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ÚSTAV MATEMATIKY

STABILITY OF TIME DELAY FEEDBACK CONTROLS OF DYNAMICAL SYSTEMS

STABILITA ZPĚTNĚ-VAZEBNÍCH ŘÍZENÍ DYNAMICKÝCH SYSTÉMŮ S ČASOVÝM ZPOŽDĚNÍM

MASTER'S THESIS

DIPLOMOVÁ PRÁCE

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Specification Master's Thesis

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Pursuant to Act no. 111/1998 concerning universities and the BUT study and examination rules, you have been assigned the following topic by the institute director Master's Thesis:

Stability of Time Delay Feedback Controls of Dynamical Systems

Concise characteristic of the task:

The control of dynamical systems by use of state–dependent values belongs among the most common ways of control. In many cases, current values of state variables cannot be involved into a control term. Therefore, these values are considered with a time delay. To achive aims of the controlled process, the correct choice of this delay (and other control parameters) is necessary.

Goals Master's Thesis:

1. A survey of stability conditions for time delay systems.
2. A suggestion of feedback control terms and their qualitative analysis.
3. The use of these controls in stabilization and synchronization of some unstable dynamical systems (e. g. oscillators).
4. Numerical and graphical experiments.

Recommended bibliography:

HÖVEL, P. Control of Complex Nonlinear Systems with Delay. Springer-Verlag Berlin Heidelberg, 2010. ISBN 978-3-642-14109-6.

MICHIELS, W., NICULESCU, S. I. Stability and Stabilization of Time-delay Systems: An Eigenvalue-based Approach. SIAM, Philadelphia, 2007. ISBN 978-0-898-71864-5.

Deadline for submission Master's Thesis is given by the Schedule of the Academic year 2019/20

In Brno,

L. S.

prof. RNDr. Josef Šlapal, CSc. Director of the Institute	doc. Ing. Jaroslav Katolický, Ph.D. FME dean

Summary
This diploma thesis deals with stability of time delay feedback controls of dynamical systems, especially controls of mechanical oscillators. Two basic types of controls are applied to linear systems. Further, synchronisation of coupled systems by use of time delay feedback controls is shown. The thesis also discusses a MATLAB solver for delay differential equations.

Abstrakt
Tato diplomová práce pojednává o stabilitě zpětně-vazebných řízení dynamických systémů s časovým zpozděním, speciálně řízení mechanických oscilátorů. Dva základní druhy řízení jsou užity v lineárních systémech. Dále je zde ukázána synchronizace dvouprvkových systémů užitím zpětně-vazebných řízení. Práce se také zabývá funkcí v MATLABu pro řešení zpožděných diferenciálních rovnic.

Keywords
Time delay feedback control, stability, mechanical oscillator, Pyragas control, controlled systems, synchronisation of systems, dde23 MATLAB solver

Klíčová slova
Zpětně-vazební řízení, stabilita, mechanický oscilátor, Pyragasova kontrola, řízené systémy, synchronizace systémů, MATLAB funkce dde23

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I declare that I have written the diploma thesis *Stability of Time Delay Feedback Controls of Dynamical Systems* on my own under the guidance of my supervisor prof. RNDr. Jan Čermák, CSc., and that I used the sources listed in references.

Bc. David Lovas

I would like to thank my supervisor prof. RNDr. Jan Čermák, CSc. for his guidance, advice and useful comments. Further, I thank my family, especially my parents, and friends.

Bc. David Lovas

Contents

1 Introduction

Oscillation is one of the basic properties of solutions of ordinary differential equations (ODEs). Harmonic oscillations are a basic prototype of oscillations. By adding more forces to the system, one can obtain damped oscillations and driven oscillations. All these systems can be solved analytically with a basic knowledge of ODEs. Problems appear in the case of controlled systems involving the factor of a time delay. Such systems are modelled by delay differential equations (DDEs). In this case, there are no analytical methods of solving and thus we need to discuss basic qualitative properties of solutions, like stability and oscillation.

One of the basic controls that use this delay factor is the time delay feedback control involving the state variable at a past time. This delay symbolises the time necessary for data processing of a machine. Thus, systems with delays are more realistic and can be used in many branches, for example robotics, flight and fluid dynamics or intelligent houses (very popular nowadays). The aim of this thesis is to apply some stability results of DDEs to control of mechanical oscillators or systems of mechanical oscillators.

Several types of delay feedback controls are known nowadays. The basic one consists of just one element which is proportional to a delayed state of the system itself. This creates an additional force influencing the motion of the system. The second widely used delay feedback control is Pyragas control. This control is based on an idea of a stabilisation of periodic solutions. If the delay is well chosen, the control vanishes. The same happens also for large times in case of a stable system. Such a type of control is also called a non-invasive control.

DDEs are a powerful branch of mathematics for controlling of systems. A widely studied problem is that of stability of DDEs. Since all the roots of the characteristic equation cannot be obtained as easily as in the case of ODEs, this problem is much more difficult than in systems without a delay part. Also an analytical solution cannot be computed in general cases of DDEs and so the equations have to be solved numerically.

There exist some methods of deciding whether the system is stable or not. Some of these ideas come from the second half of the nineteenth century when control theory was first conceived by the Russian mathematician Lev Pontryagin. The methods are based on recognising signs of roots' real parts without the explicit knowledge of the roots or even the total number of roots.

The real parts of the roots depend on constants which characterise the system itself. Since some types of oscillators will be studied, one of the constants is definitely the frequency of the system. If more complicated systems are considered, other constants of the systems, for example friction, also influence the stability of the systems. Other constants are added to be the control. The first necessary constant is the delay. As it has been said, the feedback element is multiplied by a constant. This can also build or break the stability of the system.

The methods for solving DDEs give the solution as a system of inequalities. When visualising the results, diagrams often provide useful insights. A problem can be that a dependency of just two parameters can be visualised in a standard two-dimensional diagram. One way to create the diagram is to consider a fixed system which shall be controlled. In this case, the diagram shows the dependency of the time delay and the constant multiplying the feedback element. The second possibility is to fix one of the constants defining the controller. Typically, this fixed constant is the time delay. In this case, the control is just partly defined, but also the controlled system is not fully defined. So the aim is to find a combination of the controller and the system that will be stable. Here, the diagram shows the dependency of the missing parameters.

The thesis is divided into seven main chapters. In chapter 2, we begin with the mathematical background of uncontrolled oscillators. Basic mechanical oscillators with a delay feedback control are introduced. We also explain how to understand the concept of asymptotic stability for oscillations.

Chapter 3 discusses a harmonic oscillator and a damped oscillator which are taken as the fundamental part of the discussed models. To these systems, two types of a controller are added, namely that with just one feedback element, and Pyragas control. Also, we state here Pontryagin's theorem on stability of DDEs and apply it to the controlled oscillators' systems. In some cases, this theorem yields general necessary and sufficient stability conditions for the studied systems. On the other hand, the existing literature does not answer all stability issues connected with controlled oscillations. In such a case, only necessary or sufficient conditions are known. To verify the quality of obtained results, we visualise them for some fixed and some varying parameters.

The results of chapter 4 originate from the well known fact that linear higher-order ordinary differential equations can be easily transformed to systems of ODEs of the first order. The same procedure may be done for DDEs as well. If the system of equations is split into an uncontrolled part and a controlled part, two system matrices are obtained. Thus, we are able to convert our problem to a matrix form. Discussing this form, we present a stability theorem which gives necessary and sufficient conditions in terms of matrices' eigenvalues. This theorem is also simplified in some particular cases. The controls discussed above are used as the delay feedback control.

In the next chapter, basic notions connected with synchronisation are introduced. Synchronising objects are coupled by use of states of objects involved in the system. In this thesis, we consider the coupled parts in delayed forms. The system is synchronised if an auxiliary DDE is stabilised. Three different ways of synchronisation are shown. In each of the cases, the algorithm for solving is based on the conversion of the model to a form involving simpler harmonic oscillators, possibly with a delay. Various illustrations are shown as well.

Chapter 6 focuses on numerical solutions of DDEs. A solver in MATLAB is fully introduced. Using this solver, two systems from the previous chapter are solved numerically. These numerical solutions and corresponding graphs are also compared to the obtained theoretical results.

Final remarks commenting on the results and possible future research conclude the thesis.

2 Some essentials on oscillator's controls and stability

This section introduces the basic form of the studied problems. We focus on the motion of the system and its related mathematical apparatus, in particular stability.

Since we consider linear homogeneous differential models only, the notion of asymptotic stability or stability is defined for any linear homogeneous differential equation by the requirement that any its solution is tending to the zero solution, or its bounded, respectively. Due to linearity, the notion of asymptotic stability or stability is usually related to the zero solution, to any other solution or to the equation itself. If there exists an unbounded solution of the equation, then we say that the solution (or the equation itself) is unstable.

2.1 The mechanical oscillator and its control

A mechanical oscillator is a system of a point mass moving in time repetitively. During this movement, position, velocity (the first derivation of position) and acceleration (the second derivation of position) are changing. The modelling of such equations is one of a typical application of ODEs.

If we consider the simplest model of an one-dimensional mechanical oscillator, the only force in the system is the force created by the spring. We describe this system by the equation

$$m\ddot{y}(t) + ky(t) = 0$$

where m is the mass of the point mass and k is the stiffness of the spring in the system. Both m, k are real and positive. The equation can be equivalently modified to the form

$$\ddot{y}(t) + \omega^2 y(t) = 0, \quad \omega^2 = \frac{k}{m} > 0. \tag{2.1}$$

This mechanical system is called the harmonic oscillator. Further, we can consider more forces acting to the mass point. If a frictional force is added to (2.1), we get a damped oscillator given by

$$\ddot{y}(t) + 2b\dot{y}(t) + \omega^2 y(t) = 0, \quad \omega^2 = \frac{k}{m}, \quad b = \frac{l}{2m}. \tag{2.2}$$

In case of a driven oscillator, a periodic external force is added to the system (2.1). This model is described as

$$\ddot{y}(t) + \omega^2 y(t) = \frac{P}{m}\sin(\omega t), \quad \omega^2 = \frac{k}{m} > 0.$$

where P is a real amplitude of the external force. It is also possible to create a damped driven oscillator. All of these models can be solved analytically, so we get a general solution and, together with initial conditions, we can get a particular solution (see [1]).

If we consider a controller of an oscillator, it is an external force depending on time and acting on the mass point for getting the system to a required motion. In general, the controlled harmonic oscillator's differential equation is

$$\ddot{y}(t) + \omega^2 y(t) + u(t) = 0$$

where $u(t)$ is a general function depending on time.

In a feedback controller case, the external force depends on the motion of the point mass. So now the control function is $u(y(t))$. This means that the controller reacts to the actual motion of the mass point and drives the mass point at the same time. The problem is that the controller of this type can not be produced in practise because we would get an external mechanism which is able to drive the mass point without any data processing.

For solving this problem, we consider a feedback (time) delay controller $u(y(t-\tau))$ where $\tau > 0$ is called the delay of the control. This form of the control means that the external mechanism receives some information about the motion of the point mass, has some time τ to data processing and drives the point mass with respect to these information from past. The equation with this type of the controller is call the delay differential equation (DDE). However, even the simplest form of DDEs can not be solved analytically and numerical methods are used for solving only. Similarly, it is possible to control a damped oscillator, a driven oscillator and a damped driven oscillator.

In the next chapters, we will consider the feedback delay control in the forms

$$u(y(t)) = cy(t-\tau), \quad \text{or} \quad u(y(t)) = c(y(t) - y(t-\tau))$$

where $\tau > 0$, $c \in \mathbb{R}$. These controls will be added to different types of mechanical oscillators and we will study stability of each system depending on coefficients.

We shall note that stability of the system depends just on the homogeneous part of the (delay) differential equation. The non-homogeneous part does not break (asymptotic) stability and also it can not make the system stable. Thus, the case of the driven oscillator and the damped driven oscillator will not be studied because the results would be equivalent to the results from the cases of the harmonic oscillator and the damped oscillator, respectively.

2.2 Basics of stability of harmonic oscillators and their delay feedback control

We briefly recall the algorithm for studying of stability of higher order linear ODEs. This problem will be solved similarly in the case of DDEs.

We consider an autonomous linear ODE of n-th order ($n \geq 2$). These differential equations are solved by the characteristic equation which is obtained by the assumption that the solution has the form

$$y(t) = e^{\lambda t}.$$

If we substitute this assumption into the linear ODE (including its derivatives), then we obtain the characteristic equation where λ is called the root of the characteristic equation. The characteristic equation of linear ODEs of n-th order is a polynomial of order n. We solve this polynomial analytically or by numerical methods and we get n complex roots λ. It is well known that the studied linear ODE is asymptotically stable if and only if all characteristics roots λ have negative real parts, and unstable if at least one characteristics root has a positive real part. The equation is stable if all characteristic roots have non-positive real parts and those with zero real parts are simple. See [2] for more details, proofs and examples.

In a particular case of the harmonic oscillator, the ODE describing the system is given by (2.1) where ω is called the frequency of the mechanical system. The corresponding characteristics equation is

$$\lambda^2 + \omega^2 = 0.$$

Thus, we obtain the polynomial of the second order which may be solved easily. The roots are

$$\lambda = \pm \mathrm{i}\omega.$$

Since the roots are simple and their real parts are zero, this equation is stable. Similarly, the damped oscillator's asymptotic stability would be determined using its characteristic equation.

The system of the harmonic oscillator with the simple feedback delayed control has the form

$$\ddot{y}(t) + \omega^2 y(t) + cy(t-\tau) = 0, \quad t > 0.$$

Similarly as for ODE, we need a initial condition of the system. The motion at the initial time $t = 0$ is not sufficient because the function $y(t)$ is defined for time $t \geq 0$, so the component $y(t-\tau)$ would be undefined in the whole time interval $t \in \langle -\tau; 0\rangle$. In case of DDEs, we need a continuous function which defines the behaviour of the system at time $t \in \langle -\tau; 0\rangle$. The system is not controlled on this time interval and the controller does not act on the system. Hence, the fully defined harmonic oscillator with the feedback delay control is given by

$$\begin{aligned} &\ddot{y}(t) + \omega^2 y(t) + cy(t-\tau) = 0, \quad t > 0 \\ &y(t) = \psi(t), \quad -\tau \leq t \leq 0, \end{aligned} \tag{2.3}$$

where $\psi(t) \in C\left(\langle -\tau; 0\rangle; \mathbb{R}\right)$ is an initial function.

As for the linear ODEs, we assume that the solution of the system is

$$y(t) = e^{\lambda t}$$

with $\lambda \in \mathbb{C}$, so clearly

$$y(t-\tau) = e^{\lambda(t-\tau)}.$$

Substituting these assumptions in the differential equation (2.3) gives

$$\lambda^2 e^{\lambda t} + \omega^2 e^{\lambda t} + c e^{\lambda(t-\tau)} = 0,$$

equivalently,

$$\lambda^2 + \omega^2 + c e^{-\lambda\tau} = 0. \tag{2.4}$$

This equation is called the characteristic equation of the system (2.3). Similarly to the linear non-delayed case, the equation's asymptotic stability can be equivalently expressed by the requirement that all the roots of (2.4) have negative real parts. See [3].

Now we would find the roots of the characteristic equation in the linear ODEs case. However, in our case, the equation is not a polynomial and we do not know neither the exact number of the roots in general or the signs of their real parts.

3 The application of the Pontryagin's method

In this chapter, one of the most widely used method for the determination of stability will be stated and used. First, we give one necessary definition. Monic real-rooted polynomials $f(x)$ of degree n with roots $\alpha_n \leq \cdots \leq \alpha_1$ and $g(x)$ of degree $n-1$ or n with roots $\beta_{n-1} \leq \cdots \leq \beta_1$ (possibly $\beta_n \leq \beta_{n-1}$) are said to be interlacing if $\alpha_n \leq \beta_{n-1} \leq \alpha_{n-1} \leq \cdots \leq \alpha_1 \leq \beta_0 \leq \alpha_0$ (possibly $\beta_n \leq \alpha_n$). We also say that $g(x)$ interlaces $f(x)$.

For using Pontryagin's method, the exponential polynomial $H(\lambda)$ must be stated. This function is defined as the left-hand side of the characteristic equation where the right-hand side is zero. Computing this function in the critical point $\lambda = \mathrm{i}y$, $H(\mathrm{i}y)$ may be split into the real and the complex part as

$$H(\mathrm{i}y) = F(y) + \mathrm{i}G(y).$$

The following properties show when all roots of (2.4) have negative real part. The theory goes from Pontryagin's theorem which can be seen with its proof in [4]. The theorem needs some conditions on $H(y), G(y)$ and $F(y)$. These functions for delay controls of mechanical oscillators satisfy all of the conditions. Now we must define the function

$$\Delta(y) := G'(y)F(y) - G(y)F'(y).$$

The most important part of Pontryagin's theorem for our problem says that all the zeros of the function $H(\lambda)$ are in the open left half plane if and only if one of the following holds:

(i) All the zeros of the functions $F(y), G(y)$ are real, $F(y), G(y)$ are interlace and $\Delta(y) > 0$ for at least one value of y.

