Selected topics in fuzzy orderings

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I hereby declare that the thesis is my original work.
Parts of this thesis are based on the outcomes of joint scientific work with Radim Bělohlávek.

Abstract An ordering relation is a central concept in many areas of human activity. This work is concerned with ordering relations in the setting of fuzzy logic. We consider the notion of fuzzy order, where antisymmetry is inherently linked to a many-valued equality on the underlying universe. We thoroughly examine the origins of this concept, including the seemingly different point of view used in some works; provide remarks and observations on the existing studies; and prove new results. Then we offer a unifying concept of antisymmetry in the setting of fuzzy logic and thus also unified notion of fuzzy order. In particular, we prove that all the definitions of fuzzy order, we are concerned with, are mutually equivalent and also equivalent to the proposed generalized view. By doing so, we uncover that the link between fuzzy order and underlying fuzzy equality is even deeper than usually assumed. Finally, we utilize these new observations on the role of fuzzy equality by reconsidering the problem of Szpilrajn-like extension of fuzzy order and by providing a way to extend any fuzzy order into a linear fuzzy order in a broad class of fuzzy logics.

Mé ženě Veronice a
našemu synovi Jonášovi.
S láskou.

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Contents

Preface

An ordering relation is a central concept in many areas of human activity. In 1970s Zadeh (1971) coined generalizations of ordinary similarity and ordering relations into his, in that time novel, setting of fuzzy sets (Zadeh, 1965). Since this seminal paper appeared, many deep, theoretical results and applications were described and implemented. This thesis focuses on some basic aspects of the theory of fuzzy orderings. Namely the concept of fuzzy order itself, related axioms, a link to similarity relations, and a possibility of an extension of fuzzy ordering into a linear fuzzy ordering. We are interested in, arguably up-to-date most developed, approach where antisymmetry is defined with respect to underlying fuzzy equality and approaches which turned out to be equivalent. Note that this thesis does not reflect on other definitions of fuzzy order although many may be found in the literature. We focus only on the point of view where underlying similarity is taken into account, as this approach proved to be useful by great number of studies.

The thesis consists of three research papers (attached as Appendices A, B, and C) and this accompanying text with a brief summary of the obtained results, some additional observations, historical context, and plans for the future. The first and second papers, concerned with the concept of fuzzy order itself, are based on the outcomes of joint scientific work with my supervisor, Radim Belohlavek, without whom they would not be possible. The third paper is then devoted to a possibility of linearization of fuzzy order, i.e. to an extension of fuzzy order in a Szpilrajn-like way. Presented observations offer unifying view on up-to-date available definitions of fuzzy order with respect to fuzzy equality and some new arguments for equality-order connection to be taken into account even when studying further properties and applications of such fuzzy orders.

Note that this document is to be taken as accompanying text to the aforementioned studies. As such, it only briefly summarizes the most important results we obtained on the concept of fuzzy order itself, its definitions, and various aspects of its connection with underlying fuzzy equality. In particular all the proofs, auxiliary lemmas, many remarks, comments, and also some of the obtained results are omitted. If the reader is interested in some particular result, its proof, or some related information, it can be found in the attached papers in full detail.

Chapter 1

Preliminaries

We start by basics of ordinal order theory, fuzzy logic, fuzzy sets, and fuzzy relations. The hope is that the text is self contained and accessible even for a reader who does not work in the setting of fuzzy logic and order theory on the daily basis. If the reader is familiar with these topics then appendices of the first and third papers attached to this text may be used as a brief summary of this chapter.

One of the most fundamental concepts in mathematics is a relation, the formal counterpart of a relationship between entities in our world. We are concerned with particular type of relations – binary relations on a set. Such relations capture relationships between pairs of elements in a given situation. Arguably, the most important relationships in our perception of the world are of two kinds: the ones, which groups similar things together, and the ones, which compare objects to each other. Corresponding binary relations are called equivalences and orders, respectively.

In this chapter, we first briefly summarize the well known definitions and properties of binary relations on a set in general and of equivalences and orders in particular. Then, we move our attention to basics of fuzzy logic, especially to the way fuzzy relations and their properties are defined. The last part of present chapter is then devoted to fuzzy equivalences and in particular fuzzy equalities.

1.1 Binary relations on a set

Let U be a set. Any subset R of $U \times U$ is called a *binary relation* on U. For any $u, v \in U$ we say that u is related to v by R if $\langle u, v\rangle \in R$ – this is often denoted simply by $R(u, v)$ or uRv.

As relations are just special kinds of sets, we may carry out the well known set operations in a straightforward way. Moreover, we call a relation E an *extension of a relation* R if $R \subseteq E$.

There are numerous intriguing properties of binary relations on a given set of which the following will be of importance in subsequent chapters.

Definition 1.1.1. For a binary relation R on a set U, we define the following well-known properties:

$$R(u,u), \tag{reflexivity}$$
$$R(u,v) \Rightarrow R(v,u), \tag{symmetry}$$
$$R(u,v) \wedge R(v,u) \Rightarrow u = v, \tag{antisymmetry}$$
$$R(u,v) \Rightarrow \neg R(v,u), \tag{asymmetry}$$
$$R(u,v) \wedge R(v,w) \Rightarrow R(u,w), \tag{transitivity}$$
$$u \neq v \Rightarrow R(u,v) \vee R(v,u), \tag{completeness}$$
$$R(u,v) \vee R(v,u), \tag{strong completeness}$$

for each $u, v, w \in U$.

We call R reflexive, symmetric, antisymmetric, asymmetric, transitive, complete, and strongly complete if it fulfills the respective property.

It is worth noting that different terms, such as linear, connex, connected, total, and trichotomic, are used in the literature to describe (strong) complete relations, depending on the context.

All of these and many more properties of relations together with their interrelationships may be found e.g. in (Toth, 2020). Using some of the properties above, we may define various interesting classes of binary relations on a set.

Definition 1.1.2. Binary relation R on U is called:
- *preorder (or quasiorder)* if it is reflexive and transitive;
- *equivalence* if it is a symmetric preorder, i.e. a reflexive, transitive, and symmetric binary relation;
- *order (also partial order, ordering)* if it is an antisymmetric preorder, i.e. a reflexive, transitive, and antisymmetric binary relation.

We denote preorders by $\prec$, equivalences by $\equiv$ and orders by $\leq$, possibly with sub- or superscripts.

Equivalences and equality

As noted above, equivalences are of utmost importance as they allow us to model indistinguishability of objects in the given situation. Arguably, the most prominent of all the equivalences on any set U is the equality relation.

Definition 1.1.3. An *equality (or identity) on U* is an equivalence $\equiv$ on U, which moreover satisfies

$$u \equiv v \quad \text{implies} \quad u = v \tag{separability}$$

for any u, v in U. Here, $u = v$ means that u and v are the same object.

Note: The form of the definition above may feel overcomplicated as the notion of eqaulity is well-known and can be defined in a more straightforward manner. Nevertheless, we use this form to highlight the analogy between definitions of equality in the classical setting and the setting of fuzzy logic (see below).

Equality is of such importance that it is often distinguished from all other predicates on the level of language of first order logic – the language is then called *language with equality.* That is there is the symbol $=$ reserved in the language, which should always be interpreted by the equality relation. Introduction of this symbol into the language comes hand in hand with extra axioms – called equality axioms – whose meaning goes back to Leibniz's considerations. For more information, we refer the reader to standard textbooks on mathematical logic, e.g. Cori and Lascar (2000). In accordance with this practice, we use the symbol $=$ only for the identity relation on the respective set. It is also worth noting that the equality is the only reflexive relation on a set which is symmetric and antisymmetric at the same time.

In the following sections, we will see that, contrary to the Boolean case, there is an abundance of equalities within a fuzzy logic framework. This well known observation leads to various possible generalizations of many classical concepts and properties which are in the Boolean case defined with respect to the identity. In Chapter 3, we focus on antisymmetry and its interrelationship with separability of underlying equality as these properties are crucial for the concept of order in the setting of fuzzy logic.

Orders

The other prominent kind of binary relations is ordering on a set, i.e. relations modeling comparison between objects. There are two common views on an ordering on a set, the first one as per Definition 1.1.2, the second one known as a strict order.

Definition 1.1.4. A *strict order* on a set U is a binary relation on U which is transitive and asymmetric.

It is a well known fact that both definitions delineate same class of relations.

Proposition 1.1.5. *If $\leq$ is an order on a set U then a binary relation $<$ on U defined by $u < v = u \leq v \wedge u \neq v$ for each $u, v \in U$ is a strict order on U.*

If $<$ is a strict order on U then a binary relation $\leq$ on U defined by $u \leq v = u < v \vee u = v$ for each $u, v \in U$ is an order on U.

The constructions are mutually inverse.

The structure $\langle U, \leq \rangle$, consisting of a set U and an order relation $\leq$ defined on U, is commonly referred to as an *ordered set*. In the subsequent chapters, we extensively utilize two distinguished classes of orders – linear orders and lattices.

Linear orders

Definition 1.1.6. An order relation $\leq$ on U is called a *linear order (or chain)* if it is moreover strong complete.

In other words, order is linear if for any pair of objects we can decide which object is a predecessor and which object is a successor in the given sense, e.g. which is smaller, better, further, ... Note that such concept is utterly natural – many common orders are linear, e.g. numbers or anything that can be numbered.

One of the most fundamental results in the field of order theory is an extension theorem proved by Szpilrajn (1930).[1]

Theorem 1.1.7 (Szpilrajn's extension theorem)**.** *For any order $\leq$ on a set U there is a linear order on U which contains $\leq$.*

That is every order can be extended into a linear order while preserving the original comparisons between objects. For finite cases, this extension is straightforward – decide for every pair of uncomparable elements, pair by pair, what the resulting order should be. There is always at least one option to do so without breaking properties necessary for a relation to be an order and after finite number of steps we obtain desired linear order. In general case, this theorem only holds if we accept the axiom of choice.

Using Szpilrajn's result, Dushnik and Miller (1941) introduced so called *realizers* of an order and the concept of an *order dimension*.

[1] Szpilrajn acknowledges the prior existence of unpublished proofs by Banach, Kuratowski, and Tarski.

Definition 1.1.8. Let $\leq$ be an order on U. A collection $\mathcal{K}$ of linear orders on U is called a *realizer of* $\leq$ if for any two elements u, v in U we have $u \leq v$ if and only if $u \leq' v$ holds for every $\leq'$ in $\mathcal{K}$. That is we have $u \leq v = \bigwedge_{\leq' \in \mathcal{K}} u \leq' v$ for each $u, v \in U$.

Alternatively we say that $\mathcal{K}$ *realizes* $\leq$ or $\leq$ *is realized by (linear orders of)* $\mathcal{K}$.

Theorem 1.1.9 (Dushnik and Miller 1941, Theorem 2.32)**.** *If* $\leq$ *is any partial order on a set* U *then there exists a collection* $\mathcal{K}$ *of linear orders on* U *which realize* $\leq$.

Definition 1.1.10. A *dimension* of an order $\leq$ on U is the smallest cardinal number $\mathbf{m}$ such that $\leq$ is realized by $\mathbf{m}$ linear orders on U. Dimension of $\leq$ is often denoted by $\dim(\leq)$.

These outcomes initiated the development of dimension theory and led to many useful applications, e.g Arrow's and Suzumura's extension theorems used in theory of social choice (Arrow, 2012; Suzumura, 1983), Schnyder's characterization of planar graphs (Schnyder, 1989), effective storage of finite orderings in computer memory by the set of its realizers, and many more. Today, the dimension theory is a well-established field in the study of ordered sets, as it enables us to characterize any order by using the most prevalent type of orderings – chains.

Lattices

This section contains few selected results from lattice theory. In this work, lattices are employed in two ways. First, particular type of lattices is used as a structure of truth degrees in fuzzy logic while some of the obtained results depends on further properties of this structure. Second, lattices are the most understood types of orders in setting of fuzzy logic, including deeply developed applications (Belohlavek, 2001, 2002, 2004; Höhle, 1987). As such, they serve as one of justifications for our choice of an approach to fuzzy orders and a source of motivation.

It is a well known, yet still captivating, fact that there are two equivalent definitions of a lattice structure. One characterizes a lattice as a special type of an order while the other defines it as an algebra.

Definition 1.1.11. Let L be a non-empty set. An ordered set $\langle L, \leq \rangle$ is called a *lattice* if every pair of elements from L has an infimum, i.e. greatest lower bound, and a supremum, i.e. least upper bound, in $\langle L, \leq \rangle$.

Alternatively, *lattice* is an algebra $\langle L, \vee, \wedge \rangle$ where $\vee$ and $\wedge$ are two binary operations on L such that both $\vee$ and $\wedge$ are commutative and associative and where absorption laws – $a \vee (a \wedge b) = a = a \wedge (a \vee b)$, for every a, b in L – hold. The operations $\vee$ and $\wedge$ are then called *join* and *meet*, respectively.

We say that a lattice is *complete* if every subset of L has supremum and infimum in $\langle L, \leq \rangle$.

The transition between the two definitions is straightforward. Given a lattice as an ordered set $\langle L, \leq \rangle$, for every a and b in L, defining $a \vee b = \sup(a, b)$ and $a \wedge b = \inf(a, b)$ transforms it into a lattice as an algebra. Conversely, for a lattice as an algebra, if we set $a \leq b$ to be true if and only if $a \wedge b = a$, we obtain a lattice as an ordered set.

In Chapter 4, we discuss the possibility of linear extension of any fuzzy order on a set. It turns out that such possibility is dependent on extra properties of the underlying residuated lattice (see below). We therefore define the following concept of join-irreducibility.

Definition 1.1.12. (Davey and Priestley, 2002)
Let L be a lattice. An element $x \in L$ is *join-irreducible* [2] if

1. $x \neq 0$ (in case L has a zero)
2. $x = y \vee z$ implies $x = y$ or $x = z$ for all $y, z \in L$.

[2] Also called *supremum-irreducible* or *sup-irreducible*.

Note 1.1.13. It is also possible to define related concept of *irreducibility by arbitrary joins*, i.e. $x \in L$ is irreducible by arbitrary joins if there is no subset K of L such that $x \notin K$ and $\bigvee K = x$.

We use the term join-(ir)reducibility in the sense of definition above. If there is a need for the notion of (ir)reducibility by arbitrary joins then it is clearly stated.

The join-irreducibility turns out to be crucial for the top element of a lattice as this element plays the role of full truth in fuzzy logic. Therefore, we often utilize the following lemma.

Lemma 1.1.14. *In any lattice L with top element 1 and bottom element 0 the element 1 is join-irreducible if and only if for every finite set $K \subseteq L \setminus \{1\}$ we have $\bigvee K \neq 1$.*

Proof. If 1 is join-irreducible in L then for every such finite set K we have $\bigvee K \neq 1$ by induction. That is $\bigvee \emptyset = 0$, for K with $K = \{a\}$ we have $\bigvee K = a \neq 1$ and for $K_n = \{a_1, \dots, a_n\}$, i.e. with $|K_n| = n$, we have $\bigvee K_n = \bigvee K_{n-1} \vee a_i$ for some K_{n-1} with $|K_{n-1}| = n - 1$ and $i \in \{1, ..., n\}$, i.e. $\bigvee K_n = a \vee b$ for some $a, b \in L \setminus \{1\}$ therefore $\bigvee K_n \neq 1$ by 1 being join-irreducible.

If any finite K has supremum lower than 1 then also every K with $|K| = 2$ has supremum lower than 1. That is 1 is join-irreducible in L. □

Since the concepts mentioned above have been introduced, a lot has been done in areas related to order dimension (Trotter, 1992), lattices (Birkhoff, 1940; Davey and Priestley, 2002; Grätzer, 2002), and in the theory of ordered sets in general (Caspard et al., 2012; Schröder, 2003).

1.2 Fuzzy logic and residuated lattices

In contrast to classical logic, which relies on a fixed two-element set of truth values $L = \{0, 1\}$ and classical truth functions for logical connectives, fuzzy logic takes a different approach. In fuzzy logic, neither the set of truth degrees nor the truth functions for logical connectives are fixed. Instead, fuzzy logic operates with a general set of truth degrees, usually denoted by L, and allows for general truth functions of logical connectives, which are subject to natural basic conditions. Essentially, fuzzy logic embraces a general structure of truth degrees with appropriate generalized connectives which allows for more nuanced and flexible reasoning compared to classical logic.

Since the seminal work by Goguen (1967, 1969), the structure $\mathbf{L}$ of truth degrees is usually assumed to form a complete residuated lattice (Belohlavek, 2002; Belohlavek et al., 2017; Gottwald, 2001; Hájek, 1998; Novák et al., 1999). A given theory is then often developed for the general complete residuated lattice $\mathbf{L}$ and is thus valid also for all the particular cases.

This way, we have class of structures at hand, which includes various particular cases such as the real unit interval $L = [0, 1]$ equipped with the Łukasiewicz connectives, Heyting algebras, or even two-element Boolean algebra $\mathbf{2}$ of classical logic. Each of these structures then forms a basis of particular case of fuzzy logic.

Definition 1.2.1. A *complete residuated lattice* is an algebra

$$\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$$

such that $\langle L, \wedge, \vee, 0, 1 \rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1 \rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy the so-called *adjointness property*:

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \rightarrow c \tag{1.1}$$

for each $a, b, c \in L$. The elements a of L are called *truth degrees* and $\otimes$ and $\rightarrow$ are considered as the truth functions of *(many-valued) conjunction* and *implication*[3], respectively.

[3]The operation $\rightarrow$ is also called residuum.

Often, one additional connective, biresiduum, is defined. Its interpretation is the truth function of (many-valued) equivalence.

Definition 1.2.2. The *biresiduum* in $\mathbf{L}$ is the binary operation defined by

$$a \leftrightarrow b = (a \to b) \wedge (b \to a), \tag{1.2}$$

for every a, b in L.

There are various, well known, examples of complete residuated lattices, particularly those with L being a chain. A common choice of $\mathbf{L}$ is a structure with L being unit interval, $\wedge$ and $\vee$ being minimum and maximum, and $\otimes$ being a continuous (or at least left-continuous) t-norm (i.e. a commutative, associative, and isotone operation on $[0, 1]$ with 1 acting as a neutral element). The corresponding $\to$ is then given by

$$a \to b = \max\{c \,|\, a \otimes c \leq b\}.$$

The three most important pairs of adjoint operations on the unit interval are[4]:

$$\text{Łukasiewicz:} \quad \begin{aligned} a \otimes b &= \max(a + b - 1, 0), \\ a \to b &= \min(1 - a + b, 1), \end{aligned} \tag{1.3}$$

$$\text{Gödel:} \quad \begin{aligned} a \otimes b &= \min(a, b), \\ a \to b &= \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise,} \end{cases} \end{aligned} \tag{1.4}$$

$$\text{Goguen:} \quad \begin{aligned} a \otimes b &= a \cdot b, \\ a \to b &= \begin{cases} 1 & \text{if } a \leq b, \\ \frac{b}{a} & \text{otherwise.} \end{cases} \end{aligned} \tag{1.5}$$

Another common choice for $\mathbf{L}$ is a finite chain. For example on $L = \{a_0 = 0, a_1, \ldots, a_n = 1\} \subseteq [0, 1]$ $(a_0 < \cdots < a_n)$ we can define $\otimes$ by $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and $\to$ by $a_k \to a_l = a_{\min(n-k+l,n)}$. Such defined $\mathbf{L}$ is called a *finite Łukasiewicz chain.* Similarly we can define a *finite Gödel chain* using same $L = \{a_0 = 0, a_1, \ldots, a_n = 1\} \subseteq [0, 1]$ with the operations $\otimes$ and $\to$ given as restrictions of the Gödel operations from $[0, 1]$ to L.

As noted above, even two-element Boolean algebra $\mathbf{2} = \langle\{0, 1\}, \wedge, \vee, \otimes, \to, 0, 1\rangle$, i.e. the structure of truth degrees of classical logic, is a particular case of a complete residuated lattice. This is vital because when considering the specific case $\mathbf{L} = \mathbf{2}$, the established concepts and outcomes align with those developed in classical setting. Specifically, the concepts related to fuzzy sets and fuzzy relations (see the subsequent section) may be identified with their counterparts in the theory of classical sets and relations.

1.3 Fuzzy sets and relations

Given a complete residuated lattice $\mathbf{L}$, the basic set-theoretic notions are generalized into logical framework defined by $\mathbf{L}$. We briefly survey the fundamental principles of fuzzy set theory, focusing particularly on binary fuzzy relations on a set, such as preorders, equivalences, and equalities. If the used complete residuated lattice is obvious from the context or if the given proposition is valid for any complete residuated lattice, we usually use terms such as fuzzy set, fuzzy relation, fuzzy order, etc. On the other hand, if we consider some particular complete residuated lattice, we denote it by $\mathbf{L}$ and then talk about $\mathbf{L}$-set, $\mathbf{L}$-relation, $\mathbf{L}$-order, etc.

[4]Derived from the operations used as $\otimes$, the term "minimum structure" is commonly used when referring to a Gödel structure, whereas a Goguen structure is commonly referred to as a "product structure".

Fuzzy sets

Definition 1.3.1. A *fuzzy set* (or **L**-*set*) A in a universe U is a mapping $A\colon U \to L$. The value $A(u)$ is interpreted as "the degree to which u belongs to A."

The collection of all **L**-sets in U is denoted by L^U. A fuzzy set $A \in L^U$ is called *crisp* if $A(u) = 0$ or $A(u) = 1$ for each $u \in U$. Every crisp fuzzy set $A \in L^U$ may be easily recognized as equivalent to the classical subset $\{u \in U \mid A(u) = 1\}$ of U. In fact, a crisp fuzzy set represents the characteristic function of the corresponding subset of U. It is customary to treat crisp fuzzy sets in U and their corresponding subsets of U interchangeably, as long as there is no danger of confusion.

For $a \in L$ and $u \in U$, we denote by $\{^a/u\}$ the fuzzy set A in U, called a *singleton*, for which $A(x) = a$ if $x = u$ and $A(x) = 0$ if $x \neq u$. A crisp singleton $\{^1/u\}$ may be identified with a one-element ordinary subset $\{u\}$ of U.

An *a-cut* of fuzzy set A in U is a set ${}^aA = \{u \in U \mid A(u) \geq a\}$. A crisp set A may be identified with its 1-cut. The basic operations with fuzzy sets are based on the residuated lattice operations and are defined componentwise.

Definition 1.3.2. Let A, B be fuzzy sets in U. We define the following operations derived from those of used complete residuated lattice:

$$
\begin{aligned}
(A \cap B)(u) &= A(u) \wedge B(u),\\
(A \cup B)(u) &= A(u) \vee B(u),\\
(A \otimes B)(u) &= A(u) \otimes B(u),\\
(A \to B)(u) &= A(u) \to B(u),\\
(\bigcap_{i\in I} A_i)(u) &= \textstyle\bigwedge_{i\in I} A_i(u),\\
(\bigcup_{i\in I} A_i)(u) &= \textstyle\bigvee_{i\in I} A_i(u),
\end{aligned}
$$

for each $u \in U$.

It follows from previous paragraphs that all **2**-sets are crisp fuzzy sets, i.e. these operations on **2**-sets are to be identified with their ordinary counterparts.

Given $A, B \in L^U$, we define the *degree* $A \mathrel{\underset{\sim}{\subseteq}} B$ *of inclusion of A in B* by

$$A \mathrel{\underset{\sim}{\subseteq}} B \quad = \quad \textstyle\bigwedge_{u\in U}\big(A(u) \to B(u)\big) \tag{1.6}$$

and the *degree of equality of A and B* by

$$A \approx B \quad = \quad \textstyle\bigwedge_{u\in U}\big(A(u) \leftrightarrow B(u)\big). \tag{1.7}$$

Note that (1.6) generalizes the ordinary subsethood relation $\subseteq$ and (1.7) generalizes the ordinary equality $=$ of sets.

Binary fuzzy relations

Binary fuzzy relation R between U and V is just a fuzzy set in the universe $U \times V$.

Definition 1.3.3. A *binary fuzzy relation* (or *binary* **L**-*relation*) R between U and V is any mapping $R\colon U \times V \to L$. [5]

[5]If $U = V$ then R is called a binary fuzzy relation on U.

The definition is a straightforward generalization of the definition of classical binary relation. Similarly, the basic properties of binary fuzzy relations are generalizations of their classical counterparts. But contrary to the case of the definition, these generalizations do not have to be so straightforward for each property. Generalizing reflexivity, symmetry, and transitivity appears immediate:

Definition 1.3.4. For a binary fuzzy relation R on a set U, we define following well known properties:

$$R(u,u) = 1, \qquad \text{(reflexivity)}$$
$$R(u,v) \leq R(v,u), \qquad \text{(symmetry)}$$
$$R(u,v) \otimes R(v,w) \leq R(u,w), \qquad \text{(transitivity)}$$

for each $u, v, w \in U$.

We say that R is reflexive, symmetric, and transitive if it fulfills the respective property.

These definitions have been proven useful and naturally behaving by a great number of studies. Generalizing antisymmetry and completeness, however, is much less immediate. Using the properties above we may instantly define preorders and equivalences in the setting of fuzzy logic. We postpone the discussion of antisymmetry, fuzzy order, and linear fuzzy order to Chapters 3 and 4 where we analyze them thoroughly.

Definition 1.3.5. Binary fuzzy relation R on U is called:

- *fuzzy preorder (or fuzzy quasiorder)* if it is reflexive and transitive;
- *fuzzy equivalence* if it is symmetric fuzzy preorder, i.e. reflexive, transitive, and symmetric binary fuzzy relation;

We denote fuzzy preorders by $\prec$ and fuzzy equivalences by $\approx$, possibly with subscripts or superscripts. We also use terms **L**-*preorder* and **L**-*equivalence* if **L** is to be empahsized.

Transitive closures

Transitivity is a crucial property both for equalities and orders – the main subjects of this work. Therefore, we often discuss various consequences of extending some relation into its transitive closure.

Definition 1.3.6. *Transitive closure* $\mathit{Tra}(R)$ of a binary fuzzy relation R on U is the least transitive binary fuzzy relation on U containing R.

It is well known fact that transitive closure may be formed using only composition and union.

Lemma 1.3.7. *For any binary fuzzy relation* $R\colon U \times U \to L$ *we have* $\mathit{Tra}(R) = \bigvee_{n=1}^{\infty} R^n = R \cup R \circ R \cup R \circ R \circ R \cup \cdots$.

For further details on general theory of fuzzy sets and relations we refer to the books by Belohlavek (2002); Belohlavek et al. (2017); Gottwald (2001); Hájek (1998); Novák et al. (1999).

Fuzzy equivalences and fuzzy equalities

Expressing the similarity to some extent between two objects is a common practice in natural language, as exemplified by the sentence: "These two options are quite different, but there is yet another one, which is, in a way, similar to both." Modeling such propositions by means of classical logic is possible, but it has some drawbacks. For example, we can not easily use theory of preorders, equivalences, and related concepts, as the described similarity relationship is not even transitive. On the other hand, fuzzy logic offers a convenient way to handle gradual information and, moreover, the properties of fuzzy equivalences and equalities are just the properties one naturally expects from such similarity. For this reason among others, fuzzy equivalences and equalities have been deeply developed and widely utilized.

The basic notion in presence of fuzzy equivalence $\approx$ on a set is the compatibility[6] of a set or a relation with $\approx$.

Definition 1.3.8. A fuzzy set A in a universe U is *compatible with a fuzzy equivalence $\approx$ on U* if

$$A(u) \otimes u \approx v \leq A(v) \tag{1.8}$$

for every u, v in U.

A binary fuzzy relation $R\colon U \times U \to L$ is *compatible with a fuzzy equivalence $\approx$ on U* if

$$R(u_1, v_1) \otimes (u_1 \approx u_2) \otimes (v_1 \approx v_2) \leq R(u_2, v_2) \tag{1.9}$$

for every u_1, u_2, v_1, v_2 in U.

In words, compatibility of a fuzzy set A with $\approx$ means that if u is in A and u and v are equivalent, then v is in A as well. Similarly, compatibility of binary fuzzy relation R reads that if u_1 and v_1 are related by R, u_1 is equivalent to u_2, and v_1 is equivalent to v_2, then u_2 and v_2 are related by R as well. That is compatibility generalizes the classical axiom of equality.

In the end, we briefly turn our attention to fuzzy equalities, as their properties are crucial for the definition and utilization of fuzzy orders. Similarly to the classical case, fuzzy equalities are defined as separable fuzzy equivalences. However, unlike in the classical setting, there may exist multiple fuzzy equalities on a given set. We will often discuss various properties of fuzzy equalities in subsequent chapters.

Definition 1.3.9. A *fuzzy equality (or* **L**-*equality)* is a fuzzy equivalence, which moreover satisfies

$$u \approx v = 1 \quad \text{implies} \quad u = v \tag{separability}$$

for each $u, v \in U$.

To emphasize that $\approx$ is a fuzzy equality, not a mere fuzzy equivalence, we use the symbol $\eqsim$, possibly with subscripts or superscripts.

A comprehensive treatment of fuzzy equivalences, equalities, and related topics may be found in (Recasens, 2011, 2022).

[6] Often the term extensionality or congruence with respect to a fuzzy equivalence $\approx$ is used.

Chapter 2

Historical notes

Any abstract concept may be fully grasped only if we know initial motivations and historical aspects of its development. Therefore, this chapter briefly discusses these topics for the case of fuzzy order defined with respect to fuzzy similarity and related relations. We also pay some attention to the works on fuzzy lattices, as this particular type of fuzzy order was often the driving force behind new results on the concept of fuzzy order itself.

The story of fuzzy order starts with Zadeh's seminal paper (Zadeh, 1971). Since this work, a lot has been done in the fields of order theory and in particular lattice theory in the setting of fuzzy logic. Table 1 shows number of papers devoted to fuzzy order and lattice-type fuzzy order indexed by Scopus for various time frames including individual decades starting from 1970s. We find interesting that, according to this data, almost exact half of the papers devoted to these topics was written in the last 10 years and almost three quarters in the last 15 years. On the other hand, one has to be careful with such interpretations as this increase of paper count may go hand in hand with better online databases and overall better internet access in last 20 years or so. Also it may be related to the phenomenon of inflation in publishing as described by Belohlavek (2022).

Time frame	Order or lattice	Order	Lattice
1971-1980	6	6	0
1981-1990	30	21	9
1991-2000	94	48	46
2001-2010	241	151	92
2011-2020	522	388	140
2021-2023 (April)	134	104	31
2008-2023 (April)	754	554	207
2013-2023 (April)	560	423	143
1971-2023 (April)	1027	718	318

Table 1: Number of papers devoted to fuzzy orders or fuzzy lattices by time frames (mostly decades) according to Scopus. Second column contains count of papers for the given period and query "fuzzy order*" OR "fuzzy lattice" in abstract, keywords, and title. The third and fourth columns contain similar information only for "fuzzy order*" resp. "fuzzy lattice" queries. The asterisk symbol in Scopus query represents wildcard – in this case the word "order" may have any suffix.

2.1 The concept of fuzzy order

Now, we briefly cover the history of the concept of fuzzy order defined with respect to an underlying similarity by summarizing the results obtained in some works on the topic. We choose the works which were, in our opinion, the most essential or influential. As such choice may be regarded as opinionated, we support it by notes on the later influence of obtained results and also by citation count of each of the papers, which usually serves as one of metrics of the paper's influence. By doing so, we note that in some cases it may take time, further development, and possibly luck for the results to be actually recognized by the community in a form of citations. Therefore here, the citation counts are to be taken just as supplement to the notes on historical development.

The works are listed in chronological order by years of their publication. The citation counts are according to Scopus database in the end of April, 2023.

Zadeh (1971)

The first, and also most influential (2000 citations in Scopus), work on the topic was done by Zadeh (1971), where the author coined the concepts of fuzzy order[1] and fuzzy similarity.

The motivation was a study of concepts of equivalences and orders in the fuzzy setting – an emerging theory in that time. Various properties of such similarity relations and fuzzy orderings are investigated and some applications are outlined. In the end, a Szpilrajn's extension theorem is extended into the setting of fuzzy logic as an example of usefulness and depth of the theory. The utilized axioms of antisymmetry and linearity are different from today's perspective and also from the point of view of this thesis.

Blanchard (1983)

The second work, although overlooked by community (1 citation in Scopus), is very interesting from today's point of view. It is the first paper, which considers definition of fuzzy order in a sense equivalent to those used nowadays.

The motivation of this study was purely theoretical – to asses various candidates for the definition of fuzzy orders. The validity of some form of Szpilrajn's extension theorem is used as the touchstone of worthiness of the given axiom system. In total, four systems are described and then assessed in this way. Out of these candidates, the so called 4-fuzzy orderings are the ones, we will be concerned with (among different definitions) in later chapters.

Höhle and Blanchard (1985)

The next work we mention offers an important observation of a link between a fuzzy ordering and an underlying fuzzy similarity on the given set. Nowadays, this observation is crucial in utilization of fuzzy orderings, but the work was again overlooked by the community for a long time. It has 60 citations in Scopus where all but one are from year 2002 or later. The reason is that around year 2000 this link between order and similarity has been rediscovered independently of this contribution (see below).

The purpose of the paper was to improve initial results on fuzzy ordering obtained by Zadeh (1971). The link described above is captured in this excerpt from the abstract of the work: "In opposition to Zadeh's, our point of view is that an axiom of antisymmetry without a reference to a concept of equality is meaningless." Their setting is that of residuated lattice and they define all the notions in terms of category theory. In spirit of Zadeh's paper, the soundness of their approach is demonstrated by the validity of Szpilrajn's extension theorem generalization.

Interestingly until lately, no connection between both versions of fuzzy order definitions from Blanchard (1983) and Höhle and Blanchard (1985) was established, even though both works had one author in common and were published close in time to each other (see Chapter 3).

[1]It is worth noting that before Zadeh, many-valued orders were considered by Menger (1951) as part of his probabilistic approach to relations.

Höhle (1987)

The fourth work, we find important for the development of fuzzy orders, is concerned with defining fuzzy real numbers as Dedekind cuts. It has 44 citations in Scopus, only seven of which are before the year 2002. Its importance lies in being the first paper defining complete fuzzy lattices as a special kind of fuzzy order respecting the link to underlying fuzzy similarity relation. Interestingly, the used definition of fuzzy order is slightly different than the one by Höhle and Blanchard (1985), but reasons for such modification of the definition are not explained. The difference lies in antisymmetry axiom and we discuss it in more detail in Chapter 3.

Among other notions, the obtained results include Dedekind-MacNeille style completion of any fuzzy order, i.e. embedding of a fuzzy order to a reasonably constrained fuzzy lattice. These results are then applied to a generalization of real numbers into the setting of fuzzy logic, which turns the results into another convincing argument for reasonability and applicability of fuzzy orders defined in this way. It is of interest that almost the same definition was later independently proposed by Belohlavek and led to a significant development of theory of lattice-type fuzzy orders by means of formal concept analysis in the setting of fuzzy logic (see below).

Fuzzy Sets Theory and its Applications conference (1998)

After a long time, two authors – Radim Belohlavek and Ulrich Bodenhofer – came up with the concept of fuzzy order defined with respect to underlying similarity, again. They were not aware of each others research nor the works described above, albeit they were both strongly influenced by Höhle's work on fuzzy logic. Still, they announced their preliminary results on the same conference – Fourth Fuzzy Sets Theory and its Applications conference in Liptovský Ján, 1998. Their definitions are slightly different, but the core idea is same. We cover both definitions in detail in Chapter 3. After this conference, both authors published several papers devoted to their respective notions, although they never got to compare them directly.

Belohlavek (1998 and beyond)

As noted above, Belohlavek published several papers on the topic since 1998, e.g. (Belohlavek, 2001, 2002, 2004). Out of all these works, we cover in some detail (Belohlavek, 2004).[2] Its main topic is the theory of complete lattice-type fuzzy orders, while examples and motivations are based on concept lattices (i.e. hierarchical structures of concepts) generalized into the setting of fuzzy logic. The notions of fuzzy partial order, lattice-type fuzzy order, and fuzzy formal concept are introduced. Also, as a particular application of the approach, Dedekind–MacNeille completion of a partial fuzzy order is described.

Although the results were obtained independently, the used definition of fuzzy order is almost the same as the one utilized by Höhle (1987). That is, similarly to previous two cases, this work follows its specific motivations and arrives to almost the same concept of fuzzy ordering.

The work was highly influential in the community around formal concept analysis, where it sprung the research on its fuzzy counterpart, complete lattice-type fuzzy orders, and related topics. Up to date, it has 399 citations in Scopus.

Bodenhofer (1998 and beyond)

Also Bodenhofer published several papers on the topic since 1998, e.g. (Bodenhofer, 1999a, 2000, 2003). In his case, we mention some details of (Bodenhofer, 2000). The work is devoted to the various notions of fuzzy orders available at that time and shows what they are lacking by means of natural examples such as subsethood relation or implication-induced order. Then the author proceeds by discussion of involved axioms and notes their connection to underlying similarity.

[2] We note that this Belohlavek's first paper on the topic got stuck in the production process: As is apparent from the acknowledgement in this paper and from (Belohlavek, 2001), the 2004 paper was submitted in 2000.

Following this link, he finally obtains the definition of fuzzy order with respect to the underlying similarity relation which is, although obtained independently and in slightly different framework, same as the definition obtained by Höhle and Blanchard (1985). Bodenhofer was apparently not aware of previous work by Höhle and Blanchard in that time, but he acknowledges their historical priority later (Bodenhofer, 2003). The 2000 paper has 92 citations in Scopus so far.

Fan (2001)

Finally, the last contribution we include in this list is (Fan, 2001). This work is concerned with category theoretical research on the so-called Ω-categories. They may seem to be out of the scope of our work, but objects of such categories are just the fuzzy orders defined in the same way as in (Blanchard, 1983). Therefore, although approached with different motivations, the fuzzy orders were independently defined in an equivalent way again. According to Scopus, this paper has been cited 91 times so far.

Note 2.1.1. (a) Although all the mentioned works were independent and had motivations of their own, they arrived to two classes of definitions of fuzzy order. In Chapter 3, these definitions will be studied in some detail. In the end, we will see that all of them have common generalization and that they in fact describe the same class of binary fuzzy relations on a set with some possible limitations given by the context they are utilized in.

(b) We find interesting that there were two independent periods of time, where same alternative definitions of fuzzy order were proposed. First time, it was in the 80s due to Blanchard and Höhle, second time, at the turn of the century due to Belohlavek, Bodenhofer, and Fan.

(c) If we examine an impact these two periods had on fuzzy order research activity, we may see another interesting phenomenon. The first appearance of the definitions remained more or less unnoticed for many years, while the second appearance caused reignition of research on fuzzy orders, their theory, and their applications in other branches of mathematics. This seems to be an another reason why number of new papers on the topic spiked in last 15 years or so. Moreover, thanks to this renewed interest in the topic, also the older works became much more appreciated by the community.

2.2 Szpilrajn-like extension theorem for fuzzy orders

Szpilrajn-like extension theorem in the setting of fuzzy logic was considered already by Zadeh in his seminal paper on fuzzy equivalences and fuzzy orderings (Zadeh, 1971, Theorem 8). This version of the theorem was stated with respect to different concepts of antisymmetry and linearity. See Chapter 3 or (Belohlavek et al., 2017) for in-depth analysis of differences between Zadeh's and our setting. More results on Szpilrajn-like extension principle in the setting of fuzzy logic emerged soon, e.g. (Blanchard, 1983; Chakraborty and Sarkar, 1987; Hashimoto, 1983). Of these works, we once again highlight (Blanchard, 1983) where one of outlined views on fuzzy orders was lately shown to be in a sense equivalent to our view on fuzzy orderings (see Chapter 3). The main distinction lies in the different setting[3] and the fact that Blanchard in general defines the notion of a fuzzy order on a fuzzy set $A \in L^U$.

For the approach to fuzzy orders we utilize, i.e. the one which considers fuzzy equality on the underlying set, the first version of Szpilrajn-like theorem was stated already in (Höhle and Blanchard, 1985) – the work which coined this approach – see their Theorem II.7 and its corollaries. This version of the theorem was stated with respect to $\otimes$-linearity and sligthly different definition of a fuzzy order (see Chapter 3 for in detail comparison of various definitions).

As far as we know, the most detailed study on linearity of fuzzy orderings and related concepts so far is (Bodenhofer and Klawonn, 2004). This study builds upon research on the concept of

[3]That is particular type of residuated lattices where $L = [0, 1]$ and $\otimes = \wedge$.

fuzzy order itself, reignited by Belohlavek and Bodenhofer in the late 1990s to early 2000s. It analyzes several notions of linearity proposed by various authors in the setting of fuzzy order on the set with fixed fuzzy equivalence. The fixing of underlying similarity is the most important difference between their approach and the one utilized in this thesis. In the end, achievability of Szpilrajn-like theorem is studied for several situations, given by used t-norm and axiom of linearity (see their Table 1). Their main results include following observations mentioned in the conclusion of their work: The strong completeness can only serve as an appropriate concept of linearity in the setting of fuzzy logic, if $\otimes = \wedge$; The $\otimes$-linearity coined in (Höhle and Blanchard, 1985) provides preservation of the most important properties of order extension in the setting of residuated lattices on $[0, 1]$. However, it is very weak, non-intuitive, and poorly expressive concept if $\mathbf{L}$ does not have a strong negation.

In a sense, our work on the topic of linear extensions of fuzzy orders builds upon this study. To compare the approaches with fixed underlying similarity and with possibility to modify it together with the order, some of our observations throughout the Chapter 4 are related to their results.

Chapter 3

What is fuzzy order?

As it was indicated in previous parts, the first topic of this thesis is to sum, sort, and scrutinize the various approaches to fuzzy order defined with respect to underlying similarity relation found in the literature. This chapter contains summary of main results obtained in (Belohlavek and Urbanec, 2023a,b) – a two-part study on the concept of fuzzy order itself conducted jointly with Radim Belohlavek.

We focus only on the essential results regarding the concept of fuzzy order in general and its interplay with underlying fuzzy equality in particular. Therefore, we consider only part of the study's content here. Namely, although they are very interesting, we do not cover the results regarding graded point of view on the various properties of fuzzy relations. We also omit all the proofs, auxiliary lemmas, many remarks, and comments which may be of interest to reader later. For this case whole study is attached to this document (see Appendices A and B). All the definitions, theorems, etc. are accompanied with an exact references into these appendices. We present them here in their original form with only exception being a different symbol for a fuzzy equality (see Preliminaries),

3.1 Aim of the chapter

The central topic of the study is same as the one of this thesis – the arguably most developed approach to fuzzy orders, pursued originally by Ulrich Höhle, Nicole Blanchard, Ulrich Bodenhofer, and Radim Belohlavek. This approach is distinctive and significant by its treatment of antisymmetry. It assumes that the underlying universe, the fuzzy order is defined on, is already equipped with a fuzzy similarity relation, i.e. some fuzzy relation which generalizes the concept of classical equality. In fact, the above mentioned authors proposed several definitions of fuzzy order in this sense, where difference between them is mainly in the used axiom of antisymmetry.

Although many papers on fuzzy orders and their properties were published since these pioneering works (see Table 1 in Chapter 2), some basic questions on the concept of fuzzy order itself still remain open. The arguably most important of them is the question of what is an appropriate definition of fuzzy order?

All the above mentioned definitions are examined in detail and their mutual relationships described. Note also that the purpose of the study is not a quest for "the right" definition of fuzzy order which might be considered naive, or even ill-posed. Rather, the study should be approached as an exploration of an approach to fuzzy orders involving antisymmetry with respect to fuzzy equality, possible definitions of such fuzzy order, their common bits, differences, benefits, and drawbacks.

The chapter is organized as follows. We start by examining the definitions per se (Sections 3.2 to 3.5). The rest of the chapter (Sections 3.6 to 3.9) is then devoted to the axiom of antisymmetry.

3.2 Definitions of fuzzy order

Two definitions of fuzzy order on a set equipped with a generalized equality follow. We provide them in the forms used in the works of Bodenhofer and Belohlavek, as these are mostly refered to in literature. There are some mild differences in the forms present in the works by Höhle. We comment on the differences in appropriate places.

Definition 3.2.1 (Appendix A, Definition 1; Höhle, Blanchard, Bodenhofer)**.** A *fuzzy order on a set U equipped with a fuzzy equality relation $\approx$* is a binary fuzzy relation $\lesssim$ on U satisfying

$$\begin{aligned} u \approx v &\leq u \lesssim v, && (\approx\text{-reflexivity}) \\ (u \lesssim v) \otimes (v \lesssim w) &\leq u \lesssim w, && (\text{transitivity}) \\ (u \lesssim v) \otimes (v \lesssim u) &\leq u \approx v, && (\otimes\text{-antisymmetry}) \end{aligned}$$

for each $u, v, w \in U$. (Note: Höhle's and Blanchard's as well as Bodenhofer's original definitions actually assume, more generally, that $\approx$ is a fuzzy equivalence rather than fuzzy equality; this is discussed below.)

Definition 3.2.2 (Appendix A, Definition 2; Höhle, Belohlavek)**.** A *fuzzy order on a set U equipped with a fuzzy equality relation $\approx$* is a binary fuzzy relation $\lesssim$ on U compatible with $\approx$, i.e. fulfilling

$$(u_1 \lesssim v_1) \otimes (u_1 \approx u_2) \otimes (v_1 \approx v_2) \leq u_2 \lesssim v_2$$

for every $u_1, u_2, v_1, v_2 \in U$, which satisfies

$$\begin{aligned} u \lesssim u &= 1, && (\text{reflexivity}) \\ (u \lesssim v) \otimes (v \lesssim w) &\leq u \lesssim w, && (\text{transitivity}) \\ (u \lesssim v) \wedge (v \lesssim u) &\leq u \approx v, && (\wedge\text{-antisymmetry}) \end{aligned}$$

for each $u, v, w \in U$.

If distinction is needed, we shall call fuzzy orders according to Definitions 3.2.1 and 3.2.2 fuzzy orders with $\otimes$-antisymmetry and fuzzy orders with $\wedge$-antisymmetry, respectively. As noted in Chapter 2, both the Definitions 3.2.1 and 3.2.2 were introduced twice in two different time periods.

Definition 3.2.1 was in both cases defined by same conditions as listed above but with respect to a general fuzzy equivalence rather than fuzzy equality. First appearance is due to Höhle and Blanchard (1985) motivated by further study and improvement of the notion of order in the framework of fuzzy logic. The exactly same definition, but in slightly different framework, was later reinvented by Bodenhofer, who was apparently not aware of Höhle and Blanchard's work.

Definition 3.2.2 appeared, though in a little different setting, for the first time in the work by Höhle (1987), where it was stated in the framework of complete residuated lattices on $[0, 1]$ and with the concept of similarity interpreted by general fuzzy equivalence instead of fuzzy equality. It was later reinvented by Belohlavek who was not aware of Höhle's paper, this time in the exactly same form as Definition 3.2.2. See Chapter 2 for more details regarding history of the notion.

There are three obvious distinctions when comparing Definitions 3.2.1 and 3.2.2. First, Definition 3.2.2 assumes compatibility of $\lesssim$ with $\approx$. Second, the Definition 3.2.1 requires $\lesssim$ to be $\approx$-reflexive, while Definition 3.2.2 assumes reflexivity of $\lesssim$ instead. And third, the definitions use different form of antisymmetry where $\otimes$-antisymmetry of Definition 3.2.1 seems to be weaker, i.e. more general, than $\wedge$-antisymmetry of Definition 3.2.2. The aspect of one definition being seemingly more general than the other one is also explored in some detail in subsequent sections.

3.3 Fuzzy equivalence vs. fuzzy equality

As noted in Definition 3.2.1, the original definitions of Höhle, Blanchard, and Bodenhofer assume that $\approx$ is a fuzzy equivalence rather than a fuzzy equality. Fuzzy equality is a particular case of fuzzy equivalence, i.e. fuzzy equivalence moreover satisfying separability. We assume that $\approx$ is fuzzy equality in the Definition 3.2.1 for two reasons. Above all, it provides cleaner generalization of the concept of order into setting of fuzzy logic. Moreover, it allows better comparison of both definitions, as both kinds of fuzzy orders are then considered in the same context. To avoid confusion, we also note that Höhle and Blanchard (1985) use name L-equality for fuzzy equivalence relation. In the rest of the section, this distinction between definitions and reasons for our choice are briefly examined.

In our view, assuming fuzzy equivalence instead of fuzzy equality in Definition 3.2.1 represents generalization among two lines at once. First, the framework of the two-element Boolean algebra is replaced by more general framework of a complete residuated lattice. Second, the identity, i.e. the only equality in Boolean case, is replaced by a fuzzy equivalence.

The essential justification is done by considering both versions of the Definition 3.2.1, i.e. the current one with an equality and the original one with an equivalence, in the setting of classical logic.

On the one hand, the notion resulting from Definition 3.2.1 coincides with the classical notion of order. Namely, fuzzy equality becomes classical equality – identity. The defining conditions then become classical reflexivity, transitivity, and antisymmetry.

On the other hand, the notion emerging from the definition of a fuzzy order on a set with a fuzzy equivalence is not the notion of a classical order. Rather, such relation becomes a slightly restricted classical preorder (i.e. reflexive and transitive binary relation on a set limited by the choice of equivalence). The argumentation is as follows. Fuzzy equivalence becomes classical equivalence $\equiv$. Then, on the set U equipped with $\equiv$, the classical relation $\leq$ is defined, such that $\leq$ contains $\equiv$, is transitive, and satisfies antisymmetry generalized with respect to the equivalence: $u \leq v$ and $v \leq u$ implies $u \equiv v$. The relation $\leq$ is obviously reflexive and transitive, i.e. a preorder. Moreover, since $\equiv$ is contained in $\leq$, we obtain that

$$u \equiv v \quad \text{if and only if} \quad u \leq v \text{ and } v \leq u.$$

That is $\leq$ makes some elements to be lower or equal to each other if and only if the underlying equivalence $\equiv$ makes them equivalent to each other.

In the standard terminology of ordered sets, the relation $\leq$ is a preorder which moreover induces a fixed equivalence $\equiv$. As such, the concept is obviously more general than the concept of classical order which demands $\equiv$ to be the identity.

Let us point out that it is clear from Bodenhofer's papers that he was aware of this property of the definition of fuzzy order assuming fuzzy equivalence as may be seen from Bodenhofer (2000, 2003). His point of view differs from ours as he considers it to be a feature of order-preorder relationship rather than a problem. See the attached full version of the study for more details.

