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DISSERTATION THESIS

Operators on Ordered Algebras

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Abstrakt: Topologické Booleovské algebry (uzávěrové algebry resp. vnitřkové algebry) představují zobecnění topologických prostorů definovaných pomocí topologických uzávěrových a vnitřkových operátorů. Je známo, že MV-algebry představují algebraický protějšek Łukasiewiczovy nekonečně hodnotové logiky podobně jako Booleovské algebry plní tuto funkci pro klasickou dvouhodnotovou logiku. Residuované svazy tvoří širokou třídu algeber obsahující jak MV-algebry tak také další algebry, které lze považovat za algebraické sémantiky obecnějších logik než je klasická. Basic algebry byly zavedeny jakožto neasociativní zobecnění MV-algeber a představují společný základ pro MV-algebry a ortomodulární svazy. Aditivní uzávěrové a multiplikativní vnitřkové operátory na MV-algebrách byly zavedeny jakožto zobecnění topologických Booleovských algeber. V práci zavádíme a zkoumáme aditivní uzávěrové a multiplikativní vnitřkové operátory na residuovaných svazech (komutativních i nekomutativních) a na komutativních basic algebrách. Dále studujeme vlastnosti modálních operátorů (představujících speciální případ uzávěrových operátorů) na residuovaných svazech a na komutativních basic algebrách.

Klíčová slova: residuovaný svaz, basic algebra, uzávěrový operátor, vnitřkový operátor, modální operátor

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Abstract: Topological Boolean algebras (closure algebras, resp. interior algebras) are generalizations of topological spaces defined by means of topological closure and interior operators. It is well known that MV-algebras are an algebraic counterpart of the Łukasiewicz infinite valued propositional logic as well as Boolean algebras play this role for classical two valued logic. Residuated lattices form a wide class of algebras, which contains the class of MV-algebras as well as other algebras that can be taken as algebraic semantics ofa a more general logic than the classic logic. Basic algebras have been introduced as non-associative generalizations of MV-algebras. Basic algebras are in a sense a common base for MV-algebras and orthomodular lattices. Additive closure and multiplicative interior operators on MV-algebras were introduced as generalization of topological Boolean algebras. We introduce and investigate additive closure and multiplicative interior operators on residuated lattices (both in the commutative and non-commutative case) and on commutative basic algebras. Moreover, we study modal operators (special cases of closure operators) on residuated lattices and on commutative basic algebras.

Key words: residuated lattice, basic algebra, closure operator, interior operator, modal operator

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Prohlášení

Prohlašuji, že jsem disertační práci zpracoval samostatně pod vedením pana prof. RNDr. Jiřího Rachůnka, DrSc. a všechny použité zdroje jsem uvedl v seznamu literatury.

V Olomouci dne ....
podpis

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Poděkování

Rád bych poděkoval svému školiteli prof. RNDr. Jiřímu Rachůnkovi, DrSc. za jeho vedení, vstřícnost a za čas, který mi věnoval v průběhu celého studia. Velice si toho vážím.

Chapter 1

Introduction

Topological Boolean algebras (closure algebras, resp. interior algebras) are generalizations of topological spaces defined by means of topological closure and interior operators [35].

Recall that if B is a Boolean algebra and $g : B \to B$ is a mapping then g is called a *topological closure operator* on B if for any $x, y \in B$,

1. $g(x \vee y) = g(x) \vee g(y)$,

2. $x \leq g(x)$,

3. $g(g(x)) = g(x)$,

4. $g(0) = 0$.

A *topological interior operator* is defined dually.

In [33], additive closure and multiplicative interior operators on MV-algebras were introduced as generalization of topological Boolean algebras. It is well known that MV-algebras are an algebraic counterpart of the Łukasiewicz infinite valued propositional logic as well as Boolean algebras play this role for classical two valued logic. Every Boolean algebra is in fact an MV-algebra and conversely, every MV-algebra A contains the greatest Boolean subalgebra $B(A)$ formed by complemented (i.e. additive, resp. multiplicative, idempotent) elements. According to [33], the restriction of each additive closure operator of an MV-algebra is a topological closure operator on the Boolean algebra $B(A)$.

Moreover, in every complete MV-algebra, each topological closure operator on $B(A)$ can be extended to an additive closure operator on A.

The Łukasiewicz logic is one of the most important logics in the theory of fuzzy sets. Hájek's basic fuzzy logic generalizes many of such logics. It is known that BL-algebras introduced also by Hájek are an algebraic counterpart of the basic fuzzy logic. Bounded residuated lattices form a wide class of algebras, which contains not only the class of all BL-algebras, but also the class of all Heyting algebras. Therefore bounded residuated lattices can be taken as an algebraic semantics of a more general logic than the basic logic.

In MV-algebras there are two binary operations $\oplus$ and $\odot$ which are mutually dual. Therefore by [33], for the MV-algebras the research of additive closure operators (ac-operators) is equivalent with that of multiplicative interior operators (mi-operators). Nevertheless, in the case of $R\ell$-monoids and then also in more general bounded residuated lattices an operation with dual properties to the binary operation $\odot$ does not generally exist.

The commutative residuated lattices were first introduced by M. Ward and R.P. Dilworth [36] as generalization of ideal lattices of rings. Non-commutative residuated lattices, sometimes called pseudo-residuated lattices, biresiduated lattices or generalized residuated lattices are algebraic counterparts of substructural logics, that is, logics which lack some of the three structural rules, namely contraction, weakening and exchange. Complete studies on residuated latticfes were developed by N. Galatos, P. Jipsen, T. Kowalski and H. Ono [18], C. Tsinakis [22] and others.

Non-commutative bounded integral residuated lattices form a large class of algebras containing some classes of algebras behind many-valued and fuzzy logics, such as pseudo MV-algebras [19] (or equivalently GMV-algebras [27]), pseudo BL-algebras [9], pseudo MTL-algebras [16] and $R\ell$-monoids [14], and consequently the classes of their commutative cases, i.e. MV-algebras [7], BL-algebras [20], MTL-algebras [15] and commutative $R\ell$-monoids [11]. Moreover, Heyting algebras [1] which are algebras of the intuitionistic logic can be also considered as residuated lattices.

A *bounded integral residuated lattice* is an algebra $M = (M; \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$ of type $(2, 2, 2, 2, 2, 0, 0)$ satisfying the following conditions:

(i) $(M; \odot, 1)$ is a monoid,

(ii) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice,

(iii) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$ for any $x, y \in M$.

In what follows, by a *residuated lattice* we will mean a bounded integral residuated lattice. If the operation $\odot$ on a residuated lattice M is commutative then M is called a *commutative residuated lattice*. In such a case the operations $\rightarrow$ and $\rightsquigarrow$ coincide.

In a residuated lattice M we define two unary operations (negations) "$^-$" and "$^\sim$" on M such that $x^- := x \rightarrow 0$ and $x^\sim := x \rightsquigarrow 0$ for each $x \in M$.

Recall that the mentioned algebras of many-valued and fuzzy logics are characterized in the class of residuated lattices as follows:

A residuated lattice M is

(a) a pseudo MTL-algebra if M satisfies the identities of pre-linearity

  (iv) $(x \rightarrow y) \vee (y \rightarrow x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow x)$;

(b) an $R\ell$-monoid if M satisfies the identities of divisibility

  (v) $(x \rightarrow y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x)$;

(c) a pseudo BL-algebra if M satisfies both (iv) and (v);

(d) involutive if M satisfies the identities

  (vi) $x^{-\sim} = x = x^{\sim-}$;

(e) a GMV-algebra (or equivalently a pseudo MV-algebra) if M satisfies (iv), (v) and (vi);

(f) a Heyting algebra if the operations "$\odot$" and "$\wedge$" coincide.

A residuated lattice M is called *good*, if M satisfies the identity $x^{-\sim} = x^{\sim -}$. For example, every commutative residuated lattice, every GMV-algebra and every pseudo BL-algebra which is a subdirect product of linearly ordered pseudo BL-algebras [12] is good.

By [8], every good residuated lattice satisfies the identity $(x^- \odot y^-)^\sim = (x^\sim \odot y^\sim)^-$. If M is good, we define binary operation "$\oplus$" on M as follows:

$$x \oplus y = (y^- \odot x^-)^\sim.$$

A residuated lattice M is called *normal* if it satisfies the identities

$$(x \odot y)^{-\sim} = x^{-\sim} \odot y^{-\sim}, \quad (x \odot y)^{\sim -} = x^{\sim -} \odot y^{\sim -}.$$

For example, every Heyting algebra and every good pseudo BL-algebra is normal [28], [13].

We introduce multiplicative interior operators (mi-operators) on bounded commutative residuated lattices as the generalization of analogous operators on MV-algebras and $R\ell$-monoids and we show their properties. The binary operation $\oplus$, which need not to be dual to $\odot$ in general, but it makes possible to introduce some analogy of an additive closure operator (ac-operator) from the theory of MV-algebras. We show mutual relationships between mi- and ac-operators, especially for the case of normal residuated lattices. Further, we describe mi- and ac-operators induced by operators on the quotient residuated lattice $M/D(M)$ of a residuated lattice M by the filter $D(M)$ of dense elements in M and on the residuated lattice of regular elements in M.

The second class of algebras on which we investigate the properties of interior and closure operators are basic algebras. Basic algebras have been introduced in [3] as non-associative generalizations of MV-algebras. The name "basic algebra" was selected because these algebras are in a sense a common base for MV-algebras and orthomodular lattices [3], and should not be confused with BL- algebras as the intersection of classes of basic algebras and BL-algebras is just the class of MV-algebras.

Definition. A *basic algebra* is an algebra $\langle A; \oplus, \neg, 0\rangle$ of type $\langle 2, 1, 0\rangle$ that satisfies the identities

(i) $x \oplus 0 = x$,

(ii) $\neg\neg x = x$,

(iii) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$,

(iv) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg 0$.

Moreover, if $x \oplus y = y \oplus x$ for any $x, y \in A$, then A is called a *commutative basic algebra*.

If $A = \langle A; \oplus, \neg, 0\rangle$ is a basic algebra, then $(A, \wedge, \vee, 1, 0)$, where

$$x \vee y := \neg(\neg x \oplus y) \oplus y$$
$$x \wedge y := \neg(\neg x \vee \neg y)$$
$$1 := \neg 0,$$

is a bounded lattice whose induced order is given by

$$x \leq y \iff \neg x \oplus y = 1.$$

If A is commutative, then this lattice is distributive [3]. Moreover [4], this lattice $(A; \wedge, \vee)$ is endowed by a set $(^a)_{a\in A}$ of so-called *sectional antitone involutions*, i.e. for every $a \in A$ there is a mapping $x \mapsto x^a$ of the interval $[a, 1]$ into intself such that for any $x, y \in [a, 1]$

$$x^{aa} = x, \quad x \leq y \Longrightarrow y^a \leq x^a.$$

This system $\mathcal{L}(\mathcal{A}) = (L; \wedge, \vee, (^a)_{a\in L}, 0, 1)$ is called a lattice with sectional antitone involutions assigned to $\mathcal{A} = (A; \oplus, \neg, 0)$. Also conversely, starting with a bounded lattice with sectional antitone involutions $\mathcal{L} = (L; \wedge, \vee, (^a)_{a\in L}, 0, 1)$, one can convert it into a basic algebra $\mathcal{A}(\mathcal{L}) = (L; \oplus, \neg, 0)$, where

$$\neg x = x^0, \quad x \oplus y = (\neg x \vee y)^y.$$

Moreover, the assignments $\mathcal{A} \to \mathcal{L}(\mathcal{A})$ and $\mathcal{L} \to \mathcal{A}(\mathcal{L})$ are one-to-one correspondences, i.e. $\mathcal{A}(\mathcal{L}(\mathcal{A})) = \mathcal{A}$ and $(\mathcal{L}(\mathcal{A}(\mathcal{L})) = \mathcal{L}$.

Note that analogously as MV-algebras are an algebraic counterpart of the propositional infinite-valued Łukasiewicz logic (and Boolean algebras are a counterpart of the propositional classical two-valued logic), commutative basic algebras constitute an algebraic semantics of the propositional logic $\mathcal{L}_{CBA}$ [2] which is a non-associative generalization of the Łukasiewicz logic.

We introduce and investigate additive closure and multiplicative interior operators on commutative basic algebras and describe connections between such operators. Further we show that (additively) idempotent elements of any commutative basic algebra A form a subalgebra $B(A)$ of A which is a Boolean algebra, and we give relations between e.g. additive closure operators on A and topological operators on $B(A)$. Moreover, we study operators on quotient commutative basic algebras.

Another type of operators that we investigate on the above-mentioned algebras are so called *modal operators*. Modal operators are special cases of closure operators. Recall [26] that the notion of a modal operator has its main source in the theory of topoi and sheafification (see [17], [24], [25], [37]). Moreover, modal operators have become also from the theory of frames, where frame maps can be recognized as modal operators on a complete Heyting algebra (see [10]).

Chapter 2

Main results

2.1. Interior and closure operators

In this section we describe the results from the papers [A], [B] and [C].

2.1.1. Operators on commutative residuated lattices

A *commutative bounded integral residuated lattice* is an algebra $M = (M; \odot, \vee, \wedge, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ satisfying the following conditions:

(i) $(M; \odot, 1)$ is a commutative monoid,

(ii) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice,

(iii) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for all $x, y, z \in M$.

In what follows, by a *residuated lattice* we will mean *a commutative bounded integral residuated lattice.*

Let M be a residuated lattice. We define a unary operation $^-$ on M such that $x^- := x \rightarrow 0$. Furthermore, we define a binary operation $\oplus$ on M as follows:

$$x \oplus y = (x^- \odot y^-)^-.$$

Definition. Let M be a residuated lattice. A mapping $f : M \rightarrow M$ is called a *multiplicative interior operator (mi-operator)* on M if for any $x, y \in M$

1. $f(x \odot y) = f(x) \odot f(y)$,

2. $f(x) \leq x$,

3. $f(f(x)) = f(x)$,

4. $f(1) = 1$.

5. $x \leq y \Longrightarrow f(x) \leq f(y)$.

If M is an $R\ell$-monoid, i.e. a residuated lattice satisfying $x \odot (x \to y) = x \wedge y$ for any $x, y \in M$, then it can be shown [32] that the property 5 from the definition follows from properties 1 - 4. This is not true for the more general setting of residuated lattices. One can find an example (clA, 3.2) of a mapping f on a residuated lattice M that satisfies the conditions 1 - 4 from the definition of an multiplicative interior operator, but the mapping f is not monotone.

Definition. Let M be a residuated lattice. A mapping $g : M \to M$ is called an *additive closure operator (ac-operator)* on M if for any $x, y \in M$

1. $g(x \oplus y) = g(x) \oplus g(y)$,

2. $x \leq g(x)$,

3. $g(g(x)) = g(x)$,

4. $g(0) = 0$,

5. $x \leq y \Longrightarrow g(x) \leq g(y)$.

Let $f : M \to M$ be a mapping on a residuated lattice M. We define a mapping $f^- : M \to M$ such that

$$f^-(x) = (f(x^-))^-,$$

for any $x \in M$.

We call a residuated lattice M *normal* if it satisfies the identity

$$(x \odot y)^{--} = x^{--} \odot y^{--}.$$

Proposition (clA, 3.8)**.** If M is a normal residuated lattice and f is an mi-operator on M, then the mapping f^- is an ac-operator on M.

If g is an ac-operator on a normal residuated lattice M, then g^- need not be an mi-operator, i.e. condition 2 from the definition of an mi- operator need not be satisfied on M as we can see in the Example ([A], 3.12).

A residuated lattice M is called *involutive* if it satisfies $x^{--} = x$ for any $x \in M$. One can see that any involutive residuated lattice is normal. Hence by previous proposition, if f is an mi-operator on such a residuated lattice M, then f^- is an ac-operator on M. Furthermore, if g is an ac-operator on an involutive residuated lattice M, then by Proposition ([A], 3.10), g^- is an mi-operator on M. Moreover, $f \mapsto f^-$ and $g \mapsto g^-$ are one-to-one correspondences between mi-operators and ac-operators on an involutive residuated lattice.

The situation for normal residuated lattices which are not involutive is more complicated. Namely, although f^- is still an ac-operator for any mi-operator f on a residuated lattice M, for ac-operator g on M, g^- need not be an mi-operator. Furthermore, if f is an mi-operator on M then f^- satisfies in fact a condition that is stronger than axiom 2 in the definition of an ac-operator on M. Therefore, we will introduce now the notions of wmi- and sac- operators on normal residuated lattices.

Definition. Let M be a residuated lattice and $f : M \to M$. Then f is called a *weak mi-operator (a wmi-operator)* on M if it satisfies conditions 1 and 3 - 5 of the definition of an mi-operator and for any $x \in M$

2a. $f(x) \leq x^{--}$.

Definition. Let M be a normal residuated lattice and $g : M \to M$. Then g is called a *strong ac-operator (an sac-operator)* on M if it satisfies conditions 1 and 3 - 5 of the definition of an ac-operator and for any $x \in M$

2b. $x^{--} \leq g(x)$.

Now we will describe connections among mi-, ac-, wmi- and sac-operators on normal residuated lattices.

Proposition (A, 3.16)**.** Let M be a normal residuated lattice.

(i) If f is a wmi-operator on M, then f^- is an sac-operator on M.

(ii) If g is an sac-operator on M, then g^- is a wmi-operator on M.

If M is a normal residuated lattice, denote by $wmi(M)$ the set of wmi-operators on M and by $sac(M)$ the set of sac-operators on M. Suppose that $wmi(M)$ and $sac(M)$ are pointwise ordered.

Let $\alpha : wmi(M) \to sac(M)$ be the mapping such that $\alpha(f) = f^-$, for any $f \in wmi(M)$, and $\beta : sac(M) \to wmi(M)$ be the mapping such that $\beta(g) = g^-$, for any $g \in sac(M)$.

Theorem (A, 3.17)**.** If M is a normal residuated lattice, then α and β form an antitone Galois connection, i.e. $f \leq \beta(g)$ if and only if $g \leq \alpha(f)$, for any $f \in wmi(M)$ and $g \in sac(M)$.

Definition. Let M be a residuated lattice. A nonempty subset F of M is called a *filter* of M if the following conditions hold

1. $x, y \in F \Longrightarrow x \odot y \in F$,

2. $x \in F$, $y \in M$, $x \leq y \Longrightarrow y \in F$.

By [22], filters of commutative residuated lattices are in a one-to-one correspondence with their congruences. If F is a filter of a commutative residuated lattice M, then for the corresponding congruence Θ_F we have:

$$\begin{aligned}\langle x, y\rangle \in \Theta_F &\iff (x \to y) \wedge (y \to x) \in F \iff (x \to y) \odot (y \to x) \in F \\ &\iff x \to y, y \to x \in F,\end{aligned}$$

for each $x, y \in M$. In such a case, $F = \{x \in M : \langle x, 1\rangle \in \Theta_F\}$. For any filter F of M we put $M/F := M/\Theta_F$.

If M is a residuated lattice, denote $D(M) = \{x \in M : x^{--} = 1\}$ the set of *dense elements* in M.

Proposition (A, 4.6)**.** If M is a residuated lattice, then $D(M)$ is a filter of M.

We say that a residuated lattice M has *Glivenko property* [6] if for any $x, y \in M$

$$(x \to y)^{--} = x \to y^{--}.$$

Recall that the notion of a residuated lattice with Glivenko property was introduced and investigated in [6].

Proposition ([6])**.** A residuated lattice M has Glivenko property if and only if M satisfies the identity

$$(x^{--} \to x)^{--} = 1.$$

An element x of a residuated lattice M is called *regular* if $x^{--} = x$. Denote by $Reg(M)$ the set of all regular elements in M. If $x, y \in Reg(M)$, put $x \vee_* y := (x \vee y)^{--}, x \wedge_* y := (x \wedge y)^{--}$, $x \odot_* y := (x \odot y)^{--}$ and $x \oplus_* y = (x \oplus y)^{--}$.

Theorem. [6] For any residuated lattice M the following conditions are equivalent.

(i) M has Glivenko property,

(ii) $(Reg(M); \vee_*, \wedge_*, \odot_*, \to, 0, 1)$ is an involutive residuated lattice and the mapping $^{--} : M \to Reg(M)$ such that $^{--} : x \mapsto x^{--}$ is a surjective homomorphism of residuated lattices.

Notice that if M is a normal residuated lattice and $x, y \in Reg(M)$, then $x \odot_* y = (x \odot y)^{--} = x^{--} \odot y^{--} = x \odot y$. For arbitrary residuated lattice we have $x \oplus_* y = x \oplus y$.

The following assertions concerning connections between $D(M)$ and $Reg(M)$ are consequences of the previous Theorem:

Theorem (A, 4.7)**.** If M is a residuated lattice with Glivenko property, then for any $x, y \in M$ we have $\langle x, y \rangle \in \Theta_{D(M)}$ if and only if $x^{--} = y^{--}$. Moreover, the quotient residuated lattice $M/D(M)$ is involutive.

Theorem (A, 4.8)**.** If M is a residuated lattice with Glivenko property, then the residuated lattices $Reg(M)$ and $M/D(M)$ are isomorphic.

Theorem (A, 4.10)**.** Let M be a normal residuated lattice with Glivenko property, f an mi-operator (resp. an ac-operator) on M and $f^* : M/D(M) \rightarrow M/D(M)$ the mapping such that $f^*(x/D(M)) = f(x^{--})/D(M)$. Then f^* is an mi-operator (resp. an ac-operator) on $M/D(M)$.

Theorem (A, 4.11)**.** If M is a normal residuated lattice with Glivenko property and f is an mi-operator (resp. an ac-operator) on M, then the mapping $f^{\#}$ such that $f^{\#}(x) = f(x)^{--}$ for any $x \in Reg(M)$ is an mi-operator (resp. an ac-operator) on the residuated lattice $Reg(M)$.

Theorem (A, 4.12)**.** Let M be a normal residuated lattice with Glivenko property. If $g : Reg(M) \rightarrow Reg(M)$ is an mi-operator on the involutive residuated lattice $Reg(M)$, then the mapping $g^+ : M \rightarrow M$ such that $g^+(x) := g(x^{--})$ for any $x \in M$, is a wmi-operator on M.

2.1.2. Interior and closure operators on residuated lattices

In this section we ivestigate properties of additive closure operators and multiplicative interior operators on bounded integral residuated lattices that need not be commutative.

Recall that a *bounded integral residuated lattice* is an algebra $M = (M; \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$ of type $(2, 2, 2, 2, 2, 0, 0)$ satisfying the following conditions:

(i) $(M; \odot, 1)$ is a monoid,

(ii) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice,

(iii) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$ for any $x, y \in M$.

In what follows, by a *residuated lattice* we will mean a bounded integral residuated lattice.

A residuated lattice M is called *good*, if M satisfies the identity $x^{-\sim} = x^{\sim -}$. For example, every commutative residuated lattice, every GMV-algebra and every pseudo BL-algebra which is a subdirect product of linearly ordered pseudo BL-algebras [12] is good.

By [8], every good residuated lattice satisfies the identity $(x^- \odot y^-)^\sim = (x^\sim \odot y^\sim)^-$. If M is good, we define binary operation "$\oplus$" on M as follows:

$$x \oplus y = (y^- \odot x^-)^\sim.$$

Let M be a residuated lattice. We define interior multiplicative operators and additive closure operators on M in the same manner as in the case of commutative residuated latttices.

Let $f : M \to M$ be a mapping, and consider two new mappings

$$f_-^\sim : M \to M,\ f_\sim^- : M \to M,$$

such that for each $x \in M$

$$f_-^\sim(x) := (f(x^-))^\sim$$

and

$$f_\sim^-(x) := (f(x^\sim))^-.$$

Proposition (B, 3.4)**.** If $f : M \to M$ is a monotone mapping on a residuated lattice M, then both mappings $f_\sim^-, f_-^\sim$ are monotone.

Theorem (B, 3.7)**.** If M is a good normal residuated lattice and f is an mi-operator on M, then the mappings $f_\sim^-$ and $f_-^\sim$ are ac-operators on M.

Theorem (B, 3.9)**.** Let M be a good normal residuated lattice and let g be an ac-operator on M. Then the mappings $g_-^\sim, g_\sim^-$ satisfy identities 1, 3, 4, 5 from definition of an mi-operator.

If g is an ac-operator on a good normal residuated lattice M, then $g_-^\sim$ need not be an mi-operator, i.e. condition 2 from the definition of an mi- operator need not be satisfied on M as we can see in the Example (B, 3.11) of a commutative residuated lattice.

Definition. Let M be a residuated lattice and $f : M \to M$. Then f is called a *weak mi-operator (a wmi-operator)* on M if it satisfies conditions 1 and 3 - 5 of the definition of an mi-operator, and for any $x \in M$

2a. $f(x) \leq x^{-\sim}$.

Definition. Let M be a good normal residuated lattice and $g : M \to M$. Then g is called a *strong ac-operator (an sac-operator)* on M if it satisfies conditions 1 and 3 - 5 of the definition of an ac-operator, and for any $x \in M$

2b. $x^{-\sim} \leq g(x)$.

We have that if f is an mi-operator, then $f^{-}_{\sim}$ and $f^{\sim}_{-}$ are sac-operators and consequently ac-operators, and if g is an ac-operator then $g^{\sim}_{-}$ and $g^{-}_{\sim}$ are wmi-operators. Now we will describe connections among mi-, ac-, wmi- and sac-operators on good normal residuated lattices.

Proposition. Let M be a good normal residuated lattice.

(i) If f is a wmi-operator on M, then $f^{-}_{\sim}$ and $f^{\sim}_{-}$ are sac-operators on M.

(ii) If g is an sac-operator on M, then $g^{-}_{\sim}$ and $g^{\sim}_{-}$ are wmi-operators on M.

If M is a normal residuated lattice, denote by $wmi(M)$ the set of wmi-operators on M and by $sac(M)$ the set of sac-operators on M. Suppose that $wmi(M)$ and $sac(M)$ are pointwise ordered.

Let $\alpha_1, \alpha_2 : wmi(M) \to sac(M)$ be the mappings such that $\alpha_1(f) = f^{\sim}_{-}$, and $\alpha_2(f) = f^{-}_{\sim}$ for any $f \in wmi(M)$, and $\beta_1, \beta_2 : sac(M) \to wmi(M)$ be the mappings such that $\beta_1(g) = g^{\sim}_{-}$, and $\beta_2(g) = g^{-}_{\sim}$ for any $g \in sac(M)$.

Theorem (B, 3.14)**.** Let M be a normal residuated lattice.

(i) α_1 and β_2 form an antitone Galois connection, i.e. $f \leq \beta_2(g)$ if and only if $g \leq \alpha_1(f)$, for any $f \in wmi(M)$ and $g \in sac(M)$.

(ii) α_2 and β_1 form an antitone Galois connection, i.e. $f \leq \beta_1(g)$ if and only if $g \leq \alpha_2(f)$, for any $f \in wmi(M)$ and $g \in sac(M)$.