(ii) All the zeros of the function $F(y)$ are real and for each of these zeros $y = y_0$ is $\Delta(y_0) > 0$.

(iii) All the zeros of the function $G(y)$ are real and for each of these zeros $y = y_0$ is $\Delta(y_0) > 0$.

3.1 Harmonic oscillator

Let's apply the theorem in our system. Multiplying $e^{\lambda\tau}$ on both sides of the characteristic equation (2.4), the exponential polynomial

$$H(\lambda) := \lambda^2 e^{\lambda\tau} + \omega^2 e^{\lambda\tau} + c = 0$$

for the system (2.3) is obtained.

For applying Pontryagin's theorem, the functions $F(y)$ and $G(y)$ are needed. Thus, we calculate $H(\lambda)$ at the value $\lambda = \mathrm{i}y$, i.e. at the critical point where asymptotic stability becomes instability and vice versa. The function $H(\mathrm{i}y)$ is

$$H(\mathrm{i}y) = (\mathrm{i}y)^2 e^{\mathrm{i}y\tau} + \omega^2 e^{\mathrm{i}y\tau} + c = -y^2\left(\cos(y\tau) + \mathrm{i}\sin(y\tau)\right) + \omega^2\left(\cos(y\tau) + \mathrm{i}\sin(y\tau)\right) + c.$$

The functions $F(y), G(y)$ are

$$\begin{aligned} F(y) &= -y^2\cos(y\tau) + \omega^2\cos(y\tau) + c \\ G(y) &= -y^2\sin(y\tau) + \omega^2\sin(y\tau), \end{aligned}$$

with their derivatives

$$\begin{aligned} F'(y) &= -2y\cos(y\tau) + (y^2 - \omega^2)\tau\sin(y\tau) \\ G'(y) &= -2y\sin(y\tau) + (\omega^2 - y^2)\tau\cos(y\tau). \end{aligned}$$

The computation of these functions at $y = 0$ gives

$$F(0) = \omega^2 + c, \quad G(0) = 0, \quad F'(0) = 0, \quad G'(0) = \tau\omega^2.$$

So

$$\Delta(0) = \tau\omega^2(\omega^2 + c).$$

The statement (iii) of Pontryagin's theorem says that if the zero solution of the system (2.3) is asymptotically stable, then $\Delta(0) > 0$. That implies

$$c > -\omega^2 \tag{3.1}$$

because the coefficients ω^2 and τ must be positive.

3.1.1 Negative constant of the controller

The case $c < 0$ will be studied now. We want to find zeros of $G(y)$:

$$\begin{gathered} G(y) = 0 \implies -y^2\sin(y\tau) + \omega^2\sin(y\tau) = 0 \\ y_0 = \pm\omega, \quad y_0 = \frac{n\pi}{\tau}, \quad n \in \mathbb{Z}. \end{gathered}$$

All zeros of $G(y)$ are real. The condition $\Delta(y_0) > 0$ must be satisfied for all y_0. Due to $G(y_0) = 0$, $\Delta(y_0)$ is

$$\Delta(y_0) = F(y_0)G'(y_0).$$

First take $y_0 = \omega$:

$$\begin{aligned} &F(\omega)G'(\omega) > 0, \quad \text{i.e.} \\ &\left(-\omega^2\cos(\omega\tau) + \omega^2\cos(\omega\tau) + c\right)\left(-2\omega\sin(\omega\tau) + (\omega^2 - \omega^2)\tau\cos(\omega\tau)\right) > 0, \\ &-2c\omega\sin(\omega\tau) > 0. \end{aligned}$$

Recall, ω is always positive and c is negative in this case. The inequality holds if

$$2k\pi < \omega\tau < (2k+1)\pi, \quad k \in \mathbb{N} \cup \{0\}.$$

Similarly for $y_0 = -\omega$:

$$\begin{aligned} &F(-\omega)G'(-\omega) > 0, \quad \text{i.e.} \\ &\left(-\omega^2\cos(-\omega\tau) + \omega^2\cos(-\omega\tau) + c\right)\left(2\omega\sin(-\omega\tau) + (\omega^2 - \omega^2)\tau\cos(-\omega\tau)\right) > 0, \\ &2c\omega\sin(-\omega\tau) > 0, \\ &-2c\omega\sin(\omega\tau) > 0. \end{aligned}$$

We obtain the same condition

$$2k\pi < \omega\tau < (2k+1)\pi, \quad k \in \mathbb{N} \cup \{0\}.$$

The third case, namely $y_0 = n\pi/\tau$:

$$F\left(\frac{n\pi}{\tau}\right) G'\left(\frac{n\pi}{\tau}\right) > 0, \quad \text{i.e.}$$

$$\left(-\left(\frac{n\pi}{\tau}\right)^2 \cos(n\pi) + \omega^2 \cos(n\pi) + c\right)\left(-2\frac{n\pi}{\tau}\sin(n\pi) + \left(\left(\frac{n\pi}{\tau}\right)^2 - \omega^2\right)\tau\cos(n\pi)\right) > 0,$$

$$\left(\left(\omega^2 - \left(\frac{n\pi}{\tau}\right)^2\right)(-1)^n + c\right)\left(\omega^2 - \left(\frac{n\pi}{\tau}\right)^2\right)\tau(-1)^n > 0,$$

$$\left(\omega^2 - \left(\frac{n\pi}{\tau}\right)^2\right)^2 + \left(\omega^2 - \left(\frac{n\pi}{\tau}\right)^2\right)c\tau(-1)^n > 0.$$

We rewrite the inequality for the future work in the form

$$\left(\omega^2 - \left(\frac{n\pi}{\tau}\right)^2\right)^2 > \left(\left(\frac{n\pi}{\tau}\right)^2 - \omega^2\right)c\tau(-1)^n. \tag{3.2}$$

Clearly, there exists $\bar{k} \in \mathbb{N} \cup \{0\}$, for which the inequalities

$$2\bar{k}\pi < \omega\tau < (2\bar{k}+1)\pi \tag{3.3}$$

hold. Two cases must be studied.

First, assume $0 \le n \le 2\bar{k}$. Multiply the inequality by π and consider (3.3):

$$0 \le n\pi \le 2\bar{k}\pi < \omega\tau$$

All terms are non-negative, so we may square them without any changes of the directions of the inequalities. Thus

$$0 \le (n\pi)^2 \le (2\bar{k}\pi)^2 < (\omega\tau)^2 \quad \Longrightarrow \quad \left(\frac{n\pi}{\tau}\right)^2 - \omega^2 < 0.$$

The inequality (3.2) under conditions above holds always for an odd n. For an even n, (3.2) is satisfied if

$$\left(\frac{n\pi}{\tau}\right)^2 - \omega^2 < c.$$

This holds for all even n such that $0 < n < 2\bar{k}$. We may take $n = 2\bar{k}$ for the maximization of the interval.

Second, assume $2\bar{k} < n$. Recall that both $n, \bar{k}$ are natural here. Hence, also $2\bar{k}+1 \le n$ holds. We multiply the inequality by π and consider (3.3):

$$2k\pi < \omega\tau < (2k+1)\pi \le n\pi.$$

All terms are non-negative, so we may square them without any changes of the directions of the inequalities. Thus

$$(2k\pi)^2 < (\omega\tau)^2 < (2k+1)^2\pi^2 \le (n\pi)^2 \quad \Longrightarrow \quad \omega^2 - \left(\frac{n\pi}{\tau}\right)^2 < 0.$$

Now the inequality (3.2) holds always for even n and it is satisfied for odd n if

$$\omega^2 - \left(\frac{n\pi}{\tau}\right)^2 < c.$$

This holds for all odd n such that $2\bar{k} < n$. The interval is maximal if $n = 2\bar{k} + 1$.

It follows from these two cases that

$$c > \max\left\{\left(\frac{2\bar{k}\pi}{\tau}\right)^2 - \omega^2, \omega^2 - \left(\frac{(2\bar{k}+1)\pi}{\tau}\right)^2\right\}.$$

With this constant c and $\omega\tau$ such that $2k\pi < \omega\tau < (2k+1)\pi, k \in \mathbb{N} \cup \{0\}$, the function $G(y)$ has all roots y_0 with negative real parts and $\Delta(y_0) > 0$ for these roots. Since we have used expressions odd and even n which are defined for non-negative integers only, we should study the case of negative n. It follows from the even function $\Delta(y)$. Hence, the case of negative n is trivial. As the consequence of these computations, every root of the function $H(y)$ has negative a real part by Pontryagin's theorem if

$$\begin{aligned} &\frac{2k\pi}{\omega} < \tau < \frac{(2k+1)\pi}{\omega}, \quad k \in \mathbb{N} \cup \{0\}, \\ 0 > c > \max &\left\{\left(\frac{2k\pi}{\tau}\right)^2 - \omega^2, \omega^2 - \left(\frac{(2k+1)\pi}{\tau}\right)^2\right\} \end{aligned} \tag{3.4}$$

and the system (2.3) is asymptotically stable.

3.1.2 Positive constant of the controller

Now we consider $c > 0$. The whole algorithm will be similar to the computations before. First, zeros of $G(y)$ have to be found. The zeros are

$$y_0 = \pm\omega, \quad y_0 = \frac{n\pi}{\tau}, \quad n \in \mathbb{Z}$$

and they are real. Recall that we need

$$\Delta(y_0) = F(y_0)G'(y_0) > 0$$

for all discovered zeros.

For $y_0 = -\omega$ the condition is

$$\begin{aligned} &F(-\omega)G'(-\omega) > 0, \quad \text{i.e.} \\ &\big(-\omega^2\cos(-\omega\tau) + \omega^2\cos(-\omega\tau) + c\big)\big(2\omega\sin(-\omega\tau) + (\omega^2 - \omega^2\tau\cos(-\omega\tau)\big) > 0, \\ &2c\omega\sin(-\omega\tau) > 0, \\ &2c\omega\sin(\omega\tau) < 0. \end{aligned}$$

Since both ω, c are positive now, the inequality holds for

$$(2k+1)\pi < \omega\tau < (2k+2)\pi.$$

If $y_0 = \omega$

$$\begin{aligned}
&F(\omega)G'(\omega) > 0, \quad \text{i.e.}\\
&\left(-\omega^2\cos(\omega\tau) + \omega^2\cos(\omega\tau) + c\right)\left(-2\omega\sin(\omega\tau) + (\omega^2 - \omega^2\tau\cos(\omega\tau)\right) > 0,\\
&-2c\omega\sin(\omega\tau) > 0,\\
&2c\omega\sin(\omega\tau) < 0.
\end{aligned}$$

We obtain the same condition

$$(2k+1)\pi < \omega\tau < (2k+2)\pi.$$

Now the case $y_0 = n\pi/\tau$ is studied. The computations are analogous as in the section 3.1.1. The final result of this case is the inequality (3.2), i.e.

$$\left(\omega^2 - \left(\frac{n\pi}{\tau}\right)^2\right)^2 > \left(\left(\frac{n\pi}{\tau}\right)^2 - \omega^2\right)c\tau(-1)^n. \tag{3.2}$$

Clearly, there exists $\bar{k} \in \mathbb{N} \cup \{0\}$ such that the inequalities

$$(2\bar{k}+1)\pi < \omega\tau < (2\bar{k}+2)\pi \tag{3.5}$$

hold. The next work has to be split into two parts.

First, consider $0 \le n \le 2\bar{k}+1$. We multiply it by π and, together with (3.5), we obtain

$$0 \le n\pi \le (2\bar{k}+1)\pi < \omega\tau.$$

All terms are positive, so we may square them. Hence

$$0 \le (n\pi)^2 \le (2\bar{k}+1)^2\pi^2 < (\omega\tau)^2 \quad \Longrightarrow \quad \left(\frac{n\pi}{\tau}\right)^2 - \omega^2 < 0.$$

The inequality (3.2) under conditions above holds for even n. For odd n, (3.2) is satisfied if

$$\omega - \left(\frac{n\pi}{\tau}\right)^2 > c.$$

This holds for all odd n such that $0 < n < 2\bar{k}+1$. We may take the maximal value of the interval, namely $n = 2\bar{k}+1$.

Now we consider $2\bar{k}+1 < n$. Since both $n, \bar{k}$ are natural, also $2\bar{k}+2 \le n$ holds. Similarly, the inequality is multiplied by π and together with (3.5)

$$(2\bar{k}+1)\pi < \omega\tau < (2\bar{k}+2)\pi \le n\pi.$$

Since the terms are positive, they may be squared without a change of the inequality signs. Hence

$$(2\bar{k}+1)^2\pi^2 < (\omega\tau)^2 < (2\bar{k}+2)^2\pi^2 \le (n\pi)^2 \quad \Longrightarrow \quad \left(\frac{n\pi}{\tau}\right)^2 - \omega^2 > 0.$$

For n odd, (3.2) is always satisfied. For n even, it holds if

$$\left(\frac{n\pi}{\tau}\right)^2 - \omega^2 > c.$$

The value of even n maximising the interval is $n = 2\bar{k} + 2$.

The final conclusion is that c must be

$$c < \min\left\{\omega^2 - \left(\frac{(2\bar{k}+1)\pi}{\tau}\right)^2, \left(\frac{(2\bar{k}+2)\pi}{\tau}\right)^2 - \omega^2\right\}.$$

Under this condition on c and $(2\bar{k}+1)\pi < \omega\tau < 2(\bar{k}+1)\pi, k \in \mathbb{N} \cup \{0\}$, the function $G(y)$ has all zeros y_0 with negative real parts and the inequality $\Delta(y_0) > 0$ holds for all these zeros. Again, the condition for negative n follows from the even function $\Delta(y)$. Our final result of this section is that every zero of $H(y)$ has a negative real part by Pontryagin's theorem if

$$\begin{gathered}\frac{(2k+1)\pi}{\omega} < \tau < \frac{2(k+1)\pi}{\omega}, \quad k \in \mathbb{N} \cup \{0\}, \\ 0 < c < \min\left\{\omega^2 - \left(\frac{(2k+1)\pi}{\tau}\right)^2, \left(\frac{2(k+1)\pi}{\tau}\right)^2 - \omega^2\right\}\end{gathered} \tag{3.6}$$

and the system (2.3) is asymptotically stable.

3.1.3 Stability region

We have to note that some values of ω, c have not been studied. The case $\omega \leq 0$ is not determined because ω is a frequency of the mechanical oscillator (2.3) and the frequency cannot be non-positive. The case $c = 0$ is not studied in the theme of controlled systems either. The controller would vanish in this case and we would get the standard harmonic oscillator. Clearly, if one would assume $\tau = 0$, the DDE system would become an ODE system and its stability would be solved by a standard method by checking the negativity of the real parts of all roots of the characteristic equation.

The graphical representation of the stability of the system may be done in two ways. The first possibility is to consider couples (ω, c) for some τ; the second possibility is to consider couples (τ, c), for some ω. For the (ω, c) representation, we consider τ as an unchangeable variable defining the main character of the controller for which c represents an acting force. The following graphs of the asymptotic stability depending on ω, c will be shown for some chosen values of τ.

The asymptotic stability region is given by the computations in the section 3.1.1 and 3.1.2 for the lower half plane and the upper half plane, respectively. This region consists of infinitely many triangles A_k, each for one particular $k \in \mathbb{N} \cup \{0\}$. The union of these triangles is called the asymptotic stability region denoted by $S = \bigcup_{k=0}^{\infty} A_k$. It follows from Pontryagin's theorem that if $(\omega, c) \in S$, the solution $y = 0$ of the system (2.3) is asymptotically stable. The triangles alternate around the axis $c = 0$ and grow as $\omega \to +\infty$.