Moreover, we note that using a fuzzy equivalence instead of a fuzzy equality also leads to possibly not unique distinguished elements, such as a largest and a smallest element in an ordered set or a supremum and an infimum of some of its subsets. This sort of problems is illustrated by the following example.

Example 3.3.1 (Appendix A, Example 1)**.** Let $U = \{u, v, w\}$, let a classical equivalence $\equiv$ be given by the equivalence classes $\{u\}$ and $\{v, w\}$. Then the relation $\leq$ given by $u \leq u$, $v \leq v$, $w \leq w$, $u \leq v$, $u \leq w$, $v \leq w$, and $w \leq v$ is an order on a set with an equivalence in the sense of Höhle, Blanchard, and Bodenhofer. Defining naturally a smallest element x as an element such that $x \leq y$ for every y, and dually for a largest element, it is immediate that u is the only smallest element. On the other hand, both v and w are largest, even though these are two distinct elements.

3.4 Reflexivity and compatibility

Another immediate difference between Definitions 3.2.1 and 3.2.2 is in the axiom of reflexivity. As it turns out, there is hidden interplay of $\approx$-reflexivity and compatibility with respect to $\approx$. We therefore examine these, seemingly unrelated, variances together. We are in the situation where $\approx$-reflexivity required by Definition 3.2.1 is stronger than reflexivity of Definition 3.2.2, while Definition 3.2.2 moreover requires compatibility of the fuzzy order $\lesssim$ with $\approx$.

We start by a rather epistemic observation. In the classical setting, an identity relation is always implicitly given on the universe U and axioms of equality are taken as valid. First part is translated into setting of fuzzy logic by defining fuzzy order with respect to fuzzy equality, but the second part is present only in Definition 3.2.2 – compatibility, i.e. a generalization of the axiom of equality of classical logic.

Nevertheless, it is already known that, given the context of Definitions 3.2.1 and 3.2.2, both options, i.e. $\approx$ reflexivity or reflexivity and compatibility, are equivalent. The argument was observed for the first time by Belohlavek and Vychodil (2005, Lemma 1.82) in the context of fuzzy equivalences on sets with fuzzy equalities and later, independently, by Bodenhofer and Demirci (2008) in the context of fuzzy orders.

Proposition 3.4.1 (Appendix A, Corollary 2, (Belohlavek and Vychodil, 2005), (Bodenhofer and Demirci, 2008))**.** *Let $\approx$ be a fuzzy equality and $\lesssim$ be transitive. Then $\lesssim$ is $\approx$-reflexive if and only if $\lesssim$ is reflexive and compatible with $\approx$.*

In the study, we examine this relationship thoroughly, taking the graded point of view on all related properties of fuzzy relations. The outcome is general observation (Lemma 1 in Appendix A) whose particular corollary – by strengthening initial assumptions – is the proposition above.

Note that an alternative point of view offers itself – assume classical identity is given on each set and define all the other relations, including fuzzy equality, with respect to the identity. Then fuzzy order on U would be defined rather as a tuple $\langle \approx, \lesssim \rangle$ of fuzzy relations on U, where $\approx$ is fuzzy equality and $\lesssim$ meets all the properties required in the Definition 3.2.1 or 3.2.2. Although this point of view is also valid, we prefer to align with the classical situation as much as possible.

3.5 Antisymmetry and constraints regarding fuzzy equality

The last difference between Definitions 3.2.1 and 3.2.2 is the form of antisymmetry axiom. In this section, we take a point of view where the form of antisymmetry is considered as a lower bound for the fuzzy equality. An alternative perspective to consider is, which fuzzy equalities enable the given relation to be fuzzy order on the given set. This alternative perspective is of importance in Chapter 4.

The question is what are the limitations on fuzzy equality $\approx$. The basic answers were already stated for both definitions by different authors. For the case of Definition 3.2.1 it was provided by Bodenhofer (2000) who proved the following proposition.[1]

Proposition 3.5.1 (Bodenhofer 2000, Theorem 18)**.** *If $\lesssim$ is a reflexive and transitive fuzzy relation on U and $\approx$ is a fuzzy equality on U, then $\lesssim$ is a fuzzy order according to Definition 3.2.1 if and only if*

$$(u \lesssim v) \otimes (v \lesssim u) \ \leq \ u \approx v \ \leq \ (u \lesssim v) \wedge (v \lesssim u) \tag{3.1}$$

for every $u, v \in U$.

For the case of Definition 3.2.2 the corresponding result was obtained by Belohlavek (2002) replacing the equality 3.1 by

$$u \approx v \ = \ (u \lesssim v) \wedge (v \lesssim u).$$

[1]In Bodenhofer's setting $\approx$ is general fuzzy equivalence. See last but one section for more details.

In our study, we follow this line up to the lemma (Appendix A, Lemma 2) whose corollary is a stronger version of both propositions above.

Proposition 3.5.2 (Appendix A, Corollary 3)**.** *Let $\lesssim$ be a fuzzy relation and $\approx$ be a fuzzy equality on U.*
(a) $\lesssim$ is $\approx$-reflexive and $\otimes$-antisymmetric iff

$$(u \lesssim v) \otimes (v \lesssim u) \;\leq\; u \approx v \;\leq\; (u \lesssim v) \wedge (v \lesssim u).$$

(b) $\lesssim$ is $\approx$-reflexive and $\wedge$-antisymmetric iff

$$u \approx v \;=\; (u \lesssim v) \wedge (v \lesssim u).$$

This proposition together with the equivalence of $\approx$-reflexivity to reflexivity and compatibility in case of transitive relations (cf. Proposition 3.4.1) leads to non-redundant generalization of both the result by Bodenhofer (2000, Theorem 18) and its counterpart for fuzzy orders according to Definition 3.2.2 mentioned above.

Theorem 3.5.3 (Appendix A, Theorem 4)**.** *Let $\lesssim$ be a transitive fuzzy relation and $\approx$ be a fuzzy equality on U.*
(a) The following conditions are equivalent:
(a1) $\lesssim$ is a fuzzy order according to Definition 3.2.1.
(a2) $\lesssim$ is reflexive, $\otimes$-antisymmetric, and compatible with $\approx$.
(a3) $(u \lesssim v) \otimes (v \lesssim u) \;\leq\; u \approx v \;\leq\; (u \lesssim v) \wedge (v \lesssim u)$.
(b) The following conditions are equivalent:
(b1) $\lesssim$ is a fuzzy order according to Definition 3.2.2.
(b2) $\lesssim$ is $\approx$-reflexive and $\wedge$-antisymmetric.
(b3) $u \approx v \;=\; (u \lesssim v) \wedge (v \lesssim u)$.

Thanks to Theorem 3.5.3 we now have various equivalent conditions for a transitive fuzzy relation $\lesssim$ to become a fuzzy order with respect to Definition 3.2.1 (resp. 3.2.2). In particular, one of these conditions (a3 resp. b3) is expressed only by a relationship between $\lesssim$ and the fuzzy equality $\approx$. The theorem is also a little bit stronger than previous obtained results, as its assumptions do not contain redundancy anymore. Again the study continues in the direction of the theorem above by considering it in the graded setting.

3.6 Alternative definition of antisymmetry and fuzzy order

Considering results obtained in previous sections, the only essential difference between the two concepts of fuzzy order described in Definitions 3.2.1 and 3.2.2 is antisymmetry. In this section, we first examine an alternative form of antisymmetry, called crisp antisymmetry, which is used in the literature in definitions of fuzzy order without reference to underlying fuzzy equality. Then, we continue by stating a common generalization of all the considered forms of antisymmetry and by its means also common generalization of all the considered definitions of fuzzy order.

Crisp antisymmetry appeared for the first time in work by Blanchard (1983) and then was independently rediscovered by Fan (2001). See Chapter 2 for more historical details. As the setting of these works is different than ours, we state it in the form of obvious generalization into the framework of general complete residuated lattices. This generalization appeared in the works of Yao (Yao, 2010; Yao and Lu, 2009).

Definition 3.6.1 (Appendix B, Definition 3; Blanchard, Fan)**.** A *fuzzy order on a set U* is a binary fuzzy relation $\lesssim$ on U satisfying

$$u \lesssim u \;=\; 1, \qquad \text{(reflexivity)}$$

$$(u \lesssim v) \otimes (v \lesssim w) \;\leq\; u \lesssim w, \qquad \text{(transitivity)}$$

$$(u \lesssim v) = 1 \text{ and } (v \lesssim u) = 1 \text{ imply } u = v, \qquad \text{(crisp antisymmetry)}$$

for each $u, v, w \in U$. The pair $\langle U, \lesssim\rangle$ shall be called a *fuzzy ordered set* (according to Definition 3.6.1).

The rest of the section is devoted to the relationship between Definition 3.6.1 and Definitions 3.2.1 and 3.2.2. Already Bodenhofer (2003) and Belohlavek (2001, 2002, 2004) observed that fuzzy equality may be avoided in Definitions 3.2.1 and 3.2.2, respectively. Belohlavek utilized that for a fuzzy order according to Definition 3.2.2 $\approx$ is uniquely determined by $\lesssim$; and Bodenhofer made various observations on the relationship between $\lesssim$ and $\approx$ as regards Definition 3.2.1.

These considerations are tightly related to the results of previous section. Namely, from theorem 3.5.3 (a3 and b3) we may immediately derive same results as authors above. That is, for Definition 3.2.2, $\approx$ is uniquely determined by $\lesssim$; and for Definition 3.2.1, it follows that $\approx$ is limited by relations induced by $\lesssim$ from both sides.

To examine a relationship of the Definition 3.6.1 to other definitions, we start by observation formulated by Xie et al. (2009) for $\otimes = \wedge$ and Yao (2010) for general complete residuated lattices:

Proposition 3.6.2 (Appendix B, Lemma 1). *(a) If $\langle U, \approx, \lesssim\rangle$ is a fuzzy ordered set according to Definition 3.2.2, then $\langle U, \lesssim\rangle$ is a fuzzy ordered set according to Definition 3.6.1.*

(b) If $\langle U, \lesssim\rangle$ is a fuzzy ordered set according to Definition 3.6.1, then $\approx$ defined by

$$u \approx v = (u \lesssim v) \wedge (v \lesssim u) \tag{3.2}$$

is a fuzzy equality and $\langle U, \approx, \lesssim\rangle$ is a fuzzy ordered set according to Definition 3.2.2.

That is situation between Definitions 3.2.2 and 3.6.1 is clear. We now provide another proposition in the spirit of Proposition 3.6.2 regarding the relationship between Definition 3.2.1 and Definition 3.6.1:

Proposition 3.6.3 (Appendix B, Lemma 2). *(a) If $\langle U, \approx, \lesssim\rangle$ is a fuzzy ordered set according to Definition 3.2.1, then $\langle U, \lesssim\rangle$ is a fuzzy ordered set according to Definition 3.6.1.*

(b) If $\langle U, \lesssim\rangle$ is a fuzzy ordered set according to Definition 3.6.1, then $\approx$ defined by

$$u \approx v = (u \lesssim v) \otimes (v \lesssim u) \tag{3.3}$$

is a fuzzy equality and $\langle U, \approx, \lesssim\rangle$ is a fuzzy ordered set according to Definition 3.2.1.

Important difference between Propositions 3.6.2 and 3.6.3 is that in the former, the constructions from (a) and (b) are mutually inverse, while in the latter, $\approx$ defined by (3.3) is but one of the possible fuzzy equalities described by (a3) of Theorem 3.5.3.

3.7 A unifying concept of antisymmetry

Having considered the three variants of antisymmetry, namely the $\otimes$-antisymmetry, $\wedge$-antisymmetry, and crisp antisymmetry, we now present their common generalization.

Starting with a complete residuated lattice $\langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$, we consider three different conjunction-like operators on L. First and second are, well known, possible choices of conjunction in complete residuated lattice. Namely, a generalized t-norm (also called strong conjunction or, lately, just t-norm) $\otimes$ and a lattice meet (also called weak conjunction) $\wedge$. For the third, we employ more general conjunction-like operations $\odot$ on L which satisfy

$$a \odot b = b \odot a, \tag{3.4}$$

$$a_1 \odot a_2 \leq b_1 \odot b_2, \text{ whenever } a_1 \leq b_1 \text{ and } a_2 \leq b_2, \tag{3.5}$$

$$a \odot 1 \leq a, \text{ and} \tag{3.6}$$

$$1 \odot 1 = 1. \tag{3.7}$$

Obviously, every generalized t-norm (including $\wedge$) satisfies these conditions. The operator $\odot$ and its defining conditions may be found in Appendix B on page 4. Using $\odot$, we define the following notion of antisymmetry:

Definition 3.7.1 (Appendix B, page 5)**.** Let $\odot$ satisfy (3.4)–(3.7). A binary fuzzy relation $\lesssim$ on a set U equipped with a fuzzy equality $\approx$ satisfies $\odot$-*antisymmetry* if

$$(u \lesssim v) \odot (v \lesssim u) \leq u \approx v \tag{3.8}$$

for each $u, v \in U$.

It is obvious that both $\otimes$-antisymmetry and $\wedge$-antisymmetry are particular cases of $\odot$-antisymmetry. Surprisingly, the same holds true for notion of crisp antisymmetry which seems to be different at the first sight.

Proposition 3.7.2 (Appendix B, Lemma 4)**.** *Consider the binary operation $\bullet$ on L and the fuzzy relation $\approx$ on U defined by*

$$a \bullet b = \begin{cases} 1 & \text{for } a = 1 \text{ and } b = 1, \\ 0 & \text{otherwise;} \end{cases} \qquad u \approx v = \begin{cases} 1 & \text{for } u = v, \\ 0 & \text{otherwise.} \end{cases} \tag{3.9}$$

Then $\bullet$ satisfies (3.4)–(3.7) and $\approx$ is a fuzzy equality (the crisp fuzzy equality). Moreover, a binary fuzzy relation $\lesssim$ on U satisfies crisp antisymmetry if and only if it satisfies $\bullet$-antisymmetry.

3.8 Equivalence of definitions of fuzzy order

Having generalized notion of antisymmetry at hand, we utilize it to state another definition of fuzzy order which subsumes all the previous ones, i.e. Definitions 3.2.1, 3.2.2, and 3.6.1.

For this purpose we consider the following fuzzy relations on a given universe U:

$\lesssim$ ... a reflexive and transitive fuzzy relation on U,

$\equiv_\odot$ ... a fuzzy relation defined by

$$u \equiv_\odot v \;=\; (u \lesssim v) \odot (v \lesssim u), \tag{3.10}$$

$\approx_\odot$ ... the transitive closure of $\equiv_\odot$, i.e.

$$u \approx_\odot v = [\mathrm{Tra}(\equiv_\odot)](u, v). \tag{3.11}$$

Note that Appendix B contains a thorough analysis of properties of $\equiv_\odot$ and $\approx_\odot$, which are often utilized in proofs of observations in this section. The most important of these observation are summarized in following proposition.

Proposition 3.8.1 (Appendix B, Lemma 9)**.** *Let $\odot$ satisfy (3.4)–(3.7) and $\lesssim$ be a reflexive and transitive fuzzy relation on U.*

(a) $\approx_\odot$ is a fuzzy equivalence on U.
(b) The following conditions are equivalent:
  (b1) $\approx_\odot$ is a fuzzy equality;
  (b2) $\equiv_\odot$ is separable;
  (b3) $\lesssim$ satisfies crisp antisymmetry.
(c) If $\odot$ is a t-norm which dominates $\otimes$, then $\equiv_\odot = \approx_\odot$.

We are ready to state the aforementioned generalized definition of fuzzy order.

Definition 3.8.2 (Appendix B, Definition 4)**.** Let $\odot$ satisfy (3.4)–(3.7). A *fuzzy order on a set U equipped with a fuzzy equality relation* $\approx$ is a binary fuzzy relation $\lesssim$ on U compatible with $\approx$, i.e. satisfying

$$(u_1 \lesssim v_1) \otimes (u_1 \approx u_2) \otimes (v_1 \approx v_2) \ \leq \ u_2 \lesssim v_2,$$

for every $u_1, u_2, v_1, v_2 \in U$, which, moreover, fulfills

$$\begin{aligned} u \lesssim u \ &= \ 1, && \text{(reflexivity)} \\ (u \lesssim v) \otimes (v \lesssim w) \ &\leq \ u \lesssim w, && \text{(transitivity)} \\ (u \lesssim v) \odot (v \lesssim u) \ &\leq \ u \approx v, && (\odot\text{-antisymmetry}) \end{aligned}$$

for each $u, v, w \in U$.

First note that Definition 3.8.2 indeed encompasses the notion of fuzzy order according to Definition 3.2.2 and, by (a) of Theorem 3.5.3, also Definition 3.2.1. Moreover, as for the crisp fuzzy equality the compatibility condition is trivially satisfied, it also generalizes Definition 3.6.1 (cf. Proposition 3.7.2). Namely,

- for $\odot = \otimes$, Definition 3.8.2 yields Definition 3.2.1;
- for $\odot = \wedge$, Definition 3.8.2 yields Definition 3.2.2;
- for $\odot = \bullet$, Definition 3.8.2 yields Definition 3.6.1.

The last two theorems of this section examine the mutual relationships between all the definitions of fuzzy order. We first state a theorem in a spirit of Theorem 3.5.3 for the concept defined by Definition 3.8.2.

Theorem 3.8.3 (Appendix B, Theorem 1)**.** *Let $\lesssim$ be a reflexive and transitive fuzzy relation on U. The following conditions are equivalent:*

(a) There exists $\odot$ satisfying (3.4)–(3.7) and a fuzzy equality $\approx$ such that $\lesssim$ is a fuzzy order on U equipped with $\approx$ according to Definition 3.8.2.

(b) For each $\odot$ satisfying (3.4)–(3.7) there exists a fuzzy equality $\approx$ such that $\lesssim$ is a fuzzy order on U equipped with $\approx$ according to Definition 3.8.2.

(c) There exists $\odot$ satisfying (3.4)–(3.7) such that $\lesssim$ is a fuzzy order on U equipped with $\approx_\odot$ according to Definition 3.8.2.

(d) For each $\odot$ satisfying (3.4)–(3.7), $\lesssim$ is a fuzzy order on U equipped with $\approx_\odot$ according to Definition 3.8.2.

(e) There exists $\odot$ satisfying (3.4)–(3.7) and a fuzzy equality $\approx$ on U such that $\equiv_\odot \leq \approx \leq \equiv_\wedge$.

(f) For each $\odot$ satisfying (3.4)–(3.7) there exists a fuzzy equality $\approx$ on U such that $\equiv_\odot \leq \approx \leq \equiv_\wedge$.

Finally, the notions of fuzzy order according to Definitions 3.2.1, 3.2.2, 3.6.1, and 3.8.2 are essentially mutually equivalent.

Theorem 3.8.4 (Appendix B, Theorem 2)**.** *Let $\lesssim$ be a reflexive and transitive fuzzy relation on U. Each of the following conditions is equivalent to any of conditions (a)–(f) in Theorem 3.8.3. (Thus, in particular, the following conditions are mutually equivalent.)*

(a) $\lesssim$ is a fuzzy order according to Definition 3.2.1 for some fuzzy equality $\approx$.

(b) $\lesssim$ is a fuzzy order according to Definition 3.2.2 for some fuzzy equality $\approx$.

(c) $\lesssim$ is a fuzzy order according to Definition 3.6.1.

As a concluding note of this section, let us remark that other definitions of the general notion of fuzzy order may be formulated. For example, it is easy to verify using previous results that the following conditions are equivalent for a fuzzy relation $\lesssim$ on U and for any $\odot$ satisfying (3.4)–(3.7).

- $\lesssim$ is a fuzzy order according to Definition 3.8.2 for some fuzzy equality $\approx$;
- $\lesssim$ is transitive and the induced fuzzy relation $\equiv_\odot$ is reflexive and separable;
- $\lesssim$ is transitive and the induced fuzzy relation $\approx_\odot$ is a fuzzy equality.

3.9 Distinctive properties of the variants of antisymmetry and fuzzy order

We now know that the choice of a variant of antisymmetry condition – and therefore of fuzzy order definition – is to some extent a matter of taste. Still, it is of importance to know advantages and disadvantages of each such choice. We state only four theorems obtained in the study, as they are self explaining. Reader interested in more details may consult Appendix B, which contains not only the proofs, but also some additional remarks to each of the following results.

Theorem 3.9.1 (Appendix B, Theorem 3). *Let $\odot$ satisfy (3.4)–(3.7) and let $\lesssim$ be a fuzzy order according to Definition 3.8.2 for some fuzzy equality $\approx$.*

(a) The operation $\bullet$ defined by (3.9) is the smallest operation satisfying (3.4)–(3.7);

hence $\bullet$ is the smallest operation $\odot$ for which $\lesssim$ is a fuzzy order according to Definition 3.8.2 for some fuzzy equality $\approx$.

(b) The operation $\wedge$ is the largest operation $\odot$ satisfying (3.4)–(3.7);

hence $\wedge$ is the largest operation $\odot$ for which $\lesssim$ is a fuzzy order according to Definition 3.8.2 for some fuzzy equality $\approx$.

Theorem 3.9.2 (Appendix B, Theorem 4). *Of all the operations $\odot$ satisfying (3.4)–(3.7) for which a given fuzzy relation $\lesssim$ is a fuzzy order according to Definition 3.8.2 for some fuzzy equality $\approx$, $\otimes$ is the only one that satisfies adjointness w.r.t. $\to$, i.e.*

$$a \odot b \leq c \quad \text{iff} \quad a \leq b \to c \quad \text{for every } a, b, c \in L.$$

Theorem 3.9.3 (Appendix B, Theorem 5). *Let $\mathbf{L}$ be an arbitrary complete residuated lattice and let U have at least two elements. Then $\wedge$ is the only operation $\odot$ satisfying (3.4)–(3.7) such that for each fuzzy order $\lesssim$ according to Definition 3.8.2, the interval $\mathcal{I}_\odot$ is a singleton. Hence, $\wedge$ is the only operation satisfying (3.4)–(3.7) for which $\approx$ is uniquely determined by $\lesssim$.*

Theorem 3.9.4 (Appendix B, Theorem 6). *Let $\lesssim$ be reflexive and transitive fuzzy relation on U.*

(a) The largest reflexive and symmetric fuzzy relation contained in $\lesssim$ (i.e. the most informative indistinguishability w.r.t. $\lesssim$ in the sense above) is $\equiv_\wedge$, which is also the largest reflexive, symmetric, and transitive fuzzy relation contained in $\lesssim$.

(b) The least reflexive, symmetric, and transitive fuzzy relation contained in $\lesssim$ is $\equiv_\bullet$.

Chapter 4

Linear extensions of fuzzy orders

Extending a partial order into a chain is a classical problem in order theory. For fuzzy orders, such Szpilrajn-like completion was considered already by Zadeh (1971) when he introduced the concept of fuzzy order itself. These consideration were soon to be followed by others but many questions still remain open. One of the most recent study (Bodenhofer and Klawonn, 2004) on the topic analyzes different axioms for linearity of a fuzzy order in some detail. Surprisingly, the outcome of the study is that a completion of a fuzzy order with desirable properties is reachable only for very weak axiom of $\otimes$-linearity. We show that this is related to structure of fuzzy equalities on a set which is much richer than its counterpart in the Boolean case. Moreover, we propose a solution to fuzzy order completion problem by manipulating both entities, i.e. a fuzzy order and its induced fuzzy equality together, in a compatible way. Using this idea, which may be regarded as further extension of reflections on the role of fuzzy equality in the definition of fuzzy order in the spirit of Chapter 3, we obtain a way to extend any fuzzy order into linear fuzzy order in a broad class of fuzzy logics.

In this chapter, we summarize the results obtained in (Urbanec, 2023), i.e. the last study this thesis is built upon. Again, we present only the essential results and omit a lot of other content, such as auxiliary propositions and proofs. The full study is attached to this text as Appendix C.

4.1 A structure of fuzzy equalities on a finite set

The first theorems describe the structure of all fuzzy equalities on a finite set. This structure is more intricate in the setting of fuzzy logic than in the classical case as it is not limited to a single equality, i.e. the identity. Although it is interesting by itself, our primary objective is to examine the properties of linear fuzzy order extensions. We show in further sections that there is a connection between this structure of fuzzy equalities and possibility of extending general fuzzy order into a linear one. Here, we focus only on conditions under which the structure of all fuzzy equalities on a finite set forms a lattice. In Section 4.3, we will see that the same conditions characterize the class of residuated lattices which admits linear extension of arbitrary fuzzy order for a particular form of linearity.

Theorem 4.1.1 (Appendix C, Theorem 1)**.** *Let U be a finite set with at least two elements. The set of all* $\mathbf{L}$*-equalities on U equipped with subsethood relation forms a lattice if and only if* $\mathbf{L}$ *has a join-irreducible unit.*

In case U has less than two elements, such structure is a one-element complete lattice.

To ensure that a lattice of all $\mathbf{L}$-equalities on a finite set is a complete one, even stronger conditions must be imposed on $\mathbf{L}$.

Theorem 4.1.2 (Appendix C, Theorem 2)**.** *Let U be a finite set with at least two elements. The set of all* $\mathbf{L}$*-equalities on U equipped with subsethood relation forms a complete lattice if and* *only if* **L** *has unit irreducible by arbitrary joins, i.e. if and only if there is no set* D *of degrees from* $L \setminus \{1\}$ *with* $\bigvee D = 1$.

4.2 Completeness and linearity of binary fuzzy relation

There are various notions of completeness and linearity used in the theory of binary fuzzy relations on a set. Here, we are interested in linear fuzzy orderings, i.e. we focus on completeness of a fuzzy order relation in a sense of arbitrary two elements in a set being fully comparable. Even in this sense, there are multiple approaches to the concept of linearity in the literature. We discuss only strong completeness – the most widespread of these properties – and so-called crisp linearity (see below), here. For some other options and their mutual relationships see full results in Appendix C.

Definition 4.2.1 (Appendix C, Definition 4)**.** Binary fuzzy relation R on a set U is *strong complete* if

$$R(u, v) \vee R(v, u) = 1 \qquad \text{(strong completeness)}$$

holds for every $u, v \in U$.

Usually, the works utilizing the notion of linearity of fuzzy orders use only linear residuated lattices, especially the ones given by (left) continuous t-norms. As we use general complete residuated lattices, we need to discuss another aspect of linearity. Namely the expected meaning of linearity. Assume the strong completeness in some residuated lattice **L** with join-reducible unit. Then, for some $a, b \in L \setminus \{1\}$ such that $a \vee b = 1$, even relation R on $U = \{u, v\}$ where $R(u, u) = R(v, v) = 1$, $R(u, v) = a$, and $R(v, u) = b$ is strong complete, i.e. linear in the given setting. This situation might be considered as unnatural – R is a linear ordering where no element of pair u, v is fully above the other one. Therefore, we define yet another concept of linearity, which assures that such situation does not arise.

Definition 4.2.2 (Appendix C, Definition 5)**.** Binary fuzzy relation R on a set U is *crisp linear* if

$$R(u, v) = 1 \text{ or } R(v, u) = 1 \qquad \text{(crisp linearity)}$$

holds for every $u, v \in U$.

As every crisp linear binary fuzzy relation is obviously strong complete, the existence of crisp linear extension of a relation R implies the existence of strong complete extension of R. Note also that in case of residuated lattice with join-irreducible unit, in particular in any residuated lattice on $[0, 1]$, a binary fuzzy relation is crisp linear if and only if it is strong complete.[1] In the rest of the chapter, we examine when a fuzzy order extension into crisp linear fuzzy order exists and some derived notions for these cases.

4.3 Extensions and Szpilrajn-like theorem for fuzzy orders

In this section, we discuss the existence of a linear extension of any fuzzy order. The core idea differentiating our approach from previous studies is considering also the induced fuzzy equality in the extension process. It may be seen as further extension of reflections on the role of fuzzy equality in the definition of fuzzy order as presented in Chapter 3. There are two main reasons why we do so.

[1] The up to date most detailed study of linearity axioms for fuzzy orderings (Bodenhofer and Klawonn, 2004) use the setting of left-continuous t-norms on the interval $[0, 1]$, therefore some of our results may be easily related to the results obtained there.

First, fixing the fuzzy equality is, in our opinion, point of view which comes from Boolean setting where there is only one equality and therefore no reason to think about its modifications together with other entities in the given situation. We think that there is no general justification of the same view when there is many fuzzy equalities available on the given universe. That is a possibility of strengthening or weakening the given equality may be taken as new and advantageous aspect in the setting of fuzzy logic which is degenerated in the Boolean case.

Second reason has same root cause but immediate practical consequences: Fixing the fuzzy equality in the beginning of an extension process limits the situation by a great deal. In fact the main reason, why the results on linear extensions of fuzzy orders are quite pessimistic so far (Bodenhofer and Klawonn, 2004), is that a fuzzy equality[2] induced by a resulting linear fuzzy order has to obey same limits as the one induced by an initial fuzzy order. We start by recalling the definition of an extension of a binary fuzzy relation, in particular of a fuzzy order.[3]

Definition 4.3.1 (Appendix C, Definition 6)**.** Let R, S, and $\lesssim$ be binary fuzzy relations on U.

- We call S an *extension of* R if $R \subseteq S$. If $R \subset S$ we call S a *proper extension of* R.
- If $\lesssim$ is a fuzzy order on a set with fuzzy equality $\langle U, \approx \rangle$, we call a fuzzy order $\lesssim'$ on a set with fuzzy equality $\langle U, \approx' \rangle$ a *fuzzy order extension of* $\lesssim$ if $\lesssim'$ is an extension of $\lesssim$ and $\approx'$ is an extension of $\approx$.

Utilizing the idea described above, we arrive to conclusion that in broad class of complete residuated lattices, including every complete residuated lattice on $[0, 1]$, every fuzzy order may be extended into a crisp linear fuzzy order.

Theorem 4.3.2 (Appendix C, Theorem 5)**.** *A residuated lattice* $\mathbf{L}$ *has a join-irreducible unit if and only if for every set equipped with* $\mathbf{L}$*-equality* $\langle U, \approx \rangle$*, for any* $u, v \in U$*, and for each* $\mathbf{L}$*-order* $\lesssim$ *on* $\langle U, \approx \rangle$ *there is a crisp linear fuzzy order extension* $\lesssim'$ *of* $\lesssim$ *on* U *such that* $u \lesssim' v = u \lesssim v$*.*

Note that the condition of keeping comparability degree of u to v unchanged is of importance later, in Section 4.4, where the intersection representation of fuzzy orders is discussed. But if we omit it now, we obtain a straightforward generalization of classical Szpilrajn's extension theorem to the setting of residuated lattices and crisp linearity. Note that assumption of join-irreducibility of the residuated lattice's unit can not be dropped for crisp linearity (cf. Appendix C).

Corollary 4.3.3 (Appendix C, Corollary 6; Extension theorem for crisp linearity)**.** *Let* $\mathbf{L}$ *be a residuated lattice with join-irreducible unit. For any set* U *and any* $\mathbf{L}$*-order* $\lesssim$ *on* U *there is a crisp linear* $\mathbf{L}$*-order extension of* $\lesssim$*.*

If one tries to implement similar construction in the setting, where an underlying similarity is interpreted by a general fuzzy equivalence, it becomes rather trivial. The reason is separability of induced relation being the only limiting factor here. In such case, every fuzzy order has a linearization fulfilling any reasonable property of completeness as every fuzzy order may be extended into full relation on the given set. In the spirit of Section 3.3, we consider it to be another manifestation of fuzzy equivalences being inappropriate choice for the interpretation of underlying similarity.

We conclude this section by Example 4.3.4, which shows natural fuzzy orders without crisp linear extensions in the setting of residuated lattices with join-reducible unit.

Example 4.3.4 (Appendix C, Example 2)**.** It is well known (Belohlavek, 2002; Bodenhofer, 1999a; Höhle and Blanchard, 1985) that for any complete residuated lattice, the function $\rightarrow$ is a fuzzy order on the set L of truth degrees equipped with the fuzzy equality induced by $\leftrightarrow$

[2]Fuzzy equivalence in the case of (Bodenhofer and Klawonn, 2004). The idea remains the same, though.

[3]Note that thanks to the idea of manipulating both $\lesssim$ and $\approx$ together and to the alternative point of view on definitions of fuzzy order we presented in Chapter 3, it does not matter which of considered definitions of fuzzy order we use in this chapter.

on L. Such fuzzy order is moreover isomorphic to L^U for a singleton $U = \{u\}$ where $\rightarrow$ is lifted to $\subsetneq$ on L^U and $\leftrightarrow$ becomes $\approx$ on L^U. That is, from one point of view, this fuzzy order is a generalization of an important order induced by truth function of implication known from classical logic, and from another point of view, it is a generalization of classical power set ordered by the set inclusion.

Let $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ be any residuated lattice with unit join-reducible by $a, b \in L\setminus\{1\}$, i.e. where $a \vee b = 1$. Such residuated lattices exist, e.g. Heyting algebra on $L = \{0, a, b, 1\}$ with $0 \leq a \leq 1$, $0 \leq b \leq 1$ and $x \leq x$ for each $x \in L$. Now in case of $\mathbf{L}$, neither of the $\mathbf{L}$-orders described above can be extended into crisp linear fuzzy order unless two elements of L (resp. L^U), namely a and b (resp. $\{^a/u\}$ and $\{^b/u\}$), are factorized into one.

4.4 Intersection representation of fuzzy orders

Another important property of ordering relations in the Boolean case is an intersection representation of any order in the spirit of (Dushnik and Miller, 1941, Theorem 2.32) which was breifly described in Preliminaries. Utilizing the generalized version of Szpilrajn theorem from previous section, we obtain the similar intersection representation of fuzzy orders in a straightforward manner.

Theorem 4.4.1 (Appendix C, Theorem 8)**.** *A residuated lattice* $\mathbf{L}$ *has a join-irreducible unit if and only if for every set* U *equipped with* $\mathbf{L}$*-equality* $\approx$ *and every* $\mathbf{L}$*-order* $\lesssim$ *on* $\langle U, \approx\rangle$*, there is a set* $Ext(\lesssim)$ *of crisp linear* $\mathbf{L}$*-order extensions of* $\lesssim$ *such that* $[\bigcap Ext(\lesssim)](u, v) = u \lesssim v$ *for each* $u, v \in U$*.*

Two final corollaries of the results proven in previous sections, i.e. version of Szpilrajn's theorem for crisp linearity and other related results, are: First, equivalent characterizations of all residuated lattices $\mathbf{L}$ with join-irreducible unit; Second, generalizations of two well known theorems of classical order theory – Szpilrajn's extension theorem (Szpilrajn, 1930) and intersection representation theorem (Dushnik and Miller, 1941, Theorem 2.32), see Preliminaries – into the setting of complete residuated lattices on $[0, 1]$ and strong completeness.[4]

Corollary 4.4.2 (Appendix C, Corollary 9)**.** *The following propositions are equivalent:*

1. *The residuated lattice* $\mathbf{L}$ *has a join-irreducible unit.*
2. *For any finite set* U*, the set of all* $\mathbf{L}$*-equalities on* U *ordered by set inclusion forms a lattice.*
3. *Any finite* $\mathbf{L}$*-order may be extended into crisp linear* $\mathbf{L}$*-order.*
4. *An arbitrary* $\mathbf{L}$*-order may be extended into crisp linear* $\mathbf{L}$*-order.*
5. *An arbitrary* $\mathbf{L}$*-order may be represented as an intersection of some set of its crisp linear fuzzy order extensions.*

Corollary 4.4.3 (Appendix C, Corollary 10)**.** *Let* $\mathbf{L}$ *be a complete residuated lattice on* $[0, 1]$ *and* U *an arbitrary set equipped with an* $\mathbf{L}$*-equality* $\approx$*. Then for each* $\mathbf{L}$*-order* $\lesssim$ *on* $\langle U, \approx\rangle$ *we have*

1. *There is a strong complete* $\mathbf{L}$*-order extension of* $\lesssim$ *on* U*.*
2. *There is a set* $Ext(\lesssim)$ *of strong complete* $\mathbf{L}$*-orders on* U *such that* $u \lesssim v = [\bigcap Ext(\lesssim)](u, v)$ *for each* $u, v \in U$*.*

4.5 A note on the essential properties of chains

Last class of results obtained in (Urbanec, 2023), we present here, are essential properties of extension process in the classical setting and their translation into the setting of fuzzy logic.

[4] The conditions of crisp linearity and strong completeness coincide in any complete residuated lattice on $[0, 1]$. Thus we preffer to call the condition strong completeness here because it is a well established name.

In their study, Bodenhofer and Klawonn (2004) have identified three essential properties of partial orderings which are desirable also in the setting of fuzzy logic: an existence of linear extension; a possibility of an order representation by an intersection of its linear extensions; and the equivalence between maximality and linearity of an order. In addition, they have shown that if linearity is interpreted by the strong completeness then none of these properties can be attained unless we use the Gödel t-norm logic.[5] As their setting is the one of left continuous t-norms on $[0,1]$, i.e. particular linear residuated lattices, the concepts of strong completeness and crisp linearity coincide.

We have already seen how our approach improves these results by realizing that both order and equality relations have to be manipulated together. A short comment on each of these properties follows.

Existence of complete extension As we have already seen in Corollary 4.3.3, for suitable residuated lattices (including all the t-norm logics on $[0,1]$) each fuzzy order may be extended into a crisp linear fuzzy order. That is in the given setting the situation is analogous to the Boolean case.

Intersection representation Theorem 4.4.1 describes a representation of any fuzzy order by an intersection of its crisp linear fuzzy order extensions for suitable residuated lattices. Again, there is an obvious analogy to the Boolean case.

Maximality vs linearity The last relationship is more complex in the setting of fuzzy logic than in the Boolean case. The difference can be seen already from the following definition of maximality of fuzzy equality and fuzzy order.

Definition 4.5.1 (Appendix C, Definition 7)**.**
A fuzzy equality $\approx$ on U is *maximal* if there is no fuzzy equality $\approx'$ properly extending $\approx$ on U. A fuzzy order $\lesssim$ on $\langle U, \approx\rangle$ is *maximal on* U if there is no fuzzy order $\lesssim'$ on $\langle U, \approx'\rangle$ properly extending $\lesssim$.

Utilizing results on representation of strong complete (pre)orders obtained by Bodenhofer (1999b, Theorem 4), we may derive that maximality of crisp linear fuzzy order is given by maximality of its induced fuzzy equality.

Theorem 4.5.2 (Appendix C, Theorem 11)**.** *Let* $\mathbf{L}$ *be a residuated lattice with join-irreducible unit. Then following propositions hold for every* $\mathbf{L}$*-order* $\lesssim$ *on a set with an* $\mathbf{L}$*-equality* $\langle U, \approx\rangle$*:*

1. *If* $\lesssim$ *is a maximal* $\mathbf{L}$*-order on* U *then it is crisp linear.*
2. *If* $\lesssim$ *is a crisp linear* $\mathbf{L}$*-order on* $\langle U, \approx\rangle$ *then it is a maximal* $\mathbf{L}$*-order on* U *if and only if* $\approx$ *is a maximal* $\mathbf{L}$*-equality on* U*.*

Therefore we see that, because of a much more complex structure of all equalities on a set, the one-to-one relationship between linear and maximal orders from the Boolean case is lost in the setting of fuzzy logic.

[5] Their definition of fuzzy order assumes fuzzy equivalence on the set (cf. Section 3.3), but the core idea remains same even in the case of fuzzy equality.

Chapter 5

Conclusions and further topics

In this thesis, we summarized our considerations on the existing approaches to fuzzy order defined with respect to an underlying generalized equality. We first set up a historical context and then examined the definitions and their mutual relationships in some detail. We provided various observations to enhance the current understanding of the concept of fuzzy order and proposed the generalized point of view. Then we moved our attention to a classical problem of order theory, a Szpilrajn-like extension theorem, generalized for fuzzy orders. By doing so, we shed more light on the problem present in the literature since the inception of the concept of fuzzy order itself.

There are two categories of results we consider most important. First, a unifying concept of antisymmetry together with the resulting generalized notion of fuzzy order. These considerations yielded the theorems showing that the existing variants of the notion of fuzzy order defined with respect to a fuzzy equality are in a sense mutually equivalent and are moreover equivalent to our generalized concept of fuzzy order. This is in contrast to current understanding that the definitions are different, some of them being more general than others. An alternative perspective one can adopt is that the various available definitions differ only in the limits they impose on the underlying similarity relation; however, despite these different limits, the class of fuzzy relations they describe is always the same.

The other category is then determined by the idea that the dependence of fuzzy order on underlying equality is only half of the story because the dependence is actually mutual. That is both relations should always be considered and manipulated together. First manifestation of this perspective is apparent in the equivalence of the various definitions of fuzzy order. Continuing the line of this perspective, we arrived to the Szpilrajn-like extension theorem for fuzzy orders. Here, we showed that thanks to manipulating both the entities together, we may extend any fuzzy order into a crisp linear one in a broad class of residuated lattices, including all residuated lattices on $[0, 1]$.

For the future, quite many lines of research offers themselves naturally. The most interesting is a dimension theory for fuzzy orders. We have seen a small taste of classical dimension theory together with its most famous results in Preliminaries. Generalizing these results into setting of fuzzy logic seems to be a good starting point in this direction. We already have some preliminary results and shall present them in future publications.

The second topic worth of further attention is that of lattice-type fuzzy orders and how our observations affect them. As mentioned in Chapter 2, lattice-type fuzzy orders are, similarly to the classical case, one of the main driving forces behind research conducted on fuzzy orders. Also in this area, we already have some interesting preliminary observations.

Among the topics, which we would like to focus on in the long term, are deeper applications of fuzzy order outside of formal concept analysis, as there are not many of them now. We feel that various applications may attract further attention and thus help to broaden the knowledge of fuzzy orders. The second long term topic we mention, in a sense related to applications, is considering our results from the perspective of category theory as fuzzy (pre)orders appear in the categorical works quite often.

Finally, one very interesting observation, rather methodological than mathematical, is hidden between the lines of this thesis. All the results were actually being developed simultaneously and were affecting each other. Often, they were gradually updated by switching there and back between theoretical and applied side of the central question "What is fuzzy order?". Many of them were scratched on the way, completely rebuilt, or suddenly appeared from nowhere. Only then I have fully appraised the idea that my supervisor often mentions: mathematics is in a sense "experimental" science where one has to "experiment" and "play" with the concepts. Indeed, one has to test the concepts he is considering by "playing" with them in as many contexts as he can and update his understanding of the theory accordingly, even if it means starting from scratch again. For me personally, this is one of most important lessons I take from the time spent working on this topic and I am grateful to my supervisor for it.

Shrnutí v českém jazyce

V této práci jsme se zabývali existujícími přístupy k fuzzy uspořádáním definovaným vzhledem ke zobecněné rovnosti na uvažovaném univerzu. Nejprve jsme stručně popsali původ pojmu a jeho historii. Poté jsme shrnuli existující přístupy, přidali k nim nová pozorování a postřehy a nakonec je zastřešili novým, obecnějším pohledem. Svá pozorování, zejména ta o těsnější vazbě mezi fuzzy uspořádáním a rovností na uvažovaném univerzu, jsme dále využili k získání nových poznatků o rozšiřování fuzzy uspořádání ve stylu Szpilrajnovy věty.

Za nejdůležitější považujeme dva typy výsledků dosažených v této práci. Prvním je již zmíněný zobecněný pohled na antisymetrii a tedy i na fuzzy uspořádání jako takové. Tato pozorování vyústila v sérii vět ukazujících, že všechny námi uvažované pohledy na fuzzy uspořádání, včetně nově navrženého, jsou v jistém smyslu ekvivalentní. Tento poznatek rozporuje často přijímaný pohled, kde jsou některé z definic považovány za obecnější než jiné. Alternativně lze tyto výsledky interpretovat jako pozorování, že uvažované, dosud dostupné definice fuzzy uspořádání se vzájemně liší pouze v omezeních, která kladou právě na fuzzy rovnost uvažovanou na univerzu. Všechny ale popisují stejnou množinu fuzzy relací.

Druhá třída výsledků je poté odvozena od souvisejícího pozorování, že závislost mezi fuzzy uspořádáním a fuzzy rovností je vzájemná. Tedy chceme-li fuzzy uspořádání v dané situaci nějakým způsobem upravit, tak se tyto úpravy musí vhodně odrážet i na příslušné fuzzy rovnosti. Toto pozorování je do jisté míry vidět již na ekvivalenci definic uvedené výše. V plné šíři jsme jej ale využili při úvahách o rozšiřování fuzzy uspořádání v duchu Szpilrajnových výsledků. Díky těmto úpravám obou relací zároveň jsme popsali postup pro získání lineárního rozšíření fuzzy uspořádání v mnoha různých fuzzy logikách, zejména pak ve všech, kde jsou stupně pravdivosti interpretovány intervalem $[0, 1]$.

Dosažené výsledky nabízejí několik směrů pro budoucí výzkum. Nejzajímavějším z nich je dimenze fuzzy uspořádání, v duchu výsledků dosažených Dushnikem a Millerem (1941) pro klasická uspořádání. Dalším, neméně důležitým tématem je vliv našich pozorování na svazová fuzzy uspořádání. Svazová fuzzy uspořádání jsou pravděpodobně nejvíce prozkoumaným typem fuzzy uspořádání a mají mnoho aplikací zejména v kontextu formální konceptuální analýzy nad fuzzy logikou. V obou těchto směrech již máme základní výsledky, které plánujeme představit v budoucích pracích.

Z dlouhodobějšího pohledu bychom se chtěli věnovat i dalším, různorodým, hlouběji zpracovaným aplikacím fuzzy uspořádání, neboť tyto dle našeho názoru zatím chybí, zejména ve srovnání s množstvím aplikací klasických uspořádání. Naší nadějí je, že důkladný popis zajímavých aplikací povede k dalšímu rozvoji fuzzy uspořádání i v teoretické rovině. Druhý dlouhodobější cíl do jisté míry souvisí s tím prvním – zvážit dosažené výsledky z pohledu teorie kategorií, kde jsou fuzzy (před)uspořádání poměrně často uvažována v různých kontextech.

Závěrem vyzdvihneme jedno pozorování, které je spíše metodologické nežli matematické. Všechny dosažené výsledky byly ve skutečnosti budovány zároveň a často se vzájemně ovlivňovaly. Zejména posun v jednom směru často způsobil výrazné změny v uvažování o směru druhém – již dosažené výsledky musely být znovu zváženy, upraveny, či dokonce zahozeny; některé myšlenky se pak díky změně kontextu objevily jakoby z ničeho. Až při těchto momentech jsem plně docenil myšlenku často zmiňovanou mým vedoucím: i v matematice mají experimenty své místo. Vskutku, teoretické výsledky se projasňovaly a zpřesňovaly s každým kontextem, ve kterém jsme dané koncepty uvažovali a experimentovali s nimi. A naopak, nové úvahy o aplikacích se samy nabízely s každým posunem v teoretických poznatcích. Osobně, tuto zkušenost považuji za jednu z nejdůležitějších, kterou si z práce na tomto tématu odnáším, a jsem za ni svému vedoucímu vděčný.

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Appendix A

On the concept of fuzzy order I: Remarks and observation

The first part of the two-part-study on the concept of fuzzy order defined with respect to a fuzzy equality. The study was conducted together with my supervisor, Radim Belohlavek, and published in International Journal of General Systems (Belohlavek and Urbanec, 2023a,b). Chapter 3 contains summary of the main results obtained in this study.

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On the concept of fuzzy order I: Remarks and observations

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We consider the concept of fuzzy order in which antisymmetry is intrinsically connected to a many-valued equality on the underlying universe. We examine the origins of this concept, provide remarks and observations on the existing studies, and prove new results. In part I, we scrutinize the existing approaches to the examined concept of fuzzy order and present remarks and results to elucidate the available notions and findings, as well as to provide a deeper insight into several issues. In part II, we explore antisymmetry.

Keywords: order; fuzzy logic; fuzzy equality; antisymmetry

1. Aim of this paper

The concept of order is one of the basic concepts accompanying human reasoning. Correspondingly, orders – known also as partial orders or orderings – became a widely studied kind of relations, which are utilized across a variety of fields. Recall that a classical order on a set U is a binary relation $\leq$ on U that is reflexive, antisymmetric, and transitive, i.e. satisfies $u \leq u$; $u \leq v$ and $v \leq u$ implies $u = v$; and $u \leq v$ and $v \leq w$ implies $u \leq w$ for all $u, v, w \in U$. In addition to a classical, bivalent setting, the concept of order makes a good sense in a more general setting, in which bivalence is replaced by graduality. For instance, instead of conceiving inclusion, which is a particular example of order, as bivalent, one may consider an entity as being included in another entity to a certain degree. It hence comes as no surprise that generalized orders – known as fuzzy orders – in which ordering is a matter of degree represent a thoroughly studied subject.

Since the pioneering paper by Zadeh (1971),[1] a number of definitions of the concept of fuzzy order have been proposed.[2] In our paper, we are concerned with the arguably most developed approach to fuzzy orders, pursued originally by Ulrich Höhle, Nicole Blanchard, Ulrich Bodenhofer, and Radim Belohlavek. The distinctive feature of this approach is the treatment of antisymmetry: The approach assumes that the set on which a fuzzy order is defined is equipped with a fuzzy relation that generalizes ordinary equality, which is involved in classical antisymmetry. This approach actually subsumes two particular definitions of antisymmetry, which shall

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[1] It is worth noting that before Zadeh, many-valued orders were considered by Menger (1951) as part of his probabilistic approach to relations.
[2] We identified over a thousand papers on fuzzy order in Scopus (papers containing "fuzzy order" or "fuzzy lattice" in the title, abstract, or keywords).

be examined in detail below.

Even though a number of papers on fuzzy orders have been published after the pioneering works by the above-mentioned authors, some basic questions still remain open or answered partially only. Most important among them is the basic question of what is actually an appropriate definition of fuzzy order.

Note at this point that such question should not be regarded as a quest for "the right" definition of fuzzy order, which might rightfully be considered as ill posed. Namely, in the more general setting of fuzzy logic, different situations may require different definitions of fuzzy order, each of which may serve the intended purpose in the particular situation. Yet, all of these definitions may indeed be proper generalizations of the classical concept of order.[3]

Correspondingly, rather than looking for "the right" definition of fuzzy order, the question mentioned in the previous paragraph is to be understood as a question of ramifications of and relationships between possible definitions of fuzzy orders. Exploration of this question as regards the above approach to fuzzy order, i.e. involving antisymmetry with respect to generalized equality, is the primary purpose of our paper.