The following theorem is now an immediate consequence.

Theorem (B, 3.15)**.** Let M be a good normal residuated lattice.

(i) If f is an mi-operator on M and $h = (f^{-}_{\sim})^{\sim}_{-} = (f^{\sim}_{-})^{-}_{\sim}$, then $f^{\sim}_{-} = h^{\sim}_{-}$ and $f^{-}_{\sim} = h^{-}_{\sim}$.

(ii) If g is an ac-operator on M and $k = (g^{-}_{\sim})^{\sim}_{-} = (g^{\sim}_{-})^{-}_{\sim}$, then $g^{-}_{\sim} = k^{-}_{\sim}$ and $g^{\sim}_{-} = k^{\sim}_{-}$.

We introduce Glivenko property of a residuated lattice as the noncommutative generalization of Glivenko property which was investigated in the case of commutative residuated lattices.

Definition. We say that a residuated lattice M has *Glivenko property (GP)* if for any $x, y \in M$ we have

$$(x \to y)^{-\sim} = x \to y^{-\sim},\ (x \rightsquigarrow y)^{\sim-} = x \rightsquigarrow y^{\sim-}.$$

It can be seen in Lemma (B, 4.2) that in the case of good residuated lattices the equalities required in Glivenko property are in fact equivalent to these conditions:

(i) $(x^{-\sim} \to x)^{-\sim} = 1 = (x^{\sim-} \rightsquigarrow x)^{\sim-}$, for any $x \in M$,

(ii) $(x \to y)^{-\sim} = x^{-\sim} \to y^{-\sim},\ (x \rightsquigarrow y)^{\sim-} = x^{\sim-} \rightsquigarrow y^{\sim-}$, for any $x, y \in M$.

Definition. Let M be a residuated lattice. A nonempty set F of M is called a *filter* of M if the following conditions hold

(i) $x, y \in F$ imply $x \odot y \in F$,

(ii) $x \in F, x \leq y \in M$ imply $y \in F$.

Definition. A subset $D \subseteq M$ is called a *deductive system* of M if

(i) $1 \in D$,

(ii) $x \in D, x \to y \in D$ imply $y \in D$.

Proposition (B, 4.4)**.** If $H \subseteq M$, then H is a filter in M if and only if H is a deductive system in M.

A filter H of M is called *normal* [34] if $x \to y \in H$ iff $x \rightsquigarrow y \in H$ for each $x, y \in M$. Normal filters of any residuated lattice M are in one-to-one correspondence with the congruences on M. If H is a normal filter of M, then H is the kernel of the unique congruence θ_H such that $\langle x, y\rangle \in \theta_H$ if and only if $(x \to y) \odot (y \to x) \in H$ if and only if $(x \rightsquigarrow y) \odot (y \rightsquigarrow x) \in H$.

Hence we will consider quotient residuated lattices M/H of residuated lattices M by their normal filters. If $x \in M$ then we will denote by x/H the class of M/H containing x.

If M is a residuated lattice, denote $D(M) = \{x \in M; x^{-\sim} = 1 = x^{\sim-}\}$ the set of *dense elements* in M.

Theorem (B, 4.5)**.** (i) If M is a good residuated lattice, then $D(M)$ is a filter in M.

(ii) If, moreover, M satisfies (GP), then $D(M)$ is a normal filter in M.

Theorem. [B, 4.6] Let M be a good residuated lattice satisfying (GP). Then $\langle x, y\rangle \in \theta_{D(M)}$ if and only if $x^{-\sim} = y^{-\sim}$ for all $x, y \in M$. Moreover, $M/D(M)$ is an involutive residuated lattice.

An element x of a residuated lattice M is called *regular* if $x^{-\sim} = x = x^{\sim-}$. Denote by $Reg(M)$ the set of all regular elements in M. Clearly $0, 1 \in Reg(M)$. If $x, y \in M$, put $x \vee_* y := (x \vee y)^{-\sim}, x \wedge_* y := (x \wedge y)^{-\sim}, x \odot_* y := (x \odot y)^{-\sim}$.

Theorem. [B, 4.7] Let M be a good normal residuated lattice satisfying (GP). Then $Reg(M) = (Reg(M); \odot_*, \vee_*, \wedge_*, \to, \rightsquigarrow, 0, 1)$ is an involutive residuated lattice and the mapping ${}^{-\sim} : M \to Reg(M)$ such that ${}^{-\sim} : x \mapsto x^{-\sim}$ is a retract of the reduct $(M; \odot, \to, \rightsquigarrow, 0, 1)$ onto $(Reg(M); \odot, \to, \rightsquigarrow, 0, 1)$.

Theorem (B, 4.8)**.** If M is a good normal residuated lattice such that $Reg(M) = (Reg(M); \odot, \to, \rightsquigarrow, 0, 1)$ is an involutive residuated lattice and the mapping ${}^{-\sim}$ is a retract of $(M; \to, \rightsquigarrow)$ onto $(Reg(M); \to, \rightsquigarrow)$, then M satisfies (GP).

Theorem (B, 4.9)**.** Let M be a good normal residuated lattice. Then the following statements are equivalent:

1. M satisfies (GP).

2. $(Reg(M); \odot, \vee_*, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$ is an involutive residuated lattice and the mapping $^{-\sim} : M \rightarrow Reg(M)$ such that $^{-\sim} : x \mapsto x^{-\sim}$ is a retract of $(M; \odot, \rightarrow, \rightsquigarrow, 0, 1)$ onto $(Reg(M); \odot, \rightarrow, \rightsquigarrow, 0, 1)$.

The following assertion is now an immediate consequence.

If M is a good normal residuated lattice satisfying (GP), then $(\odot, \rightarrow, \rightsquigarrow, 0, 1)$-reducts of $M/D(M)$ and $Reg(M)$ are isomorphic.

Theorem (B, 4.11)**.** If M is a good normal residuated lattice satisfying (GP) and f is an mi-operator (an ac-operator) on M, then the mapping $f^* : Reg(M) \rightarrow Reg(M)$ such that $f^*(x) = f(x)^{-\sim}$, for any $x \in Reg(M)$, is an mi-operator (an ac-operator) on the residuated lattice $Reg(M)$.

Theorem (B, 4.12)**.** If M is a good normal residuated lattice satisfying (GP) and f is an mi-operator on the residuated lattice $Reg(M)$, then the mapping $f^+ : M \rightarrow M$ such that $f^+(x) = f(x^{-\sim})$, for any $x \in M$, is a wmi-operator on M.

Theorem (B, 4.13)**.** Let M be a good residuated lattice satisfying (GP) and $g : Reg(M) \rightarrow Reg(M)$ be an ac-operator on $Reg(M)$. Then the mapping $g^+ : M \rightarrow M$ such that $g^+(x) = g(x^{-\sim})$, for any $x \in M$, is an sac-operator on M.

2.1.3. Interior and closure operators on basic algebras

Recall that a *basic algebra* is an algebra $\langle A; \oplus, \neg, 0\rangle$ of type $\langle 2, 1, 0\rangle$ that satisfies the identities

(i) $x \oplus 0 = x$,

(ii) $\neg\neg x = x$,

(iii) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$,

(iv) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg 0$.

Moreover, if $x \oplus y = y \oplus x$ for any $x, y \in A$, then A is called a *commutative basic algebra.*

In a basic algebra A we define a binary operation (subtraction) such that

$$x \ominus y := \neg(\neg x \oplus y).$$

Moreover, define for any $x, y \in A$

$$x \odot y := \neg(\neg x \oplus \neg y).$$

Definition. Let A be a commutative basic algebra. A mapping $g : A \to A$ is called an *additive closure operator (ac-operator)* on A if for any $x, y \in A$

1. $g(x \oplus y) = g(x) \oplus g(y)$,

2. $x \leq g(x)$,

3. $g(g(x)) = g(x)$,

4. $g(0) = 0$.

Proposition (C, X)**.** Let $g : A \to A$ be an ac-operator on a commutative basic algebra A. Then g is a monotone mapping.

Definition. Let A be a commutative basic algebra. A mapping $f : A \to A$ is called a *multiplicative interior operator (mi-operator)* on A if for any $x, y \in A$

1. $f(x \odot y) = f(x) \odot f(y)$,

2. $f(x) \leq x$,

3. $f(f(x)) = f(x)$,

4. $f(1) = 1$.

Let $f : A \to A$ be a mapping, and consider the mapping

$$f^{\neg} : A \to A,$$

such that for each $x \in A$

$$f^{\neg}(x) := \neg(f(\neg x)).$$

Theorem (C, 3.1)**.** If $g : A \to A$ is an ac-operator on a commutative basic algebra A, then the mapping $g^{\neg} : A \to A$ is an mi-operator on A.

Theorem (C, 3.2)**.** If $f : A \to A$ is an mi-operator on a commutative basic algebra A, then the mapping $f^{\neg} : A \to A$ is an ac-operator on A.

If A is a commutative basic algebra, denote by $mi(A)$ the set of mi-operators on A and by $ac(A)$ the set of ac-operators on A. Suppose that $mi(A)$ and $ac(A)$ are pointwise ordered.

Let $\alpha : mi(A) \to ac(A)$ be the mapping such that $\alpha(f) = f^{\neg}$, for any $f \in mi(A)$, and $\beta : ac(A) \to mi(A)$ be the mapping such that $\beta(g) = g^{\neg}$, for any $g \in ac(A)$.

Theorem. [C, 3.3] If A is a commutative basic algebra, then α and β form an antitone Galois connection, i.e. $f \leq \beta(g)$ if and only if $g \leq \alpha(f)$, for any $f \in mi(A)$ and $g \in ac(A)$.

The following theorem is now an immediate consequence.

Theorem. [C, 3.4] Let A be a commutative basic algebra.

(i) If f is an mi-operator on A and $h = (f^{\neg})^{\neg}$ is the corresponding mi-operator on A, then the induced ac-operators $f^{\neg}$ and $h^{\neg}$ are the same.

(ii) If g is an ac-operator on A and $k = (g^{\neg})^{\neg}$ is the corresponding ac-operator on A, then the induced mi-operators $g^{\neg}$ and $k^{\neg}$ are the same.

Let A be a basic algebra. Denote by $B(A) := \{x \in A : x \oplus x = x\}$ the set of all idempotent elements of A.

Let A be a commutative basic algebra, C a subalgebra of A and $g : A \to A$ ($f : A \to A$) an ac-operator (an mi-operator) on A. Then C is called a *closure subalgebra* (an *interior subalgebra*) with respect to g (to f) if $g(x) \in C$ ($f(x) \in C$) for any $x \in C$.

Proposition (C, 4.2)**.** If A is a commutative basic algebra, then $B(A)$ is a subalgebra of A.

Theorem (C, 4.1)**.** If A is a commutative basic algebra, then $B(A)$ is a Boolean algebra.

Proposition (C, 4.3)**.** Let A be a commutative basic algebra. Then the Boolean subalgebra $B(A)$ of A is a closure subalgebra (an interior subalgebra) with respect to any ac-operator (any mi-operator) on A.

Theorem (C, 4.2)**.** Let A be a commutative basic algebra and $g : A \to A$ an ac-operator ($f : A \to A$ an mi-operator). Then the restriction of g to $B(A)$ (f to $B(A)$) is a topological closure (topological interior) operator on the Boolean algebra $B(A)$.

A commutative basic algebra is called *complete* if the underlying lattice $(A; \vee, \wedge)$ is complete.

Theorem (C, 4.3)**.** Let A be a complete commutative basic algebra and g a topological closure operator on the Boolean algebra $B(A)$. Then there is an ac-operator g^* on A such that the restriction of g^* to $B(A)$ is equal to g.

Let A be a basic algebra. A subset $J \subseteq A$ is called an *ideal* of A [5], if it contains 0 and satisfies the following conditions:

1. if $a \ominus b \in J$ and $b \in J$, then $a \in J$,

2. if $a \ominus b \in J$ and $a \geq b$, then $(c \ominus b) \ominus (c \ominus a) \in J$ for every $c \in A$,

3. if $a \ominus b \in J$ and $b \ominus a \in J$, then $(a \ominus c) \ominus (b \ominus c) \in J$ for every $c \in A$.

Theorem. [5] Let A be a commutative basic algebra and $I \subseteq A$ be an ideal. Then the relation Θ_I defined by

$$\langle a, b\rangle \in \Theta_I \iff a \ominus b \in I \text{ and } b \ominus a \in I.$$

is a congruence on A such that $[0]_{\Theta_I} = I$.

Theorem (C, 5.2)**.** Let A be a commutative basic algebra, $g : A \to A$ an ac-operator and I a g-ideal in A. Then the mapping $g^* : A/I \to A/I$ such that $g^*(x/I) = g(x)/I$ is an ac-operator on the commutative quotient algebra A/I.

2.2. Modal operators

Modal operators (special cases of closure operators) were introduced and investigated on Heyting algebras in [26], on MV-algebras in [21], on commutative $R\ell$-monoids in [31] and on (non-commutative) $R\ell$-monoids in [30]. Moreover, monotone modal operators on commutative bounded residuated lattices were studied in [23].

We define and study monotone modal operators on general (not necessarily commutative) residuated lattices and on commutative basic algebras, and describe the results from the papers [D] and [E].

2.2.1. Modal operators on residuated lattices

Recall that a *bounded integral residuated lattice* is an algebra $M = (M; \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ of type $(2, 2, 2, 2, 2, 2, 0, 0)$ satisfying the following conditions:

(i) $(M; \odot, 1)$ is a monoid,

(ii) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice,

(iii) $x \odot y \leq z$ iff $x \leq y \to z$ iff $y \leq x \rightsquigarrow z$ for any $x, y \in M$.

In what follows, by a *residuated lattice* we will mean a bounded integral residuated lattice. A residuated lattice M is called *good*, if M satisfies the identity

$x^{-\sim} = x^{\sim -}$. If M is good, we define binary operation "$\oplus$" on M as follows:

$$x \oplus y = (y^- \odot x^-)^\sim.$$

Definition. Let M be a residuated lattice. A mapping $f : M \longrightarrow M$ is called a *modal operator* on M if for any $x, y \in M$

(M1) $x \leq f(x)$,

(M2) $f(f(x)) = f(x)$,

(M3) $f(x \odot y) = f(x) \odot f(y)$.

A modal operator f is called *monotone*, if for any $x, y \in M$

(M4) $x \leq y \Longrightarrow f(x) \leq f(y)$.

If M is a good residuated lattice and for any $x, y \in M$

(M5) $f(x \oplus y) = f(x \oplus f(y)) = f(f(x) \oplus y)$,

then f is called *strong*.

In all cases of $R\ell$-monoids every modal operator is already monotone. However, in general residuated lattices the converse need not hold. An example of a modal operator that is not monotone is given in [23].

Proposition (D, 5)**.** Let f be a monotone modal operator on a good residuated lattice M. Then it is strong if and only if for any $x \in M$

$$x \oplus f(0) = f(x^{-\sim}) = f(0) \oplus x.$$

Theorem (D, 6)**.** Let M be a residuated lattice and $f : M \longrightarrow M$ be a mapping. Then f is a monotone modal operator on M if and only if for any $x, y \in M$:

(i) $x \rightarrow f(y) = f(x) \rightarrow f(y)$,

(ii) $x \rightsquigarrow f(y) = f(x) \rightsquigarrow f(y)$,

(iii) $f(x) \odot f(y) \geq f(x \odot y)$.

In general, if f is a monotone modal operator, the equation $f(0) = 0$ need not hold. An example of such modal operator is shown in [23]. Thus we will investigate under which condition this equality holds.

Proposition (D, 7)**.** Let M be a residuated lattice and f be a monotone modal operator. Then the following conditions are equivalent.

(i) $f(0) = 0$,

(ii) $f(x^{\sim}) = x^{\sim}$, for all $x \in M$,

(iii) $f(x^{-}) = x^{-}$, for all $x \in M$.

As a consequence of the previous proposition we obtain the following result. Let M be a good residuated lattice satisfying $x^{-\sim} = x$ for all $x \in M$. Let f be a monotone modal operator on M such that $f(0) = 0$. Then f is the identity on M.

Let M be a residuated lattice and f be a modal operator on M. We denote by

$$\mathrm{Fix}(f) = \{x \in M; f(x) = x\}$$

the set of all fixed elements of the operator f. By the definition of a modal operator it is obvious that $\mathrm{Fix}(f) = \mathrm{Im}(f)$.

Proposition (D, 18)**.** If f is a monotone modal operator on a residuated lattice M, then $\mathit{Fix}(f) = (\mathit{Fix}(f); \odot, \vee_{\mathit{Fix}(f)}, \wedge, \rightarrow, \rightsquigarrow, f(0), 1)$, where $x \vee_{\mathrm{Fix}(f)} y = f(x \vee y)$ for any $x, y \in \mathit{Fix}(f)$, and $\wedge, \rightarrow, \rightsquigarrow$ are the restrictions of the binary operations from M on $\mathit{Fix}(f)$, is a residuated lattice.

2.2.2. Modal operators on commutative basic algebras

Recall that a *basic algebra* is an algebra $\langle A; \oplus, \neg, 0\rangle$ of type $\langle 2, 1, 0\rangle$ that satisfies the identities

(i) $x \oplus 0 = x$,

(ii) $\neg\neg x = x$,

(iii) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$,

(iv) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg 0$,

(v) $x \oplus y = y \oplus x$.

Moreover, if $x \oplus y = y \oplus x$ for any $x, y \in A$, then A is called a *commutative basic algebra*. In a commutative basic algebra A we define a binary operation such that for any $x, y \in A$

$$x \odot y := \neg(\neg x \oplus \neg y).$$

Definition. Let A be a commutative basic algebra. A mapping $f : A \to A$ is called a *modal operator* on A if for any $x, y \in A$

1. $x \leq f(x)$,

2. $f(f(x)) = f(x)$,

3. $f(x \odot y) = f(x) \odot f(y)$.

A modal operator f is called *strong*, if for any $x, y \in A$

4. $f(x \oplus y) = f(x \oplus f(y))$.

Let A be a basic algebra. Denote by $B(A) := \{x \in A : x \oplus x = x\}$ the set of all idempotent elements of A.

Proposition. [29] If A is a commutative basic algebra, then $B(A)$ is a subalgebra of A.

Theorem. [29] If A is a commutative basic algebra, then $B(A)$ is a Boolean algebra.

For an arbitrary element $a \in B(A)$ denote by $g_a : A \to A$ the mapping such that $g_a(x) = a \oplus x$ for any $x \in A$.

Theorem (E, 3.5)**.** Let A be a commutative basic algebra, and $a \in B(A)$. Then $g_a : A \to A$ is a modal operator on A.

For an element $a \in B(A)$ consider mappings $h_a : A \to A$ and $k_a : A \to A$ such that for any $x \in A$

$$h_a(x) := a \to x, \quad k_a(x) := (x \to a) \to a.$$

Proposition (E, 3.6)**.** If A is a commutative basic algebra and $a \in B(A)$, then the mappings h_a and k_a are modal operators on A.

Let A be a commutative basic algebra. Denote by $M(A)$ and $M_s(A)$ the set of all modal and all strong modal operators on A.

Theorem (E, 3.7)**.** If $f_1, f_2 \in M(A)$, or $f_1, f_2 \in M_s(A)$, then $f_1 f_2 \in M(A)$, or $f_1 f_2 \in M_s(A)$, respectively, if and only if $f_1 f_2 = f_2 f_1$.

Proposition (E, 3.8)**.** Let A be a commutative basic algebra, $a \in B(A)$ and $f \in M(A)$. If $f(x) \leq g_a(x)$ for any $x \in A$, then $f(a) = a$.

Theorem (E, 3.12)**.** Let A be a commutative basic algebra, and $f : A \to A$ be a mapping. Then f is a modal operator on A if and only if for any $x, y \in A$ it satisfies:

(i) $x \to f(y) = f(x) \to f(y)$,

(ii) $f(x) \odot f(y) \geq f(x \odot y)$.

Chapter 3

Papers

[A] Rachůnek J., Svoboda Z.: *Interior and closure operators on commutative bounded residuated lattices*, Acta Univ. Palacki. Olomuc., Fac. Rerum Nat., Math. 52, No. 1 (2013), 121–134.

[B] Rachůnek J., Svoboda Z.: *Interior and closure operators on bounded residuated lattices*, Cent. Eur. J. Math. 12, No. 3 (2014), 534–544.

[C] Rachůnek J., Svoboda Z.: *Interior and closure operators on commutative basic algebras*, Math. Slovaca, to appear.

[D] Rachůnek J., Svoboda Z.: *Monotone modal operators on bounded integral residuated lattices*, Math. Bohem. 137, No. 3 (2012), 333–345.

[E] Svoboda Z.: *Modal operators on commutative basic algebras*, Math. Slovaca, in review

Bibliography

[1] Balbes R., Dwinger P.: Distributive Lattices, University Missouri Press, Columbia, 1974.

[2] Botur M., Halaš R.: *Commutative basic algebras and non-associative fuzzy logics*, Arch. Math. Logic 48 (2009), 243–255.

[3] Chajda I., Halaš R., Kühr, J.: *Many valued quantum algebras*, Algebra Univers. 60 (2009), 63–90.

[4] Chajda I., Halaš R., Kühr, J.: Semilattice Structures, Heldermann Verlag, 2007.

[5] Chajda I., Kühr J.: *Ideals and congruences of basic algebras*, Soft. Comput. 17 (2013), 401-410.

[6] Cignoli, R., Torrens, A.: *Glivenko like theorems in natural expansions of BCK-logic*. Math. Log. Quart. 50 (2004), 111–125.

[7] Cignoli R. L. O., Itala M. L., Mundici D., Algebraic Foundations of Many-valued Reasoning, Kluwer Academic Publishers, Dordrecht, 2000.

[8] Ciungu L. C., *Classes of residuated lattices*, Annals of University of Craiova. Math. Comp. Sci. Ser. 33 (2006), 180–207.

[9] DiNola A., Georgescu G., Iorgulescu A., *Pseudo-BL algebras; Part I*, Multiple Val. Logic 8 (2002), 673–714.

[10] Dowker C.H., Papert D.: Quotient Frames and Subspaces, Proc. London Math. Soc. 16 (1966), 275–296.

[11] Dvurečenskij A., Rachůnek J., *Probabilistic averaging in bounded commutative residuated ℓ-monoids*, Discrete Math. 306 (2006), 1317–1326.

[12] Dvurečenskij A., *Every linear pseudo BL-algebra admits a state*, Soft Comput. 11 (2007), 495–501.

[13] Dvurečenskij A., Rachůnek J., *On Riečan and Bosbach states for bounded $R\ell$-monoids*, Math. Slovaca 56 (2006), 487–500.

[14] Dvurečenskij A., Rachůnek J., *Probabilistic averaging in bounded Rℓ-monoids*, Semigroup Forum 72 (2006), 191–206.

[15] Esteva F., Godo L., *Monoidal t-norm based logic: towards a logic for left-continuous t-norms*, Fuzzy Sets Syst. 124 (2001), 271–288.

[16] Flondor P., Georgescu G., Iorgulescu A., *Pseudo-t-norms and pseudo-BL algebras*, Soft Comput. 5 (2001), 355–371.

[17] Freyd P.J.: *Aspects of topoi*, Bull. Austral. Math. Soc. 7 (1972), 1–76.

[18] Galatos N., Jipsen P., Kowalski T., Ono H.: Residuated Lattices: An Algebraic Glimpse at Substructural Logics. *Elsevier, Amsterdam*, 2007.

[19] Georgescu G., Iorgulescu A., *Pseudo-MV algebras*, Multiple Val. Logic 6 (2001), 95–135.

[20] Hájek P., Metamathematics of Fuzzy Logic, Springer, Dordrecht, 1998.

[21] Harlenderová M., Rachůnek J.: *Modal operators on MV-algebras*, Math. Bohem. 131 (2006), 39–48.

[22] Jipsen P., Tsinakis C.: A Survey of Residuated Lattices. In: Ordered Algebraic Structures, *Kluwer, Dordrecht*, 2006, 19–56.

[23] Kondo M.: *Modal operators on commutative residuated lattices*, Math. Slovaca 61 (2011), 1–14.

[24] Lawvere F.W.: *Quantifiers and Sheaves*, Actes Congres Intern. Math., Tome 1, 1970, 329–334.

[25] Lawvere F.W.: Toposes, Algebraic Geometry and Logic, Springer Lecture Notes 274, Berlin, 1972.

[26] Macnab D.S.: *Modal operators on Heyting algebras*, Alg. Univ. 12 (1981), 5–29.

[27] Rachůnek J., *A non-commutative generalization of MV-algebras*, Czechoslovak Math. J. 52 (2002), 255–273.

[28] Rachůnek J., Slezák, V., *Negation in bounded commutative DRℓ-monoids*, Czechoslovak Math. J. 56 (2007), 755–763.

[29] Rachůnek J., Svoboda Z.: *Interior and closure operators on commutative basic algebras*, Math. Slovaca, to appear.

[30] Rachůnek J., Šalounová D.: *Modal operators on bounded residuated l-monoids*, Math. Bohemica 133 (2008), 299–311.

[31] Rachůnek J., Šalounová D.: *Modal operators on bounded commutative residuated l-monoids*, Math. Slovaca 57 (2007), 321–332.

[32] Rachůnek, J., Švrček, F.: *Interior and closure operators on bounded commutative residuated ℓ-monoids.* Discuss. Math., Gen. Alg. Appl. 28 (2008), 11–27.

[33] Rachůnek, J., Švrček, F.: *MV-algebras with additive closure operators.* Acta Univ. Palacki. Olomouc. Fac. Rer. Nat. Math. 39 (2000), 183 – 189.

[34] Rachůnek J., Šalounová, D., *States on Generalizations of Fuzzy Structures*, Palacký Univ. Press, Olomouc, 2011.

[35] Rasiowa H, Sikorski R. The Mathematics of Metamathematics, Panstw. Wyd. Nauk., Warszawa, 1963.

[36] Ward M, Dilworth RP., *Residuated lattices.* Trans. Amer. Math. Soc., 1939, 45 (1939), 335–354.

[37] Wraith G.C.: Lectures on Elementary Topoi, in Model Theory and Topoi, Springer Lecture Notes 445, Berlin (1975).

INTERIOR AND CLOSURE OPERATORS ON COMMUTATIVE BOUNDED RESIDUATED LATTICES*

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Abstract. Commutative bounded integral residuated lattices form a large class of algebras containing some classes of algebras behind many valued and fuzzy logics. In the paper we introduce and investigate additive closure and multiplicative interior operators on this class of algebras.

Keywords: residuated lattice, bounded integral residuated lattice, interior operator, closure operator

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1. Introduction

Commutative bounded integral residuated lattices form a large class of algebras containing some classes of algebras behind many valued and fuzzy logics, such as MV-algebras [2], BL-algebras [9], MTL-algebras [7] and commutative $R\ell$-monoids [12], [6]. Moreover, Heyting algebras [1] which are algebras of the intuitionistic logic can be also viewed as commutative bounded integral lattices.