The boundaries of the region S are curves given by

$$(-1)^k \left(\left(\frac{k\pi}{\tau}\right)^2 - \omega^2\right) - c = 0. \tag{3.7}$$

In the figure 3.1, the curves show the dependence of c on ω and we obtain a collection of parabolas. In the figure 3.2, the curves show the dependence of c on ω^2. Here, the curves

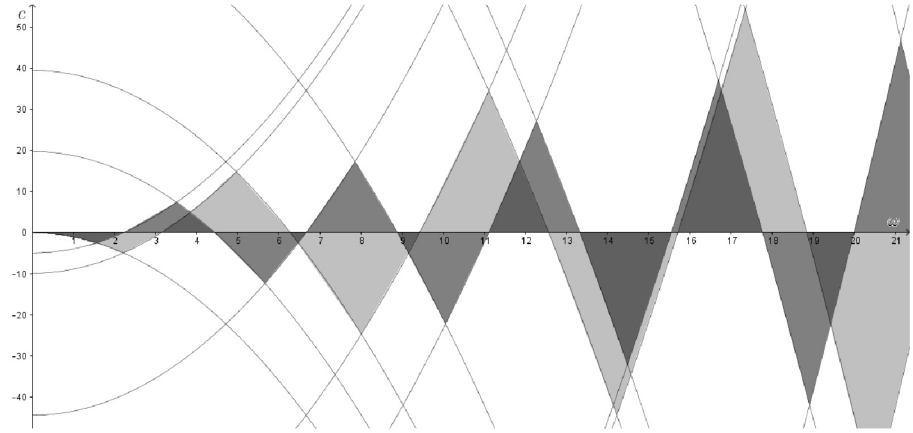


Figure 3.1: The region of asymptotic stability for the (ω, c) couples and $\tau = 1$ (light grey), $\tau = \sqrt{2}$ (dark grey)

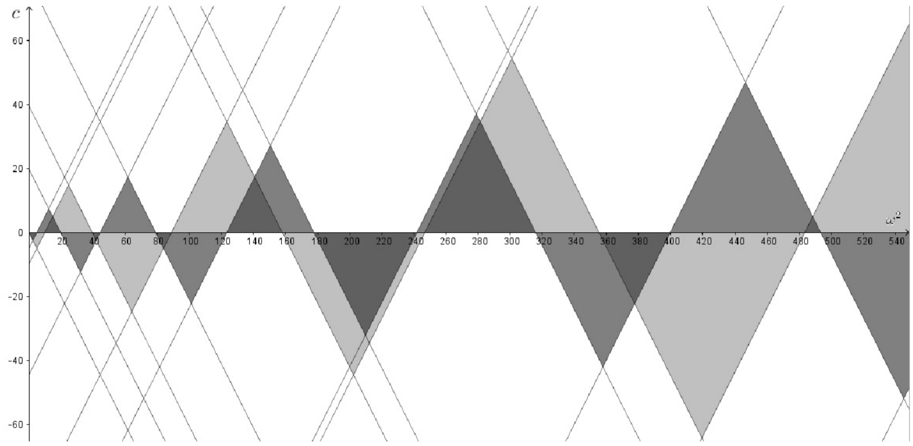


Figure 3.2: The region of asymptotic stability for the (ω^2, c) couples and $\tau = 1$ (light grey), $\tau = \sqrt{2}$ (dark grey)

are straight lines. The whole theory up until now describes the system with roots with negative real parts and so the discovered region S is the asymptotically stable region.

Now suppose that at least one root of the system is purely imaginary. The system can not be asymptotically stable. We substitute this root $\lambda = \mathrm{i}w, w \in \mathbb{R}$ into the characteristic equation (2.4) and we obtain

$$-w^2 + \omega^2 + ce^{-\mathrm{i}w\tau} = 0.$$

This may be split into the real and the imaginary part

$$\begin{aligned} Re: &\quad -w^2 + \omega^2 + c\cos(-w\tau) = -w^2 + \omega^2 + c\cos(w\tau) = 0 \\ Im: &\quad c\sin(-w\tau) = -c\sin(w\tau) = 0. \end{aligned}$$

Recall that $c \neq 0$. For satisfying the imaginary part,

$$w = \frac{k\pi}{\tau}, \quad k \in \mathbb{Z}$$

is needed. Substituting this form of w into the real part of the characteristic equation gives

$$-\left(\frac{k\pi}{\tau}\right)^2 + \omega^2 + c(-1)^k = 0$$

which can be rewritten as

$$(-1)^k \left(\left(\frac{k\pi}{\tau} \right)^2 - \omega^2 \right) - c = 0.$$

These are the boundaries (3.7) of the region S. If $\lambda = \mathrm{i}w$ is known to be a purely imaginary root of the system, we may study its complex conjugate $\bar{\lambda}$ as well. Substituting $\bar{\lambda}$ into the characteristic equation, we obtain

$$\left(\bar{\lambda}\right)^2 + \omega + ce^{-\bar{\lambda}\tau} = 0.$$

Recall that ω, τ and c are real constants. From basic formulas from complex analysis, we obtain

$$\overline{(\lambda)^2} + \omega^2 + \overline{e^{-\lambda\tau}} = 0$$

and

$$\overline{\lambda^2 + \omega^2 + e^{-\lambda\tau}} = 0.$$

Because the term

$$\lambda^2 + \omega^2 + e^{-\lambda\tau}$$

with a purely imaginary λ is a solution of the characteristic equation, we get a trivial equation $\bar{0} = 0$, which holds. Thus, if a purely imaginary term is a solution of (2.4), its complex conjugate is a solution as well.

The conclusion is that the system has at least two purely imaginary roots if ω, c are taken such that (3.7) is satisfied. We denote these curves as

$$B = \bigcup_{k=0}^{\infty} B_k = \bigcup_{k=0}^{\infty} \left\{ (x, y) \in \mathbb{R} : y = (-1)^k \left(\left(\frac{k\pi}{\tau} \right)^2 - x^2 \right) \right\}.$$

Note that B is not just the boundary of S, it is the set of all the curves in the half plane.

However, this statement does not tell us any information about the stability or instability of the system with $(\omega, c) \in B$. A partial result is given by Theorem 3.9. in [5], which states and proves the behaviour of the system for $(\omega, c) \in \partial S \setminus \{(0, 0)\}$ where ∂S is the boundary of S and also is a particular subset of B. The solution $y = 0$ for the system (2.3) with $(\omega, c) \in \partial S \setminus \{(0, 0)\}$ is stable.

Further, a characterisation of the region $U = \mathbb{R}^2 \setminus (S \cup B)$ can be given. Recall that if we take $(\omega, c) \in S$, every root of the system has a negative real part. If we take $(\omega, c) \in B$, there exist at least two purely imaginary roots, i.e. roots with their real parts equal to zero. Hence, if we take $(\omega, c) \in U$, there are no purely imaginary roots and there exists at least one root with a non-negative real part. This implies that the system has at least one root with a positive real part. Clearly, this system is unstable.

The last missing undetermined region is $B \setminus \partial S$. Until this day no theorem about the behaviour of the system in this region has been given. The most probable hypothesis is that the system is unstable. This tip comes from numerical methods, but no proof of this claim has been given.

The second possibility of showing the stability region in the 2D-plane is to consider the couples (τ, c). This interpretation gives all combinations for designing a control of the form

$$u(y(t)) = cy(t - \tau)$$

for a given harmonic oscillator, i.e. given frequency ω. The boundaries of the stable region are

$$c = (-1)^{k+1}\left(\left(\frac{k\pi}{\tau}\right)^2 - \omega^2\right), \quad k \in \mathbb{N} \cup \{0\}.$$

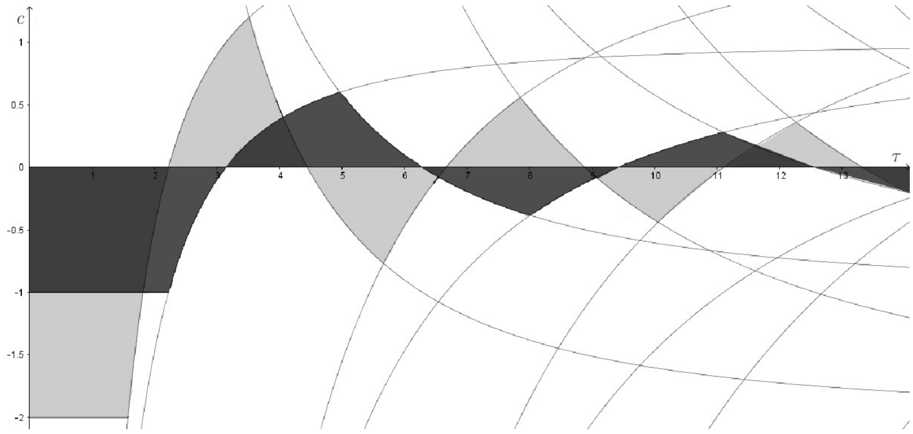


Figure 3.3: The region of asymptotic stability for the (τ, c) couples and $\omega = 1$ (dark grey), $\omega = \sqrt{2}$ (light grey)

The regions have shapes similar to triangles. Different to triangles in figures 3.1, 3.2, the triangles become smaller with growing values of τ on the x-axis. It means that one has fewer options for choosing the control constant c with a large time delay τ.

The stability region is bounded from below (see the leftmost constant lines $c = -1$ and $c = -2$ for $\omega = 1$ and $\omega = \sqrt{2}$, respectively in the figure 3.3). These boundaries are given by the condition (3.1) from Pontryagin's theorem. This inequality also holds for representations shown in figures 3.1, 3.2 but it does not provide any limitation of the region because the condition is contained in the derived boundaries of the region. The regions similar to triangles are higher and denser for growing frequency ω.

At the end of this section, we should note that a substitution of the delay τ is used in some bibliographies. The substitution is $s = t/\tau$. Then the initial problem is defined by

$$y(t) = \tilde{y}(s(t))$$

and its derivatives are

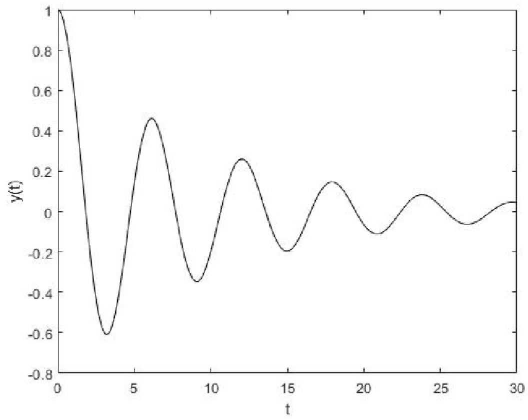
$$\begin{aligned}
\dot{y}(t) &= \frac{d}{ds}\tilde{y}(s(t))\frac{d}{dt}s(t) = \frac{1}{\tau}\dot{\tilde{y}}(s) \\
\ddot{y}(t) &= \frac{1}{\tau}\frac{d}{ds}\dot{\tilde{y}}(s(t))\frac{d}{dt}s(t) = \frac{1}{\tau^2}\ddot{\tilde{y}}(s).
\end{aligned}$$

The transformed delayed term is

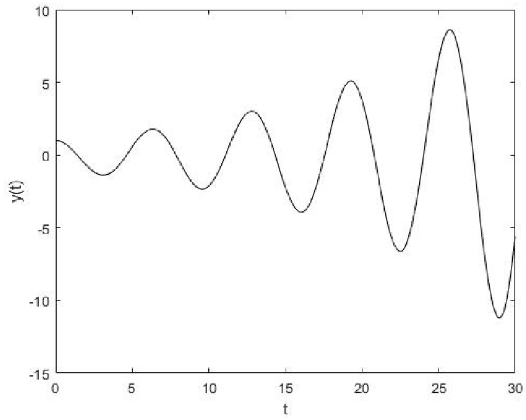
$$y(t-\tau) = \tilde{y}\left(\frac{t-\tau}{\tau}\right) = \tilde{y}(s-1).$$

Thus, the DDE (2.3) is transformed into

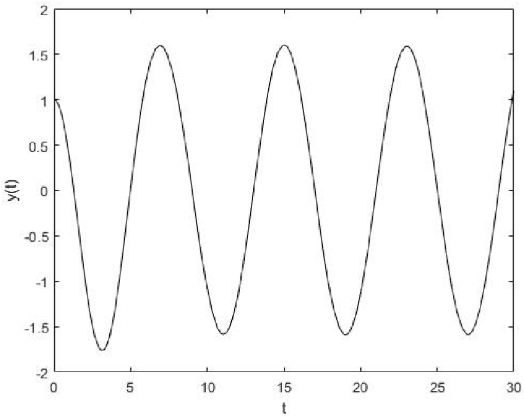
$$\frac{1}{\tau^2}\ddot{\tilde{y}}(s) + \omega^2\tilde{y}(s) + c\tilde{y}(s-1) = 0, \quad s \geq 0.$$



(a) $\omega = 1; \tau = 1; c = -0.2$



(b) $\omega = 1; \tau = 1; c = 0.2$



(c) $\omega = 1; \tau = 4; c = -\left(\left(\frac{\pi}{\tau}\right)^2 - \omega^2\right)$

Figure 3.4: Oscillation by (2.3) with initial conditions $y(t) = 1, \dot{y}(t) = 0$ for $-\tau \leq t \leq 0$

Equivalently,

$$\ddot{\tilde{y}}(s) + \omega^2\tau^2\tilde{y}(s) + c\tau^2\tilde{y}(s-1) = 0, \quad s \geq 0.$$

The initial function ψ is also transformed and it is given on the interval $-1 \leq s \leq 0$. Substitutions $\tilde{a}^2 = \omega^2\tau^2$ and $\tilde{c} = c\tau^2$ transform (2.3) to the final form

$$\ddot{\tilde{y}}(s) + \tilde{a}^2\tilde{y}(s) + \tilde{c}\tilde{y}(s-1) = 0, \quad s \geq 0.$$

This form of the last DDE is similar to the DDE in (2.3). The computations are similar but they become easier now because τ vanishes from all equations formally. In fact, all graphs are also easier to be drawn because the system depends on two constants $\tilde{a}, \tilde{c}$ instead of ω, c, τ and a particular value of τ does not have to be chosen before drawing a graph of the stability regions. Anyway, the delay τ is still a part of the system and it is "hidden" inside the constants $\tilde{a}, \tilde{c}$. Hence, under the substitution above, all constants must be chosen carefully for (asymptotic) stability of the initial system.

3.2 Damped oscillator

The system of a damped oscillator is described as

$$m\ddot{y}(t) + l\dot{y}(t) + ky(t) = 0.$$

Compared to the case of the harmonic oscillator, a friction is considered now. The constant l is called the coefficient of friction and depends on touched materials and on surrounds. The equation can also be written as

$$\ddot{y}(t) + b\dot{y}(t) + \omega^2 y(t) = 0, \quad \omega^2 = \frac{k}{m} > 0, \quad b = \frac{l}{m}.$$

The system becomes the harmonic oscillator if $b = 0$. It follows from [1], the system is asymptotically stable for any $b > 0$ (equivalently $l > 0$). What is going on if $l < 0$? This case is called an oscillator with a negative friction. The system with a negative friction without any control forces is unstable. This system can be seen as a reversed pendulum but it is not trivial to model a real mechanical oscillator as a point mass on a string with a negative friction.

The system of a damped oscillator with a feedback delay control is given as

$$\begin{aligned} &\ddot{y}(t) + b\dot{y}(t) + \omega^2 y(t) + cy(t-\tau) = 0, \quad t > 0 \\ &y(t) = \psi(t), \quad -\tau \le t \le 0 \end{aligned} \tag{3.8}$$

where $\psi(t) \in C\left(\langle -\tau; 0\rangle; \mathbb{R}\right)$ is an initial function. In the following, Pontryagin's theorem will be used to derive a stability region depending on ω, τ, c and b newly.

3.2.1 Sufficient stability conditions

An intuitive way for solving the damped system is to use the algorithm described in the section 3.1. Note, there are no restrictions just on the harmonic oscillation case and Pontryagin's theorem works for any general problem.

Begin with the exponential polynomial $H(\lambda)$ created by the substitution $y(t) = e^{\lambda t}$ into the DDE of the system (3.8):

$$\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + \omega^2 e^{\lambda t} + ce^{\lambda(t-\tau)} = 0.$$

Equivalently,

$$\lambda^2 + b\lambda + \omega^2 + ce^{-\lambda\tau} = 0.$$

Finally, the exponential polynomial is

$$H(\lambda) := \lambda^2 e^{\lambda\tau} + b\lambda e^{\lambda\tau} + \omega^2 e^{\lambda\tau} + c = 0. \tag{3.9}$$

Further, the function $\Delta(y)$ must be defined. For doing that, $H(\mathrm{i}y)$ has to be computed:

$$\begin{aligned} H(\mathrm{i}y) &= -y^2 e^{\mathrm{i}y\tau} + \mathrm{i}bye^{\mathrm{i}y\tau} + \omega^2 e^{\mathrm{i}y\tau} + c \\ &= -y^2\left(\cos(y\tau) + \mathrm{i}\sin(y\tau)\right) + \mathrm{i}by\left(\cos(y\tau) + \mathrm{i}\sin(y\tau)\right) + \omega^2\left(\cos(y\tau) + \mathrm{i}\sin(y\tau)\right) + c. \end{aligned}$$

Now split it into real and complex part,

$$\begin{aligned}H(\mathrm{i}y) &= F(y) + \mathrm{i}G(y)\\ F(y) &= -y^2\cos(y\tau) - by\sin(y\tau) + \omega^2\cos(y\tau) + c\\ G(y) &= -y^2\sin(y\tau) + by\cos(y\tau) + \omega^2\sin(y\tau).\end{aligned}$$

The derivatives of both $F(y), G(y)$ are required:

$$\begin{aligned}F'(y) &= -2y\cos(y\tau) + y^2\tau\sin(y\tau) - b\sin(y\tau) - by\tau\cos(y\tau) - \omega^2\tau\sin(y\tau)\\ G'(y) &= -2y\sin(y\tau) - y^2\tau\cos(y\tau) + b\cos(y\tau) - by\tau\sin(y\tau) - \omega^2\tau\cos(y\tau).\end{aligned}$$

Recall, the function $\Delta(y)$ is defined as

$$\Delta(y) := G'(y)F(y) - G(y)F'(y).$$

At first, $\Delta(y)$ at $y = 0$ has to be computed for Pontryagin's theorem. Since

$$F(0) = \omega^2 + c, \quad G(0) = 0, \quad F'(0) = 0, \quad G'(0) = b + \omega\tau,$$

it implies

$$\Delta(0) = (\omega^2 + c)(b + \omega\tau).$$

The first statement of Pontryagin's theorem says that if the zero solution of the system (3.8) is asymptotically stable, then $\Delta(0) > 0$. This holds if either

$$c > -\omega^2 \wedge b > -\omega^2\tau$$

or

$$c < -\omega^2 \wedge b < -\omega^2\tau.$$

The result can be interpreted as following: If the coefficient of friction is "big enough", i.e. the case of positive friction or weak negative friction, the condition on control coefficient is the same as in the section 3.1. On the other hand, if the b is "too small", the only possibility to get a stable system is to choose a negative c.