2. Definitions of fuzzy order

Throughout the rest of this paper, we assume a framework for dealing with fuzzy sets and fuzzy relations that is based on complete residuated lattices used as the structures of truth degrees. For details, we refer to Appendix. In particular, we denote an arbitrary complete residuated lattice by $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$.

2.1. *Preliminary considerations*

A transfer of an ordinary concept to a fuzzy setting, i.e. a generalization of a concept defined in the framework of classical logic to a fuzzy logic framework, involves two aspects: The first one is obvious and requires that the generalized concept indeed be a generalization of the ordinary concept. The second one, which is somewhat vague and much less trivial, asks that the generalized concept be useful and behave naturally. In this broader perspective, which forms the starting point of our considerations, the first aspect concerns mathematical correctness, while the second pertains to mathematical practice.

However trivial the imperative of the first aspect appears, it is worth noting that it may be understood several ways. Our understanding, which has become common in the past two decades or so, is as follows. A definition of the generalized concept assumes a general structure $\mathbf{L}$ of truth degrees and is expressed by appropriate conditions. For instance, a binary fuzzy relation $R : U \times U \rightarrow L$ is called transitive if the condition

$$R(u, v) \otimes R(v, w) \leq R(u, w) \tag{1}$$

holds true for each $u, v, w \in U$. Now, one may consider the structure $\mathbf{L}$ of truth degrees as a parameter and consider the definition for arbitrary $\mathbf{L}$. Those $\mathbf{L}$'s include, e.g., the real unit interval $L = [0, 1]$ and Łukasiewicz operations, but also – as a

[3]This is a common situation encountered in many areas of mathematics: A given concept defined in a given framework might have several different meaningful generalizations in a more general framework.

very particular case – the two-valued Boolean algebra $\mathbf{L} = \mathbf{2}$ of classical logic in which $L = \{0, 1\}$, i.e. the classical truth values 0 and 1 are the only recognized degrees of truth. Taking $\mathbf{L} = \mathbf{2}$ means that the fuzzy relation R is in fact a two-valued relation and may thus be identified with the corresponding ordinary relation $o(R) = \{\langle u, v\rangle \mid R(u, v) = 1\}$. As one then easily checks, R is transitive in the sense of definition (1) if and only if the ordinary relation $o(R)$ is classically transitive, i.e. $\langle u, v\rangle \in o(R)$ and $\langle v, w\rangle \in o(R)$ imply $\langle u, w\rangle \in o(R)$ for each $u, v, w \in U$. It is in this sense that the definition (1) of transitivity of a fuzzy relation *generalizes* the classical definition.[4]

Usefulness and natural behavior, i.e. the second aspect of generalizing a classical concept to a fuzzy setting, basically implies that the generalized concept be useful in modeling of reality, have nice properties, and be connected to other concepts in the generalized framework in an analogous way the classical concept is in the classical framework. For instance, when generalizing the concept of an equivalence relation to a fuzzy setting, it is desired that the generalized concept of fuzzy equivalence provides a reasonable model of indistinguishability in a setting that allows for gradual indistinguishability, and that it is naturally connected to an appropriately defined concept of a fuzzy partition.

It is immediate that meeting the requirement for the generalized concept to be indeed a generalization of the corresponding ordinary concept does not imply that the second requirement is satisfied, i.e. usefulness and natural behavior of the generalized concept. To meet both of the above-outlined criteria, one needs to "experiment" and "play" with the generalized concept, i.e. explore its properties in the generalized framework and possibly modify its definition, until a generalized concept comes up that is useful and behaves naturally from the viewpoint of the concerned needs.[5]

According to the above rationale, to define a reasonable concept of fuzzy order not only requires to provide a generalization of classical orders but also to examine thoroughly the properties of such generalization with regard to notions which are relevant when considering gradual ordering. Since classical orders are reflexive, antisymmetric, and transitive relations, it appears reasonable to define generalized conditions of reflexivity, antisymmetry, and transitivity, and define fuzzy orders as fuzzy relations that satisfy these generalized conditions. Generalizing reflexivity and transitivity appears immediate: reflexivity of a fuzzy relation $R : U \times U \to L$ means $R(u, u) = 1$ for each $u \in U$, while transitivity of R is defined by (1). These two definitions have been proven useful and naturally behaving by a great number of studies. Generalizing antisymmetry, however, is much less immediate.

To illustrate our point, let us recall the pioneering paper by Zadeh (1971), in which a fuzzy relation R is considered antisymmetric if

$$R(u, v) > 0 \text{ and } R(v, u) > 0 \text{ imply } u = v \tag{2}$$

for each $u, v \in U$. As one easily checks, this definition generalizes classical antisymmetry, and hence is mathematically correct. However, it has serious drawbacks, of which we present the following one.

In the classical setting, one of the most important examples of orders is represented by inclusion $\subseteq$ of sets: $\subseteq$ is a reflexive, antisymmetric, and transitive relation, i.e. the pair $\langle 2^U, \subseteq\rangle$ is an ordered set, for any set U. However, the graded inclusion

[4] See Belohlavek, Dauben, and Klir (2017) for a detailed exposition of generalization to the framework of fuzzy logic.
[5] Clearly, for different purposes, the concerned needs may be different. It may hence well be that there co-exist several different generalized concepts, each of which is a generalization of the given classical concept, is useful and behaves naturally for the particular purpose.

$\underset{\sim}{\subseteq}$ of fuzzy sets on U, defined by (26) in Appendix, does not satisfy Zadeh's antisymmetry (2).[6] From the above viewpoint, the reason is that Zadeh did not put his definition to a proper test, i.e. did not derive his definition from natural examples and did not consider it in a proper context of relevant mathematical considerations. In a sense, (2) provides a formalistic approach to antisymmetry, which is mathematically correct but has a rather limited use.

This example is not to criticize Zadeh, who typically had been deriving his notions from natural examples, nor to criticize several others paper and even textbooks, such as the widely circulated Klir and Yuan (1995), which adopted Zadeh's definition, or proposed different definitions, with similar drawbacks.[7] Rather, we intend to emphasize the importance of putting notions generalized to the setting of fuzzy logic to proper tests involving concepts and theories with which the generalized notions shall interact properly.

2.2. *Definitions of fuzzy order on a set with generalized equality*

From today's perspective, the approach we examine in our paper may be regarded as alleviating the drawbacks of formalistic attempts as the one outlined above. The key idea of this approach is to consider as fuzzy (graded, many-valued) not only the order relation on the universe set U but also the equality relation on U. That is to say, one considers a universe U, a fuzzy relation $\lesssim$ generalizing classical order, and a fuzzy relation $\approx$ generalizing classical equality. Even without further exploration, such approach appears well thought out because the classical theory of ordered sets refers to equality on many occasions including the definition of antisymmetry.

Remark 1. From an epistemic viewpoint, it is even tempting – when considering a fuzzy order $\lesssim$ on U – to assume that $\lesssim$ is defined on a set U on which a generalized equality $\approx$ is given already. This view agrees with the classical situation in which equality $=$ is implicitly understood as being given on the considered universe set. In drawing conclusions of this sort, though, one has to be careful because an alternative view is also possible in which even in a fuzzy setting, U may be regarded as equipped with classical equality $=$ only, and both $\lesssim$ and $\approx$ may be understood as further entities with the provision that in the classical case, $=$ coincides with $\approx$.[8]

The approach we explore has been initiated in the pioneering works of Höhle, Blanchard, Bodenhofer, and Belohlavek; see e.g. Belohlavek (2001, 2002, 2004); Blanchard (1989); Bodenhofer (2000, 2003); Bodenhofer and Klawonn (2004); Höhle (1987); Höhle and Blanchard (1985). It appeared for the first time in the paper by Höhle and Blanchard (1985) and was apparently rediscovered later by Bodenhofer and Belohlavek, whose notions of fuzzy order differ from each other in the antisymmetry condition. While Bodenhofer's antisymmetry coincides with that of Höhle and Blanchard (1985), Belohlavek's antisymmetry is different and essentially coincides with antisymmetry proposed in yet another paper by Höhle (1987) whose purpose is a study of Dedekind's construction of real numbers in a fuzzy setting. Both Bodenhofer and Belohlavek studied their notions of fuzzy order in several sub-

[6] This is now well known: Take e.g. the Łukasiewicz structure on $L = [0,1]$, $U = \{u, v\}$, $A = \{{}^{0.1}/u, {}^{0.9}/v\}$, and $B = \{{}^{0.9}/u, {}^{0.1}/v\}$. Then $A \underset{\sim}{\subseteq} B = 0.2 > 0$ and $B \underset{\sim}{\subseteq} A = 0.2 > 0$ but $A \neq B$. Several other notions of degree of inclusion of fuzzy sets proposed in the literature violate Zadeh's antisymmetry as well.

[7] A slightly more general condition for antisymmetry is used e.g. by Gottwald (1993) and by Fodor and Roubens (1994), where antisymmetry asks that $R(u,v) \otimes R(v,u) > 0$ imply $u = v$ for each $u, v \in U$, where $\otimes$ is a chosen t-norm. As with Zadeh's antisymmetry, graded inclusion does not satisfy this condition.

[8] By coincidence of $=$ with $\approx$ we mean that $u \approx v = 1$ for $u = v$ and $u \approx v = 0$ for $u \neq v$.

sequent papers. They both were motivated by rather different goals and have not examined the relationships of their two notions of fuzzy order to any deeper extent.

Let us note at this point that a related notion of fuzzy order has independently been introduced by Blanchard (1983) and Fan (2001). This notion does not involve fuzzy equality on the underlying universe set U, and is formulated in a slightly restricted framework. Nevertheless, its generalization to the framework of complete residuated lattices turns out to be equivalent in a sense with the approach utilizing fuzzy equalities. We present details on this topic in part II (Belohlavek and Urbanec 2023).

We now provide the two definitions of fuzzy order on a set with a generalized equality. We provide them basically in the forms present in the works of Bodenhofer and Belohlavek, respectively, since these forms are most common in the literature; the definitions which appeared in the works by Höhle are just mild variations of the definitions we present. Detailed comments on the definitions are presented below.

Definition 1 (Höhle, Blanchard, Bodenhofer)**.** A fuzzy order on a set U equipped with a fuzzy equality relation $\approx$ is a binary fuzzy relation $\lesssim$ on U satisfying

$$
\begin{aligned}
u \approx v &\le u \lesssim v && (\approx\text{-reflexivity}),\\
(u \lesssim v) \otimes (v \lesssim w) &\le u \lesssim w && (\text{transitivity}),\\
(u \lesssim v) \otimes (v \lesssim u) &\le u \approx v && (\otimes\text{-antisymmetry}),
\end{aligned}
$$

for each $u, v, w \in U$. (Note: Höhle and Blanchard's as well as Bodenhofer's original definitions actually assume, more generally, that $\approx$ is a fuzzy equivalence rather than fuzzy equality; this is discussed below.)

Definition 2 (Höhle, Belohlavek)**.** A fuzzy order on a set U equipped with a fuzzy equality relation $\approx$ is a binary fuzzy relation $\lesssim$ on U compatible with $\approx$, i.e. fulfilling

$$
(u_1 \lesssim v_1) \otimes (u_1 \approx u_2) \otimes (v_1 \approx v_2) \le u_2 \lesssim v_2,
$$

for every $u_1, u_2, v_1, v_2 \in U$, which satisfies

$$
\begin{aligned}
u \lesssim u &= 1 && (\text{reflexivity}),\\
(u \lesssim v) \otimes (v \lesssim w) &\le u \lesssim w && (\text{transitivity}),\\
(u \lesssim v) \wedge (v \lesssim u) &\le u \approx v && (\wedge\text{-antisymmetry}),
\end{aligned}
$$

for each $u, v, w \in U$.

Remark 2 (nomenclature)**.** (a) If distinction is needed, we shall call fuzzy orders according to Definitions 1 and 2 fuzzy orders with $\otimes$-antisymmetry and fuzzy orders with $\wedge$-antisymmetry, respectively.

(b) Various terms are used in the literature, e.g. partial ordering in L-underdeterminate sets (Höhle and Blanchard 1985), T-E-ordering (Bodenhofer),[9] partial ordering on an I-valued set (Höhle 1987), and **L**-order on a set with **L**-equality (Belohlavek). Also note that instead of complete residuated lattices, Höhle and Blanchard (1985) use somewhat more particular structures.[10] Höhle (1987) and

[9] Bodenhofer uses T and E for $\otimes$ and $\approx$, respectively.

[10] They use completely lattice-ordered commutative semigroups whose identity element is the largest element (such structures are equivalent to complete residuated lattices) satisfying additionally for non-empty $A \subseteq L$ that $\bigvee A = 1$ implies $\bigvee\{a \otimes a \mid a \in A\} = 1$.

Bodenhofer use $L = [0, 1]$ with a left-continuous t-norm.[11] Let us also mention that Höhle (1987) uses a more general concept of fuzzy equality inspired by category-theoretical considerations, in which the degree $u \approx u$ may be strictly smaller than 1 and is interpreted as an extent to which u exists.

(c) The structure consisting of U, $\approx$, and $\precsim$ as described in the definitions is called a fuzzy ordered set, and is denoted $\langle U, \approx, \precsim\rangle$, or $\langle\langle U, \approx\rangle, \precsim\rangle$ if the distinct role of $\approx$, e.g. resulting from epistemic preference as discussed in Remark 1, is to be emphasized. In the latter case, one naturally speaks of a fuzzy order $\precsim$ on a set with fuzzy equality $\langle U, \approx\rangle$.

Remark 3 (basic relationships)**.** (a) One may observe two distinctions when comparing Definition 1 with Definition 2. First, the definitions use different forms of antisymmetry, with the stronger $\wedge$-antisymmetry implying the weaker $\otimes$-antisymmetry. Basic relationships of these two forms of antisymmetry are addressed in Section 3.3; a thorough consideration of antisymmetry is presented in the second part of this paper. Second, Definition 1 requires $\approx$-reflexivity of the fuzzy order $\precsim$ while Definition 2 requires that $\precsim$ be reflexive and compatible with $\approx$. Note at this point that in presence of the other conditions, these two requirements are equivalent and that we discuss this relationship in Section 3.2. It hence follows from the facts just mentioned that Definition 1 delineates a more general notion of fuzzy order than Definition 2 in the following sense: If $\langle U, \approx, \precsim\rangle$ is a fuzzy ordered set according to Definition 2, it also is a fuzzy ordered set according to Definition 1, but not vice versa.[12] This view, however, is thoroughly reconsidered in part II of our paper, in which an alternative view is provided.

(b) As noted in Definition 1, the original definitions of Höhle, Blanchard, and Bodenhofer assume that $\approx$ is a fuzzy equivalence rather than a fuzzy equality. A fuzzy equality is a more particular concept than a fuzzy equivalence since it additionally satisfies separation; see Appendix. We nevertheless assume that $\approx$ is a fuzzy equality in Definition 1 since this assumption yields, in a sense, a cleaner generalization of the notion of order to the setting of fuzzy logic, which may moreover be better compared to the notion of fuzzy order from Definition 2; see Section 3.1 for details. Still, we shall speak of Definition 1 as the definition by Höhle, Blanchard, and Bodenhofer, as no confusion arises in view of the present remark.

Note also that in their definition of fuzzy order, Höhle and Blanchard (1985) in fact use – somewhat misleadingly – the term "L-equality" to denote a fuzzy equivalence.

(c) With respect to the problem with Zadeh's antisymmetry and graded inclusion, it is immediate and nowadays well known that for any set X, graded inclusion $\precsim$ becomes a fuzzy order with $\otimes$-antisymmetry on $U = L^X$ when one considers $A \approx B = (A \subsetsim B) \otimes (B \subsetsim A)$ for fuzzy sets $A, B \in L^X$, and becomes a fuzzy order with $\wedge$-antisymmetry on L^X when one considers $A \approx B = (A \subsetsim B) \wedge (B \subsetsim A)$. Note that in both cases, $\approx$ is a fuzzy equality on L^X. These observations appear in their respective forms in Belohlavek (2002); Bodenhofer (1999); Höhle (1987).

(d) It is well known and important for algebraic investigations of logic that the truth function $\to$ of classical implication defines an order $\leq$ on the set $L = \{0, 1\}$ of classical truth values by letting $a \leq b$ iff $a \to b = 1$. In other words, $\to$ is the

[11] Note that $\langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ with L being the real unit interval $[0, 1]$ is a complete residuated lattice if and only if $\otimes$ is a left-continuous t-norm and $a \to b = \max\{c \mid a \otimes c \leq b\}$. Hence, the only restriction compared to the framework of complete residuated lattices is $L = [0, 1]$.

[12] Put $U = \{u, v\}$ and let $\approx$ and $\precsim$ be defined by $u \approx u = v \approx v = 1$, $u \approx v = v \approx u = 0.5$, $u \precsim u = v \precsim v = 1$, $u \precsim v = 0.8$, and $v \precsim u = 0.6$. Then for the Łukasiewicz structure on $L = [0, 1]$, $\precsim$ is a fuzzy order according to Definition 1 but not according to Definition 2, as $(u \precsim v) \wedge (v \precsim u) = 0.8 \wedge 0.6 = 0.6 \not\leq 0.5 = u \approx v$.

characteristic function of $\leq$. A natural generalization of this property holds true in the present framework (recall: the two-valued Boolean algebra is a particular case of a complete residuated lattice): For any complete residuated lattice $\mathbf{L}$, the function $\rightarrow$ (i.e. residuum, or truth function of implication) is a fuzzy order with $\otimes$-antisymmetry on the set L of truth degrees equipped with the fuzzy equality defined by $a \leftrightarrow_{\otimes} b = (a \rightarrow b) \otimes (b \rightarrow a)$. Furthermore, $\rightarrow$ is a fuzzy order with $\wedge$-antisymmetry when the fuzzy equality is defined by $a \leftrightarrow_{\wedge} b = (a \rightarrow b) \wedge (b \rightarrow a)$. These observations appear in works by Belohlavek (2002); Bodenhofer (1999); Höhle and Blanchard (1985).[13] Notice that the examples in the present condition (d) may be regarded as special cases of those of condition (c) of the present remark, because L may be identified with L^U for a singleton $U = \{u\}$, in which case $\subsetneq$ becomes $\rightarrow$ and $\approx$ becomes $\leftrightarrow_{\otimes}$ or $\leftrightarrow_{\wedge}$, respectively.

(e) For $L = [0, 1]$ (or, more generally, a linearly ordered L), $\otimes = \wedge$, and $\approx$ coinciding with ordinary equality (i.e. $u \approx v = 1$ for $u = v$ and $u \approx v = 0$ for $u \neq v$), both $\otimes$-antisymmetry and $\wedge$-antisymmetry are equivalent to Zadeh's antisymmetry (2), which – for $\otimes$-antisymmetry – is mentioned by Bodenhofer (1999) and Höhle and Blanchard (1985).

Remark 4 (historical comments)**.** (a) Definition 1 was – with the conditions listed above but, as noted, with fuzzy equivalence rather than fuzzy equality – proposed for the first time by Höhle and Blanchard (1985), who aimed to improve and further study the concept of order in the setting of fuzzy logic originally introduced by Zadeh (1971). This definition was later reinvented by Bodenhofer, who was apparently not aware of Höhle and Blanchard's work. Bodenhofer does not cite this work in his first papers (1999; 2000), but cites it in his next paper (2003), in which he acknowledges Höhle and Blanchard's historical priority.

(b) Definition 2 was proposed for the first time by Höhle (1987) with a more particular choice of structures $\mathbf{L}$ of truth degrees (namely, complete residuated lattices on $[0, 1]$) but with a more general concept of fuzzy equality; cf. Remark 2 (b). It was later reinvented by Belohlavek who was not aware of Höhle's paper.

(c) It is worth noting that the motivation in the works investigating the notion of fuzzy order according to Definition 1, i.e. by Höhle and Blanchard (1985) and by Bodenhofer, was basically a general study of fuzzy order. The motivation in the first works exploring fuzzy orders according to Definition 2, i.e. by Höhle (1987) and by Belohlavek, was more particular, namely to study certain ordered structures determined by binary fuzzy relations. In particular, Höhle studied the so-called Dedekind cuts, while Belohlavek studied so-called concept lattices; both of these structures are strongly related (put briefly, Dedekind cuts are a particular case of concept lattices).

(d) Interestingly, Höhle (1987) does not comment on and does not cite his previous definition of fuzzy order (Höhle and Blanchard 1985), i.e. does not mention why he changed his definition for the purpose of his 1987 paper.

(e) Even though neither Bodenhofer nor Belohlavek were initially familiar with Höhle's work on fuzzy orders, both were strongly influenced by Höhle's work on fuzzy logic.

(f) Bodenhofer and Belohlavek discussed their works on fuzzy order at the FSTA 1998 conference in Liptovský Ján, at which point most of the results of their first papers were worked out, but they never got to comparing their approaches. Note also that Belohlavek's first paper (Belohlavek 2004) got stuck in the production process: As is apparent from the acknowledgment in this paper and from Belohlavek (2001),

[13] It is worth noting that Höhle and Blanchard (1985) consider $\otimes$-antisymmetry but take $\leftrightarrow_{\wedge}$ as the fuzzy equality.

the 2004 paper was submitted in 2000.

In the remainder of this paper we shall examine the properties, relationships, and ramifications of the two notions of fuzzy order in detail.

3. Observations and results

We now present our observations on the two notions of fuzzy order. We also aim at providing and putting in context the existing results and, in particular, attempt to clarify relationships between the various conditions involved by providing clean statements.

To provide a deeper insight, we not only consider the conditions involved in the definitions of fuzzy order but also consider truth degrees to which these conditions are satisfied, such as the degree to which a fuzzy relation $\precsim$ is reflexive or transitive. This is because considerations of these degrees and the respective relationships provide a deeper understanding of the concerned notions. We therefore start by recalling the definition of the degrees of relevant properties of fuzzy relations.

For binary fuzzy relations R and $\approx$ on U we define:

$$\mathrm{ref}(R) = \bigwedge_{u \in U} R(u,u), \tag{3}$$

$$\mathrm{ref}_{\approx}(R) = \bigwedge_{u,v \in U} ((u \approx v) \to R(u,v)), \tag{4}$$

$$\mathrm{sym}(R) = \bigwedge_{u,v \in U} (R(u,v) \to R(v,u)), \tag{5}$$

$$\mathrm{tra}(R) = \bigwedge_{u,v,w \in U} ((R(u,v) \otimes R(v,w)) \to R(u,w)), \tag{6}$$

$$\wedge\text{-ant}(R) = \bigwedge_{u,v \in U} ((R(u,v) \wedge R(v,u)) \to (u \approx v)), \tag{7}$$

$$\otimes\text{-ant}(R) = \bigwedge_{u,v \in U} ((R(u,v) \otimes R(v,u)) \to (u \approx v)), \tag{8}$$

$$\mathrm{comp}(R) = \bigwedge_{u_1,u_2,v_1,v_2 \in U} ((R(u_1,v_1) \otimes (u_1 \approx u_2) \otimes (v_1 \approx v_2)) \to R(u_2,v_2)). \tag{9}$$

The degrees $\mathrm{ref}(R)$, $\mathrm{ref}_{\approx}(R)$, $\mathrm{sym}(R)$, $\mathrm{tra}(R)$, $\wedge$-ant(R), $\otimes$-ant(R), and $\mathrm{comp}(R)$ are called the degree of reflexivity, $\approx$-reflexivity, symmetry, transitivity, $\wedge$-antisymmetry, and $\otimes$-antisymmetry of R, and the compatibility of R with $\approx$, respectively.

Remark 5. (a) The above degrees have a clear meaning. For instance, $\mathrm{ref}(R)$ and $\mathrm{sym}(R)$ are just the truth degrees of the first-order formulas[14]

$$(\forall x) r(x,x) \quad \text{and} \quad (\forall x)(\forall y)(r(x,y) \Rightarrow r(y,x)),$$

respectively, i.e. formulas verbally described as "for each x, x is related to x" and

[14] More precisely, truth degrees in a first order structure in which the relation symbol r is interpreted by the fuzzy relation R.

"for each x and y, if x is related to y then y is related to x"; similarly for the other degrees.

(b) Observe that the degree $\mathrm{ref}_{\approx}(R)$ is just the degree $\approx \subsetapprox R$ to which $\approx$ is included in R; cf. (26) in Appendix.

(c) Since $\bigwedge_{j\in J} a_j = 1$ iff $a_j = 1$ for each $j \in J$, we obtain that $\mathrm{ref}(R) = 1$ if and only if R is reflexive. The same holds for the other properties, hence the degrees of the properties naturally generalize the respective bivalent properties.

(d) Grades of properties (or graded properties) of fuzzy relations were studied by Gottwald (1993, 2001) and Belohlavek (2002), and were later systematically examined within the effort by Běhounek and Cintula (2006).

(e) Alternatively, one can consider a fuzzy relation a-reflexive for a given truth degree $a \in L$ if $a \leq R(u,u)$ for each $u \in U$. One may then check that

$$\mathrm{ref}(R) = \bigvee\{a \in L \mid R \text{ is } a\text{-reflexive}\};$$

the same holds true for the other properties.

(f) The notations $\wedge$-ant(R) and $\otimes$-ant(R) assume that $\approx$ is obvious from the context; alternatively, one could use "$\wedge$-$\approx$-ant(R)" and "$\otimes$-$\approx$-ant(R)." The same applies to comp$(\precsim)$.

3.1. *Fuzzy order on a set with fuzzy equality vs. fuzzy equivalence*

As briefly discussed in Remark 3 (b), Definition 1 differs from the original definition of fuzzy order by Bodenhofer (1999) and Höhle and Blanchard (1985) in that it assumes that $\approx$ is a fuzzy equivalence rather than fuzzy equality. We now briefly examine this distinction since it is conceptually significant and has not been properly addressed in the literature.

In our view, assuming a fuzzy equivalence instead of fuzzy equality in Definition 1 does not represent a direct generalization of the notion of order to a fuzzy setting. Rather, it represents a generalization that proceeds along two lines simultaneously. First, the two-valued Boolean algebra is replaced by the more general complete residuated lattice. Second, equality is replaced by equivalence.

This is also apparent when one examines what results from these two notions – i.e. fuzzy order on a set with a fuzzy equality per Definition 1 and fuzzy order on a set with a fuzzy equivalence per the original definition by Höhle, Blanchard, and Bodenhofer – if the definitions are considered within the classical setting. Consider thus both definitions with the structure $\mathbf{L}$ of truth degrees being the two-element Boolean algebra $\mathbf{2}$ of classical logic.

On the one hand, the notion resulting from Definition 1 coincides with the classical notion of order because a fuzzy equality becomes classical equality, and the defining conditions become classical reflexivity, transitivity, and antisymmetry.

On the other hand, the notion which results from the definition of a fuzzy order on a set with a fuzzy equivalence is not the notion of a classical order. Rather, it is a notion of classical relation $\leq$ on a set U, on which a classical equivalence $\equiv$ is defined, such that $\leq$ contains $\equiv$, is transitive, and satisfies a generalized form of antisymmetry in that $u \leq v$ and $v \leq u$ implies $u \equiv v$. As $\leq$ contains $\equiv$ and as $\equiv$ is reflexive, $\leq$ is reflexive as well. Moreover, since $\equiv$ is contained in $\leq$, we obtain that

$$u \equiv v \quad \text{if and only if} \quad u \leq v \text{ and } v \leq u.$$

In terms of standard notions of ordered sets (Birkhoff 1967; Blyth 2005; Davey and

Priestley 2002; Grätzer 2007), this means that $\leq$ is a quasiorder (preorder in an alternative terminology; i.e. is reflexive and transitive) and $\equiv$ is just the equivalence that is used to make the quasiorder to an order by a well-known factorization.

Let us point out that it is clear from Bodenhofer's papers that he was aware of this property of the definition of fuzzy order assuming fuzzy equivalence. He addresses this topic in Bodenhofer (2000, 2003). In particular, Bodenhofer (2003, p. 123) says:

> Although this is most often not mentioned explicitly, many orderings in classical mathematics are in fact only preorderings that may be understood as orderings by considering some factorization. ... In contrast to most classical cases, however, we do not use the projection of a given preordering to the factor set with respect to the underlying equivalence relation defined by the symmetric kernel, but include the equivalence relation in the axioms of the ordering explicitly. This might look like a significant deviation from the classical formulation, however, the two ways are logically equivalent.

Although we basically agree with Bodenhofer's remarks, we find it necessary to obey the maxim, according to which a generalization of a classical concept to the setting of fuzzy logic needs to behave as explained above. That is, the generalized concept needs to become the original classical concept when considered in the classical setting, i.e. when the considered structure of truth degrees is the two-element Boolean algebra. This is why we prefer Definition 1 assuming a fuzzy equality instead of fuzzy equivalence. In addition to the above reason, ramifications of Definition 1 and Definition 2 may more directly be compared when both definitions assume a fuzzy equality.

The notion of a (fuzzy) order on a set with a (fuzzy) equivalence also implies some inconvenient properties compared to the ordinary notion of order. An example is the fact that important distinguished elements, such as largest and smallest elements or suprema and infima are then not unique. Rather, they are unique just up to the equivalence. We illustrate this property by the following example in the classical setting.

Example 1. Let $U = \{u, v, w\}$, let a classical equivalence $\equiv$ be given by the equivalence classes $\{u\}$ and $\{v, w\}$. Then the relation $\leq$ given by $u \leq u$, $v \leq v$, $w \leq w$, $u \leq v$, $u \leq w$, $v \leq w$, and $w \leq v$ is an order on a set with an equivalence in the sense of Höhle, Blanchard, and Bodenhofer. Defining naturally a smallest element x as an element such that $x \leq y$ for every y, and dually for a largest element, it is immediate that u is the only smallest element. On the other hand, both v and w are largest, even though these are two distinct elements.

With respect to the last paragraph, let us note that the non-uniqueness of distinguished elements can be handled by developing the theory in an appropriate manner, but the resulting theory is not likely to be straightforward. This is apparent e.g. from studies of the notion of a lattice in quasiordered sets initiated by Chajda (1992).

Remark 6. It is easy to check the following claim: If $\precsim$ is a fuzzy order on the set U with a fuzzy equivalence $\approx$ in the sense of Höhle, Blanchard, and Bodenhofer, one may – generalizing in a straightforward manner the well-known classical construction of order from a quasiorder – consider the factor set $U' = U/E$ of U by the ordinary equivalence E defined by

$$\langle u, v \rangle \in E \quad \text{if and only if} \quad u \approx v = 1,$$

and define fuzzy relations $\lesssim'$ and $\approx'$ on U' by

$$[u]_E \lesssim' [v]_E \;=\; u \lesssim v \quad \text{and} \quad [u]_E \approx' [v]_E \;=\; u \approx v,$$

for any equivalence classes $[u]_E$ and $[v]_E$ in U'. Then $\lesssim'$ is a fuzzy order on the set U' equipped with a fuzzy equality $\approx'$ according to Definition 1.

3.2. *Reflexivity and compatibility*

An immediate difference between Definitions 1 and 2 consists in their condition of reflexivity. Both generalize classical reflexivity in that in the classical setting, i.e. **L** being the two-element Boolean algebra, both coincide with classical reflexivity. However, while $\approx$-reflexivity required by Definition 1 is stronger than reflexivity of Definition 2, the latter requires compatibility of the fuzzy order $\lesssim$ with $\approx$.[15]

Remark 7. (a) From the epistemic viewpoint mentioned in Remark 1, it seems natural, if not necessary, to assume compatibility of $\lesssim$ with $\approx$. Compatibility generalizes the axiom of equality of classical logic, which in the context of order relations, reads: if u_1 is less than or equal to v_1, u_1 equals u_2, and v_1 equals v_2, then u_2 is less than or equal to v_2. In the setting involving degrees, compatibility is compelling particularly when degrees of equality are interpreted as degrees of indistinguishability. Compatibility then says that the following formula is true (i.e. its truth degree equals 1): if u_1 is less than or equal to v_1, u_1 is indistinguishable from u_2, and v_1 is indistinguishable from v_2, then u_2 is less than or equal to v_2. Validity of such formula seems an unavoidable condition.

(b) Interestingly, compatibility has not been mentioned by Höhle and Blanchard (1985), nor in the first works by Bodenhofer; Bodenhofer actually considers compatibility considerably later (Bodenhofer and Demirci 2008). On the other hand, compatibility has been a common condition utilized in modern studies of fuzzy relational systems in the early 2000s; see e.g. the books by Belohlavek (2002), Gottwald (2001), and Hájek (1998).

In spite of the seemingly different conditions, i.e. $\approx$-reflexivity vs. reflexivity and compatibility, $\approx$-reflexivity turns out to be equivalent to reflexivity and compatibility given the context of both definitions.[16] The argument was observed for the first time by Belohlavek and Vychodil (2005, Lemma 1.82) in the context of fuzzy equivalences on sets with fuzzy equalities and later, independently, by Bodenhofer and Demirci (2008) in the context of fuzzy orders. Since, as we shall see below, this relationship is of considerable importance, we consider it thoroughly, namely by taking into account the degrees of the properties of fuzzy relations. We start by the following lemma, in which $\mathrm{tra}(\lesssim)^2$ stands for $\mathrm{tra}(\lesssim) \otimes \mathrm{tra}(\lesssim)$ and analogously for $\mathrm{ref}_{\approx}(\lesssim)^2$.

Lemma 1. *Let $\lesssim$ and $\approx$ be arbitrary fuzzy relations on a given set U. Then*

---

[15] For a fuzzy equality $\approx$ on $U = \{u, v\}$ defined by $u \approx u = v \approx v = 1$ and $u \approx v = v \approx u = 0.5$, the fuzzy relation $\lesssim$ defined by $u \lesssim u = v \lesssim v = 1$ and $u \lesssim v = v \lesssim u = 0$ is reflexive but not $\approx$-reflexive, demonstrating that $\approx$-reflexivity of $\lesssim$ is stronger than reflexivity.

[16] We observed in n. 15 that reflexivity of $\lesssim$ does not imply $\approx$-reflexivity. Observe, moreover, that compatibility of a (possibly transitive, and $\otimes$-antisymmetric or $\wedge$-antisymmetric) fuzzy relation $\lesssim$ with a fuzzy equality $\approx$ does not imply $\approx$-reflexivity of $\lesssim$ either (just take U and $\approx$ as in n. 15 and the empty fuzzy relation $\emptyset$ for $\lesssim$).

$$\mathrm{ref}(\approx) \otimes \mathrm{ref}_{\approx}(\lesssim) \leq \mathrm{ref}(\lesssim), \tag{10}$$

$$\mathrm{sym}(\approx) \otimes \mathrm{tra}(\lesssim)^2 \otimes \mathrm{ref}_{\approx}(\lesssim)^2 \leq \mathrm{comp}(\lesssim), \tag{11}$$

$$\mathrm{ref}(\approx) \otimes \mathrm{ref}(\lesssim) \otimes \mathrm{comp}(\lesssim) \leq \mathrm{ref}_{\approx}(\lesssim). \tag{12}$$

In the following as well as in the subsequent proofs, we shall use – with no further notice – common properties of infima and suprema, as well as properties of complete residuated lattices (Belohlavek 2002; Gottwald 2001; Novák, Perfilieva, and Močkoř 1999).

Proof. Inequality (10) holds true iff for each $u \in U$, $\mathrm{ref}(\approx) \otimes \mathrm{ref}_{\approx}(\lesssim) \leq u \lesssim u$, which is indeed the case, as

$$\mathrm{ref}(\approx) \otimes \mathrm{ref}_{\approx}(\lesssim) \leq (u \approx u) \otimes ((u \approx u) \to (u \lesssim u)) \leq u \lesssim u.$$

To check (11), we need to verify that for each $u_1, u_2, v_1, v_2 \in U$ one has

$$\mathrm{sym}(\approx) \otimes \mathrm{tra}(\lesssim)^2 \otimes \mathrm{ref}_{\approx}(\lesssim)^2 \leq ((u_1 \lesssim v_1) \otimes (u_1 \approx u_2) \otimes (v_1 \approx v_2)) \to u_2 \lesssim v_2,$$

which is equivalent to

$$\mathrm{sym}(\approx) \otimes \mathrm{tra}(\lesssim)^2 \otimes \mathrm{ref}_{\approx}(\lesssim)^2 \otimes (u_1 \lesssim v_1) \otimes (u_1 \approx u_2) \otimes (v_1 \approx v_2) \leq u_2 \lesssim v_2,$$

which holds true. Indeed, observe first (easy, by standard arguments) that

$$\begin{aligned}
(u \approx v) \otimes \mathrm{sym}(\approx) &\leq v \approx u, \\
(u \approx v) \otimes \mathrm{ref}_{\approx}(\lesssim) &\leq u \lesssim v, \text{ and} \\
(u \approx v) \otimes (v \approx w) \otimes \mathrm{tra}(\lesssim) &\leq u \approx w.
\end{aligned}$$

Now,

$$\begin{aligned}
&\mathrm{sym}(\approx) \otimes \mathrm{tra}(\lesssim)^2 \otimes \mathrm{ref}_{\approx}(\lesssim)^2 \otimes (u_1 \lesssim v_1) \otimes (u_1 \approx u_2) \otimes (v_1 \approx v_2) \\
&= (u_1 \approx u_2) \otimes \mathrm{sym}(\approx) \otimes \mathrm{ref}_{\approx}(\lesssim) \otimes (u_1 \lesssim v_1) \otimes (v_1 \approx v_2) \otimes \mathrm{ref}_{\approx}(\lesssim) \otimes \mathrm{tra}(\lesssim)^2 \\
&\leq (u_2 \approx u_1) \otimes \mathrm{ref}_{\approx}(\lesssim) \otimes (u_1 \lesssim v_1) \otimes (v_1 \lesssim v_2) \otimes \mathrm{tra}(\lesssim)^2 \\
&\leq (u_2 \lesssim u_1) \otimes (u_1 \lesssim v_2) \otimes \mathrm{tra}(\lesssim) \\
&\leq u_2 \lesssim v_2,
\end{aligned}$$

completing the proof of (11).

Verifying (12) amounts to checking

$$\mathrm{ref}(\approx) \otimes \mathrm{ref}(\lesssim) \otimes \mathrm{comp}(\lesssim) \leq (u \approx v) \to (u \lesssim v),$$

i.e.

$$\mathrm{ref}(\lesssim) \otimes \mathrm{ref}(\approx) \otimes (u \approx v) \otimes \mathrm{comp}(\lesssim) \leq u \lesssim v,$$

for each $u, v \in U$, which is easy as

$$\begin{aligned} & \mathrm{ref}(\lesssim) \otimes \mathrm{ref}(\approx) \otimes (u \approx v) \otimes \mathrm{comp}(\lesssim) \\ \leq\ & (u \lesssim u) \otimes (u \approx u) \otimes (u \approx v) \otimes \mathrm{comp}(\lesssim) \leq u \lesssim v, \end{aligned}$$

with the last equality holding due to the definition of $\mathrm{comp}(\lesssim)$. □

Remark 8. (a) The meaning of inequalities (10), (11), and (12) may easily be described verbally as follows; justification may either be intuitive or formal as explained in (b) below. In particular, (10) means that if $\approx$ is reflexive and $\lesssim$ is $\approx$-reflexive, then $\lesssim$ is reflexive, even when this implication is interpreted in a way in which degrees of being reflexive and $\approx$-reflexive are taken into account. In the same vein, (11) is interpreted as claiming that if $\approx$ is symmetric, $\lesssim$ is transitive and $\approx$-reflexive, then $\lesssim$ is compatible with $\approx$.[17] Finally, (12) means that if $\approx$ is reflexive and $\lesssim$ is reflexive and compatible with $\approx$, then $\lesssim$ is $\approx$-reflexive.

(b) To explain a formal justification of the meaning of the above inequalities, consider (10); for the other inequalities, one proceeds analogously. There is a first-order formula expressing inequality (10) syntactically, which is:

$$[(\forall x)(x = x) \& (\forall x)(x = x \Rightarrow x \leq x)] \Rightarrow (\forall x)(x \leq x). \tag{13}$$

Namely, consider a first-order structure with universe U such that the relation symbols $=$ and $\leq$ are interpreted by the fuzzy relations $\approx$ and $\lesssim$, and connectives & and $\Rightarrow$ are interpreted by $\otimes$ and $\to$. Due to basic semantic rules of first-order fuzzy logic (Belohlavek 2002; Gottwald 2001; Hájek 1998), the truth degrees of the subformulas $(\forall x)(x = x)$, $(\forall x)(x = x \Rightarrow x \leq x)$, and $(\forall x)(x \leq x)$ are then just equal to $\mathrm{ref}(\approx)$, $\mathrm{ref}_{\approx}(\lesssim)$, and $\mathrm{ref}(\lesssim)$, respectively. Hence, the truth degree of formula (13) is equal to

$$[\mathrm{ref}(\approx) \otimes \mathrm{ref}_{\approx}(\lesssim)] \to \mathrm{ref}(\lesssim). \tag{14}$$

From the properties of $\to$, it now follows that formula (13) is true, i.e. the truth degree (14) equals 1, if and only if inequality (10) is satisfied.

(c) One may easily observe that the exponents in the inequality (11) tell us how many times the degree of the respective property is used in the proof of the inequality. For instance, the exponent 2 in $\mathrm{tra}(\lesssim)^2$ indicates that the degree of transitivity is used twice in the proof. This demonstrates an interesting added value of analyzing relationships among the concerned properties of fuzzy relations by looking at the degrees to which the properties are satisfied.

(d) In (11), neither of the exponents of 2 in $\mathrm{tra}(\lesssim)^2$ and in $\mathrm{ref}_{\approx}(\lesssim)^2$ may be reduced to 1. Indeed, consider the residuated lattice $\mathbf{L}$ to be the real unit interval $L = [0, 1]$ equipped with the Łukasiewicz connectives (cf. Appendix). Let $U =$

[17] To take the exponents ^{2} properly into account, the precise meaning of (11) is: if $\approx$ is symmetric and $\lesssim$ is transitive and $\lesssim$ is transitive and $\lesssim$ is $\approx$-reflexive and $\lesssim$ is $\approx$-reflexive, then $\lesssim$ is compatible with $\approx$. Namely, conjunction is not idempotent in general in fuzzy logic, hence the two appearances of "$\lesssim$ is transitive" as well as "$\lesssim$ is $\approx$-reflexive" (two appearances because the exponents in $\mathrm{tra}(\lesssim)^2$ and $\mathrm{ref}_{\approx}(\lesssim)^2$ are equal to 2).

$\{u_1, u_2, u_3, u_4\}$ and let $\approx$ and $\lesssim$ be given as:

$\approx$	u_1	u_2	u_3	u_4
u_1	1	1	0	0
u_2	1	1	0	0
u_3	0	0	1	1
u_4	0	0	1	1

$\lesssim$	u_1	u_2	u_3	u_4
u_1	1	0.8	0.6	0.2
u_2	0.8	1	1	0.6
u_3	0	0	1	0.8
u_4	0	0	0.8	1

As one may verify, $\mathrm{sym}(\approx) = 1$, $\mathrm{tra}(\lesssim) = 0.8$, $\mathrm{ref}_{\approx}(\lesssim) = 0.8$, and $\mathrm{comp}(\lesssim) = 0.2$. Now we have

$$\mathrm{sym}(\approx) \otimes \mathrm{tra}(\lesssim) \otimes \mathrm{ref}_{\approx}(\lesssim)^2 = 1 \otimes 0.8 \otimes 0.8^2 = 0.4 \not\leq 0.2 = \mathrm{comp}(\lesssim)$$

and

$$\mathrm{sym}(\approx) \otimes \mathrm{tra}(\lesssim)^2 \otimes \mathrm{ref}_{\approx}(\lesssim) = 1 \otimes 0.8^2 \otimes 0.8 = 0.4 \not\leq 0.2 = \mathrm{comp}(\lesssim).$$

Let us now consider some corollaries of Lemma 1, which result by strengthening the assumptions.

Corollary 1. *Let $\approx$ be reflexive and symmetric (in particular, a fuzzy equality) and $\lesssim$ be transitive. Then:*

$$\mathrm{ref}_{\approx}(\lesssim) \leq \mathrm{ref}(\lesssim),$$
$$\mathrm{ref}_{\approx}(\lesssim)^2 \leq \mathrm{comp}(\lesssim),$$
$$\mathrm{ref}(\lesssim) \otimes \mathrm{comp}(\lesssim) \leq \mathrm{ref}_{\approx}(\lesssim).$$

Proof. Trivial given Lemma 1, because reflexivity and symmetry of $\approx$ is equivalent to $\mathrm{ref}(\approx) = 1$ and $\mathrm{sym}(\approx) = 1$, respectively, and transitivity of $\lesssim$ is equivalent to $\mathrm{tra}(\lesssim) = 1$. □

Considering as a particular case the full satisfaction of the properties involved in Corollary 1, we obtain the above-mentioned claim:

Corollary 2 ((Belohlavek and Vychodil 2005), (Bodenhofer and Demirci 2008))**.** *Let $\approx$ be a fuzzy equality and $\lesssim$ be transitive. Then $\lesssim$ is $\approx$-reflexive if and only if $\lesssim$ is reflexive and compatible with $\approx$.*

Proof. By a moment's reflection from Corollary 1 taking into account that $\lesssim$ is $\approx$-reflexive, reflexive, and compatible with $\approx$ iff $\mathrm{ref}_{\approx}(\lesssim) = 1$, $\mathrm{ref}(\lesssim) = 1$, and $\mathrm{comp}(\lesssim) = 1$, respectively. □

Remark 9. (a) Observe that Corollary 2 is in the form of a logical equivalence, which may be rephrased as follows: Given the assumptions (i.e. $\approx$ a fuzzy equality and $\lesssim$ transitive), we have $\mathrm{ref}_{\approx}(\lesssim) = 1$ if and only if $\mathrm{ref}(\lesssim) = 1$ and $\mathrm{comp}(\lesssim) = 1$.

(b) On the other hand, Corollary 1, i.e. a direct "graded generalization" of Corollary 2, is in the form of three inequalities of truth degrees expressing three implications regarding graded properties of fuzzy relations. Namely, corresponding to the three inequalities of Corollary 1 are three implications regarding graded properties of $\lesssim$ and $\approx$, respectively, which have their truth degree equal to 1. In particular, using the basic rules of semantics of first-order fuzzy logic, it may be shown that:

- $\mathrm{ref}_{\approx}(\lesssim) \leq \mathrm{ref}(\lesssim)$ holds true iff the truth degree of the formula "if $\lesssim$ is $\approx$-reflexive then $\lesssim$ is reflexive" equals 1, i.e. the formula is fully true;

- $\mathrm{ref}_{\approx}(\lesssim)^2 \leq \mathrm{comp}(\lesssim)$ holds true iff the truth degree of the formula "if $\lesssim$ is $\approx$-reflexive and $\lesssim$ is $\approx$-reflexive then $\lesssim$ is compatible" equals 1; and
- $\mathrm{ref}(\lesssim) \otimes \mathrm{comp}(\lesssim) \leq \mathrm{ref}_{\approx}(\lesssim)$ holds true iff the truth degree of the formula "if $\lesssim$ is reflexive and $\lesssim$ is compatible then $\lesssim$ is $\approx$-reflexive" equals 1.

(c) Observe that if $\otimes$ is idempotent, which is e.g. the case of the two-element Boolean algebra, then the three inequalities may readily be replaced by a single equality, namely

$$\mathrm{ref}_{\approx}(\lesssim) = \mathrm{ref}(\lesssim) \otimes \mathrm{comp}(\lesssim). \tag{15}$$

This equality expresses the fact that the formula "$\lesssim$ is $\approx$-reflexive if and only if $\lesssim$ is reflexive and $\lesssim$ is compatible" regarding graded properties of $\lesssim$ and $\approx$ has its truth degree equal to 1.

(d) In general, however, the three inequalities of Corollary 1 may not be expressed by the single equality (15); for instance the fuzzy relations in Remark 8 (d) do not satisfy (15).

Remark 10. One may formulate other corollaries of Lemma 1. As an example, the following corollary concerns crisp properties of $\lesssim$ and $\approx$, as does Corollary 2, but is more informative than Corollary 2:

Let $\lesssim$ and $\approx$ be arbitrary fuzzy relations. Then

(a) if $\approx$ is reflexive and $\lesssim$ is $\approx$-reflexive, then $\lesssim$ is reflexive; if $\approx$ is symmetric and $\lesssim$ is transitive and $\approx$-reflexive, then $\lesssim$ is compatible with $\approx$;
(b) if $\approx$ is reflexive and $\lesssim$ is reflexive and compatible with $\approx$ then $\lesssim$ is $\approx$-reflexive.

Remark 11. Clearly, Corollary 2 implies that both Definition 1 and Definition 2 of fuzzy order may be rephrased so that they differ in the condition of antisymmetry only. That is, the condition of $\approx$-reflexivity in Definition 1 may equivalently be replaced by reflexivity and compatibility, and, conversely, the latter two conditions in Definition 2 may be replaced by $\approx$-reflexivity (cf. Theorem 4 below).

3.3. *Constraints regarding fuzzy equality*

Both the $\otimes$-antisymmetry and $\wedge$-antisymmetry may be regarded as lower bounds for the fuzzy equality $\approx$. This view opens the question of exploring constraints pertaining to $\approx$. A basic answer was provided by Bodenhofer (2000, Theorem 18) who proved the following claim, which he phrased for fuzzy equivalences instead of equalities:

If $\lesssim$ is a reflexive and transitive fuzzy relation on U and $\approx$ is a fuzzy equality on U, then $\lesssim$ is a fuzzy order according to Definition 1 if and only if

$$(u \lesssim v) \otimes (v \lesssim u) \;\leq\; u \approx v \;\leq\; (u \lesssim v) \wedge (v \lesssim u) \tag{16}$$

for every $u, v \in U$. An easy inspection of the proof reveals that a corresponding theorem for the notion of fuzzy order with $\wedge$-antisymmetry is obtained when replacing (16) by the equality

$$u \approx v \;=\; (u \lesssim v) \wedge (v \lesssim u),$$

the validity of which for fuzzy orders according to Definition 2 was observed by Belohlavek (2002).

The results we just mentioned, though, involve a redundancy in that the assumption of reflexivity of $\lesssim$ may be dropped. The redundancy may be regarded as resulting from a lack of awareness of the relationship of reflexivity and compatibility vs. $\approx$-reflexivity (cf. Section 3.2). In fact, some easy observations on the notions involved render a non-redundant generalization of the above-mentioned results. Below we provide such observations in a more general manner which, as in Section 3.2, take into account the degrees of the properties of fuzzy relations. Then we obtain a proper formulation of the above-mentioned results as simple consequences. We start with the following observation.