Topological Boolean algebras, i.e. closure or interior algebras [15], are generalizations of topological spaces defined by means of topological closure and interior operators. In [13] closure and interior MV-algebras as generalizations of topological Boolean algebras were introduced by means of so-called additive closure and multiplicative interior operators. It is known that every MV-algebra M contains the greatest Boolean subalgebra $B(M)$ of all complemented elements. By [13], the restriction of any additive closure operator on M onto $B(M)$ is a topological closure operator on $B(M)$. Moreover, if M is a complete MV-algebra, then every topological closure operator on $B(M)$ can be extended to an additive closure operator on M. Since the addition and multiplication of MV-algebras are mutually dual operations, analogous properties are also true for multiplicative interior operators on M and $B(M)$.

The notions of additive closure and multiplicative interior operators (ac- and mi- operators, for short) were generalized in [14] to commutative residuated ℓ-monoids (= commutative $R\ell$-monoids), i.e. commutative bounded integral residuated lattices satisfying divisibility [11], [8]. But the dual operation to multiplication in such residuated lattices does not exist in general. Hence, connections between mi- and ac-operators are more complicated than those in the case of MV-algebras.

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In the paper we introduce and investigate analogous operators on arbitrary commutative bounded integral residuated lattices. We describe connections between mi-operators and ac-operators in this general setting. Moreover, we generalize the notions of mi- and ac-operators to so-called weak mi-operators and strong ac-operators and show that there is an antitone Galois connection between them. Furthermore, we describe, for residuated lattices with Glivenko property, connections between mi- and ac- operators on them and on the residuated lattices of their regular elements.

2. Preliminaries

A *commutative bounded integral residuated lattice* is an algebra $M = (M; \odot, \vee, \wedge, \to, 0, 1)$ of type $(2,2,2,2,0,0)$ satisfying the following conditions:

(i) $(M; \odot, 1)$ is a commutative monoid,
(ii) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice,
(iii) $x \odot y \le z$ iff $x \le y \to z$ for all $x, y, z \in M$.

In what follows, by a *residuated lattice* we will mean *a commutative bounded integral residuated lattice*.

For any residuated lattice M we define a unary operation (negation) $^-$ on M such that $x^- := x \to 0$.

Recall that algebras of logics mentioned in Introduction are characterized in the class of residuated lattices as follows:

A residuated lattice M is

(a) an MTL-algebra if M satisfies the identity of pre-linearity
  (iv) $(x \to y) \vee (y \to x) = 1$;
(b) involutive if M satisfies the identity of double negation
  (v) $x^{--} = x$;
(c) an $R\ell$-monoid (or a bounded commutative GBL-algebra) if M satisfies the identity of divisibility
  (vi) $(x \to y) \odot x = x \wedge y$;
(d) a BL-algebra if M satisfies both (iv) and (vi);
(e) an MV-algebra if M is an involutive BL-algebra;
(f) a Heyting algebra if the operations "$\odot$" and "$\wedge$" coincide.

Proposition 2.1 ([4, 11]). *Let M be a residuated lattice. Then for any $x, y, z \in M$ we have:*

(i) $x \le y \Longrightarrow y^- \le x^-$,
(ii) $x \odot y \le x \wedge y$,
(iii) $(x \to y) \odot x \le y$,
(iv) $x \le x^{--}$,
(v) $x^{---} = x^-$,
(vi) $x \to (y \to z) = y \to (x \to z)$,
(vii) $x \to (y \to z) = (x \odot y) \to z$,
(viii) $x \le y \Longrightarrow z \to x \le z \to y$,
(ix) $x \le y \Longrightarrow y \to z \le x \to z$,

(x) $y \to z \leq (x \to y) \to (x \to z)$,

(xi) $x \to y \leq (y \to z) \to (x \to z)$.

(xii) $x^{--} \to y^{--} = x \to y^{--}$,

(xiii) $(x \to y^{--})^{--} = x \to y^{--}$,

(xiv) $(x \odot y)^{-} = y \to x^{-} = x \to y^{-} = x^{--} \to y^{-} = y^{--} \to x^{-}$,

(xv) $(x \odot y)^{--} \geq x^{--} \odot y^{--}$.

Let M be a residuated lattice. We define a binary operation $\oplus$ on M as follows:

$$x \oplus y = (x^{-} \odot y^{-})^{-}.$$

Lemma 2.2 ([4]). *Let M be a residuated lattice. For any $x, y \in M$ we have*

(i) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(ii) $x \oplus y \geq x^{--} \vee y^{--} \geq x \vee y$,

(iii) $x \oplus 0 = x^{--}$,

(iv) $(x \oplus y)^{--} = x^{--} \oplus y^{--} = x \oplus y$,

(v) $x \odot x^{-} = 0, \quad x \oplus x^{-} = 1$.

We call a residuated lattice M *normal* if it satisfies the identity

$$(x \odot y)^{--} = x^{--} \odot y^{--}.$$

For example, every involutive residuated lattice, every Heyting algebra and every BL-algebra is normal [5] (note that the name "normal" is sometimes used for non-commutative residuated lattices where all filters are normal, see [10]).

Similarly as in [14] for residuated ℓ-monoids we can prove the following identities.

Lemma 2.3. *Let M be a normal residuated lattice. Then for any $x, y \in M$*

(i) $(x \oplus y)^{-} = x^{-} \odot y^{-}$,

(ii) $(x \odot y)^{-} = x^{-} \oplus y^{-}$.

Proof. (i) Since M is normal, we have $(x \oplus y)^{-} = (x^{-} \odot y^{-})^{--} = x^{---} \odot y^{---} = x^{-} \odot y^{-}$.

(ii) By Lemma 2.2 (iv), we have $x^{-} \oplus y^{-} = (x^{-} \oplus y^{-})^{--} = ((x^{--} \odot y^{--})^{-})^{--} = (x^{--} \odot y^{--})^{-} = (x \odot y)^{---} = (x \odot y)^{-}$. □

3. Connections between interior and closure operators

Definition. Let M be a residuated lattice. A mapping $f : M \to M$ is called a *multiplicative interior operator (mi-operator)* on M if for any $x, y \in M$

(1) $f(x \odot y) = f(x) \odot f(y)$,

(2) $f(x) \leq x$,

(3) $f(f(x)) = f(x)$,

(4) $f(1) = 1$.

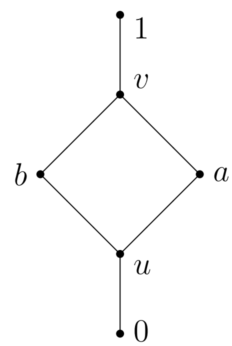
(5) $x \leq y \Longrightarrow f(x) \leq f(y)$.

Remark 3.1. If M is an $R\ell$-monoid, i.e. a residuated lattice satisfying $x\odot(x \to y) = x\wedge y$ for any $x, y \in M$, then it can be shown [14] that the property 5 from the definition follows from properties 1 - 4.

Example 3.2. Let $M_1 = \{0, u, a, b, v, 1\}$. We define the operations $\odot$ and $\to$ on M_1 as follows:

$\odot$	0	u	a	b	v	1
0	0	0	0	0	0	0
u	0	0	0	0	0	u
a	0	0	a	0	a	a
b	0	0	0	b	b	b
v	0	0	a	b	v	v
1	0	u	a	b	v	1

$\to$	0	u	a	b	v	1
0	1	1	1	1	1	1
u	v	1	1	1	1	1
a	b	b	1	b	1	1
b	a	a	a	1	1	1
v	u	u	a	b	1	1
1	0	u	a	b	v	1



Then M_1 is an involutive normal residuated lattice in which pre-linearity and divisibility are not satisfied since we have $(a \to b) \vee (b \to a) = b \vee a \neq 1$, and $v \odot (v \to u) = v \odot u = 0 \neq u = v \wedge u$. However, we get $x^{--} = x$ for all $x \in M$.

Let $f_1 : M_1 \to M_1$ be the mapping such that $f_1(0) = 0, f_1(u) = u, f_1(a) = a, f_1(b) = 0, f_1(v) = v, f_1(1) = 1$. Then the mapping f_1 satisfies the conditions 1 - 4 from the definition of an mi-operator, but the mapping f_1 is not monotone since $u < b$, whereas $f_1(u) \nleq f_1(b)$.

Example 3.3. Let M be the residuated lattice from Example 3.2. Let us consider the mapping $f_2 : M \to M$ such that $f_2(0) = f_2(u) = f_2(a) = f_2(b) = 0, f_2(v) = v, f_2(1) = 1$. Then f_2 is an mi-operator on M.

Lemma 3.4. *Let f be an mi-operator on a residuated lattice M. Then for any $x, y \in M$*

$$f(x \to y) \leq f(x) \to f(y).$$

Proof. Let $x, y \in M$. Then $(x \to y) \odot x \leq y$ and we have $f(x \to y) \odot f(x) = f((x \to y) \odot x) \leq f(y)$. Thus $f(x \to y) \leq f(x) \to f(y)$. □

Let $f : M \to M$ be a mapping on a residuated lattice M. We define a mapping $f^- : M \to M$ such that

$$f^-(x) = (f(x^-))^-,$$

for any $x \in M$.

Proposition 3.5. *If $f : M \to M$ is a monotone mapping on a residuated lattice M, then the mapping f^- is monotone, too.*

Proof. Let $x, y \in M$ be such that $x \leq y$. Then by Proposition 2.1 $y^- \leq x^-$, so $f(y^-) \leq f(x^-)$. Therefore $(f(x^-))^- \leq (f(y^-))^-$ or equivalently $f^-(x) \leq f^-(y)$. □

Proposition 3.6. *Let M be a residuated lattice. If f is an mi-operator on M and $x, y \in M$, then*

(i) $x \leq f^-(x)$,
(ii) $f^-(f^-(x)) = f^-(x)$,
(iii) $f^-(0) = 0$,
(iv) $x \leq y \Longrightarrow f^-(x) \leq f^-(y)$.

Proof. (i): If $x \in M$, then $f^-(x) = (f(x^-))^- \geq x^{--} \geq x$.

(ii): For any $x \in M$ we have $f^-(f^-(x)) = f^-((f(x^-))^-) = (f(f(x^-))^{--})^-$ and $f(x^-) \leq (f(x^-))^{--}$ by Proposition 2.1. Since f is monotone $f(f(x^-)) = f(x^-) \leq f((f(x^-))^{--})$, thus $(f(x^-))^- \geq (f((f(x^-))^{--}))^-$, and $f^-(x) \geq f^-(f^-(x))$. By (i) we also have $f^-(x) \leq f^-(f^-(x))$. Thus $f^-(f^-(x)) = f^-(x)$.

(iii): $f^-(0) = (f(0^-))^- = (f(1))^- = 1^- = 0$.

(iv): It follows from Proposition 3.5. □

Proposition 3.7. *Let M be a normal residuated lattice and f be an mi-operator on M. Then the mapping f^- satisfies the identity*

$$f^-(x \oplus y) = f^-(x) \oplus f^-(y).$$

Proof. Let $x, y \in M$. Then $f^-(x) \oplus f^-(y) = ((f^-(x))^- \odot (f^-(y))^-)^- = ((f(x^-))^{--} \odot (f(y^-))^{--})^- = (f(x^-) \odot f(y^-))^{---} = (f(x^-) \odot f(y^-))^- = (f(x^- \odot y^-))^- = (f((x \oplus y)^-))^- = f^-(x \oplus y)$. □

Definition. Let M be a residuated lattice. A mapping $g : M \to M$ is called an *additive closure operator (ac-operator)* on M if for any $x, y \in M$

(1) $g(x \oplus y) = g(x) \oplus g(y)$,
(2) $x \leq g(x)$,
(3) $g(g(x)) = g(x)$,
(4) $g(0) = 0$,
(5) $x \leq y \Longrightarrow g(x) \leq g(y)$.

Proposition 3.8. *If M is a normal residuated lattice and f is an mi-operator on M, then the mapping f^- is an ac-operator on M.*

Proof. It follows from Propositions 3.6 and 3.7. □

Lemma 3.9. *If M is a residuated lattice and g is an ac-operator on M, then g satisfies the identity*

$$g(x^{--}) = (g(x))^{--}.$$

Proof. By Lemma 2.2 (iii), we have $g(x^{--}) = g(x \oplus 0) = g(x) \oplus g(0) = g(x) \oplus 0 = (g(x))^{--}$. □

Proposition 3.10. *Let M be a normal residuated lattice and g be an ac-operator on M. Then we have for any $x, y \in M$*

(i) $g^-(x \odot y) = g^-(x) \odot g^-(y)$,

(ii) $g^-(x) \leq x^{--}$,

(iii) $g^-(g^-(x)) = g^-(x)$,

(iv) $g^-(1) = 1$,

(v) $x \leq y \Longrightarrow g^-(x) \leq g^-(y)$.

Proof. (i) Let $x, y \in M$. Then we have $g^-(x \odot y) = (g((x \odot y)^-))^-$, and by Lemma 2.3 we get $(g((x \odot y)^-))^- = (g(x^-) \oplus g(y^-))^- = (g(x^-))^- \odot (g(y^-))^- = g^-(x) \odot g^-(y)$.

(ii) Since $x^- \leq g(x^-)$, we have $(g(x^-))^- = g^-(x) \leq x^{--}$.

(iii) By Lemma 3.9, $g^-(g^-(x)) = (g((g(x^-))^{--}))^- = (g(g(x^-)))^{---} = (g(x^-))^- = g^-(x)$.

(iv) $g^-(1) = (g(1^-))^- = (g(0))^- = 0^- = 1$.

(v) For any $x, y \in M$ such that $x \leq y$ we have $y^- \leq x^-$, thus $g(y^-) \leq g(x^-)$ and $g^-(x) = (g(x^-))^- \leq (g(y^-))^- = g^-(y)$. □

Remark 3.11. If g is an ac-operator on a normal residuated lattice M, then g^- need not be an mi-operator, i.e. condition 2 from the definition of an mi- operator need not be satisfied on M as we can see in the following example.

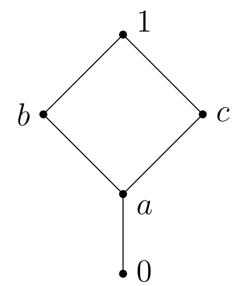
Example 3.12. Let $M_2 = \{0, a, b, c, 1\}$. Let the operations $\odot$ and $\rightarrow$ be defined on M_2 as follows.

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	a	b
c	0	a	a	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

Then $M_2 = (M_2; \odot, \vee, \wedge, \rightarrow, 0, 1)$ is a residuated lattice which is both BL-algebra and Heyting algebra with the derived operation $\oplus$:

$\oplus$	0	a	b	c	1
0	0	1	1	1	1
a	1	1	1	1	1
b	1	1	1	1	1
c	1	1	1	1	1
1	1	1	1	1	1



Let $g : M_2 \rightarrow M_2$ be the mapping such that $g(0) = 0, g(a) = g(b) = b, g(c) = 1, g(1) = 1$. Then we can easily verify that g is an ac-operator on M_2. However, the inequality $g^-(x) \leq x$ does not hold for all $x \in M_2$, since, for instance, $g^-(a) = (g(a^-))^- = (g(0))^- = 0^- = 1 \nleq a$.

Recall that a residuated lattice M is called *involutive* if it satisfies $x^{--} = x$ for any $x \in M$.

Remark 3.13. It is obvious that any involutive residuated lattice is normal. Hence by Proposition 3.8, if f is an mi-operator on such a residuated lattice M, then f^- is an ac-operator on M. Furthermore, if g is an ac-operator on an involutive residuated lattice M, then by Proposition 3.10, g^- is an mi-operator on M. Moreover, $f \mapsto f^-$ and $g \mapsto g^-$ are one-to-one correspondences between mi-operators and ac-operators on an involutive residuated lattice.

Remark 3.14. The situation for normal residuated lattices which are not involutive is more complicated. Namely, although f^- is still an ac-operator for any mi-operator f on a residuated lattice M, for ac-operator g on M, g^- need not be an mi-operator. Furthermore, if f is an mi-operator on M, then by the proof of Proposition 3.6 (i), f^- satisfies in fact a condition that is stronger than axiom 2 in the definition of an ac-operator on M. Therefore, we will introduce now the notions of wmi- and sac- operators on normal residuated lattices.

Definition. Let M be a residuated lattice and $f : M \rightarrow M$. Then f is called a *weak mi-operator (a wmi-operator)* on M if it satisfies conditions 1 and 3 - 5 of the definition of an mi-operator and for any $x \in M$

2a. $f(x) \leq x^{--}$.

Definition. Let M be a normal residuated lattice and $g : M \rightarrow M$. Then g is called a *strong ac-operator (an sac-operator)* on M if it satisfies conditions 1 and 3 - 5 of the definition of an ac-operator and for any $x \in M$

2b. $x^{--} \leq g(x)$.

Remark 3.15. We have that if f is an mi-operator, then f^- is an sac-operator and if g is an ac-operator, then g^- is a wmi-operator.

Now we will describe connections among mi-, ac-, wmi- and sac-operators on normal residuated lattices.

Proposition 3.16. *Let M be a normal residuated lattice.*

(i) If f is a wmi-operator on M, then f^- is an sac-operator on M.

(ii) If g is an sac-operator on M, then g^- is a wmi-operator on M.

Proof. (i) It suffices to prove condition 2b. If $x \in M$, then by 2a., $f(x^-) \leq x^{---} = x^-$, hence $(f(x^-))^- = f^-(x) \geq x^{--}$.

(ii) Analogously we will only verify condition 2a. If $x \in M$, then $x^- = (x^-)^{--} \leq g(x^-)$, thus $x^{--} \geq (g(x^-))^- = g^-(x)$. □

If M is a normal residuated lattice, denote by $wmi(M)$ the set of wmi-operators on M and by $sac(M)$ the set of sac-operators on M. Suppose that $wmi(M)$ and $sac(M)$ are pointwise ordered.

Let $\alpha : wmi(M) \to sac(M)$ be the mapping such that $\alpha(f) = f^-$, for any $f \in wmi(M)$, and $\beta : sac(M) \to wmi(M)$ be the mapping such that $\beta(g) = g^-$, for any $g \in sac(M)$.

Theorem 3.17. *If M is a normal residuated lattice, then α and β form an antitone Galois connection, i.e. $f \leq \beta(g)$ if and only if $g \leq \alpha(f)$, for any $f \in wmi(M)$ and $g \in sac(M)$.*

Proof. Let $f \in wmi(M), g \in sac(M)$ and $f \leq \beta(g) = g^-$. Then $f(x) \leq g^-(x) = (g(x^-))^-$, thus $f(x)^- \geq (g(x^-))^{--}$, for any $x \in M$. Therefore $(f(x^-))^- \geq (g(x^{--}))^{--} \geq (g(x))^{--} \geq g(x)$, thus $\alpha(f)(x) \geq g(x)$, for any $x \in M$. That means $g \leq \alpha(f)$.

Conversely, let $g \leq \alpha(f)$. Then $f^-(x) \geq g(x)$, i.e. $(f(x^-))^- \geq g(x)$, and so $(f(x^-))^{--} \leq (g(x))^-$, for any $x \in M$. Hence $(f(x^{--}))^{--} \leq (g(x^-))^- = g^-(x)$, and $(f(x^{--}))^{--} \geq (f(x))^{--} \geq f(x)$. That means $\beta(g)(x) = g^-(x) \geq (f(x^{--}))^{--} \geq f(x)$, for any $x \in M$, and thus $f \leq \beta(g)$. □

The following theorem is now an immediate consequence.

Theorem 3.18. *Let M be a normal residuated lattice.*

(i) If f is an mi-operator on M and $h = (f^-)^-$ is the corresponding wmi-operator on M, then the induced sac-operators f^- and h^- are the same.

(ii) If g is an ac-operator on M and $k = (g^-)^-$ is the corresponding sac-operator on M, then the induced wmi-operators g^- and k^- are the same.

4. Operators on residuated lattices with Glivenko property

Definition. Let M be a residuated lattice. A nonempty subset F of M is called a *filter* of M if the following conditions hold

(1) $x, y \in F \Longrightarrow x \odot y \in F$,
(2) $x \in F$, $y \in M$, $x \leq y \Longrightarrow y \in F$.

A subset D of M is called a *deductive system* of M if
(3) $1 \in D$,
(4) $x, x \to y \in D \Longrightarrow y \in D$.

It is known that a nonempty subset of M is a filter of M if and only if it is a deductive system of M.

By [11], filters of commutative residuated lattices are in a one-to-one correspondence with their congruences. If F is a filter of a commutative residuated lattice M, then for the corresponding congruence Θ_F we have:

$$\begin{aligned}\langle x, y\rangle \in \Theta_F &\iff (x \to y) \wedge (y \to x) \in F \iff (x \to y) \odot (y \to x) \in F \\ &\iff x \to y, y \to x \in F,\end{aligned}$$

for each $x, y \in M$. In such a case, $F = \{x \in M : \langle x, 1\rangle \in \Theta_F\}$. For any filter F of M we put $M/F := M/\Theta_F$.

If M is a residuated lattice, denote $D(M) = \{x \in M : x^{--} = 1\}$ the set of *dense elements* in M.

We say that a residuated lattice M has *Glivenko property* [3] if for any $x, y \in M$

$$(x \to y)^{--} = x \to y^{--}.$$

Proposition 4.1 ([3]). *A residuated lattice M has Glivenko property if and only if M satisfies the identity*

$$(x^{--} \to x)^{--} = 1.$$

An element x of a residuated lattice M is called *regular* if $x^{--} = x$. Denote by $Reg(M)$ the set of all regular elements in M. If $x, y \in Reg(M)$, put $x \vee_* y := (x \vee y)^{--}, x \wedge_* y := (x \wedge y)^{--}$, $x \odot_* y := (x \odot y)^{--}$ and $x \oplus_* y = (x \oplus y)^{--}$.

Theorem 4.2 ([3]). *For any residuated lattice M the following conditions are equivalent.*

(i) M has Glivenko property,
(ii) $(Reg(M); \vee_*, \wedge_*, \odot_*, \to, 0, 1)$ is an involutive residuated lattice and the mapping $^{--}$: $M \to Reg(M)$ such that $^{--} : x \mapsto x^{--}$ is a surjective homomorphism of residuated lattices.

Remark 4.3. If M is a normal residuated lattice and $x, y \in Reg(M)$, then $x \odot_* y = (x \odot y)^{--} = x^{--} \odot y^{--} = x \odot y$. For arbitrary residuated lattice we have $x \oplus_* y = x \oplus y$.

Proposition 4.4. *If a residuated lattice M has Glivenko property if and only if $(x \to y)^{--} = x^{--} \to y^{--}$, for any $x, y \in M$.*

Proof. It follows from Proposition 2.1 (xii). □

Remark 4.5. Every $R\ell$-monoid has Glivenko property because by [12] it satisfies the identity $(x \to y)^{--} = x^{--} \to y^{--}$.

Proposition 4.6. *If M is a residuated lattice, then $D(M)$ is a filter of M.*

Proof. Let $x, y \in D(M)$, i.e. $x^{--} = 1 = y^{--}$. Then by Proposition 2.1, $(x \odot y)^{--} \geq x^{--} \odot y^{--} = 1$, hence $(x \odot y)^{--} = 1$, and so $x \odot y \in D(M)$.

If $x \in D(M), z \in M$ and $x \leq z$, then obviously $z \in D(M)$. □

The following assertions concerning connections between $D(M)$ and $Reg(M)$ are consequences of Theorem 4.2.

Theorem 4.7. *If M is a residuated lattice with Glivenko property, then for any $x, y \in M$ we have $\langle x, y\rangle \in \Theta_{D(M)}$ if and only if $x^{--} = y^{--}$. Moreover, the quotient residuated lattice $M/D(M)$ is involutive.*

Proof. Let $x, y \in M$. Then $\langle x, y\rangle \in \Theta_{D(M)} \iff x \to y, y \to x \in D(M) \iff (x \to y)^{--} = 1 = (y \to x)^{--} \iff x^{--} \to y^{--} = 1 = y^{--} \to x^{--} \iff x^{--} \leq y^{--}, y^{--} \leq x^{--} \iff x^{--} = y^{--}$.

Therefore, $(x/D(M))^{--} = x^{--}/D(M) = x/D(M)$. □

Theorem 4.8. *If M is a residuated lattice with Glivenko property, then the residuated lattices $Reg(M)$ and $M/D(M)$ are isomorphic.*

Remark 4.9. It is obvious that the mappings $\varphi : Reg(M) \to M/D(M)$ and $\psi : M/D(M) \to Reg(M)$ such that $\varphi : x \mapsto x/D(M)$ and $\psi : y/D(M) \mapsto y^{--}$ are mutually inverse isomorphisms between $Reg(M)$ and $M/D(M)$.

Theorem 4.10. *Let M be a normal residuated lattice with Glivenko property, f an mi-operator (resp. an ac-operator) on M and $f^* : M/D(M) \to M/D(M)$ the mapping such that $f^*(x/D(M)) = f(x^{--})/D(M)$. Then f^* is an mi-operator (resp. an ac-operator) on $M/D(M)$.*

Proof. Let f be an mi-operator on M and $x, y \in M$ be elements such that $x/D(M) = y/D(M)$. Then

$$f^*(x/D(M)) = f(x^{--})/D(M) = f(y^{--})/D(M) = f^*(y)/D(M).$$

Therefore f^* is defined correctly. We will verify that it is an mi-operator.

(1) $f^*(x/D(M)) \odot f^*(y/D(M)) = f(x^{--})/D(M) \odot f(y^{--})/D(M) = (f(x^{--} \odot y^{--}))/D(M) = f((x \odot y)^{--})/D(M) = f^*((x \odot y)/D(M)) = f^*((x/D(M)) \odot (y/D(M)))$.

(2) $f^*(x/D(M)) = f(x^{--})/D(M) \leq x^{--}/D(M) = x/D(M)$.

(3) $f^*(f^*(x/D(M))) = f^*(f(x^{--})/D(M)) = f((f(x^{--}))^{--})/D(M) \leq (f(x^{--}))^{--}/D(M) = f(x^{--})/D(M) = f^*(x/D(M))$. Conversely, $(f(x^{--}))^{--}/D(M) \geq f(x^{--})/D(M) \Longrightarrow f((f(x^{--}))^{--})/D(M) \geq f(f(x^{--}))/D(M) = f(x^{--})/D(M) \Longrightarrow f^*(f^*(x/D(M))) \geq f^*(x/D(M))$. Hence, $f^*(f^*(x/D(M))) = f^*(x/D(M))$.

(4) $f^*(1/D(M)) = f(1^{--})/D(M) = f(1)/D(M) = 1/D(M)$.

(5) $x/D(M) \leq y/D(M) \Longrightarrow x^{--}/D(M) \leq y^{--}/D(M) \Longrightarrow f(x^{--})/D(M) \leq f(y^{--})/D(M) \Longrightarrow f^*(x/D(M)) \leq f^*(y/D(M))$.