Following the algorithm from the section 3.1, the problem shall be split into cases of positive and negative coefficient c. All zeros of $G(y)$ have to be computed in both cases. So we have to solve

$$(\omega^2 - y^2)\sin(y\tau) + by\cos(y\tau) = 0.$$

The equation can be solved numerically but it would not give general forms of all zeros of the function $G(y)$. Thus, a substitution transforming the damped system to the harmonic oscillator case will be used.

The aim is to find a substitution which eliminates the term $\dot{y}(t)$. The wanted substitution is

$$y(t) = e^{-\frac{b}{2}t}z(t), \tag{3.10}$$

where $z(t)$ is a smooth function. To prove this choice, the first and the second derivatives are required:

$$\begin{aligned}\dot{y}(t) &= -\frac{b}{2}e^{-\frac{b}{2}t}z(t) + e^{-\frac{b}{2}t}\dot{z}(t)\\ \ddot{y}(t) &= \frac{b^2}{4}e^{-\frac{b}{2}t}z(t) - be^{-\frac{b}{2}t}\dot{z}(t) + e^{-\frac{b}{2}t}\ddot{z}(t).\end{aligned}$$

Together with the original form of $y(t)$, they give

$$e^{-\frac{b}{2}t}\ddot{z}(t) + (b-b)e^{-\frac{b}{2}t}\dot{z}(t) + \left(\frac{b^2}{4} - \frac{b^2}{2} + \omega^2\right) e^{-\frac{b}{2}t}z(t) + ce^{-\frac{b}{2}(t-\tau)}z(t-\tau) = 0.$$

Clearly, the once derived term is vanishing. By multiplying $e^{\frac{b}{2}t}$ and a correction of the constants, the differential equation

$$\ddot{z}(t) + \left(\omega^2 - \frac{b^2}{4}\right) z(t) + ce^{\frac{b}{2}\tau}z(t-\tau) = 0$$

is obtained. Now every term of the equation has a form of the function $z(t)$ multiplied by a constant. The constants can be denoted as

$$\left(\omega^2 - \frac{b^2}{4}\right) = \hat{a}, \quad ce^{\frac{b}{2}\tau} = \hat{c}. \tag{3.11}$$

These substitutions give a new form of the DDE describing the damped feedback oscillator

$$\ddot{z}(t) + \hat{a}z(t) + \hat{c}z(t-\tau) = 0. \tag{3.12}$$

This equation is similar to the harmonic oscillator case (2.3).

The only difference comes if $b^2 > 4\omega^2$. In such a situation, the constant $\hat{a}$ is negative and a similar case have not been studied in the section 3.1. Fortunately, it is not a big deal to study this problem.

Assume $\hat{a} < 0$. Since the algorithm from the harmonic oscillator is followed, the exponential polynomial $H(\lambda)$ must be found and also the corresponding functions $F(z), G(z)$. For now, just the function

$$G(z) = (\hat{a} - z^2)\sin(z\tau)$$

is required. The zeros of $G(z)$ with the negative real coefficient $\hat{a}$ are

$$z_0 = \pm i\sqrt{|\hat{a}|}, \quad z_0 = \frac{n\pi}{\tau}, \quad n \in \mathbb{Z}.$$

So there are complex zeros and Pontryagin's theorem says that the exponential polynomial $H(\lambda)$ has a root with a negative real part. This implies the system is not asymptotically stable whenever $\hat{a} < 0$, i.e. $b^2 > 4\omega^2$.

The next results may be obtained from the section 3.1. The solution has to be split into the cases of a negative and a positive $\hat{c}$. By the previous results, the system of the damped oscillator described by (3.12) with (3.10) is asymptotically stable if and only if

$$\frac{2k\pi}{\sqrt{\hat{a}}} < \tau < \frac{(2k+1)\pi}{\sqrt{\hat{a}}}, \quad k \in \mathbb{N} \cup \{0\},$$
$$0 > \hat{c} > \max\left\{\left(\frac{2k\pi}{\tau}\right)^2 - \hat{a}, \hat{a} - \left(\frac{(2k+1)\pi}{\tau}\right)^2\right\}$$

or

$$\frac{(2k+1)\pi}{\sqrt{\hat{a}}} < \tau < \frac{2(k+1)\pi}{\sqrt{\hat{a}}}, \quad k \in \mathbb{N} \cup \{0\},$$
$$0 < \hat{c} < \min\left\{\hat{a} - \left(\frac{(2k+1)\pi}{\tau}\right)^2, \left(\frac{2(k+1)\pi}{\tau}\right)^2 - \hat{a}\right\}.$$

Additional condition for both cases is $\hat{a} > 0$. In terms of the original variables ω, τ, b, c given from (3.11), the inequalities, where ω, b are chosen and τ, c dependent, become

$$\frac{2k\pi}{\sqrt{\omega^2 - \frac{b^2}{4}}} < \tau < \frac{(2k+1)\pi}{\sqrt{\omega^2 - \frac{b^2}{4}}}, \quad k \in \mathbb{N} \cup \{0\},$$

$$0 > c > \max\left\{\left(\frac{2k\pi}{\tau}\right)^2 - \omega^2 + \frac{b^2}{4}, \omega^2 - \frac{b^2}{4} - \left(\frac{(2k+1)\pi}{\tau}\right)^2\right\} e^{-\frac{b}{2}\tau}$$

or

$$\frac{(2k+1)\pi}{\sqrt{\omega^2 - \frac{b^2}{4}}} < \tau < \frac{2(k+1)\pi}{\sqrt{\omega^2 - \frac{b^2}{4}}}, \quad k \in \mathbb{N} \cup \{0\},$$

$$0 < c < \min\left\{\omega^2 - \frac{b^2}{4} - \left(\frac{(2k+1)\pi}{\tau}\right)^2, \left(\frac{2(k+1)\pi}{\tau}\right)^2 - \omega^2 + \frac{b^2}{4}\right\} e^{-\frac{b}{2}\tau}$$

and for both cases also $4\omega^2 > b^2$ must hold.

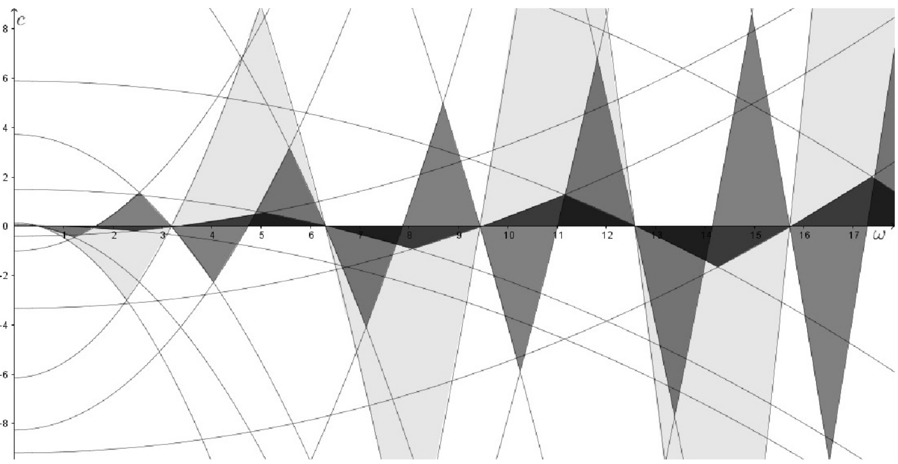


Figure 3.5: The (ω, c) plane with the stability region of (3.8) for $\tau = 1, b = 1$ (light grey), $\tau = 2, b = 1$ (middle grey) and $\tau = 1, b = 2$ (dark grey)

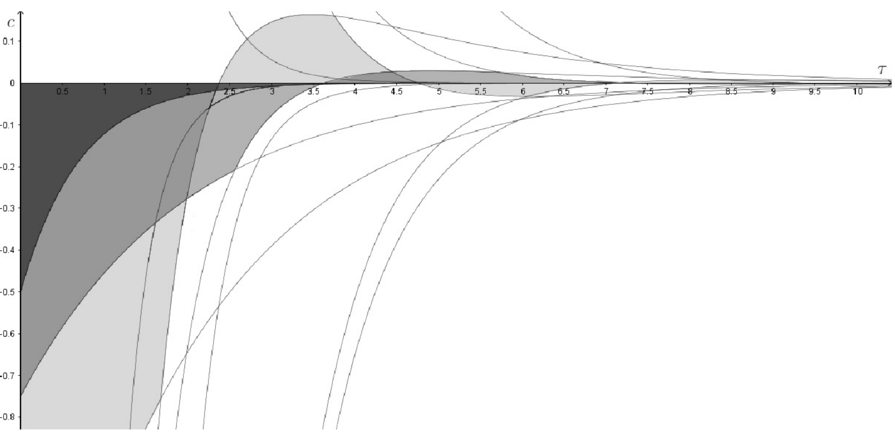


Figure 3.6: The (τ, c) plane with the stability region of (3.8) for $\omega = \sqrt{2}, b = 1$ (light grey), $\omega = 1, b = 1$ (middle grey) and $\omega = 1, b = \sqrt{2}$ (dark grey)

The solution is similar to the harmonic oscillator case. The region bounded by the conditions above can be drawn in both (ω, c) and (τ, c) planes. Choosing the parameters from this region, the function $z(t)$ derived from (3.10) as

$$z(t) = e^{\frac{b}{2}t}y(t)$$

is eventually tending to zero. The function $z(t)$ is the product of two functions, namely $e^{\frac{b}{2}t}$ and $y(t)$. Here, the discussion has to be split into two parts depending on the sign of b.

If $b > 0$, the argument of $e^{\frac{b}{2}t}$ is positive and the function itself is unbounded. Now suppose, the derived condition on τ and c hold, i.e. $z(t)$ is tending to zero. This means the product of the unbounded function $e^{\frac{b}{2}t}$ and the searched function $y(t)$ is tending to zero. A situation like this is possible if $y(t)$ is tending to zero "more strongly" then $e^{\frac{b}{2}t}$ is tending to infinity. This situation is described by the derived conditions.

On the other hand, there are parameters for which $y(t)$ is tending to zero but $z(t)$ is not. These parameters are not determined by the conditions on τ and c of $z(t)$. In this situation, $y(t)$ converges to zero but it is not "enough strong" to defeat the unboundness of $e^{\frac{b}{2}t}$.

The conclusion of this discussion is that the conditions derived in this section for $b > 0$ are sufficient conditions only, not necessary. Under them, the function $y(t)$ is tending to zero but there are other combinations of τ and c making $y(t)$ asymptotically stable.

Now assume $b < 0$. The function $e^{\frac{b}{2}t}$ is tending to zero. Choosing parameters out of the stability region of $z(t)$, the function $y(t)$ is clearly unbounded, even so strongly unbounded that it is able to break the boundness of $e^{\frac{b}{2}t}$. In the case of bounded $z(t)$, we do not know any sufficient information about $y(t)$. The product of bounded $e^{\frac{b}{2}t}$ and unbounded $y(t)$ may be bounded. Clearly, also the product of $e^{\frac{b}{2}t}$ and bounded $y(t)$ is bounded. Since it is not possible to extract $y(t)$ from $z(t)$ analytically, we can not state any sufficient conclusion for $b < 0$ now. The conditions are only the necessary conditions.

3.2.2 Necessary and sufficient stability conditions

The aim of this section is to state necessary conditions for asymptotic stability of the system (3.8). We will use Pontryagin's theorem again, but the algorithm will be changed.

The main difference comes right in the beginning in the exponential polynomial (3.9). Recall, $H(\lambda)$ in the section 3.2.1 has been considered as

$$H(\lambda) := \lambda^2 e^{\lambda\tau} + b\lambda e^{\lambda\tau} + \omega^2 e^{\lambda\tau} + c = 0.$$

Now we will substitute terms in this function to simplify the argument of $e^{\lambda\tau}$. Set a new variable s as

$$s = \lambda\tau.$$

Clearly,

$$\lambda = \frac{s}{\tau}$$

with positive constant τ may be substituted as an argument of $H(\lambda)$. Multiplying by τ^2, we obtain a new function $\bar{H}(s)$ depending on the variable s and characteristics of the mechanical system. Step by step,

$$\begin{aligned} H(\lambda) &= \lambda^2 e^{\lambda\tau} + b\lambda e^{\lambda\tau} + \omega^2 e^{\lambda\tau} + c = 0 \\ H\left(\frac{s}{\tau}\right) &= \left(\frac{s}{\tau}\right)^2 e^{\frac{s}{\tau}\tau} + b\frac{s}{\tau}e^{\frac{s}{\tau}\tau} + \omega^2 e^{\frac{s}{\tau}\tau} + c = 0 \\ \bar{H}(s) := \tau^2 H\left(\frac{s}{\tau}\right) &= s^2 e^s + b\tau s e^s + \omega^2\tau^2 e^s + c\tau^2 = 0. \end{aligned}$$

The function $\bar{H}(s)$ is also an exponential polynomial based on the characteristic equation of the system (3.8). This means that Pontryagin's theorem may be applied with $\bar{H}(s)$. Before doing that, the real part and the complex part of the exponential polynomial at the critical value $s = \mathrm{i}y$ have to be computed:

$$\begin{aligned} \bar{H}(\mathrm{i}y) &= -y^2 e^{\mathrm{i}y} + \mathrm{i}b\tau y e^{\mathrm{i}y} + \omega^2\tau^2 e^{\mathrm{i}y} + c\tau^2 \\ &= -y^2(\cos(y) + \mathrm{i}\sin(y)) + \mathrm{i}b\tau y(\cos(y) + \mathrm{i}\sin(y)) + \omega^2\tau^2(\cos(y) + \mathrm{i}\sin(y)) + c\tau^2 \\ &= \bar{F}(y) + \mathrm{i}\bar{G}(y) \end{aligned}$$

$$\bar{F}(y) = -y^2\cos(y) - b\tau y\sin(y) + \omega^2\tau^2\cos(y) + c\tau^2 \tag{3.13}$$

$$\bar{G}(y) = -y^2\sin(y) + b\tau y\cos(y) + \omega^2\tau^2\sin(y) \tag{3.14}$$

A similar delay system solved by Pontryagin's theorem has been discussed in [6]. Theorems here are done even for more general second-order delay problem. In this thesis, we still focus on controlled mechanical oscillations only. One of results from this paper may be used for our system (3.8).

First, a conclusion of a theorem in [6] states necessary conditions for asymptotic stability of the controlled damped system satisfying $b \neq 0$ and $\omega^2 c < 0$. The necessary conditions are

$$b > 0 \quad \text{and} \quad \omega^2 + c > 0. \tag{3.15}$$

Moreover, also necessary sufficient conditions have been stated here. It follows from them, the system (3.8) with the parameters $b \neq 0$ and $\omega^2 c < 0$ is asymptotically stable if and only if

$$b > 0, \quad \omega^2 + c > 0 \tag{3.16}$$

and

$$\begin{array}{lllll} \text{(a)} & \text{if} \quad \omega^2 > 0, & \text{then} & \bar{F}(r_{2k}) > 0 & k = 1, 2, \dots \\ \text{(b)} & \text{if} \quad \omega^2 < 0, & \text{then} & \bar{F}(r_{2k+1}) < 0 & k = 0, 1, 2, \dots \end{array} \tag{3.17}$$

where the function $\bar{F}(y)$ is defined in (3.13) and r_i are positive roots of the function $\bar{G}(y)$ defined in (3.14). The proofs of both statements may be seen in [6].

As it has been said, this thesis focuses on problems of mechanical oscillations. It restricts the work to cases of the positive real phase frequencies ω. Thus, the case (3.17b) can not be used. Considering this fact, the assumptions of both necessary and necessary sufficient conditions are modified to $b \neq 0$ and $c < 0$. This also means, that the derived necessary sufficient conditions may be used for case of the negative control constant c. The conditions (3.15), (3.16) may be equivalently written as

$$b > 0, \quad 0 > c > -\omega^2.$$

An important result of the theorem above is that the damped system (3.8) with $c < 0$ can not be stabilised with a negative constant b. This rejects the idea of damped oscillators with negative frictions under this particular assumptions of this statement.