Lemma 2. *Let $\approx$ and $\lesssim$ be arbitrary binary fuzzy relations on U.*

(a) One has

$$\mathrm{sym}(\approx) \ \leq\ \mathrm{ref}_{\approx}(\lesssim) \leftrightarrow \bigwedge_{u,v\in U} [u \approx v \rightarrow ((u \lesssim v) \wedge (v \lesssim u))]. \tag{17}$$

(b) If $\approx$ is symmetric (in particular, a fuzzy equality), then

$$\mathrm{ref}_{\approx}(\lesssim) = \bigwedge_{u,v\in U} [u \approx v \rightarrow ((u \lesssim v) \wedge (v \lesssim u))]. \tag{18}$$

(c) If $\approx$ is symmetric (in particular, a fuzzy equality), then

$$\lesssim \text{ is } \approx\text{-reflexive} \quad \text{iff} \quad u \approx v \leq (u \lesssim v) \wedge (v \lesssim u). \tag{19}$$

Proof. (a) Since for any $a, b, c \in L$, $a \leq b \leftrightarrow c$ is equivalent to $a \leq b \rightarrow c$ and $a \leq c \rightarrow b$, i.e. – due to adjointness – to $a \otimes b \leq c$ and $a \otimes c \leq b$, we need to verify

$$\mathrm{sym}(\approx) \otimes \mathrm{ref}_{\approx}(\lesssim) \leq \bigwedge_{u,v\in U} [u \approx v \rightarrow ((u \lesssim v) \wedge (v \lesssim u))] \tag{20}$$

and

$$\mathrm{sym}(\approx) \otimes \bigwedge_{u,v\in U} [u \approx v \rightarrow ((u \lesssim v) \wedge (v \lesssim u))] \leq \mathrm{ref}_{\approx}(\lesssim). \tag{21}$$

Check (20) first. Since

$$[u \approx v \rightarrow ((u \lesssim v) \wedge (v \lesssim u))] = (u \approx v \rightarrow u \lesssim v) \wedge (u \approx v \rightarrow v \lesssim u),$$

(20) holds true iff for each $u, v \in U$,

$$\mathrm{sym}(\approx) \otimes \mathrm{ref}_{\approx}(\lesssim) \leq (u \approx v \rightarrow u \lesssim v) \tag{22}$$

and

$$\mathrm{sym}(\approx) \otimes \mathrm{ref}_{\approx}(\lesssim) \leq (u \approx v \rightarrow v \lesssim u). \tag{23}$$

While (22) is trivial due to the definition of $\mathrm{ref}_{\approx}(\lesssim)$, (23) is equivalent to

$$(u \approx v) \otimes \mathrm{sym}(\approx) \otimes \mathrm{ref}_{\approx}(\lesssim) \leq v \lesssim u,$$

which is true because

$$
\begin{aligned}
&(u \approx v) \otimes \mathrm{sym}(\approx) \otimes \mathrm{ref}_{\approx}(\lesssim) \\
&\leq (u \approx v) \otimes (u \approx v \to v \approx u) \otimes (v \approx u \to v \lesssim u) \leq v \lesssim u.
\end{aligned}
$$

Checking (21) is straightforward: As $a \to (b \wedge c) \leq a \to b$, we have

$$
\mathrm{sym}(\approx) \otimes \bigwedge_{u,v \in U} [u \approx v \to ((u \lesssim v) \wedge (v \lesssim u))] \leq \bigwedge_{u,v \in U} [u \approx v \to u \lesssim v] = \mathrm{ref}_{\approx}(\lesssim).
$$

(b) follows from (a) because if $\approx$ is symmetric, we have $\mathrm{sym}(\approx) = 1$, hence

$$
\mathrm{ref}_{\approx}(\lesssim) \leftrightarrow \bigwedge_{u,v \in U} [u \approx v \to ((u \lesssim v) \wedge (v \lesssim u))] = 1.
$$

Now, since $a \leftrightarrow b = 1$ iff $a = b$, equality (18) readily follows.

(c) follows from (b) because $\approx$-reflexivity of $\lesssim$ means $\mathrm{ref}_{\approx}(\lesssim) = 1$ and because $a \to b = 1$ is equivalent to $a \leq b$. □

As an immediate consequence of Lemma 2 (c) and the definition of $\otimes$- and $\wedge$-antisymmetry we have:

Corollary 3. *Let $\lesssim$ be a fuzzy relation and $\approx$ be a fuzzy equality on U.*

(a) $\lesssim$ is $\approx$-reflexive and $\otimes$-antisymmetric iff

$$
(u \lesssim v) \otimes (v \lesssim u) \ \leq \ u \approx v \ \leq \ (u \lesssim v) \wedge (v \lesssim u).
$$

(b) $\lesssim$ is $\approx$-reflexive and $\wedge$-antisymmetric iff

$$
u \approx v \ = \ (u \lesssim v) \wedge (v \lesssim u).
$$

The preceding corollary along with the equivalence of $\approx$-reflexivity to reflexivity and compatibility for transitive fuzzy relations (cf. Corollary 2) immediately yield the announced non-redundant and hence more informative rephrasement of the result by Bodenhofer (2000, Theorem 18) and its counterpart for fuzzy orders according to Definition 2, which we mentioned above.

Theorem 4. *Let $\lesssim$ be a transitive fuzzy relation and $\approx$ be a fuzzy equality on U.*

(a) The following conditions are equivalent:
  (a1) $\lesssim$ is a fuzzy order according to Definition 1.
  (a2) $\lesssim$ is reflexive, $\otimes$-antisymmetric, and compatible with $\approx$.
  (a3) $(u \lesssim v) \otimes (v \lesssim u) \ \leq \ u \approx v \ \leq \ (u \lesssim v) \wedge (v \lesssim u)$.

(b) The following conditions are equivalent:
  (b1) $\lesssim$ is a fuzzy order according to Definition 2.
  (b2) $\lesssim$ is $\approx$-reflexive and $\wedge$-antisymmetric.
  (b3) $u \approx v \ = \ (u \lesssim v) \wedge (v \lesssim u)$.

Remark 12. (a) Theorem 4 basically presents equivalent conditions for a transitive fuzzy relation $\lesssim$ to become a fuzzy order. In particular, it shows that one such condition is expressed by a simple constraint regarding $\lesssim$ and the fuzzy equality $\approx$. Note also that Theorem 4 lets us regard the claims as trivial consequences of the definitions of and previous observations on the individual properties of fuzzy orders.

(b) Compared to Theorem 18 by Bodenhofer (2000), part (a) of Theorem 4 is stronger. Namely, as mentioned in the beginning of this section, Bodenhofer claims in his Theorem 18 that being a fuzzy order (according to Definition 1) is equivalent to the inequality (a3) of Theorem 4 for any *reflexive and transitive* $\lesssim$. Theorem 4, on the other hand, makes it explicit that the assumption of reflexivity for $\lesssim$ may be dropped. This is worth mentioning because reflexivity is actually implied by one of the properties of fuzzy orders, namely $\approx$-reflexivity. In this respect, Theorem 4 is nonredundant and properly separates the role of the individual properties.

(c) On the other hand, let us note that as is clear from the proof of his Theorem 18, Bodenhofer (2000) was aware of the fact that the inequality $u \approx v \leq (u \lesssim v) \wedge (v \lesssim u)$ is equivalent to $\approx$-reflexivity for any transitive $\lesssim$, rather than any reflexive and transitive $\lesssim$.

We conclude this section by presenting a possible generalization of the previous theorem which takes the degrees of validity into account. We start with the following graded generalization of Corollary 3.

Lemma 3. *Let $\approx$ and $\lesssim$ be fuzzy relations, and let $\approx$ be symmetric (in particular, a fuzzy equality).*

(a)

$$\mathrm{ref}_{\approx}(\lesssim) \;\wedge\; \otimes\text{-ant}(\lesssim) = \bigwedge_{u,v\in U} \big([((u \lesssim v) \otimes (v \lesssim u)) \to u \approx v] \\ \wedge [u \approx v \to ((u \lesssim v) \wedge (v \lesssim u))]\big).$$

(b)

$$\mathrm{ref}_{\approx}(\lesssim) \;\wedge\; \wedge\text{-ant}(\lesssim) = \bigwedge_{u,v\in U} [u \approx v \leftrightarrow ((u \lesssim v) \wedge (v \lesssim u))].$$

Proof. (a) Due to Lemma 2 (b) and the definition of $\otimes$-antisymmetry,

$$\mathrm{ref}_{\approx}(\lesssim) \wedge \otimes\text{-ant}(\lesssim) = \otimes\text{-ant}(\lesssim) \wedge \mathrm{ref}_{\approx}(\lesssim) = \\ = \bigwedge_{u,v\in U} ((u \lesssim v) \otimes (v \lesssim u)) \to u \approx v] \wedge \bigwedge_{u,v\in U} [u \approx v \to ((u \lesssim v) \wedge (v \lesssim u))].$$

The required equality now follows because

$$\textstyle\bigwedge_{i\in I}(a_i \to b_i) \wedge \bigwedge_{i\in I}(b_i \to c_i) = \bigwedge_{i\in I}[(a_i \to b_i) \wedge (b_i \to c_i)]$$

holds in any complete residuated lattice.

(b) Lemma 2 (b) again and the definition of $\wedge$-antisymmetry yield

$$\mathrm{ref}_{\approx}(\lesssim) \;\wedge\; \wedge\text{-ant}(\lesssim) \\ = \bigwedge_{u,v\in U} [u \approx v \to ((u \lesssim v) \wedge (v \lesssim u))] \wedge \bigwedge_{u,v\in U} [((u \lesssim v) \wedge (v \lesssim u)) \to u \approx v],$$

from which the required equality follows because

$$\textstyle\bigwedge_{i\in I}(a_i \to b_i) \wedge \bigwedge_{i\in I}(b_i \to a_i) = \bigwedge_{i\in I}(a_i \leftrightarrow b_i).$$

□

For brevity, we now only consider a graded generalization of part (b) in Theorem 4. Note that in this respect, Lemma 3 may be interpreted as claiming that the degree to which $u \approx v$ equals $(u \lesssim v) \wedge (v \lesssim u)$ coincides with the infimum of the degree of $\approx$-reflexivity of $\lesssim$ and the degree of $\wedge$-antisymmetry of $\lesssim$. We now obtain the following possible generalization of Theorem 4 (b):

Lemma 4 (generalization of Theorem 4 (b))**.** *Let $\lesssim$ be a transitive fuzzy relation and $\approx$ be symmetric (in particular, a fuzzy equality).*

(b_{12})

$$(\mathrm{ref}(\lesssim) \otimes \mathrm{comp}(\lesssim)) \ \wedge \ \wedge\text{-ant}(\lesssim) \leq \mathrm{ref}_{\approx}(\lesssim) \ \wedge \ \wedge\text{-ant}(\lesssim)$$
$$\mathrm{ref}_{\approx}(\lesssim)^2 \ \wedge \ \wedge\text{-ant}(\lesssim) \leq (\mathrm{ref}(\lesssim) \otimes \mathrm{comp}(\lesssim)) \ \wedge \ \wedge\text{-ant}(\lesssim)$$

(b_{23})

$$\mathrm{ref}_{\approx}(\lesssim) \ \wedge \ \wedge\text{-ant}(\lesssim) = \bigwedge_{u,v \in U} [u \approx v \leftrightarrow ((u \lesssim v) \wedge (v \lesssim u))].$$

(b_{13})

$$(\mathrm{ref}(\lesssim) \otimes \mathrm{comp}(\lesssim)) \ \wedge \ \wedge\text{-ant}(\lesssim) \leq \bigwedge_{u,v \in U} [u \approx v \leftrightarrow ((u \lesssim v) \wedge (v \lesssim u))]$$
$$\Big(\bigwedge_{u,v \in U} [u \approx v \leftrightarrow ((u \lesssim v) \wedge (v \lesssim u))]\Big)^2 \leq (\mathrm{ref}(\lesssim) \otimes \mathrm{comp}(\lesssim)) \ \wedge \ \wedge\text{-ant}(\lesssim)$$

Proof. The claims of direct consequences of Lemma 1, Lemma 3 (b), and the properties of complete residuated lattices. □

Remark 13. (a) Notice that Lemma 4 implies (b) of Theorem 4. In particular, part (b_{12}) of Lemma 4 implies the equivalence of (b1) with (b2) in Theorem 4. In a similar manner, part (b_{23}) and (b_{13}) of Lemma 4 imply the equivalence of (b2) with (b3) and of (b1) with (b3) in Theorem 4, respectively.

Indeed, if $\lesssim$ satisfies (b1), i.e. is reflexive, transitive, $\wedge$-antisymmetric, and compatible with $\approx$ then $\mathrm{ref}(\lesssim) = 1$, $\mathrm{comp}(\lesssim) = 1$, and $\wedge\text{-ant}(\lesssim) = 1$, from which it follows by the first inequality in (b_{12}) that $\mathrm{ref}_{\approx}(\lesssim) = 1$ and $\wedge\text{-ant}(\lesssim) = 1$, i.e. $\lesssim$ is $\approx$-reflexive and $\wedge$-antisymmetric, establishing (b2). Similarly, the second inequality in (b_{12}) implies that (b2) implies (b1). For (b_{23}) and (b_{13}), one proceeds analogously.

(b) Clearly, other graded generalizations of Theorem 4 may be obtained, e.g. generalizations employing a general degree $\mathrm{tra}(\lesssim)$ of transitivity of $\lesssim$ instead of assuming transitivity of $\lesssim$ as in Lemma 4. Like Lemma 4, such generalizations would have a form of a set of inequalities, since it is unclear how the two pairs of inequalities in (b_{12}) and (b_{13}) might be expressed by two equalities, one implying the equivalence of (b1) with (b2) and the second the equivalence of (b1) with (b3).

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Appendix: Residuated lattices, fuzzy sets, and fuzzy relations

Structures of truth degrees

Unlike classical logic, which uses a two-element set $L = \{0, 1\}$ of truth values and classical truth functions of logical connectives, i.e. uses a fixed structure of truth values, neither the set of truth degrees nor the truth functions of logical connectives are fixed in fuzzy logic. A modern approach in fuzzy logic assumes a general set L of truth degrees and general truth functions of logical connectives satisfying some natural basic conditions, i.e. assumes a general structure $\mathbf{L}$ of truth degrees. This assumption thus delineates a class of structures, which includes various particular structures such as the real unit interval $L = [0, 1]$ equipped with the Łukasiewicz connectives. A given theory or method, such as a theory of fuzzy equivalence relations, is then developed for the general assumptions, i.e. for a general $\mathbf{L}$, and is hence valid also for any of the particular structures.

Since the seminal work by Goguen (1967, 1969), it proved useful to assume that the structure $\mathbf{L}$ of truth degrees forms a complete residuated lattice (Belohlavek 2002; Belohlavek, Dauben, and Klir 2017; Gottwald 2001; Hájek 1998; Novák, Perfilieva, and Močkoř 1999), i.e. an algebra

$$\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$$

such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy the so-called adjointness property:

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \rightarrow c \tag{24}$$

for each $a, b, c \in L$. The elements a of L are called truth degrees and $\otimes$ and $\rightarrow$ are considered as the truth functions of (many-valued) conjunction and implication, respectively. The biresiduum in $\mathbf{L}$ is the binary operation defined by

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a), \tag{25}$$

and is interpreted as the truth function of (many-valued) equivalence.

Examples of complete residuated lattices, particularly those with L being $[0, 1]$ or a finite subchain of $[0, 1]$ which are based on t-norms and their residua, are well known. A common choice of $\mathbf{L}$ is a structure with $L = [0, 1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a continuous (or at least left-continuous) t-norm (i.e. a commutative, associative, and isotone operation on $[0, 1]$ with 1 acting as a neutral element) with the corresponding $\rightarrow$ given by

$$a \rightarrow b = \max\{c \mid a \otimes c \leq b\}.$$

The three most important pairs of adjoint operations on the unit interval are: Łukasiewicz ($a \otimes b = \max(a+b-1,0)$, $a \to b = \min(1-a+b,1)$); Gödel ($a \otimes b = \min(a,b)$, $a \to b = 1$ if $a \leq b$ and $a \to b = b$ if $a > b$); and Goguen ($a \otimes b = a \cdot b$, $a \to b = 1$ if $a \leq b$ and $a \to b = b/a$ if $a > b$).

Another common choice is a finite linearly ordered $\mathbf{L}$. For instance, one can put $L = \{a_0 = 0, a_1, \ldots, a_n = 1\} \subseteq [0,1]$ ($a_0 < \cdots < a_n$) with $\otimes$ given by $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and the corresponding $\to$ given by $a_k \to a_l = a_{\min(n-k+l,n)}$. Such an $\mathbf{L}$ is called a finite Łukasiewicz chain. Another possibility is a finite Gödel chain which consists of $L = \{a_0 = 0, a_1, \ldots, a_n = 1\} \subseteq [0,1]$ and the restrictions of the Gödel operations from $[0,1]$ to L.

Importantly, a special case of a complete residuated lattice is the two-element Boolean algebra $\langle\{0,1\}, \wedge, \vee, \otimes, \to, 0, 1\rangle$, denoted by $\mathbf{2}$, which is the structure of truth degrees of classical logic. This is important because for the particular case $\mathbf{L} = \mathbf{2}$, the developed notions and results essentially become the ordinary notions. In particular, the notions regarding fuzzy sets and fuzzy relations (cf. the next section) may be identified with the corresponding notions regarding classical sets and classical relations.

Fuzzy sets and relations

Given a complete residuated lattice $\mathbf{L}$, we define the usual notions: An $\mathbf{L}$-set (fuzzy set, or L-set if one need not emphasize the operations on L) A in a universe U is a mapping $A\colon U \to L$, $A(u)$ being interpreted as "the degree to which u belongs to A." Let L^U denote the collection of all $\mathbf{L}$-sets in U. Binary $\mathbf{L}$-relations (binary fuzzy relations) between U and V are naturally just $\mathbf{L}$-sets in the universe $U \times V$, i.e. mappings $R : U \times V \to L$.

The basic operations with $\mathbf{L}$-sets are based on the residuated lattice operations and are defined componentwise. For instance, the intersection of $\mathbf{L}$-sets $A, B \in L^U$ is an $\mathbf{L}$-set $A \cap B$ in U such that $(A \cap B)(u) = A(u) \wedge B(u)$ for each $u \in U$; to emphasize that $\cap$ arises from $\wedge$, one also writes $A \wedge B$ instead of $A \cap B$.

A fuzzy set $A \in L^U$ is called crisp if $A(u) = 0$ or $A(u) = 1$ for each $u \in U$. Each crisp fuzzy set $A \in L^U$ may be obviously identified with the ordinary subset $\{u \in U \mid A(u) = 1\}$ of U; a crisp fuzzy set is in fact the characteristic function of the corresponding ordinary subset of U. Note also that all $\mathbf{2}$-sets are crisp and hence $\mathbf{2}$-sets and operations with $\mathbf{2}$-sets can be identified with ordinary sets and operations with ordinary sets, respectively. It is a common practice not to distinguish crisp fuzzy sets in U from the corresponding ordinary subsets of U if there is no danger of confusion.

For $a \in L$ and $u \in U$, we denote by $\{a/u\}$ the $\mathbf{L}$-set A in U, called a singleton, for which $A(x) = a$ if $x = u$ and $A(x) = 0$ if $x \neq u$. A crisp singleton $\{1/u\}$ may be identified with a one-element ordinary subset $\{u\}$ of U.

Given $A, B \in L^U$, we define the degree $A \mathrel{\underset{\sim}{\subset}} B$ of inclusion of A in B by

$$A \mathrel{\underset{\sim}{\subset}} B = \textstyle\bigwedge_{u \in U} \big(A(u) \to B(u)\big), \tag{26}$$

which is also denoted $S(A,B)$ in the literature, and the degree of equality of A and B by

$$A \approx B = \textstyle\bigwedge_{u \in U} \big(A(u) \leftrightarrow B(u)\big). \tag{27}$$

Note that (26) generalizes the ordinary subsethood relation $\subseteq$. Described verbally,

$A \underset{\sim}{\subseteq} B$ represents the degree to which every element of A is an element of B. In particular, we write $A \subseteq B$ iff $A \underset{\sim}{\subseteq} B = 1$. As a consequence, $A \subseteq B$ iff $A(u) \leq B(u)$ for each $u \in U$. Likewise, (27) generalizes the ordinary equality $=$ of sets, and $A \approx B$ represents the degree to which every element belongs to A iff it belongs to B. Clearly, $A = B$ iff $A \approx B = 1$.

An **L**-equivalence (fuzzy equivalence) on U is a binary fuzzy relation $\approx$ on U, i.e. $\approx: U \times U \to L$, satisfying for each $u, v, w \in U$ the conditions

$$u \approx u \;=\; 1, \tag{28}$$

$$u \approx v \;=\; v \approx u, \tag{29}$$

$$(u \approx v) \otimes (v \approx w) \;\leq\; u \approx w, \tag{30}$$

called reflexivity, symmetry, and transitivity, respectively. An **L**-equality is an **L**-equivalence satisfying the condition of separation, i.e.

$$u \approx v = 1 \quad \text{implies} \quad u = v, \tag{31}$$

for each $u, v \in U$.

A binary fuzzy relation $R : U \times U \to L$ is called compatible with a fuzzy equivalence $\approx$ on U if

$$R(u_1, v_1) \otimes (u_1 \approx u_2) \otimes (v_1 \approx v_2) \leq R(u_2, v_2). \tag{32}$$

Put verbally, compatibility reads that if u_1 and v_1 are related by R, u_1 is equivalent to u_2, and v_1 is equivalent to v_2, then u_2 and v_2 are related by R as well.

In some contexts, it is convenient to speak of $\otimes$-transitivity, rather than transitivity, of a fuzzy relation to emphasize that the connective "and" is interpreted by the truth function $\otimes$, and to distinguish this condition from, e.g. $\wedge$-transitivity, which would read $(u \approx v) \wedge (v \approx w) \leq u \approx w$. This manner of emphasizing the truth functions is common in the literature and we adopt it when needed. For further details on fuzzy sets we refer to the books by Belohlavek (2002); Belohlavek, Dauben, and Klir (2017); Gottwald (2001); Hájek (1998); Novák, Perfilieva, and Močkoř (1999).

Appendix B

On the concept of fuzzy order II: Antisymmetry

The second part of the two-part-study on the concept of fuzzy order defined with respect to fuzzy equality. The study was conducted together with my supervisor, Radim Belohlavek, and published in International Journal of General Systems (Belohlavek and Urbanec, 2023a,b). Chapter 3 contains summary of the main results obtained in this study.

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On the concept of fuzzy order II: Antisymmetry

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In the second part of our paper, we explore antisymmetry of fuzzy orders. We provide a unifying definition of antisymmetry, which generalizes three existing variants of antisymmetry examined in the literature, along with the corresponding generalized definition of fuzzy order. We prove that all the particular instances of the generalized definition, which include the three basic ones, are mutually equivalent. We also examine distinctive properties of the three basic notions of fuzzy order.

Keywords: order; fuzzy logic; fuzzy equality; antisymmetry

1. Preliminaries

We assume that the reader is familiar with the first part of our paper (Belohlavek and Urbanec 2023), to which we refer simply by "part I." Part I contains preliminaries in fuzzy logic in its Appendix and the notions and results we use in the present paper. We only recall the two definitions of fuzzy order analyzed in part I:

Definition 1 (Höhle, Blanchard, Bodenhofer)**.** A fuzzy order on a set U equipped with a fuzzy equality relation $\approx$ is a binary fuzzy relation $\lesssim$ on U satisfying

$$\begin{aligned} u \approx v &\leq u \lesssim v && (\approx\text{-reflexivity}),\\ (u \lesssim v) \otimes (v \lesssim w) &\leq u \lesssim w && (\text{transitivity}),\\ (u \lesssim v) \otimes (v \lesssim u) &\leq u \approx v && (\otimes\text{-antisymmetry}), \end{aligned}$$

for each $u, v, w \in U$. (Note: Höhle and Blanchard's as well as Bodenhofer's original definitions actually assume, more generally, that $\approx$ is a fuzzy equivalence rather than fuzzy equality; this is discussed in part I.)

Definition 2 (Höhle, Belohlavek)**.** A fuzzy order on a set U equipped with a fuzzy equality relation $\approx$ is a binary fuzzy relation $\lesssim$ on U compatible with $\approx$, i.e. fulfilling

$$(u_1 \lesssim v_1) \otimes (u_1 \approx u_2) \otimes (v_1 \approx v_2) \leq u_2 \lesssim v_2,$$

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for every $u_1, u_2, v_1, v_2 \in U$, which satisfies

$$\begin{aligned} u \lesssim u &= 1 && \text{(reflexivity)}, \\ (u \lesssim v) \otimes (v \lesssim w) &\leq u \lesssim w && \text{(transitivity)}, \\ (u \lesssim v) \wedge (v \lesssim u) &\leq u \approx v && \text{($\wedge$-antisymmetry)}, \end{aligned}$$

for each $u, v, w \in U$.

2. Antisymmetry reconsidered

In view of part I (cf. Remark 11), antisymmetry represents the only essential difference between the two notions of fuzzy order expressed by Definitions 1 and 2. In this section, we explore antisymmetry in detail.

We first consider what we call crisp antisymmetry, a version of antisymmetry used in the literature in definitions of fuzzy order which do not employ fuzzy equality. Given the three variants of antisymmetry, namely the $\otimes$-antisymmetry, $\wedge$-antisymmetry, and crisp antisymmetry, we then provide a generalization of these variants. It turns out that in addition to the three variants, the generalized notion of antisymmetry renders a variety of other particular forms of antisymmetry. Importantly, we prove that all these forms are, in a sense, equivalent, and hence it is basically a matter of one's preference which concept of antisymmetry to use in the definition of fuzzy order. We then provide considerations of distinguishing properties of the various versions of antisymmetry, and thus various notions of fuzzy order. We conclude by a discussion regarding future research in fuzzy order.

2.1. *Crisp antisymmetry and avoiding fuzzy equality*

We now examine in detail a possible approach to fuzzy orders that avoids explicit reliance on the notion of fuzzy equality. This approach turns out to be almost equivalent to the approach utilizing the notion of fuzzy equality as codified by Definitions 1 and 2. Its possible shortcoming, in our view, consists in that it is not as clean compared to the approach utilizing the notion of fuzzy equality from a logical and an epistemic viewpoint, both of which have been explained in part I. Its advantage, however, is that the corresponding definition is simpler compared to Definitions 1 and 2.

The approach seems to have appeared for the first time in a study by Blanchard (1983), who examined Szpilrajn's embedding theorem in a fuzzy setting and introduced for this purpose several notions of fuzzy order. In particular, the notion Blanchard calls 4-fuzzy ordering is that of a fuzzy relation $\lesssim$ on a universe U satisfying reflexivity, i.e. $u \lesssim u = 1$, transitivity w.r.t. $\wedge$, i.e. $(u \lesssim v) \wedge (v \lesssim w) \leq u \lesssim w$, and the following form of antisymmetry, we shall call *crisp antisymmetry*:

$$(u \lesssim v) = 1 \text{ and } (v \lesssim u) = 1 \text{ imply } u = v, \tag{1}$$

for any $u, v \in U$.[1] In (1), $u = v$ means that u equals v, hence crisp antisymmetry provides a straightforward generalization of ordinary antisymmetry. Note that

[1]In fact, Blanchard in general defines the notion of a fuzzy order on a fuzzy set defined on the universe U. The notion we describe corresponds to the case of a fuzzy order on a set, i.e. when the fuzzy set on U is identified with U.

Blanchard only used the real unit interval $[0,1]$ as the set of truth degrees and the minimum $\wedge$ on $[0,1]$, i.e. infimum, as a truth function of conjunction, hence the employment of $\wedge$ in Blanchard's definition of transitivity. Blanchard seems not to have continued this approach to fuzzy order in her further work. Instead, she later employed the notion of fuzzy order proposed in her paper (Höhle and Blanchard 1985), which we discussed in part I.

Independently, the same notion of fuzzy order, i.e. not referring to fuzzy equality and using crisp antisymmetry has been proposed by Fan (2001), who used it in his further studies (Zhang and Fan 2005; Zhang, Xie, and Fan 2009; Xie, Zhang, and Fan 2009).[2] Fan uses the so-called frames as the structures of truth degrees, and hence uses the infimum $\wedge$ as the truth function of conjunction, as Blanchard does, rather than a more general $\otimes$ employed in our framework of residuated lattices.[3]

The following is the obvious generalization of the definition by Blanchard and Fan to the framework of general complete residuated lattices; it appeared in the works of Yao (Yao and Lu 2009; Yao 2010):

Definition 3 (Blanchard, Fan)**.** A fuzzy order on a set U is a binary fuzzy relation $\precsim$ on U satisfying

$$\begin{aligned} u \precsim u &= 1 && \text{(reflexivity)},\\ (u \precsim v) \otimes (v \precsim w) &\leq u \precsim w && \text{(transitivity)},\\ (u \precsim v) = 1 \text{ and } (v \precsim u) = 1 &\text{ implies } u = v && \text{(crisp antisymmetry)}, \end{aligned}$$

for each $u, v, w \in U$. The pair $\langle U, \precsim \rangle$ shall be called a fuzzy ordered set (according to Definition 3).

Let us now consider the relationship of Definition 3 to Definitions 1 and 2. The possibility to avoid fuzzy equality in Definitions 1 and 2 has been observed in the respective early papers by Belohlavek and Bodenhofer. Thus, Belohlavek (2001, 2002, 2004) observed and utilized the observation that a fuzzy order according to Definition 2 satisfies

$$u \approx v = (u \precsim v) \wedge (v \precsim u), \tag{2}$$

i.e. $\approx$ is uniquely determined by $\precsim$. Bodenhofer made various observations on the relationship between $\precsim$ and $\approx$ as regards Definition 1 too (see Section 3.3 in part I) and made comments regarding a possible omission of fuzzy equality (Bodenhofer 2003, end of Section 5).

Later on, Xie, Zhang, and Fan (2009) for $\otimes = \wedge$ and Yao (2010) for general complete residuated lattices made the following observation on the relationship between Definition 2 and Definition 3:

Lemma 1. *(a) If $\langle U, \approx, \precsim \rangle$ is a fuzzy ordered set according to Definition 2, then $\langle U, \precsim \rangle$ is a fuzzy ordered set according to Definition 3.*

(b) If $\langle U, \precsim \rangle$ is a fuzzy ordered set according to Definition 3, then $\approx$ defined by (2) is a fuzzy equality and $\langle U, \approx, \precsim \rangle$ is a fuzzy ordered set according to Definition 2.

[2] With respect to this notion of fuzzy order, Zhang and Fan (2005) cite an earlier paper by L. Fan, Q.-Y. Zhang, W.-Y. Xiang, and C.-Y. Zheng, "An L-fuzzy approach to quantitative domain (I) (generalized ordered set valued in frame and adjunction theory)," *Fuzzy Systems Math.*14 (2000), 6–7, written in Chinese, which we were not able to obtain.

[3] A frame, or a complete Heyting algebra, is a complete lattice satisfying $a \wedge (\bigvee_j b_j) = \bigvee_j (a \wedge b_j)$. That is, a frame may be regarded as a complete residuated lattice in which $\otimes$ coincides with the infimum $\wedge$.

We now provide an observation analogous to Lemma 1 regarding the relationship between Definition 1 and Definition 3:

Lemma 2. *(a) If $\langle U, \approx, \lesssim \rangle$ is a fuzzy ordered set according to Definition 1, then $\langle U, \lesssim \rangle$ is a fuzzy ordered set according to Definition 3.*

(b) If $\langle U, \lesssim \rangle$ is a fuzzy ordered set according to Definition 3, then $\approx$ defined by

$$u \approx v = (u \lesssim v) \otimes (v \lesssim u) \tag{3}$$

is a fuzzy equality and $\langle U, \approx, \lesssim \rangle$ is a fuzzy ordered set according to Definition 1.

Proof. (a): Since reflexivity of $\lesssim$ follows from $\approx$-reflexivity of $\lesssim$ and reflexivity of $\approx$, it remains to verify crisp antisymmetry. If $u \lesssim v = 1$ and $v \lesssim u = 1$ then $\otimes$-antisymmetry yields

$$1 = 1 \otimes 1 = (u \lesssim v) \otimes (v \lesssim u) \leq u \approx v,$$

i.e. $u \approx v = 1$. Since $\approx$ is a fuzzy equality, it is separable, whence $u = v$.

(b): It is straightforward to check that $\approx$ defined by (3) is a fuzzy equivalence (this also follows from Lemma 6 below). If $u \approx v = 1$ then $(u \lesssim v) \otimes (v \lesssim u) = 1$, hence $u \lesssim v = 1$ and $v \lesssim u = 1$, from which $u = v$ follows due to crisp antisymmetry, verifying that $\approx$ is separable, and thus a fuzzy equality. The claim now follows from Theorem 4 (a) in part I. □

Remark 1. It is clear that the two constructions in (a) and (b) of Lemma 1, bringing $\langle U, \approx, \lesssim \rangle$ to $\langle U, \lesssim \rangle$ and vice versa, are mutually inverse.

On the other hand, the constructions in (a) and (b) of Lemma 2 are not mutually inverse because $\approx$ defined by (3) is but one of the possible fuzzy equalities described by Theorem 4 (a3) in part I. In this regard, one may generalize (b) in Lemma 2 as follows:

(b') If $\langle U, \lesssim \rangle$ is a fuzzy ordered set according to Definition 3, then if $\approx$ is a fuzzy equality satisfying (a3) of Theorem 4 in part I, then $\langle U, \approx, \lesssim \rangle$ is a fuzzy ordered set according to Definition 1.

2.2. *A unifying concept of antisymmetry*

2.2.1. Unification of $\otimes$-, $\wedge$-, and crisp antisymmetry

We shall consider binary operations on a given complete lattice $\langle L, \leq, 0, 1 \rangle$. Following a recent common practice, we call a t-norm on $\langle L, \leq, 0, 1 \rangle$ a binary operation $\otimes : L \times L \to L$ which is commutative, associative, order-preserving, and has 1 as its neutral element, i.e. $1 \otimes a = a$ for each $a \in L$. In this generalized meaning, classical t-norms are just t-norms on $\langle [0, 1], \leq, 0, 1 \rangle$; moreover, the operation $\otimes$ of any complete residuated lattice $\langle L, \wedge, \vee, \otimes, \to, 0, 1 \rangle$ is a t-norm on $\langle L, \leq, 0, 1 \rangle$.

In addition, we employ more general conjunction-like operations $\odot$ which satisfy

$$a \odot b = b \odot a, \tag{4}$$

$$a_1 \odot a_2 \leq b_1 \odot b_2, \text{ whenever } a_1 \leq b_1 \text{ and } a_2 \leq b_2, \tag{5}$$

$$a \odot 1 \leq a, \text{ and} \tag{6}$$

$$1 \odot 1 = 1. \tag{7}$$

Obviously, every t-norm satisfies these conditions. We need the following properties.

Lemma 3. *Assume (4)–(7). Then*

$$a \odot b \leq a \wedge b, \tag{8}$$

$$a \odot b = 1 \textit{ implies } a = 1 \textit{ and } b = 1. \tag{9}$$

Proof. (8): (5) and (6) imply $a \odot b \leq a \odot 1 \leq a$. Using (4), one similarly obtains $a \odot b \leq 1 \odot b = b \odot 1 \leq b$. Putting these together, we get $a \odot b \leq a \wedge b$.

(9): In view of (8), if $a \odot b = 1$ then $a \wedge b = 1$, from which $a = 1 = b$ readily follows. □

Consider now the following notion. Let $\odot$ satisfy (4)–(7). A binary fuzzy relation $\lesssim$ on a set U equipped with a fuzzy equality $\approx$ satisfies $\odot$-*antisymmetry* if

$$(u \lesssim v) \odot (v \lesssim u) \leq u \approx v \tag{10}$$

for each $u, v \in U$.

While both $\otimes$-antisymmetry and $\wedge$-antisymmetry are obviously particular cases of $\odot$-antisymmetry, the same holds true for the seemingly different notion of crisp antisymmetry:

Lemma 4. *Consider the binary operation $\bullet$ on L and the fuzzy relation $\approx$ on U defined by*

$$a \bullet b = \begin{cases} 1 & \textit{for } a = 1 \textit{ and } b = 1, \\ 0 & \textit{otherwise;} \end{cases} \qquad u \approx v = \begin{cases} 1 & \textit{for } u = v, \\ 0 & \textit{otherwise.} \end{cases} \tag{11}$$

Then $\bullet$ satisfies (4)–(7) and $\approx$ is a fuzzy equality (the crisp fuzzy equality). Moreover, a binary fuzzy relation $\lesssim$ on U satisfies crisp antisymmetry if and only if it satisfies $\bullet$-antisymmetry.

Proof. Straightforward by a direct verification of the conditions involved. □

Remark 2. The operation $\bullet$ defined by (11) is the smallest operation satisfying (4)–(7) in that any $\odot$ verifying (4)–(7) satisfies $a \bullet b \leq a \odot b$ for any $a, b \in L$.

2.3. *Constructing a fuzzy equality from* $\lesssim \odot \lesssim^{-1}$

Notice that the crisp fuzzy equality $\approx$ in Lemma 4, which is involved in the condition of $\bullet$-antisymmetry, may in fact be obtained from $\lesssim$ by

$$u \approx v = (u \lesssim v) \bullet (v \lesssim u). \tag{12}$$

In view of Lemma 1, Lemma 2, and Lemma 4, and in particular the relationships (2), (3), and (12), respectively, we now explore – for the subsequent considerations on antisymmetry in general – the role of the fuzzy relation $\lesssim \odot \lesssim^{-1}$, which we denote $\equiv_\odot$, i.e.

$$u \equiv_\odot v \;=\; (u \lesssim v) \odot (v \lesssim u).$$

The following observation is immediate.

Lemma 5. *For $\odot$ satisfying (4)–(7), $\equiv_\odot$ is separable if and only if $\lesssim$ satisfies crisp antisymmetry.*

Proof. The $\Rightarrow$-part follows from (7). The $\Leftarrow$-part follows from (9). $\square$

Note that Lemma 5, which holds true for any fuzzy relation $\lesssim$, in fact provides a reformulation of crisp antisymmetry in terms of the fuzzy relation $\equiv_\odot$ derived from $\lesssim$.

We now recall a result by Bodenhofer (2000), which is related to our problem. For this purpose, recall the concept of dominance and an important result by De Baets and Mesiar (1998), on which Bodenhofer's result is based. A t-norm $\odot$ dominates a t-norm $\otimes$ (Klement, Mesiar, and Pap 2000) if

$$(a \odot b) \otimes (c \odot d) \leq (a \otimes c) \odot (b \otimes d) \tag{13}$$

for every $a, b, c, d \in L$; this is denoted by $\otimes \ll \odot$. De Baets and Mesiar proved that $\odot$ dominates $\otimes$ if and only if the $\odot$-intersection of any two $\otimes$-transitive fuzzy relations is $\otimes$-transitive. Here, the $\odot$-intersection $R \odot S$ of R and S is defined by $(R \odot S)(x, y) = R(x, y) \odot S(x, y)$, and $\otimes$-transitivity of R means $R(x, y) \otimes R(y, z) \leq R(x, z)$. The following lemma presents the above-mentioned result by Bodenhofer (2000, Theorem 17):

Lemma 6. *Let $\lesssim$ be a reflexive and transitive fuzzy relation on U and let $\odot$ be a t-norm dominating $\otimes$. Then $\lesssim$ is a fuzzy order on U equipped with a fuzzy equivalence $\equiv_\odot$ in the sense of Definition 1 (i.e., the original variant with fuzzy equivalence instead of fuzzy equality).*

Remark 3. (a) While Bodenhofer (2000) proves his Theorem 17 (i.e. Lemma 6) directly, the theorem follows from the equivalence of conditions (a1) and (a3) in Theorem 4 in part I, which is, as mentioned in part I, essentially the content of Bodenhofer's Theorem 18. Namely, since $\odot$ dominates $\otimes$, we have $\otimes \leq \odot$, hence $(u \lesssim v) \otimes (v \lesssim u) \leq (u \lesssim v) \odot (v \lesssim u) = u \approx_\odot v$, verifying (a3).

(b) In view of Lemma 5, the claim of Lemma 6 may be altered to fit Definition 1: Let $\lesssim$ be a reflexive and transitive fuzzy relation on U and let $\odot$ be a t-norm dominating $\otimes$. If $\lesssim$ satisfies crisp antisymmetry then $\equiv_\odot$ is a fuzzy equality and $\lesssim$ is a fuzzy order on U equipped with $\equiv_\odot$ according to Definition 1.

The obstacle we now face in proceeding with $\odot$-antisymmetry for a general $\odot$ satisfying (4)–(7) is that the fuzzy relation $\equiv_\odot$ need not be transitive if $\odot$ does not dominate $\otimes$.

Example 1. Let $U = \{u, v, w\}$ and let $L = [0, 1]$ with $\otimes$ being any of the Gödel, Goguen, and Łukasiewicz t-norm, and let $\odot$ be the drastic product $\otimes_D$, i.e.

$$a \otimes_D b = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{else.} \end{cases}$$

Notice that $\otimes_D$ satisfies (4)–(7). Let now $\lesssim$ be defined as follows:

$\lesssim$	u	v	w
u	1	1	0.7
v	0.7	1	0.7
w	0.7	1	1

As one easily checks, $\lesssim$ is $\otimes$-transitive. Nevertheless, the fuzzy relation $\equiv_{\otimes_D}$ is not

$\otimes$-transitive. Namely,

$$(u \equiv_{\otimes_D} v) \otimes (v \equiv_{\otimes_D} w) = 0.7 \otimes 0.7 \not\leq 0 = 0.7 \otimes_D 0.7 = (u \lesssim w) \otimes_D (w \lesssim u) = (u \equiv_{\otimes_D} w).$$

A natural way out is to consider the transitive closure of $\equiv_{\odot}$ rather than $\equiv_{\odot}$. Recall that the transitive closure $\text{Tra}(R)$ of a binary fuzzy relation $R : U \times U \to L$, i.e. the least transitive fuzzy relation containing R, satisfies

$$\text{Tra}(R) = \bigvee_{n=1}^{\infty} R^n = R \vee R \circ R \vee R \circ R \circ R \vee \cdots$$

where $(R \circ S)(u, v) = \bigvee_{x \in U} R(u, x) \otimes S(x, v)$.

Clearly, $\equiv_{\odot}$ is symmetric and since $1 \odot 1 = 1$, reflexivity of $\lesssim$ implies reflexivity of $\equiv_{\odot}$. Now, since the transitive closure preserves reflexivity and symmetry, we obtain:

Lemma 7. $\text{Tra}(\equiv_{\odot})$ *is reflexive and symmetric, whenever* $\lesssim$ *is reflexive.*

Since we are interested in fuzzy equalities, i.e. require separability, the following example demonstrating that the transitive closure does not preserve separability seems to present a problem:

Example 2. Consider $L = [0, 1]$ and the fuzzy relation R on the set

$$U = \{u, v\} \cup \{x_{11}\} \cup \{x_{21}, x_{22}\} \cup \{x_{31}, x_{32}, x_{33}\} \cup \cdots \cup \{x_{i1}, \ldots, x_{ii}\} \cup \cdots$$

defined by $R(y, z) = 0$ for every $y, z \in U$ except for

$$R(u, x_{i1}) = R(x_{i1}, x_{i2}) = \cdots = R(x_{ii}, v) = 1 - {}^1\!/_{i+1} \quad \text{for each } i = 1, 2, \ldots$$

For $R^n = R \circ \cdots \circ R$ (n times), one easily checks that for $\otimes$ being the Gödel t-norm,

$$R(u, v) = 0,\ R^2(u, v) = 1 - {}^1\!/_2,\ R^3(u, v) = 1 - {}^1\!/_3, \ldots,\ R^n(u, v) = 1 - {}^1\!/_n, \ldots,$$

and thus

$$[\text{Tra}(R)](u, v) = \left(\bigvee_{n=1}^{\infty} R^n\right)(u, v) = \bigvee_{n=1}^{\infty} (1 - {}^1\!/_n) = 1.$$

Hence, while R is separable, $\text{Tra}(R)$ is not. A similar example may be obtained for the Goguen and the Łukasiewicz t-norm.

Now, the particular structure of $\equiv_{\odot}$ enables us to prove that the possible problem of losing separability by the transitive closure does not materialize in our setting:

Lemma 8. *Let* $\lesssim$ *be transitive. If* $\equiv_{\odot}$ *is separable then* $\text{Tra}(\equiv_{\odot})$ *is separable.*

Proof. Let us first check that for each $i = 1, 2, \ldots$, one has.

$$u \equiv_{\odot}^{i} v \ \leq\ (u \lesssim v) \wedge (v \lesssim u). \tag{14}$$

Indeed, due to (5) and (6), $x \equiv_{\odot} y \leq x \lesssim y$, which along with the transitivity of $\lesssim$

yields

$$
\begin{aligned}
u \equiv_{\odot}^{i} v &= \bigvee_{x_1,\ldots,x_{i-1}\in U} ((u \equiv_{\odot} x_1) \otimes (x_1 \equiv_{\odot} x_2) \otimes \cdots \otimes (x_{i-1} \equiv_{\odot} v)) \\
&\le \bigvee_{x_1,\ldots,x_{i-1}\in U} ((u \lesssim x_1) \otimes (x_1 \lesssim x_2) \otimes \cdots \otimes (x_{i-1} \lesssim v)) \\
&\le \bigvee_{x_1,\ldots,x_{i-1}\in U} u \lesssim v \\
&= u \lesssim v.
\end{aligned}
$$

In a similar manner one obtains $u \equiv_{\odot}^{i} v \le v \lesssim u$, from which (14) readily follows. Now,

$$
[\mathrm{Tra}(\equiv_{\odot})](u,v) = \bigvee_{i=1}^{\infty} u \equiv_{\odot}^{i} v \le \bigvee_{i=1}^{\infty}((u \lesssim v) \wedge (v \lesssim u)) \le (u \lesssim v) \wedge (v \lesssim u).
$$

It follows that if $[\mathrm{Tra}(\equiv_{\odot})](u,v) = 1$ then $(u \lesssim v) \wedge (v \lesssim u) = 1$, whence $u \lesssim v = 1$ and $v \lesssim u = 1$. Condition (7) then yields $u \equiv_{\odot} v = (u \lesssim v) \odot (v \lesssim u) = 1 \odot 1 = 1$, from which $u = v$ follows by the separability of $\equiv_{\odot}$. □

2.4. *Main result: Equivalence of definitions of fuzzy order*

In view of the notions and observations in the preceding paragraphs, we now proceed toward a general concept of fuzzy order and our main result in this section. For this purpose we consider the following fuzzy relations on a given universe U:

$\lesssim$ ... a reflexive and transitive fuzzy relation on U,

$\equiv_{\odot}$ ... a fuzzy relation defined by

$$
u \equiv_{\odot} v \;=\; (u \lesssim v) \odot (v \lesssim u), \tag{15}
$$

$\approx_{\odot}$ ... the transitive closure of $\equiv_{\odot}$, i.e.

$$
u \approx_{\odot} v = [\mathrm{Tra}(\equiv_{\odot})](u,v). \tag{16}
$$

We first summarize and extend the previous observations regarding $\equiv_{\odot}$ and $\approx_{\odot}$:

Lemma 9. *Let $\odot$ satisfy (4)–(7) and $\lesssim$ be a reflexive and transitive fuzzy relation on U.*

(a) $\approx_{\odot}$ is a fuzzy equivalence on U.
(b) The following conditions are equivalent:
- *(b1) $\approx_{\odot}$ is a fuzzy equality;*
- *(b2) $\equiv_{\odot}$ is separable;*
- *(b3) $\lesssim$ satisfies crisp antisymmetry.*

(c) If $\odot$ is a t-norm which dominates $\otimes$, then $\equiv_{\odot} = \approx_{\odot}$.

Proof. (a): The claim follows from Lemma 7.

(b1) ⇒ (b3): Let $u \lesssim v = 1$ and $v \lesssim u = 1$. Due to (7), $u \equiv_{\odot} v = (u \lesssim v) \odot (v \lesssim u) = 1 \odot 1 = 1$. Since $u \equiv_{\odot} v \le [\mathrm{Tra}(\equiv_{\odot})](u,v) = u \approx_{\odot} v$, we obtain $u \approx_{\odot} v = 1$, whence $u = v$ due to separability of $\approx_\odot$.

(b3) $\Rightarrow$ (b2): If $u \equiv_\odot v = 1$ then (9) yields $u \lesssim v = 1$ and $v \lesssim u = 1$, hence $u = v$ using crisp antisymmetry of $\lesssim$.

(b2) $\Rightarrow$ (b1): The claim follows from (a) and Lemma 8.

(c): This is Bodenhofer's observation based on De Baets and Mesiar (1998); cf. Lemma 6. □

Remark 4. For $\bullet$ defined by (11), $\equiv_\bullet$ is transitive, which is obvious because $\equiv_\bullet$ is the crisp equality, cf. Lemma 4. Hence, $\equiv_\bullet = \approx_\bullet$, even though $\bullet$ does not meet the assumption of Lemma 9 (c) because $\bullet$ is not a t-norm. Nevertheless, $\bullet$ still satisfies the dominance condition (13), with $\odot = \bullet$ and any t-norm $\otimes$, which is easily seen to imply transitivity of $\bullet$-intersection of arbitrary $\otimes$-transitive fuzzy relations. Notice that in this sense, not only $\bullet$ dominates $\otimes$, but also $\otimes$ dominates $\bullet$. Yet $\bullet \neq \otimes$, which cannot happen with t-norms because if a t-norm $\odot$ both dominates and is dominated by a t-norm $\otimes$, then $\odot = \otimes$.