Similarly for ac-operators on M. □

Theorem 4.11. *If M is a normal residuated lattice with Glivenko property and f is an mi-operator (resp. an ac-operator) on M, then the mapping $f^\#$ such that $f^\#(x) = f(x)^{--}$ for any $x \in Reg(M)$ is an mi-operator (resp. an ac-operator) on the residuated lattice $Reg(M)$.*

Proof. If $x \in Reg(M)$, then also $f(x)^{--} \in Reg(M)$. The assertion is hence a direct consequence of the preceeding theorem because the mapping ψ from Remark 4.9 is an isomorphism of residuated lattices. □

Theorem 4.12. *Let M be a normal residuated lattice with Glivenko property. If $g : Reg(M) \to Reg(M)$ is an mi-operator on the involutive residuated lattice $Reg(M)$, then the mapping $g^+ : M \to M$ such that $g^+(x) := g(x^{--})$ for any $x \in M$, is a wmi-operator on M.*

Proof. Let g be an mi-operator on $Reg(M)$ and $g^+(x) = g(x^{--})$ for any $x \in M$.

(1) $g^+(x \odot y) = g((x \odot y)^{--}) = g(x^{--} \odot y^{--}) = g(x^{--} \odot_* y^{--}) = g(x^{--}) \odot_* g(y^{--}) = g(x^{--}) \odot g(y^{--}) = g^+(x) \odot g^+(y)$.

(2) $g^+(x) = g(x^{--}) \leq x^{--}$.

(3) $g^+(g^+(x)) = g((g^+(x))^{--}) = g((g(x^{--}))^{--}) = g(g(x^{--})) = g(x^{--}) = g^+(x)$.

(4) $g^+(1) = g(1^{--}) = g(1) = 1$.

(5) $x \leq y \Longrightarrow g^+(x) = g(x^{--}) \leq g(y^{--}) = g^+(y)$.

Hence g is an mi-operator on M. □

Theorem 4.13. *Let M be a residuated lattice with Glivenko property. If $h : Reg(M) \to Reg(M)$ is an ac-operator on $Reg(M)$, then the mapping $\hat{h}(x) = h(x^{--})$ for any $x \in M$, is an sac-operator on M.*

Proof. 1. $\hat{h}(x \oplus y) = h((x \oplus y)^{--}) = h(x^{--} \oplus y^{--}) = h(x^{--} \oplus_* y^{--}) = h(x^{--}) \oplus_* h(y^{--}) = h(x^{--}) \oplus h(y^{--}) = \hat{h}(x) \oplus \hat{h}(y)$.

2. $\hat{h}(x) = h(x^{--}) \geq x^{--}$.

3. - 5. Similarly as in the proof of Theorem 4.12. □

References

[1] Balbes, R., Dwinger, P.: Distributive Lattices. *University Missouri Press, Columbia*, 1974.

[2] Cignoli, R. L. O., D'Ottaviano, M. L., Mundici, D.: Algebraic Foundations of Many-valued Reasoning. *Kluwer Academic Publishers, Dordrecht*, 2000.

[3] Cignoli, R., Torrens, A.: *Glivenko like theorems in natural expansions of BCK-logic.* Math. Log. Quart. **50** (2004), 111–125.

[4] Ciungu, L. C.: *Classes of residuated lattices.* Annals of University of Craiova. Math. Comp. Sci. Ser. **33** (2006), 180–207.

[5] Dvurečenskij, A., Rachůnek, J.: *On Riečan and Bosbach states for bounded $R\ell$-monoids.* Math. Slovaca **56** (2006), 487–500.

[6] Dvurečenskij, A., Rachůnek, J.: *Probabilistic averaging in bounded commutative residuated ℓ-monoids.* Discrete Math. **306** (2006), 1317–1326.

[7] Esteva, F., Godo, L.: *Monoidal t-norm based logic: towards a logic for left-continuous t-norms.* Fuzzy Sets Syst. **124** (2001), 271–288.

[8] Galatos, N., Jipsen, P., Kowalski, T., Ono, H.: Residuated Lattices: An Algebraic Glimpse at Substructural Logics. *Elsevier, Amsterdam*, 2007.

[9] Hájek, P.: Metamathematics of Fuzzy Logic. *Springer, Dordrecht*, 1998.

[10] Jipsen, P., Montana, A.: *The Blok-Ferreirim theorem for normal GBL-algebras and its application.* Algebra Universalis **60** (2009), 381–404.

[11] Jipsen, P., Tsinakis, C.: A Survey of Residuated Lattices. In: Ordered Algebraic Structures, *Kluwer, Dordrecht*, (2006), 19–56.

[12] Rachůnek, J., Slezák, V.: *Negation in bounded commutative $DR\ell$-monoids.* Czechoslovak Math. J. **56** (2007), 755–763.

[13] Rachůnek, J., Švrček, F.: *MV-algebras with additive closure operators.* Acta Univ. Palacki. Olomouc. Fac. Rer. Nat. Math. **39** (2000), 183 – 189.

[14] Rachůnek, J., Švrček, F.: *Interior and closure operators on bounded commutative residuated ℓ-monoids.* Discuss. Math., Gen. Alg. Appl. **28** (2008), 11–27.

[15] Sikorski, R.: Boolean Algebras, Second Edition. *Springer-Verlag, Berlin-Göttingen-Heidelbeg-New York*, 1963.

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Interior and closure operators on bounded residuated lattices

Research Article

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Abstract: Bounded integral residuated lattices form a large class of algebras containing some classes of algebras behind many valued and fuzzy logics. In the paper we introduce and investigate multiplicative interior and additive closure operators (mi- and ac-operators) generalizing topological interior and closure operators on such algebras. We describe connections between mi- and ac-operators, and for residuated lattices with Glivenko property we give connections between operators on them and on the residuated lattices of their regular elements.

MSC: 03G10, 06D35, 06A15, 06F05

Keywords: Residuated lattice • Bounded integral residuated lattice • Interior operator • Closure operator

1. Introduction

Bounded integral residuated lattices form a large class of algebras containing some classes of algebras behind many-valued and fuzzy logics, such as pseudo MV-algebras [12] (or equivalently GMV-algebras [16]), pseudo BL-algebras [5], pseudo MTL-algebras [11] and $R\ell$-monoids [9], and consequently the classes of their commutative cases, i.e. MV-algebras [2], BL-algebras [13], MTL-algebras [10] and commutative $R\ell$-monoids [8]. Moreover, Heyting algebras [1] which are algebras of the intuitionistic logic can be also considered as residuated lattices.

Topological Boolean algebras, i.e. closure or interior algebras [22], are generalizations of topological spaces defined by means of topological closure and interior operators. In [20] closure and interior MV-algebras as generalizations of topological Boolean algebras were introduced by means of so-called additive closure and multiplicative interior operators. It is known that every MV-algebra M contains the greatest Boolean subalgebra $B(M)$ of all complemented elements. By [20], the restriction of any additive closure operator on M onto $B(M)$ is a topological closure

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operator on $B(M)$. Moreover, if M is a complete MV-algebra, then every topological closure operator on $B(M)$ can be extended to an additive closure operator on M. Since the addition and multiplication of MV-algebras are mutually dual operations, analogous properties are also true for multiplicative interior operators on M and $B(M)$.

The notions of additive closure and multiplicative interior operators (ac- and mi- operators, for short) were generalized in [21] to commutative residuated ℓ-monoids (= commutative $R\ell$-monoids), i.e. commutative bounded integral residuated lattices satisfying divisibility [14], [15]. But the dual operation to multiplication in such residuated lattices does not exist in general. Hence, connections between mi- and ac- operators are more complicated than those in the case of MV-algebras. Note that mi- and ac- operators on bounded residuated lattice ordered monoids were studied in [23].

In the paper we introduce and investigate analogous operators on arbitrary bounded integral residuated lattices. We describe connections between mi-operators and ac-operators in this general setting. Moreover, we generalize the notions of mi- and ac- operators to so-called weak mi-operators and strong ac-operators and show that there is an antitone Galois connection between them. Furthermore, we describe, for residuated lattices with Glivenko property, connections between mi- and ac- operators on them and on the residuated lattices of their regular elements.

2. Preliminaries

A *bounded integral residuated lattice* is an algebra $M = (M; \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ of type $(2, 2, 2, 2, 2, 0, 0)$ satisfying the following conditions:

(i) $(M; \odot, 1)$ is a monoid,

(ii) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice,

(iii) $x \odot y \leq z$ iff $x \leq y \to z$ iff $y \leq x \rightsquigarrow z$ for any $x, y \in M$.

In what follows, by a *residuated lattice* we will mean a bounded integral residuated lattice. If the operation $\odot$ on a residuated lattice M is commutative then M is called a *commutative residuated lattice*. In such a case the operations $\to$ and $\rightsquigarrow$ coincide.

In a residuated lattice M we define two unary operations (negations) "$^-$" and "$^\sim$" on M such that $x^- := x \to 0$ and $x^\sim := x \rightsquigarrow 0$ for each $x \in M$.

Recall that the mentioned algebras of many-valued and fuzzy logics are characterized in the class of residuated lattices as follows:

A residuated lattice M is

(a) a pseudo MTL-algebra if M satisfies the identities of pre-linearity

(iv) $(x \to y) \vee (y \to x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow x)$;

(b) an $R\ell$-monoid if M satisfies the identities of divisibility

(v) $(x \to y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x)$;

(c) a pseudo BL-algebra if M satisfies both (iv) and (v);

(d) involutive if M satisfies the identities

(vi) $x^{-\sim} = x = x^{\sim -}$;

(e) a GMV-algebra (or equivalently a pseudo MV-algebra) if M satisfies (iv), (v) and (vi);

(f) a Heyting algebra if the operations "$\odot$" and "$\wedge$" coincide.

A residuated lattice M is called *good*, if M satisfies the identity $x^{-\sim} = x^{\sim -}$. For example, every commutative residuated lattice, every GMV-algebra and every pseudo BL-algebra which is a subdirect product of linearly ordered pseudo BL-algebras [6] is good.

By [4], every good residuated lattice satisfies the identity $(x^- \odot y^-)^\sim = (x^\sim \odot y^\sim)^-$. If M is good, we define binary operation "$\oplus$" on M as follows:

$$x \oplus y = (y^- \odot x^-)^\sim.$$

In the next proposition we recall some basic properties of residuated lattices.

Proposition 2.1 ([4],[15],[14]).
Let M be a residuated lattice. Then for any $x, y, z \in M$ we have:

(i) $x \odot y \le x \wedge y$,

(ii) $x \le y \Longrightarrow x \odot z \le y \odot z,\ z \odot x \le z \odot y$,

(iii) $x \le y \Longrightarrow z \to x \le z \to y,\ z \rightsquigarrow x \le z \rightsquigarrow y$,

(iv) $x \le y \Longrightarrow x \to z \ge y \to z,\ x \rightsquigarrow z \ge y \rightsquigarrow z$,

(v) $(x \odot y) \to z = x \to (y \to z),\ (y \odot x) \rightsquigarrow z = x \rightsquigarrow (y \rightsquigarrow z)$,

(vi) $(y \to z) \odot (x \to y) \le x \to z,\ (x \rightsquigarrow y) \odot (y \rightsquigarrow z) \le x \rightsquigarrow z$,

(vii) $x \le x^{-\sim},\ x \le x^{\sim -}$,

(viii) $x^{-\sim -} = x^-,\ x^{\sim - \sim} = x^\sim$,

(ix) $x \le y \Longrightarrow y^- \le x^-,\ y^\sim \le x^\sim$,

(x) $x \odot (x \rightsquigarrow y) \le y,\ (x \to y) \odot x \le y$,

(xi) $y \le x \to y,\ y \le x \rightsquigarrow y$,

(xii) $x \to y \le y^- \rightsquigarrow x^-,\ x \rightsquigarrow y \le y^\sim \rightsquigarrow x^\sim$,

(xiii) $(x \odot y)^- = x \to y^-,\ (x \odot y)^\sim = y \rightsquigarrow x^\sim$,

(xiv) if M is good, $(x \odot y)^{-\sim} \ge x^{-\sim} \odot y^{-\sim}, (x \odot y)^{\sim -} \ge x^{\sim -} \odot y^{\sim -}$,

(xv) $x \to y \leq (z \to x) \to (z \to y)$,

(xvi) $x \rightsquigarrow y \leq (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)$,

(xvii) $y^{-} \rightsquigarrow x^{-} = x^{-\sim} \to y^{-\sim} = x \to y^{-\sim}$,

(xviii) $y^{\sim} \to x^{\sim} = x^{\sim-} \rightsquigarrow y^{\sim-} = x \rightsquigarrow y^{\sim-}$.

Moreover, if M is good, then

(xv) $x^{-\sim} \oplus y^{-\sim} = x^{-\sim} \oplus y = x \oplus y^{-\sim} = x \oplus y$,

(xvi) $x \oplus 0 = x^{-\sim} = 0 \oplus x$,

(xvii) $x \oplus y = x^{-} \rightsquigarrow y^{-\sim} = y^{\sim} \to x^{-\sim}$,

(xviii) $y \oplus x^{-} = x \to y^{-\sim}$, $x^{\sim} \oplus y = x \rightsquigarrow y^{-\sim}$,

(xix) $(x \oplus y) \oplus 0 = x \oplus y$,

(xx) $x \leq y \Longrightarrow z \oplus x \leq z \oplus y$, $x \oplus z \leq y \oplus z$,

(xxi) $\oplus$ is associative.

A residuated lattice M is called *normal* if it satisfies the identities

$$(x \odot y)^{-\sim} = x^{-\sim} \odot y^{-\sim}, \quad (x \odot y)^{\sim-} = x^{\sim-} \odot y^{\sim-}.$$

For example, every Heyting algebra and every good pseudo BL-algebra is normal [19], [7].

Proposition 2.2 ([17]).
Let M be a good and normal residuated lattice. Then for any $x, y \in M$

(i) $(x \oplus y)^{-} = y^{-} \odot x^{-}$, $(x \oplus y)^{\sim} = y^{\sim} \odot x^{\sim}$,

(ii) $x^{-} \oplus y^{-} = (y \odot x)^{-}$, $x^{\sim} \oplus y^{\sim} = (y \odot x)^{\sim}$.

3. Connections between interior and closure operators

Definition .
Let M be a residuated lattice. A mapping $f : M \to M$ is called a *multiplicative interior operator (mi-operator)* on M if for any $x, y \in M$

1. $f(x \odot y) = f(x) \odot f(y)$,
2. $f(x) \leq x$,
3. $f(f(x)) = f(x)$,
4. $f(1) = 1$,
5. $x \leq y \Longrightarrow f(x) \leq f(y)$.

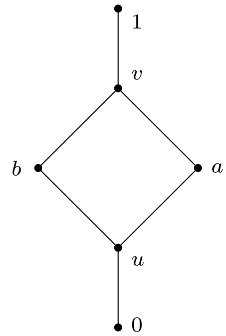
Remark 3.1.
If M is a $R\ell$-monoid, i.e. a residuated lattice satisfying $x \odot (x \to y) = x \wedge y$ for any $x, y \in M$, then it can be shown [21] that the property 5 from the definition follows from properties 1 - 4.

Example 3.2.

Let $M_1 = \{0, u, a, b, v, 1\}$. We define the operations $\odot$ and $\rightarrow$ on M_1 as follows:

$\odot$	0	u	a	b	v	1
0	0	0	0	0	0	0
u	0	0	0	0	0	u
a	0	0	a	0	a	a
b	0	0	0	b	b	b
v	0	0	a	b	v	v
1	0	u	a	b	v	1

$\rightarrow$	0	u	a	b	v	1
0	1	1	1	1	1	1
u	v	1	1	1	1	1
a	b	b	1	b	1	1
b	a	a	a	1	1	1
v	u	u	a	b	1	1
1	0	u	a	b	v	1



Then M_1 is a commutative involutive normal residuated lattice in which pre-linearity and divisibility are not satisfied since we have $(a \rightarrow b) \vee (b \rightarrow a) = b \vee a \neq 1$, and $v \odot (v \rightarrow u) = v \odot u = 0 \neq u = v \wedge u$. However, we get $x^{--} = x$ for all $x \in M$.

Let $f_1 : M_1 \rightarrow M_1$ be the mapping such that $f_1(0) = 0, f_1(u) = u, f_1(a) = a, f_1(b) = 0, f_1(v) = v, f_1(1) = 1$. Then the mapping f_1 satisfies the conditions 1 - 4 from the definition of an mi-operator, but the mapping f_1 is not monotone since $u < b$, whereas $f_1(u) \nleq f_1(b)$.

Lemma 3.3.

Let f be an mi-operator on a residuated lattice M. Then for any $x, y \in M$

$$f(x \rightarrow y) \leq f(x) \rightarrow f(y), \quad f(x \rightsquigarrow y) \leq f(x) \rightsquigarrow f(y).$$

Proof. Let $x, y \in M$. By Proposition 2.1 we have $x \odot (x \rightsquigarrow y) \leq y$, and by monotony of f we have $f(x) \odot f(x \rightsquigarrow y) \leq f(y)$. Thus $f(x \rightsquigarrow y) \leq f(x) \rightsquigarrow f(y)$. Similarly, since by Proposition 2.1 $(x \rightarrow y) \odot x \leq y$, $f(x \rightarrow y) \leq f(x) \rightarrow f(y)$. □

Let $f : M \rightarrow M$ be a mapping, and consider two new mappings

$$f_{-}^{\sim} : M \rightarrow M, \; f_{\sim}^{-} : M \rightarrow M,$$

such that for each $x \in M$

$$f_{-}^{\sim}(x) := (f(x^{-}))^{\sim}$$

and

$$f_{\sim}^{-}(x) := (f(x^{\sim}))^{-}.$$

Proposition 3.4.

If $f : M \rightarrow M$ is a monotone mapping on a residuated lattice M, then both mappings $f_{\sim}^{-}, f_{-}^{\sim}$ are monotone.

Proof. Let $x, y \in M$ be such that $x \leq y$. Then $y^{-} \leq x^{-}$ and $f(y^{-}) \leq f(x^{-})$. Therefore $f_{-}^{\sim}(x) = (f(x^{-}))^{\sim} \leq (f(y^{-}))^{\sim} = f_{-}^{\sim}(y)$. Analogously for $f_{\sim}^{-}$. □

Proposition 3.5.

Let $f : M \to M$ be an mi-operator on a residuated lattice M. Then for any $x \in M$ we have

(i) $x \le f_-^\sim(x)$,

(ii) $f_-^\sim(f_-^\sim(x)) = f_-^\sim(x)$,

(iii) $f_-^\sim(0) = 0$,

(iv) $x \le y \Longrightarrow f_-^\sim(x) \le f_-^\sim(y)$.

Proof. (i): If $x \in M$ then $f_-^\sim(x) = (f(x^-))^\sim \ge x^{-\sim} \ge x$.

(ii): By (i), for any $x \in M$ we have $f_-^\sim(x) \le f_-^\sim(f_-^\sim(x))$. Further we know that $f(x^-) \le (f(x^-))^{\sim-}$ and so

$$f_-^\sim(f_-^\sim(x)) = f_-^\sim((f(x^-))^\sim) = (f((f(x^-))^{\sim-}))^\sim \le (f(f(x^-)))^\sim = (f(x^-))^\sim = f_-^\sim(x).$$

(iii): $f_-^\sim(0) = (f(0^-))^\sim = (f(1))^\sim = 1^\sim = 0$.

(iv): It follows from Proposition 3.4. □

Remark 3.6.

It can be readily shown that analogous properties hold for the operator $f_\sim^-$.

Definition .

Let M be a good residuated lattice. A mapping $g : M \to M$ is called an *additive closure operator (ac-operator)* on M if for any $x, y \in M$

1. $g(x \oplus y) = g(x) \oplus g(y)$,
2. $x \le g(x)$,
3. $g(g(x)) = g(x)$,
4. $g(0) = 0$,
5. $x \le y \Longrightarrow g(x) \le g(y)$.

Theorem 3.7.

If M is a good normal residuated lattice and f is an mi-operator on M, then the mappings $f_\sim^-$ and $f_-^\sim$ are ac-operators on M.

Proof. By Propositions 3.4 and 3.5, we need only verify the identity 1 from the definition of an ac-operator. Let $x, y \in M$. Then

$$\begin{aligned} f_-^\sim(x \oplus y) &= (f((x \oplus y)^-))^\sim = (f(y^- \odot x^-))^\sim = (f(y^-) \odot f(x^-))^\sim \\ &= (f(x^-))^\sim \oplus (f(y^-))^\sim = f_-^\sim(x) \oplus f_-^\sim(y). \end{aligned}$$

□

Lemma 3.8.

If M is a good normal residuated lattice and g is an ac-operator on M then g satisfies the identity

$$g(x^{-\sim}) = (g(x))^{-\sim}.$$

Proof. We have $g(x^{-\sim}) = g(x \oplus 0) = g(x) \oplus g(0) = g(x) \oplus 0 = (g(x))^{-\sim}$. □

Theorem 3.9.

Let M be a good normal residuated lattice and let g be an ac-operator on M. Then the mappings $g_{-}^{\sim}, g_{\sim}^{-}$ satisfy identities 1, 3, 4, 5 from definition of an mi-operator.

Proof. Let $x, y \in M$. Then we have for $g_{-}^{\sim}$:

1. $g_{-}^{\sim}(x \odot y) = (g((x \odot y)^{-}))^{\sim} = (g(y^{-} \oplus x^{-}))^{\sim} = (g(y^{-}) \oplus g(x^{-}))^{\sim} = (g(x^{-}))^{\sim} \odot (g(y^{-}))^{\sim} = g_{-}^{\sim}(x) \odot g_{-}^{\sim}(y)$,

3. $g_{-}^{\sim}(g_{-}^{\sim}(x)) = g_{-}^{\sim}((g(x^{-}))^{\sim}) = (g((g(x^{-}))^{\sim -}))^{\sim} = (g(g(x^{-\sim -})))^{\sim} = (g(g(x^{-})))^{\sim} = (g(x^{-}))^{\sim} = g_{-}^{\sim}(x)$,

4. $g_{-}^{\sim}(1) = (g(1^{-}))^{\sim} = (g(0))^{\sim} = 0^{\sim} = 1$.

5. Similarly as in Proposition 3.4.

Analogously for the mapping $g_{\sim}^{-}$. □

Remark 3.10.

If g is an ac-operator on a good normal residuated lattice M, then $g^{-\sim}$ need not be an mi-operator, i.e. condition 2 from the definition of an mi- operator need not be satisfied on M as we can see in the following example of a commutative residuated lattice.

Example 3.11.

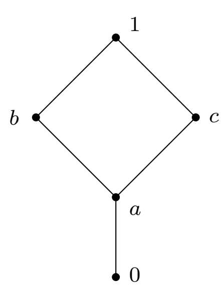
Let $M_2 = \{0, a, b, c, 1\}$. Let the operations $\odot$ and $\to$ be defined on M_2 as follows.

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	a	b
c	0	a	a	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

Then $M_2 = (M_2; \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ is a commutative $R\ell$-monoid which is both BL-algebra and Heyting algebra with the derived operation $\oplus$:

$\oplus$	0	a	b	c	1
0	0	1	1	1	1
a	1	1	1	1	1
b	1	1	1	1	1
c	1	1	1	1	1
1	1	1	1	1	1



Let $g : M_2 \to M_2$ be the mapping such that $g(0) = 0, g(a) = g(b) = b, g(c) = 1, g(1) = 1$. Put $g^{-} = g_{-}^{\sim} = g_{\sim}^{-}$. Then we can easily verify that g is an ac-operator on M_2. However, the inequality $g^{-}(x) \leq x$ does not hold for all $x \in M_2$, since, for instance, $g^{-}(a) = (g(a^{-}))^{-} = (g(0))^{-} = 0^{-} = 1 \nleq a$.

Definition .

Let M be a residuated lattice and $f : M \to M$. Then f is called a *weak mi-operator (a wmi-operator)* on M if it satisfies conditions 1 and 3 - 5 of the definition of an mi-operator, and for any $x \in M$

2a. $f(x) \leq x^{-\sim}$.

Definition .

Let M be a good normal residuated lattice and $g : M \to M$. Then g is called a *strong ac-operator (an sac-operator)* on M if it satisfies conditions 1 and 3 - 5 of the definition of an ac-operator, and for any $x \in M$

2b. $x^{-\sim} \leq g(x)$.

Remark 3.12.

We have that if f is an mi-operator, then $f_{\sim}^{-}$ and $f_{-}^{\sim}$ are sac-operators and consequently ac-operators, and if g is an ac-operator then $g_{-}^{\sim}$ and $g_{\sim}^{-}$ are wmi-operators.

Now we will describe connections among mi-, ac-, wmi- and sac-operators on good normal residuated lattices.

Proposition 3.13.

Let M be a good normal residuated lattice.

(i) If f is a wmi-operator on M, then $f_{\sim}^{-}$ and $f_{-}^{\sim}$ are sac-operators on M.

(ii) If g is an sac-operator on M, then $g_{\sim}^{-}$ and $g_{-}^{\sim}$ are wmi-operators on M.

Proof. (i) It suffices to prove condition 2b. If $x \in M$ then by 2a, $f(x^{-}) \leq (x^{-})^{-\sim} = (x^{-})^{\sim -} = x^{-}$, hence $f_{-}^{\sim}(x) = (f(x^{-}))^{\sim} \geq x^{-\sim}$. Similarly for $f_{\sim}^{-}$.

(ii) Analogously, we will only verify condition 2a. If $x \in M$ then $x^{\sim} = (x^{\sim})^{-\sim} \leq g(x^{\sim})$, thus $x^{\sim -} \geq (g(x^{\sim}))^{-} = g_{\sim}^{-}(x)$. □

If M is a normal residuated lattice, denote by $wmi(M)$ the set of wmi-operators on M and by $sac(M)$ the set of sac-operators on M. Suppose that $wmi(M)$ and $sac(M)$ are pointwise ordered.

Let $\alpha_1, \alpha_2 : wmi(M) \to sac(M)$ be the mappings such that $\alpha_1(f) = f_{-}^{\sim}$, and $\alpha_2(f) = f_{\sim}^{-}$ for any $f \in wmi(M)$, and $\beta_1, \beta_2 : sac(M) \to wmi(M)$ be the mappings such that $\beta_1(g) = g_{-}^{\sim}$, and $\beta_2(g) = g_{\sim}^{-}$ for any $g \in sac(M)$.

Theorem 3.14.

Let M be a normal residuated lattice.

(i) α_1 and β_2 form an antitone Galois connection, i.e. $f \leq \beta_2(g)$ if and only if $g \leq \alpha_1(f)$, for any $f \in wmi(M)$ and $g \in sac(M)$.

(ii) α_2 and β_1 form an antitone Galois connection, i.e. $f \leq \beta_1(g)$ if and only if $g \leq \alpha_2(f)$, for any $f \in wmi(M)$ and $g \in sac(M)$.