Further, this expression of Pontryagin's theorem does not give any information about the system with a positive control constant $c > 0$. This case is not allowed by the essential of mechanical oscillations $\omega^2 > 0$. Anyway the statement does not set the equation (3.8) with positive control constant to be unstable. Conditions for $c > 0$ have not been derived in this sections because this case does not fulfil our assumptions. Following the results from the section 3.2.1, there exist combinations of the parameter with $c > 0$ stabilising the equation (3.8). Unfortunately, we have not derived any necessary sufficient conditions. In a general DDE problem, it is not a big deal to consider a system described by an equation where would be a negative constant instead of our ω^2. In such a case, Pontryagin's theorem might state necessary sufficient conditions even for controls with positive control constants.

The most difficult problem appears from the conditions (3.17). Since $\bar{G}(y)$ has, in general, infinitely many positive roots, it is not possible to check signs of $\bar{F}(r_i)$ for each $i \in \mathbb{N}$. Fortunately, the paper [6] brings a solution. It has been proved there that if

$$\omega^2 > 0, \quad \bar{F}(r_{2k}) > 0, \quad k = 1, 2, \dots, n,$$

where

$$\frac{3}{2}\pi > [r_{2n}] > \pi \quad \text{and} \quad (-r_{2n}^2 + \omega^2\tau^2)\cos(r_{2n}) + c\tau^2 > 0, \tag{3.18}$$

then

$$\bar{F}(r_{2k}) > 0$$

for all k. The function $[\alpha]$ is defined as the residue modulo 2π of a real number α. Clearly, $[\alpha]$ is also real and $0 \leq [\alpha] < 2\pi$. Similar statement could be given also for (3.17b) if $\omega^2 < 0$ but it is not needed for the system (3.8). This gives a significant simplification, because only finitely many positive roots of $\bar{G}(y)$ have to be considered.

A big disadvantage of the algorithm described in this section is that stability regions in any plane cannot be drawn. If one wants to check the stability of the damped oscillator, all parameters must be chosen in advance. If the conditions (3.16) hold, roots of $\bar{G}(y)$ have to be computed and check the condition (3.17a). It is enough to consider the roots until the one satisfying (3.18).

The case $b < 0$ cannot be stabilised using this control (by (3.16)). The case $b > 0$ means that already uncontrolled system is asymptotically stable. Here, the control may be used for speeding up the convergence, or it can have a destabilising effect.

A similar algorithm may be used for harmonic oscillators, too. Some results are given in [6]. In the beginning, the exponential polynomial have to be recomputed to the form of $\bar{H}(s)$ from the current section and the function $F(y), G(y)$ as well. Results would be equivalent to the stability regions derived in the section 3.1.

3.3 Pyragas control

A special case of controls are so called noninvasive controls. They can be used for a control of a periodic function. The characteristic of this method is that the control vanishes for a particular choice of the delay τ. Since the aim of the control is to obtain a controlled stable periodic system , the noninvasive control vanishes for $t \to \infty$, too.

The well known type of the noninvasive control is Pyragas control. The control of this type has the form

$$u(y(t)) = c(y(t) - y(t - \tau)). \tag{3.19}$$

This method was originally developed by physicist Kestutis Pyragas from the Russian Federation in 1992 [7]. Clearly, the function $y(t)$ describing a movement of a point mass (oscillation) is a periodic function. The period of $y(t)$ will be denoted as T. Recall, the period of a function is the least positive constant such that $y(t) = y(t + T)$. If the time delay τ is chosen properly, the controller (3.19) is identically equal zero. In this case, the proper choice is

$$\tau = kT, \quad k \in \mathbb{N}.$$

It is easy to show

$$u(y(t)) = c(y(t) - y(t - kT)) = c(y(t) - y(t)) \equiv 0.$$

Since this chapter's aim is to find conditions for asymptotic stability of $y(t)$, the oscillation under these conditions shall tend to the zero state. Thus, we may consider

$$\lim_{t\to\infty} y(t) \to 0.$$

Clearly, also

$$\lim_{t\to\infty} y(t - \tau) \to 0.$$

It shows that the control also vanishes for any choice of τ as the time increases and so

$$\lim_{t\to\infty} u(y(t)) = \lim_{t\to\infty} c(y(t) - y(t - \tau)) \to c(0 - 0) = 0.$$

. Note, this property does not hold in general, an asymptotic stable function is necessary.

3.3.1 Pyragas control on harmonic oscillators

A harmonic oscillator with Pyragas control is given by

$$\begin{aligned} &\ddot{y}(t) + \omega^2 y(t) + c\left(y(t) - y(t - \tau)\right) = 0, \quad t > 0 \\ &y(t) = \psi(t), \quad -\tau \le t \le 0, \end{aligned} \tag{3.20}$$

where $\psi(t) \in C\left(\langle -\tau; 0\rangle; \mathbb{R}\right)$ is an initial function. In some references, the control has the form

$$u(y(t)) = c\left(y(t - \tau) - y(t)\right).$$

Since $c \in \mathbb{R}$, this control can be transform to the form from (3.20) by changing the sign of c. The results will be similar.

To study asymptotic stability of the system, Pontryagin's theorem will be used. Hence, (3.20) must be converted to a form similar to (2.3). Thus,

$$\ddot{y}(t) + \left(\omega^2 + c\right) y(t) - cy(t - \tau) = 0$$

and do substitutions

$$\left(\omega^2 + c\right) = \hat{a}, \quad -c = \hat{c}. \tag{3.21}$$

So the equation

$$\ddot{y}(t) + \hat{a}y(t) + \hat{c}y(t - \tau) = 0 \tag{3.22}$$

is obtained. This is a DDE similar to delay equation of the harmonic oscillator with the simple feedback control $u(y(t)) = cy(t - \tau)$ which has been studied in the section 3.1. Final results for (3.20) are given briefly referring to the section 3.1 where the complete computations have been done.

Following a partial result from the section 3.2, the system (3.20) (equivalently (3.22)) is not asymptotically stable if $\hat{a} < 0$, i.e. $c < -\omega^2$. The condition $\Delta(0) > 0$ following the notation from Pontryagin's theorem gives the condition

$$\tau\hat{a}(\hat{a} + \hat{c}) > 0.$$

In the sense of the original constants,

$$\begin{aligned} \tau\left(\omega^2 + c\right)\left(\omega^2 + c - c\right) &> 0 \\ \tau\omega^2\left(\omega^2 + c\right) &> 0. \end{aligned}$$

Since both τ and ω^2 are positive constants and $(\omega^2 + c) > 0$ holds by the derivation above, the inequality is satisfied without any other restriction. By the final results from 3.1.1 and 3.1.2, the system of a harmonic oscillator with Pyragas control (3.20), equivalent to (3.22) with (3.21), is asymptotically stable if and only if

$$\begin{aligned} \frac{2k\pi}{\sqrt{\hat{a}}} < \tau < \frac{(2k+1)\pi}{\sqrt{\hat{a}}}, &\quad k \in \mathbb{N} \cup \{0\}, \\ 0 > \hat{c} > \max\left\{\left(\frac{2k\pi}{\tau}\right)^2 - \hat{a}, \hat{a} - \left(\frac{(2k+1)\pi}{\tau}\right)^2\right\} \end{aligned}$$

or

$$\begin{aligned} \frac{(2k+1)\pi}{\sqrt{\hat{a}}} < \tau < \frac{2(k+1)\pi}{\sqrt{\hat{a}}}, &\quad k \in \mathbb{N} \cup \{0\}, \\ 0 < \hat{c} < \min\left\{\hat{a} - \left(\frac{(2k+1)\pi}{\tau}\right)^2, \left(\frac{2(k+1)\pi}{\tau}\right)^2 - \hat{a}\right\}. \end{aligned}$$

In both cases, also $\hat{a} > 0$ must be satisfied. By the backward substitutions (3.21), the conditions become

$$\begin{aligned} \frac{2k\pi}{\sqrt{\omega^2 + c}} < \tau < \frac{(2k+1)\pi}{\sqrt{\omega^2 + c}}, &\quad k \in \mathbb{N} \cup \{0\}, \\ 0 < c < \min\left\{\omega^2 + c - \left(\frac{2k\pi}{\tau}\right)^2, \left(\frac{(2k+1)\pi}{\tau}\right)^2 - \omega^2 - c\right\} \end{aligned}$$

or

$$\frac{(2k+1)\pi}{\sqrt{\omega^2+c}} < \tau < \frac{2(k+1)\pi}{\sqrt{\omega^2+c}}, \quad k \in \mathbb{N} \cup \{0\},$$
$$0 > c > \max\left\{\left(\frac{(2k+1)\pi}{\tau}\right)^2 - \omega^2 - c, \omega^2 + c - \left(\frac{2(k+1)\pi}{\tau}\right)^2\right\},$$

and also $c > -\omega^2$ must be satisfied in both cases. Note, inequalities' signs in the conditions for c and operations $\max, \min$ are changed because the conditions are multiplied by a negative number.

Since the searched constant c is in the both sides of the inequalities for c, the obtained conditions must by rewritten as following four conditions:

(i)

$$\frac{2k\pi}{\sqrt{\omega^2+c}} < \tau < \frac{(2k+1)\pi}{\sqrt{\omega^2+c}}, \quad k \in \mathbb{N} \cup \{0\},$$
$$c < \omega^2 + c - \left(\frac{2k\pi}{\tau}\right)^2 \leq \left(\frac{(2k+1)\pi}{\tau}\right)^2 - \omega^2 - c$$

(ii)

$$\frac{2k\pi}{\sqrt{\omega^2+c}} < \tau < \frac{(2k+1)\pi}{\sqrt{\omega^2+c}}, \quad k \in \mathbb{N} \cup \{0\},$$
$$c < \left(\frac{(2k+1)\pi}{\tau}\right)^2 - \omega^2 - c \leq \omega^2 + c - \left(\frac{2k\pi}{\tau}\right)^2$$

(iii)

$$\frac{(2k+1)\pi}{\sqrt{\omega^2+c}} < \tau < \frac{2(k+1)\pi}{\sqrt{\omega^2+c}}, \quad k \in \mathbb{N} \cup \{0\},$$
$$c > \omega^2 + c - \left(\frac{2(k+1)\pi}{\tau}\right)^2 \geq \left(\frac{(2k+1)\pi}{\tau}\right)^2 - \omega^2 - c$$

(iv)

$$\frac{(2k+1)\pi}{\sqrt{\omega^2+c}} < \tau < \frac{2(k+1)\pi}{\sqrt{\omega^2+c}}, \quad k \in \mathbb{N} \cup \{0\},$$
$$c > \left(\frac{(2k+1)\pi}{\tau}\right)^2 - \omega^2 - c \geq \omega^2 + c - \left(\frac{2(k+1)\pi}{\tau}\right)^2$$

In the second step by splitting the c conditions' triple inequalities, these four pairs of the conditions become the following triples

(i)

$$\begin{aligned}\frac{2k\pi}{\sqrt{\omega^2+c}} < \tau <& \frac{(2k+1)\pi}{\sqrt{\omega^2+c}}, \quad k \in \mathbb{N}\cup\{0\},\\ \omega^2+c-\left(\frac{2k\pi}{\tau}\right)^2 \leq& \left(\frac{(2k+1)\pi}{\tau}\right)^2-\omega^2-c,\\ 0<c<&\omega^2+c-\left(\frac{2k\pi}{\tau}\right)^2\end{aligned}$$

(ii)

$$\begin{aligned}\frac{2k\pi}{\sqrt{\omega^2+c}} < \tau <& \frac{(2k+1)\pi}{\sqrt{\omega^2+c}}, \quad k \in \mathbb{N}\cup\{0\},\\ \left(\frac{(2k+1)\pi}{\tau}\right)^2-\omega^2-c \leq& \omega^2+c-\left(\frac{2k\pi}{\tau}\right)^2,\\ 0<c<& \left(\frac{(2k+1)\pi}{\tau}\right)^2-\omega^2-c\end{aligned}$$

(iii)

$$\begin{aligned}\frac{(2k+1)\pi}{\sqrt{\omega^2+c}} < \tau <& \frac{2(k+1)\pi}{\sqrt{\omega^2+c}}, \quad k \in \mathbb{N}\cup\{0\},\\ \omega^2+c-\left(\frac{2(k+1)\pi}{\tau}\right)^2 \geq& \left(\frac{(2k+1)\pi}{\tau}\right)^2-\omega^2-c,\\ 0>c>& \omega^2+c-\left(\frac{(2k+2)\pi}{\tau}\right)^2\end{aligned}$$

(iv)

$$\begin{aligned}\frac{(2k+1)\pi}{\sqrt{\omega^2+c}} < \tau <& \frac{2(k+1)\pi}{\sqrt{\omega^2+c}}, \quad k \in \mathbb{N}\cup\{0\},\\ \left(\frac{(2k+1)\pi}{\tau}\right)^2-\omega^2-c \geq& \omega^2+c-\left(\frac{2(k+1)\pi}{\tau}\right)^2,\\ 0>c>& \left(\frac{(2k+1)\pi}{\tau}\right)^2-\omega^2-c.\end{aligned}$$

The second and the new third inequalities of cases (i) – (iv) can be simplified and so the final forms of the conditions is obtained as

(i)

$$\begin{aligned}\frac{2k\pi}{\sqrt{\omega^2+c}} < \tau <& \frac{(2k+1)\pi}{\sqrt{\omega^2+c}}, \quad k \in \mathbb{N}\cup\{0\},\\ \omega^2+c \leq& \frac{(8k^2+4k+1)\pi^2}{2\tau^2},\\ \tau >& \frac{2k\pi}{\omega}\end{aligned}$$

(ii)

$$\frac{2k\pi}{\sqrt{\omega^2+c}} < \tau < \frac{(2k+1)\pi}{\sqrt{\omega^2+c}}, \quad k \in \mathbb{N} \cup \{0\},$$
$$\omega^2 + c \geq \frac{(8k^2+4k+1)\pi^2}{2\tau^2},$$
$$0 < c < \frac{1}{2}\left[\left(\frac{(2k+1)\pi}{\tau}\right)^2 - \omega^2\right]$$

(iii)

$$\frac{(2k+1)\pi}{\sqrt{\omega^2+c}} < \tau < \frac{2(k+1)\pi}{\sqrt{\omega^2+c}}, \quad k \in \mathbb{N} \cup \{0\},$$
$$\omega^2 + c \geq \frac{(8k^2+12k+5)\pi^2}{2\tau^2},$$
$$\tau < \frac{2(k+1)\pi}{\omega}$$

(iv)

$$\frac{(2k+1)\pi}{\sqrt{\omega^2+c}} < \tau < \frac{2(k+1)\pi}{\sqrt{\omega^2+c}}, \quad k \in \mathbb{N} \cup \{0\},$$
$$\omega^2 + c \leq \frac{(8k^2+12k+5)\pi^2}{2\tau^2},$$
$$0 > c > \frac{1}{2}\left[\left(\frac{(2k+1)\pi}{\tau}\right)^2 - \omega^2\right].$$

In fact, not all of these inequalities are necessary. Look at the following figure 3.7. Here, the curves are