In the present perspective, the following concept provides a natural generalization of the three notions of fuzzy order presented in Definitions 1, 2, and 3:

Definition 4. Let $\odot$ satisfy (4)–(7). A fuzzy order on a set U equipped with a fuzzy equality relation $\approx$ is a binary fuzzy relation $\lesssim$ on U compatible with $\approx$, i.e. satisfying

$$(u_1 \lesssim v_1) \otimes (u_1 \approx u_2) \otimes (v_1 \approx v_2) \ \leq \ u_2 \lesssim v_2,$$

for every $u_1, u_2, v_1, v_2 \in U$, which, moreover, fulfills

$$\begin{aligned} u \lesssim u \ &= \ 1 && \text{(reflexivity)},\\ (u \lesssim v) \otimes (v \lesssim w) \ &\leq \ u \lesssim w && \text{(transitivity)},\\ (u \lesssim v) \odot (v \lesssim u) \ &\leq \ u \approx v && \text{($\odot$-antisymmetry)}, \end{aligned}$$

for each $u, v, w \in U$.

Remark 5. Definition 4 generalizes the notion of fuzzy order according to Definition 2 and, in view of Theorem 4 (a) in part I, the notion of fuzzy order according to Definition 1 as well. Since for a crisp fuzzy equality, compatibility is trivially satisfied, it also essentially generalizes Definition 3 (cf. Lemma 4). Namely,

- for $\odot = \otimes$, Definition 4 yields Definition 1;
- for $\odot = \wedge$, Definition 4 yields Definition 2;
- for $\odot = \bullet$, Definition 4 yields Definition 3.

To describe relationships among the discussed definitions of fuzzy orders, as well as among the respective variants of antisymmetry, we first present a theorem providing a number of mutually equivalent possibilities to define the general notion of fuzzy order according to Definition 4. Next, we present a theorem claiming that the notions of fuzzy order according to Definitions 1, 2, 3, and 4 are essentially mutually equivalent.[4]

Theorem 1. *Let $\lesssim$ be a reflexive and transitive fuzzy relation on U. The following conditions are equivalent:*

[4]As is easily seen, further variations of the claims of the two theorems may be formulated. For instance, a variation of Theorem 1 may be proved for fuzzy orders according to Definition 1 as well as for fuzzy orders according to Definition 2.

(a) *There exists $\odot$ satisfying (4)–(7) and a fuzzy equality $\approx$ such that $\lesssim$ is a fuzzy order on U equipped with $\approx$ according to Definition 4.*

(b) *For each $\odot$ satisfying (4)–(7) there exists a fuzzy equality $\approx$ such that $\lesssim$ is a fuzzy order on U equipped with $\approx$ according to Definition 4.*

(c) *There exists $\odot$ satisfying (4)–(7) such that $\lesssim$ is a fuzzy order on U equipped with $\approx_\odot$ according to Definition 4.*

(d) *For each $\odot$ satisfying (4)–(7), $\lesssim$ is a fuzzy order on U equipped with $\approx_\odot$ according to Definition 4.*

(e) *There exists $\odot$ satisfying (4)–(7) and a fuzzy equality $\approx$ on U such that $\equiv_\odot \leq \approx \leq \equiv_\wedge$.*

(f) *For each $\odot$ satisfying (4)–(7) there exists a fuzzy equality $\approx$ on U such that $\equiv_\odot \leq \approx \leq \equiv_\wedge$.*

Proof. We prove the claim by verifying the following implications.

(b) $\Rightarrow$ (a) and (d) $\Rightarrow$ (c) are obvious.

(a) $\Rightarrow$ (e): For $\odot$ and $\approx$ from (a), $\equiv_\odot \leq \approx$ is just the $\odot$-antisymmetry of $\lesssim$ while $\approx \leq \equiv_\wedge$ is a consequence of $\approx$-reflexivity of $\lesssim$ due to Lemma 2 (c) in part I. Note that $\approx$-reflexivity of $\lesssim$ follows from the reflexivity and compatibility of $\lesssim$ due to Corollary 2 in part I.

(f) $\Rightarrow$ (b): For an arbitrary $\odot$ satisfying (4)–(7), consider a fuzzy equality $\approx$ implied by (f). Like in the proof of "(a) $\Rightarrow$ (e)," the $\odot$-antisymmetry of $\lesssim$ is expressed by $\equiv_\odot \leq \approx$, while the compatibility of $\lesssim$ with $\approx$ follows – due to Corollary 2 in part I – from $\approx$-reflexivity of $\approx$, which itself is expressed by $\approx \leq \equiv_\wedge$.

(c) $\Rightarrow$ (e): Consider an $\odot$ implied by (c). We verify the two inequalities in (e) for $\approx$ being $\approx_\odot$. Clearly, $\equiv_\odot \leq \mathrm{Tra}(\equiv_\odot) = \approx_\odot$, checking the first inequality in (e). Now, (8) clearly implies $\equiv_\odot \leq \equiv_\wedge$, hence $\mathrm{Tra}(\equiv_\odot) \leq \mathrm{Tra}(\equiv_\wedge)$. Since Lemma 9 (c) implies $\mathrm{Tra}(\equiv_\wedge) = \equiv_\wedge$, we obtain

$$\approx_\odot = \mathrm{Tra}(\equiv_\odot) \leq \mathrm{Tra}(\equiv_\wedge) = \equiv_\wedge,$$

verifying the second inequality in (e).

(f) $\Rightarrow$ (d): Consider an arbitrary $\odot$ satisfying (4)–(7) and a fuzzy equality $\approx$ implied by (f). First, $\equiv_\odot \leq \approx$, which holds due to (f), expresses the $\odot$-antisymmetry of $\lesssim$. Secondly, as $\approx_\odot = \mathrm{Tra}(\equiv_\odot) \leq \mathrm{Tra}(\approx) = \approx$, the second inequality of (f), and Lemma 2 (c) in part I, imply

$$\approx_\odot \leq \approx \leq \equiv_\wedge,$$

hence $\lesssim$ is $\approx_\odot$-reflexive. The compatibility of $\lesssim$ and $\approx_\odot$ now follows from Corollary 2 in part I.

(e) $\Rightarrow$ (f): Consider an arbitrary $\odot$ satisfying (4)–(7). We check that the fuzzy relation $\approx = \approx_\odot$ satisfies the conditions in (f).

Due to (8), $(u \lesssim v) \odot (v \lesssim u) \leq (u \lesssim v) \wedge (v \lesssim u)$, i.e. $\equiv_\odot \leq \approx_\wedge$, whence $\mathrm{Tra}(\equiv_\odot) \leq \mathrm{Tra}(\equiv_\wedge)$. Since $\approx_\odot = \mathrm{Tra}(\equiv_\odot)$ and $\approx_\wedge = \mathrm{Tra}(\equiv_\wedge)$, using $\equiv_\odot \leq \mathrm{Tra}(\equiv_\odot)$ and Lemma 9 (c) we obtain

$$\equiv_\odot \leq \approx_\odot \leq \approx_\wedge = \equiv_\wedge,$$

verifying the required inequality in (f). It remains to check that $\approx_\odot$ is indeed a fuzzy equality. Since $\approx_\odot$ is a fuzzy equivalence due to Lemma 7 and $\approx_\odot = \mathrm{Tra}(\equiv_\odot)$, it remains to verify the separability of $\approx_\odot$.

Consider an operation $*$ and a fuzzy equality $\sim$ implied by (e). Due to Lemma 8, it suffices to check the separability of $\equiv_\odot$. Let thus $u \equiv_\odot v = 1$, i.e. $(u \lesssim v) \odot (v \lesssim u) = 1$. Due to (9), $(u \lesssim v) = 1$ and $(v \lesssim u) = 1$, whence (7) yields

$$1 = (u \lesssim v) * (v \lesssim u) = u \equiv_* v.$$

The first inequality in (e) now implies $u \equiv_* v \ \leq \ u \equiv v$, whence $u \approx_\odot v = 1$, from which $u = v$ follows by the separability of $\approx_\odot$.

□

The second theorem reveals the equivalence of the four definitions of a fuzzy order:

Theorem 2. *Let $\lesssim$ be a reflexive and transitive fuzzy relation on U. Each of the following conditions is equivalent to any of conditions (a)–(f) in Theorem 1. (Thus, in particular, the following conditions are mutually equivalent.)*

(a) $\lesssim$ is a fuzzy order according to Definition 1 for some fuzzy equality $\approx$.

(b) $\lesssim$ is a fuzzy order according to Definition 2 for some fuzzy equality $\approx$.

(c) $\lesssim$ is a fuzzy order according to Definition 3.

Proof. Obviously, any of (a), (b), and (c) implies condition (a) of Theorem 1. Conversely, condition (b) of Theorem 1 obviously implies (a) and (b) of the present theorem. Using Lemma 4, a moment's reflection shows that it also implies (c) of the present theorem. The claim now follows from the mutual equivalence of conditions (a) and (b) of Theorem 1. □

Remark 6. (a) It is apparent that in addition to the above mutually equivalent conditions for $\lesssim$ to form a fuzzy order, other conditions may be obtained.

(b) Other definitions of the general notion of fuzzy order may be formulated. For instance, in view of the above results, one may verify that the following conditions are equivalent for a fuzzy relation $\lesssim$ on U for any $\odot$ satisfying (4)–(7):

(b1) $\lesssim$ is a fuzzy order according to Definition 4 for some fuzzy equality $\approx$;
(b2) $\lesssim$ is transitive and the induced fuzzy relation $\equiv_\odot$ is reflexive and separable;
(b3) $\lesssim$ is transitive and the induced fuzzy relation $\approx_\odot$ is a fuzzy equality.

3. Distinctive properties of the various notions of antisymmetry and fuzzy order

In view of the results of the preceding section, the choice of the operation $\odot$, which is employed in the general concept of $\odot$-antisymmetry and the notion of fuzzy order according to Definition 4, does not essentially matter and is rather a matter of one's preference. In this section, though, we look at the question of which significant properties distinguish the three basic notions of fuzzy order codified by Definitions 1, 2, and 3, which correspond to $\otimes$-antisymmetry, $\wedge$-antisymmetry, and $\bullet$-antisymmetry (or, equivalently, crisp antisymmetry), respectively.

First view: Ordering of aggregation operations $\odot$

Clearly, a partial order $\leq$ can be defined on the class of all operations $\odot$ on L satisfying (4)–(7) by putting

$$\odot_1 \ \leq \ \odot_2 \quad \text{if and only if} \quad a \odot_1 b \ \leq \ a \odot_2 b \ \text{ for every } a, b \in L.$$

The following claim implies that from this viewpoint, $\wedge$-antisymmetry and $\bullet$-antisymmetry, and hence fuzzy orders according to Definitions 2 and 3, have distinct roles:

Theorem 3. *Let $\odot$ satisfy (4)–(7) and let $\lesssim$ be a fuzzy order according to Definition 4 for some fuzzy equality $\approx$.*

(a) The operation $\bullet$ defined by (11) is the smallest operation satisfying (4)–(7); hence $\bullet$ is the smallest operation $\odot$ for which $\lesssim$ is a fuzzy order according to Definition 4 for some fuzzy equality $\approx$.

(b) The operation $\wedge$ is the largest operation $\odot$ satisfying (4)–(7); hence $\wedge$ is the largest operation $\odot$ for which $\lesssim$ is a fuzzy order according to Definition 4 for some fuzzy equality $\approx$.

Proof. (a): The first part follows from the definition of $\bullet$ and property (7) of the considered operations $\odot$. The second part is a direct consequence of Theorem 2 and the first part.

(b): The first part follows from property (8) in Lemma 3. The second part follows again from Theorem 2 and the first part. □

Second view: $\odot$ as logical connective

Since the degree $(u \lesssim v) \odot (v \lesssim u)$ is interpreted as a degree to which u is less than or equal to v *and* v is less than or equal to u, the operation $\odot$ satisfying (4)–(7) is naturally interpreted as conjunctive aggregation. Since it is well established that adjointness of conjunction w.r.t. implication is an essential property from a logical view (Belohlavek 2002; Goguen 1969; Gottwald 2001; Hájek 1998), the following immediate observation points out a distinguished position of $\otimes$-antisymmetry and of the notion of fuzzy order according to Definition 1:

Theorem 4. *Of all the operations $\odot$ satisfying (4)–(7) for which a given fuzzy relation $\lesssim$ is a fuzzy order according to Definition 4 for some fuzzy equality $\approx$, $\otimes$ is the only one that satisfies adjointness w.r.t. $\to$, i.e.*

$$a \odot b \leq c \quad \textit{iff} \quad a \leq b \to c \quad \textit{for every } a, b, c \in L.$$

Proof. The proof follows from Theorem 1 and the following well-known argument showing that in each residuated lattice, $\otimes$ is the only binary operation satisfying adjointness w.r.t. $\to$: First, $\otimes$ satisfies adjointness due to the definition of a complete residuated lattice; second, if $\odot$ satisfies adjointness, then for each $a, b, c \in L$, $a \otimes b \leq c$ iff $a \leq b \to c$ iff $a \odot b \leq c$, from which it follows that $a \otimes b = a \odot b$. □

Note also that for fuzzy orders according to Definitions 1 and 2, which employ $\otimes$- and $\wedge$-antisymmetry, respectively, one need not extend the language of residuated lattices because both $\otimes$ and $\wedge$ are residuated lattice operations. For fuzzy orders according to Definition 4, which employs $\odot$-antisymmetry for a general $\odot$, the presence of $\odot$ means that the language of residuated lattices needs to be extended unless $\odot$ is definable by the residuated lattice operations. That is to say, while fuzzy orders according to Definitions 1 and 2 may be developed within the framework of complete residuated lattices, fuzzy orders according to Definition 4 require a richer framework of complete residuated lattices equipped with an additional operation.

Third view: Uniqueness of fuzzy equality

Let $\odot$ satisfy (4)–(7) and let $\precsim$ be a fuzzy order on U equipped with $\approx$ according to Definition 4. According to Theorem 1, the set of all fuzzy equalities $\sim$ for which $\precsim$ is a fuzzy order on U equipped with $\sim$ forms the interval

$$\mathcal{I}_\odot = \{\sim \mid \sim \text{ is a fuzzy equality and } a \approx_\odot b \leq a \sim b \leq a \approx_\wedge b \text{ for every } a, b \in L\}$$

in the set of all fuzzy equalities on U partially ordered by inclusion of fuzzy relations. The following theorem reveals another distinct feature of $\wedge$ and fuzzy orders with $\wedge$-antisymmetry according to Definition 2:

Theorem 5. *Let* $\mathbf{L}$ *be an arbitrary complete residuated lattice and let* U *have at least two elements. Then* $\wedge$ *is the only operation* $\odot$ *satisfying (4)–(7) such that for each fuzzy order* $\precsim$ *according to Definition 4, the interval* $\mathcal{I}_\odot$ *is a singleton. Hence,* $\wedge$ *is the only operation satisfying (4)–(7) for which* $\approx$ *is uniquely determined by* $\precsim$*.*

Proof. Due to Theorem 1, $\mathcal{I}_\wedge$ is a singleton. On the other hand, let $\odot$ be different from $\wedge$. We prove the claim by constructing a fuzzy order for which $\mathcal{I}_\odot$ is not a singleton.

Since $\odot \neq \wedge$, there exist $a, b \in L$ such that

$$a \odot b < a \wedge b.$$

Pick two distinct elements $u, v \in U$ and consider the fuzzy relation $\precsim$ on U defined by

$$x \precsim x = 1 \text{ for each } x \in U, \quad u \precsim v = a, \quad v \precsim u = b, \quad \text{and} \quad x \precsim y = 0 \text{ otherwise.}$$

Define fuzzy relations $\sim_1$ and $\sim_2$ on U by

$$x \sim_1 y = (x \precsim y) \odot (y \precsim x) \quad \text{and} \quad x \sim_2 y = (x \precsim y) \wedge (y \precsim x),$$

for any $x, y \in U$. One may observe that $\sim_1$ and $\sim_2$ are two distinct fuzzy equalities on U. Note that separability of $\sim_1$ and $\sim_2$ follows from $1 \odot 1 = 1$ and $a \odot b < a \wedge b$, since these assumptions imply that $a \neq 1$ or $b \neq 1$, hence $u \sim_1 v \neq 1$ and $u \sim_2 v \neq 1$, verifying that $x \sim_1 y = 1$ implies $x = y$ and $x \sim_2 y = 1$ implies $x = y$ for any $x, y \in U$.

As $\precsim$ is clearly $\sim_1$-reflexive and $\sim_2$-reflexive, $\precsim$ is reflexive and compatible with $\sim_1$ as well as with $\sim_2$ due to Corollary 2 of part I. Now, $\precsim$ is obviously transitive, satisfies $\odot$-antisymmetry w.r.t. $\sim_1$, and due to (8), also w.r.t. $\sim_2$. We obtained that $\precsim$ is a fuzzy order on U equipped with $\sim_1$ as well as a fuzzy order on U equipped with $\sim_2$ according to Definition 4. Therefore, $\mathcal{I}_\odot$ contains both $\sim_1$ and $\sim_2$, and is hence not a singleton. ∎

It is well known and trivial fact that for any ordinary order $\leq$, the equality relation $=$ is determined by $\leq$ as follows:

$$u = v \quad \text{if and only if} \quad u \leq v \text{ and } v \leq u.$$

In view of Theorem 5, a generalization of this property is satisfied only for the notion of fuzzy order according to Definition 2, revealing thus a distinct role of $\wedge$-antisymmetry.

Fourth view: Indistinguishability with respect to hierarchy

In addition to the distinct features of $\wedge$ and $\bullet$ established above, one may derive further distinct properties of these two aggregation operations by the following rationale.

A fuzzy order $\precsim$ represents a graded hierarchy of the objects on the underlying universe U. It is hence natural to ask which objects are indistinguishable with respect to the hierarchy. Such an indistinguishability is naturally conceived as a fuzzy relation $\sim$ on U satisfying at least the following properties: $\sim$ is reflexive, symmetric, and is included in $\precsim$. Reflexivity and symmetry are implied by the obvious requirements that any $u \in U$ is indistinguishable from itself and that if u is indistinguishable from v, then v is indistinguishable from u. Inclusion of $\sim$ in $\precsim$ is crucial for our argument below and we derive this requirement intuitively as follows: Since $\precsim$ is reflexive, u is less than or equal to u for each u. One hence expects that if u is indistinguishable from v, then u is less than or equal to v as well, since the other possibility, i.e. u not being less than or equal to v, would distinguish u from v.

Now, of all the possible indistinguishabilities w.r.t. the hierarchy represented by $\precsim$, one is naturally interested in the largest one, which is most informative (the least one is intuitively expected to be the crisp identity).

In a fuzzy setting, reflexivity, symmetry and inclusion of $\sim$ in $\precsim$ mean $u \sim u = 1$, $u \sim v = v \sim u$, and $u \sim v \leq u \precsim v$ for any $u, v \in U$. The following observation reveals distinct roles of $\wedge$ and $\bullet$ from the present viewpoint:

Theorem 6. *Let $\precsim$ be reflexive and transitive fuzzy relation on U.*

(a) The largest reflexive and symmetric fuzzy relation contained in $\precsim$ (i.e. the most informative indistinguishability w.r.t. $\precsim$ in the sense above) is $\equiv_\wedge$, which is also the largest reflexive, symmetric, and transitive fuzzy relation contained in $\precsim$.

(b) The least reflexive, symmetric, and transitive fuzzy relation contained in $\precsim$ is $\equiv_\bullet$.

Proof. Since by definition, $u \equiv_\wedge v = (u \precsim v) \wedge (v \precsim u)$, the first part in (a) follows from the following claim, which is easy to verify: For any binary fuzzy relation R, the relation S_R defined by

$$S_R(u, v) = R(u, v) \wedge R(v, u)$$

is the largest symmetric fuzzy relation contained in R. The second part is due to the fact established above that $\equiv_\wedge$ is reflexive and transitive.

(b) is trivial because $\approx_\bullet$ is the crisp equality. $\square$

4. Conclusions and future topics

4.1. *Conclusions*

In our two-part paper, we thoroughly consider the existing definitions of fuzzy order in which antisymmetry is formulated with respect to a generalized equality on the underlying universe. We review the current approaches, which exist in the literature for quite some time (Belohlavek 2001, 2002, 2004; Blanchard 1983; Bodenhofer 1999, 2000, 2003; Höhle 1987; Höhle and Blanchard 1985) but have not been examined from the perspective we provide in our treatment.

We first present a detailed account of the development of the variants of the considered notion of fuzzy order along with a number of historical remarks starting with the initial paper by Zadeh (1971). Secondly, we provide various kinds of observations to enhance the current understanding of the examined notion of fuzzy order, and analyze relationships between the existing variants of this notion. Thirdly, we study in detail the notion of antisymmetry, which is arguably the least understood of the conditions required by the existing definitions of fuzzy order.

The most important results regarding antisymmetry is a unifying concept of antisymmetry along with the resulting generalization of the concept of fuzzy order and our theorems according to which – contrary to the present understanding – the existing variants of the notion of fuzzy order are mutually equivalent and are equivalent to our generalized concept of fuzzy order. The latter is due to a new perspective that we present, which is different from the current view according to which fuzzy orders with $\otimes$-antisymmetry are more general than fuzzy orders with $\wedge$-antisymmetry. The new perspective consists in asking:

Which fuzzy relations may be regarded as fuzzy orders?

We regard such a perspective more suitable compared to the one considered implicitly in some previous works, namely one based on the question: Given a fixed fuzzy equality, which fuzzy relations may be regarded as fuzzy orders? We also identify several properties that distinguish the existing variants of the notion of fuzzy order.

4.2. *Future topics*

The present results open a general problem of whether and to what extent it matters which of the variants of the notion of fuzzy order examined in this paper one employs in the development of further areas involving the notion of fuzzy order. For instance, whether and to what extent this matters in the development of complete lattices, closure structures, fixed point theory, and other topics in the setting of fuzzy logic. We obtained several results along these lines already and shall present them in future publications.

For illustration, let us consider the concept of a complete lattice in the setting of fuzzy logic as developed by Belohlavek (2001, 2002, 2004); see also Höhle (1987) for a closely related earlier approach. Let $\lesssim$ be a fuzzy order on a set U equipped with a fuzzy equality $\approx$ in the sense of Definition 2, which notion represents the framework for the considerations on complete lattices we are about to recall (Belohlavek 2001, 2002, 2004).

For any fuzzy set $A \in L^U$ define the fuzzy sets $\mathcal{L}(A) \in L^U$ and $\mathcal{U}(A) \in L^U$ of lower and upper cones of A, respectively, by

$$[\mathcal{L}(A)](u) = \bigwedge_{v \in U} (A(v) \to u \lesssim v) \quad \text{and} \quad [\mathcal{U}(A)](u) = \bigwedge_{v \in U} (A(v) \to v \lesssim u).$$

Furthermore, define for any $A \in L^U$ the fuzzy sets $\inf(A)$ and $\sup(A)$ of infima and suprema by

$$\inf(A) = \mathcal{L}(A) \wedge \mathcal{UL}(A) \quad \text{and} \quad \sup(A) = \mathcal{U}(A) \wedge \mathcal{LU}(A), \tag{17}$$

where $\mathcal{UL}(A)$ and $\mathcal{LU}(A)$ stand for $\mathcal{U}(\mathcal{L}(A))$ and $\mathcal{L}(\mathcal{U}(A))$, respectively. That is, $[\inf(A)](u) = [\mathcal{L}(A)](u) \wedge [\mathcal{UL}(A)](u)$ for each $u \in U$ and analogously for $\sup(A)$.

Now, a fuzzy ordered set $\langle U, \approx, \lesssim \rangle$ in the sense of Definition 2 is called a complete lattice if for every $A \in L^U$, both $\inf(A)$ and $\sup(A)$ are $\approx$-singletons (Belohlavek 2001, 2002, 2004). Note that a $\approx$-singleton is a fuzzy set $A \in L^U$ for which there exists $u \in U$ such that for each $v \in U$ one has $A(v) = u \approx v$; there exist other, equivalent definitions of $\approx$-singletons. If A is a $\approx$-singleton, u is the only element for which $A(u) = 1$, hence one may also speak of the $\approx$-singleton determined by u. If $\inf(A)$ is a $\approx$-singleton, the unique element $u \in U$ for which $[\inf(A)](u) = 1$ is called the infimum of A; the same applies to suprema. Note that in the original works (Belohlavek 2001, 2002, 2004), complete lattices as defined above are called completely lattice **L**-ordered sets and that in some subsequent works, they are called simply fuzzy lattices by other authors.

While the theory of complete lattices and related structures in a fuzzy setting has been advanced considerably, the purpose of the present illustration is to briefly point out a natural possibility to reconsider the above notions from the viewpoint of the general definition of a fuzzy order provided by Definition 4. For the sake of our illustration we refrain to the case in which $\lesssim$ is a fuzzy order on a set U equipped with the fuzzy equality $\approx_\otimes$ according to Definition 1, and hence also a fuzzy order according Definition 4 (for $\odot = \otimes$). Such a setting is very close to the one of fuzzy orders according to Definition 2, i.e. to the setting in which the theory of complete lattices has been developed as mentioned above. For instance, $u \approx_\otimes v = (u \lesssim v) \otimes (v \lesssim u)$ is analogous to the equality $u \approx v = (u \lesssim v) \wedge (v \lesssim u)$ implied by Definition 2. Yet, this setting does not impose any restriction on $\lesssim$ itself; cf. Theorem 1, its part (d), and Theorem 2.

In order to obtain a sound variant of the above notion of a complete lattice for our setting with a fuzzy ordered set $\langle U, \approx_\otimes, \lesssim \rangle$ according to Definition 1, one may proceed in several ways, of which we present the following one. For a fuzzy set $A \in L^U$, put

$$\inf_\otimes(A) = \mathcal{L}(A) \otimes \mathcal{UL}(A) \quad \text{and} \quad \sup_\otimes(A) = \mathcal{U}(A) \otimes \mathcal{LU}(A),$$

with $\mathcal{L}$ and $\mathcal{U}$ defined as above. Let us call $\langle U, \approx_\otimes, \lesssim \rangle$ a complete lattice if both $\inf_\otimes(A)$ and $\sup_\otimes(A)$ are $\approx_\otimes$-singletons for each $A \in L^U$. This definition is directly analogous to (17); in a sense, $\otimes$ replaces $\wedge$ in appropriate places.

It has been established (Belohlavek 2004, Lemma 11) that for a fuzzy ordered set $\langle U, \approx, \lesssim \rangle$ according to Definition 2, the following conditions are equivalent for any fuzzy set $A \in L^U$:

(a) $\inf(A)$ is a $\approx$-singleton;
(b) there exists $u \in U$ such that $[\inf(A)](u) = 1$.

Adopting the proof of this Lemma 11, we obtain an analogous property for the present setting: For a fuzzy ordered set $\langle U, \approx_\otimes, \lesssim \rangle$ according to Definition 1, the following conditions are equivalent for any fuzzy set $A \in L^U$:

(a') $\inf(A)$ is a $\approx_\otimes$-singleton;
(b') there exists $u \in U$ such that $[\inf_\otimes(A)](u) = 1$.

In view of Theorem 1 and Theorem 2, and the obvious equivalence of the above conditions (b) and (b'), we obtain the following result: For a reflexive and transitive relation $\lesssim$ on U, the following conditions are equivalent:

(i) $\langle U, \approx, \lesssim \rangle$ is a fuzzy ordered set according to Definition 2 that forms a complete lattice in the sense of (Belohlavek 2001, 2002, 2004).

(ii) $\langle U, \approx_{\otimes}, \lesssim \rangle$ is a fuzzy ordered set according to Definition 1 that forms a complete lattice in the sense of the above definition with $\inf_{\otimes}$ and $\sup_{\otimes}$.

This result is one of several possible ways expressing the fact that being a complete lattice is invariant with respect to the two possible notions of a fuzzy order involved. A proper study of such an invariance in general and of its ramifications for the theory of complete lattices thus presents a topic for further research. Note that relevant results, which need to be reconsidered in the present perspective, have been obtained by Martinek (2008, 2011).

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Appendix C

On linear extensions of fuzzy orders

The paper (Urbanec, 2023) concerned with the possibility of Szpilrajn-like extensions of fuzzy orders. It is the source of the results discussed in Chapter 4 and is currently under review in Fuzzy Sets and Systems. The topic was also one of the main sources of an inspiration for many considerations on the definition of the concept of fuzzy order itself.

On linear extensions of fuzzy orders

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Abstract

We reevaluate the strength of a link between fuzzy order and fuzzy equality on an underlying universe. We first observe that compared to the Boolean setting, the sitauation is much more interesting in the setting of fuzzy logic where there may be many fuzzy equalities on the given set. Then, we show that the link is deeper and most importantly bidirectional, i.e. defining fuzzy order w. r. t. a fuzzy equality is not enough; the fuzzy equality should moroever mirror all the adjustments made to the fuzzy order accordingly. Utilizing this idea, we provide a generalization of Szpilrajn's extension theorem within the framework of fuzzy logic, which further alleviates the drawbacks that such generalizations possessed in the previous studies.

Keywords: order, fuzzy logic, linearity, Szpilrajn extension

1. Aim of this paper

Extending a partial order into a chain is a classical problem in order theory. For fuzzy orders, such Szpilrajn-like completion was considered already by Zadeh [33] when he introduced the concept of fuzzy order itself. These considerations were soon to be followed by others but many questions still remain open. One of the most recent study [11] on the topic analyzes different axioms for linearity of a fuzzy order in some detail. Surprisingly, the outcome of the study is that a completion of any fuzzy order with desirable properties exists only for a very weak axiom of $\otimes$-linearity. We show that this is related to a structure of fuzzy equalities on a set which is much richer than its counterpart in Boolean logic. Moreover, we propose a solution to the fuzzy order completion problem by manipulating both entities, i.e. a fuzzy order and its induced fuzzy equality together, in a compatible way. Using this idea, which may be regarded as a further extension of reflections on the role of a fuzzy equality in the definition of fuzzy order in the spirit of [5], we obtain a way to extend any fuzzy order into a linear fuzzy order in a broad class of fuzzy logics.

Our paper is organized as follows. In Section 2, we briefly summarize preliminary notions with emphasis on fuzzy equalities and fuzzy orders. Section 3 scrutinizes different perspectives on linearity of a binary fuzzy relation and introduces our point of view. In Section 4, we state generalizations of Szpilrajn-like extension theorem enriching the up to date available results. Section 5 contains some observations on an intersection representation of fuzzy orders. Finally, in Section 6 we assess our results via their relationship to the essential properties of chains as identified in [11]. Moreover, Appendix contains introductions to order theory and fuzzy logic, followed by a brief history of the extension theorems in setting of fuzzy logic.

2. Preliminaries

The most important notions, the reader should be familiar with, are those of fuzzy equality and fuzzy order. We therefore describe them in some detail now. To keep the pace of the text, all other general preliminary notions, such as residuated lattices, fuzzy sets, classical orders, etc., are covered in Appendix.[1] Here, we only introduce some notation conventions for these general notions.

[1] Auxiliary definitions and propositions stated in Appendix are denoted by number prefixed by letter A when referenced in the text.

If the used complete residuated lattice is obvious from the context or if the given proposition is valid for any complete residuated lattice, we usually use terms such as fuzzy set, fuzzy relation, fuzzy order, etc. On the other hand, if we consider some particular complete residuated lattice, we denote it by **L** and then talk about **L**-set, **L**-relation, **L**-order, etc. Moreover, the symbol **2** is used for the two element Boolean algebra of classical logic, i.e. a very particular case of complete residuated lattice.

2.1. Fuzzy equivalences and equalities

Fuzzy equivalences and fuzzy equalities are binary fuzzy relations on a set often used to model indistinguishability. As such, they play a significant role in formalizations of many fields of human reasoning. In this section, we cover definitions and some relevant properties. For further details, we refer reader to the books [1, 29].

Definition 1. A *fuzzy equivalence (**L**-equivalence) on U* is a binary fuzzy relation $\approx$ on U, i.e. $\approx\colon U \times U \to L$, satisfying for each $u, v, w \in U$ the conditions

$$\begin{aligned} u \approx u &= 1, \\ u \approx v &= v \approx u, \\ (u \approx v) \otimes (v \approx w) &\leq u \approx w, \end{aligned}$$

called *reflexivity*, *symmetry*, and *transitivity*, respectively. A *fuzzy equality (**L**-equality)* is a fuzzy equivalence moreover satisfying the separability condition, i.e.

$$u \approx v = 1 \quad \text{implies} \quad u = v,$$

for each $u, v \in U$.

We denote a fuzzy equality by $\eqsim$ to distinguish it from a general fuzzy equivalence as the difference between them has a central role in this work.

An important concept regarding fuzzy equivalences is that of compatibility, i.e. a generalization of the axiom of equality of classical logic [1, 22].

Definition 2. A binary fuzzy relation $R\colon U \times U \to L$ is called *compatible with a fuzzy equivalence* $\approx$ on U if

$$R(u_1, v_1) \otimes (u_1 \approx u_2) \otimes (v_1 \approx v_2) \leq R(u_2, v_2) \tag{1}$$

for each $u_1, u_2, v_1, v_2 \in U$.

A structure of fuzzy equalities on a finite set

The following paragraphs describe the structure of all fuzzy equalities on a finite set. This structure is more intricate in the setting of fuzzy logic than in the classical case as it is not limited to a single equality, i.e. the identity. Although it is interesting by itself, our primary objective is to examine the properties of linear fuzzy order extensions. We show in further sections that there is a connection between this structure of fuzzy equalities and a possibility of extending general fuzzy order into a linear one. Here, we focus only on conditions under which the structure of all fuzzy equalities on a finite set forms a lattice. In Sections 4 and 5, we will see that same conditions characterize the class of residuated lattices which admits a linear extension of arbitrary fuzzy order for a particular form of linearity.

Lemma 1. *For any set U the set of all fuzzy equalities on U equipped with the (crisp) subsethood relation is a meet semilattice where meet is the set intersection.*

PROOF. It is well known that all fuzzy equivalences on a set U form a lattice where the meet is given by set intersection. It remains to show that intersection of any two separable binary fuzzy relations on U is a separable binary fuzzy relation on U. But this is obvious as for any two such relations R and S on U and any elements u and v in U we have $[R \cap S](u, v) \leq R(u, v)$, i.e. $R \cap S$ is separable by separability of R.

A question remains under which conditions there are also suprema of all pairs of fuzzy equalities, i.e. under which conditions the structure forms a lattice. General answer is that it depends on the properties of residuated lattice $\mathbf{L}$. This claim is the subject of the following propositions.

Lemma 2. *The transitive closure of any separable and symmetric binary $\mathbf{L}$-relation R on a finite set U with $|U| \geq 2$ is separable and symmetric binary $\mathbf{L}$-relation on U if and only if $\mathbf{L}$ is a residuated lattice with join-irreducible unit.*

PROOF. As R is separable and symmetric, there is no pair $u', v' \in U$ such that $u' \neq v'$ and $R(u', v') = 1$. By well known properties of transitive closure of fuzzy relation (see e.g. [1, 29]), we have that for any $u, v \in U$ the degree $[\mathrm{Tra}(R)](u, v)$ is equal to $\bigvee_{i=1}^{\infty} R^i(u, v)$. Moreover finiteness of U implies existence of n such that $[\mathrm{Tra}(R)](u, v) = \bigvee_{i=1}^{n} R^i(u, v)$. But this, together with nonexistence of pair of distinct u' and v' with $R(u', v') = 1$, implies $[\mathrm{Tra}(R)](u, v) = 1$ if and only if $\bigvee_{i=1}^{n} R^i(u, v) = 1$ while $R^i(u, v) < 1$ for each $i \in \{1, 2, \ldots, n\}$. Using Lemma A1, we get $[\mathrm{Tra}(R)](u, v) = 1$ if and only if L has a join-reducible unit.

That is there is some symmetric and separable $\mathbf{L}$-relation R on finite set U with inseparable transitive closure $\mathrm{Tra}(R)$ if and only if $\mathbf{L}$ has a join-reducible unit.

Remark 1. In case of infinite sets there may be a symmetric and separable $\mathbf{L}$-relation whose transitive closure is not separable even when considering $\mathbf{L}$ with join-irreducible unit. For example, let $\mathbf{L}$ be interval $[0, 1]$ equipped with some t-norm $\otimes$ and the corresponding residuum $\rightarrow$ and let $U = \{u, v\} \cup \{w_i | i \in \mathbb{N}\}$. Then, by putting $R(u, w_1) = R(w_1, u) = R(v, w_1) = R(w_1, v) = 0.9, R(u, w_2) = R(w_2, u) = R(v, w_2) = R(w_2, v) = 0.99, R(u, w_3) = R(w_3, u) = R(v, w_3) = R(w_3, v) = 0.999$, $\ldots$, we obtain a symmetric and separable relation R whose transitive closure $\mathrm{Tra}(R)$ contains $[\mathrm{Tra}(R)](u, v) = \bigvee_{i=1}^{\infty}(R(u, w_i) \otimes R(w_i, v)) = 1 = \bigvee_{i=1}^{\infty}(R(v, w_i) \otimes R(w_i, u)) = [\mathrm{Tra}(R)](v, u)$, i.e. which is not separable.

Lemma 3. *Let $\mathbf{L}$ be a residuated lattice with join-irreducible unit. Then for any finite set U and any pair of $\mathbf{L}$-equalities $\approx$, $\approx'$ on U, the supremum $\approx \vee \approx'$ in the set of all $\mathbf{L}$-equalities on U equipped with subsethood relation exists and is given by $\mathrm{Tra}(\approx \cup \approx')$.*

PROOF. First note that $\approx \cup \approx'$ is reflexive (resp. symmetric) by reflexivity (resp. symmetry) of both $\approx$ and $\approx'$. Moreover $\approx \cup \approx'$ is separable. Namely by symmetry and separability of both $\approx$ and $\approx'$, we have $u \approx v \neq 1$ and $u \approx' v \neq 1$; and by properties of $\mathbf{L}$ we have $u \approx v \neq 1$ and $u \approx' v \neq 1$ implies $u \approx v \vee u \approx' v \neq 1$ for any pair of distinct $u, v \in U$.

Therefore by Lemma 2, $\mathrm{Tra}(\approx \cup \approx')$ is reflexive, symmetric, transitive, and separable binary $\mathbf{L}$-relation, i.e. $\mathbf{L}$-equality. As the transitive closure $\mathrm{Tra}(R)$ is moreover the least transitive relation containing R for any binary $\mathbf{L}$-relation R on U, $\mathrm{Tra}(\approx \cup \approx')$ is the least such $\mathbf{L}$-equality on U. Which means that it is the supremum of $\approx$ and $\approx'$ in the ordered set of all $\mathbf{L}$-equalities on U.

Having suprema for all pairs of $\mathbf{L}$-equalities on U, we can easily obtain a supremum for any finite set of $\mathbf{L}$-equalities on U. Following theorem moreover shows that these suprema exist if and only if $\mathbf{L}$ has a join-irreducible unit.

Theorem 1. *Let U be a finite set with at least two elements. The set of all $\mathbf{L}$-equalities on U equipped with subsethood relation forms a lattice if and only if $\mathbf{L}$ has a join-irreducible unit.*

In case U has less than two elements, such structure is a one-element complete lattice.

PROOF. The forward implication derives from the following construction. Suppose that the structure of $\mathbf{L}$-equalities on U is a lattice. Then any pair of $\mathbf{L}$-equalities on U has a supremum in this lattice. In particular any pair of $\mathbf{L}$-equalities $\approx$, $\approx'$ on U, given as $u \approx v = v \approx u = a$, $u \approx' v = v \approx' u = b$ for some distinct u, v from U and distinct a, b from L different than 1, and $w \approx w = w \approx' w = 1$ for any $w \in U$, has a supremum. By Lemma 3, this supremum is $\mathbf{L}$-equality $\mathrm{Tra}(\approx \cup \approx')$, i.e. $1 > [\mathrm{Tra}(\approx \cup \approx')](u, v) \geq u \approx v \vee u \approx' v = a \vee b$. But as choice of a and b is arbitrary, we get $a \vee b < 1$ for any a, b in $L \setminus \{1\}$, i.e. 1 is join-irreducible in $\mathbf{L}$.

On the other hand if $\mathbf{L}$ has a join-irreducible unit then existence of supremum of any pair of $\mathbf{L}$-equalities on U follows from Lemma 3 and existence of infimum of such pair follows from Lemma 1.

The part for the sets with less than two elements is trivial as there is only one $\mathbf{L}$-equality on each such set.

To ensure that a lattice of all **L**-equalities on a finite set is a complete one, even stronger conditions must be imposed on **L**.

Theorem 2. *Let U be a finite set with at least two elements. The set of all* **L***-equalities on U equipped with subsethood relation forms a complete lattice if and only if* **L** *has unit irreducible by arbitrary joins, i.e. if and only if there is no set D of degrees from $L \setminus \{1\}$ with $\bigvee D = 1$.*

PROOF. The forward implication is shown by contraposition. If L is a complete residuated lattice where some such set D exists then we can create set E of **L**-equalities on any set U containing at least two distinct elements u and v in the following way. The set E contains one **L**-equality $\approx_d$ for each of $d \in D$ defined by $u \approx_d v = d$ and $w \approx_d w = 1$ for any $w \in U$. We get $[\bigvee E](u, v) = 1$, i.e. $\bigvee E$ is not a separable relation. That is if **L** is not a complete residuated lattice with a unit irreducible by arbitrary joins then the set of all fuzzy equalities on U is not a complete lattice.

The converse implication follows from the fact that the supremum s of any set D' of degrees from $L \setminus \{1\}$ exists and we have $s < 1$. As no pair of distinct elements is fully equal in any **L**-equality, it is not fully equal in the union of any set of **L**-equalities (by $s < 1$ for any D') and it is not fully equal in the transitive closure of the union (by the same argument). Therefore the transitive closure of the union is reflexive, symmetric, transitive, and separable **L**-relation on U, that is a fuzzy equality. Moreover, as the transitive closure is the least transitive relation containing the union of original **L**-equalities, it is also the least **L**-equality containing all the original **L**-equalities, that is their supremum.

Remark 2. Obviously **2** is one of residuated lattices where the unit is join-irreducible. By putting **L**=**2** we obtain a rather trivial situation with only one possible **2**-equality – the identity. Therefore the set of all **2**–equalities on any U is just a singleton, that is, if equipped with the subsethood relation, trivial complete lattice.

For further details on fuzzy equivalences and related concepts we refer to the books [1, 29].

2.2. Fuzzy ordering

In this work, we define fuzzy order on a set equipped with fuzzy equality $\langle U, \approx\rangle$ in the spirit of definitions by Belohlavek, Bodenhofer, Blanchard, and Höhle [2, 9, 24, 25].

Definition 3. A *fuzzy order (***L***-order) on a set U equipped with a fuzzy equality relation* $\approx$ is a binary fuzzy relation $\lesssim$ on U, i.e. $\lesssim : U \times U \to L$, compatible with $\approx$ and satisfying

$$\begin{aligned} u \lesssim u &= 1, && \text{(reflexivity)} \\ (u \lesssim v) \otimes (v \lesssim w) &\leq u \lesssim w, && \text{(transitivity)} \\ (u \lesssim v) \otimes (v \lesssim u) &\leq u \approx v, && \text{(antisymmetry)} \end{aligned}$$

for each $u, v, w \in U$.

One of the consequences of [4, 5] is a possibility to characterize fuzzy orders by means of transitivity of the given relation and properties of its intersection with the dual relation only.

Lemma 4. *A transitive binary fuzzy relation R on U is a fuzzy order on U if and only if $R \otimes R^{-1}$ is reflexive and separable binary fuzzy relation on U. That is in such case $R \otimes R^{-1}$ is a fuzzy equality and R is fuzzy order on $\langle U, R \otimes R^{-1}\rangle$.*

PROOF. Follows immediately from Remark 6 of [5].

This observation lets us pinpoint one fuzzy equality, we are relating the order to, and it moreover emphasizes the relationship between these two relations. We often leverage it to simplify the proofs in the following sections. Note that using $\otimes$ as an intersection interpretation is a matter of taste. We might as well use $\wedge$ or even some more general conjunction-like operation in its place (see [5]).

Note 1. (a) It is easy to show that in case $\mathbf{L}$=$\mathbf{2}$ we obtain the classical notion of order, i.e. reflexive, transitive, and antisymmetric relation.

(b) Moreover as was shown in [5, Theorems 1 and 2], binary fuzzy relation $\lesssim$ is a fuzzy order on $\langle U, \approx\rangle$ according to our definition if and only if, for some fuzzy equality $\approx'$ such that $(\lesssim \otimes \lesssim^{-1}) \leq \approx' \leq (\lesssim \wedge \lesssim^{-1})$, it is a fuzzy order on $\langle U, \approx'\rangle$ according to the definition[2] due to Höhle, Blanchard, and Bodenhofer [9, 25] if and only if it is a fuzzy order on $\langle U, \lesssim \wedge \lesssim^{-1}\rangle$ according to the definition due to Höhle and Belohlavek [2, 24] if and only if it is a fuzzy order on U according to the definition due to Blanchard and Fan [7, 17].

3. Completeness and linearity of binary fuzzy relation

There are various notions of completeness and linearity used in the theory of binary fuzzy relations on a set. In this work, we are interested in linear fuzzy orderings, i.e. we focus on completeness of a fuzzy order relation in a sense of arbitrary two elements in a set being fully comparable. Even in this sense, there are multiple approaches to the concept of linearity in the literature. The used naming conventions are mild variation of those in [11].[3]

Definition 4. Let R be a binary $\mathbf{L}$-relation on a set U. We define the following well known properties of R:

$$R(u,v) \vee R(v,u) = 1, \qquad \text{(strong completeness)}$$
$$R(u,v) \oplus R(v,u) = 1, \qquad \text{($\oplus$-completeness)}$$
$$\neg R(u,v) \leq R(v,u), \qquad \text{($\otimes$-linearity)}$$
$$u \neq v \text{ implies } R(u,v) > 0 \text{ or } R(v,u) > 0, \qquad \text{(Zadeh's linearity)}$$

for any $u, v \in U$.

One may see that each of these properties is a generalization of some property equivalent to the classical completeness of a binary relation on a set into the setting of fuzzy logic. As it is often the case in the setting of fuzzy logic, these generalizations of equivalent properties of classical logic are not equivalent anymore.

The most widespread of these properties is the strong completeness, which is a straightforward generalization of the classical linearity. The option of $\oplus$-completeness represents same idea using a strong conjunction, if it is available in $\mathbf{L}$. The $\otimes$-linearity was coined in [25] and generalizes a different property equivalent to the completeness in the classical case, namely $\neg R(u,v) \Rightarrow R(v,u)$ for each $u, v \in U$. Zadeh's linearity is then the original definition of linearity of fuzzy orders which appeared already in Zadeh's seminal paper on fuzzy equivalences and fuzzy orders [33]. There are still other versions of linearity, e.g. the one in classic book [27], which are mild alterations of those described above.

Interestingly, the most recent study [11] on linearity axioms for fuzzy orders shows that, when one is concerned with linear extensions of fuzzy orders, the $\otimes$-linearity may be the preferred option because it preserves the most of desired properties of the classical extension process. See the last section of Appendix for more details.

Usually, the works utilizing the notion of linearity of fuzzy orders use only linear residuated lattices, especially the ones given by (left) continuous t-norms. As we use general complete residuated lattices, we need to discuss another aspect of linearity. Namely, the expected meaning of the axiom. Take for example the strong completeness in some residuated lattice $\mathbf{L}$ with join-reducible unit. Then, for some $a, b \in L \setminus \{1\}$ such that $a \vee b = 1$, even relation R on $U = \{u, v\}$ where $R(u,u) = R(v,v) = 1$, $R(u,v) = a$, and $R(v,u) = b$ is strong complete, i.e. linear in the given setting. This situation might be considered as unnatural – R is a linear ordering where no element of pair u, v is fully above the other one. Therefore we define yet another concept of linearity, which assures that such situation does not arise.

[2] This holds only with the provision that we demand $\approx$ to be a fuzzy equality, not mere fuzzy equivalence. We refer once more to [4] for more details.

[3] In fact Bodenhofer and Klawonn use names T-linearity and S-completeness for $\otimes$-linearity and $\oplus$-completeness in [11] as they denote $\otimes$ and $\oplus$ by T and S respectively.

Definition 5. Binary fuzzy relation R on a set U is *crisp linear* if

$$R(u, v) = 1 \text{ or } R(v, u) = 1 \qquad \text{(crisp linearity)}$$

holds for every $u, v \in U$.

As every crisp linear binary fuzzy relation is obviously strong complete and every strong complete binary fuzzy relation is $\oplus$-complete, $\otimes$-linear, and satisfies Zadeh's linearity, the existence of crisp linear extension of a relation R implies the existence of extension of R fulfilling all the other mentioned linearity conditions. Note also that in case of residuated lattice with join-irreducible unit, in particular in any residuated lattice on $[0, 1]$, a binary fuzzy relation is crisp linear if and only if it is strong complete.[4] In the rest of the paper, we examine when a fuzzy order extension into crisp linear fuzzy order exists and some derived notions for these cases.

4. Extensions and Szpilrajn-like theorem for fuzzy orders

We are finally ready to discuss the existence of a linear extension of any fuzzy order. We start by recalling the definition of a fuzzy order extension and related concepts.

Definition 6. Let R, S, and $\lesssim$ be binary fuzzy relations on U:

- We call S an *extension of* R if $R \subseteq S$. If $R \subset S$ we call S a *proper extension of* R.
- If $\lesssim$ is a fuzzy order on a set with fuzzy equality $\langle U, \approx \rangle$, we call a fuzzy order $\lesssim'$ on a set with fuzzy equality $\langle U, \approx' \rangle$ a *fuzzy order extension of* $\lesssim$ if $\lesssim'$ is an extension of $\lesssim$ and $\approx'$ is an extension of $\approx$.

In the rest of the section, we are going to show that, given $\mathbf{L}$ has a join-irreducible unit, each fuzzy order may be extended into a crisp linear fuzzy order; i.e. a variant of Szpilrajn's extension theorem generalized to setting of fuzzy logic and crisp linearity.

The core idea differentiating our approach from previous studies is considering also the induced fuzzy equality in the extension process. It may be seen as further extension of reflections on the role of fuzzy equality in the definition of fuzzy order as presented in [5]. There are two main reasons why we do so.

First, fixing the fuzzy equality is, in our opinion, point of view which comes from Boolean setting where there is only one equality and therefore no reason to think about its modifications together with other entities in the given situation. We think that there is no general justification of the same view when there is many fuzzy equalities available on the given universe. That is a possibility of strengthening or weakening the given equality may be taken as new and advantageous property of fuzzy setting which is degenerated in Boolean case.

Second reason has same root cause but immediate practical consequences: Fixing the fuzzy equality in the beginning of an extension process limits the situation by a great deal. In fact the main reason, why the results on linear extensions of fuzzy orders are quite pessimistic so far [11], is that a fuzzy equality[5] induced by a resulting linear fuzzy order has to obey same limits as the one induced by an initial fuzzy order.