Proof. (i) Let $f \in wmi(M), g \in sac(M)$ and $f \leq \beta_2(g) = g_{\sim}^{-}$. Then $f(x) \leq g_{\sim}^{-}(x) = (g(x^{\sim}))^{-}$, thus $f(x)^{\sim} \geq (g(x^{\sim}))^{-\sim}$, for any $x \in M$. Therefore $(f(x^{-}))^{\sim} \geq (g(x^{-\sim}))^{-\sim} \geq g(x^{-\sim}) \geq g(x)$, thus $\alpha_1(f)(x) \geq g(x)$, for any $x \in M$. That means $g \leq \alpha_1(f)$.

Conversely, let $g \leq \alpha_1(f)$. Then $f^{\sim}_{-}(x) \geq g(x)$, i.e. $(f(x^{-}))^{\sim} \geq g(x)$, and so $(f(x^{-}))^{\sim -} \leq (g(x))^{-}$, for any $x \in M$. Hence $(f(x^{\sim -}))^{\sim -} \leq (g(x^{\sim}))^{-} = g^{-}_{\sim}(x)$, and $(f(x^{\sim -}))^{\sim -} \geq f(x^{\sim -}) \geq f(x)$. That means $\beta_2(g)(x) = g^{-}(x^{\sim}) \geq (f(x^{\sim -}))^{\sim -} \geq f(x)$, for any $x \in M$, and thus $f \leq \beta_2(g)$.

(ii): Analogously. □

The following theorem is now an immediate consequence.

Theorem 3.15.
Let M be a good normal residuated lattice.

(i) If f is an mi-operator on M and $h = (f^{-}_{\sim})^{\sim}_{-} = (f^{\sim}_{-})^{-}_{\sim}$, then $f^{\sim}_{-} = h^{\sim}_{-}$ and $f^{-}_{\sim} = h^{-}_{\sim}$.

(ii) If g is an ac-operator on M and $k = (g^{-}_{\sim})^{\sim}_{-} = (g^{\sim}_{-})^{-}_{\sim}$, then $g^{-}_{\sim} = k^{-}_{\sim}$ and $g^{\sim}_{-} = k^{\sim}_{-}$.

4. Operators on residuated lattices with Glivenko properties

Lemma 4.1.
Let M be a residuated lattice. For any $x, y \in M$ we have

$$(x \to y^{\sim -})^{\sim -} = x \to y^{\sim -},\ (x \rightsquigarrow y^{-\sim})^{-\sim} = x \rightsquigarrow y^{-\sim}.$$

Proof. Let $x, y \in M$. Then $(x \to y^{\sim -})^{\sim -} = ((x \odot y^{\sim})^{-})^{\sim -} = (x \odot y^{\sim})^{-} = x \to y^{\sim -}$. Analogously for the second identity. □

As a corollary we obtain that if M is a good residuated lattice, then for any $x, y \in M$

$$(x \to y^{-\sim})^{-\sim} = x \to y^{-\sim}.$$

Lemma 4.2.
Let M be a good residuated lattice. Then the following conditions are equivalent:

(i) $(x \to y)^{-\sim} = x \to y^{-\sim}$, $(x \rightsquigarrow y)^{\sim -} = x \rightsquigarrow y^{\sim -}$, for any $x, y \in M$.

(ii) $(x^{-\sim} \to x)^{-\sim} = 1 = (x^{\sim -} \rightsquigarrow x)^{\sim -}$, for any $x \in M$.

(iii) $(x \to y)^{-\sim} = x^{-\sim} \to y^{-\sim}$, $(x \rightsquigarrow y)^{\sim -} = x^{\sim -} \rightsquigarrow y^{\sim -}$, for any $x, y \in M$.

Proof. (i) $\Longrightarrow$ (ii): Let M satisfy (i) and $x \in M$. Then $(x^{-\sim} \to x)^{-\sim} = x^{-\sim} \to x^{-\sim} = 1$, and similarly $(x^{\sim -} \rightsquigarrow x)^{\sim -} = 1$.

(ii) $\Longrightarrow$ (i): Let M satisfy (ii) and $x, y \in M$. Then $y^{-\sim} \to y \leq (x \to y^{-\sim}) \to (x \to y)$, hence $1 = (y^{-\sim} \to y)^{-\sim} \leq ((x \to y^{-\sim}) \to (x \to y))^{-\sim} \leq ((x \to y^{-\sim}) \to (x \to y)^{-\sim})^{-\sim} = (x \to y^{-\sim}) \to (x \to y)^{-\sim}$, therefore $x \to y^{-\sim} \leq (x \to y)^{-\sim}$. Conversely, $(x \to y)^{-\sim} \leq (x \to y^{-\sim})^{-\sim} = x \to y^{-\sim}$.

(i) $\Longrightarrow$ (iii): We have $(x \to y)^{-\sim} = x \to y^{-\sim} = x^{-\sim} \to y^{-\sim}$. Analogously for the second identity.

(iii) $\Longrightarrow$ (ii): $(x^{\sim -} \to x)^{-\sim} = x^{-\sim} \to x^{-\sim} = 1$. Analogously $(x^{\sim -} \rightsquigarrow x)^{\sim -} = 1$.

□

Definition .
We say that a residuated lattice M has *Glivenko property (GP)* if M satisfies the equivalent conditions in Lemma 4.2.

Remark 4.3.
Recall that the notion of a residuated lattice with Glivenko property in the commutative case (as a residuated lattice satisfying the identity $(x \to y)^{--} = x \to y^{--}$) was introduced and investigated in [3].

Definition .
Let M be a residuated lattice. A nonempty set F of M is called a *filter* of M if the following conditions hold

(i) $x, y \in F$ imply $x \odot y \in F$,

(ii) $x \in F, x \leq y \in M$ imply $y \in F$.

Definition .
A subset $D \subseteq M$ is called a *deductive system* of M if

(i) $1 \in D$,

(ii) $x \in D, x \to y \in D$ imply $y \in D$.

Proposition 4.4.
If $H \subseteq M$, then H is a filter in M if and only if H is a deductive system in M.

Proof. Let H be a filter. Then clearly $1 \in H$. Now let $x \in H, x \to y \in H$. Then $(x \to y) \odot x \in H$, and since $(x \to y) \odot x \leq x \wedge y$ it follows that $y \in H$.

Conversely, let H be a deductive system and let $x, y \in H$. Then $x \to (y \to (x \odot y)) = (x \odot y) \to (x \odot y) = 1$, thus $y \to (x \odot y) \in H$ and hence $x \odot y \in H$. Let $x \in H$ and $z \in M$ be such that $x \leq z$. Then $x \to z = 1 \in H$, therefore $z \in H$. □

Now it can be readily shown that H is a filter in M if and only if

(i) $1 \in H$,

(ii) $x \in H, x \rightsquigarrow y \in H$ imply $y \in H$.

A filter H of M is called *normal* [18] if $x \to y \in H$ iff $x \rightsquigarrow y \in H$ for each $x, y \in M$. Normal filters of any residuated lattice M are in one-to-one correspondence with the congruences on M. If H is a normal filter of M, then H is the kernel of the unique congruence θ_H such that $\langle x, y \rangle \in \theta_H$ if and only if $(x \to y) \odot (y \to x) \in H$ if and only if $(x \rightsquigarrow y) \odot (y \rightsquigarrow x) \in H$.

Hence we will consider quotient residuated lattices M/H of residuated lattices M by their normal filters. If $x \in M$ then we will denote by x/H the class of M/H containing x.

If M is a residuated lattice, denote $D(M) = \{x \in M; x^{-\sim} = 1 = x^{\sim -}\}$ the set of *dense elements* in M.

Theorem 4.5. (i) If M is a good residuated lattice, then $D(M)$ is a filter in M.

(ii) If, moreover, M satisfies (GP), then $D(M)$ is a normal filter in M.

Proof. (i): Clearly $1 \in D(M)$. Let $x, y \in D(M)$, i.e. $x^{-\sim} = 1 = y^{-\sim}$. Then $(x \odot y)^{-\sim} \geq x^{-\sim} \odot y^{-\sim} = 1 \odot 1 = 1$, hence $x \odot y \in D(M)$. If $x \in D(M), z \in M$ and $x \leq z$, then obviously $z \in D(M)$.

(ii): Let now M satisfy (GP), $x, y \in M$ and $x \to y \in D(M)$, i.e. $(x \to y)^{-\sim} = 1$. Then $x^{-\sim} \to y^{-\sim} = 1$, thus $x^{-\sim} \leq y^{-\sim}$, and since M is good we have $(x \rightsquigarrow y)^{-\sim} = x^{-\sim} \rightsquigarrow y^{-\sim} = 1$. Therefore $x \rightsquigarrow y \in D(M)$.

It can be shown in a similar manner that $x \rightsquigarrow y \in D(M)$ implies $x \to y \in D(M)$. Hence the filter $D(M)$ is normal. □

Theorem 4.6.

Let M be a good residuated lattice satisfying (GP). Then $\langle x, y \rangle \in \theta_{D(M)}$ if and only if $x^{-\sim} = y^{-\sim}$ for all $x, y \in M$. Moreover, $M/D(M)$ is an involutive residuated lattice.

Proof. Let $x, y \in M$. Then

$\langle x, y \rangle \in \theta_{D(M)} \iff x \to y, y \to x \in D(M) \iff (x \to y)^{-\sim} = 1 = (y \to x)^{-\sim}$. Since $x \to y \leq x \to y^{-\sim}$ we get $(x \to y)^{-\sim} \leq (x \to y^{-\sim})^{-\sim}$, and thus $(x \to y^{-\sim})^{-\sim} = 1$. By Lemma 4.1, $1 = (x \to y^{-\sim})^{-\sim} = x \to y^{-\sim}$, hence $x \leq y^{-\sim}$, and consequently $x^{-\sim} \leq y^{-\sim}$. Analogously we get $y^{-\sim} \leq x^{-\sim}$. Moreover, $(x/D(M))^{-\sim} = x^{-\sim}/D(M) = x/D(M)$.

□

An element x of a residuated lattice M is called *regular* if $x^{-\sim} = x = x^{\sim -}$. Denote by $Reg(M)$ the set of all regular elements in M. Clearly $0, 1 \in Reg(M)$. If $x, y \in M$, put $x \vee_* y := (x \vee y)^{-\sim}, x \wedge_* y := (x \wedge y)^{-\sim}, x \odot_* y := (x \odot y)^{-\sim}$.

Theorem 4.7.

Let M be a good normal residuated lattice satisfying (GP). Then $Reg(M) = (Reg(M); \odot_*, \vee_*, \wedge_*, \to, \rightsquigarrow, 0, 1)$ is an involutive residuated lattice and the mapping ${}^{-\sim} : M \to Reg(M)$ such that ${}^{-\sim} : x \mapsto x^{-\sim}$ is a retract of the reduct $(M; \odot, \to, \rightsquigarrow, 0, 1)$ onto $(Reg(M); \odot, \to, \rightsquigarrow, 0, 1)$.

Proof. The mapping ${}^{-\sim} : M \to M$ is a closure operator on the lattice $(M; \wedge, \vee)$ and $Reg(M)$ is the set of all fixed elements of ${}^{-\sim}$. Therefore $Reg(M)$ is a lattice with respect to the induced ordering on M, and for the lattice operations $\vee_*$ and $\wedge_*$ we have $x \wedge_* y = x \wedge y$ and $x \vee_* y = (x \vee y)^{-\sim}$ for all $x, y \in Reg(M)$.

Let $x, y \in Reg(M)$. Since M is normal we have $x \odot_* y = (x \odot y)^{-\sim} = x^{-\sim} \odot y^{-\sim} = x \odot y$, thus $x \odot_* y = x \odot y$.

Since M satisfies (GP) we get for any $x, y \in Reg(M)$

$$(x \to y)^{-\sim} = x^{-\sim} \to y^{-\sim} = x \to y,$$

$$(x \rightsquigarrow y)^{\sim -} = x^{\sim -} \rightsquigarrow y^{\sim -} = x \rightsquigarrow y$$

Hence the restriction of $\odot$ onto $Reg(M)$ has left and right adjunctions, therefore $Reg(M) = (Reg(M); \odot, \wedge, \vee_*, \to, \rightsquigarrow, 0, 1)$ is a residuated lattice.

Finally, it is clear that ${}^{-\sim}$ is a surjective homomorphism of $(M; \odot, \to, \rightsquigarrow, 0, 1)$ onto $(Reg(M); \odot, \to, \rightsquigarrow, 0, 1)$. □

From Therem 4.6 and Theorem 4.7 we obtain the following.

Theorem 4.8.
If M is a good normal residuated lattice such that $Reg(M) = (Reg(M); \odot, \to, \rightsquigarrow, 0, 1)$ is an involutive residuated lattice and the mapping ${}^{-\sim}$ is a retract of $(M; \to, \rightsquigarrow)$ onto $(Reg(M); \to, \rightsquigarrow)$, then M satisfies (GP).

Proof. We have $(x \to y)^{-\sim} = x^{-\sim} \to y^{-\sim}$ for any $x, y \in M$. Therefore $(x^{-\sim} \to x)^{-\sim} = x^{-\sim} \to x^{-\sim} = 1$ for any $x \in M$. Moreover, $(x \rightsquigarrow y)^{\sim-} = (x \rightsquigarrow y)^{-\sim} = x^{-\sim} \rightsquigarrow y^{-\sim} = x^{\sim-} \rightsquigarrow y^{-\sim}$. Hence M satisfies (GP). □

Theorem 4.9.
Let M be a good normal residuated lattice. Then the following statements are equivalent:

1. M satisfies (GP).
2. $(Reg(M); \odot, \vee_*, \wedge, \to, \rightsquigarrow, 0, 1)$ is an involutive residuated lattice and the mapping ${}^{-\sim} : M \to Reg(M)$ such that ${}^{-\sim} : x \mapsto x^{-\sim}$ is a retract of $(M; \odot, \to, \rightsquigarrow, 0, 1)$ onto $(Reg(M); \odot, \to, \rightsquigarrow, 0, 1)$.

The following assertion is now an immediate consequence.

Corollary 4.10.
If M is a good normal residuated lattice satisfying (GP), then $(\odot, \to, \rightsquigarrow, 0, 1)$-reducts of $M/D(M)$ and $Reg(M)$ are isomorphic.

Theorem 4.11.
If M is a good normal residuated lattice satisfying (GP) and f is an mi-operator (an ac-operator) on M, then the mapping $f^* : Reg(M) \to Reg(M)$ such that $f^*(x) = f(x)^{-\sim}$, for any $x \in Reg(M)$, is an mi-operator (an ac-operator) on the residuated lattice $Reg(M)$.

Proof. Let f be an mi-operator on M and $x, y \in Reg(M)$.

(1) $f^*(x \odot y) = f(x \odot y)^{-\sim} = (f(x) \odot f(y))^{-\sim} = f(x)^{-\sim} \odot f(y)^{-\sim} = f^*(x) \odot f^*(y)$.

(2) $f^*(x) = f(x)^{-\sim} \leq x^{-\sim} = x$.

(3) $f^*(f^*(x)) = f^*(f(x)^{-\sim}) = (f(f((x))^{-\sim}))^{-\sim} \geq (f(f(x)))^{-\sim} = f(x)^{-\sim} = f^*(x)$. Conversely, $f^*(f^*(x)) = f^*(f(x)^{-\sim}) \leq f(x)^{-\sim} = f^*(x)$.

(4) $f^*(1) = f(1)^{-\sim} = 1^{-\sim} = 1$.

(5) $x \leq y \Longrightarrow f^*(x) = f(x)^{-\sim} \leq f(y)^{-\sim} = f^*(y)$.

Similarly for ac-operators on M.

□

Theorem 4.12.
If M is a good normal residuated lattice satisfying (GP) and f is an mi-operator on the residuated lattice $Reg(M)$, then the mapping $f^+ : M \to M$ such that $f^+(x) = f(x^{-\sim})$, for any $x \in M$, is a wmi-operator on M.

Proof. Let f be an mi-operator on $Reg(M)$.

(1) $f^+(x \odot y) = f((x \odot y)^{-\sim}) = f(x^{-\sim} \odot y^{-\sim}) = f(x^{-\sim} \odot_* y^{-\sim}) = f(x^{-\sim}) \odot_* f(y^{-\sim}) = f(x^{-\sim}) \odot f(y^{-\sim}) = f^+(x) \odot f^+(y)$.

(2) $f^+(x) = f(x^{-\sim}) \leq x^{-\sim}$.

(3) $f^+(f^+(x)) = f((f^+(x))^{-\sim}) = f(f(x^{-\sim})) = f(x^{-\sim}) = f^+(x)$.

(4) $f^+(1) = f(1^{-\sim}) = f(1) = 1$.

(5) $x \leq y \Longrightarrow f^+(x) = f(x^{-\sim}) \leq f(y^{-\sim}) = f^+(y)$. □

Theorem 4.13.
Let M be a good residuated lattice satisfying (GP) and $g : Reg(M) \to Reg(M)$ be an ac-operator on $Reg(M)$. Then the mapping $g^+ : M \to M$ such that $g^+(x) = g(x^{-\sim})$, for any $x \in M$, is an sac-operator on M.

Proof. Let $x, y \in M$.

(1) By Proposition 2.1, $x^{-\sim} \oplus y^{-\sim} = x \oplus y$. Hence $(x \oplus y)^{-\sim} = (x^{-\sim} \oplus y^{-\sim})^{-\sim} = (x^{-\sim-} \odot y^{-\sim-})^{\sim-\sim} = (x^{-\sim-} \odot y^{-\sim-})^{\sim} = x^{-\sim} \oplus y^{-\sim}$, thus $g^+(x \oplus y) = g((x \oplus y)^{-\sim}) = g(x^{-\sim} \oplus y^{-\sim}) = g(x^{-\sim} \oplus_* y^{-\sim}) = g(x^{-\sim}) \oplus_* g(y^{-\sim}) = g(x^{-\sim}) \oplus g(x^{-\sim}) = g^+(x) \oplus g^+(y)$.

(2) $g^+(x) = g(x^{-\sim}) \geq x^{-\sim}$.

(3) $g^+(g^+(x)) = g((g^+(x))^{-\sim}) = g((g(x^{-\sim}))^{-\sim}) = g(g(x^{-\sim})) = g(x^{-\sim}) = g^+(x)$.

(4) - (5) Similarly as in the proof of preceeding theorem. □

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References

[1] Balbes R., Dwinger P., Distributive Lattices, University Missouri Press, Columbia, 1974

[2] Cignoli R. L. O., Itala M. L., Mundici D., Algebraic Foundations of Many-valued Reasoning, Kluwer Academic Publishers, Dordrecht, 2000

[3] Cignoli, R., Torrens, A., Glivenko like theorems in natural expansions of BCK-logic, Math. Log. Quart., 2004, 50, 111–125

[4] Ciungu L. C., Classes of residuated lattices, Annals of University of Craiova. Math. Comp. Sci. Ser., 2006, 33, 180–207

[5] DiNola A., Georgescu G., Iorgulescu A., Pseudo-BL algebras; Part I, Multiple Val. Logic, 2002, 8, 673–714

[6] Dvurečenskij A., Every linear pseudo BL-algebra admits a state, Soft Comput., 2007, 11, 495–501

[7] Dvurečenskij A., Rachůnek J., On Riečan and Bosbach states for bounded Rℓ-monoids, Math. Slovaca, 2006, 56, 487–500

[8] Dvurečenskij A., Rachůnek J., Probabilistic averaging in bounded commutative residuated ℓ-monoids, Discrete Math., 2006, 306, 1317–1326

[9] Dvurečenskij A., Rachůnek J., Probabilistic averaging in bounded Rℓl-monoids, Semigroup Forum, 2006, 72, 191–206

[10] Esteva F., Godo L., Monoidal t-norm based logic: towards a logic for left-continuous t-norms, Fuzzy Sets Syst., 2001, 124, 271–288

[11] Flondor P., Georgescu G., Iorgulescu A., Pseudo-t-norms and pseudo-BL algebras, Soft Comput., 2001, 5, 355–371

[12] Georgescu G., Iorgulescu A., Pseudo-MV algebras, Multiple Val. Logic, 2001, 6, 95–135

[13] Hájek P., Metamathematics of Fuzzy Logic, Springer, Dordrecht, 1998

[14] Jipsen P., Tsinakis C., A Survey of Residuated Lattices, In: Martinez J. (Ed.) Ordered Algebraic Structures, Kluwer, Dordrecht, 2006, 19–56

[15] Galatos N., Jipsen P., Kowalski T., Ono H., Residuated Lattices: An Algebraic Glimpse at Substructural Logics, Elsevier, Amsterdam, 2007

[16] Rachůnek J., A non-commutative generalization of MV-algebras, Czechoslovak Math. J., 2002, 52, 255–273

[17] Rachůnek J., Šalounová, D., A generalization of local fuzzy structures, Soft Comput., 2007, 11, 565–571

[18] Rachůnek J., Šalounová, D., States on Generalizations of Fuzzy structures, Palacký Univ. Press, Olomouc, 2011

[19] Rachůnek J., Slezák, V., Negation in bounded commutative DRℓ-monoids, Czechoslovak Math. J., 2007, 56, 755–763

[20] Rachůnek J., Švrček, F., MV-algebras with additive closure operators, Acta Univ. Palacki. Olomouc. Fac.

Rer. Nat. Math., 2000, 39, 183–189

[21] Rachůnek J., Švrček, F., Interior and closure operators on bounded commutative residuated ℓ-monoids. Discuss. Math., Gen. Alg. Appl., 2008, 28, 11–27

[22] Sikorski R., Boolean Algebras, , 2nd edition, Springer-Verlag, Berlin-Gttingen-Heidelbeg-New York, 1963

[23] Švrček F., Interior and closure operators on bounded residuated lattice ordered monoids, Czechoslovak Math. J., 2008, 58, 345–357

INTERIOR AND CLOSURE OPERATORS ON COMMUTATIVE BASIC ALGEBRAS

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Abstract. Commutative basic algebras are non-associative generalizations of MV-algebras and form an algebraic semantics of a non-associative generalization of the propositional infinite-valued Łukasiewicz logic. In the paper we investigate additive closure and multiplicative interior operators on commutative basic algebras as a generalization of topological operators.

1. Introduction

Topological Boolean algebras, i.e. closure or interior algebras [10], are generalizations of topological spaces defined by means of topological closure and interior operators. In [9] closure and interior MV-algebras as generalizations of topological Boolean algebras were introduced and investigated by means of so-called additive closure and multiplicative interior operators.

Commutative basic algebras have been introduced in [4] as non-associative generalizations of MV-algebras. (The name "basic algebra" was selected because these algebras are in a sense a common base for the structures that were dealt with in [4].) Note that analogously as MV-algebras are an algebraic counterpart of the propositional infinite-valued Łukasiewicz logic (and Boolean algebras are a counterpart of the propositional classical two-valued logic), commutative basic algebras constitute an algebraic semantics of the propositional logic $\mathcal{L}_{CBA}$ [1] which is a non-associative generalization of the Łukasiewicz logic.

In the paper we introduce and investigate additive closure and multiplicative interior operators on commutative basic algebras and describe connections between such operators. Further we show that (additively) idempotent elements of any commutative basic algebra A form a subalgebra $B(A)$ of A which is a Boolean algebra, and we give relations between e.g. additive closure operators on A and topological operators on $B(A)$. Moreover, we study operators on quotient commutative basic algebras.

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2. Preliminaries

Definition 2.1. A *basic algebra* is an algebra $\langle A; \oplus, \neg, 0\rangle$ of type $\langle 2, 1, 0\rangle$ that satisfies the identities

(i) $x \oplus 0 = x$,
(ii) $\neg\neg x = x$,
(iii) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$,
(iv) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg 0$.

Moreover, if $x \oplus y = y \oplus x$ for any $x, y \in A$, then A is called a *commutative basic algebra*.

If $A = \langle A; \oplus, \neg, 0\rangle$ is a basic algebra, then $(A, \wedge, \vee, 1, 0)$, where

$$\begin{aligned} x \vee y &:= \neg(\neg x \oplus y) \oplus y \\ x \wedge y &:= \neg(\neg x \vee \neg y) \\ 1 &:= \neg 0 \end{aligned}$$

is a bounded lattice whose induced order is given by

$$x \leq y \iff \neg x \oplus y = 1.$$

If A is commutative, then this lattice is distributive [4].

In a basic algebra A we define a binary operation (subtraction) such that

$$x \ominus y := \neg(\neg x \oplus y).$$

Moreover, define for any $x, y \in A$

$$x \odot y := \neg(\neg x \oplus \neg y).$$

Lemma 2.1. [2][8] *Let A be a commutative basic algebra. Then for any $x, y, z \in A$ we have:*

(i) *if $x \leq y$, then $x \oplus z \leq y \oplus z, z \ominus y \leq z \ominus x$ and $x \ominus z \leq y \ominus z$,*
(ii) $(x \wedge y) \oplus z = (x \oplus z) \wedge (y \oplus z)$,
(iii) $x \oplus y \geq x \vee y$,
(iv) $x \odot y \leq x \wedge y$,
(v) $\neg(x \wedge y) = \neg x \vee \neg y$,
(vi) $\neg(x \vee y) = \neg x \wedge \neg y$,
(vii) $(x \vee y) \oplus z = (x \oplus z) \vee (y \oplus z)$.

3. Operators on basic algebras

In this section we introduce additive closure and multiplicative interior operators on commutative basic algebras which are generalizations of topological operators on Boolean algebras.

Definition 3.1. Let A be a commutative basic algebra. A mapping $g : A \to A$ is called an *additive closure operator (ac-operator)* on A if for any $x, y \in A$

1. $g(x \oplus y) = g(x) \oplus g(y)$,
2. $x \le g(x)$,
3. $g(g(x)) = g(x)$,
4. $g(0) = 0$.

Proposition 3.1. Let $g : A \to A$ be an ac-operator on a commutative basic algebra A. Then g is a monotone mapping.

Proof. Let $x, y \in A$ such that $x \le y$. Then
$x \le y \Longrightarrow x \vee y = y \Longrightarrow g(x \vee y) = g(y) \Longrightarrow g(\neg(\neg y \oplus x) \oplus x) = g(y) \Longrightarrow g(\neg(\neg y \oplus x)) \oplus g(x) = g(y) \Longrightarrow g(x) \le g(y)$.
□

Let $f : A \to A$ be a mapping, and consider the mapping

$$f^{\neg} : A \to A,$$

such that for each $x \in A$

$$f^{\neg}(x) := \neg(f(\neg x)).$$

Proposition 3.2. Let $g : A \to A$ be an ac-operator on a commutative basic algebra A. Then for any $x, y \in A$ we have

(i) $g^{\neg}(x \odot y) = g^{\neg}(x) \odot g^{\neg}(y)$,
(ii) $g^{\neg}(x) \le x$,
(iii) $g^{\neg}(g^{\neg}(x)) = g^{\neg}(x)$,
(iv) $g^{\neg}(1) = 1$.

