iA_j}$ of modes A_i and A_j will be characterized by CM $\gamma_{A_iA_j}$ and then the GR2EoF will read as [8]

$$E_{F,2}^{G}\left(\rho_{A_iA_j}\right) = \frac{1}{2}\ln g_k, \tag{5.49}$$

where [30]

$$g_k = \begin{cases} 1, & \text{if } a_k \geq \sqrt{a_i^2 + a_j^2 - 1}; \\ \frac{\zeta}{8a_k^2}, & \text{if } \alpha_k < a_k < \sqrt{a_i^2 + a_j^2 - 1}; \\ \left(\frac{a_i^2 - a_j^2}{a_k^2 - 1}\right)^2, & \text{if } a_k \leq \alpha_k. \end{cases} \tag{5.50}$$

Further,

$$\begin{gathered} \alpha_k = \left\{1 + \frac{\left(a_i^2 - a_j^2\right)^2}{2\left(a_i^2 + a_j^2\right)} + \frac{|a_i^2 - a_j^2|}{2\left(a_i^2 + a_j^2\right)}\left[\left(a_i^2 - a_j^2\right)^2 + 8\left(a_i^2 + a_j^2\right)\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}, \\ \delta = \left[(a_1 - a_2 - a_3)^2 - 1\right]\left[(a_1 + a_2 - a_3)^2 - 1\right]\left[(a_1 - a_2 + a_3)^2 - 1\right]\left[(a_1 + a_2 + a_3)^2 - 1\right], \\ \zeta = 2a_1^2 + 2a_2^2 + 2a_3^2 + 2a_1^2a_2^2 + 2a_1^2a_3^2 + 2a_2^2a_3^2 - a_1^4 - a_2^4a_3^4 - \sqrt{\delta} - 1. \end{gathered} \tag{5.51}$$

Thereafter, if we apply this to our state $\rho_{AB}^{(4)}$, then for the parameters of CM (5.46) $a_1 = a$, $a_2 = b$ and $a_3 = \nu$ occur for the case and noticeably $A_1 \equiv A$, $A_2 \equiv B$ and $A \equiv E$ are corresponding modes. Examining the Eq. (5.50), one will conclude that only the second branch in the equation applies and after some algebra, the result of GR2EoF is

$$E_{F,2}^{G}\left(\rho_{AB}^{(4)}\right) = \ln\left(\frac{a}{\nu}\right) \tag{5.52}$$

just like in the case of GIE in Eq. (5.26).

For the state $\rho_{AB}^{(5)}$ in the CM (5.46) we will use equations $a_1 = a$, $a_2 = b$ and $a_3 = \tilde{\nu}$. Again, in the Eq. (5.50) the second branch occurs and the final result is

$$E_{F,2}^{G}\left(\rho_{AB}^{(5)}\right) = \ln\left(\frac{b}{\tilde{\nu}}\right). \tag{5.53}$$

Once again, in accordance with Eq.(5.30), the result coincidents with the result of GIE.

Next, we calculated GR2EoF for the same concrete test state of the class $\rho_{AB}^{(6)}$, i.e. the state $\rho_{AB}^{\prime(6)}$ with parameters $a = 2\sqrt{2}$, $b = \sqrt{2}$ and $k_{x,p} = (\sqrt{97} \pm 1)/8$. For the CM (5.46) applies $a_1 = a$, $a_2 = b$ and $a_3 = \nu = \nu_1$. One finds out that $a_k < \alpha_k$ and thus the third branch of Eq. (5.50) has to be taken. Finally, the GR2EoF is equal to

$$E_{F,2}^{G}\left(\rho_{AB}^{\prime(6)}\right) = \ln\left(\frac{6}{5}\right). \tag{5.54}$$

Comparing the result with Eq. (5.32), one can see that GIE and GR2EoF are equal. Finally, using the same process, one finds out that GIE and GR2EoF are equal for the test state $\rho_{AB}^{\prime(7)}$ as well.
To sum it up, for all the new investigated cases, GIE showed up to be equal to GR2EoF just like in the previous investigations. This fact strongly supports the conjecture, that these two entanglement measures are equivalent.