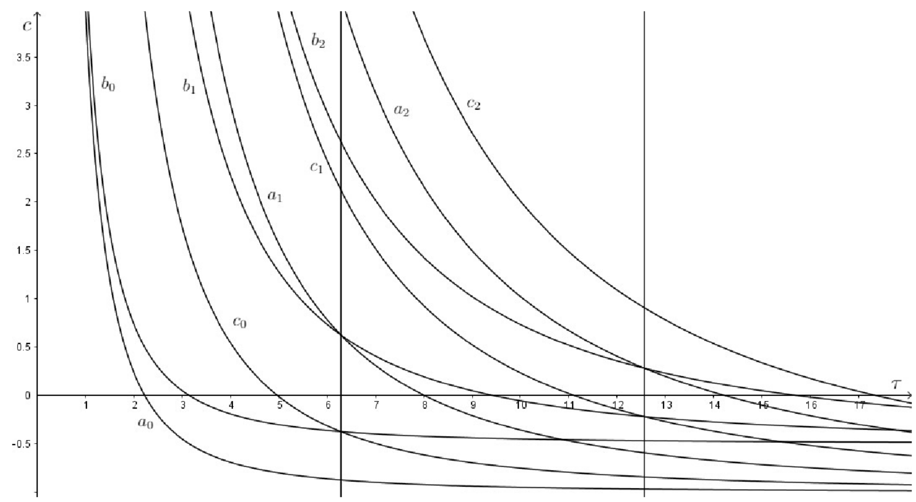
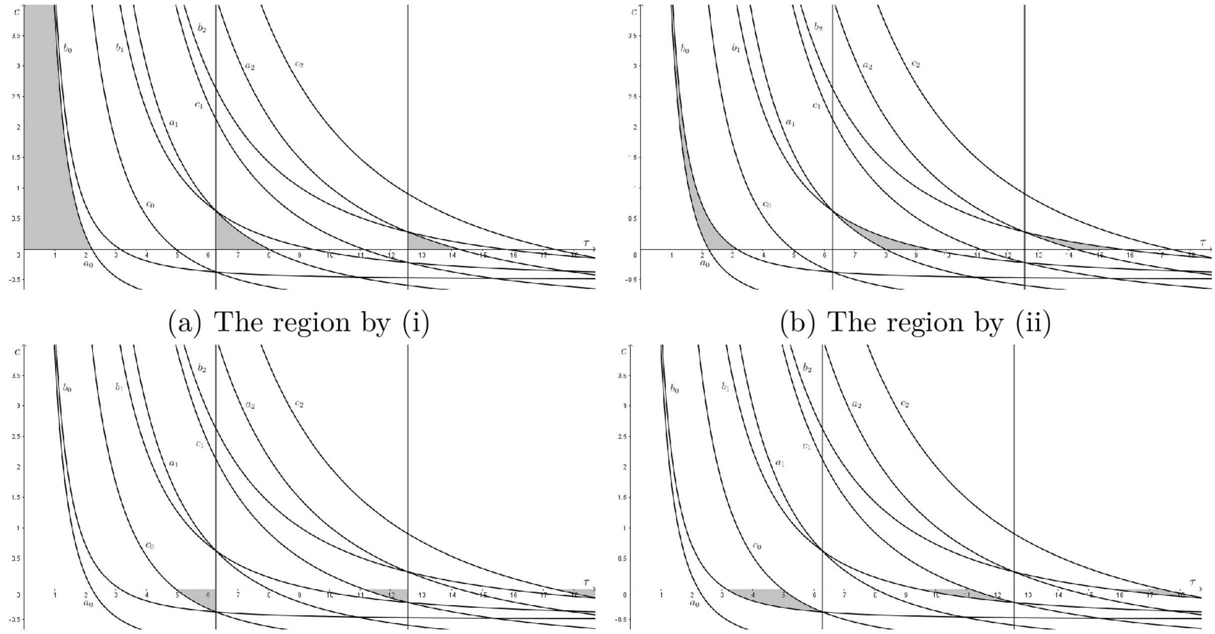


Figure 3.7: The (τ, c) plane with curves given by inequalities from (i) – (iv) for $k = 0, 1, 2$



(a) The region by (i) (b) The region by (ii)

(c) The region by (iii) (d) The region by (iv)

Figure 3.8: The regions of asymptotic stability for (3.20) with $\omega = 1$

$$\begin{aligned} a_k: &\quad \omega^2 + c = \frac{(8k^2+4k+1)\pi^2}{2\tau^2} \\ b_k: &\quad c = \frac{1}{2}\left[\left(\frac{(2k+1)\pi}{\tau}\right)^2 - \omega^2\right] \\ c_k: &\quad \omega^2 + c = \frac{(8k^2+12k+5)\pi^2}{2\tau^2} \end{aligned}$$

with $\omega = 1$.

The conditions (i) set the asymptotically stable region as the region bounded by $\tau = \frac{2k\pi}{\omega}$ from the left and by a_k from the right on the upper half-plane. The condition (ii) set the asymptotically stable region as the region bounded by a_k from the left and by b_k from the right on the upper half-plane. Similarly for the conditions (iii): by c_k from the left and by $\tau = \frac{2(k+1)\pi}{\omega}$ from the right. Now the region is in the lower half-plane. Finally, the region from the conditions (iv) is bounded by b_k from the left, by c_k form the right on the lower half-plane. See the figure 3.8.

The final asymptotic stability region (presented in the figure 3.9) is an union of the four regions above. It is obvious that the four regions for each k become just two regions, one in the upper half-plane and one in the lower half-plane. Hence, the curves a_k, c_k can be neglected as boundaries of the stability region. Even if it has been proved from the pictures for particular $\omega = 1$, this fact is true for arbitrary positive ω and every $k \in \mathbb{N} \cup \{0\}$.

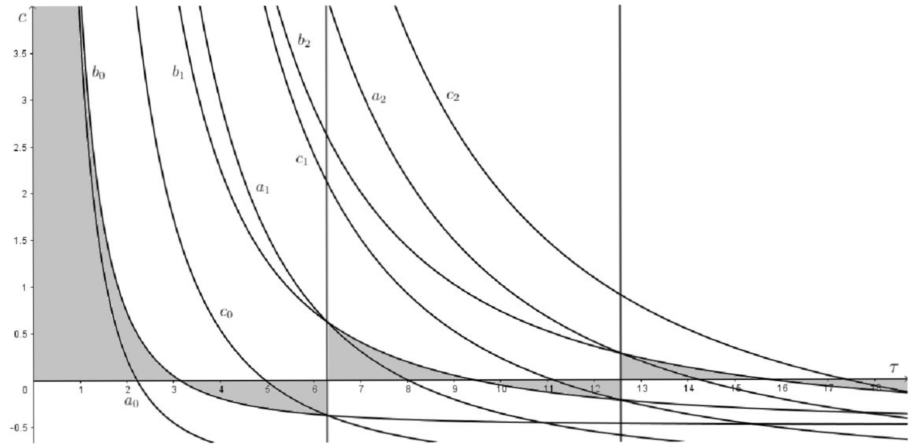


Figure 3.9: The union of the stability regions

The conclusion is that the conditions (i) – (iv) can be simplified as the following: The system (3.20) is asymptotically stable if and only if

$$\begin{aligned} \frac{2k\pi}{\omega} < \tau < \frac{(2k+1)\pi}{\omega}, \quad k \in \mathbb{N} \cup \{0\}, \\ 0 < c < \frac{1}{2}\left[\left(\frac{(2k+1)\pi}{\tau}\right)^2 - \omega^2\right] \end{aligned} \tag{3.23}$$

or

$$\begin{aligned} \frac{(2k+1)\pi}{\omega} < \tau < \frac{2(k+1)\pi}{\omega}, \quad k \in \mathbb{N} \cup \{0\}, \\ 0 > c > \frac{1}{2}\left[\left(\frac{(2k+1)\pi}{\tau}\right)^2 - \omega^2\right]. \end{aligned} \tag{3.24}$$

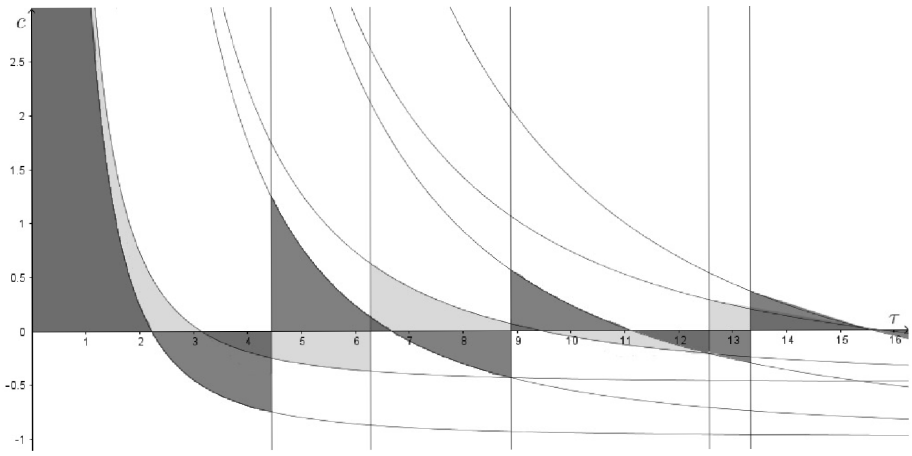
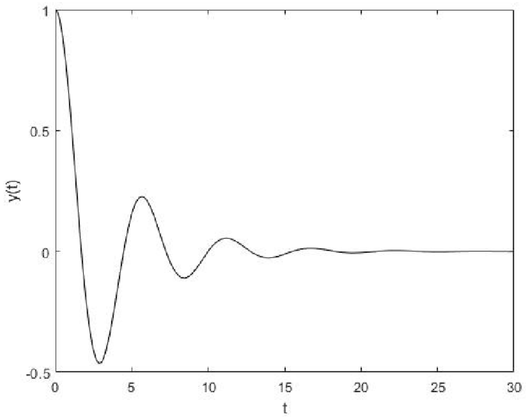


Figure 3.10: The asymptotic stable region of (3.20) with $\omega = 1$ (light grey) and $\omega = \sqrt{2}$ (dark grey)

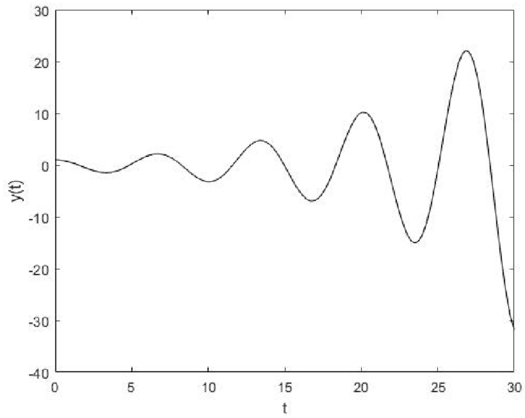
Note, the condition (3.1) is trivial in this case because

$$\hat{c} > -\hat{a} \quad \Longrightarrow \quad -c > -\omega^2 - c \quad \Longrightarrow \quad \omega^2 > 0$$

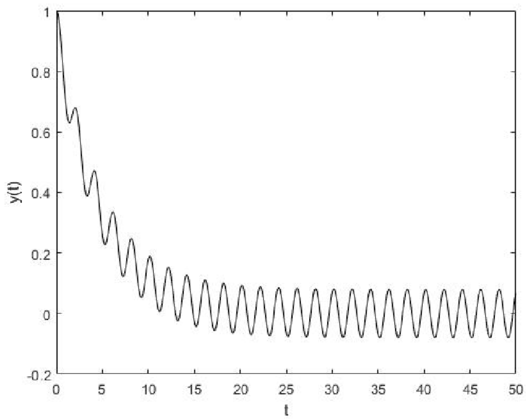
with $\hat{a}, \hat{c}$ from (3.21) holds always. Also the condition $c > -\omega^2$ is neglected because it is automatically satisfied by the inequalities above.



(a) $\omega = 1; \tau = 1; c = 0.5$



(b) $\omega = 1; \tau = 1; c = -0.3$



(c) $\omega = 1; \tau = 1; c = \frac{1}{2}\left(\left(\frac{\pi}{\tau}\right)^2 - \omega^2\right)$

Figure 3.11: Oscillation by (3.20) with initial conditions $y(t) = 1, \dot{y}(t) = 0$ for $-\tau \leq t \leq 0$

The regions of asymptotic stability are again similar to triangles. With growing ω, they are higher and denser.

3.3.2 Pyragas control on damped oscillators

Pyragas control can be used in a system of damped oscillation as well. This general system has the form

$$\begin{aligned} &\ddot{y}(t) + b\dot{y}(t) + \omega^2 y(t) + c\left(y(t) - y(t-\tau)\right) = 0, \quad t > 0 \\ &y(t) = \psi(t), \quad -\tau \leq t \leq 0, \end{aligned} \tag{3.25}$$

with $\psi(t)$ an initial function. We suppose b to be any real number except zero. The task of asymptotic stability will be answered by the algorithm from the section 3.2.2. Before its using, the delay differential equation of the damped oscillation with Pyragas control (3.25) must be converted to the form of (3.8). Thus, the equation

$$\ddot{y}(t) + b\dot{y}(t) + (\omega^2 + c)y(t) - cy(t-\tau) = 0$$

is obtained.

This equation gives the characteristic equation

$$\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + (\omega^2 + c)e^{\lambda t} + ce^{\lambda(t-\tau)} = 0.$$

The corresponding exponential polynomial is

$$H(\lambda) := \lambda^2 e^{\tau\lambda} + b\lambda e^{\tau\lambda} + (\omega^2 + c)e^{\tau\lambda} + c = 0.$$

Using the substitution

$$\lambda = \frac{s}{\tau}$$

as in the section 3.2.2, the exponential polynomial can be equivalently expressed by

$$\bar{H}(s) = s^2 e^s + b\tau s e^s + (\omega^2 + c)\tau^2 e^s - c\tau^2 = 0.$$

The functions $\bar{F}(y)$ and $\bar{G}(y)$ come from $\bar{H}(s)$ at the critical value $s = \mathrm{i}y$:

$$\begin{aligned}
\bar{H}(\mathrm{i}y) &= \bar{F}(y) + \mathrm{i}\bar{G}(y) \\
&= (\mathrm{i}y)^2 e^{\mathrm{i}y} + b\tau(\mathrm{i}y)e^{\mathrm{i}y} + (\omega^2 + c)\tau^2 e^{\mathrm{i}y} - c\tau^2 \\
&= -y^2(\cos(y) + \mathrm{i}\sin(y)) + \mathrm{i}b\tau y(\cos(y) + \mathrm{i}\sin(y)) + (\omega^2 + c)\tau^2(\cos(y) + \mathrm{i}\sin(y)) - c\tau^2
\end{aligned}$$

$$\bar{F}(y) = -y^2\cos(y) - b\tau y\sin(y) + (\omega^2 + c)\tau^2\cos(y) - c\tau^2 \tag{3.26}$$

$$\bar{G}(y) = -y^2\sin(y) + b\tau y\cos(y) + (\omega^2 + c)\tau^2\sin(y). \tag{3.27}$$

The above functions are necessary for using the results from [6]. A theorem from this paper states the necessary conditions for asymptotic stability of (3.25) with parameters satisfying

$$b \neq 0, \quad (\omega^2 + c)c > 0 \tag{3.28}$$

as

$$b > 0, \quad \omega^2 > 0. \tag{3.29}$$

The second inequality from the assumptions (3.28) holds if

$$\omega^2 + c > 0 \quad \wedge \quad c > 0 \quad \Longrightarrow \quad c > 0$$

or

$$\omega^2 + c < 0 \quad \wedge \quad c < 0 \quad \Longrightarrow \quad c < -\omega^2.$$

The first inequality from (3.29) allows the case of positive damped constant b only. This means that the system with (3.28) and negative friction can not by stabilised. The second inequality from 3.29 holds automatically.

The paper [6] gives the necessary sufficient conditions with proofs as well. The equations (3.25) with (3.28) is asymptotically stable if and only if (3.29) hold and

$$\begin{array}{lllll}
\text{(a)} & \text{if} \quad c > -\omega^2, & \text{then} & \bar{F}(r_{2k}) > 0 & k = 1, 2, \ldots \\
\text{(b)} & \text{if} \quad c < -\omega^2, & \text{then} & \bar{F}(r_{2k+1}) < 0 & k = 0, 1, 2, \ldots
\end{array} \tag{3.30}$$

where the function $\bar{F}(y)$ is defined in (3.26) and r_i are positive roots of the function $\bar{G}(y)$ defined in (3.27). Considering the requirement (3.28), the statement (3.30a) shall be written as the following:

$$\text{if} \quad c > 0, \quad \text{then} \quad \bar{F}(r_{2k}) > 0 \quad k = 1, 2, \ldots$$

The conditions (3.30) expect to check all roots of $\bar{G}(y)$. A problem comes with infinite number of roots of $\bar{G}(y)$. This conditions may be weakened. It has been proved in [6] that if

$$c > -\omega^2, \quad \bar{F}(r_{2k}) > 0, \quad k = 1, 2, \dots, n,$$

where

$$\frac{3}{2}\pi > [r_{2n}] > \pi \quad \text{and} \quad (-r_{2n}^2 + (\omega^2 + c)\tau^2)\cos(r_{2n}) - c\tau^2 > 0, \tag{3.31}$$

then

$$\bar{F}(r_{2k}) > 0$$

for all k. The function $[\alpha]$ is defined as the residue modulo 2π of a real number α $(0 \leq [\alpha] < 2\pi)$. Similarly for the statement (3.30b), if

$$c < -\omega^2, \quad \bar{F}(r_{2k+1}) < 0, \quad k = 1, 2, \dots, m,$$

where

$$(-r_{2m+1}^2 + (\omega^2 + c)\tau^2)\cos(r_{2m+1}) - c\tau^2 > 0, \tag{3.32}$$

then

$$\bar{F}(r_{2k+1}) < 0$$

for all k.This gives a significant simplification, because only finitely many positive roots of $\bar{G}(y)$ have to be considered.

As in the section 3.2.2, stability regions can not be drawn in any plane. For designing an asymptotic stable damped oscillator under the conditions derived in this section, one have to choose the parameters satisfying the assumptions (3.28) and the necessary conditions (3.29). With this particular choice of ω, b and c, the time delay τ must be taken with respect to the conditions (3.30). It is enough to consider finitely many roots. It holds for (3.30a) and (3.30b) from (3.31) and (3.32), respectively.