Proof. (i): Let $x, y \in A$. Then $g^{\neg}(x \odot y) = g^{\neg}(\neg(\neg x \oplus \neg y)) = \neg g(\neg\neg(\neg x \oplus \neg y)) = \neg(g(\neg x \oplus \neg y)) = \neg(g(\neg x) \oplus g(\neg y)) = \neg(\neg(\neg g(\neg x)) \oplus \neg(\neg g(\neg y))) = \neg(g(\neg x)) \odot \neg(g(\neg y)) = g^{\neg}(x) \odot g^{\neg}(y)$.

(ii): $g^{\neg}(x) = \neg(g(\neg x)) \le \neg\neg x = x$.

(iii): $g^{\neg}(g^{\neg}(x)) = g^{\neg}(\neg(g(\neg x))) = \neg g(\neg\neg(g(\neg x))) = \neg(g(g(\neg x))) = \neg(g(\neg x)) = g^{\neg}(x)$.

(iv): $g^{\neg}(1) = \neg(g(\neg 1)) = \neg g(0) = \neg 0 = 1$. □

Definition 3.2. Let A be a commutative basic algebra. A mapping $f : A \to A$ is called a *multiplicative interior operator (mi-operator)* on A if for any $x, y \in A$

1. $f(x \odot y) = f(x) \odot f(y)$,
2. $f(x) \le x$,
3. $f(f(x)) = f(x)$,
4. $f(1) = 1$.

Theorem 3.1. *If $g : A \to A$ is an ac-operator on a commutative basic algebra A, then the mapping $g^{\neg} : A \to A$ is an mi-operator on A.*

Proof. It follows from Proposition 3.2. □

Proposition 3.3. Let $f : A \to A$ be an mi-operator on a commutative basic algebra A. Then for any $x \in A$ we have

(i) $f^{\neg}(x \oplus y) = f^{\neg}(x) \oplus f^{\neg}(y)$,
(ii) $x \leq f^{\neg}(x)$,
(iii) $f^{\neg}(f^{\neg}(x)) = f^{\neg}(x)$,
(iv) $f^{\neg}(0) = 0$.

Proof. Let f be an mi-operator on A and let $x \in A$.

(i): $f^{\neg}(x \oplus y) = \neg(f(\neg(x \oplus y))) = \neg(f(\neg(\neg\neg x \oplus \neg\neg y))) = \neg(f(\neg x \odot \neg y)) = \neg(f(\neg x) \odot f(\neg y)) = \neg(f(\neg x)) \oplus \neg(f(\neg y)) = f^{\neg}(x) \oplus f^{\neg}(y)$.

(ii): $f^{\neg}(x) = \neg(f(\neg x)) \geq \neg(\neg x) = x$.

(iii): $f^{\neg}(f^{\neg}(x)) = f^{\neg}(\neg(f(\neg x))) = \neg(f(\neg\neg(f(\neg x)))) = \neg(f(f(\neg x))) = \neg(f(\neg x)) = f^{\neg}(x)$.

(iv): $f^{\neg}(0) = \neg(f(\neg 0)) = \neg(f(1)) = \neg 1 = 0$.

□

Theorem 3.2. *If $f : A \to A$ is an mi-operator on a commutative basic algebra A, then the mapping $f^{\neg} : A \to A$ is an ac-operator on A.*

Proof. It follows from Proposition 3.3. □

Proposition 3.4. Let $g : A \to A$ be an ac-operator on a commutative basic algebra A. Then for any $x \in A$ we have $g^{\neg}(x \ominus y) = g^{\neg}(x) \ominus g(y)$.

Proof. Let $x, y \in A$. Then $g^{\neg}(x \ominus y) = \neg(g(\neg(x \ominus y))) = \neg(g(\neg\neg(y \oplus \neg x))) = \neg(g(y) \oplus g(\neg x)) = \neg(g(\neg x)) \ominus g(y) = g^{\neg}(x) \ominus g(y)$. □

If A is a commutative basic algebra, denote by $mi(A)$ the set of mi-operators on A and by $ac(A)$ the set of ac-operators on A. Suppose that $mi(A)$ and $ac(A)$ are pointwise ordered.

Let $\alpha : mi(A) \to ac(A)$ be the mapping such that $\alpha(f) = f^{\neg}$, for any $f \in mi(A)$, and $\beta : ac(A) \to mi(A)$ be the mapping such that $\beta(g) = g^{\neg}$, for any $g \in ac(A)$.

Theorem 3.3. *If A is a commutative basic algebra, then α and β form an antitone Galois connection, i.e. $f \leq \beta(g)$ if and only if $g \leq \alpha(f)$, for any $f \in mi(A)$ and $g \in ac(A)$.*

Proof. Let $f \in mi(A), g \in ac(A)$ and $f \leq \beta(g) = g^{\neg}$. Then $f(x) \leq g^{\neg}(x) = \neg(g(\neg x))$, thus $\neg f(x) \geq \neg\neg(g(\neg x))$, for any $x \in A$. Therefore $\neg(f(\neg x)) \geq \neg\neg(g(\neg\neg x)) = g(x)$, thus $\alpha(f)(x) \geq g(x)$, for any $x \in A$. That means $g \leq \alpha(f)$.

Conversely, let $g \leq \alpha(f)$. Then $f^{\neg}(x) \geq g(x)$, i.e. $\neg(f(\neg x)) \geq g(x)$, and so $\neg\neg(f(\neg x)) \leq \neg(g(x))$, for any $x \in A$. Hence $\neg\neg(f(\neg\neg x)) = f(x) \leq \neg(g(\neg x)) = g^{\neg}(x)$. That means $\beta(g)(x) = g^{\neg}(x) \geq f(x)$, for any $x \in A$, and thus $f \leq \beta(g)$. □

The following theorem is now an immediate consequence.

Theorem 3.4. *Let A be a commutative basic algebra.*

(i) *If f is an mi-operator on A and $h = (f^\neg)^\neg$ is the corresponding mi-operator on A, then the induced ac-operators $f^\neg$ and $h^\neg$ are the same.*
(ii) *If g is an ac-operator on A and $k = (g^\neg)^\neg$ is the corresponding ac-operator on A, then the induced mi-operators $g^\neg$ and $k^\neg$ are the same.*

4. Boolean subalgebras of commutative basic algebras

Lemma 4.1. *Let A be a commutative basic algebra. Then for any $x, y, z \in A$*

$$x \odot (y \vee z) = (x \odot y) \vee (x \odot z).$$

Proof. Let $x, y, z \in A$. Then $x \odot (y \vee z) = \neg(\neg x \oplus \neg(y \vee z)) = \neg(\neg x \oplus (\neg y \wedge \neg z)) = \neg((\neg x \oplus \neg y) \wedge (\neg x \oplus \neg z)) = \neg\neg(x \odot y) \vee \neg\neg(x \odot z) = (x \odot y) \vee (x \odot z)$. □

Lemma 4.2. *Let A be a commutative basic algebra, and $x, y \in A$. Then the following statements are equivalent:*

(i) $x \oplus y = y$,
(ii) $x \odot y = x$,
(iii) $y \vee \neg x = 1$,
(iv) $x \wedge \neg y = 0$.

Proof. Let $x, y \in A$.

(ii) $\iff$ (iii): If $x \odot y = x$, then $\neg x \vee y = y \vee \neg x = \neg(\neg y \oplus \neg x) \oplus \neg x = (y \odot x) \oplus \neg x = x \oplus \neg x = 1$. Conversely, if $y \vee \neg x = 1$, then $x = x \odot 1 = x \odot (\neg x \vee y) = (x \odot \neg x) \vee (x \odot y) = 0 \vee (x \odot y) = x \odot y$.

(iii) $\iff$ (iv): If $y \vee \neg x = 1$, then $x \wedge \neg y = \neg(\neg x \vee \neg\neg y) = \neg(\neg(\neg\neg x \oplus y) \oplus y) = \neg(\neg(x \oplus y) \oplus y) = \neg(\neg x \vee y) = 0$. Conversely, if $x \wedge \neg y = 0$, then $\neg x \vee y = \neg(\neg\neg x \oplus y) \oplus y = \neg(x \oplus y) \oplus y = \neg x \vee y = \neg(x \wedge \neg y) = 1$.

(iv) $\iff$ (i): Dual to (ii) $\iff$ (iii). □

From the previous lemma we obtain the following.

Lemma 4.3. *Let A be a commutative basic algebra, and $x \in A$. Then the following statements are equivalent.*

(i) $x \oplus x = x$,
(ii) $x \odot x = x$,
(iii) $\neg x \oplus \neg x = \neg x$,
(iv) $\neg x \odot \neg x = \neg x$,
(v) $x \vee \neg x = 1$,
(vi) $x \wedge \neg x = 0$.

Let A be a basic algebra. Denote by $B(A) := \{x \in A : x \oplus x = x\}$ the set of all idempotent elements of A.

Lemma 4.4. *Let A be a commutative basic algebra. Then for any $a \in B(A)$ and $x, y \in A$*

(i) $x \odot a = x \wedge a$,
(ii) $a \odot (x \oplus y) = (a \odot x) \oplus (a \odot y)$,
(iii) $x \oplus a = x \vee a$,
(iv) $a \oplus (x \odot y) = (a \oplus x) \odot (a \oplus y)$.

Proof. (i): Let $a \in B(A), x \in A$. Then

$$x \leq a \Longrightarrow a \leq x \oplus a \leq a \oplus a = a \Longrightarrow x \oplus a = a \Longrightarrow x \odot a = x = x \wedge a.$$

Let $y \in A$. We have $y \odot a \leq y, a$. Let $z \in A, z \leq y, a$. Then $z = z \odot a \leq y \odot a$, thus $y \odot a = y \wedge a$.

(ii): Let $a \in B(A)$ and $x, y \in A$. Then $(a \wedge x) \oplus (a \wedge y) = (a \oplus a) \wedge (x \oplus a) \wedge (a \oplus y) \wedge (x \oplus y) = a \wedge (x \oplus y)$, thus $a \odot (x \oplus y) = (a \odot x) \oplus (a \odot y)$.

(iii), (iv): Similarly. □

Let A be a commutative basic algebra, C a subalgebra of A and $g : A \to A$ ($f : A \to A$) an ac-operator (an mi-operator) on A. Then C is called a *closure subalgebra* (an *interior subalgebra*) with respect to g (to f) if $g(x) \in C$ ($f(x) \in C$) for any $x \in C$.

Proposition 4.1. A subalgebra C is a closure (interior) subalgebra with respect to an ac-operator g (an mi-operator f) if and only if C is an interior (closure) subalgebra with respect to the mi-operator $g^{\neg}$ (ac-operator $f^{\neg}$).

Proof. Let C be a closure subalgebra with respect to an ac-operator g. If $x \in C$, then $\neg x \in C$ and $g(\neg x) \in C$. Therefore $g^{\neg}(x) = \neg(g(\neg x)) \in C$, and C is an interior subalgebra with respect to the mi-operator $g^{\neg}$.

Analogously we can show that if D is a interior subalgebra with respect to an mi-operator f, then D is a closure subalgebra with respect to $f^{\neg}$. □

Proposition 4.2. If A is a commutative basic algebra, then $B(A)$ is a subalgebra of A.

Proof. Let $x, y \in B(A)$. By Lemma 4.3, $\neg x \in B(A)$. Moreover, by Lemma 2.1(vii), $(x \oplus y) \oplus (x \oplus y) = (x \vee y) \oplus (x \vee y) = ((x \vee y) \oplus x) \vee ((x \vee y) \oplus y) = (x \oplus x) \vee (y \oplus x) \vee (x \oplus y) \vee (y \oplus y) = x \vee (x \vee y) \vee (x \vee y) \vee y = x \vee y = x \oplus y$, thus $x \oplus y \in B(A)$. Further we can see that $0 \in B(A)$.

□

Theorem 4.1. *If A is a commutative basic algebra, then $B(A)$ is a Boolean algebra.*

Proof. If $x, y \in B(A)$, then $\neg x, \neg y \in B(A)$, thus $\neg x \vee \neg y \in B(A)$, and $x \wedge y = \neg(\neg x \vee \neg y) \in B(A)$. Therefore $B(A) = (B(A); \vee, \wedge, 0, 1)$ is a bounded lattice. Since A is commutative, the underlying lattice $(A; \vee, \wedge)$ is distributive, and it follows that the lattice $B(A)$ is distributive. Moreover, for any $x \in B(A)$ we have that $\neg x$ is the complement of x in $B(A)$. □

Proposition 4.3. Let A be a commutative basic algebra. Then the Boolean subalgebra $B(A)$ of A is a closure subalgebra (an interior subalgebra) with respect to any ac-operator (any mi-operator) on A.

Proof. Let $g : A \to A$ be an ac-operator, and $x \in B(A)$. Since $g(x) \oplus g(x) = g(x \oplus x) = g(x)$, we have $g(x) \in B(A)$.

Analogously for any mi-operator on A. □

Recall that if B is a Boolean algebra and $g : B \to B$ is a mapping then g is called a *topological closure operator* on B if for any $x, y \in B$,

1. $g(x \vee y) = g(x) \vee g(y)$,
2. $x \leq g(x)$,
3. $g(g(x)) = g(x)$,
4. $g(0) = 0$.

A *topological interior operator* is defined dually.

Theorem 4.2. *Let A be a commutative basic algebra and $g : A \to A$ an ac-operator ($f : A \to A$ an mi-operator). Then the restriction of g to $B(A)$ (f to $B(A)$) is a topological closure (topological interior) operator on the Boolean algebra $B(A)$.*

A commutative basic algebra is called *complete* if the underlying lattice $(A; \vee, \wedge)$ is complete.

Theorem 4.3. *Let A be a complete commutative basic algebra and g a topological closure operator on the Boolean algebra $B(A)$. Then there is an ac-operator g^* on A such that the restriction of g^* to $B(A)$ is equal to g.*

Proof. First we show that the lattice $B(A)$ is a complete sublattice of A.

Let $x_i \in B(A), i \in I$, and $x = \bigwedge(x_i : i \in I)$ in the lattice A. Then $x \oplus x = \bigwedge(x_i : i \in I) \oplus \bigwedge(x_i : i \in I)$, hence $x \oplus x \leq x_j \oplus x_j$ for any $j \in I$ and $x \oplus x \leq x_j \oplus x_j = x_j$ for any $j \in I$. Therefore $x \oplus x \leq \bigwedge(x_i : i \in I) = x$, which implies $x \in B(A)$. Thus $(B(A); \vee, \wedge)$ is a complete sublattice of $(A; \vee, \wedge)$.

Now let g be a topological closure operator on $B(A)$. Let $g^* : A \to A$ be a mapping such that $g^*(x) = g(\bigwedge(a \in B(A) : x \leq a))$ for any $x \in A$. To verify that g^* is an ac-operator on A, let $x, y \in A$:

1. Let $a \in B(A)$ such that $x \oplus y \leq a$. Then $\bigwedge(b \in B(A) : x \leq b) \leq a$ and $\bigwedge(c \in B(A) : y \leq c) \leq a$, hence $\bigwedge(b \in B(A) : x \leq b) \oplus \bigwedge(c \in B(A) : y \leq c) \leq a \oplus a = a$. Therefore $\bigwedge(b \in B(A) : x \leq b) \oplus \bigwedge(c \in B(A) : y \leq c) \leq \bigwedge(a \in B(A) : x \oplus y \leq a)$. Now we have $g^*(x) \oplus g^*(y) = g(\bigwedge(b \in B(A) : x \leq b)) \oplus g(\bigwedge(c \in B(A) : y \leq c)) = g(\bigwedge(b \in B(A) : x \leq b) \oplus \bigwedge(c \in B(A) : y \leq c)) \leq g(\bigwedge(a \in B(A) : x \oplus y \leq a)) = g^*(x \oplus y)$.

Conversely, $x \oplus y \le \bigwedge(b \in B(A) : x \le b) \oplus \bigwedge(c \in B(A) : y \le c)$, hence $\bigwedge(a \in B(A) : x \oplus y \le a) \le \bigwedge(b \in B(A) : x \le b) \oplus \bigwedge(c \in B(A) : y \le c)$. Thus we obtain $g(\bigwedge(a \in B(A) : x \oplus y \le a)) \le g(\bigwedge(b \in B(A) : x \le b) \oplus \bigwedge(c \in B(A) : y \le c)) = g(\bigwedge(b \in B(A) : x \le b)) \oplus g(\bigwedge(c \in B(A) : x \le c))$, that is $g^*(x \oplus y) \le g^*(x) \oplus g^*(y)$.

2. By the definition, $x \le g^*(x)$ for any $x \in A$.

3. $g^*(g^*(x)) = g^*(g(\bigwedge(a \in B(A) : x \le a))) = g(g(\bigwedge(a \in B(A) : x \le a))) = g(\bigwedge(a \in B(A) : x \le a)) = g^*(x)$.

4. Since $0 \in B(A)$, we have $g^*(0) = g(0) = 0$. □

5. Operators on quotient commutative basic algebras

Recall that a *commutative residuated l-groupoid* (see e.g. [3]) is an algebra $L = (L; \wedge, \vee, \odot, \to, 0, 1)$ of type (2, 2, 2, 2, 0, 0) such that

(i) $(L; \wedge, \vee, 0, 1)$ is a bounded lattice;
(ii) $(L; \odot, 1)$ is a commutative groupoid with identity 1;
(iii) the operation $\odot$ and $\to$ satisfy the adjointness property

$$x \odot y \le z \iff x \le y \to z.$$

The notion of a commutative residuated l-groupoid is a generalization of that of a commutative bounded integral residuated lattice (see e.g. [7], [6]) in which the multiplication $\odot$ need not be associative.

We can introduce the dual notion called *commutative dually residuated l-groupoid*, which is an algebra $L = (L; \wedge, \vee, \oplus, -, 1, 0)$ again of type (2, 2, 2, 2, 0, 0) such that

(i) $(L; \wedge, \vee, 1, 0)$ is a bounded lattice,
(ii) $(L; \oplus, 0)$ is a commutative groupoid with zero 0;
(iii) the operations $\oplus$ and $-$ satisfy the dual adjointness property

$$x \oplus y \ge z \iff x \ge z - y.$$

Let $A = (A; \oplus, \neg, 0)$ be a commutative basic algebra and $x \to y = y \oplus \neg x$ for any $x, y \in A$. Then by [3], $(A; \wedge, \vee, \odot, \to, 0, 1)$ is a commutative residuated l-groupoid.

Recall that in each commutative basic algebra $A = (A; \oplus, \neg, 0)$ the binary operation $\odot$ such that $x \odot y := \neg(\neg x \oplus \neg y)$, for any $x, y \in A$, has been introduced. At the same time, $x \oplus y = \neg(\neg x \odot \neg y)$, hence the operations $\oplus$ and $\odot$ are mutually dual.

Moreover one can see that in commutative basic algebras, the connections between the operations $\oplus$ and $-$ are dual to those between the operations $\odot$ and $\to$. Therefore in any commutative basic algebra A, $y - x = y \odot \neg x = y \ominus x$, for any $x, y \in A$, thus $x \oplus y \ge z \iff x \ge z \ominus y$. Hence $(A; \wedge, \vee, \oplus, \ominus, 1, 0)$ is a commutative dually residuated ℓ-groupoid.

Let A be a basic algebra. A subset $J \subseteq A$ is called an *ideal* of A [5], if it contains 0 and satisfies the following conditions:

(1) if $a \ominus b \in J$ and $b \in J$, then $a \in J$,
(2) if $a \ominus b \in J$ and $a \geq b$, then $(c \ominus b) \ominus (c \ominus a) \in J$ for every $c \in A$,
(3) if $a \ominus b \in J$ and $b \ominus a \in J$, then $(a \ominus c) \ominus (b \ominus c) \in J$ for every $c \in A$.

Theorem 5.1. [5] *Let A be a commutative basic algebra and $I \subseteq A$ be an ideal. Then the relation Θ_I defined by*

$$\langle a, b\rangle \in \Theta_I \iff a \ominus b \in I \text{ and } b \ominus a \in I.$$

is a congruence on A such that $[0]_{\Theta_I} = I$.

Moreover, according to [5], the ideals of basic algebras are, in fact, in a one-to-one correspondence with their congruences. Therefore we can write A/I instead of A/Θ_I.

Let A be a commutative basic algebra, $g : A \to A$ an ac-operator on A and $I \subseteq A$ an ideal of A. Then I is called a *g-ideal* if $g(x) \in I$ for any $x \in I$.

Theorem 5.2. *Let A be a commutative basic algebra, $g : A \to A$ an ac-operator and I a g-ideal in A. Then the mapping $g^* : A/I \to A/I$ such that $g^*(x/I) = g(x)/I$ is an ac-operator on the commutative quotient algebra A/I.*

Proof. First we will show that the mapping g^* is correctly defined. Let $x/I = y/I$ i.e. $\langle x, y\rangle \in \Theta_I$. Then $x \odot \neg y, \neg x \odot y \in I$, hence $g(x \odot \neg y), g(\neg x \odot y) \in I$. Since we have $g(y) \oplus g(x \odot \neg y) = g(y \oplus (x \odot \neg y)) = g(x \vee y) \geq g(x)$, it follows, by the definition of a commutative dually residuated l-groupoid, that $g(x \odot \neg y) \geq g(x) \ominus g(y) = g(x) \odot \neg g(y)$. Since $g(x \odot \neg y) \in I$, (and since by [5] every ideal of a basic algebra is downwards closed) we obtain $g(x) \odot \neg g(y) \in I$. It can be proved similarly that $\neg g(x) \odot g(y) \in I$, thus $\langle g(x), g(y)\rangle \in \Theta_I$, i.e. $g(x)/I = g(y)/I$. Moreover, we have shown that Θ_I is a congruence with respect to the unary operation g on A.

Now we will verify that g^* satisfies the conditions from the definition of a ac-operator. Let $x, y \in A$.

1. $g^*(x/I \oplus y/I) = g^*((x \oplus y)/I) = (g(x \oplus y))/I = (g(x) \oplus g(y))/I = g(x)/I \oplus g(y)/I = g^*(x/I) \oplus g^*(y/I)$.
2. Since $x \leq g(x)$, we have $g(x) = x \vee g(x)$. Thus $x/I \vee g^*(x/I) = x/I \vee g(x)/I = (x \vee g(x))/I = g(x)/I = g^*(x/I)$. Therefore $x/I \leq g^*(x/I)$.
3. $g^*(g^*(x/I)) = g^*(g(x)/I) = g(g(x))/I = g(x)/I = g^*(x/I)$.
4. $g^*(0/I) = g(0)/I = 0/I$.

□

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References

[1] BOTUR, M.—HALAŠ, R.: *Commutative basic algebras and non-associative fuzzy logics*, Arch. Math. Logic **48** (2009), 243–255.
[2] BOTUR, M.—HALAŠ, R.—KÜHR, J.: *States on commutative basic algebras*, Fuzzy Sets and Systems **187** (2012), 77–91.
[3] BOTUR, M.—CHAJDA, I.—HALAŠ, R.: *Are basic algebras residuated structures?*, Soft Comput **14** (2010), 251–255.
[4] CHAJDA, I.—HALAŠ, R.—KÜHR, J.: *Many valued quantum algebras*, Algebra Univers. **60** (2009), 63–90.
[5] CHAJDA, I.—KÜHR, J.: *Ideals and congruences of basic algebras*, Soft. Comput. **17** (2013), 401–410.
[6] GALATOS, N.—JIPSEN, P.—KOWALSKI, T.—ONO H.: *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*, Elsevier, Amsterdam, 2007.
[7] JIPSEN, P.—TSINAKIS, C.: *A Survey of Residuated Lattices*, In: Ordered Algebraic Structures (J. Martinez, Ed.), Kluwer, Dordrecht, 2006, pp. 19–56.
[8] RACHŮNEK, J.—ŠALOUNOVÁ, D.: *State operators on commutative basic algebras*, WCCI 2012 IEEE World Congress on Computational Intelligence, June, 10-15, 2012 - Brisbane, Australia, 1511–1516.
[9] RACHŮNEK, J.—ŠVRČEK, F.: *MV-algebras with additive closure operators*, Acta Univ. Palacki. Olomouc., Fac. rer. nat., Math. **39** (2000), 183–189.
[10] RASWIOWA, H.—SIKORSKI, R.: *The Mathematics of Metamathematics*, Panstw. Wyd. Nauk, Warszawa, 1963.

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MONOTONE MODAL OPERATORS ON BOUNDED INTEGRAL RESIDUATED LATTICES

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Abstract. Bounded integral residuated lattices form a large class of algebras containing some classes of commutative and noncommutative algebras behind many-valued and fuzzy logics. In the paper, monotone modal operators (special cases of closure operators) are introduced and studied.

Keywords: residuated lattice, bounded integral residuated lattice, modal operator, closure operator

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Bounded integral residuated lattices form a large class of algebras containing some classes of algebras behind many-valued and fuzzy logics, such as pseudo MV-algebras [15] (or equivalently GMV-algebras [23]), pseudo BL-algebras [5], pseudo MTL-algebras [12] and $R\ell$-monoids [10], and consequently, the classes of their commutative cases, i. e. MV-algebras [3], BL-algebras [16], MTL-algebras [11] and commutative $R\ell$-monoids [9]. Moreover, Heyting algebras [2] which are algebras of the intuitionistic logic can be also viewed as residuated lattices.

Modal operators (special cases of closure operators) were introduced and investigated on Heyting algebras in [22], on MV-algebras in [17], on commutative $R\ell$-monoids in [24] and on (non-commutative) $R\ell$-monoids in [26]. Moreover, monotone modal operators on commutative bounded residuated lattices were studied in [19].

In the paper we define and study monotone modal operators on general (not necessarily commutative) residuated lattices.

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A *bounded integral residuated lattice* is an algebra $M = (M; \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$ of type $(2, 2, 2, 2, 2, 0, 0)$ satisfying the following conditions:

(i) $(M; \odot, 1)$ is a monoid,
(ii) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice,
(iii) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$ for any $x, y \in M$.

In what follows, by a *residuated lattice* we will mean a bounded integral residuated lattice. If the operation "$\odot$" on a residuated lattice M is commutative then M is called a *commutative residuated lattice*.

In a residuated lattice M we define two unary operations "$^-$" and "$^\sim$" on M such that $x^- := x \rightarrow 0$ and $x^\sim := x \rightsquigarrow 0$ for each $x \in M$.

Recall that the above mentioned algebras of many-valued and fuzzy logics are characterized in the class of residuated lattices as follows:

A residuated lattice M is

(a) a pseudo MTL-algebra if M satisfies the identities of pre-linearity
 (iv) $(x \rightarrow y) \vee (y \rightarrow x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow x)$;
(b) an $R\ell$-monoid if M satisfies the identities of divisibility
 (v) $(x \rightarrow y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x)$;
(c) a pseudo BL-algebra if M satisfies both (iv) and (v);
(d) a GMV-algebra (or equivalently a pseudo MV-algebra) if M satisfies (iv), (v) and the identities
 (vi) $x^{-\sim} = x = x^{\sim-}$;
(e) a Heyting algebra if the operations "$\odot$" and "$\wedge$" coincide.