We start by considering only finite universe U, because as byproduct we uncover interesting connection between possibility of extending $\mathbf{L}$-order on U and structure of $\mathbf{L}$-equalities on U. The proof for finite case utilizes the classical, i.e. formulated in classical logic, version of Szpilrajn's theorem to obtain sought extension. Then we move our attention to a general, possibly infinite, case where we use a different approach. We prove slightly generalized version of the Szpilrajn's extension theorem in the setting of fuzzy logic and the expected form of the extension theorem for fuzzy orders becomes its consequence.

[4] The up to date most detailed study of linearity axioms for fuzzy orderings [11] use the setting of left-continuous t-norms on the interval $[0, 1]$, thus some of our results may be easily related to the results obtained there.

[5] Fuzzy equivalence in the case of [11]. The idea remains the same, though.

Note 2. The proofs in next two sections are rather technical and combine standard techniques used in the setting of residuated lattices and common ideas used in the classical order theory. To avoid burdening the reader with immediate technical details, we occasionally provide a concise explanation of the proof idea, focusing only on most important or nonstandard thoughts. This is because these details do not convey anything new or inventive; thus delving into them immediately may be of little benefit to the reader. A full proof always follows right after such sketch.

4.1. The finite case

The reason, why the following approach may be used only in the finite case, is a richer structure of fuzzy equalities in comparison with the classical setting. Namely, each fuzzy equality is a fuzzy order, therefore many properties of the structure of all fuzzy equalities on the given set give rise to similar properties in the structure of all fuzzy orders on same set. Now, the structure of fuzzy equalities forbids us from leveraging Zorn's lemma because, in general, not every chain of fuzzy equalities has an upper bound in the set of all fuzzy equalities on a finite set ordered by set inclusion (see Example 1). This problem translates to the structure of all fuzzy orders on the same set in a straightforward way. Therefore, as Zorn's lemma is a cornerstone of transition to infinite sets in a classical proof of standard Szpilrajn's theorem, we can not generalize this idea to setting of fuzzy logic in a straightforward manner.

Example 1. Let $\mathbf{L}$ be the Łukasiewicz structure on $[0,1]$ and let $U = \{u, v\}$. For each $a \in [0,1)$ define $\eqsim_a$ on U by: $u \eqsim_a u = v \eqsim_a v = 1$ and $u \eqsim_a v = v \eqsim_a u = a$. Obviously each $\eqsim_a$ is an $\mathbf{L}$-equality on U and the set of all such $\eqsim_a$ is a chain if ordered by set inclusion. Moreover $[\bigvee_{a\in[0,1)} \eqsim_a](u,v) = [\bigvee_{a\in[0,1)} \eqsim_a](v,u) = 1$, i.e. $\bigvee_{a\in[0,1)} \eqsim_a$ is not separable. Therefore we have chain of $\mathbf{L}$-equalities on a finite set U, which does not have an upper bound in the set of all fuzzy equalities on a finite set ordered by set inclusion. The same chain is one of examples of chain of $\mathbf{L}$-orders without an upper bound in the set of all $\mathbf{L}$-orders on U.

Lemma 5. *A fuzzy preorder $\prec$ on a set U is a fuzzy order on U w. r. t. some fuzzy equality if and only if its 1-cut is antisymmetric crisp binary relation.*

Proof. Follows from [5, Lemma 9 – points (b1) and (b3)] and a fact that a binary fuzzy relation is crisp antisymmetric if and only if its 1-cut is an antisymmetric crisp binary relation.

Theorem 3. *A residuated lattice $\mathbf{L}$ has a join-irreducible unit if and only if for every finite set equipped with $\mathbf{L}$-equality $\langle U, \eqsim\rangle$ and for each $\mathbf{L}$-order $\precsim$ on $\langle U, \eqsim\rangle$ there is a crisp linear fuzzy order $\precsim_S$ on $\langle U, \eqsim_S\rangle$ such that $\precsim_S$ is a fuzzy order extension of $\precsim$.*

Proof idea. The proof is mostly technical. For the forward implication we construct a sought extension by leveraging the classical Szpilrajn's theorem and verify its properties by Lemma 4 and standard techniques. The converse implication is then shown by contraposition, i.e. by construction of counter example for residuated lattices with reducible unit. Again, properties of the counter example are verified by standard manipulations with fuzzy relations.

Proof. We prove forward implication by a construction of such extension.

Let $\leq$ be the 1-cut of $\precsim$. Then $\leq$ is a crisp order, therefore by classical Szpilrajn's theorem we may extend it into a linear order $\leq_C$. Now put $\precsim_C \;=\; \leq_C \cup \precsim$. The relation $\precsim_C$ is obviously reflexive and crisp linear.

The transitive closure $\mathrm{Tra}(\precsim_C)$ of $\precsim_C$ is reflexive (because $\precsim_C \subseteq \mathrm{Tra}(\precsim_C)$), crisp linear (same reason), and transitive (it is transitive closure of $\precsim_C$). That is, $\mathrm{Tra}(\precsim_C)$ is a crisp linear $\mathbf{L}$-preorder on U.

Now by Lemma 4, it is enough to show that $\eqsim_C = (\mathrm{Tra}(\precsim_C) \otimes \mathrm{Tra}(\precsim_C)^{-1})$ is an $\mathbf{L}$-equality on U, i.e., by same Lemma, that $\eqsim_C$ is reflexive and separable. Reflexivity is immediate. Separability follows from following observation.

Leveraging that $\mathbf{L}$ has join-irreducible unit and U is finite, we show for any $u, v \in U$ that $[\mathrm{Tra}(\precsim_C)](u,v) = 1$ if and only if $u \leq_C v = 1$. More precisely, if we choose any pair u, v then we obtain $[\mathrm{Tra}(\precsim_C)](u,v) = \bigvee_{w_i\in U}(u \precsim_C w_1 \otimes w_1 \precsim_C w_2 \otimes \cdots \otimes w_n \precsim_C v)$. That is $[\mathrm{Tra}(\precsim_C)](u,v) = 1$ if and only if

- either $u \lesssim_C w_1 = w_1 \lesssim_C w_2 = \cdots = w_n \lesssim_C v = 1$ for some $w_1, \ldots, w_n \in U$,
- or there is a finite set of degrees $a_j \in L \setminus \{1\} (j = 1, \ldots, m)$ such that $a_j = u \lesssim_C w_1^j \otimes w_1^j \lesssim_C w_2^j \otimes \cdots \otimes w_n^j \lesssim_C v$ and $\bigvee_{j=1,\ldots,m} \{a_j\} = 1$.

The first case means that already $u \leq_C w = 1$ by transitivity of $\leq_C$ and the construction of $\lesssim_C$. While the second case is impossible by join-irreducibility of unit in **L**.

That is $[\mathrm{Tra}(\lesssim_C)](u, v) = 1$ if and only if $u \lesssim_C v = 1$ if and only if $u \leq_C v = 1$. The separability of $\approx_C$ therefore follows from the antisymmetry of $\leq_C$. This finishes the proof of the forward direction.

Now, we show the converse implication by contraposition, i.e. by construction of a counterexample for any residuated lattice **L**' with join-reducible unit. That is suppose **L**' contains two elements a, b different from 1 such that $a \vee b = 1$, let $U = \{u_1, u_2, u_3\}$, and let $\lesssim$ be a binary **L**'-relation on U given by the following table:

$\lesssim$	u_1	u_2	u_3
u_1	1	b	1
u_2	a	1	1
u_3	a	b	1

By routine computation, we can see that $\lesssim$ is transitive and that its induced relation $\approx = (\lesssim \otimes \lesssim^{-1})$ is reflexive and separable, i.e. $\lesssim$ is an **L**'-order on $\langle U, \approx \rangle$ by Lemma 4.

Recall that for any extension S of binary fuzzy relation R on U and any $u, v \in U$ we have $R(u, v) \leq S(u, v)$. That is for any transitive extension $\lesssim'$ of $\lesssim$ where $u_1 \lesssim' u_2 = 1$ we have $(u_3 \lesssim' u_2) \geq (u_3 \lesssim u_2 \vee (u_3 \lesssim' u_1 \otimes u_1 \lesssim' u_2)) \geq (u_3 \lesssim u_2 \vee (u_3 \lesssim u_1 \otimes u_1 \lesssim' u_2)) = b \vee a = 1$ by the fact above, transitivity of $\lesssim'$, and properties of **L**'. On the other hand, for any transitive extension $\lesssim''$ of $\lesssim$ with $u_2 \lesssim'' u_1 = 1$ we have $(u_3 \lesssim'' u_1) \geq (u_3 \lesssim u_1 \vee (u_3 \lesssim'' u_2 \otimes u_2 \lesssim'' u_1)) \geq (u_3 \lesssim u_1 \vee (u_3 \lesssim u_2 \otimes u_2 \lesssim'' u_1)) = a \vee b = 1$ by analogous arguments. Therefore neither $\lesssim'$ nor $\lesssim''$ is an **L**'-order on U by Lemma 4 and the fact that none of $\lesssim' \otimes \lesssim'^{-1}$ and $\lesssim'' \otimes \lesssim''^{-1}$ is an **L**'-equality on U.

In conclusion, there is no **L**'-order extension of $\lesssim$ with u and v being fully comparable, in particular there is no crisp linear **L**'-order extension of $\lesssim$.

In addition, one may note that the conditions imposed on **L** are same for existence of a crisp linear **L**-order extension of an arbitrary **L**-order on a finite set equipped with **L**-equality $\langle U, \approx \rangle$ and for structure of all **L**-equalities on a finite set U to be a lattice. This is not a coincidence as may be seen from proofs of Theorems 1 and 3, which both leverage same idea – given **L** has unit reducible by a and b, two elements can not be equal both in degrees a and b and remain separated at the same time.

We also note that join-reducibility of unit by infinite number of degrees is not a problem here as we work only with finite universe in this section. Moreover, as we will see in the subsequent part, thanks to the properties of fuzzy orders, it is not an issue even in the case of arbitrary universe.

Remark 3. If one tries to implement similar construction in the setting, where an underlying similarity is interpreted as a general fuzzy equivalence, it becomes rather trivial. The reason is separability of an induced relation being the only limiting factor here. In such case, every fuzzy order has a linearization fulfilling any reasonable property of completeness as every fuzzy order may be extended into full relation on the given set. In the spirit of [4, 5], we consider it to be another manifestation of a fuzzy equivalences being an inappropriate choice for the interpretation of an underlying similarity.

4.2. Generalized Szpilrajn's theorem and the infinite case

As already mentioned above, in general case we are going to show a somehow stronger proposition and then get the expected form of Szpilrajn-like theorem as its consequence. We start by showing that if we increase a degree of comparability of some u and v in some fuzzy preorder while preserving transitivity, the comparability degree of any $w \in U$ to u (resp. v to any $w \in U$) does not change. Next, we utilize this observation to show that for any pair $m, n \in U$ we can extend any fuzzy preorder into its crisp linear extension where the comparability degree of m to n does not increase. Note that the condition of keeping the comparability degree of m to n unchanged is of importance later, in Section 5, where an intersection representation of fuzzy orders is discussed.

Lemma 6. *Let U be an arbitrary set, $u, v \in U$, R a transitive binary fuzzy relation on U, and $R' = R \cup \{{}^1/\langle u, v\rangle\}$. Then for any $w \in U$ we have*

1. $[\mathit{Tra}(R')](v, w) = R(v, w)$
2. $[\mathit{Tra}(R')](w, u) = R(w, u)$

Proof idea. The proof is again mostly technical. We leverage that input relation is transitive and that there is only one degree of comparability between elements increased by external means. Thus the sole source of changes during the transitive closure is this incremented degree and the transitive closure may be constructed in just a few steps.

Proof. From the construction and properties of $\mathrm{Tra}(R')$, we have $[\mathrm{Tra}(R')](v, w) = \bigvee_{y,x_1,\dots\in U} R'(v, x_1) \otimes \cdots \otimes R'(y, w)$ for any $v, w \in U$. We show that for each such sequence $R'(v, x_1), \dots, R'(y, w)$ the degree $R'(v, x_1) \otimes \cdots \otimes R'(y, w)$ is lower or equal to $R(v, w)$. There are two possibilities:

- Either $R'(v, x_1), \dots, R'(y, w)$ does not contain $R'(u, v)$, then it is same as the sequence $R(v, x_1), \dots, R(y, w)$ and $R'(v, x_1) \otimes \cdots \otimes R'(y, w) = R(v, x_1) \otimes \cdots \otimes R(y, w) \leq R(v, w)$ by transitivity of R;
- Or $R'(v, x_1), \dots, R'(y, w)$ contains $R'(u, v)$ at least once. Take suffix of the sequence starting after last such occurrence of $R'(u, v)$. This suffix has to contain sequence $R'(v, y_1), \dots, R'(y, w)$ for some $y_1, \dots, y \in U$ for which we have $R'(v, x_1) \otimes \cdots \otimes R'(y, w) \leq R'(v, y_1) \otimes \cdots \otimes R'(y, w) = R(v, y_1) \otimes \cdots \otimes R(y, w) \leq R(v, w)$ by transitivity of R and construction of R'.

That is $[\mathrm{Tra}(R')](v, w) = \bigvee_{y,x_1,\dots\in U} R'(v, x_1) \otimes \cdots \otimes R'(y, w) \leq R(v, w)$. Moreover $R \subseteq \mathrm{Tra}(R')$ by the construction of $\mathrm{Tra}(R')$. Therefore $[\mathrm{Tra}(R')](v, w) = R(v, w)$.

The proof of the second equality, i.e. $[\mathrm{Tra}(R')](w, u) = R(w, u)$, is analogous.

Lemma 7. *For any fuzzy preorder $\prec$ on arbitrary set U and any pair $m, n \in U$ at least one of the following propositions holds for any pair $u, v \in U$:*

1. *There is a fuzzy preorder $\prec'$ extending $\prec$ such that $u \prec' v = 1$ and $m \prec' n = m \prec n$.*
2. *There is a fuzzy preorder $\prec''$ extending $\prec$ such that $v \prec'' u = 1$ and $m \prec'' n = m \prec n$.*

Proof idea. This is another technical proof. The core idea is showing that if one the propositions is not valid then the other one is. Moreover, we utilize Lemma 6 and transitivity of input relation to analyze how increasing comparability of u and v affects comparability of m and n.

Proof. Take any $m, n, u, v \in U$. If $u \prec v = 1$ (resp. $v \prec u = 1$) then it is enough to put $\prec' = \prec$ (resp. $\prec'' = \prec$).

Otherwise we show that $\neg 1 \Rightarrow 2$ and $\neg 2 \Rightarrow 1$.

Let $\prec' = \mathrm{Tra}(\prec \cup \{{}^1/\langle u, v\rangle\})$ and $\prec'' = \mathrm{Tra}(\prec \cup \{{}^1/\langle v, u\rangle\})$. We show that these relations are the extensions described in 1 and 2 respectively. That is we need to show that $m \prec' n > m \prec n$ implies $m \prec'' n = m \prec n$ and that $m \prec'' n > m \prec n$ implies $m \prec' n = m \prec n$.

Suppose $\neg 1$, that is $m \prec' n > m \prec n$. Then $m \prec n < m \prec' n = m \prec u \otimes v \prec n$ by Lemma 6, transitivity of $\prec$, and construction of $\prec'$. As $\prec$ is transitive we also have $m \prec n \geq m \prec v \otimes v \prec n$ and $m \prec n \geq m \prec u \otimes u \prec n$. Putting these facts together, we obtain the following inequalities:

$$m \prec u \otimes u \prec n \leq m \prec n < m \prec u \otimes v \prec n,$$

$$m \prec v \otimes v \prec n \leq m \prec n < m \prec u \otimes v \prec n.$$

From monotony of $\otimes$, we get $u \prec n < v \prec n$ and $m \prec v < m \prec u$.

By transitivity of $\prec$, we infer $m \prec n \geq m \prec v \otimes v \prec n > m \prec v \otimes u \prec n$, i.e. $m \prec n > m \prec v \otimes u \prec n$. This implies that putting $v \prec'' u = 1$ does not break $m \prec'' n = m \prec n$ (using

$m \prec'' v = m \prec v$ and $u \prec'' n = u \prec n$ by Lemma 6 and by transitive closure being the least transitive relation containing original relation). In particular, $\prec''$ is fuzzy preorder extending $\prec$ with $m \prec'' n = m \prec n$, i.e. 2 holds.

The implication $\neg 2 \Rightarrow 1$ is shown dually.

As a straightforward consequence of these lemmas, we get the following theorem.

Theorem 4. *Let $\prec$ be a fuzzy preorder on an arbitrary set U. For any pair $u, v \in U$ there is a crisp linear fuzzy preorder $\prec'$ extending $\prec$ such that $u \prec' v = u \prec v$.*

Proof. Suppose the proposition is not valid. Then for some set U, some **L**-preorder $\prec$ on U, and some elements u, v in U there are at least two elements u', v' in U such that neither $u' \prec v'$ nor $v' \prec u'$ can be increased to 1 while maintaining transitivity and keeping $u \prec v = u \prec' v$. But this contradicts the previous lemma.

We now have quiet strong results on extensions of fuzzy preorders. The obvious question is how these extensions behave, when the initial fuzzy preorder is fuzzy order on a set. The following lemma gives us the answer and it is the last step needed for the proof of the final version of Szpilrajn–like theorem for fuzzy orders and crisp linearity.

Lemma 8. *Let **L** be a residuated lattice with join-irreducible unit. For any **L**-order $\lesssim$ on an arbitrary set with **L**-equality $\langle U, \approx \rangle$ and any pair $m, n \in U$ at least one of the following propositions holds for any pair $u, v \in U$:*

1. *There is an **L**-order extension $\lesssim'$ of $\lesssim$ on U such that $u \lesssim' v = 1$ and $m \lesssim' n = m \lesssim n$.*
2. *There is an **L**-order extension $\lesssim''$ of $\lesssim$ on U such that $v \lesssim'' u = 1$ and $m \lesssim'' n = m \lesssim n$.*

Proof idea. This is the last rather technical proof with mostly standard techniques utilized. We leverage Lemma 7 and prove that if we start with fuzzy order, we end with fuzzy order extension. To verify that the result is indeed fuzzy order, we combine previously obtained results, mainly Lemma 4 and 6, with properties of initial fuzzy order.

Proof. By Lemma 7, we know that $\lesssim$ may be extended into at least one of **L**-preorders $\lesssim'$ and $\lesssim''$ such that $u \lesssim' v = 1$ and $m \lesssim' n = m \lesssim n$ or $u \lesssim'' v = 1$ and $m \lesssim'' n = m \lesssim n$. We moreover show that in case of $\lesssim$ being an **L**-order on U, at least one of these extensions is an **L**-order on U. For clarity, we denote $\lesssim' \otimes \lesssim'^{-1}$ by $\approx'$ and $\lesssim'' \otimes \lesssim''^{-1}$ by $\approx''$.

We start by showing that if $\lesssim'$ is not an **L**-order on $\langle U, \approx' \rangle$ then $\lesssim''$ is an **L**-order on $\langle U, \approx'' \rangle$. Suppose that $\lesssim'$ is not an **L**-order on $\langle U, \approx \rangle$. We know that it is reflexive and transitive as it is an **L**-preorder, i.e., by Lemma 4, $\approx'$ is not separable, which means there is a pair $u', v' \in U$ such that $u' \lesssim' v' = v' \lesssim' u' = 1$.

There are three possibilities for relationship of such u' and v' in the initial order $\lesssim$:

(a) $u' \lesssim v' < 1$ and $v' \lesssim u' < 1$
(b) $u' \lesssim v' = 1$ and $v' \lesssim u' < 1$
(c) $u' \lesssim v' < 1$ and $v' \lesssim u' = 1$

In the case of (a), $u' \lesssim' v' = v' \lesssim' u' = 1$ implies $v' \lesssim u = v \lesssim u' = u' \lesssim u = v \lesssim v' = 1$ (by Lemma 6, transitivity of $\lesssim$, and the construction of $\lesssim'$). But then from $v \lesssim v' = v' \lesssim u = 1$ we get $v \lesssim u = 1$ by transitivity of $\lesssim$. This means that $\lesssim'' = \lesssim$, i.e. the proposition 2 holds trivially.

The cases of (b) and (c) are dual, we therefore show just the case of (b). By the properties of transitive closure, the construction of $\lesssim'$, and Lemma 6, we know that $1 = v' \lesssim' u' = v' \lesssim u' \vee (v' \lesssim u \otimes u \lesssim' v \otimes v \lesssim u')$. By join-irreducibility of 1 in **L** and assumption of $v' \lesssim u' < 1$, we then have $v' \lesssim u \otimes u \lesssim' v \otimes v \lesssim u' = 1$. By properties of $\otimes$ and assumption of $u' \lesssim v' = 1$, we therefore get $v' \lesssim u = v \lesssim u' = 1$ and $(v \lesssim u) \geq (v \lesssim u' \otimes u' \lesssim v' \otimes v' \lesssim u) = 1$. This again means that $\lesssim'' = \lesssim$, i.e. the proposition 2 holds trivially.

The other implication, i.e. if $\lesssim''$ is not **L**-order on $\langle U, \approx'' \rangle$ then $\lesssim'$ is **L**-order on $\langle U, \approx' \rangle$, is shown dually.

Putting both the implications together finishes the proof.

Theorem 5. *A residuated lattice* $\mathbf{L}$ *has a join-irreducible unit if and only if for every set equipped with* $\mathbf{L}$*-equality* $\langle U, \approx\rangle$, *for any* $u, v \in U$, *and for each* $\mathbf{L}$*-order* $\lesssim$ *on* $\langle U, \approx\rangle$ *there is a crisp linear fuzzy order extension* $\lesssim'$ *of* $\lesssim$ *on* U *such that* $u \lesssim' v = u \lesssim v$.

Proof. The proof of the first implication is analogous to the proof of Theorem 4. Suppose $\mathbf{L}$ has a join-irreducible unit and that there exists some $\mathbf{L}$-order on some set which can not be extended into crisp linear $\mathbf{L}$-order with the given properties. Then it contradicts Lemma 8.

The other implication is shown by contraposition, i.e. by constructing a counterexample for any residuated lattice $\mathbf{L}$' with a join-reducible unit. One such counterexample is the one from the proof of the finite case (see the proof of the Theorem 3).

As noted sooner, the requirement of fixed degree of comparability of some pair will be crucial in the next section, where we consider representation of a fuzzy order by an intersection of its linear extensions. But if we omit it now, we obtain a straightforward generalization of classical Szpilrajn's extension theorem to the setting of residuated lattices and crisp linearity. Note that assumption of join-irreducibility of the residuated lattice's unit can not be dropped for crisp linearity as may be easily seen from the proof of the theorem above.

Corollary 6 (Extension theorem for crisp linearity). *Let* $\mathbf{L}$ *be a residuated lattice with join-irreducible unit. For any set* U *and any* $\mathbf{L}$*-order* $\lesssim$ *on* U *there is a crisp linear* $\mathbf{L}$*-order extension of* $\lesssim$.

Proof. Just pick one element $w \in U$ for both u and v. The extension obtained in Theorem 5 is then one of possible crisp linear $\mathbf{L}$-order extensions of $\lesssim$.

Note 3. It is immediate now, that in our setting extending fuzzy order implies extending the underlying fuzzy equality. This is crucial difference between our approach and approach presented in [11] (see beginning of Section 4). We will return to this observation once more in the last part of the work.

As the setting of (left) continuous t-norms on $[0, 1]$ is the most common setting for applications of fuzzy logic, we present the theorem for this setting as a corollary of the results of previous sections. We state it not just for crisp linearity, but also for all the other mentioned linearity conditions.

Corollary 7 (Extension theorem for residuated lattices on $[0, 1]$**).** *Let* $\mathbf{L}$ *be a residuated lattice on the interval* $[0, 1]$. *For any set* U *and any* $\mathbf{L}$*-order* $\lesssim$ *on* U *there is a crisp linear (resp. strong complete,* $\oplus$*-complete,* $\otimes$*-linear, Zadeh's linearity satisfying)* $\mathbf{L}$*-order extension of* $\lesssim$.

Proof. Follows straight from the relationships between the linearity axioms in the case of linear residuated lattices (see Section 3).

Note 4. One may be tempted to use linear completeness of MTL [16, 26] to generalize the results to all prelinear residuated lattices. Doing so would obviously contradict one direction in Theorem 5 above. Upon closer examination, it turns out that this is not possible when working with fuzzy orders in our sense – i.e. defined on a set with fuzzy equality, as a transitive relation inducing a fuzzy equality, or using crisp antisymmetry (see [4, 5]) – because the properties corresponding to such definitions of fuzzy order are not expressed in the language of first order MTL.

In fact our results show that the concept of such fuzzy order itself is not expressible in first order MTL, i.e. there is no definition of this concept in the given language. More precisely, the problem is the quality described in the property of separability of fuzzy equality – if a definition w. r. t. fuzzy equality is used – or property of antisymmetry itself – if crisp antisymmetry is used (cf. [4, 5]). Because if one is able to express any of these properties in first order MTL then, by linear completeness of MTL, the propositions above are valid in every prelinear residuated lattice. But they are not – any prelinear, not linear residuated lattice $\mathbf{L}$ with join-reducible unit, e.g. Heyting algebra on $L = \{0, a, b, 1\}$ with $0 \leq a \leq 1$, $0 \leq b \leq 1$ and $x \leq x$ for each $x \in L$, can be used as counterexample by the same construction as in proofs above.

We conclude this section by Example 2, which shows natural fuzzy orders without crisp linear extensions in the setting of residuated lattices with join-reducible unit.

Example 2. It is well known [1, 9, 25] that for any complete residuated lattice, the function $\rightarrow$ is a fuzzy order on the set L of truth degrees equipped with the fuzzy equality induced by $\leftrightarrow$ on L. Such fuzzy order is moreover isomorphic to L^U for a singleton $U = \{u\}$ where $\rightarrow$ is lifted to $\subsetneq$ on L^U and $\leftrightarrow$ becomes $\approx$ on L^U. That is, from one point of view, this fuzzy order is a generalization of an important order induced by truth function of implication known from classical logic, and from another point of view, it is a generalization of classical powerset ordered by set inclusion.

Let $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ be any residuated lattice with unit join-reducible by $a, b \in L \setminus \{1\}$, i.e. where $a \vee b = 1$. Such residuated lattices exist, e.g. Heyting algebra from the Note 4 above. Now in case of $\mathbf{L}$, neither of the $\mathbf{L}$-orders described above can be extended into crisp linear fuzzy order unless two elements of L (resp. L^U), namely a and b (resp. $\{^a/u\}$ and $\{^b/u\}$), are factorized into one.

5. Intersection representation of fuzzy orders

Another important property of ordering relations in the Boolean case is an intersection representation of any order in the spirit of [15, Theorem 2.32]. See Theorem A2 in Appendix for classical version of this theorem. Utilizing the generalized version of Szpilrajn's theorem from previous section, we obtain a similar intersection representation of fuzzy orders in a straightforward manner.

Theorem 8. *A residuated lattice* $\mathbf{L}$ *has a join-irreducible unit if and only if for every set* U *equipped with* $\mathbf{L}$*-equality* $\approx$ *and every* $\mathbf{L}$*-order* $\lesssim$ *on* $\langle U, \approx\rangle$*, there is a set* $Ext(\lesssim)$ *of crisp linear* $\mathbf{L}$*-order extensions of* $\lesssim$ *such that* $[\bigcap Ext(\lesssim)](u, v) = u \lesssim v$ *for each* $u, v \in U$.

PROOF. For forward implication, we have that for every pair $u, v \in U$ there is a crisp linear $\mathbf{L}$-order extension $\lesssim_{uv}$ such that $u \lesssim_{uv} v = u \lesssim v$ by Theorem 5. If we put $Ext(\lesssim) = \{\lesssim_{uv} | u, v \in U\}$ and $\lesssim' = \bigcap Ext(\lesssim)$ then we get for all $u, v \in U : u \lesssim' v \leq u \lesssim v$ from $\lesssim_{uv} \in Ext(\lesssim)$ while we also have $u \lesssim' v \geq u \lesssim v$ as each member of $Ext(\lesssim)$ is extension of $\lesssim$. Together, we get $\forall u, v \in U : u \lesssim v = u \lesssim' v$, i.e. $[\bigcap Ext(\lesssim)](u, v) = u \lesssim v$ for each $u, v \in U$.

The converse implication is shown as follows. If for any set U and any $\mathbf{L}$-order $\lesssim$ on $\langle U, \approx\rangle$ there is a set $Ext(\lesssim)$ of crisp linear $\mathbf{L}$-order extensions of $\lesssim$ such that $[\bigcap Ext(\lesssim)](u, v) = u \lesssim v$ for each $u, v \in U$ then for any such $\lesssim$ there is some crisp linear $\mathbf{L}$-order extension and therefore, by Theorem 5, the lattice $\mathbf{L}$ has a join-irreducible unit.

As there are two conjunctions in any residuated lattice $\mathbf{L}$ ($\otimes$ and $\wedge$) and possibly many more conjunction-like operators may be defined (see e.g. [5]), one may be interested in intersection representation for the intersection operation induced by different conjunction-like operation, for example $\otimes$. Unfortunately, this is in general not possible as is shown in Example 3 below.

Example 3. Let $\mathbf{L}$ be a three element Łukasiewicz chain on $L = \{0, \frac{1}{2}, 1\}$ and let U be a set with six elements, i.e. $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$. Set $\lesssim : U \times U \rightarrow L$ as depicted below. Such $\lesssim$ is an $\mathbf{L}$-order on $\langle U, \lesssim \otimes \lesssim^{-1}\rangle$ with no representation by $\otimes$-intersection of crisp linear $\mathbf{L}$-order extensions of $\lesssim$.

$\lesssim$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	1	0	0	0	1	1
u_2	0	1	0	1	0	1
u_3	$\frac{1}{2}$	0	1	1	1	$\frac{1}{2}$
u_4	0	0	0	1	0	0
u_5	$\frac{1}{2}$	0	0	0	1	$\frac{1}{2}$
u_6	0	0	0	0	0	1

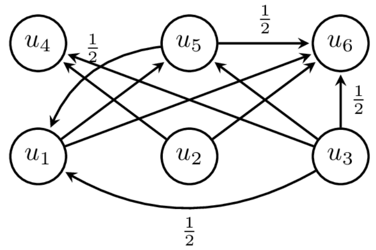

Figure 1: A fuzzy order derived from a standard six element crown graph with no representation by $\otimes$-intersection of crisp linear $\mathbf{L}$-order extensions shown in a tabular and a simplified graph representation – the loops and the labels on edges with the degree 1 are omitted.

The reasoning is as follows: It is well known that a crown graph with even number of vertices and its edges oriented from one side of the bipartition to the other forms the graph of a partially ordered set with order dimension equal to half of the vertices count. We can easily see that graph of 1-cut of $\lesssim$ is the six element crown graph with edges oriented as described above. Therefore, we need at least three crisp linear **L**-orders in the representing set just to "cover the ones". But then we are not able to represent the comparability degree of $u_5 \lesssim u_1$, as only option is setting it to $\frac{1}{2}$ in one of the chosen extensions and to 1 in all the others (by the properties of $\otimes$). But there is no **L**-order extension $\lesssim'$ of $\lesssim$ with $u_5 \lesssim' u_1 = 1$ because $u_5 \lesssim' u_1 = 1 = u_1 \lesssim' u_5$ breaks separability of $\lesssim' \otimes \lesssim'^{-1}$. Therefore no such set of extensions exists.

Two final corollaries of the results proven in previous sections, i.e. versions of Szpilrajn's theorem for crisp linearity and other related results, are equivalent characterizations of all residuated lattices **L** with join-irreducible unit and generalizations of two well known theorems of classical order theory – Szpilrajn's extension theorem [31] and intersection representation theorem [15, Theorem 2.32]; see Appendix – into setting of complete residuated lattices on $[0, 1]$ and strong completeness.[6]

Corollary 9. *The following propositions are equivalent:*

1. *The residuated lattice* **L** *has a join-irreducible unit.*
2. *For any finite set U, the set of all* **L***-equalities on U ordered by set inclusion forms a lattice.*
3. *Any finite* **L***-order may be extended into crisp linear* **L***-order.*
4. *An arbitrary* **L***-order may be extended into crisp linear* **L***-order.*
5. *An arbitrary* **L***-order may be represented as an intersection of some set of its crisp linear fuzzy order extensions.*

Corollary 10. *Let* **L** *be a complete residuated lattice on $[0, 1]$ and U an arbitrary set equipped with an* **L***-equality $\approx$. Then for each* **L***-order $\lesssim$ on $\langle U, \approx \rangle$ we have*

1. *There is a strong complete* **L***-order extension of $\lesssim$ on U.*
2. *There is a set $Ext(\lesssim)$ of strong complete* **L***-orders on U such that $u \lesssim v = [\bigcap Ext(\lesssim)](u, v)$ for each $u, v \in U$.*

Proof. Both corollaries follow immediately from Theorems 1, 3, 5, and 8.

6. A note on the essential properties of chains

In their study [11], Bodenhofer and Klawonn have identified three essential properties of partial orderings which are desirable even in the setting of fuzzy logic: an exitence of a linear extension; a possibility of an order representation by an intersection of its linear extensions; and the equivalence between maximality and linearity of an order. In addition, they have shown that if linearity is interpreted by the strong completeness then none of these properties can be attained unless we use the Gödel t-norm logic[7]. As their setting is the one of left continuous t-norms on $[0, 1]$, i.e. particular linear residuated lattices, the concepts of strong completeness and crisp linearity coincide.

We have already seen how our approach improves these results by realizing that both order and equality relations have to be manipulated together. A short comment on each of these properties follows.

Existence of complete extension. As we have already seen in Corollary 6, for suitable residuated lattices (including all the t-norm logics on $[0, 1]$) each fuzzy order may be extended into a crisp linear fuzzy order. That is in the given setting the situation is analogous to the Boolean case.

[6] The conditions of crisp linearity and strong completeness coincide in any complete residuated lattice on $[0, 1]$. Thus we preffer to call the condition strong completeness here because it is a well established name.

[7] Their definition of fuzzy order assumes fuzzy equivalence on the set (cf. Section 2.2), but the core idea remains same even in the case of fuzzy equality.

Intersection representation. Section 5 contains all the steps necessary to obtain a representation of any fuzzy order by an intersection of its crisp linear fuzzy order extensions for suitable residuated lattices. Again, there is an obvious analogy to the Boolean case.

Maximality vs linearity. The last relationship is more complex in the setting of fuzzy logic than in the Boolean case. The difference can be seen already from the following definition of maximality of fuzzy equality and fuzzy order.

Definition 7.
A fuzzy equality $\approx$ on U is *maximal* if there is no fuzzy equality $\approx'$ properly extending $\approx$ on U. A fuzzy order $\lesssim$ on $\langle U, \approx\rangle$ is *maximal on* U if there is no fuzzy order $\lesssim'$ on $\langle U, \approx'\rangle$ properly extending $\lesssim$.

Assume residuated lattice with join-irreducible unit. From the previous sections it follows that if there is a maximal extension of a fuzzy order then it is also crisp linear. Namely if some fuzzy order $\lesssim$ is maximal then it does not have any other fuzzy order extension than itself. But as in our setting every fuzzy order has a crisp linear extension, we get that $\lesssim$ is crisp linear.

The converse implication generally does not hold. In [10, Theorem 4] Bodenhofer shows that every crisp linear fuzzy order is union of some fuzzy equality and some linear crisp order.[8] Therefore, we may derive that maximality of crisp linear fuzzy order is given by maximality of its induced fuzzy equality.

Theorem 11. *Let* $\mathbf{L}$ *be a residuated lattice with join-irreducible unit. Then following propositions hold for every* $\mathbf{L}$*-order* $\lesssim$ *on a set with an* $\mathbf{L}$*-equality* $\langle U, \approx\rangle$*:*

1. *If* $\lesssim$ *is a maximal* $\mathbf{L}$*-order on* U *then it is crisp linear.*
2. *If* $\lesssim$ *is a crisp linear* $\mathbf{L}$*-order on* $\langle U, \approx\rangle$ *then it is a maximal* $\mathbf{L}$*-order on* U *if and only if* $\approx$ *is a maximal* $\mathbf{L}$*-equality on* U*.*

Proof. First point follows from the fact that every $\mathbf{L}$-order in our setting has a crisp linear $\mathbf{L}$-order extension.

Second point then follows from the fact that any $\mathbf{L}$-order $\lesssim$ is crisp linear if and only if it is a union of linear crisp order $\leq$ and $\mathbf{L}$-equality $\approx$ due to [10, Theorem 4].[9] Therefore, as each linear crisp order $\leq$ is maximal, $\lesssim$ is maximal if and only if $\approx$ is maximal.

In conclusion we see that, because of a much more complex structure of all equalities on a set, the one-to-one relationship between linear and maximal orders from the Boolean case is lost in the setting of fuzzy logic.

7. Conclusion

Although this work was focused mainly on the construction of linear extensions of a fuzzy order, one very important idea arose during the development of this construction. Namely, we have seen that altering fuzzy order inevitably leads to alteration of its induced fuzzy equality. This phenomenon is degenerated in Boolean case, as there we have only one possible equality, while there are, possibly uncountably, many fuzzy equalities on the given set in the setting of fuzzy logic. That is a possibility of strengthening or weakening the given fuzzy equality may be taken as a new and advantageous property in the setting of fuzzy logic, which is degenerated in Boolean case. This idea may be seen as further extension of reflections on the role of fuzzy equality in the definition of fuzzy order as presented in [5].

The up to date results on a linearization of a fuzzy order [11] show that without this observation we either lose all the important properties of such completion or we have to weaken the concept

[8] Again, in [10] Bodenhofer uses slightly different definition of fuzzy order but the propositions translates easily to our setting.

[9] Although Bodenhofer states this result only for complete residuated lattices on $[0, 1]$, a careful inspection of its proof (see [8]) reveals that it utilizes only general properties of complete residuated lattices.

of linearity by a great deal. On the other hand, utilizing this idea, we were able to obtain results similar to those in classical setting even for quite strong condition of crisp linearity in the broad class of complete residuated lattices. In particular, the results hold in all complete residuated lattices on $[0, 1]$ where crisp linearity moreover merges with the widely used condition of strong completeness.

We consider this observation together with the generalized extension theorem and related results as another step in understanding fuzzy orders. That is, following the direction pioneered by Belohlavek, Blanchard, Bodenhofer, and Höhle, fuzzy orders should be defined with respect to fuzzy equality (be it explicitly or implicitly by crisp antisymmetry). And furthermore, every adjustment of fuzzy order should mirror on the equality in an appropriate manner.

The results presented in this paper have some obvious consequences we will pursue further – the most important is the concept of dimension of fuzzy order in the spirit of work by Dushnik and Miller [15] and other related concepts of dimension. There are also many interesting applications of extensions and intersection representations of orders in classical settings. We hope that examining these ideas in the setting of fuzzy logic brings further understanding of fuzzy orders and their properties.

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Appendix: General preliminaries and historical notes

Classical equivalences, equality, and orders

The arguably most prominent binary relations on a set in classical setting are of two kinds – equivalences and orders. *Equivalence on a set* U is a binary relation $\equiv$ on U, i.e. $\equiv: U \times U \to \mathbf{2}$, which is reflexive, symmetric, and transitive. A notable equivalence on every set U is the *identity*, i.e. the only *equality* – separable equivalence – on U.

On the other hand, an *order* (also *partial order*) on U is a binary relation $\leq$ on U, i.e. $\leq: U \times U \to \mathbf{2}$, such that it is reflexive, antisymmetric, and transitive. If $\leq$ is an order on a set U then a pair $\langle U, \leq \rangle$ is called an *ordered set*. Moreover, if $\leq$ is a total order on U, i.e. it is an order where $u \leq v \vee v \leq u = 1$ for every u, v in U, we call $\leq$ a *linear order* and $\langle U, \leq \rangle$ a *linearly order set* or a *chain*.

Common generalization of both of these concepts is a *preorder* (also called quasiorder) on U, i.e. a reflexive and transitive binary relation $\prec$ on U. One may notice that the only relation on any set, which is both equivalence and order on U, is again the identity.

One of the most famous results in order theory is Szpilrajn's extension theorem. It was proven by Edward Szpilrajn in 1930 [31], although he acknowledges the prior existence of unpublished proofs by Banach, Kuratowski, and Tarski. Many interesting results are built upon this theorem, e.g. the intersection representation of orders and the basis of dimension theory [15].

Theorem A1 (Szpilrajn's extension theorem [31]). *For any partial order $\leq$ on any set U there is a linear order $\leq'$ on U such that $\leq'$ is an extension of $\leq$.*

Theorem A2 ([15, Theorem 2.32]). *If $\leq$ is any partial order on a set U then there exists a collection $Ext(\leq)$ of linear orders on U which realizes $\leq$, i.e. such that $u \leq v = [\bigcap Ext(\leq)](u, v)$ for each $u, v \in U$.*

For further information on general order theory we refer to the books [12, 30, 32].

Few notes on lattices

A lattice is a special kind of an ordered set, that is an ordered set $\langle L, \leq \rangle$ such that there exist supremum and infimum of any pair of elements x, y from L in $\langle L, \leq \rangle$. If suprema and infima moreover exist for any subset K of L then $\langle L, \leq \rangle$ is called a *complete lattice*. As the structure of truth vales for fuzzy logic typically forms a complete residuated lattice (see below), we briefly discuss some lattice properties which are related to our use case.

Definition A1. *[14]*
Let $\langle L, \leq \rangle$ be a lattice. An element $x \in L$ is *join-irreducible* (also called *supremum-irreducible* or *sup-irreducible*) if

1. $x \neq 0$ (in case L has a zero)
2. $x = y \vee z$ implies $x = y$ or $x = z$ for all $y, z \in L$.

It is also possible to define a related concept of *irreducibility by arbitrary joins*, i.e. $x \in L$ being irreducible by arbitrary joins if there is no subset K of L such that $x \notin K$ and $\bigvee K = x$. We use the term join-(ir)reducibility in the sense of the definition above. If there is a need for the notion of (ir)reducibility by arbitrary joins then it is clearly stated.

The join-irreducibility turns out to be crucial for the top element of a lattice as this element is expressing the full truth when a residuated lattice is used as structure of truth degrees for fuzzy logic. Therefore, we often utilize the following well known lemma.

Lemma A1. *In any lattice L with top element 1 and bottom element 0 the element 1 is join-irreducible if and only if for every finite set $K \subseteq L \setminus \{1\}$ we have $\bigvee K \neq 1$.*

For more details on general lattice theory we refer to the books [6, 14, 21].

Residuated lattices, fuzzy sets, and fuzzy relations

In contrast to classical logic, which relies on a fixed two-element set of truth values $L = \{0, 1\}$ and classical truth functions for logical connectives, fuzzy logic takes a different approach. In fuzzy logic, neither the set of truth degrees nor the truth functions for logical connectives are fixed. Instead, fuzzy logic operates with a general set of truth degrees, usually denoted by L, and allows for general truth functions of logical connectives, which are subject to natural basic conditions. Essentially, fuzzy logic embraces a general structure of truth degrees with appropriate generalized connectives, which allows for more nuanced and flexible reasoning compared to classical logic.

Since the seminal work by Goguen [18, 19], the structure $\mathbf{L}$ of truth degrees is usually assumed to form a complete residuated lattice [1, 3, 20, 22, 28]. A given theory is then often developed for the general complete residuated lattice $\mathbf{L}$ and is thus valid also for all the particular cases.

This way, we have class of structures at hand, which includes various particular cases such as the real unit interval $L = [0, 1]$ equipped with the Łukasiewicz connectives, Heyting algebras, or even two-element Boolean algebra **2** of classical logic (see below). Each of these structures then forms a basis of a particular case of fuzzy logic.

Definition A2. A *complete residuated lattice* is an algebra

$$\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$$

such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy the so-called *adjointness property*:

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \rightarrow c \tag{2}$$

for each $a, b, c \in L$. The elements a of L are called *truth degrees* and $\otimes$ and $\rightarrow$ are considered as the truth functions of *(many-valued) conjunction* and *implication*[10], respectively.

Often, one additional connective, biresiduum, is defined. Its interpretation is the truth function of (many-valued) equivalence.

Definition A3. The *biresiduum* in $\mathbf{L}$ is the binary operation defined by

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a), \tag{3}$$

for every a, b in L.

There are various, well known, examples of complete residuated lattices, particularly those with L being a chain. A common choice of $\mathbf{L}$ is a structure with L being the unit interval, $\wedge$ and $\vee$ being minimum and maximum, and $\otimes$ being a continuous t-norm (i.e. a commutative, associative, and isotone operation on $[0, 1]$ with 1 acting as a neutral element). The corresponding $\rightarrow$ is then given by

$$a \rightarrow b = \max\{c \,|\, a \otimes c \leq b\}.$$

The three most important pairs of adjoint operations on the unit interval are[11]:

$$\text{Łukasiewicz:} \quad \begin{aligned} a \otimes b &= \max(a + b - 1, 0), \\ a \rightarrow b &= \min(1 - a + b, 1), \end{aligned} \tag{4}$$

$$\text{Gödel:} \quad \begin{aligned} a \otimes b &= \min(a, b), \\ a \rightarrow b &= \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise,} \end{cases} \end{aligned} \tag{5}$$

$$\text{Goguen:} \quad \begin{aligned} a \otimes b &= a \cdot b, \\ a \rightarrow b &= \begin{cases} 1 & \text{if } a \leq b, \\ \frac{b}{a} & \text{otherwise.} \end{cases} \end{aligned} \tag{6}$$

[10] The operation $\rightarrow$ is also called residuum.

[11] Derived from the operations used as $\otimes$, the term "minimum structure" is commonly used when referring to a Gödel structure, whereas a Goguen structure is commonly referred to as a "product structure".

Another common choice for **L** is a finite chain. For example on $L = \{a_0 = 0, a_1, \ldots, a_n = 1\} \subseteq [0,1]$ ($a_0 < \cdots < a_n$) we can define $\otimes$ by $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and $\to$ by $a_k \to a_l = a_{\min(n-k+l,n)}$. Such defined **L** is then called a *finite Łukasiewicz chain*. Similarly, we can define a *finite Gödel chain* using same $L = \{a_0 = 0, a_1, \ldots, a_n = 1\} \subseteq [0,1]$ with the operations $\otimes$ and $\to$ given as restrictions of the Gödel operations from $[0,1]$ to L.

As noted above, even two-element Boolean algebra $\mathbf{2} = \langle\{0,1\}, \wedge, \vee, \otimes, \to, 0, 1\rangle$, i.e. the structure of truth degrees of classical logic, is a – very particular – case of a complete residuated lattice. This is vital because when considering the specific case $\mathbf{L} = \mathbf{2}$, the established concepts and outcomes align with those developed in the classical setting. Specifically, the concepts related to fuzzy sets and fuzzy relations (see the subsequent section) may be identified with their counterparts in the theory of classical sets and relations.

Fuzzy sets and fuzzy relations

Given a complete residuated lattice **L**, the basic set-theoretic notions are generalized into logical framework defined by **L**. We briefly survey the fundamental principles of fuzzy set theory, focusing particularly on binary fuzzy relations on a set, such as preorders, equivalences, and equalities.

A *fuzzy set* (or **L**-*set*) A in a universe U is a mapping $A\colon U \to L$. The value $A(u)$ is interpreted as "the degree to which u belongs to A." The collection of all **L**-sets in U is denoted by L^U. A fuzzy set $A \in L^U$ is called *crisp* if $A(u) = 0$ or $A(u) = 1$ for each $u \in U$. Every crisp fuzzy set $A \in L^U$ may be easily recognized as equivalent to the classical subset $\{u \in U \mid A(u) = 1\}$ of U. In fact, a crisp fuzzy set represents the characteristic function of the corresponding subset of U. It is customary to treat crisp fuzzy sets in U and their corresponding subsets of U interchangeably, as long as there is no danger of confusion.

For $a \in L$ and $u \in U$, we denote by $\{{}^a/u\}$ the fuzzy set A in U, called a *singleton*, for which $A(x) = a$ if $x = u$ and $A(x) = 0$ if $x \neq u$. A crisp singleton $\{{}^1/u\}$ may be identified with a one-element ordinary subset $\{u\}$ of U.

An *a-cut* of fuzzy set A in U is a set ${}^aA = \{u \in U \mid A(u) \geq a\}$. A crisp set A may be identified with its 1-cut.

The basic operations with fuzzy sets are based on the residuated lattice operations and are defined componentwise.

Definition A4. Let A, B be fuzzy sets in U. We define

$$\begin{aligned}
(A \cap B)(u) &= A(u) \wedge B(u) \\
(A \cup B)(u) &= A(u) \vee B(u) \\
(A \otimes B)(u) &= A(u) \otimes B(u) \\
(A \to B)(u) &= A(u) \to B(u) \\
(\bigcap_{i\in I} A_i)(u) &= \textstyle\bigwedge_{i\in I} A_i(u) \\
(\bigcup_{i\in I} A_i)(u) &= \textstyle\bigvee_{i\in I} A_i(u)
\end{aligned}$$

It follows from previous paragraphs that all **2**-sets are crisp fuzzy sets, i.e. these operations on **2**-sets are to be identified with their ordinary counterparts.

Given $A, B \in L^U$, we define the *degree* $A \subsetneq B$ *of inclusion of* A *in* B by

$$A \subsetneq B \quad = \quad \textstyle\bigwedge_{u\in U}\big(A(u) \to B(u)\big) \tag{7}$$

and the *degree of equality* of A and B by

$$A \approx B \quad = \quad \textstyle\bigwedge_{u\in U}\big(A(u) \leftrightarrow B(u)\big). \tag{8}$$

Note that (7) generalizes the ordinary subsethood relation $\subseteq$ and (8) generalizes the ordinary equality $=$ of sets.

Binary fuzzy relation R between U and V is just a fuzzy set in the universe $U \times V$. The basic properties of binary fuzzy relations are generalizations of their classical counterparts. As such a generalized forms of reflexivity, symmetry, and transitivity appear immediate:

Definition A5. For a binary fuzzy relation R on a set U, we define following well known properties:

$$R(u,u) = 1, \qquad \text{(reflexivity)}$$
$$R(u,v) \leq R(v,u), \qquad \text{(symmetry)}$$
$$R(u,v) \otimes R(v,w) \leq R(u,w) \qquad \text{(transitivity)}$$

for each $u, v, w \in U$.