Conclusion

The aim of the Thesis was investigation of GIE for a specific class of Gaussian states called GLEMS. After we introduced the general definition of GIE, we explained the way of derivation GIE for GLEMS and applied it to individual cases. The results we reached are summarized in Subsection 5.2.
Further, we compared GIE to the other two entanglement measures, namely logarithmic negativity and GR2EoF. Our results of GR2EoF for the investigated classes of states were equal to the result of GIE, which supported the hypothesis, that GIE and GR2EoF are equivalent entanglement measures.
There are many toppics following up on our results that are left open for future investigation, such as the proof of equivalence between GIE and GR2EoF, analytical formulas for the most generic GLEMS or the investigation of GIE for bipartite multimode states.

Bibliography

[1] L. Mišta, Jr. and R. Tatham, Gaussian Intrinsic Entanglement, Phys. Rev. Lett. **117**, 240505 (2016).

[2] G. Adesso, A. Safarini, and F. Illuminati, Determination of Continuous Variable Entanglement by Purity Measurements, Phys. Rev. Lett. **92**, 087901 (2004).

[3] G. Adesso, D. Girolami, and A. Serafini, Measuring Gaussian Quantum Information and Correlations Using the Rényi Entropy of Order 2, Phys. Rev. Lett. **109**, 190502 (2012).

[4] G. Vidal, Entanglement monotones, J. Mod. Opt. **47**, 355 (2000).

[5] N. Gisin and S. Wolf, Linking Classical and Quantum Key Agreement: Is there "Bound Information?", in Proceedings of CRYPTO 2000, Lecture Notes in Computer Science 1880 (Springer, Berlin, 2000) p. 482.

[6] U. M. Maurer, Secret Key Agreement by Public Discussion from Common Information, IEEE Trans. Inf. Theor. **39**, 733 (1993).

[7] U. M. Maurer and S. Wolf, Unconditionally secure key agreement and the intrinsic conditional information, IEEE Trans. Inf. Theor. **45**, 499 (1999).

[8] G. Adesso, D. Girolami, and A. Serafini, Measuring Gaussian Quantum Information and Correlations Using the Rényi Entropy of Order 2, Phys. Rev. Lett. **109**, 190502 (2012).

[9] V. Coffman, J. Kundu, and W. K. Wootters, Distributed entanglement, Phys. Rev. A **61**, 052306 (2000).

[10] M. Koashi and A. Winter, Monogamy of quantum entanglement and other correlations, Phys. Rev. A **69**, 022309 (2004).

[11] M. B. Plenio, The logarithmic negativity: A full entanglement monotone that is not convex, Phys. Rev. Lett. **95,** 090503 (2005).

[12] K. Audenaert, M. B. Plenio, and J. Eisert, Phys. Rev. Lett. **90,** 027901 (2003).

[13] J. Eirset, Entanglement in quantum information theory, Ph.D. thesis, University of Potsdam, 2001.

[14] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, Purification of Noisy Entanglement and Faithful Teleportation via Noisy Channels, Phys. Rev. Lett. **76**, 722 (1996).

[15] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Quantifying Entanglement, Phys. Rev. Lett. **78**, 2275 (1997).

[16] M. Christandl and A. Witner, "Squashed entanglement": An additive entanglement measure, J. Math. Phys. **45**, 829 (2004).

[17] Quantum Information with Continuous Variables of Atoms and Light, edited by N. J. Cerf, G. Leuchs, and E. S. Polzik (Imperial College Press, London, 2007).

[18] R. Simon, Peres-Horodecki Separability Criterion for Continuous Variable Systems, Phys. Rev. Lett. **84**, 2726 (2000).