A residuated lattice M is called *good*, if M satisfies the identity $x^{-\sim} = x^{\sim-}$. For example, every commutative residuated lattice, every GMV-algebra and every pseudo BL-algebra which is a subdirect product of linearly ordered pseudo BL-algebras [7] are good.

By [4], every good residuated lattice satisfies the identity $(x^- \odot y^-)^\sim = (x^\sim \odot y^\sim)^-$. If M is good, we define a binary operation "$\oplus$" on M as

$$x \oplus y = (y^- \odot x^-)^\sim.$$

In the following proposition we recall some necessary basic properties of residuated lattices.

Proposition 1 ([1],[4],[14],[18]). *Let M be a residuated lattice. For all $x, y, z \in M$ we have*

(1) $x \odot y \leq x \wedge y$,

(2) $x \leq y \Longrightarrow x \odot z \leq y \odot z,\ z \odot x \leq z \odot y$,

(3) $x \leq y \Longrightarrow z \to x \leq z \to y,\ z \rightsquigarrow x \leq z \rightsquigarrow y$,

(4) $x \leq y \Longrightarrow x \to z \geq y \to z,\ x \rightsquigarrow z \geq y \rightsquigarrow z$,

(5) $(x \odot y) \to z = x \to (y \to z),\ (y \odot x) \rightsquigarrow z = x \rightsquigarrow (y \rightsquigarrow z)$,

(6) $(y \to z) \odot (x \to y) \leq x \to z,\ (x \rightsquigarrow y) \odot (y \rightsquigarrow z) \leq x \rightsquigarrow z$,

(7) $x \leq x^{-\sim},\ x \leq x^{\sim-}$,

(8) $x^{-\sim-} = x^{-},\ x^{\sim-\sim} = x^{\sim}$,

(9) $x \leq y \Longrightarrow y^{-} \leq x^{-},\ y^{\sim} \leq x^{\sim}$,

(10) $x \odot (x \rightsquigarrow y) \leq y,\ (x \to y) \odot x \leq y$,

(11) $y \leq x \to y,\ y \leq x \rightsquigarrow y$,

(12) $x \to y \leq y^{-} \to x^{-},\ x \to y \leq y^{\sim} \rightsquigarrow x^{\sim}$.

Moreover, if M is good, then

(13) $(x \odot y)^{-} = x \to y^{-}$.

(14) $x^{-\sim} \oplus y^{-\sim} = x^{-\sim} \oplus y = x \oplus y^{-\sim} = x \oplus y$,

(15) $x \oplus 0 = x^{-\sim} = 0 \oplus x$,

(16) $x \oplus y = x^{-} \rightsquigarrow y^{-\sim} = y^{\sim} \to x^{-\sim}$,

(17) $y \oplus x^{-} = x \to y^{-\sim},\ x^{\sim} \oplus y = x \rightsquigarrow y^{-\sim}$,

(18) $(x \oplus y) \oplus 0 = x \oplus y$,

(19) $x \leq y \Longrightarrow z \oplus x \leq z \oplus y,\ x \oplus z \leq y \oplus z$,

(20) $\oplus$ *is associative.*

Definition. Let M be a residuated lattice. A mapping $f : M \longrightarrow M$ is called a *modal operator* on M if for any $x, y \in M$

(M1) $x \leq f(x)$,
(M2) $f(f(x)) = f(x)$,
(M3) $f(x \odot y) = f(x) \odot f(y)$.

A modal operator f is called *monotone*, if for any $x, y \in M$

(M4) $x \leq y \Longrightarrow f(x) \leq f(y)$.

If M is a good residuated lattice and for any $x, y \in M$

(M5) $f(x \oplus y) = f(x \oplus f(y)) = f(f(x) \oplus y)$,

then f is called *strong*.

In all cases of $R\ell$-monoids every modal operator is already monotone. However, in general residuated lattices the converse need not hold. The example below was given in [19].

Example 1. Let $X = (\{x/10 | 0 \leq x \leq 10, x \in Z\}, \wedge, \vee, 0, 1)$ be a bounded lattice where $x \wedge y = min\{x, y\}$ and $x \vee y = max\{x, y\}$. If we define operators $\odot$ and $\to$ on X as

$$x \odot y = \begin{cases} x & \text{if } y = 1 \\ y & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } x = 1 \\ 0.9 & \text{otherwise} \end{cases}$$

then it is easy to show that the structure $(X, \wedge, \vee, \odot, \to, 0, 1)$ is a bounded commutative integral residuated lattice. We define an operator $f : X \to X$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 - x & \text{if } 0 < x \leq 0.5 \\ x & \text{if } x > 0.5 \end{cases}$$

Although f is a modal operator it is not monotone, because we have $0.2 < 0.4$ but $f(0.2) = 0.8 \not\leq 0.6 = f(0.4)$.

Now we will show examples of monotone modal operators.

Example 2. Let $M_1 = \{0, a, b, c, 1\}$. We define the operations $\odot$ and $\to$ on M_1 as follows:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	a	b
c	0	a	a	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

Then $M_1 = (M_1; \odot, \vee, \wedge, \to, 0, 1)$ is a commutative $R\ell$-monoid which is both a BL-algebra and a Heyting algebra (i. e. a Gödel algebra). Since M_1 is commutative, we can also consider the operation $\oplus$.

Let now $f_1 : M_1 \to M_1$ be the mapping such that $f_1(0) = 0, f_1(a) = f_1(b) = b$ and $f_1(c) = f_1(1) = 1$. Then f_1 is a strong monotone modal operator on M_1.

Example 3. Let $M_2 = \{0, a, b, c, 1\}$ and let the operations $\odot, \to, \rightsquigarrow$ on M_2 be defined as follows:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	a	b	a	b
c	0	0	0	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	c	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	0	c	1	c	1
c	b	b	b	1	1
1	0	a	b	c	1

Then $M_2 = (M_2; \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ is a non-commutative residuated lattice which is a pseudo MTL-algebra but not an $R\ell$-monoid beause $(b \to a) \odot b = c \odot b = 0 \neq a = a \wedge b$. (Notice that the lattices $(M_1; \vee, \wedge)$ and $(M_2; \vee, \wedge)$ are isomorphic.)

Let us consider the mapping $f_2 : M_2 \to M_2$ such that $f_2(0) = f_2(a) = f_2(b) = b$ and $f_2(c) = f_2(1) = 1$. Then f_2 is a monotone modal operator on M_2.

Since $a^{-\sim} = b \neq c = a^{\sim -}$, the residuated lattice M_2 is not good, hence the addition on M_2 does not exist.

Example 4. Let $M_3 = \{0, a, b, c, 1\}$. We define operations $\odot, \to, \rightsquigarrow$ as follows:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	a	b	b
c	0	a	a	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	b	1	1	1
c	0	b	b	1	1
1	0	a	b	c	1

Then $M_3 = (M_3; \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ is a linearly ordered (non-commutative) residuated lattice, which is a pseudo MTL-algebra. Since $c \odot (c \rightsquigarrow b) = c \odot 1 = c \neq b = b \wedge c$, M_3 is not an $R\ell$-monoid.

Let $f_3 : M_3 \to M_3$ be the mapping such that $f_3(0) = f_3(a) = a, f_3(b) = b, f_3(c) = c$ and $f_3(1) = 1$. Then f_3 is a monotone modal operator on M_3. Moreover, the residuated lattice M_3 is good, hence the operation $\oplus$ exists and one can easily see that the operator f_3 is strong.

Remark. Recall [22] that the notion of a modal operator has its main source in the theory of topoi and sheafification (see [13], [20], [21], [28]). Moreover, modal operators have come also from the theory of frames, where frame maps can be recognized as modal operators on a complete Heyting algebra (see [6]). Therefore the modal operators do not have direct and explicit connections to modal logics. Moreover, modal operators have some diferent properties than e. g. the logic operator "necessarily". Among other, we show that for every modal operator f

on any good residuated lattice satisfying the identity $x^{-\sim} = x$, $f(0) = 0$ if and only if f is the identity.

Proposition 2. *Let M be a residuated lattice. If f is a monotone modal operator on M and $x, y \in M$, then*

(i) $f(x \to y) \leq f(x) \to f(y) = f(f(x) \to f(y)) = x \to f(y) =$
$= f(x \to f(y))$,
$f(x \rightsquigarrow y) \leq f(x) \rightsquigarrow f(y) = f(f(x) \rightsquigarrow f(y)) = x \rightsquigarrow f(y) =$
$= f(x \rightsquigarrow f(y))$,

(ii) $f(x) \leq (x \rightsquigarrow f(0)) \to f(0)$, $f(x) \leq (x \to f(0)) \rightsquigarrow f(0)$,

(iii) $x^- \odot f(x) \leq f(0)$, $f(x) \odot x^\sim \leq f(0)$,

(iv) $f(x \vee y) = f(x \vee f(y)) = f(f(x) \vee f(y))$.

Moreover, if M is good, then for any $x \in M$

(v) $x \oplus f(0) \geq f(x^{-\sim}) \geq f(x)$, $f(0) \oplus x \geq f(x^{-\sim}) \geq f(x)$.

Proof. (i) By Proposition 1 (10), $(x \to y) \odot x \leq y$. It follows immediately that $f((x \to y) \odot x) = f(x \to y) \odot f(x) \leq f(y)$. Thus we have $f(x \to y) \leq f(x) \to f(y)$. By Proposition 1, $f(x) \to f(y) \leq x \to f(y) \leq f(x \to f(y)) \leq f(x) \to f(f(y)) = f(x) \to f(y)$, therefore $f(x) \to f(y) = x \to f(y) = f(x \to f(y))$.

Moreover, $f(x) \to f(y) \leq f(f(x) \to f(y)) \leq f(f(x)) \to f(f(y)) = f(x) \to f(y)$, which implies that $f(f(x) \to f(y)) = f(x) \to f(y)$. The proof can be done similarly for "$\rightsquigarrow$".

(ii) By (i), $f(x) \rightsquigarrow f(0) = x \rightsquigarrow f(0)$ and by Proposition 1(10), $f(x) \odot (f(x) \rightsquigarrow f(0)) \leq f(0)$. Thus we have $f(x) \leq (f(x) \rightsquigarrow f(0)) \to f(0) = (x \rightsquigarrow f(0)) \to f(0)$.

(iii) Since $0 \leq f(0)$, it follows that $x^- = x \to 0 \leq x \to f(0) = f(x) \to f(0)$. Therefore $x^- \odot f(x) \leq f(0)$. In a similar way we get $f(x) \odot x^\sim \leq f(0)$.

(iv) By the monotony of f we get $f(x \vee y) \leq f(x \vee f(y)) \leq f(f(x) \vee f(y)) \leq f(f(x \vee y)) = f(x \vee y)$.

(v) By Proposition 1 and by (i), $x \oplus f(0) = x^- \rightsquigarrow f(0)^{-\sim} \geq x^- \rightsquigarrow f(0) = f(x^- \rightsquigarrow f(0)) \geq f(x^- \rightsquigarrow 0) = f(x^{-\sim}) \geq f(x)$.

Analogously we prove the remaining inequalities. □

Proposition 3. *If M is a good residuated lattice and f is a strong monotone modal operator on M, then for any $x, y \in M$*

(i) $f(x \oplus y) = f(f(x) \oplus f(y))$,

(ii) $x \oplus f(0) = f(x^{-\sim}) = f(0) \oplus x$.

Proof. (i) Obvious.

(ii) Since f is strong, we have $f(x \oplus f(0)) = f(x \oplus 0) = f(x^{-\sim})$. This means that by Proposition 2 (v), $f(x^{-\sim}) = f(x \oplus f(0)) \geq x \oplus f(0) \geq f(x^{-\sim})$. The proof of $f(x^{-\sim}) = f(0) \oplus x$ follows in the same manner. □

Proposition 4. *Let M be a good residuated lattice and f a monotone modal operator on M.*

(1) If for any $x \in M$ we have $x \oplus f(0) = f(x \oplus 0)$, then

a) $f(x) \oplus f(0) = x \oplus f(0)$,

b) $f(0) \oplus f(x) = f(0) \oplus x$.

(2) If for any $x \in M$ we have $f(0) \oplus x = f(0 \oplus x)$, then

a) $f(x) \oplus f(0) = f(0) \oplus x$,

b) $f(x) \oplus f(0) = x \oplus f(0)$.

Proof. Let f be a monotone modal operator on a good residuated lattice M.

(1) It follows from Proposition 2 (v) that $f(x) \leq x \oplus f(0)$. Thus $f(x) \oplus f(0) \leq x \oplus f(0) \oplus f(0)$. By the assumption, we have $f(0) \oplus f(0) = f(f(0) \oplus 0) = f(0 \oplus f(0)) = f(f(0 \oplus 0)) = f(0 \oplus 0) = f(0)$. Therefore $f(x) \oplus f(0) \leq x \oplus f(0)$. Conversely, it is obvious that $x \oplus f(0) \leq f(x) \oplus f(0)$. Thus we get $f(x) \oplus f(0) = x \oplus f(0)$. It can be shown in a similar manner that $f(0) \oplus f(x) = f(0) \oplus x$.

(2) Analogously. □

From the above proposition we get a characterization of strong modal operators.

Proposition 5. *Let f be a monotone modal operator on a good residuated lattice M. Then it is strong if and only if for any $x \in M$*

$$x \oplus f(0) = f(x^{-\sim}) = f(0) \oplus x.$$

Proof. If f is strong, then by Proposition 3(ii) $x \oplus f(0) = f(x^{-\sim}) = f(0) \oplus x$.

Conversely, suppose that $x \oplus f(0) = f(x^{-\sim}) = f(x \oplus 0)$. By Proposition 1 (18), $x \oplus y = x \oplus y \oplus 0$ holds for all $x, y \in M$, and by Proposition 4 we have

$$\begin{aligned} f(x \oplus f(y)) &= f((x \oplus f(y)) \oplus 0) \\ &= x \oplus f(y) \oplus f(0) \\ &= x \oplus y \oplus f(0) \\ &= f(x \oplus y \oplus 0) \\ &= f(x \oplus y). \end{aligned}$$

By Proposition 4 we can find in the same manner that $f(f(x) \oplus y) = f(x \oplus y)$. Therefore f is a strong modal operator.

□

Theorem 6. *Let M be a residuated lattice and $f : M \longrightarrow M$ a mapping. Then f is a monotone modal operator on M if and only if we have for any $x, y \in M$:*

(i) $x \to f(y) = f(x) \to f(y)$,
(ii) $x \rightsquigarrow f(y) = f(x) \rightsquigarrow f(y)$,
(iii) $f(x) \odot f(y) \geq f(x \odot y)$.

Proof. Suppose a mapping f satisfies (i) - (iii). We will show that f also satisfies the conditions (M1) - (M4) from the definition of a monotone modal operator.

(M1) By (i), $x \to f(x) = f(x) \to f(x) = 1$, which implies that $x \leq f(x)$.

(M2) Since $1 = f(x) \to f(x) = f(f(x)) \to f(x)$, it follows that $f(f(x)) \leq f(x)$, thus by (1) we have $f(f(x)) = f(x)$.

(M3) By (M1) , $x \odot y \leq f(x \odot y)$, and it follows that $y \leq x \rightsquigarrow f(x \odot y) = f(x) \rightsquigarrow f(x \odot y)$ and $f(x) \odot y \leq f(x \odot y)$. Thus we get $f(x) \leq y \to f(x \odot y) = f(y) \to f(x \odot y)$ and $f(x) \odot f(y) \leq f(x \odot y)$. Therefore $f(x) \odot f(y) = f(x \odot y)$.

(M4) Note that if $x \leq y$, then $x \leq f(y)$. From the fact that $1 = x \to f(y) = f(x) \to f(y)$ we obtain $f(x) \leq f(y)$. □

In general, if f is a monotone modal operator, the equation $f(0) = 0$ need not hold. An example is shown in [19]. Thus we will investigate under which condition this equality holds.

Proposition 7. *Let M be a residuated lattice and f a monotone modal operator. Then the following conditions are equivalent.*

(i) $f(0) = 0$,

(ii) $f(x^{\sim}) = x^{\sim}$, *for all* $x \in M$,

(iii) $f(x^{-}) = x^{-}$, *for all* $x \in M$.

Proof. (i) $\Longrightarrow$ (ii): Suppose that $f(0) = 0$. It follows from Proposition 2 (ii) that $f(x) \leq (x \to f(0)) \rightsquigarrow f(0) = (x \to 0) \rightsquigarrow 0 = x^{-\sim}$. Therefore $f(x) \leq x^{-\sim}$ and $f(x^{\sim}) \leq (x^{\sim})^{-\sim} = x^{\sim}$. Since $x^{\sim} \leq f(x^{\sim})$, we have that $f(x^{\sim}) = x^{\sim}$ for all $x \in M$.

(ii) $\Longrightarrow$ (i): Suppose that $f(x^{\sim}) = x^{\sim}$ for all $x \in M$. Then we get $f(0) = f(1^{\sim}) = 1^{\sim} = 0$.

It can be proved in a similar manner that (i) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (i). □

Corollary 8. *Let M be a good residuated lattice satisfying $x^{-\sim} = x$ for all $x \in M$. Let f be a monotone modal operator on M such that $f(0) = 0$. Then f is the identity on M.*

A residuated lattice M is called *normal* if it satisfies the identities

$$(x \odot y)^{-\sim} = x^{-\sim} \odot y^{-\sim},$$
$$(x \odot y)^{\sim -} = x^{\sim -} \odot y^{\sim -}.$$

For example, every Heyting algebra and every good pseudo BL-algebra is normal [27], [8].

Proposition 9 ([25])**.** *Let M be a good and normal residuated lattice. Then for any $x, y \in M$*

(i) $(x \oplus y)^{-} = y^{-} \odot x^{-},\ (x \oplus y)^{\sim} = y^{\sim} \odot x^{\sim}$,

(ii) $x^{-} \oplus y^{-} = (y \odot x)^{-},\ x^{\sim} \oplus y^{\sim} = (y \odot x)^{\sim}$.

Denote by

$$I(M) = \{a \in M; a \odot a = a\}$$

the set of all multiplicative idempotents in a residuated lattice M. Clearly $0, 1 \in M$.

Proposition 10. *Let M be a good and normal residuated lattice. Then the following conditions are equivalent.*

(i) $a^{-} \in I(M)$,

(ii) $a^{\sim} \in I(M)$,

(iii) $a \oplus a = a^{-\sim}$.

Proof. (ii) $\Longleftrightarrow$ (iii): If $a^{\sim} \in I(M)$, then $a \oplus a = (a^{\sim} \odot a^{\sim})^{-} = (a^{\sim})^{-} = a^{-\sim}$. Conversely, suppose that $a \oplus a = a^{-\sim}$. By Proposition 9(i), we have $a^{\sim} = (a^{-\sim})^{\sim} = (a \oplus a)^{\sim} = a^{\sim} \odot a^{\sim}$. Therefore $a^{\sim} \in \mathrm{I}(M)$.

(i) $\Longleftrightarrow$ (iii): Analogously. □

Let M be a good residuated lattice and $a \in M$. We denote by $\varphi_a : M \to M$ the mapping such that $\varphi_a(x) = a \oplus x$ for all $x \in M$.

Proposition 11. *Let M be a good and normal residuated lattice and let $a \in M$. If φ_a is a strong monotone modal operator on M, then $a^{-}, a^{\sim}, a^{-\sim} \in I(M)$.*

Proof. Since $\varphi_a(x \odot y) = \varphi_a(x) \odot \varphi_a(y)$, we have $a \oplus (x \odot y) = (a \oplus x) \odot (a \oplus y)$ for any $x, y \in M$. By setting $x = y = 0$, we obtain $a \oplus 0 = (a \oplus 0) \odot (a \oplus 0)$, thus $a^{-\sim} = a^{-\sim} \odot a^{-\sim}$, which implies that $a^{-\sim} \in I(M)$.

Further, $a \oplus (x \oplus y) = \varphi_a(x \oplus y) = \varphi_a(x \oplus \varphi_a(y)) = a \oplus (x \oplus (a \oplus y))$ for any $x, y \in M$. For $x = y = 0$ we have $a^{-\sim} = a \oplus 0 = a \oplus (0 \oplus 0) = a \oplus (0 \oplus (a \oplus 0)) = (a \oplus 0) \oplus a^{-\sim} = a^{-\sim} \oplus a^{-\sim}$, thus $a^{-\sim} = (a^- \odot a^-)^\sim$. This implies that $a^- = (a^- \odot a^-)^{\sim -} = a^{-\sim -} \odot a^{-\sim -} = a^- \odot a^-$ and so $a^- \in I(M)$.

Moreover, by Proposition 10, $a^\sim \in I(M)$. □

Proposition 12. *If M is a good and normal residuated lattice and $a \in M$ is such that $a^-, a^{-\sim} \in I(M)$, then φ_a satisfies conditions (M1), (M2), (M4) from the definition of a strong monotone modal operator, and*

(M5') $f(x \oplus y) = f(f(x) \oplus y)$.

Moreover, if a commutes with every $x \in M$, then φ_a satisfies (M5).

Proof. (M1) For any we have $x \in M$ $\varphi_a(x) = a \oplus x = (x^- \odot a^-)^\sim \geq x^{-\sim} \geq x$.

(M2) Since $a^- \in I(M)$, we get $\varphi_a(\varphi_a(x)) = a \oplus (a \oplus x) = a \oplus x = \varphi_a(x)$.

(M4) If $x \leq y$, then $\varphi_a(x) = a \oplus x \leq a \oplus y = \varphi_a(y)$.

(M5') Let $x, y \in M$. We have $\varphi_a(\varphi_a(x) \oplus y) = \varphi_a(a \oplus x \oplus y) = a \oplus a \oplus x \oplus y = a \oplus x \oplus y = \varphi_a(x \oplus y)$.

Now suppose that a commutes with every $x \in M$. For any $x, y \in M$ we get $\varphi_a(x \oplus \varphi_a(y)) = a \oplus (x \oplus (a \oplus y)) = ((a \oplus a) \oplus x) \oplus y = (a^{-\sim} \oplus x) \oplus y = a \oplus (x \oplus y) = \varphi_a(x \oplus y)$. □

Proposition 13. *Let M be a good and normal residuated lattice and f a monotone modal operator on M such that $f(x) = f(x^{-\sim})$ for all $x \in M$. Then f is strong if and only if $f = \varphi_{f(0)}$ and $f(0)^- \in I(M)$.*

Proof. Let f be a monotone modal operator on M satisfying the identity $f(x) = f(x^{-\sim})$.

If f is strong then by Proposition 5, $f(x) = f(x^{-\sim}) = x \oplus f(0)$ for any $x \in M$, hence $f = \varphi_{f(0)}$ and therefore, by Proposition 11, $f(0)^-, f(0)^{-\sim} \in I(M)$.

Conversely, let f be any modal operator on M. Then $f(0)^{-\sim} = f(0 \odot 0)^{-\sim} = (f(0) \odot f(0))^{-\sim} = f(0)^{-\sim} \odot f(0)^{-\sim}$, thus $f(0)^{-\sim} \in I(M)$. Let now f be monotone, $f = \varphi_{f(0)}$ and $f(0)^- \in I(M)$. Then by Proposition 11 we get that f is strong. $\square$

Let M be a residuated lattice and $a \in I(M)$. Consider the mappings $\psi_a^1 : M \longrightarrow M$ and $\psi_a^2 : M \longrightarrow M$ such that $\psi_a^1(x) = a \to x$ and $\psi_a^2(x) = a \rightsquigarrow x$.

Proposition 14. *Let M be a good residuated lattice and $a \in I(M)$. Then for any $x, y \in M$*

(1) $\psi_a^1(x \oplus y) = \psi_a^1(x \oplus \psi_a^1(y))$,
(2) $\psi_a^1(x \oplus y) \leq \psi_a^1(\psi_a^1(x) \oplus y)$,
(3) $\psi_a^2(x \oplus y) = \psi_a^2(\psi_a^2(x) \oplus y)$,
(4) $\psi_a^2(x \oplus y) \leq \psi_a^2(x \oplus \psi_a^2(y))$.

Proof. (1) We have $y \leq a \to y = \psi_a^1(y)$, thus $\psi_a^1(x \oplus y) \leq \psi_a^1(x \oplus \psi_a^1(y))$.

To prove the converse inequality first note that since $(a \to x) \odot a \leq x$, we have $(a \to x) \odot (a \odot x^\sim) \leq x \odot x^\sim = 0$, hence $a \odot x^\sim \leq (a \to x)^\sim$. Thus we have $\psi_a^1(x \oplus \psi_a^1(y)) = \psi_a^1((\psi_a^1(y)^\sim \odot x^\sim)^-) = a \to (\psi_a^1(y)^\sim \odot x^\sim)^- = (a \odot \psi_a^1(y)^\sim \odot x^\sim)^-$, hence $a \odot \psi_a^1(y)^\sim \odot x^\sim = a \odot (a \to y)^\sim \odot x^\sim \geq a \odot (a \odot y^\sim) \odot x^\sim = (a \odot a) \odot (y^\sim \odot x^\sim) = a \odot (y^\sim \odot x^\sim)$, therefore $\psi_a^1(x \oplus \psi_a^1(y)) = (a \odot \psi_a^1(y)^\sim \odot x^\sim)^- \leq (a \odot y^\sim \odot x^\sim)^- = a \to (y^\sim \odot x^\sim)^- = a \to (x \oplus y) = \psi_a^1(x \oplus y)$, i. e. $\psi_a^1(x \oplus \psi_a^1(y)) \leq \psi_a^1(x \oplus y)$.

(2) Since $x \leq a \to x = \psi_a^1(x)$, we get $x \oplus y \leq \psi_a^1(x) \oplus y$, thus $\psi_a^1(x \oplus y) \leq \psi_a^1(\psi_a^1(x) \oplus y)$.

(3) We have $x \le a \rightsquigarrow x = \psi_a^2(x)$, hence $x \oplus y \le \psi_a^2(x) \oplus y$, and so $\psi_a^2(x \oplus y) \le \psi_a^2(\psi_a^2(x) \oplus y)$. Further, since $a \odot (a \rightsquigarrow y) \le y$, we get $(y^- \odot a) \odot (a \rightsquigarrow y) \le y^- \odot y = 0$, and so $y^- \odot a \le (a \rightsquigarrow y)^-$.

We have $\psi_a^2(\psi_a^2(x) \oplus y) = \psi_a^2((y^- \odot \psi_a^2(x)^-)^\sim) = a \rightsquigarrow (y^- \odot \psi_a^2(x)^-)^\sim = ((y^- \odot \psi_a^2(x)^- \odot a)^\sim$, hence $y^- \odot \psi_a^2(x)^- \odot a = y^- \odot (a \rightsquigarrow x)^- \odot a \ge y^- \odot (x^- \odot a) \odot a = y^- \odot x^- \odot a$, thus $\psi_a^2(\psi_a^2(x) \oplus y) = (y^- \odot \psi_a^2(x)^- \odot a)^\sim \le (y^- \odot x^- \odot a)^\sim = ((y^- \odot x^-) \odot a)^\sim = a \rightsquigarrow (x \oplus y) = \psi_a^2(x \oplus y)$. Therefore $\psi_a^2(x \oplus y) = \psi_a^2(\psi_a^2(x) \oplus y)$.