These definitions have been proven useful and naturally behaving by a great number of studies. Generalizing antisymmetry, however, is much less immediate and its thorough discussion may be found in [4, 5]. It is also briefly discussed in introductory sections of this paper. Using the properties above, we may immediately define *preorder* (also *quasiorder*) in fuzzy setting as a reflexive and transitive binary fuzzy relation. The particular types of fuzzy preorders – fuzzy equivalences, fuzzy equalities, and fuzzy orders – are also discussed in the preliminary sections of the paper.
For further details on fuzzy sets we refer to the books [1, 3, 20, 22, 28].

Brief history of Szpilrajn's theorem in the setting of fuzzy logic

Szpilrajn-like extension theorem in the setting of fuzzy logic was considered already by Zadeh in his seminal paper on fuzzy equivalences and fuzzy orderings (see Theorem 8 in [33]). This version of the theorem was stated with respect to different concepts of antisymmetry and linearity. See [3, 4] for in-depth analysis of differences between Zadeh's and our setting. More results on Szpilrajn-like extension principle in the setting of fuzzy logic emerged soon, e.g. [7, 13, 23]. Of these works, we highlight [7] where one of outlined views on fuzzy orders was lately shown to be in a sense equivalent to our view on fuzzy orderings [4, 5]. The main distinction lies in a different setting[12] and the fact that Blanchard in general defines the notion of a fuzzy order on a fuzzy set $A \in L^U$.

For the approach to fuzzy orders we utilize, i.e. the one which considers fuzzy equality on the underlying set, the first version of Szpilrajn-like theorem was stated already in [25] – the work which coined this approach – see their Theorem II.7 and its corollaries. This version of the theorem was stated with respect to $\otimes$-linearity. Interestingly until lately, no connection between both versions of fuzzy order definitions from [7] and [25], and of the theorem in particular, was established, even though both works had one author in common and were published close in time to each other.

As far as we know,the most detailed study on linearity of fuzzy orderings and related concepts so far is [11]. This study builds upon research on the concept of fuzzy order itself, reignited by Belohlavek and Bodenhofer in the late 1990s to early 2000s. It analyzes several notions of linearity proposed by various authors in the setting of fuzzy order on the set with fixed fuzzy equivalence. The fixing of underlying similarity is the most important difference between their approach and the one utilized in the current study. In the end, achievability of Szpilrajn-like theorem is studied for several situations, given by used t-norm and axiom of linearity (see their Table 1). Their main results include following observations mentioned in the conclusion of their work: The strong completeness can only serve as an appropriate concept of linearity in the setting of fuzzy logic, if $\otimes = \wedge$; The $\otimes$-linearity coined in [25] provides preservation of the most important properties of order extension in the setting of residuated lattices on $[0, 1]$. However, it is very weak, non-intuitive, and poorly expressive concept if $\mathbf{L}$ does not have a strong negation.

In a sense, our paper builds upon this study. To compare the approaches with fixed underlying similarity and with possibility to modify it together with the order, some of our observations throughout the text are related to their results.

[12] That is particular type of residuated lattices where $L = [0, 1]$ and $\otimes = \wedge$.

Selected topics in fuzzy orderings

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Abstrakt – An ordering relation is a central concept in many areas of human activity. This work is concerned with ordering relations in the setting of fuzzy logic. We consider the notion of fuzzy order, where antisymmetry is inherently linked to a many-valued equality on the underlying universe. We thoroughly examine the origins of this concept, including the seemingly different point of view used in some works; provide remarks and observations on the existing studies; and prove new results. Then we offer a unifying concept of antisymmetry in the setting of fuzzy logic and thus also unified notion of fuzzy order. In particular, we prove that all the definitions of fuzzy order, we are concerned with, are mutually equivalent and also equivalent to the proposed generalized view. By doing so, we uncover that the link between fuzzy order and underlying fuzzy equality is even deeper than usually assumed. Finally, we utilize these new observations on the role of fuzzy equality by reconsidering the problem of Szpilrajn-like extension of fuzzy order and by providing a way to extend any fuzzy order into a linear fuzzy order in a broad class of fuzzy logics.

Preface

An ordering relation is a central concept in many areas of human activity. In 1970s Zadeh (1971) coined generalizations of ordinary similarity and ordering relations into his, in that time novel, setting of fuzzy sets (Zadeh, 1965). Since this seminal paper appeared, many deep, theoretical results and applications were described and implemented. This thesis focuses on some basic aspects of the theory of fuzzy orderings. Namely the concept of fuzzy order itself, related axioms, a link to similarity relations, and a possibility of an extension of fuzzy ordering into a linear fuzzy ordering. We are interested in, arguably up-to-date most developed, approach where antisymmetry is defined with respect to underlying fuzzy equality and approaches which turned out to be equivalent. Note that this thesis does not reflect on other definitions of fuzzy order although many may be found in the literature. We focus only on the point of view where underlying similarity is taken into account, as this approach proved to be useful by great number of studies.

The thesis is built upon results obtained in three research papers (Belohlavek and Urbanec, 2023a,b; Urbanec, 2023). This author paper consists of a brief summary of the obtained results, some additional observations, historical context, and plans for the future. The first and second papers, concerned with the concept of fuzzy order itself, are based on the outcomes of joint scientific work with my supervisor, Radim Belohlavek, without whom they would not be possible. The third paper is then devoted to a possibility of linearization of fuzzy order, i.e. to an extension of fuzzy order in a Szpilrajn-like way. Presented observations offer unifying view on up-to-date available definitions of fuzzy order with respect to fuzzy equality and some new arguments for equality-order connection to be taken into account even when studying further properties and applications of such fuzzy orders.

This text only briefly summarizes the most important results we obtained on the concept of fuzzy order itself, its definitions, and various aspects of its connection with underlying fuzzy equality. In particular all the proofs, auxiliary lemmas, many remarks, comments, and also some of the obtained results are omitted. If the reader is interested in some particular result, its proof, or some related information, it can be found in the full thesis text or in the papers (Belohlavek and Urbanec, 2023a,b; Urbanec, 2023).

Chapter 1

Preliminaries

We start by basics of ordinal order theory, fuzzy logic, fuzzy sets, and fuzzy relations. The hope is that the text is self contained and accessible even for a reader who does not work in the setting of fuzzy logic and order theory on the daily basis.

One of the most fundamental concepts in mathematics is a relation, the formal counterpart of a relationship between entities in our world. We are concerned with particular type of relations – binary relations on a set. Such relations capture relationships between pairs of elements in a given situation. Arguably, the most important relationships in our perception of the world are of two kinds: the ones, which groups similar things together, and the ones, which compare objects to each other. Corresponding binary relations are called equivalences and orders, respectively.

In this chapter, we first briefly summarize the well known definitions and properties of binary relations on a set in general and of equivalences and orders in particular. Then, we move our attention to basics of fuzzy logic, especially to the way fuzzy relations and their properties are defined. The last part of present chapter is then devoted to fuzzy equivalences and in particular fuzzy equalities.

1.1 Binary relations on a set

Let U be a set. Any subset R of $U \times U$ is called a *binary relation* on U. For any $u, v \in U$ we say that u is related to v by R if $\langle u, v \rangle \in R$ – this is often denoted simply by $R(u, v)$ or uRv.

As relations are just special kinds of sets, we may carry out the well known set operations in a straightforward way. Moreover, we call a relation E an *extension of a relation* R if $R \subseteq E$.

There are numerous intriguing properties of binary relations on a given set of which the following will be of importance in subsequent chapters.

Definition 1.1.1. For a binary relation R on a set U, we define the following well-known properties:

$$R(u,u), \quad \text{(reflexivity)}$$
$$R(u,v) \Rightarrow R(v,u), \quad \text{(symmetry)}$$
$$R(u,v) \wedge R(v,u) \Rightarrow u = v, \quad \text{(antisymmetry)}$$
$$R(u,v) \Rightarrow \neg R(v,u), \quad \text{(asymmetry)}$$
$$R(u,v) \wedge R(v,w) \Rightarrow R(u,w), \quad \text{(transitivity)}$$
$$u \neq v \Rightarrow R(u,v) \vee R(v,u), \quad \text{(completeness)}$$
$$R(u,v) \vee R(v,u), \quad \text{(strong completeness)}$$

for each $u, v, w \in U$.

We call R reflexive, symmetric, antisymmetric, asymmetric, transitive, complete, and strongly complete if it fulfills the respective property.

It is worth noting that different terms, such as linear, connex, connected, total, and trichotomic, are used in the literature to describe (strong) complete relations, depending on the context.

All of these and many more properties of relations together with their interrelationships may be found e.g. in (Toth, 2020). Using some of the properties above, we may define various interesting classes of binary relations on a set.

Definition 1.1.2. Binary relation R on U is called:
- *preorder (or quasiorder)* if it is reflexive and transitive;
- *equivalence* if it is a symmetric preorder, i.e. a reflexive, transitive, and symmetric binary relation;
- *order (also partial order, ordering)* if it is an antisymmetric preorder, i.e. a reflexive, transitive, and antisymmetric binary relation.

We denote preorders by $\prec$, equivalences by $\equiv$ and orders by $\leq$, possibly with sub- or superscripts.

Equivalences and equality

As noted above, equivalences are of utmost importance as they allow us to model indistinguishability of objects in the given situation. Arguably, the most prominent of all the equivalences on any set U is the equality relation.

Definition 1.1.3. An *equality (or identity) on U* is an equivalence $\equiv$ on U, which moreover satisfies

$$u \equiv v \quad \text{implies} \quad u = v \quad \text{(separability)}$$

for any u, v in U. Here, $u = v$ means that u and v are the same object.

Note: The form of the definition above may feel overcomplicated as the notion of eqaulity is well-known and can be defined in a more straightforward manner. Nevertheless, we use this form to highlight the analogy between definitions of equality in the classical setting and the setting of fuzzy logic (see below).

Equality is of such importance that it is often distinguished from all other predicates on the level of language of first order logic – the language is then called *language with equality.* That is there is the symbol $=$ reserved in the language, which should always be interpreted by the equality relation. Introduction of this symbol into the language comes hand in hand with extra axioms – called equality axioms – whose meaning goes back to Leibniz's considerations. For more information, we refer the reader to standard textbooks on mathematical logic, e.g. Cori and Lascar (2000). In accordance with this practice, we use the symbol $=$ only for the identity relation on the respective set. It is also worth noting that the equality is the only reflexive relation on a set which is symmetric and antisymmetric at the same time.

In the following sections, we will see that, contrary to the Boolean case, there is an abundance of equalities within a fuzzy logic framework. This well known observation leads to various possible generalizations of many classical concepts and properties which are in the Boolean case defined with respect to the identity. In Chapter 3, we focus on antisymmetry and its interrelationship with separability of underlying equality as these properties are crucial for the concept of order in the setting of fuzzy logic.

Orders

The other prominent kind of binary relations is ordering on a set, i.e. relations modeling comparison between objects. There are two common views on an ordering on a set, the first one as per Definition 1.1.2, the second one known as a strict order.

Definition 1.1.4. A *strict order* on a set U is a binary relation on U which is transitive and asymmetric.

It is a well known fact that both definitions delineate same class of relations.

Proposition 1.1.5. *If $\leq$ is an order on a set U then a binary relation $<$ on U defined by $u < v = u \leq v \wedge u \neq v$ for each $u, v \in U$ is a strict order on U.*

If $<$ is a strict order on U then a binary relation $\leq$ on U defined by $u \leq v = u < v \vee u = v$ for each $u, v \in U$ is an order on U.

The constructions are mutually inverse.

The structure $\langle U, \leq \rangle$, consisting of a set U and an order relation $\leq$ defined on U, is commonly referred to as an *ordered set*. In the subsequent chapters, we extensively utilize two distinguished classes of orders – linear orders and lattices.

Linear orders

Definition 1.1.6. An order relation $\leq$ on U is called a *linear order (or chain)* if it is moreover strong complete.

In other words, order is linear if for any pair of objects we can decide which object is a predecessor and which object is a successor in the given sense, e.g. which is smaller, better, further, . . . Note that such concept is utterly natural – many common orders are linear, e.g. numbers or anything that can be numbered.

One of the most fundamental results in the field of order theory is an extension theorem proved by Szpilrajn (1930).[1]

Theorem 1.1.7 (Szpilrajn's extension theorem)**.** *For any order $\leq$ on a set U there is a linear order on U which contains $\leq$.*

That is every order can be extended into a linear order while preserving the original comparisons between objects. For finite cases, this extension is straightforward – decide for every pair of uncomparable elements, pair by pair, what the resulting order should be. There is always at least one option to do so without breaking properties necessary for a relation to be an order and after finite number of steps we obtain desired linear order. In general case, this theorem only holds if we accept the axiom of choice.

Using Szpilrajn's result, Dushnik and Miller (1941) introduced so called *realizers* of an order and the concept of an *order dimension*.

[1] Szpilrajn acknowledges the prior existence of unpublished proofs by Banach, Kuratowski, and Tarski.

Definition 1.1.8. Let $\leq$ be an order on U. A collection $\mathcal{K}$ of linear orders on U is called a *realizer of* $\leq$ if for any two elements u, v in U we have $u \leq v$ if and only if $u \leq' v$ holds for every $\leq'$ in $\mathcal{K}$. That is we have $u \leq v = \bigwedge_{\leq' \in \mathcal{K}} u \leq' v$ for each $u, v \in U$.

Alternatively we say that $\mathcal{K}$ *realizes* $\leq$ or $\leq$ *is realized by (linear orders of)* $\mathcal{K}$.

Theorem 1.1.9 (Dushnik and Miller 1941, Theorem 2.32)**.** *If $\leq$ is any partial order on a set U then there exists a collection $\mathcal{K}$ of linear orders on U which realize $\leq$.*

Definition 1.1.10. A *dimension* of an order $\leq$ on U is the smallest cardinal number $\mathbf{m}$ such that $\leq$ is realized by $\mathbf{m}$ linear orders on U. Dimension of $\leq$ is often denoted by $\dim(\leq)$.

These outcomes initiated the development of dimension theory and led to many useful applications, e.g Arrow's and Suzumura's extension theorems used in theory of social choice (Arrow, 2012; Suzumura, 1983), Schnyder's characterization of planar graphs (Schnyder, 1989), effective storage of finite orderings in computer memory by the set of its realizers, and many more. Today, the dimension theory is a well-established field in the study of ordered sets, as it enables us to characterize any order by using the most prevalent type of orderings – chains.

Lattices

This section contains few selected results from lattice theory. In this work, lattices are employed in two ways. First, particular type of lattices is used as a structure of truth degrees in fuzzy logic while some of the obtained results depends on further properties of this structure. Second, lattices are the most understood types of orders in setting of fuzzy logic, including deeply developed applications (Belohlavek, 2001, 2002, 2004; Höhle, 1987). As such, they serve as one of justifications for our choice of an approach to fuzzy orders and a source of motivation.

It is a well known, yet still captivating, fact that there are two equivalent definitions of a lattice structure. One characterizes a lattice as a special type of an order while the other defines it as an algebra.

Definition 1.1.11. Let L be a non-empty set. An ordered set $\langle L, \leq \rangle$ is called a *lattice* if every pair of elements from L has an infimum, i.e. greatest lower bound, and a supremum, i.e. least upper bound, in $\langle L, \leq \rangle$.

Alternatively, *lattice* is an algebra $\langle L, \vee, \wedge \rangle$ where $\vee$ and $\wedge$ are two binary operations on L such that both $\vee$ and $\wedge$ are commutative and associative and where absorption laws – $a \vee (a \wedge b) = a = a \wedge (a \vee b)$, for every a, b in L – hold. The operations $\vee$ and $\wedge$ are then called *join* and *meet*, respectively.

We say that a lattice is *complete* if every subset of L has supremum and infimum in $\langle L, \leq \rangle$.

The transition between the two definitions is straightforward. Given a lattice as an ordered set $\langle L, \leq \rangle$, for every a and b in L, defining $a \vee b = \sup(a, b)$ and $a \wedge b = \inf(a, b)$ transforms it into a lattice as an algebra. Conversely, for a lattice as an algebra, if we set $a \leq b$ to be true if and only if $a \wedge b = a$, we obtain a lattice as an ordered set.

In Chapter 4, we discuss the possibility of linear extension of any fuzzy order on a set. It turns out that such possibility is dependent on extra properties of the underlying residuated lattice (see below). We therefore define the following concept of join-irreducibility.

Definition 1.1.12. (Davey and Priestley, 2002)
Let L be a lattice. An element $x \in L$ is *join-irreducible* [2] if

1. $x \neq 0$ (in case L has a zero)
2. $x = y \vee z$ implies $x = y$ or $x = z$ for all $y, z \in L$.

[2] Also called *supremum-irreducible* or *sup-irreducible*.

Note 1.1.13. It is also possible to define related concept of *irreducibility by arbitrary joins*, i.e. $x \in L$ is irreducible by arbitrary joins if there is no subset K of L such that $x \notin K$ and $\bigvee K = x$.

We use the term join-(ir)reducibility in the sense of definition above. If there is a need for the notion of (ir)reducibility by arbitrary joins then it is clearly stated.

The join-irreducibility turns out to be crucial for the top element of a lattice as this element plays the role of full truth in fuzzy logic. Therefore, we often utilize the following lemma.

Lemma 1.1.14. *In any lattice L with top element 1 and bottom element 0 the element 1 is join-irreducible if and only if for every finite set $K \subseteq L \setminus \{1\}$ we have $\bigvee K \neq 1$.*

Proof. If 1 is join-irreducible in L then for every such finite set K we have $\bigvee K \neq 1$ by induction. That is $\bigvee \emptyset = 0$, for K with $K = \{a\}$ we have $\bigvee K = a \neq 1$ and for $K_n = \{a_1, \ldots, a_n\}$, i.e. with $|K_n| = n$, we have $\bigvee K_n = \bigvee K_{n-1} \vee a_i$ for some K_{n-1} with $|K_{n-1}| = n - 1$ and $i \in \{1, ..., n\}$, i.e. $\bigvee K_n = a \vee b$ for some $a, b \in L \setminus \{1\}$ therefore $\bigvee K_n \neq 1$ by 1 being join-irreducible.

If any finite K has supremum lower than 1 then also every K with $|K| = 2$ has supremum lower than 1. That is 1 is join-irreducible in L. □

Since the concepts mentioned above have been introduced, a lot has been done in areas related to order dimension (Trotter, 1992), lattices (Birkhoff, 1940; Davey and Priestley, 2002; Grätzer, 2002), and in the theory of ordered sets in general (Caspard et al., 2012; Schröder, 2003).

1.2 Fuzzy logic and residuated lattices

In contrast to classical logic, which relies on a fixed two-element set of truth values $L = \{0, 1\}$ and classical truth functions for logical connectives, fuzzy logic takes a different approach. In fuzzy logic, neither the set of truth degrees nor the truth functions for logical connectives are fixed. Instead, fuzzy logic operates with a general set of truth degrees, usually denoted by L, and allows for general truth functions of logical connectives, which are subject to natural basic conditions. Essentially, fuzzy logic embraces a general structure of truth degrees with appropriate generalized connectives which allows for more nuanced and flexible reasoning compared to classical logic.

Since the seminal work by Goguen (1967, 1969), the structure $\mathbf{L}$ of truth degrees is usually assumed to form a complete residuated lattice (Belohlavek, 2002; Belohlavek et al., 2017; Gottwald, 2001; Hájek, 1998; Novák et al., 1999). A given theory is then often developed for the general complete residuated lattice $\mathbf{L}$ and is thus valid also for all the particular cases.

This way, we have class of structures at hand, which includes various particular cases such as the real unit interval $L = [0, 1]$ equipped with the Łukasiewicz connectives, Heyting algebras, or even two-element Boolean algebra $\mathbf{2}$ of classical logic. Each of these structures then forms a basis of particular case of fuzzy logic.

Definition 1.2.1. A *complete residuated lattice* is an algebra

$$\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$$

such that $\langle L, \wedge, \vee, 0, 1 \rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1 \rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy the so-called *adjointness property*:

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \rightarrow c \tag{1.1}$$

for each $a, b, c \in L$. The elements a of L are called *truth degrees* and $\otimes$ and $\rightarrow$ are considered as the truth functions of *(many-valued) conjunction* and *implication*[3], respectively.

[3]The operation $\rightarrow$ is also called residuum.

Often, one additional connective, biresiduum, is defined. Its interpretation is the truth function of (many-valued) equivalence.

Definition 1.2.2. The *biresiduum* in **L** is the binary operation defined by

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a), \tag{1.2}$$

for every a, b in L.

There are various, well known, examples of complete residuated lattices, particularly those with L being a chain. A common choice of **L** is a structure with L being unit interval, $\wedge$ and $\vee$ being minimum and maximum, and $\otimes$ being a continuous (or at least left-continuous) t-norm (i.e. a commutative, associative, and isotone operation on $[0,1]$ with 1 acting as a neutral element). The corresponding $\rightarrow$ is then given by

$$a \rightarrow b = \max\{c \mid a \otimes c \leq b\}.$$

The three most important pairs of adjoint operations on the unit interval are[4]:

$$\text{Łukasiewicz:} \quad \begin{aligned} a \otimes b &= \max(a + b - 1, 0), \\ a \rightarrow b &= \min(1 - a + b, 1), \end{aligned} \tag{1.3}$$

$$\text{Gödel:} \quad \begin{aligned} a \otimes b &= \min(a, b), \\ a \rightarrow b &= \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise,} \end{cases} \end{aligned} \tag{1.4}$$

$$\text{Goguen:} \quad \begin{aligned} a \otimes b &= a \cdot b, \\ a \rightarrow b &= \begin{cases} 1 & \text{if } a \leq b, \\ \frac{b}{a} & \text{otherwise.} \end{cases} \end{aligned} \tag{1.5}$$

Another common choice for **L** is a finite chain. For example on $L = \{a_0 = 0, a_1, \ldots, a_n = 1\} \subseteq [0,1]$ ($a_0 < \cdots < a_n$) we can define $\otimes$ by $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and $\rightarrow$ by $a_k \rightarrow a_l = a_{\min(n-k+l,n)}$. Such defined **L** is called a *finite Łukasiewicz chain*. Similarly we can define a *finite Gödel chain* using same $L = \{a_0 = 0, a_1, \ldots, a_n = 1\} \subseteq [0,1]$ with the operations $\otimes$ and $\rightarrow$ given as restrictions of the Gödel operations from $[0,1]$ to L.

As noted above, even two-element Boolean algebra $\mathbf{2} = \langle \{0,1\}, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$, i.e. the structure of truth degrees of classical logic, is a particular case of a complete residuated lattice. This is vital because when considering the specific case $\mathbf{L} = \mathbf{2}$, the established concepts and outcomes align with those developed in classical setting. Specifically, the concepts related to fuzzy sets and fuzzy relations (see the subsequent section) may be identified with their counterparts in the theory of classical sets and relations.

1.3 Fuzzy sets and relations

Given a complete residuated lattice **L**, the basic set-theoretic notions are generalized into logical framework defined by **L**. We briefly survey the fundamental principles of fuzzy set theory, focusing particularly on binary fuzzy relations on a set, such as preorders, equivalences, and equalities. If the used complete residuated lattice is obvious from the context or if the given proposition is valid for any complete residuated lattice, we usually use terms such as fuzzy set, fuzzy relation, fuzzy order, etc. On the other hand, if we consider some particular complete residuated lattice, we denote it by **L** and then talk about **L**-set, **L**-relation, **L**-order, etc.

[4]Derived from the operations used as $\otimes$, the term "minimum structure" is commonly used when referring to a Gödel structure, whereas a Goguen structure is commonly referred to as a "product structure".

Fuzzy sets

Definition 1.3.1. A *fuzzy set* (or **L**-*set*) A in a universe U is a mapping $A\colon U \to L$. The value $A(u)$ is interpreted as "the degree to which u belongs to A."

The collection of all **L**-sets in U is denoted by L^U. A fuzzy set $A \in L^U$ is called *crisp* if $A(u) = 0$ or $A(u) = 1$ for each $u \in U$. Every crisp fuzzy set $A \in L^U$ may be easily recognized as equivalent to the classical subset $\{u \in U \mid A(u) = 1\}$ of U. In fact, a crisp fuzzy set represents the characteristic function of the corresponding subset of U. It is customary to treat crisp fuzzy sets in U and their corresponding subsets of U interchangeably, as long as there is no danger of confusion.

For $a \in L$ and $u \in U$, we denote by $\{^a/u\}$ the fuzzy set A in U, called a *singleton*, for which $A(x) = a$ if $x = u$ and $A(x) = 0$ if $x \neq u$. A crisp singleton $\{^1/u\}$ may be identified with a one-element ordinary subset $\{u\}$ of U.

An a-*cut* of fuzzy set A in U is a set ${}^aA = \{u \in U \mid A(u) \geq a\}$. A crisp set A may be identified with its 1-cut. The basic operations with fuzzy sets are based on the residuated lattice operations and are defined componentwise.

Definition 1.3.2. Let A, B be fuzzy sets in U. We define the following operations derived from those of used complete residuated lattice:

$$\begin{aligned}
(A \cap B)(u) &= A(u) \wedge B(u),\\
(A \cup B)(u) &= A(u) \vee B(u),\\
(A \otimes B)(u) &= A(u) \otimes B(u),\\
(A \to B)(u) &= A(u) \to B(u),\\
(\bigcap_{i \in I} A_i)(u) &= \textstyle\bigwedge_{i \in I} A_i(u),\\
(\bigcup_{i \in I} A_i)(u) &= \textstyle\bigvee_{i \in I} A_i(u),
\end{aligned}$$

for each $u \in U$.

It follows from previous paragraphs that all **2**-sets are crisp fuzzy sets, i.e. these operations on **2**-sets are to be identified with their ordinary counterparts.

Given $A, B \in L^U$, we define the *degree* $A \underset{\sim}{\subseteq} B$ *of inclusion of* A *in* B by

$$A \underset{\sim}{\subseteq} B \quad = \quad \textstyle\bigwedge_{u \in U} \big(A(u) \to B(u)\big) \tag{1.6}$$

and the *degree of equality of* A *and* B by

$$A \approx B \quad = \quad \textstyle\bigwedge_{u \in U} \big(A(u) \leftrightarrow B(u)\big). \tag{1.7}$$

Note that (1.6) generalizes the ordinary subsethood relation $\subseteq$ and (1.7) generalizes the ordinary equality $=$ of sets.

Binary fuzzy relations

Binary fuzzy relation R between U and V is just a fuzzy set in the universe $U \times V$.

Definition 1.3.3. A *binary fuzzy relation* (or *binary* **L**-*relation*) R between U and V is any mapping $R\colon U \times V \to L$. [5]

[5] If $U = V$ then R is called a binary fuzzy relation on U.

The definition is a straightforward generalization of the definition of classical binary relation. Similarly, the basic properties of binary fuzzy relations are generalizations of their classical counterparts. But contrary to the case of the definition, these generalizations do not have to be so straightforward for each property. Generalizing reflexivity, symmetry, and transitivity appears immediate:

Definition 1.3.4. For a binary fuzzy relation R on a set U, we define following well known properties:

$$R(u,u) = 1, \qquad \text{(reflexivity)}$$
$$R(u,v) \le R(v,u), \qquad \text{(symmetry)}$$
$$R(u,v) \otimes R(v,w) \le R(u,w), \qquad \text{(transitivity)}$$

for each $u, v, w \in U$.

We say that R is reflexive, symmetric, and transitive if it fulfills the respective property.

These definitions have been proven useful and naturally behaving by a great number of studies. Generalizing antisymmetry and completeness, however, is much less immediate. Using the properties above we may instantly define preorders and equivalences in the setting of fuzzy logic. We postpone the discussion of antisymmetry, fuzzy order, and linear fuzzy order to Chapters 3 and 4 where we analyze them thoroughly.

Definition 1.3.5. Binary fuzzy relation R on U is called:

- *fuzzy preorder (or fuzzy quasiorder)* if it is reflexive and transitive;
- *fuzzy equivalence* if it is symmetric fuzzy preorder, i.e. reflexive, transitive, and symmetric binary fuzzy relation;

We denote fuzzy preorders by $\prec$ and fuzzy equivalences by $\approx$, possibly with subscripts or superscripts. We also use terms **L**-*preorder* and **L**-*equivalence* if **L** is to be empahsized.

Transitive closures

Transitivity is a crucial property both for equalities and orders – the main subjects of this work. Therefore, we often discuss various consequences of extending some relation into its transitive closure.

Definition 1.3.6. *Transitive closure Tra*(R) of a binary fuzzy relation R on U is the least transitive binary fuzzy relation on U containing R.

It is well known fact that transitive closure may be formed using only composition and union.

Lemma 1.3.7. *For any binary fuzzy relation* $R\colon U \times U \to L$ *we have* $\mathit{Tra}(R) = \bigvee_{n=1}^{\infty} R^n = R \cup R \circ R \cup R \circ R \circ R \cup \cdots$.

For further details on general theory of fuzzy sets and relations we refer to the books by Belohlavek (2002); Belohlavek et al. (2017); Gottwald (2001); Hájek (1998); Novák et al. (1999).

Fuzzy equivalences and fuzzy equalities

Expressing the similarity to some extent between two objects is a common practice in natural language, as exemplified by the sentence: "These two options are quite different, but there is yet another one, which is, in a way, similar to both." Modeling such propositions by means of classical logic is possible, but it has some drawbacks. For example, we can not easily use theory of preorders, equivalences, and related concepts, as the described similarity relationship is not even transitive. On the other hand, fuzzy logic offers a convenient way to handle gradual information and, moreover, the properties of fuzzy equivalences and equalities are just the properties one naturally expects from such similarity. For this reason among others, fuzzy equivalences and equalities have been deeply developed and widely utilized.

The basic notion in presence of fuzzy equivalence $\approx$ on a set is the compatibility[6] of a set or a relation with $\approx$.

Definition 1.3.8. A fuzzy set A in a universe U is *compatible with a fuzzy equivalence* $\approx$ *on* U if

$$A(u) \otimes u \approx v \leq A(v) \tag{1.8}$$

for every u, v in U.

A binary fuzzy relation $R\colon U \times U \to L$ is *compatible with a fuzzy equivalence* $\approx$ *on* U if

$$R(u_1, v_1) \otimes (u_1 \approx u_2) \otimes (v_1 \approx v_2) \leq R(u_2, v_2) \tag{1.9}$$

for every u_1, u_2, v_1, v_2 in U.

In words, compatibility of a fuzzy set A with $\approx$ means that if u is in A and u and v are equivalent, then v is in A as well. Similarly, compatibility of binary fuzzy relation R reads that if u_1 and v_1 are related by R, u_1 is equivalent to u_2, and v_1 is equivalent to v_2, then u_2 and v_2 are related by R as well. That is compatibility generalizes the classical axiom of equality.

In the end, we briefly turn our attention to fuzzy equalities, as their properties are crucial for the definition and utilization of fuzzy orders. Similarly to the classical case, fuzzy equalities are defined as separable fuzzy equivalences. However, unlike in the classical setting, there may exist multiple fuzzy equalities on a given set. We will often discuss various properties of fuzzy equalities in subsequent chapters.

Definition 1.3.9. A *fuzzy equality (or* **L***-equality)* is a fuzzy equivalence, which moreover satisfies

$$u \approx v = 1 \quad \text{implies} \quad u = v \tag{separability}$$

for each $u, v \in U$.

To emphasize that $\approx$ is a fuzzy equality, not a mere fuzzy equivalence, we use the symbol $\eqsim$, possibly with subscripts or superscripts.

A comprehensive treatment of fuzzy equivalences, equalities, and related topics may be found in (Recasens, 2011, 2022).

[6] Often the term extensionality or congruence with respect to a fuzzy equivalence $\approx$ is used.

Chapter 2

Historical notes

Any abstract concept may be fully grasped only if we know initial motivations and historical aspects of its development. Therefore, this chapter briefly discusses these topics for the case of fuzzy order defined with respect to fuzzy similarity and related relations. We also pay some attention to the works on fuzzy lattices, as this particular type of fuzzy order was often the driving force behind new results on the concept of fuzzy order itself.

The story of fuzzy order starts with Zadeh's seminal paper (Zadeh, 1971). Since this work, a lot has been done in the fields of order theory and in particular lattice theory in the setting of fuzzy logic. Table 1 shows number of papers devoted to fuzzy order and lattice-type fuzzy order indexed by Scopus for various time frames including individual decades starting from 1970s. We find interesting that, according to this data, almost exact half of the papers devoted to these topics was written in the last 10 years and almost three quarters in the last 15 years. On the other hand, one has to be careful with such interpretations as this increase of paper count may go hand in hand with better online databases and overall better internet access in last 20 years or so. Also it may be related to the phenomenon of inflation in publishing as described by Belohlavek (2022).

Time frame	Order or lattice	Order	Lattice
1971-1980	6	6	0
1981-1990	30	21	9
1991-2000	94	48	46
2001-2010	241	151	92
2011-2020	522	388	140
2021-2023 (April)	134	104	31
2008-2023 (April)	754	554	207
2013-2023 (April)	560	423	143
1971-2023 (April)	1027	718	318

Table 1: Number of papers devoted to fuzzy orders or fuzzy lattices by time frames (mostly decades) according to Scopus. Second column contains count of papers for the given period and query "fuzzy order*" OR "fuzzy lattice" in abstract, keywords, and title. The third and fourth columns contain similar information only for "fuzzy order*" resp. "fuzzy lattice" queries. The asterisk symbol in Scopus query represents wildcard – in this case the word "order" may have any suffix.

2.1 The concept of fuzzy order

Now, we briefly cover the history of the concept of fuzzy order defined with respect to an underlying similarity by summarizing the results obtained in some works on the topic. We choose the works which were, in our opinion, the most essential or influential. As such choice may be regarded as opinionated, we support it by notes on the later influence of obtained results and also by citation count of each of the papers, which usually serves as one of metrics of the paper's influence. By doing so, we note that in some cases it may take time, further development, and possibly luck for the results to be actually recognized by the community in a form of citations. Therefore here, the citation counts are to be taken just as supplement to the notes on historical development.

The works are listed in chronological order by years of their publication. The citation counts are according to Scopus database in the end of April, 2023.

Zadeh (1971)

The first, and also most influential (2000 citations in Scopus), work on the topic was done by Zadeh (1971), where the author coined the concepts of fuzzy order[1] and fuzzy similarity.

The motivation was a study of concepts of equivalences and orders in the fuzzy setting – an emerging theory in that time. Various properties of such similarity relations and fuzzy orderings are investigated and some applications are outlined. In the end, a Szpilrajn's extension theorem is extended into the setting of fuzzy logic as an example of usefulness and depth of the theory. The utilized axioms of antisymmetry and linearity are different from today's perspective and also from the point of view of the thesis.

Blanchard (1983)

The second work, although overlooked by community (1 citation in Scopus), is very interesting from today's point of view. It is the first paper, which considers definition of fuzzy order in a sense equivalent to those used nowadays.

The motivation of this study was purely theoretical – to asses various candidates for the definition of fuzzy orders. The validity of some form of Szpilrajn's extension theorem is used as the touchstone of worthiness of the given axiom system. In total, four systems are described and then assessed in this way. Out of these candidates, the so called 4-fuzzy orderings are the ones, we will be concerned with (among different definitions) in later chapters.

Höhle and Blanchard (1985)

The next work we mention offers an important observation of a link between a fuzzy ordering and an underlying fuzzy similarity on the given set. Nowadays, this observation is crucial in utilization of fuzzy orderings, but the work was again overlooked by the community for a long time. It has 60 citations in Scopus where all but one are from year 2002 or later. The reason is that around year 2000 this link between order and similarity has been rediscovered independently of this contribution (see below).

The purpose of the paper was to improve initial results on fuzzy ordering obtained by Zadeh (1971). The link described above is captured in this excerpt from the abstract of the work: "In opposition to Zadeh's, our point of view is that an axiom of antisymmetry without a reference to a concept of equality is meaningless." Their setting is that of residuated lattice and they define all the notions in terms of category theory. In spirit of Zadeh's paper, the soundness of their approach is demonstrated by the validity of Szpilrajn's extension theorem generalization.

Interestingly until lately, no connection between both versions of fuzzy order definitions from Blanchard (1983) and Höhle and Blanchard (1985) was established, even though both works had one author in common and were published close in time to each other (see Chapter 3).

[1] It is worth noting that before Zadeh, many-valued orders were considered by Menger (1951) as part of his probabilistic approach to relations.

Höhle (1987)

The fourth work, we find important for the development of fuzzy orders, is concerned with defining fuzzy real numbers as Dedekind cuts. It has 44 citations in Scopus, only seven of which are before the year 2002. Its importance lies in being the first paper defining complete fuzzy lattices as a special kind of fuzzy order respecting the link to underlying fuzzy similarity relation. Interestingly, the used definition of fuzzy order is slightly different than the one by Höhle and Blanchard (1985), but reasons for such modification of the definition are not explained. The difference lies in antisymmetry axiom and we discuss it in more detail in Chapter 3.

Among other notions, the obtained results include Dedekind-MacNeille style completion of any fuzzy order, i.e. embedding of a fuzzy order to a reasonably constrained fuzzy lattice. These results are then applied to a generalization of real numbers into the setting of fuzzy logic, which turns the results into another convincing argument for reasonability and applicability of fuzzy orders defined in this way. It is of interest that almost the same definition was later independently proposed by Belohlavek and led to a significant development of theory of lattice-type fuzzy orders by means of formal concept analysis in the setting of fuzzy logic (see below).

Fuzzy Sets Theory and its Applications conference (1998)

After a long time, two authors – Radim Belohlavek and Ulrich Bodenhofer – came up with the concept of fuzzy order defined with respect to underlying similarity, again. They were not aware of each others research nor the works described above, albeit they were both strongly influenced by Höhle's work on fuzzy logic. Still, they announced their preliminary results on the same conference – Fourth Fuzzy Sets Theory and its Applications conference in Liptovský Ján, 1998. Their definitions are slightly different, but the core idea is same. We cover both definitions in detail in Chapter 3. After this conference, both authors published several papers devoted to their respective notions, although they never got to compare them directly.

Belohlavek (1998 and beyond)

As noted above, Belohlavek published several papers on the topic since 1998, e.g. (Belohlavek, 2001, 2002, 2004). Out of all these works, we cover in some detail (Belohlavek, 2004).[2] Its main topic is the theory of complete lattice-type fuzzy orders, while examples and motivations are based on concept lattices (i.e. hierarchical structures of concepts) generalized into the setting of fuzzy logic. The notions of fuzzy partial order, lattice-type fuzzy order, and fuzzy formal concept are introduced. Also, as a particular application of the approach, Dedekind–MacNeille completion of a partial fuzzy order is described.

Although the results were obtained independently, the used definition of fuzzy order is almost the same as the one utilized by Höhle (1987). That is, similarly to previous two cases, this work follows its specific motivations and arrives to almost the same concept of fuzzy ordering.

The work was highly influential in the community around formal concept analysis, where it sprung the research on its fuzzy counterpart, complete lattice-type fuzzy orders, and related topics. Up to date, it has 399 citations in Scopus.

Bodenhofer (1998 and beyond)

Also Bodenhofer published several papers on the topic since 1998, e.g. (Bodenhofer, 1999a, 2000, 2003). In his case, we mention some details of (Bodenhofer, 2000). The work is devoted to the various notions of fuzzy orders available at that time and shows what they are lacking by means of natural examples such as subsethood relation or implication-induced order. Then the author proceeds by discussion of involved axioms and notes their connection to underlying similarity.

[2] We note that this Belohlavek's first paper on the topic got stuck in the production process: As is apparent from the acknowledgement in this paper and from (Belohlavek, 2001), the 2004 paper was submitted in 2000.

Following this link, he finally obtains the definition of fuzzy order with respect to the underlying similarity relation which is, although obtained independently and in slightly different framework, same as the definition obtained by Höhle and Blanchard (1985). Bodenhofer was apparently not aware of previous work by Höhle and Blanchard in that time, but he acknowledges their historical priority later (Bodenhofer, 2003). The 2000 paper has 92 citations in Scopus so far.

Fan (2001)

Finally, the last contribution we include in this list is (Fan, 2001). This work is concerned with category theoretical research on the so-called Ω-categories. They may seem to be out of the scope of our work, but objects of such categories are just the fuzzy orders defined in the same way as in (Blanchard, 1983). Therefore, although approached with different motivations, the fuzzy orders were independently defined in an equivalent way again. According to Scopus, this paper has been cited 91 times so far.

Note 2.1.1. (a) Although all the mentioned works were independent and had motivations of their own, they arrived to two classes of definitions of fuzzy order. In Chapter 3, these definitions will be studied in some detail. In the end, we will see that all of them have common generalization and that they in fact describe the same class of binary fuzzy relations on a set with some possible limitations given by the context they are utilized in.

(b) We find interesting that there were two independent periods of time, where same alternative definitions of fuzzy order were proposed. First time, it was in the 80s due to Blanchard and Höhle, second time, at the turn of the century due to Belohlavek, Bodenhofer, and Fan.

(c) If we examine an impact these two periods had on fuzzy order research activity, we may see another interesting phenomenon. The first appearance of the definitions remained more or less unnoticed for many years, while the second appearance caused reignition of research on fuzzy orders, their theory, and their applications in other branches of mathematics. This seems to be an another reason why number of new papers on the topic spiked in last 15 years or so. Moreover, thanks to this renewed interest in the topic, also the older works became much more appreciated by the community.

2.2 Szpilrajn-like extension theorem for fuzzy orders

Szpilrajn-like extension theorem in the setting of fuzzy logic was considered already by Zadeh in his seminal paper on fuzzy equivalences and fuzzy orderings (Zadeh, 1971, Theorem 8). This version of the theorem was stated with respect to different concepts of antisymmetry and linearity. See Chapter 3 or (Belohlavek et al., 2017) for in-depth analysis of differences between Zadeh's and our setting. More results on Szpilrajn-like extension principle in the setting of fuzzy logic emerged soon, e.g. (Blanchard, 1983; Chakraborty and Sarkar, 1987; Hashimoto, 1983). Of these works, we once again highlight (Blanchard, 1983) where one of outlined views on fuzzy orders was lately shown to be in a sense equivalent to our view on fuzzy orderings (see Chapter 3). The main distinction lies in the different setting[3] and the fact that Blanchard in general defines the notion of a fuzzy order on a fuzzy set $A \in L^U$.

For the approach to fuzzy orders we utilize, i.e. the one which considers fuzzy equality on the underlying set, the first version of Szpilrajn-like theorem was stated already in (Höhle and Blanchard, 1985) – the work which coined this approach – see their Theorem II.7 and its corollaries. This version of the theorem was stated with respect to $\otimes$-linearity and sligthly different definition of a fuzzy order (see Chapter 3 for in detail comparison of various definitions).

As far as we know, the most detailed study on linearity of fuzzy orderings and related concepts so far is (Bodenhofer and Klawonn, 2004). This study builds upon research on the concept of

[3]That is particular type of residuated lattices where $L = [0, 1]$ and $\otimes = \wedge$.

fuzzy order itself, reignited by Belohlavek and Bodenhofer in the late 1990s to early 2000s. It analyzes several notions of linearity proposed by various authors in the setting of fuzzy order on the set with fixed fuzzy equivalence. The fixing of underlying similarity is the most important difference between their approach and the one utilized in the thesis. In the end, achievability of Szpilrajn-like theorem is studied for several situations, given by used t-norm and axiom of linearity (see their Table 1). Their main results include following observations mentioned in the conclusion of their work: The strong completeness can only serve as an appropriate concept of linearity in the setting of fuzzy logic, if $\otimes = \wedge$; The $\otimes$-linearity coined in (Höhle and Blanchard, 1985) provides preservation of the most important properties of order extension in the setting of residuated lattices on $[0, 1]$. However, it is very weak, non-intuitive, and poorly expressive concept if $\mathbf{L}$ does not have a strong negation.

In a sense, our work on the topic of linear extensions of fuzzy orders builds upon this study. To compare the approaches with fixed underlying similarity and with possibility to modify it together with the order, some of our observations throughout the Chapter 4 are related to their results.

Chapter 3

What is fuzzy order?

As it was indicated in previous parts, the first topic of this thesis is to sum, sort, and scrutinize the various approaches to fuzzy order defined with respect to underlying similarity relation found in the literature. This chapter contains summary of main results obtained in (Belohlavek and Urbanec, 2023a,b) – a two-part study on the concept of fuzzy order itself conducted jointly with Radim Belohlavek.

We focus only on the essential results regarding the concept of fuzzy order in general and its interplay with underlying fuzzy equality in particular. Therefore, we consider only part of the study's content here. Namely, although they are very interesting, we do not cover the results regarding graded point of view on the various properties of fuzzy relations. We also omit all the proofs, auxiliary lemmas, many remarks, and comments which may be of interest to reader later. More details may be found in the thesis or in the study (Belohlavek and Urbanec, 2023a,b). All the definitions, theorems, etc. are accompanied with an exact references into these papers. We present them here in their original form with only exception being a different symbol for a fuzzy equality (see Preliminaries),

3.1 Aim of the chapter

The central topic of the study is same as the one of this thesis – the arguably most developed approach to fuzzy orders, pursued originally by Ulrich Höhle, Nicole Blanchard, Ulrich Bodenhofer, and Radim Belohlavek. This approach is distinctive and significant by its treatment of antisymmetry. It assumes that the underlying universe, the fuzzy order is defined on, is already equipped with a fuzzy similarity relation, i.e. some fuzzy relation which generalizes the concept of classical equality. In fact, the above mentioned authors proposed several definitions of fuzzy order in this sense, where difference between them is mainly in the used axiom of antisymmetry.

Although many papers on fuzzy orders and their properties were published since these pioneering works (see Table 1 in Chapter 2), some basic questions on the concept of fuzzy order itself still remain open. The arguably most important of them is the question of what is an appropriate definition of fuzzy order?

All the above mentioned definitions are examined in detail and their mutual relationships described. Note also that the purpose of the study is not a quest for "the right" definition of fuzzy order which might be considered naive, or even ill-posed. Rather, the study should be approached as an exploration of an approach to fuzzy orders involving antisymmetry with respect to fuzzy equality, possible definitions of such fuzzy order, their common bits, differences, benefits, and drawbacks.

The chapter is organized as follows. We start by examining the definitions per se (Sections 3.2 to 3.5). The rest of the chapter (Sections 3.6 to 3.9) is then devoted to the axiom of antisymmetry.

3.2 Definitions of fuzzy order

Two definitions of fuzzy order on a set equipped with a generalized equality follow. We provide them in the forms used in the works of Bodenhofer and Belohlavek, as these are mostly refered to in literature. There are some mild differences in the forms present in the works by Höhle. We comment on the differences in appropriate places.

Definition 3.2.1 (Belohlavek and Urbanec 2023a, Definition 1; Höhle, Blanchard, Bodenhofer)**.** A *fuzzy order on a set U equipped with a fuzzy equality relation* $\approx$ is a binary fuzzy relation $\lesssim$ on U satisfying

$$\begin{aligned} u \approx v &\leq u \lesssim v, && (\approx\text{-reflexivity}) \\ (u \lesssim v) \otimes (v \lesssim w) &\leq u \lesssim w, && (\text{transitivity}) \\ (u \lesssim v) \otimes (v \lesssim u) &\leq u \approx v, && (\otimes\text{-antisymmetry}) \end{aligned}$$

for each $u, v, w \in U$. (Note: Höhle's and Blanchard's as well as Bodenhofer's original definitions actually assume, more generally, that $\approx$ is a fuzzy equivalence rather than fuzzy equality; this is discussed below.)

Definition 3.2.2 (Belohlavek and Urbanec 2023a, Definition 2; Höhle, Belohlavek)**.** A *fuzzy order on a set U equipped with a fuzzy equality relation* $\approx$ is a binary fuzzy relation $\lesssim$ on U compatible with $\approx$, i.e. fulfilling

$$(u_1 \lesssim v_1) \otimes (u_1 \approx u_2) \otimes (v_1 \approx v_2) \leq u_2 \lesssim v_2$$

for every $u_1, u_2, v_1, v_2 \in U$, which satisfies

$$\begin{aligned} u \lesssim u &= 1, && (\text{reflexivity}) \\ (u \lesssim v) \otimes (v \lesssim w) &\leq u \lesssim w, && (\text{transitivity}) \\ (u \lesssim v) \wedge (v \lesssim u) &\leq u \approx v, && (\wedge\text{-antisymmetry}) \end{aligned}$$

for each $u, v, w \in U$.

If distinction is needed, we shall call fuzzy orders according to Definitions 3.2.1 and 3.2.2 fuzzy orders with $\otimes$-antisymmetry and fuzzy orders with $\wedge$-antisymmetry, respectively. As noted in Chapter 2, both the Definitions 3.2.1 and 3.2.2 were introduced twice in two different time periods.

Definition 3.2.1 was in both cases defined by same conditions as listed above but with respect to a general fuzzy equivalence rather than fuzzy equality. First appearance is due to Höhle and Blanchard (1985) motivated by further study and improvement of the notion of order in the framework of fuzzy logic. The exactly same definition, but in slightly different framework, was later reinvented by Bodenhofer, who was apparently not aware of Höhle and Blanchard's work.

Definition 3.2.2 appeared, though in a little different setting, for the first time in the work by Höhle (1987), where it was stated in the framework of complete residuated lattices on $[0, 1]$ and with the concept of similarity interpreted by general fuzzy equivalence instead of fuzzy equality. It was later reinvented by Belohlavek who was not aware of Höhle's paper, this time in the exactly same form as Definition 3.2.2. See Chapter 2 for more details regarding history of the notion.

There are three obvious distinctions when comparing Definitions 3.2.1 and 3.2.2. First, Definition 3.2.2 assumes compatibility of $\lesssim$ with $\approx$. Second, the Definition 3.2.1 requires $\lesssim$ to be $\approx$-reflexive, while Definition 3.2.2 assumes reflexivity of $\lesssim$ instead. And third, the definitions use different form of antisymmetry where $\otimes$-antisymmetry of Definition 3.2.1 seems to be weaker, i.e. more general, than $\wedge$-antisymmetry of Definition 3.2.2. The aspect of one definition being seemingly more general than the other one is also explored in some detail in subsequent sections.