3.4 Comparison of used controls

In the previous sections, we have considered two basic forms of oscillators, namely the harmonic oscillator and the damped oscillator. To each of these mechanical systems, some types of delay feedback controls have been added. First, the control of the form

$$u(y(t)) = cy(t - \tau) \tag{3.33}$$

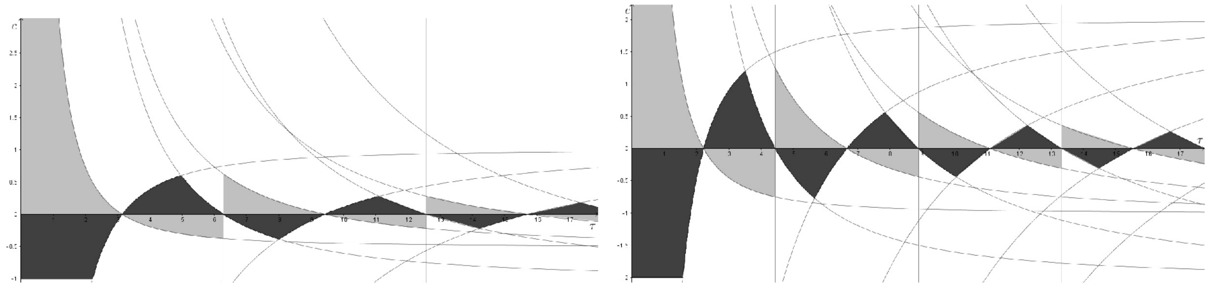
have been taken with the aim to find asymptotic stability's regions. Lately, the section 3.3 have been fully denoted to Pyragas control

$$u(y(t)) = c(y(t) - y(t - \tau)). \tag{3.34}$$

In the current section, we will put the results and see the difference of single controls.

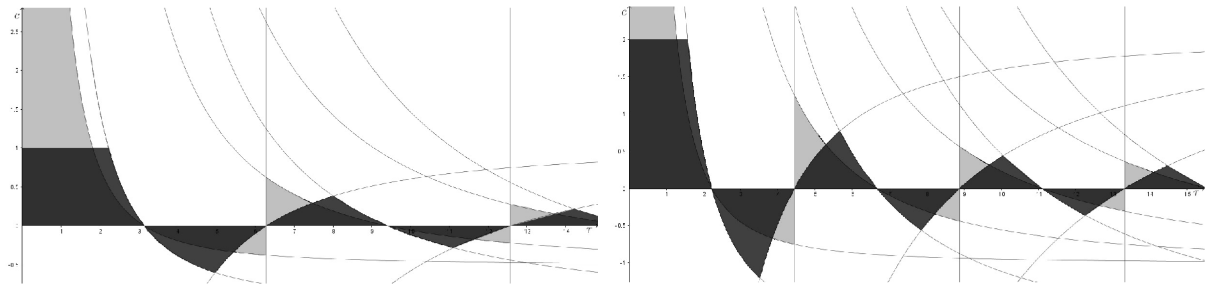
Considering harmonic oscillations (2.1), the control (3.33) gives the asymptotic stability conditions (3.4) and (3.6). Further, harmonic oscillations under the Pyragas control (3.34) is asymptotically stable if and only if (3.23) or (3.24) hold. All results are shown for particular choices of ω in the figure 3.12.

It is obvious from the figure, there are no common pairs τ, c which stabilise the harmonic oscillator with both controls. If one of the control stabilises the mechanical system



(a) $\omega = 1$ (b) $\omega = \sqrt{2}$

Figure 3.12: The regions of asymptotic stability for (2.1) with the controls (3.33) (dark grey) and (3.34) (light grey)



(a) $\omega = 1$ (b) $\omega = \sqrt{2}$

Figure 3.13: The regions of asymptotic stability for (2.1) with the controls (3.35) (dark grey) and (3.34) (light grey)

by positive control constant c in a particular interval of τ, the second control might be stabilised by a negative control constant and vice versa. Thus, any comparing of effects of each control is not possible and controls with different parameters are incomparable.

This might be managed by changing the sing of the control constant in one of controller. Doing it for (3.33), the controller becomes

$$u(y(t)) = -cy(t-\tau). \tag{3.35}$$

The conditions on asymptotic stability of a harmonic oscillator with this control can be easily transformed from the results in the section 3.1, namely the conditions (3.4) and (3.6). Thus, the DDE is asymptotically stable if and only if

$$\frac{2k\pi}{\omega} < \tau < \frac{(2k+1)\pi}{\omega}, \quad k \in \mathbb{N} \cup \{0\},$$
$$0 < c < \min\left\{\omega^2 - \left(\frac{2k\pi}{\tau}\right)^2, \left(\frac{(2k+1)\pi}{\tau}\right)^2 - \omega^2\right\}$$

or

$$\frac{(2k+1)\pi}{\omega} < \tau < \frac{2(k+1)\pi}{\omega}, \quad k \in \mathbb{N} \cup \{0\},$$
$$0 > c > \max\left\{\left(\frac{(2k+1)\pi}{\tau}\right)^2 - \omega^2, \omega^2 - \left(\frac{2(k+1)\pi}{\tau}\right)^2\right\}.$$

Looking at the figure 3.13, there is an intersection of the stability regions of the controls on the interval $\tau \in (k\pi/\omega, (k+1)\pi/\omega)$ for every $k \in \mathbb{N} \cup \{0\}$. The intersected region is given by the inequalities

$$\frac{2k\pi}{\omega} < \tau < \frac{(2k+1)\pi}{\omega}, \quad k \in \mathbb{N} \cup \{0\},$$
$$0 < c < \min\left\{\omega^2 - \left(\frac{2k\pi}{\tau}\right)^2, \frac{1}{2}\left[\left(\frac{(2k+1)\pi}{\tau}\right)^2 - \omega^2\right]\right\}$$

union

$$\frac{(2k+1)\pi}{\omega} < \tau < \frac{2(k+1)\pi}{\omega}, \quad k \in \mathbb{N} \cup \{0\},$$
$$0 > c > \max\left\{\omega^2 - \left(\frac{2(k+1)\pi}{\tau}\right)^2, \frac{1}{2}\left[\left(\frac{(2k+1)\pi}{\tau}\right)^2 - \omega^2\right]\right\}.$$

The boundary curves

$$c_k = \omega^2 - \left(\frac{2k\pi}{\tau}\right)^2 \tag{3.36}$$

come from the boundary of the system controlled by (3.35) and the boundaries

$$c_k = \frac{1}{2}\left[\left(\frac{(2k+1)\pi}{\tau}\right)^2 - \omega^2\right] \tag{3.37}$$

come from asymptotic stability of the harmonic oscillator with the control (3.34).

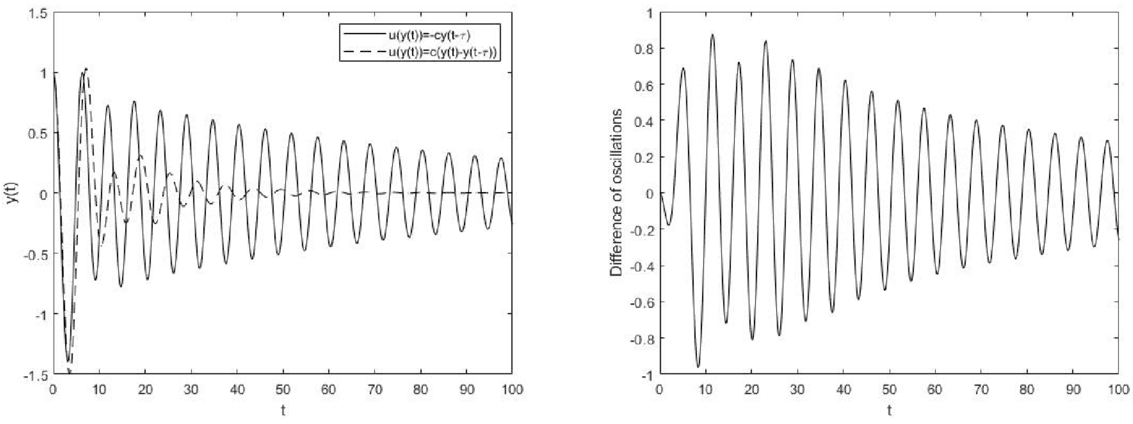
An intuitive question is which control is better (in sense of asymptotic stabilisation's speed) in the region where both systems are tending to zero. The answer strictly depends on the particular choice of the control parameters τ and c. The following breakdown holds on every interval $\tau \in (k\pi/\omega, (k+1)\pi/\omega)$.

Fix the control constant c such that there exist some $\bar{\tau}$ for which the systems with different controls are asymptotically stable. These $\bar{\tau}$ are subintervals of some $\tau \in (k\pi/\omega, (k+1)\pi/\omega)$ and each of the subintervals is bounded by values satisfying either (3.36) or (3.37). If one chooses $\bar{\tau}$ close to the boundary (3.36), the system with the control (3.35) tends to the zero slowly since it is close to the τ's value when the system would be (just) stable, i.e. the value on the curve (3.36). With this particular choice of $\bar{\tau}$ and c, the Pyragas control (3.34) is more effective. See the figure 3.14.

On the other hand, if $\bar{\tau}$ is chosen close to the boundary (3.37) with the same fixed c, the system with the control (3.34) tends to the zero slowly. This is going on because the chosen $\bar{\tau}$ is close to the τ's value when the system is controlled as a stable system. However, the system with the control (3.35) is farther from its boundary. Hence, the control (3.35) acts on the system more strongly. This situation is drawn in the figure 3.15.

If $\bar{\tau}$ is chosen from the middle of a subinterval, both controls are equivalent approximately (see the figure 3.16). Shifting $\bar{\tau}$ to the boundary (3.36), the system with (3.35) gets weaker. Similarly, $\bar{\tau}$ shifted to the boundary (3.37) makes the system with (3.34) weaker.

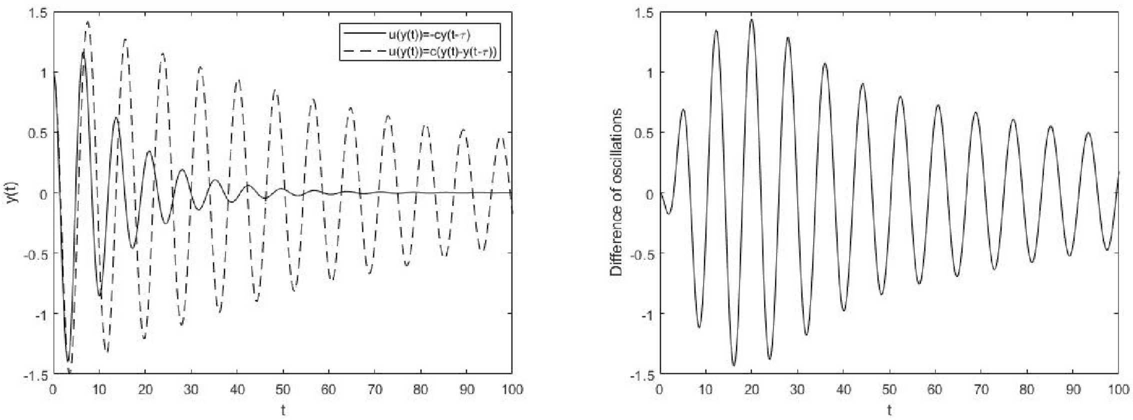
Now fix the delay τ such that $\tau \neq k\pi/\omega$ for $k = 0, 1, 2, \ldots$. Clearly, there is an interval of c asymptotically stabilising the harmonic oscillator with both the controls. The effects



(a) Oscillations under (3.35) and (3.34)

(b) The difference of oscillations

Figure 3.14: The comparison of harmonic oscillations with $\omega = 1$, $c = -0.2$, $\tau = 5.6$ and initial conditions $y(t) = 1, \dot{y}(t) = 0$ for $-\tau \leq t \leq 0$



(a) Oscillations under (3.35) and (3.34)

(b) The difference of oscillations

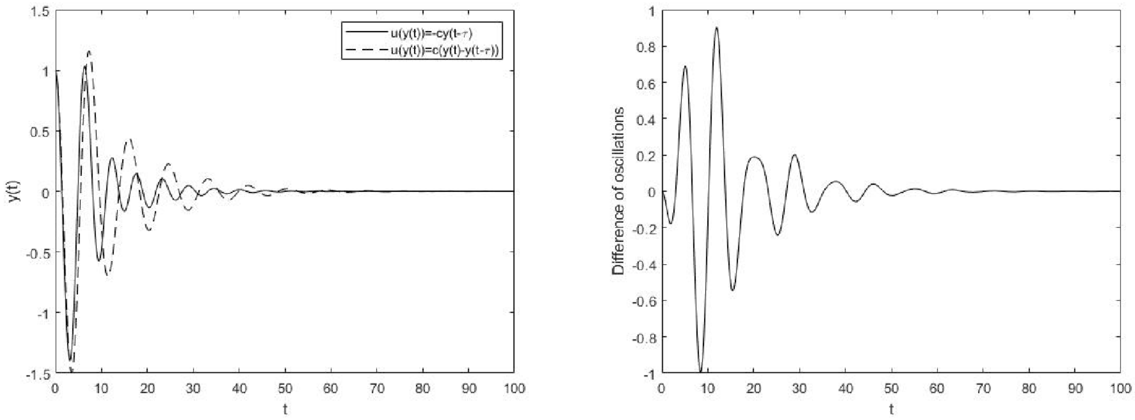
Figure 3.15: The comparison of harmonic oscillations with $\omega = 1$, $c = -0.2$, $\tau = 4.2$ and initial conditions $y(t) = 1, \dot{y}(t) = 0$ for $-\tau \leq t \leq 0$

of the controls will be described by the use of the paragraph above. Choosing a particular $\bar{c}$ from the asymptotic stable interval, we look at the fixed τ. If τ is closer to (3.36) than to (3.37), the control (3.34) is more effective and vice versa.

Suppose τ and c are chosen at the intersection of the boundary curves (3.36) and (3.37). Both controls make the system stable (see 3.17a). In general, the oscillations have different amplitudes. Also the frequencies and periods are different but the frequency of one system is a rational multiple of the second one. The same holds for the periods, too. Both facts are visible in 3.17b.

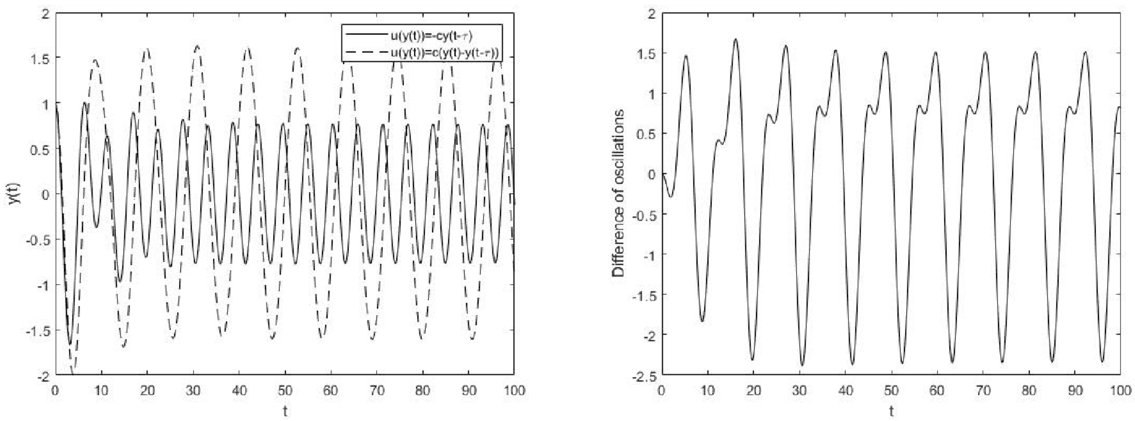
Considering the damped oscillator (2.2) and the same controls (3.33) and (3.34), the comparison is not easy as in the harmonic case. Any stability regions for the necessary sufficient conditions have not been obtained. Even if we would find some parameters asymptotically stabilising the system for both controls, we will not be able to see how far are these parameters from states of stability.

The next problem comes right from the assumptions in the sections 3.2.2 and 3.3.2. Recall, one the assumptions for the damped oscillations with (3.33) has been $c < 0$.



(a) Oscillations under (3.35) and (3.34) (b) The difference of oscillations

Figure 3.16: The comparison of harmonic oscillations with $\omega = 1$, $c = -0.2$, $\tau = 4.9$ and initial conditions $y(t) = 1, \dot{y}(t) = 0$ for $-\tau \leq t \leq 0$



(a) Oscillations under (3.35) and (3.34) (b) The difference of oscillations

Figure 3.17: The comparison of harmonic oscillations with $\omega = 1$, $c = -\frac{1}{2}$, $\tau = \pi\sqrt{3}$ and initial conditions $y(t) = 1, \dot{y}(t) = 0$ for $-\tau \leq t \leq 0$

Moreover, one of the conditions has limited this interval to $0 > c > -\omega^2$. On the other hand, the results from the section 3.3.2 give the condition for $c > 0$ or $c < -\omega^2$. Clearly, there is no intersection of control constants' possible intervals for our controls. One would get over this problem by a "smart substitution" as in the similar situation with the harmonic oscillator above in the current section.