(4) Similarly to (2).

□

Proposition 15. *If M and a are as in Proposition 14 and, moreover, a commutes with every element in M, then in (2) and (4) we have equalities.*

Proof. (2) We have $\psi_a^1(\psi_a^1(x) \oplus y) = \psi_a^1((y^\sim \odot \psi_a^1(x)^\sim)^-) = a \to (y^\sim \odot \psi_a^1(x)^\sim)^- = (a \odot y^\sim \odot \psi_a^1(x)^\sim)^-$ by Proposition 1(13), hence $a \odot y^\sim \odot \psi_a^1(x)^\sim = a \odot y^\sim \odot (a \to x)^\sim \ge a \odot y^\sim \odot (a \odot x^\sim) = (a \odot a) \odot (y^\sim \odot x^\sim) = a \odot (y^\sim \odot x^\sim)$, and similarly to the proof of (1) in Proposition 14 we get $\psi_a^1(\psi_a^1(x) \oplus y) \le \psi_a^1(x \oplus y)$.

(4) Analogously as for (2). □

Corollary 16. *If M is a commutative residuated lattice or M is a bounded $R\ell$-monoid (not necessarily commutative), and $a \in I(M)$, then in (2) and (4) we have equalities.*

Proof. For bounded $R\ell$-monoids see [26]. □

Corollary 17. *If $a \in M$ satisfies the conditions from Proposition 15 or Corollary 16, and ψ_a^1 and ψ_a^2 are monotone modal operators on M, then they are strong.*

Let M be a residuated lattice and f a modal operator on M. We denote by

$$\mathrm{Fix}(f) = \{x \in M; f(x) = x\}$$

the set of all fixed elements of the operator f. By the definition of a modal operator it is obvious that $\mathrm{Fix}(f) = \mathrm{Im}(f)$.

Proposition 18. *If f is a monotone modal operator on a residuated lattice M, then $\mathrm{Fix}(f) = (\mathrm{Fix}(f); \odot, \vee_{\mathrm{Fix}(f)}, \wedge, \rightarrow, \rightsquigarrow, f(0), 1)$, where $x \vee_{\mathrm{Fix}(f)} y = f(x \vee y)$ for any $x, y \in \mathrm{Fix}(f)$, and $\wedge, \rightarrow, \rightsquigarrow$ are the restrictions of the binary operations from M to $\mathrm{Fix}(f)$, is a residuated lattice.*

Proof. Let M be a residuated lattice and f a monotone modal operator on M.

(i) If $x, y \in \mathrm{Fix}(f)$, then $f(x \odot y) = f(x) \odot f(y) = x \odot y$, thus $x \odot y \in \mathrm{Fix}(f)$. Therefore $(\mathrm{Fix}(f); \odot, 1)$ is a residuated lattice.

(ii) Since f is a closure operator on the lattice $(M; \vee, \wedge)$, it follows that $x \wedge y \in \mathrm{Fix}(f)$ for each $x, y \in \mathrm{Fix}(f)$ and $x \vee_{\mathrm{Fix}(f)} y = f(x \vee y)$. Therefore $(\mathrm{Fix}(f); \wedge, f(0), 1)$ is a bounded lattice.

(iii) Let $x, y \in \mathrm{Fix}(f)$. Then by Proposition 2, $x \rightarrow y = f(x) \rightarrow f(y) = f(f(x) \rightarrow f(y)) = f(x \rightarrow y)$, hence $x \rightarrow y \in \mathrm{Fix}(f)$. Analogously $x \rightsquigarrow y \in \mathrm{Fix}(f)$.

(iv) Now, let $x, y, z \in \mathrm{Fix}(f)$. Then $x \odot y, y \rightarrow z, x \rightsquigarrow z \in \mathrm{Fix}(f)$, hence $x \odot_{\mathrm{Fix}(f)} y \leq z$ iff $x \leq y \rightarrow_{\mathrm{Fix}(f)} z$ iff $y \leq x \rightsquigarrow_{\mathrm{Fix}(f)} z$. □

Conclusions. In the paper we have investigated monotone modal operators, which are special cases of closure operators on bounded integral residuated lattices. The results are applicable to a wide class of algebras containing algebras of some algebras behind many-valued and fuzzy logics. One can expect that these results will also be useful for studying analogous operators on further classes of algebras, e. g. on algebras of several quantum logics.

References

[1] *P. Bahls, J. Cole, N. Galatos, P. Jipsen, C. Tsinakis*: Cancellative residuated lattices, Algebra Univers. 50 (2003), 83–106. Zbl 1092.06012

[2] *R. Balbes, P. Dwinger* : Distributive Lattices, University Missouri Press, Columbia, 1974.

[3] *R. L. O. Cignoli, I. M. L. D'Ottaviano, D. Mundici*: Algebraic Foundations of Many-valued Reasoning, Kluwer, Dordrecht, 2000. Zbl 0937.06009

[4] *L. C. Ciungu*: Classes of residuated lattices, Annals of University of Craiova. Math. Comp. Sci. Ser. 33 (2006), 180–207. Zbl 1119.03343

[5] *A. DiNola, G. Georgescu, A. Iorgulesu*: Psedo-BL algebras; Part I, Multiple Val. Logic 8 (2002), 673–714. Zbl 1028.06007

[6] *C. H. Dowker, D. Papert*: Quotient Frames and Subspaces, Proc. London Math. Soc. 16 (1966), 275–296. Zbl 0136.43405

[7] *A. Dvurečenskij*: Every linear pseudo BL-algebra admits a state, Soft Comput. 11 (2007), 495–501. Zbl 1122.06012

[8] *A. Dvurečenskij, J. Rachůnek*: On Riečan and Bosbach states for bounded Rl-monoids, Math. Slovaca 56 (2006), 487–500. Zbl 1141.06005

[9] *A. Dvurečenskij, J. Rachůnek*: Probabilistic averaging in bounded commutative residuated l-monoids, Discrete Math. 306 (2006), 1317–1326. Zbl 1105.06011

[10] *A. Dvurečenskij, J. Rachůnek*: Probabilistic averaging in bounded Rl-monoids, Semigroup Forum 72 (2006), 191–206. Zbl 1105.06010

[11] *F. Esteva, L. Godo*: Monoidal t-norm based logic: towards a logic for left-continuous t-norms, Fuzzy Sets Syst. 124 (2001), 271–288. Zbl 0994.03017

[12] *P. Flondor, G. Georgescu, A. Iorgulescu*: Pseudo-t-norms and pseudo-BL algebras, Soft Comput. 5 (2001), 355–371. Zbl 0995.03048

[13] *P. J. Freyd*: Aspects of topoi, Bull. Austral. Math. Soc. 7 (1972), 1–76. Zbl 0252.18001

[14] *N. Galatos, P. Jipsen, T. Kowalski, H. Ono*: Residuated Lattices: An Algebraic Glimpse at Substructural Logics, Elsevier, Amsterdam (2007). Zbl 1171.03001

[15] *G. Georgescu, A. Iorgulescu*: Pseudo-MV algebras, Multiple Val. Logic 6 (2001), 95–135. Zbl 1014.06008

[16] *P. Hájek*: Metamathematics of Fuzzy Logic, Kluwer, Dordrecht, 1998. Zbl 0937.03030

[17] *M. Harlenderová, J. Rachůnek*: Modal operators on MV-algebras, Math. Bohem. 131 (2006), 39–48. Zbl 1112.06014

[18] *P. Jipsen, C. Tsinakis*: A Survey of Residuated Lattices, In: Ordered Algebraic Structures, Kluwer, Dordrecht (2006), 19–56. Zbl 1070.06005

[19] *M. Kondo*: Modal operators on commutative residuated lattices, Mathematica Slovaca, to appear.

[20] *F. W. Lawvere*: Quantifiers and Sheaves, Actes Congres Intern. Math., Tome 1, 1970, 329–334.

[21] *F. W. Lawvere*: Toposes, Algebraic Geometry and Logic, Springer Lecture Notes 274, Berlin (1972).

[22] *D. S. Macnab*: Modal operators on Heyting algebras, Alg. Univ. 12 (1981), 5–29. Zbl 0459.06005

[23] *J. Rachůnek*: A non-commutative generalization of MV-algebras, Czechoslovak Math. J. 52 (2002), 255–273. Zbl 1012.06012

[24] *J. Rachůnek, D. Šalounová*: Modal operators on bounded commutative residuated l-monoids, Math. Slovaca 57 (2007), 321–332. Zbl 1150.06016

[25] *J. Rachůnek, D. Šalounová*: A generalization of local fuzzy structures, Soft Comput. 11 (2007), 565–571. Zbl 1121.06013

[26] *J. Rachůnek, D. Šalounová*: Modal operators on bounded residuated l-monoids, Math. Bohemica 133 (2008), 299–311. Zbl 05595946

[27] *J. Rachůnek, V. Slezák*: Bounded dually residuated lattice ordered monoids as a generalization of fuzzy structures, Math. Slovaca 56 (2006), 223–233. Zbl 1150.06015

[28] *G. C. Wraith*: Lectures on Elementary Topoi, in Model Theory and Topoi, Springer Lecture Notes 445, Berlin (1975).

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MODAL OPERATORS ON COMMUTATIVE BASIC ALGEBRAS

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Abstract. Commutative basic algebras are non-associative generalizations of MV-algebras and form an algebraic semantics of a non-associative generalization of the propositional infinite valued Łukasiewicz logic. In the paper modal operators (special cases of closure operators) are introduced and studied.

1. Introduction

Commutative basic algebras have been introduced in [5] as a non-associative generalizations of MV-algebras. Note that analogously as MV-algebras are an algebraic counterpart of the propositional infinite valued Łukasiewicz logic (and Boolean algebras are a counterpart of the propositional classical two-valued logic), commutative basic algebras constitute an algebraic semantices of the propositional logic $\mathcal{L}_{CBA}$ [2] which is a non-associative generalization of the Łukasiewicz logic.

Modal operators (special cases of closure operators) were introduced and investigated on Heyting algebras in [7], on MV-algebras in [6], on commutative Rℓ-monoids in [10] and on (non-commutative) Rℓ-monoids in [11]. Moreover, monotone modal operators on bounded integral residuated lattices were studied in [12].

In this paper we introduce and investigate modal operators for arbitrary commutative basic algebras.

2. Preliminaries

Definition. A *basic algebra* is an algebra $\langle A; \oplus, \neg, 0\rangle$ of type $\langle 2, 1, 0\rangle$ that satisfies the identities

(i) $x \oplus 0 = x$,
(ii) $\neg\neg x = x$,
(iii) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$,
(iv) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg 0$.

Moreover, if $x \oplus y = y \oplus x$ for any $x, y \in A$, then A is called a *commutative basic algebra.*

If $A = \langle A; \oplus, \neg, 0\rangle$ is a basic algebra, then $(A, \wedge, \vee, 1, 0)$, where

$$x \vee y := \neg(\neg x \oplus y) \oplus y$$
$$x \wedge y := \neg(\neg x \vee \neg y)$$
$$1 := \neg 0$$

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is a bounded lattice whose induced order is given by

$$x \leq y \iff \neg x \oplus y = 1.$$

If A is commutative, then this lattice is distributive [5].

In a basic algebra A we define a binary operation (subtraction) such that

$$x \ominus y := \neg(y \oplus \neg x).$$

Moreover, define for any $x, y \in A$

$$x \odot y := \neg(\neg x \oplus \neg y), \quad x \to y := \neg x \oplus y.$$

Lemma 2.1. [3],[9] *Let A be a commutative basic algebra. Then for any $x, y, z \in A$ we have:*

(i) *if $x \leq y$, then $x \oplus z \leq y \oplus z, x \odot z \leq y \odot z, z \ominus y \leq z \ominus x$ and $x \ominus z \leq y \ominus z$,*
(ii) $(x \wedge y) \oplus z = (x \oplus z) \wedge (y \oplus z)$,
(iii) $x \oplus y \geq x \vee y$,
(iv) $x \odot y \leq x \wedge y$,
(v) $\neg(x \wedge y) = \neg x \vee \neg y$,
(vi) $\neg(x \vee y) = \neg x \wedge \neg y$,
(vii) $(x \vee y) \oplus z = (x \oplus z) \vee (y \oplus z)$.

Lemma 2.2. *Let A be a commutative basic algebra. Then for any $x, y, z \in A$*

$$x \odot (y \vee z) = (x \odot y) \vee (x \odot z).$$

Proof. Let $x, y, z \in A$. Then $x \odot (y \vee z) = \neg(\neg x \oplus \neg(y \vee z)) = \neg(\neg x \oplus (\neg y \wedge \neg z)) = \neg((\neg x \oplus \neg y) \wedge (\neg x \oplus \neg z)) = \neg\neg(x \odot y) \vee \neg\neg(x \odot z) = (x \odot y) \vee (x \odot z)$. □

Lemma 2.3. *Let A be a commutative basic algebra, and $x, y \in A$. Then the following statements are equivalent:*

(i) $x \oplus y = y$,
(ii) $x \odot y = x$,
(iii) $y \vee \neg x = 1$,
(iv) $x \wedge \neg y = 0$.

Proof. Let $x, y \in A$.

(ii) $\iff$ (iii): If $x \odot y = x$, then $\neg x \vee y = y \vee \neg x = \neg(\neg y \oplus \neg x) \oplus \neg x = (y \odot x) \oplus \neg x = x \oplus \neg x = 1$. Conversely, if $y \vee \neg x = 1$, then $x = x \odot 1 = x \odot (\neg x \vee y) = (x \odot \neg x) \vee (x \odot y) = 0 \vee (x \odot y) = x \odot y$.

(iii) $\iff$ (iv): It follows directly from Lemma 2.1 (v), (vi).

(iv) $\iff$ (i): Dual to (ii) $\iff$ (iii). □

3. Modal operators on basic algebras

Definition. Let A be a commutative basic algebra. A mapping $f : A \to A$ is called an *modal operator* on A if for any $x, y \in A$

1. $x \leq f(x)$,
2. $f(f(x)) = f(x)$,
3. $f(x \odot y) = f(x) \odot f(y)$.

A modal operator f is called strong, if for any $x, y \in A$

4. $f(x \oplus y) = f(x \oplus f(y))$.

Let A be a basic algebra. Denote by $B(A) := \{x \in A : x \oplus x = x\}$ the set of all idempotent elements of A.

Proposition 3.1. [13] *If A is a commutative basic algebra, then $B(A)$ is a subalgebra of A.*

Theorem 3.2. [13] *If A is a commutative basic algebra, then $B(A)$ is a Boolean algebra.*

Corollary 3.3. *Let A be a commutative basic algebra. Then for any element $a \in A$ we have that $a \in B(A)$ if and only if $\neg a \in B(A)$.*

Lemma 3.4. *Let A be a commutative basic algebra. Then for any $a \in B(A)$ and $x, y \in A$*

(i) $x \odot a = x \wedge a$,
(ii) $a \odot (x \oplus y) = (a \odot x) \oplus (a \odot y)$,
(iii) $x \oplus a = x \vee a$,
(iv) $a \oplus (x \odot y) = (a \oplus x) \odot (a \oplus y)$.

Proof. (i): Let $a \in B(A), x \in A$. Then

$$x \leq a \Longrightarrow a \leq x \oplus a \leq a \oplus a = a \Longrightarrow x \oplus a = a.$$

Hence, by Lemma 2.3, we have $x \odot a = x$. Therefore $x \odot a = x \wedge a$. Now let $y \in A$. We have $y \odot a \leq y, a$. Let $z \in A, z \leq y, a$. Then $z = z \odot a \leq y \odot a$, thus $y \odot a = y \wedge a$.

(ii): Let $a \in B(A)$ and $x, y \in A$. Then $(a \wedge x) \oplus (a \wedge y) = (a \oplus a) \wedge (x \oplus a) \wedge (a \oplus y) \wedge (x \oplus y) = a \wedge (x \oplus y)$, thus $a \odot (x \oplus y) = (a \odot x) \oplus (a \odot y)$.

(iii): Let $a \in B(A)$ and $x \in A$. By Corollary 3.3 and part (i) we obtain $x \vee a = \neg(\neg x \wedge \neg a) = \neg(\neg x \odot \neg a) = \neg(\neg(x \oplus a)) = x \oplus a$.

(iv): Let $a \in B(A)$ and $x, y \in A$. Then $(a \oplus x) \odot (a \oplus y) = (a \vee x) \odot (a \vee y) = (a \odot a) \vee (x \odot a) \vee (a \odot y) \vee (x \odot y) = a \vee (x \odot y)$, thus $a \oplus (x \odot y) = (a \oplus x) \odot (a \oplus y)$. □

For an arbitrary element $a \in B(A)$ denote by $g_a : A \to A$ the mapping such that $g_a(x) = a \oplus x$ for any $x \in A$.

Theorem 3.5. *Let A be a commutative basic algebra, and $a \in B(A)$. Then $g_a : A \to A$ is a modal operator on A.*

Proof. a) Let $a \in B(A)$. Then for any $x, y \in A$ we have

1. $x \leq x \oplus a = g_a(x)$.
2. $g_a(g_a(x)) = a \oplus (a \oplus x) = a \vee (a \vee x) = a \vee x = a \oplus x = g_a(x)$.
3. $g_a(x \odot y) = a \oplus (x \odot y) = (a \oplus x) \odot (a \oplus y) = g_a(x) \odot g_a(y)$.

□

For an element $a \in B(A)$ consider mappings $h_a : A \to A$ and $k_a : A \to A$ such that for any $x \in A$

$$h_a(x) := a \to x, \quad k_a(x) := (x \to a) \to a.$$

Proposition 3.6. *If A is a commutative basic algebra and $a \in B(A)$, then the mappings h_a and k_a are modal operators on A.*

Proof. a) For any $x \in A$ we have $a \to x = \neg a \oplus x$, thus $h_a = g_{\neg a}$.

b) Let $x \in A$. Then $(x \to a) \to a = (\neg x \oplus a) \to a = \neg(\neg x \oplus \neg\neg a) \oplus a = (x \odot \neg a) \oplus a = (x \oplus a) \odot (\neg a \oplus a) = a \oplus x$, hence $k_a = g_a$. □

Let A be a commutative basic algebra. Denote by $M(A)$ and $M_s(A)$ the set of all modal and all strong modal operators on A.

Theorem 3.7. *If $f_1, f_2 \in M(A)$, or $f_1, f_2 \in M_s(A)$, then $f_1f_2 \in M(A)$, or $f_1f_2 \in M_s(A)$, respectively, if and only if $f_1f_2 = f_2f_1$.*

Proof. By [8], the composition of two closure operators on an arbitrary ordered set is a closure operator if and only if these operators commute. Therefore we only need to prove that for any $f_1, f_2 \in M(A)$ such that $f_1f_2 = f_2f_1$ the condition from the definition of a modal operator is satisfied.

Let $x, y \in A$. Then $f_1f_2(x \odot y) = f_1(f_2(x) \odot f_2(y)) = f_1f_2(x) \odot f_1f_2(y)$. Moreover, if $f_1f_2 \in M_s(A)$ and $f_1f_2 = f_2f_1$, then $f_1f_2(x \oplus y) = f_1f_2(x \oplus f_2(y)) = f_2f_1(x \oplus f_2(y)) = f_2f_1(x \oplus f_1f_2(y))$. Hence f_1f_2 is a strong modal operator. □

Proposition 3.8. *Let A be a commutative basic algebra, $a \in B(A)$ and $f \in M(A)$. If $f(x) \le g_a(x)$ for any $x \in A$, then $f(a) = a$.*

Proof. Let $f \in M(A)$, and $x \in A$. If $f(x) \le g_a(x)$, then $f(x) \le a \oplus x$ for any $x \in A$. Thus $f(a) \le a \oplus a = a$. Hence $f(a) = a$. □

Lemma 3.9. *Let A be a commutative basic algebra. Then for any $x, y, z \in A$ we have:*

(i) $x \odot (x \to y) = x \wedge y$,
(ii) $x \odot y \le z \iff x \le y \to z$.

Proof. Let $x, y, z \in A$. Then

(i): $x \odot (x \to y) = \neg(\neg x \oplus \neg(x \to y)) = \neg(\neg x \oplus \neg(\neg x \oplus y)) = \neg(\neg(y \oplus \neg x) \oplus \neg x) = \neg(\neg y \vee \neg x) = x \wedge y$.

(ii): If $x \le y \to z$. Then $x \odot y \le (y \to z) \odot y = y \odot (y \to z) = y \wedge z \le z$. Conversely, if $x \odot y \le z$, then $\neg y \oplus (x \odot y) \le \neg y \oplus z = y \to z$, and $\neg y \oplus (x \odot y) = \neg(\neg x \oplus \neg y) \oplus \neg y = x \vee \neg y \ge x$. □

Lemma 3.10. *Let A be a commutative basic algebra, and $f : A \to A$ be a modal operator on A. Then for any $x, y \in A$:*

(i) $x \le y \Longrightarrow f(x) \le f(y)$,
(ii) $f(x \to y) \le f(x) \to f(y) = f(f(x) \to f(y)) = x \to f(y) = f(x \to f(y))$,
(iii) $f(x) \le (x \to f(0)) \to f(0)$,
(iv) $x \oplus f(0) \ge f(x)$.

Proof. (i): Let $x \le y$. Then $f(x) = f(x \wedge y) = f(y \odot (y \to x)) = f(y) \odot f(y \to x)$, which implies $f(x) \le f(y)$.

(ii): Let $x, y \in A$. Then $f(x) \odot f(x \to y) = f(x \odot (x \to y)) = f(x \wedge y) \le f(y)$, hence by Lemma 3.9 $f(x \to y) \le f(x) \to f(y)$. Moreover we have

$$\begin{aligned} f(f(x) \to f(y)) &\le f(f(x)) \to f(f(y)) = f(x) \to f(y) \le x \to f(y) \\ &\le f(x \to f(y)) \le f(x) \to f(f(y)) = f(x) \to f(y) \le f(f(x) \to f(y)), \end{aligned}$$

hence $f(x \to f(y)) = f(f(x) \to f(y)) = f(x) \to f(y) = x \to f(y)$. □

(iii): Since $f(x) \odot (f(x) \to f(0)) = f(x) \wedge f(0) \le f(0)$, we have $f(x) \le (x \to f(0)) \to f(0)$.

(iv): For any $x \in A$ we have $x \oplus f(0) = \neg\neg x \oplus f(0) = \neg x \to f(0) = f(\neg x \to f(0)) \ge f(\neg x \to 0) = f(\neg\neg x \oplus 0) = f(x)$.

Lemma 3.11. *Let A be a commutative basic algebra, and let $f : A \to A$ be a strong modal operator. Then for any $x, y \in A$ we have:*

(i) $f(x \oplus y) = f(f(x) \oplus f(y))$,

(ii) $x \oplus f(0) = f(x)$.

Proof. (i): From the definition of a strong modal operator we obtain $f(x \oplus y) = f(x \oplus f(y)) = f(f(y) \oplus x) = f(f(y) \oplus f(x)) = f(f(x) \oplus f(y))$.

(ii): Let $x \in A$. By Lemma 3.10 (iv), we get $f(x) = f(x \oplus 0) = f(x \oplus f(0)) \geq x \oplus f(0) \geq f(x)$.

□

Theorem 3.12. *Let A be a commutative basic algebra, and $f : A \to A$ be a mapping. Then f is a modal operator on A if and only if for any $x, y \in A$ it satisfies:*

(i) $x \to f(y) = f(x) \to f(y)$,
(ii) $f(x) \odot f(y) \geq f(x \odot y)$.

Proof. $\Longleftarrow$: Let $f : A \to A$ be a mapping satisfying conditions (i) and (ii).

1. Let $x \in A$. By Lemma 3.10, $x \to f(x) = f(x) \to f(x) = \neg f(x) \oplus f(x) = 1$, hence $x \leq f(x)$.

2. By (i), for any $x \in A$ we have $1 = \neg f(x) \oplus f(x) = f(x) \to f(x) = f(f(x)) \to f(x)$, hence $f(f(x)) \leq f(x)$. Therefore $f(f(x)) = f(x)$.

3. Let $x, y \in A$. Then $x \odot y \leq f(x \odot y)$, and by Lemma 3.9 we obtain $y \leq x \to f(x \odot y) = f(x) \to f(x \odot y)$, hence $y \odot f(x) \leq f(x \odot y)$. By Lemma 3.9 and (i), we have $f(x) \leq y \to f(x \odot y) = f(y) \to f(x \odot y)$. Thus $f(x) \odot f(y) \leq f(x \odot y)$, and since f satisfies (ii), we obtain the equality $f(x) \odot f(y) = f(x \odot y)$.

$\Longrightarrow$: It follows from the definiton of a modal operator and from Lemma 3.10 (ii). □

References

[1] BOTUR, M.: *An example of a commutative basic algebra which is not an MV-algebra* Math. Slovaca 60 (2010), 171–178.

[2] BOTUR, M.—HALAŠ, R. *Commutative basic algebras and non-associative fuzzy logics*, Arch. Math. Logic 48 (2009), 243–255.

[3] BOTUR, M.—HALAŠ, R.—KÜHR, J.: *States on commutative basic algebras*, Fuzzy Sets Syst. 187 (2012), 77–91.

[4] BOTUR, M.—CHAJDA, I.—HALAŠ, R.: *Are basic algebras residuated structures?*, Soft Comput. 14 (2010), 251–255.

[5] CHAJDA, I.— HALAŠ, R.— KÜHR, J. *Many valued quantum algebras*, Algebra Univers. 60 (2009), 63–90.

[6] HARLENDEROVÁ, M.—RACHŮNEK, J.: *Modal operators on MV-algebras*, Math. Bohem. 131 (2006), 39–48.

[7] MACNAB, D. S.: *Modal operators on Heyting algebras*, Algebra Univers. 12 (1981), 5–29.

[8] RACHŮNEK, J.: *Modal operators on ordered sets*, Acta Univ. Palacki. Olomouc., Fac. Rer. Nat., Math 24 (1985), 9–14.

[9] RACHŮNEK, J.—ŠALOUNOVÁ, D.: *State operators on commutative basic algebras*, WCCI 2012 IEEE World Congress on Computational Intelligence, June, 10-15, 2012 - Brisbane, Australia, 1511–1516.

[10] RACHŮNEK, J.—ŠALOUNOVÁ, D.: *Modal operators on bounded commutative residuated l-monoids*, Math. Slovaca 57 (2007), 321–332.

[11] RACHŮNEK, J.—ŠALOUNOVÁ, D.: *Modal operators on bounded residuated l-monoids*, Math. Bohem. 133 (2008), 299–311.

[12] RACHŮNEK, J.—SVOBODA, Z.: *Monotone modal operators on bounded integral residuated lattices*, Math. Bohem. 137, No. 3 (2012), 333–345.

[13] RACHŮNEK, J.— SVOBODA, Z.: *Interior and closure operators on commutative basic algebras*, Math. Slovaca, to appear.

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