3.3 Fuzzy equivalence vs. fuzzy equality

As noted in Definition 3.2.1, the original definitions of Höhle, Blanchard, and Bodenhofer assume that $\approx$ is a fuzzy equivalence rather than a fuzzy equality. Fuzzy equality is a particular case of fuzzy equivalence, i.e. fuzzy equivalence moreover satisfying separability. We assume that $\approx$ is fuzzy equality in the Definition 3.2.1 for two reasons. Above all, it provides cleaner generalization of the concept of order into setting of fuzzy logic. Moreover, it allows better comparison of both definitions, as both kinds of fuzzy orders are then considered in the same context. To avoid confusion, we also note that Höhle and Blanchard (1985) use name L-equality for fuzzy equivalence relation. In the rest of the section, this distinction between definitions and reasons for our choice are briefly examined.

In our view, assuming fuzzy equivalence instead of fuzzy equality in Definition 3.2.1 represents generalization among two lines at once. First, the framework of the two-element Boolean algebra is replaced by more general framework of a complete residuated lattice. Second, the identity, i.e. the only equality in Boolean case, is replaced by a fuzzy equivalence.

The essential justification is done by considering both versions of the Definition 3.2.1, i.e. the current one with an equality and the original one with an equivalence, in the setting of classical logic.

On the one hand, the notion resulting from Definition 3.2.1 coincides with the classical notion of order. Namely, fuzzy equality becomes classical equality – identity. The defining conditions then become classical reflexivity, transitivity, and antisymmetry.

On the other hand, the notion emerging from the definition of a fuzzy order on a set with a fuzzy equivalence is not the notion of a classical order. Rather, such relation becomes a slightly restricted classical preorder (i.e. reflexive and transitive binary relation on a set limited by the choice of equivalence). The argumentation is as follows. Fuzzy equivalence becomes classical equivalence $\equiv$. Then, on the set U equipped with $\equiv$, the classical relation $\leq$ is defined, such that $\leq$ contains $\equiv$, is transitive, and satisfies antisymmetry generalized with respect to the equivalence: $u \leq v$ and $v \leq u$ implies $u \equiv v$. The relation $\leq$ is obviously reflexive and transitive, i.e. a preorder. Moreover, since $\equiv$ is contained in $\leq$, we obtain that

$$u \equiv v \quad \text{if and only if} \quad u \leq v \text{ and } v \leq u.$$

That is $\leq$ makes some elements to be lower or equal to each other if and only if the underlying equivalence $\equiv$ makes them equivalent to each other.

In the standard terminology of ordered sets, the relation $\leq$ is a preorder which moreover induces a fixed equivalence $\equiv$. As such, the concept is obviously more general than the concept of classical order which demands $\equiv$ to be the identity.

Let us point out that it is clear from Bodenhofer's papers that he was aware of this property of the definition of fuzzy order assuming fuzzy equivalence as may be seen from Bodenhofer (2000, 2003). His point of view differs from ours as he considers it to be a feature of order-preorder relationship rather than a problem. See (Belohlavek and Urbanec, 2023a) for more details.

Moreover, we note that using a fuzzy equivalence instead of a fuzzy equality also leads to possibly not unique distinguished elements, such as a largest and a smallest element in an ordered set or a supremum and an infimum of some of its subsets. This sort of problems is illustrated by the following example.

Example 3.3.1 (Belohlavek and Urbanec 2023a, Example 1)**.** Let $U = \{u, v, w\}$, let a classical equivalence $\equiv$ be given by the equivalence classes $\{u\}$ and $\{v, w\}$. Then the relation $\leq$ given by $u \leq u$, $v \leq v$, $w \leq w$, $u \leq v$, $u \leq w$, $v \leq w$, and $w \leq v$ is an order on a set with an equivalence in the sense of Höhle, Blanchard, and Bodenhofer. Defining naturally a smallest element x as an element such that $x \leq y$ for every y, and dually for a largest element, it is immediate that u is the only smallest element. On the other hand, both v and w are largest, even though these are two distinct elements.

3.4 Reflexivity and compatibility

Another immediate difference between Definitions 3.2.1 and 3.2.2 is in the axiom of reflexivity. As it turns out, there is hidden interplay of $\approx$-reflexivity and compatibility with respect to $\approx$. We therefore examine these, seemingly unrelated, variances together. We are in the situation where $\approx$-reflexivity required by Definition 3.2.1 is stronger than reflexivity of Definition 3.2.2, while Definition 3.2.2 moreover requires compatibility of the fuzzy order $\lesssim$ with $\approx$.

We start by a rather epistemic observation. In the classical setting, an identity relation is always implicitly given on the universe U and axioms of equality are taken as valid. First part is translated into setting of fuzzy logic by defining fuzzy order with respect to fuzzy equality, but the second part is present only in Definition 3.2.2 – compatibility, i.e. a generalization of the axiom of equality of classical logic.

Nevertheless, it is already known that, given the context of Definitions 3.2.1 and 3.2.2, both options, i.e. $\approx$ reflexivity or reflexivity and compatibility, are equivalent. The argument was observed for the first time by Belohlavek and Vychodil (2005, Lemma 1.82) in the context of fuzzy equivalences on sets with fuzzy equalities and later, independently, by Bodenhofer and Demirci (2008) in the context of fuzzy orders.

Proposition 3.4.1 (Belohlavek and Urbanec 2023a, Corollary 2; Belohlavek and Vychodil 2005; Bodenhofer and Demirci 2008)**.** *Let $\approx$ be a fuzzy equality and $\lesssim$ be transitive. Then $\lesssim$ is $\approx$-reflexive if and only if $\lesssim$ is reflexive and compatible with $\approx$.*

In the study, we examine this relationship thoroughly, taking the graded point of view on all related properties of fuzzy relations. The outcome is general observation (Belohlavek and Urbanec, 2023a, Lemma 1) whose particular corollary – by strengthening initial assumptions – is the proposition above.

Note that an alternative point of view offers itself – assume classical identity is given on each set and define all the other relations, including fuzzy equality, with respect to the identity. Then fuzzy order on U would be defined rather as a tuple $\langle \approx, \lesssim \rangle$ of fuzzy relations on U, where $\approx$ is fuzzy equality and $\lesssim$ meets all the properties required in the Definition 3.2.1 or 3.2.2. Although this point of view is also valid, we prefer to align with the classical situation as much as possible.

3.5 Antisymmetry and constraints regarding fuzzy equality

The last difference between Definitions 3.2.1 and 3.2.2 is the form of antisymmetry axiom. In this section, we take a point of view where the form of antisymmetry is considered as a lower bound for the fuzzy equality. An alternative perspective to consider is, which fuzzy equalities enable the given relation to be fuzzy order on the given set. This alternative perspective is of importance in Chapter 4.

The question is what are the limitations on fuzzy equality $\approx$. The basic answers were already stated for both definitions by different authors. For the case of Definition 3.2.1 it was provided by Bodenhofer (2000) who proved the following proposition.[1]

Proposition 3.5.1 (Bodenhofer 2000, Theorem 18)**.** *If $\lesssim$ is a reflexive and transitive fuzzy relation on U and $\approx$ is a fuzzy equality on U, then $\lesssim$ is a fuzzy order according to Definition 3.2.1 if and only if*

$$(u \lesssim v) \otimes (v \lesssim u) \ \leq \ u \approx v \ \leq \ (u \lesssim v) \wedge (v \lesssim u) \tag{3.1}$$

for every $u, v \in U$.

For the case of Definition 3.2.2 the corresponding result was obtained by Belohlavek (2002) replacing the equality 3.1 by

$$u \approx v \ = \ (u \lesssim v) \wedge (v \lesssim u).$$

[1] In Bodenhofer's setting $\approx$ is general fuzzy equivalence. See last but one section for more details.

In our study, we follow this line up to the lemma (Belohlavek and Urbanec, 2023a, Lemma 2) whose corollary is a stronger version of both propositions above.

Proposition 3.5.2 (Belohlavek and Urbanec 2023a, Corollary 3)**.** *Let* $\lesssim$ *be a fuzzy relation and* $\approx$ *be a fuzzy equality on* U*.*
(a) $\lesssim$ *is* $\approx$*-reflexive and* $\otimes$*-antisymmetric iff*

$$(u \lesssim v) \otimes (v \lesssim u) \ \leq\ u \approx v \ \leq\ (u \lesssim v) \wedge (v \lesssim u).$$

(b) $\lesssim$ *is* $\approx$*-reflexive and* $\wedge$*-antisymmetric iff*

$$u \approx v \ =\ (u \lesssim v) \wedge (v \lesssim u).$$

This proposition together with the equivalence of $\approx$-reflexivity to reflexivity and compatibility in case of transitive relations (cf. Proposition 3.4.1) leads to non-redundant generalization of both the result by Bodenhofer (2000, Theorem 18) and its counterpart for fuzzy orders according to Definition 3.2.2 mentioned above.

Theorem 3.5.3 (Belohlavek and Urbanec 2023a, Theorem 4)**.** *Let* $\lesssim$ *be a transitive fuzzy relation and* $\approx$ *be a fuzzy equality on* U*.*
(a) The following conditions are equivalent:
(a1) $\lesssim$ *is a fuzzy order according to Definition 3.2.1.*
(a2) $\lesssim$ *is reflexive,* $\otimes$*-antisymmetric, and compatible with* $\approx$*.*
(a3) $(u \lesssim v) \otimes (v \lesssim u) \ \leq\ u \approx v \ \leq\ (u \lesssim v) \wedge (v \lesssim u)$*.*
(b) The following conditions are equivalent:
(b1) $\lesssim$ *is a fuzzy order according to Definition 3.2.2.*
(b2) $\lesssim$ *is* $\approx$*-reflexive and* $\wedge$*-antisymmetric.*
(b3) $u \approx v \ =\ (u \lesssim v) \wedge (v \lesssim u)$*.*

Thanks to Theorem 3.5.3 we now have various equivalent conditions for a transitive fuzzy relation $\lesssim$ to become a fuzzy order with respect to Definition 3.2.1 (resp. 3.2.2). In particular, one of these conditions (a3 resp. b3) is expressed only by a relationship between $\lesssim$ and the fuzzy equality $\approx$. The theorem is also a little bit stronger than previous obtained results, as its assumptions do not contain redundancy anymore. Again the study continues in the direction of the theorem above by considering it in the graded setting.

3.6 Alternative definition of antisymmetry and fuzzy order

Considering results obtained in previous sections, the only essential difference between the two concepts of fuzzy order described in Definitions 3.2.1 and 3.2.2 is antisymmetry. In this section, we first examine an alternative form of antisymmetry, called crisp antisymmetry, which is used in the literature in definitions of fuzzy order without reference to underlying fuzzy equality. Then, we continue by stating a common generalization of all the considered forms of antisymmetry and by its means also common generalization of all the considered definitions of fuzzy order.

Crisp antisymmetry appeared for the first time in work by Blanchard (1983) and then was independently rediscovered by Fan (2001). See Chapter 2 for more historical details. As the setting of these works is different than ours, we state it in the form of obvious generalization into the framework of general complete residuated lattices. This generalization appeared in the works of Yao (Yao, 2010; Yao and Lu, 2009).

Definition 3.6.1 (Belohlavek and Urbanec 2023b, Definition 3; Blanchard, Fan)**.** A *fuzzy order on a set* U is a binary fuzzy relation $\lesssim$ on U satisfying

$$u \lesssim u \ =\ 1, \qquad \text{(reflexivity)}$$

$$(u \lesssim v) \otimes (v \lesssim w) \ \leq\ u \lesssim w, \qquad \text{(transitivity)}$$

$$(u \lesssim v) = 1 \text{ and } (v \lesssim u) = 1 \ \text{ imply } \ u = v, \qquad \text{(crisp antisymmetry)}$$

for each $u, v, w \in U$. The pair $\langle U, \precsim \rangle$ shall be called a *fuzzy ordered set* (according to Definition 3.6.1).

The rest of the section is devoted to the relationship between Definition 3.6.1 and Definitions 3.2.1 and 3.2.2. Already Bodenhofer (2003) and Belohlavek (2001, 2002, 2004) observed that fuzzy equality may be avoided in Definitions 3.2.1 and 3.2.2, respectively. Belohlavek utilized that for a fuzzy order according to Definition 3.2.2 $\approx$ is uniquely determined by $\precsim$; and Bodenhofer made various observations on the relationship between $\precsim$ and $\approx$ as regards Definition 3.2.1.

These considerations are tightly related to the results of previous section. Namely, from theorem 3.5.3 (a3 and b3) we may immediately derive same results as authors above. That is, for Definition 3.2.2, $\approx$ is uniquely determined by $\precsim$; and for Definition 3.2.1, it follows that $\approx$ is limited by relations induced by $\precsim$ from both sides.

To examine a relationship of the Definition 3.6.1 to other definitions, we start by observation formulated by Xie et al. (2009) for $\otimes = \wedge$ and Yao (2010) for general complete residuated lattices:

Proposition 3.6.2 (Belohlavek and Urbanec 2023b, Lemma 1)**.** *(a) If $\langle U, \approx, \precsim \rangle$ is a fuzzy ordered set according to Definition 3.2.2, then $\langle U, \precsim \rangle$ is a fuzzy ordered set according to Definition 3.6.1.*

(b) If $\langle U, \precsim \rangle$ is a fuzzy ordered set according to Definition 3.6.1, then $\approx$ defined by

$$u \approx v = (u \precsim v) \wedge (v \precsim u) \tag{3.2}$$

is a fuzzy equality and $\langle U, \approx, \precsim \rangle$ is a fuzzy ordered set according to Definition 3.2.2.

That is situation between Definitions 3.2.2 and 3.6.1 is clear. We now provide another proposition in the spirit of Proposition 3.6.2 regarding the relationship between Definition 3.2.1 and Definition 3.6.1:

Proposition 3.6.3 (Belohlavek and Urbanec 2023b, Lemma 2)**.** *(a) If $\langle U, \approx, \precsim \rangle$ is a fuzzy ordered set according to Definition 3.2.1, then $\langle U, \precsim \rangle$ is a fuzzy ordered set according to Definition 3.6.1.*

(b) If $\langle U, \precsim \rangle$ is a fuzzy ordered set according to Definition 3.6.1, then $\approx$ defined by

$$u \approx v = (u \precsim v) \otimes (v \precsim u) \tag{3.3}$$

is a fuzzy equality and $\langle U, \approx, \precsim \rangle$ is a fuzzy ordered set according to Definition 3.2.1.

Important difference between Propositions 3.6.2 and 3.6.3 is that in the former, the constructions from (a) and (b) are mutually inverse, while in the latter, $\approx$ defined by (3.3) is but one of the possible fuzzy equalities described by (a3) of Theorem 3.5.3.

3.7 A unifying concept of antisymmetry

Having considered the three variants of antisymmetry, namely the $\otimes$-antisymmetry, $\wedge$-antisymmetry, and crisp antisymmetry, we now present their common generalization.

Starting with a complete residuated lattice $\langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$, we consider three different conjunction-like operators on L. First and second are, well known, possible choices of conjunction in complete residuated lattice. Namely, a generalized t-norm (also called strong conjunction or, lately, just t-norm) $\otimes$ and a lattice meet (also called weak conjunction) $\wedge$. For the third, we employ more general conjunction-like operations $\odot$ on L which satisfy

$$\begin{aligned} a \odot b &= b \odot a, && (3.4)\\ a_1 \odot a_2 &\leq b_1 \odot b_2, \text{ whenever } a_1 \leq b_1 \text{ and } a_2 \leq b_2, && (3.5)\\ a \odot 1 &\leq a, \text{ and} && (3.6)\\ 1 \odot 1 &= 1. && (3.7) \end{aligned}$$

Obviously, every generalized t-norm (including $\wedge$) satisfies these conditions. The operator $\odot$ and its defining conditions may be found in (Belohlavek and Urbanec, 2023b) on page 4. Using $\odot$, we define the following notion of antisymmetry:

Definition 3.7.1 (Belohlavek and Urbanec 2023b, page 5)**.** Let $\odot$ satisfy (3.4)–(3.7). A binary fuzzy relation $\lesssim$ on a set U equipped with a fuzzy equality $\approx$ satisfies $\odot$-*antisymmetry* if

$$(u \lesssim v) \odot (v \lesssim u) \leq u \approx v \tag{3.8}$$

for each $u, v \in U$.

It is obvious that both $\otimes$-antisymmetry and $\wedge$-antisymmetry are particular cases of $\odot$-antisymmetry. Surprisingly, the same holds true for notion of crisp antisymmetry which seems to be different at the first sight.

Proposition 3.7.2 (Belohlavek and Urbanec 2023b, Lemma 4)**.** *Consider the binary operation $\bullet$ on L and the fuzzy relation $\approx$ on U defined by*

$$a \bullet b = \begin{cases} 1 & \text{for } a = 1 \text{ and } b = 1, \\ 0 & \text{otherwise;} \end{cases} \qquad u \approx v = \begin{cases} 1 & \text{for } u = v, \\ 0 & \text{otherwise.} \end{cases} \tag{3.9}$$

Then $\bullet$ satisfies (3.4)–(3.7) and $\approx$ is a fuzzy equality (the crisp fuzzy equality). Moreover, a binary fuzzy relation $\lesssim$ on U satisfies crisp antisymmetry if and only if it satisfies $\bullet$-antisymmetry.

3.8 Equivalence of definitions of fuzzy order

Having generalized notion of antisymmetry at hand, we utilize it to state another definition of fuzzy order which subsumes all the previous ones, i.e. Definitions 3.2.1, 3.2.2, and 3.6.1.

For this purpose we consider the following fuzzy relations on a given universe U:

$\lesssim$ ... a reflexive and transitive fuzzy relation on U,

$\equiv_\odot$ ... a fuzzy relation defined by

$$u \equiv_\odot v \; = \; (u \lesssim v) \odot (v \lesssim u), \tag{3.10}$$

$\approx_\odot$ ... the transitive closure of $\equiv_\odot$, i.e.

$$u \approx_\odot v = [\mathrm{Tra}(\equiv_\odot)](u, v). \tag{3.11}$$

Note that (Belohlavek and Urbanec, 2023b) contains a thorough analysis of properties of $\equiv_\odot$ and $\approx_\odot$, which are often utilized in proofs of observations in this section. The most important of these observation are summarized in following proposition.

Proposition 3.8.1 (Belohlavek and Urbanec 2023b, Lemma 9)**.** *Let $\odot$ satisfy (3.4)–(3.7) and $\lesssim$ be a reflexive and transitive fuzzy relation on U.*

(a) $\approx_\odot$ is a fuzzy equivalence on U.

(b) The following conditions are equivalent:

(b1) $\approx_\odot$ is a fuzzy equality;

(b2) $\equiv_\odot$ is separable;

(b3) $\lesssim$ satisfies crisp antisymmetry.

(c) If $\odot$ is a t-norm which dominates $\otimes$, then $\equiv_\odot \, = \, \approx_\odot$.

We are ready to state the aforementioned generalized definition of fuzzy order.

Definition 3.8.2 (Belohlavek and Urbanec 2023b, Definition 4)**.** Let $\odot$ satisfy (3.4)–(3.7). A *fuzzy order on a set U equipped with a fuzzy equality relation* $\approx$ is a binary fuzzy relation $\lesssim$ on U compatible with $\approx$, i.e. satisfying

$$(u_1 \lesssim v_1) \otimes (u_1 \approx u_2) \otimes (v_1 \approx v_2) \ \leq \ u_2 \lesssim v_2,$$

for every $u_1, u_2, v_1, v_2 \in U$, which, moreover, fulfills

$$u \lesssim u \ = \ 1, \qquad \text{(reflexivity)}$$
$$(u \lesssim v) \otimes (v \lesssim w) \ \leq \ u \lesssim w, \qquad \text{(transitivity)}$$
$$(u \lesssim v) \odot (v \lesssim u) \ \leq \ u \approx v, \qquad (\odot\text{-antisymmetry})$$

for each $u, v, w \in U$.

First note that Definition 3.8.2 indeed encompasses the notion of fuzzy order according to Definition 3.2.2 and, by (a) of Theorem 3.5.3, also Definition 3.2.1. Moreover, as for the crisp fuzzy equality the compatibility condition is trivially satisfied, it also generalizes Definition 3.6.1 (cf. Proposition 3.7.2). Namely,

- for $\odot = \otimes$, Definition 3.8.2 yields Definition 3.2.1;
- for $\odot = \wedge$, Definition 3.8.2 yields Definition 3.2.2;
- for $\odot = \bullet$, Definition 3.8.2 yields Definition 3.6.1.

The last two theorems of this section examine the mutual relationships between all the definitions of fuzzy order. We first state a theorem in a spirit of Theorem 3.5.3 for the concept defined by Definition 3.8.2.

Theorem 3.8.3 (Belohlavek and Urbanec 2023b, Theorem 1)**.** *Let $\lesssim$ be a reflexive and transitive fuzzy relation on U. The following conditions are equivalent:*

(a) There exists $\odot$ satisfying (3.4)–(3.7) and a fuzzy equality $\approx$ such that $\lesssim$ is a fuzzy order on U equipped with $\approx$ according to Definition 3.8.2.

(b) For each $\odot$ satisfying (3.4)–(3.7) there exists a fuzzy equality $\approx$ such that $\lesssim$ is a fuzzy order on U equipped with $\approx$ according to Definition 3.8.2.

(c) There exists $\odot$ satisfying (3.4)–(3.7) such that $\lesssim$ is a fuzzy order on U equipped with $\approx_\odot$ according to Definition 3.8.2.

(d) For each $\odot$ satisfying (3.4)–(3.7), $\lesssim$ is a fuzzy order on U equipped with $\approx_\odot$ according to Definition 3.8.2.

(e) There exists $\odot$ satisfying (3.4)–(3.7) and a fuzzy equality $\approx$ on U such that $\equiv_\odot \leq \approx \leq \equiv_\wedge$.

(f) For each $\odot$ satisfying (3.4)–(3.7) there exists a fuzzy equality $\approx$ on U such that $\equiv_\odot \leq \approx \leq \equiv_\wedge$.

Finally, the notions of fuzzy order according to Definitions 3.2.1, 3.2.2, 3.6.1, and 3.8.2 are essentially mutually equivalent.

Theorem 3.8.4 (Belohlavek and Urbanec 2023b, Theorem 2)**.** *Let $\lesssim$ be a reflexive and transitive fuzzy relation on U. Each of the following conditions is equivalent to any of conditions (a)–(f) in Theorem 3.8.3. (Thus, in particular, the following conditions are mutually equivalent.)*

(a) $\lesssim$ is a fuzzy order according to Definition 3.2.1 for some fuzzy equality $\approx$.

(b) $\lesssim$ is a fuzzy order according to Definition 3.2.2 for some fuzzy equality $\approx$.

(c) $\lesssim$ is a fuzzy order according to Definition 3.6.1.

As a concluding note of this section, let us remark that other definitions of the general notion of fuzzy order may be formulated. For example, it is easy to verify using previous results that the following conditions are equivalent for a fuzzy relation $\lesssim$ on U and for any $\odot$ satisfying (3.4)–(3.7).

- $\lesssim$ is a fuzzy order according to Definition 3.8.2 for some fuzzy equality $\approx$;
- $\lesssim$ is transitive and the induced fuzzy relation $\equiv_\odot$ is reflexive and separable;
- $\lesssim$ is transitive and the induced fuzzy relation $\approx_\odot$ is a fuzzy equality.

3.9 Distinctive properties of the variants of antisymmetry and fuzzy order

We now know that the choice of a variant of antisymmetry condition – and therefore of fuzzy order definition – is to some extent a matter of taste. Still, it is of importance to know advantages and disadvantages of each such choice. We state only four theorems obtained in the study, as they are self explaining. Reader interested in more details may consult (Belohlavek and Urbanec, 2023b), which contains not only the proofs, but also some additional remarks to each of the following results.

Theorem 3.9.1 (Belohlavek and Urbanec 2023b, Theorem 3). *Let $\odot$ satisfy (3.4)–(3.7) and let $\lesssim$ be a fuzzy order according to Definition 3.8.2 for some fuzzy equality $\approx$.*

(a) The operation $\bullet$ defined by (3.9) is the smallest operation satisfying (3.4)–(3.7);

hence $\bullet$ is the smallest operation $\odot$ for which $\lesssim$ is a fuzzy order according to Definition 3.8.2 for some fuzzy equality $\approx$.

(b) The operation $\wedge$ is the largest operation $\odot$ satisfying (3.4)–(3.7);

hence $\wedge$ is the largest operation $\odot$ for which $\lesssim$ is a fuzzy order according to Definition 3.8.2 for some fuzzy equality $\approx$.

Theorem 3.9.2 (Belohlavek and Urbanec 2023b, Theorem 4). *Of all the operations $\odot$ satisfying (3.4)–(3.7) for which a given fuzzy relation $\lesssim$ is a fuzzy order according to Definition 3.8.2 for some fuzzy equality $\approx$, $\otimes$ is the only one that satisfies adjointness w.r.t. $\to$, i.e.*

$$a \odot b \le c \quad \text{iff} \quad a \le b \to c \quad \text{for every } a, b, c \in L.$$

Theorem 3.9.3 (Belohlavek and Urbanec 2023b, Theorem 5). *Let $\mathbf{L}$ be an arbitrary complete residuated lattice and let U have at least two elements. Then $\wedge$ is the only operation $\odot$ satisfying (3.4)–(3.7) such that for each fuzzy order $\lesssim$ according to Definition 3.8.2, the interval $\mathcal{I}_\odot$ is a singleton. Hence, $\wedge$ is the only operation satisfying (3.4)–(3.7) for which $\approx$ is uniquely determined by $\lesssim$.*

Theorem 3.9.4 (Belohlavek and Urbanec 2023b, Theorem 6). *Let $\lesssim$ be reflexive and transitive fuzzy relation on U.*

(a) The largest reflexive and symmetric fuzzy relation contained in $\lesssim$ (i.e. the most informative indistinguishability w.r.t. $\lesssim$ in the sense above) is $\equiv_\wedge$, which is also the largest reflexive, symmetric, and transitive fuzzy relation contained in $\lesssim$.

(b) The least reflexive, symmetric, and transitive fuzzy relation contained in $\lesssim$ is $\equiv_\bullet$.

Chapter 4

Linear extensions of fuzzy orders

Extending a partial order into a chain is a classical problem in order theory. For fuzzy orders, such Szpilrajn-like completion was considered already by Zadeh (1971) when he introduced the concept of fuzzy order itself. These consideration were soon to be followed by others but many questions still remain open. One of the most recent study (Bodenhofer and Klawonn, 2004) on the topic analyzes different axioms for linearity of a fuzzy order in some detail. Surprisingly, the outcome of the study is that a completion of a fuzzy order with desirable properties is reachable only for very weak axiom of $\otimes$-linearity. We show that this is related to structure of fuzzy equalities on a set which is much richer than its counterpart in the Boolean case. Moreover, we propose a solution to fuzzy order completion problem by manipulating both entities, i.e. a fuzzy order and its induced fuzzy equality together, in a compatible way. Using this idea, which may be regarded as further extension of reflections on the role of fuzzy equality in the definition of fuzzy order in the spirit of Chapter 3, we obtain a way to extend any fuzzy order into linear fuzzy order in a broad class of fuzzy logics.

In this chapter, we summarize the results obtained in (Urbanec, 2023), i.e. the last study this thesis is built upon. Again, we present only the essential results and omit a lot of other content, such as auxiliary propositions and proofs. The full results may be found in the study itself.

4.1 A structure of fuzzy equalities on a finite set

The first theorems describe the structure of all fuzzy equalities on a finite set. This structure is more intricate in the setting of fuzzy logic than in the classical case as it is not limited to a single equality, i.e. the identity. Although it is interesting by itself, our primary objective is to examine the properties of linear fuzzy order extensions. We show in further sections that there is a connection between this structure of fuzzy equalities and possibility of extending general fuzzy order into a linear one. Here, we focus only on conditions under which the structure of all fuzzy equalities on a finite set forms a lattice. In Section 4.3, we will see that the same conditions characterize the class of residuated lattices which admits linear extension of arbitrary fuzzy order for a particular form of linearity.

Theorem 4.1.1 (Urbanec 2023, Theorem 1)**.** *Let U be a finite set with at least two elements. The set of all $\mathbf{L}$-equalities on U equipped with subsethood relation forms a lattice if and only if $\mathbf{L}$ has a join-irreducible unit.*

In case U has less than two elements, such structure is a one-element complete lattice.

To ensure that a lattice of all $\mathbf{L}$-equalities on a finite set is a complete one, even stronger conditions must be imposed on $\mathbf{L}$.

Theorem 4.1.2 (Urbanec 2023, Theorem 2)**.** *Let U be a finite set with at least two elements. The set of all $\mathbf{L}$-equalities on U equipped with subsethood relation forms a complete lattice if and* *only if* **L** *has unit irreducible by arbitrary joins, i.e. if and only if there is no set D of degrees from $L \setminus \{1\}$ with $\bigvee D = 1$.*

4.2 Completeness and linearity of binary fuzzy relation

There are various notions of completeness and linearity used in the theory of binary fuzzy relations on a set. Here, we are interested in linear fuzzy orderings, i.e. we focus on completeness of a fuzzy order relation in a sense of arbitrary two elements in a set being fully comparable. Even in this sense, there are multiple approaches to the concept of linearity in the literature. We discuss only strong completeness – the most widespread of these properties – and so-called crisp linearity (see below), here. For some other options and their mutual relationships see full results in (Urbanec, 2023).

Definition 4.2.1 (Urbanec 2023, Definition 4)**.** Binary fuzzy relation R on a set U is *strong complete* if

$$R(u,v) \vee R(v,u) = 1 \qquad \text{(strong completeness)}$$

holds for every $u, v \in U$.

Usually, the works utilizing the notion of linearity of fuzzy orders use only linear residuated lattices, especially the ones given by (left) continuous t-norms. As we use general complete residuated lattices, we need to discuss another aspect of linearity. Namely the expected meaning of linearity. Assume the strong completeness in some residuated lattice **L** with join-reducible unit. Then, for some $a, b \in L \setminus \{1\}$ such that $a \vee b = 1$, even relation R on $U = \{u, v\}$ where $R(u,u) = R(v,v) = 1$, $R(u,v) = a$, and $R(v,u) = b$ is strong complete, i.e. linear in the given setting. This situation might be considered as unnatural – R is a linear ordering where no element of pair u, v is fully above the other one. Therefore, we define yet another concept of linearity, which assures that such situation does not arise.

Definition 4.2.2 (Urbanec 2023, Definition 5)**.** Binary fuzzy relation R on a set U is *crisp linear* if

$$R(u,v) = 1 \text{ or } R(v,u) = 1 \qquad \text{(crisp linearity)}$$

holds for every $u, v \in U$.

As every crisp linear binary fuzzy relation is obviously strong complete, the existence of crisp linear extension of a relation R implies the existence of strong complete extension of R. Note also that in case of residuated lattice with join-irreducible unit, in particular in any residuated lattice on $[0,1]$, a binary fuzzy relation is crisp linear if and only if it is strong complete.[1] In the rest of the chapter, we examine when a fuzzy order extension into crisp linear fuzzy order exists and some derived notions for these cases.

4.3 Extensions and Szpilrajn-like theorem for fuzzy orders

In this section, we discuss the existence of a linear extension of any fuzzy order. The core idea differentiating our approach from previous studies is considering also the induced fuzzy equality in the extension process. It may be seen as further extension of reflections on the role of fuzzy equality in the definition of fuzzy order as presented in Chapter 3. There are two main reasons why we do so.

[1] The up to date most detailed study of linearity axioms for fuzzy orderings (Bodenhofer and Klawonn, 2004) use the setting of left-continuous t-norms on the interval $[0,1]$, therefore some of our results may be easily related to the results obtained there.

First, fixing the fuzzy equality is, in our opinion, point of view which comes from Boolean setting where there is only one equality and therefore no reason to think about its modifications together with other entities in the given situation. We think that there is no general justification of the same view when there is many fuzzy equalities available on the given universe. That is a possibility of strengthening or weakening the given equality may be taken as new and advantageous aspect in the setting of fuzzy logic which is degenerated in the Boolean case.

Second reason has same root cause but immediate practical consequences: Fixing the fuzzy equality in the beginning of an extension process limits the situation by a great deal. In fact the main reason, why the results on linear extensions of fuzzy orders are quite pessimistic so far (Bodenhofer and Klawonn, 2004), is that a fuzzy equality[2] induced by a resulting linear fuzzy order has to obey same limits as the one induced by an initial fuzzy order. We start by recalling the definition of an extension of a binary fuzzy relation, in particular of a fuzzy order.[3]

Definition 4.3.1 (Urbanec 2023, Definition 6)**.** Let R, S, and $\lesssim$ be binary fuzzy relations on U.

- We call S an *extension of* R if $R \subseteq S$. If $R \subset S$ we call S a *proper extension of* R.
- If $\lesssim$ is a fuzzy order on a set with fuzzy equality $\langle U, \eqsim \rangle$, we call a fuzzy order $\lesssim'$ on a set with fuzzy equality $\langle U, \eqsim' \rangle$ a *fuzzy order extension of* $\lesssim$ if $\lesssim'$ is an extension of $\lesssim$ and $\eqsim'$ is an extension of $\eqsim$.

Utilizing the idea described above, we arrive to conclusion that in broad class of complete residuated lattices, including every complete residuated lattice on $[0, 1]$, every fuzzy order may be extended into a crisp linear fuzzy order.

Theorem 4.3.2 (Urbanec 2023, Theorem 5)**.** *A residuated lattice* $\mathbf{L}$ *has a join-irreducible unit if and only if for every set equipped with* $\mathbf{L}$*-equality* $\langle U, \eqsim \rangle$*, for any* $u, v \in U$*, and for each* $\mathbf{L}$*-order* $\lesssim$ *on* $\langle U, \eqsim \rangle$ *there is a crisp linear fuzzy order extension* $\lesssim'$ *of* $\lesssim$ *on* U *such that* $u \lesssim' v = u \lesssim v$.

Note that the condition of keeping comparability degree of u to v unchanged is of importance later, in Section 4.4, where the intersection representation of fuzzy orders is discussed. But if we omit it now, we obtain a straightforward generalization of classical Szpilrajn's extension theorem to the setting of residuated lattices and crisp linearity. Note that assumption of join-irreducibility of the residuated lattice's unit can not be dropped for crisp linearity (cf. Urbanec 2023).

Corollary 4.3.3 (Urbanec 2023, Corollary 6; Extension theorem for crisp linearity)**.** *Let* $\mathbf{L}$ *be a residuated lattice with join-irreducible unit. For any set* U *and any* $\mathbf{L}$*-order* $\lesssim$ *on* U *there is a crisp linear* $\mathbf{L}$*-order extension of* $\lesssim$.

If one tries to implement similar construction in the setting, where an underlying similarity is interpreted by a general fuzzy equivalence, it becomes rather trivial. The reason is separability of induced relation being the only limiting factor here. In such case, every fuzzy order has a linearization fulfilling any reasonable property of completeness as every fuzzy order may be extended into full relation on the given set. In the spirit of Section 3.3, we consider it to be another manifestation of fuzzy equivalences being inappropriate choice for the interpretation of underlying similarity.

We conclude this section by Example 4.3.4, which shows natural fuzzy orders without crisp linear extensions in the setting of residuated lattices with join-reducible unit.

Example 4.3.4 (Urbanec 2023, Example 2)**.** It is well known (Belohlavek, 2002; Bodenhofer, 1999a; Höhle and Blanchard, 1985) that for any complete residuated lattice, the function $\to$

[2] Fuzzy equivalence in the case of (Bodenhofer and Klawonn, 2004). The idea remains the same, though.

[3] Note that thanks to the idea of manipulating both $\lesssim$ and $\eqsim$ together and to the alternative point of view on definitions of fuzzy order we presented in Chapter 3, it does not matter which of considered definitions of fuzzy order we use in this chapter.

is a fuzzy order on the set L of truth degrees equipped with the fuzzy equality induced by $\leftrightarrow$ on L. Such fuzzy order is moreover isomorphic to L^U for a singleton $U = \{u\}$ where $\to$ is lifted to $\subsetapprox$ on L^U and $\leftrightarrow$ becomes $\approx$ on L^U. That is, from one point of view, this fuzzy order is a generalization of an important order induced by truth function of implication known from classical logic, and from another point of view, it is a generalization of classical power set ordered by the set inclusion.

Let $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ be any residuated lattice with unit join-reducible by $a, b \in L\setminus\{1\}$, i.e. where $a \vee b = 1$. Such residuated lattices exist, e.g. Heyting algebra on $L = \{0, a, b, 1\}$ with $0 \leq a \leq 1$, $0 \leq b \leq 1$ and $x \leq x$ for each $x \in L$. Now in case of $\mathbf{L}$, neither of the $\mathbf{L}$-orders described above can be extended into crisp linear fuzzy order unless two elements of L (resp. L^U), namely a and b (resp. $\{{}^a/u\}$ and $\{{}^b/u\}$), are factorized into one.

4.4 Intersection representation of fuzzy orders

Another important property of ordering relations in the Boolean case is an intersection representation of any order in the spirit of (Dushnik and Miller, 1941, Theorem 2.32) which was breifly described in Preliminaries. Utilizing the generalized version of Szpilrajn theorem from previous section, we obtain the similar intersection representation of fuzzy orders in a straightforward manner.

Theorem 4.4.1 (Urbanec 2023, Theorem 8)**.** *A residuated lattice* $\mathbf{L}$ *has a join-irreducible unit if and only if for every set* U *equipped with* $\mathbf{L}$*-equality* $\approx$ *and every* $\mathbf{L}$*-order* $\lesssim$ *on* $\langle U, \approx\rangle$*, there is a set* $Ext(\lesssim)$ *of crisp linear* $\mathbf{L}$*-order extensions of* $\lesssim$ *such that* $[\bigcap Ext(\lesssim)](u, v) = u \lesssim v$ *for each* $u, v \in U$*.*

Two final corollaries of the results proven in previous sections, i.e. version of Szpilrajn's theorem for crisp linearity and other related results, are: First, equivalent characterizations of all residuated lattices $\mathbf{L}$ with join-irreducible unit; Second, generalizations of two well known theorems of classical order theory – Szpilrajn's extension theorem (Szpilrajn, 1930) and intersection representation theorem (Dushnik and Miller, 1941, Theorem 2.32), see Preliminaries – into the setting of complete residuated lattices on $[0, 1]$ and strong completeness.[4]

Corollary 4.4.2 (Urbanec 2023, Corollary 9)**.** *The following propositions are equivalent:*

1. *The residuated lattice* $\mathbf{L}$ *has a join-irreducible unit.*
2. *For any finite set* U*, the set of all* $\mathbf{L}$*-equalities on* U *ordered by set inclusion forms a lattice.*
3. *Any finite* $\mathbf{L}$*-order may be extended into crisp linear* $\mathbf{L}$*-order.*
4. *An arbitrary* $\mathbf{L}$*-order may be extended into crisp linear* $\mathbf{L}$*-order.*
5. *An arbitrary* $\mathbf{L}$*-order may be represented as an intersection of some set of its crisp linear fuzzy order extensions.*

Corollary 4.4.3 (Urbanec 2023, Corollary 10)**.** *Let* $\mathbf{L}$ *be a complete residuated lattice on* $[0, 1]$ *and* U *an arbitrary set equipped with an* $\mathbf{L}$*-equality* $\approx$*. Then for each* $\mathbf{L}$*-order* $\lesssim$ *on* $\langle U, \approx\rangle$ *we have*

1. *There is a strong complete* $\mathbf{L}$*-order extension of* $\lesssim$ *on* U*.*
2. *There is a set* $Ext(\lesssim)$ *of strong complete* $\mathbf{L}$*-orders on* U *such that* $u \lesssim v = [\bigcap Ext(\lesssim)](u, v)$ *for each* $u, v \in U$*.*

[4]The conditions of crisp linearity and strong completeness coincide in any complete residuated lattice on $[0, 1]$. Thus we preffer to call the condition strong completeness here because it is a well established name.

4.5 A note on the essential properties of chains

Last class of results obtained in (Urbanec, 2023), we present here, are essential properties of extension process in the classical setting and their translation into the setting of fuzzy logic. In their study, Bodenhofer and Klawonn (2004) have identified three essential properties of partial orderings which are desirable also in the setting of fuzzy logic: an existence of linear extension; a possibility of an order representation by an intersection of its linear extensions; and the equivalence between maximality and linearity of an order. In addition, they have shown that if linearity is interpreted by the strong completeness then none of these properties can be attained unless we use the Gödel t-norm logic.[5] As their setting is the one of left continuous t-norms on $[0,1]$, i.e. particular linear residuated lattices, the concepts of strong completeness and crisp linearity coincide.

We have already seen how our approach improves these results by realizing that both order and equality relations have to be manipulated together. A short comment on each of these properties follows.

Existence of complete extension As we have already seen in Corollary 4.3.3, for suitable residuated lattices (including all the t-norm logics on $[0,1]$) each fuzzy order may be extended into a crisp linear fuzzy order. That is in the given setting the situation is analogous to the Boolean case.

Intersection representation Theorem 4.4.1 describes a representation of any fuzzy order by an intersection of its crisp linear fuzzy order extensions for suitable residuated lattices. Again, there is an obvious analogy to the Boolean case.

Maximality vs linearity The last relationship is more complex in the setting of fuzzy logic than in the Boolean case. The difference can be seen already from the following definition of maximality of fuzzy equality and fuzzy order.

Definition 4.5.1 (Urbanec 2023, Definition 7)**.**
A fuzzy equality $\approx$ on U is *maximal* if there is no fuzzy equality $\approx'$ properly extending $\approx$ on U. A fuzzy order $\lesssim$ on $\langle U, \approx \rangle$ is *maximal on U* if there is no fuzzy order $\lesssim'$ on $\langle U, \approx' \rangle$ properly extending $\lesssim$.

Utilizing results on representation of strong complete (pre)orders obtained by Bodenhofer (1999b, Theorem 4), we may derive that maximality of crisp linear fuzzy order is given by maximality of its induced fuzzy equality.

Theorem 4.5.2 (Urbanec 2023, Theorem 11)**.** *Let $\mathbf{L}$ be a residuated lattice with join-irreducible unit. Then following propositions hold for every $\mathbf{L}$-order $\lesssim$ on a set with an $\mathbf{L}$-equality $\langle U, \approx \rangle$:*

1. *If $\lesssim$ is a maximal $\mathbf{L}$-order on U then it is crisp linear.*
2. *If $\lesssim$ is a crisp linear $\mathbf{L}$-order on $\langle U, \approx \rangle$ then it is a maximal $\mathbf{L}$-order on U if and only if $\approx$ is a maximal $\mathbf{L}$-equality on U.*

Therefore we see that, because of a much more complex structure of all equalities on a set, the one-to-one relationship between linear and maximal orders from the Boolean case is lost in the setting of fuzzy logic.

[5] Their definition of fuzzy order assumes fuzzy equivalence on the set (cf. Section 3.3), but the core idea remains same even in the case of fuzzy equality.

Chapter 5

Conclusions and further topics

In this thesis, we summarized our considerations on the existing approaches to fuzzy order defined with respect to an underlying generalized equality. We first set up a historical context and then examined the definitions and their mutual relationships in some detail. We provided various observations to enhance the current understanding of the concept of fuzzy order and proposed the generalized point of view. Then we moved our attention to a classical problem of order theory, a Szpilrajn-like extension theorem, generalized for fuzzy orders. By doing so, we shed more light on the problem present in the literature since the inception of the concept of fuzzy order itself.

There are two categories of results we consider most important. First, a unifying concept of antisymmetry together with the resulting generalized notion of fuzzy order. These considerations yielded the theorems showing that the existing variants of the notion of fuzzy order defined with respect to a fuzzy equality are in a sense mutually equivalent and are moreover equivalent to our generalized concept of fuzzy order. This is in contrast to current understanding that the definitions are different, some of them being more general than others. An alternative perspective one can adopt is that the various available definitions differ only in the limits they impose on the underlying similarity relation; however, despite these different limits, the class of fuzzy relations they describe is always the same.

The other category is then determined by the idea that the dependence of fuzzy order on underlying equality is only half of the story because the dependence is actually mutual. That is both relations should always be considered and manipulated together. First manifestation of this perspective is apparent in the equivalence of the various definitions of fuzzy order. Continuing the line of this perspective, we arrived to the Szpilrajn-like extension theorem for fuzzy orders. Here, we showed that thanks to manipulating both the entities together, we may extend any fuzzy order into a crisp linear one in a broad class of residuated lattices, including all residuated lattices on $[0, 1]$.

For the future, quite many lines of research offers themselves naturally. The most interesting is a dimension theory for fuzzy orders. We have seen a small taste of classical dimension theory together with its most famous results in Preliminaries. Generalizing these results into setting of fuzzy logic seems to be a good starting point in this direction. We already have some preliminary results and shall present them in future publications.

The second topic worth of further attention is that of lattice-type fuzzy orders and how our observations affect them. As mentioned in Chapter 2, lattice-type fuzzy orders are, similarly to the classical case, one of the main driving forces behind research conducted on fuzzy orders. Also in this area, we already have some interesting preliminary observations.

Among the topics, which we would like to focus on in the long term, are deeper applications of fuzzy order outside of formal concept analysis, as there are not many of them now. We feel that various applications may attract further attention and thus help to broaden the knowledge of fuzzy orders. The second long term topic we mention, in a sense related to applications, is considering our results from the perspective of category theory as fuzzy (pre)orders appear in the categorical works quite often.

Finally, one very interesting observation, rather methodological than mathematical, is hidden between the lines of this thesis. All the results were actually being developed simultaneously and were affecting each other. Often, they were gradually updated by switching there and back between theoretical and applied side of the central question "What is fuzzy order?". Many of them were scratched on the way, completely rebuilt, or suddenly appeared from nowhere. Only then I have fully appraised the idea that my supervisor often mentions: mathematics is in a sense "experimental" science where one has to "experiment" and "play" with the concepts. Indeed, one has to test the concepts he is considering by "playing" with them in as many contexts as he can and update his understanding of the theory accordingly, even if it means starting from scratch again. For me personally, this is one of most important lessons I take from the time spent working on this topic and I am grateful to my supervisor for it.

Shrnutí v českém jazyce

V této práci jsme se zabývali existujícími přístupy k fuzzy uspořádáním definovaným vzhledem ke zobecněné rovnosti na uvažovaném univerzu. Nejprve jsme stručně popsali původ pojmu a jeho historii. Poté jsme shrnuli existující přístupy, přidali k nim nová pozorování a postřehy a nakonec je zastřešili novým, obecnějším pohledem. Svá pozorování, zejména ta o těsnější vazbě mezi fuzzy uspořádáním a rovností na uvažovaném univerzu, jsme dále využili k získání nových poznatků o rozšiřování fuzzy uspořádání ve stylu Szpilrajnovy věty.

Za nejdůležitější považujeme dva typy výsledků dosažených v této práci. Prvním je již zmíněný zobecněný pohled na antisymetrii a tedy i na fuzzy uspořádání jako takové. Tato pozorování vyústila v sérii vět ukazujících, že všechny námi uvažované pohledy na fuzzy uspořádání, včetně nově navrženého, jsou v jistém smyslu ekvivalentní. Tento poznatek rozporuje často přijímaný pohled, kde jsou některé z definic považovány za obecnější než jiné. Alternativně lze tyto výsledky interpretovat jako pozorování, že uvažované, dosud dostupné definice fuzzy uspořádání se vzájemně liší pouze v omezeních, která kladou právě na fuzzy rovnost uvažovanou na univerzu. Všechny ale popisují stejnou množinu fuzzy relací.

Druhá třída výsledků je poté odvozena od souvisejícího pozorování, že závislost mezi fuzzy uspořádáním a fuzzy rovností je vzájemná. Tedy chceme-li fuzzy uspořádání v dané situaci nějakým způsobem upravit, tak se tyto úpravy musí vhodně odrážet i na příslušné fuzzy rovnosti. Toto pozorování je do jisté míry vidět již na ekvivalenci definic uvedené výše. V plné šíři jsme jej ale využili při úvahách o rozšiřování fuzzy uspořádání v duchu Szpilrajnových výsledků. Díky těmto úpravám obou relací zároveň jsme popsali postup pro získání lineárního rozšíření fuzzy uspořádání v mnoha různých fuzzy logikách, zejména pak ve všech, kde jsou stupně pravdivosti interpretovány intervalem $[0, 1]$.

Dosažené výsledky nabízejí několik směrů pro budoucí výzkum. Nejzajímavějším z nich je dimenze fuzzy uspořádání, v duchu výsledků dosažených Dushnikem a Millerem (1941) pro klasická uspořádání. Dalším, neméně důležitým tématem je vliv našich pozorování na svazová fuzzy uspořádání. Svazová fuzzy uspořádání jsou pravděpodobně nejvíce prozkoumaným typem fuzzy uspořádání a mají mnoho aplikací zejména v kontextu formální konceptuální analýzy nad fuzzy logikou. V obou těchto směrech již máme základní výsledky, které plánujeme představit v budoucích pracích.

Z dlouhodobějšího pohledu bychom se chtěli věnovat i dalším, různorodým, hlouběji zpracovaným aplikacím fuzzy uspořádání, neboť tyto dle našeho názoru zatím chybí, zejména ve srovnání s množstvím aplikací klasických uspořádání. Naší nadějí je, že důkladný popis zajímavých aplikací povede k dalšímu rozvoji fuzzy uspořádání i v teoretické rovině. Druhý dlouhodobější cíl do jisté míry souvisí s tím prvním – zvážit dosažené výsledky z pohledu teorie kategorií, kde jsou fuzzy (před)uspořádání poměrně často uvažována v různých kontextech.

Závěrem vyzdvihneme jedno pozorování, které je spíše metodologické nežli matematické. Všechny dosažené výsledky byly ve skutečnosti budovány zároveň a často se vzájemně ovlivňovaly. Zejména posun v jednom směru často způsobil výrazné změny v uvažování o směru druhém – již dosažené výsledky musely být znovu zváženy, upraveny, či dokonce zahozeny; některé myšlenky se pak díky změně kontextu objevily jakoby z ničeho. Až při těchto momentech jsem plně docenil myšlenku často zmiňovanou mým vedoucím: i v matematice mají experimenty své místo. Vskutku, teoretické výsledky se projasňovaly a zpřesňovaly s každým kontextem, ve kterém jsme dané koncepty uvažovali a experimentovali s nimi. A naopak, nové úvahy o aplikacích se samy nabízely s každým posunem v teoretických poznatcích. Osobně, tuto zkušenost považuji za jednu z nejdůležitějších, kterou si z práce na tomto tématu odnáším, a jsem za ni svému vedoucímu vděčný.

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