[19] G. Giedke, L.-M. Duan, J. I. Cirac, and P. Zoller, Distillability criterion for all bipartite Gaussian states, Quantum Inf. Comput. **1**, 79 (2001).

[20] G. Adesso, A. Serafini, and F. Illuminati, Extremal entanglement and mixedness in continuous variable systems, Phys. Rev. A **70**, 022318 (2004).

[21] S. Boyd and L. Vandenberghe, Convex Optimization (Cambridge University Press, Cambridge, 2004).

[22] G. Giedke and J. I. Cirac, Characterization of Gaussian operations and distillation of Gaussian states, Phys. Rev. A **66**, 032316 (2002).

[23] A. Botero and B. Reznik, Modewise entanglement of Gaussian states, Phys. Rev. A **67**, 052311 (2003).

[24] G. Giedke, M. M. Wolf, O. Krüger, R. F. Werner, and J. I. Cirac, Phys. Rev. Lett. **91**, 107901 (2003).

[25] R. Simon, Phys. Rev. Lett. **84**, 2726 (2000).

[26] L. Mišta Jr. and Klára Baksová, Gaussian intrinsic entanglement for states with partial minimum uncertainty, Phys. Rev. A **97**, 012305 (2018).

[27] D. P. DiVincenzo, M. Horodecki, D. W. Leung, J. A. Smolin, and B. M. Terhal, Locking Classical Correlations in Quantum States, Phys. Rev. Lett. **92**, 067902 (2004).

[28] H. V. Henderson and S. R. Searle, On deriving the inverse of a sum of matrices, SIAM Rev. **23**, 53 (1981).

[29] G. Adesso, A. Serafini, and F. Illuminati, Multipartite entanglement in three-mode Gaussian states of continuous-variable systems: Quantification, sharing structure, and decoherence, Phys. Rev. A **73**, 032345 (2006).

[30] G. Adesso and F. Illuminati, Gaussian measures of entanglement versus negativities: Ordering of two-mode Gaussian states, Phys. Rev. A **72**, 032334 (2005).

[31] Martin B. Plenio and Shashank Virmani, An introduction to entanglement measures. Quant.Inf.Comput.7:1-51,2007

[32] E. M. Rains, A semidefinite program for distillable entanglement, IEEE Trans. Inf. Theory **47**, 2921 (2001).

[33] J. Eisert, S. Scheel, and M. B. Plenio, Distilling Gaussian States with Gaussian Operations is Impossible, Phys. Rev. Lett. **89**, 137903 (2002).

[34] M. Ohliger, K. Kieling, and J. Eisert, Limitations of quantum computing with Gaussian cluster states, Phys. Rev. A **82**, 042336 (2010).

[35] S. P. Walborn, P. H. Souto Ribeiro, L. Davidovich, F. Mintert, and A. Buchleitner, Experimental determination of entanglement with a single measurement, Nature (London) **440**, 1022 (2006).

[36] J. L. O'Brien, G. J. Pryde, A. Gilchrist, D. F. V. James, N. K. Langford, T. C. Ralph, and A. G. White, Quantum Process Tomography of a Controlled-NOT Gate, Phys. Rev. Lett. **93**, 080502 (2004).

[37] T. Yamamoto, M. Koashi, S¸. K. Ozdemir, and N. Imoto, Experimental extraction of an entangled photon pair from two identically decohered pairs, Nature (London) **421**, 343 (2003).

[38] Jens Eisert, "Entanglement in Quantum Information Theory" Dissertation (Dr. rer. nat.), University of Potsdam, Germany, 2001.

[39] A. Serafini, G. Adesso, and F. Illuminati, Unitarily localizable entanglement of Gaussian states, Phys. Rev. A **71**, 032349 (2005).

[40] John Williamson. "On the Algebraic Problem Concerning the Normal Forms of Linear Dynamical Systems". In: American Journal of Mathematics 58.1 (1936), pp. 141– 163. doi: 10.2307/2371062