The next parts of the results deal with the functions $\bar{F}(y), \bar{G}(y)$ where we may not compare the system. The functions are different and they also give different sings for some particular values. Thus, if we would set damped controlled systems with intersecting assumptions and conditions on the control constant, we will not be able to compare the control effect without a numerical solution.

In both sections 3.2.2 and 3.3.2, the case of the negative friction $b < 0$ remains as an open problem. We know that the damped system where $b < 0$ with the control (3.33) can not be stabilised with $c < 0$, and with the control (3.34), it can not be stabilised with $c > 0$ or $c < -\omega^2$.

This problem could not be solved even by the algorithm from the section 3.2.1. Here, only necessary conditions for $b < 0$ have been obtained. Similar results might be obtained for the damped oscillator with the control (3.34). One has to eliminate the element of the friction by a similar substitution as in the section 3.2.1. Using the algorithm from the section 3.3.1, stability regions for the substituting function would be obtained. As it has been discussed in the end of the section 3.2.1, this does not guarantee necessary sufficient conditions on our original equation.

4 Feedback delay control for systems in matrix forms

In the following two chapters, systems of feedback DDEs will be studied. The system may be considered from the different point of view. The first is a mathematical point and the second is a mechanical.

The base of the mathematical system of the oscillation is the fact that every linear ODE of order n can be written as a system of n ODEs of order 1. The same may be done with DDEs, too. Clearly, if the same types of oscillators would be considered, the result will be equivalent to those from the chapter 3. Thus, different types of the control will be shown. Although the whole control will be different from the already studied cases, controls considered in a single equation of the system will be similar to the known controls form chapter 3.

The system of oscillators from a mechanical point of view is a group of, in general, n connected oscillators. This connection is called coupling. A coupled oscillator typically influences other oscillators in the system. This will be studied in the chapter 5.

For a transformation of delay systems to matrix forms, the wildly used trick from the control theory will be introduce. Most of theorems and conclusions from the control theory need systems in shape of first derivatives to each changing state [8]. In general, these systems have a form

$$\dot{\mathbf{y}}(t) = A\mathbf{y}(t) + B\mathbf{u}(t) \tag{4.1}$$

where $A \in \mathbb{R}^{n\times n}$ shows how a state of the oscillator depends on other states, $B \in \mathbb{R}^{n\times n}$ is a gain matrix showing how the states are controlled. The symbol $\dot{\mathbf{y}}(t)$ denotes a vector of the first derivative of each state $y_i(t), i = 1, \ldots, n$, with respect to time t.

In case of the feedback delay control of the harmonic oscillator as in the section 3.1, the original equation

$$\ddot{y}(t) + \omega^2 y(t) + cy(t-\tau) = 0$$

may be rewritten in the system form as

$$\begin{aligned} \dot{y_1}(t) &= y_2(t) \\ \dot{y_2}(t) &= -\omega^2 y_1(t) - cy_1(t-\tau) \end{aligned}$$

where $y_1(t)$ is the position of the point mass and $y_2(t)$ its velocity. If the notation from (4.1) is kept, 2 dimensional case is obtained with

$$A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ -c & 0 \end{pmatrix}, \quad \mathbf{u}(t) = \begin{pmatrix} y_1(t-\tau) \\ y_2(t-\tau) \end{pmatrix}.$$

This particular system has been completely studied in the previous chapter. The aim of this chapter is to find conditions on A, B of harmonic oscillations with feedback delay controls to stabilise the system.

4.1 Conditions on stability of controlled harmonic oscillators in matrix forms

Consider the general system (4.1). Conditions for asymptotic stability are similar to the case of linear DDEs. The zero solution is asymptotically stable if and only if all roots λ of the characteristic equation (exponential polynomial)

$$\det\left(\lambda I - A - Be^{-\tau\lambda}\right) = 0 \tag{4.2}$$

have negative real parts. This problem is very complicated in general. However, it becomes easier if matrices A, B are in some special forms.

First, we introduce an auxiliary motion. Matrices are said to be simultaneously triangularizable if there exists a basis which transfers matrices to an upper triangular form. So A and B are simultaneously triangularizable if there exists a matrix $P \in \mathbb{R}^{n\times n}$ such that both PAP^{-1} and PBP^{-1} are upper triangular matrices.

If A, B are simultaneously triangularizable, the problem of determining all real parts of roots of (4.2) turns to determine roots λ of

$$\prod_{i=1}^{n}\left(\lambda - \alpha_i - \beta_i e^{-\tau\lambda}\right) = 0$$

where α_i and β_i are ordered diagonal elements of PAP^{-1} and PBP^{-1}, respectively, for some P such that both PAP^{-1} and PBP^{-1} are upper triangular matrices. Coefficients α_i and β_i are also equal to eigenvalues of A and B but the order of pairs $(\alpha_i, \beta_i), i = 1, \ldots, n$ is important here. Roots with negative real parts of this equation are still required for stability. In other words, roots of

$$\lambda - \alpha - \beta e^{-\tau\lambda} = 0 \tag{4.3}$$

must have a negative real part for all pairs (α, β).

Conditions for negative real parts of all roots λ of (4.3) are given in [9]. Before stating these conditions, some additional quantities have to be set. They are

$$\begin{aligned}
\tau^* &= \frac{-\pi + |\arg(\beta)| + \arccos\left(\frac{\mathcal{R}(\alpha)}{|\beta|}\right)}{|\mathcal{I}(\alpha)| + \sqrt{|\beta|^2 - (\mathcal{R}(\alpha))^2}},\\
\tau_j^{\pm} &= \frac{(2j+1)\pi + \operatorname{sgn}(\mathcal{I}(\alpha))\arg(\beta) \pm \arccos\left(\frac{\mathcal{R}(\alpha)}{|\beta|}\right)}{|\mathcal{I}(\alpha)| \pm \sqrt{|\beta|^2 - (\mathcal{R}(\alpha))^2}},\\
D &= \frac{|\mathcal{I}(\alpha)|\arccos\left(\frac{\mathcal{R}(\alpha)}{|\beta|}\right)}{2\pi\sqrt{|\beta|^2 - (\mathcal{R}(\alpha))^2}} - \frac{\operatorname{sgn}(\mathcal{I}(\alpha))\arg(\beta)}{2\pi} - \frac{1}{2}.
\end{aligned} \tag{4.4}$$

The proposition (with a proof) from [9] giving conditions for negative real parts of all roots of (4.3) says that all roots have a negative real part if and only if at least one of the following holds:

(i) $\mathcal{R}(\alpha) + |\beta| < 0, \tau$ arbitrary

(ii) $\mathcal{R}(\alpha) + |\beta| = 0 \wedge \beta \neq 0 \wedge \tau\mathcal{I}(\alpha) - \arg(\beta) \neq 2l\pi, l \in \mathbb{Z}$

(iii) $|\mathcal{R}(\alpha)| - |\beta| < 0 \wedge \mathcal{R}(\alpha) + \mathcal{R}(\beta) < 0 \wedge \mathcal{I}(\alpha)\arg(\beta) \geq 0 \wedge$
$\wedge \big(0 < \tau < \tau^* \vee (D > 0 \wedge \tau_j^- < \tau < \tau_j^+, j = 0, 1, \ldots, \lceil D \rceil - 1)\big)$

(iv) $|\mathcal{R}(\alpha)| - |\beta| < 0 \wedge \mathcal{R}(\alpha) + \mathcal{R}(\beta) < 0 \wedge \mathcal{I}(\alpha)\arg(\beta) < 0 \wedge -1 < D < 0 \wedge 0 < \tau < \tau_0^-$

(v) $|\mathcal{R}(\alpha)| - |\beta| < 0 \wedge \mathcal{R}(\alpha) + \mathcal{R}(\beta) < 0 \wedge \mathcal{I}(\alpha)\arg(\beta) < 0 \wedge D \geq 0 \wedge$
$\wedge \big(0 < \tau < \tau_0^+ \vee (D > 1 \wedge \tau_j^- < \tau < \tau_j^+, j = 1, 2, \ldots, \lceil D \rceil - 1)\big)$

(vi) $|\mathcal{R}(\alpha)| - |\beta| < 0 \wedge \mathcal{R}(\alpha) + \mathcal{R}(\beta) \geq 0 \wedge D > 0 \wedge \tau_j^- < \tau < \tau_j^+, j = 0, 1, \ldots, \lceil D \rceil - 1.$

In this thesis, it will be enough to consider commutative matrices. Recall, A and B are commutative matrices if $AB = BA$. As a connection between simultaneously triangularizable matrices and commuting matrices, a theorem from [10] is used saying that commuting matrices are also simultaneously triangularizable.

A special case of commuting matrices A, B is when B is a diagonal matrix with a constant on the diagonal, i,e.

$$B = kI$$

where $I \in \mathbb{R}^{n \times n}$ in the identity matrix and $k \in \mathbb{R}$. It is easy to show that these A, B are commuting, since

$$AB = AkI = kAI = kA = IkA = BA.$$

The matrices are also commutative if $A = kI$.

With such a gain matrix, the above conditions can be simplified. The reason is that $B = kI$ has n real eigenvalues $\beta = k$. Also $k \neq 0$ for controlling the system. The simplifications come from facts that for relations

$$|\beta| = -\beta, \quad \arg(\beta) = \pi \quad \text{for} \quad \beta < 0$$

and

$$|\beta| = \beta, \quad \arg(\beta) = 0 \quad \text{for} \quad \beta > 0$$

and also some relations between α and β can be simplified in the conditions with this particular choice of the gain matrix.

Thus, the conditions for negativity of real parts of all roots of (4.3) with the gain matrix $B = kI$ are the following:

$$\begin{array}{rl}
\text{(i)} & \mathcal{R}(\alpha) + |\beta| < 0, \tau \text{arbitrary} \\
\text{(ii)} & \mathcal{R}(\alpha) + |\beta| = 0 \wedge \tau\mathcal{I}(\alpha) - \arg(\beta) \neq 2l\pi, l \in \mathbb{Z} \\
\text{(iii)} & \beta < 0 \wedge |\mathcal{R}(\alpha)| + \beta < 0 \wedge \mathcal{I}(\alpha) \geq 0 \wedge (0 < \tau < \tau^* \vee (D > 0 \wedge \tau_j^- < \tau < \tau_j^+, \\
& j = 0, 1, \ldots, \lceil D \rceil - 1)) \\
\text{(iv)} & \beta < 0 \wedge |\mathcal{R}(\alpha)| + \beta < 0 \wedge \mathcal{I}(\alpha) < 0 \wedge -1 < D < 0 \wedge 0 < \tau < \tau_0^- \\
\text{(v)} & \beta < 0 \wedge |\mathcal{R}(\alpha)| + \beta < 0 \wedge \mathcal{I}(\alpha) < 0 \wedge D \geq 0 \wedge (0 < \tau < \tau_0^+ \vee \\
& \vee (D > 1 \wedge \tau_j^- < \tau < \tau_j^+, j = 1, 2, \ldots, \lceil D \rceil - 1)) \\
\text{(vi)} & \beta > 0 \wedge |\mathcal{R}(\alpha)| - \beta < 0 \wedge D > 0 \wedge \tau_j^- < \tau < \tau_j^+, j = 0, 1, \ldots, \lceil D \rceil - 1.
\end{array} \tag{$\star$}$$

If one of these conditions denoted by ($\star$) holds for a pair $(\hat{\alpha}, \hat{\beta})$, then (4.3) for this pair $(\hat{\alpha}, \hat{\beta})$ has all root with negative real parts.

Moreover, the conditions may be more simplified for a particular system of (4.1). As the particular system, we take

$$\begin{aligned} \dot{y_1}(t) &= \bar{\alpha} y_2(t) - \bar{\gamma} y_1(t-\tau) \\ \dot{y_2}(t) &= \bar{\beta} y_1(t) - \bar{\gamma} y_2(t-\tau), \end{aligned} \tag{4.5}$$

where $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ are real constants. Equivalently in the matrix form,

$$\dot{\mathbf{y}}(t) = \begin{pmatrix} 0 & \bar{\alpha} \\ \bar{\beta} & 0 \end{pmatrix} \mathbf{y}(t) + \begin{pmatrix} -\bar{\gamma} & 0 \\ 0 & -\bar{\gamma} \end{pmatrix} \mathbf{y}(t-\tau).$$

It has been proved in [11] that the system (4.5) is asymptotically stable if and only if any of the following holds:

$$\begin{aligned} &\text{(i)} \quad \sqrt{\bar{\alpha}\bar{\beta}} < \bar{\gamma} \quad \wedge \quad 0 < \tau < \frac{1}{\sqrt{\bar{\gamma}^2 - \bar{\alpha}\bar{\beta}}} \arccos \frac{\sqrt{\bar{\alpha}\bar{\beta}}}{\bar{\gamma}} \\ &\text{(ii)} \quad \sqrt{-\bar{\alpha}\bar{\beta}} \leq 4\bar{\gamma} \quad \wedge \quad 0 < \tau < \frac{\pi}{2\left(\bar{\gamma} + \sqrt{-\bar{\alpha}\bar{\beta}}\right)} \\ &\text{(iii)} \quad 0 < 4\bar{\gamma} < \sqrt{-\bar{\alpha}\bar{\beta}} \quad \wedge \quad \tau \in (0, \tau_{1,0}) \cup \bigcup_{i=1}^{k} (\tau_{2,i-1}, \tau_{1,i}) \\ &\text{(iv)} \quad -\sqrt{-\bar{\alpha}\bar{\beta}} < 2\bar{\gamma} < 0 \quad \wedge \quad \tau \in \bigcup_{i=0}^{l} (\tau_{1,i}, \tau_{2,i}) \end{aligned} \tag{$\star\star$}$$

where

$$\tau_{1,n} = \frac{(4n+1)\pi}{2\left(\bar{\gamma} + \sqrt{-\bar{\alpha}\bar{\beta}}\right)}, \quad \tau_{2,n} = \frac{(4n+3)\pi}{2\left(-\bar{\gamma} + \sqrt{-\bar{\alpha}\bar{\beta}}\right)}, \quad n = 0, 1, 2, \dots,$$

$$k = \left\lceil \frac{\sqrt{-\bar{\alpha}\bar{\beta}}}{4\bar{\gamma}} \right\rceil, \quad l = \left\lceil -\frac{\sqrt{-\bar{\alpha}\bar{\beta}}}{4\bar{\gamma}} - \frac{1}{2} \right\rceil.$$

Denote this conditions by ($\star\star$). Considering the system (4.5), the conditions ($\star$) are equivalent to the conditions ($\star\star$) since [9] takes ($\star$) as a generalisation of ($\star\star$) from [11].

4.2 Two-dimensional system with a constant diagonal gain matrix

Consider the system

$$\begin{aligned} \dot{y_1}(t) &= y_2(t) + c y_1(t-\tau) \\ \dot{y_2}(t) &= -\omega^2 y_1(t) + c y_2(t-\tau) \end{aligned} \tag{4.6}$$

where $\omega, \tau \in \mathbb{R}^+$ and $c \in \mathbb{R}$. In sense of (4.1), we set

$$A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, \quad \mathbf{u}(t) = \begin{pmatrix} y_1(t-\tau) \\ y_2(t-\tau) \end{pmatrix}.$$

There are two possibilities to determine asymptotic stability. Since $B = cI$, matrices A and B are commuting, and also simultaneously triangularizable, the conditions ($\star$) can be used. The eigenvalues of both A and B are required. The eigenvalues of A are

$$\alpha = \pm i\omega.$$

For B, eigenvalues are equivalent and both are

$$\beta = c.$$

Now every combination of α and β would be studied. The system will be asymptotically stable for some parameters if some conditions from ($\star$) hold for every pair (α, β) with the chosen parameters. It is necessary to split the solution into cases with a positive and a negative c.

The easier solution is to use conditions ($\star\star$). One can simply check that the system (4.6) is equivalent to (4.5) with

$$\bar{\alpha} = 1, \quad \bar{\beta} = -\omega^2, \quad \bar{\gamma} = -c.$$

Note, the product of $\bar{\alpha}$ and $\bar{\beta}$ is negative. This means, the condition (i) from ($\star\star$) may not be used for the system (4.6).

Computation of $\tau_{1,n}$ and $\tau_{2,n}$ is necessary:

$$\tau_{1,n} = \frac{(4n+1)\pi}{2\,(\omega - c)}, \quad \tau_{2,n} = \frac{(4n+3)\pi}{2\,(\omega + c)}, \quad n = 0, 1, 2, \dots$$

Consider the negative control constant $c < 0$. The parts (ii) and (iii) from ($\star\star$) can be used.

Taking (ii), the system (4.6) is asymptotically stable if and only if

$$\begin{aligned} c &\leq -\frac{\omega}{4} \\ 0 < \tau &< \frac{\pi}{2(\omega - c)}. \end{aligned}$$

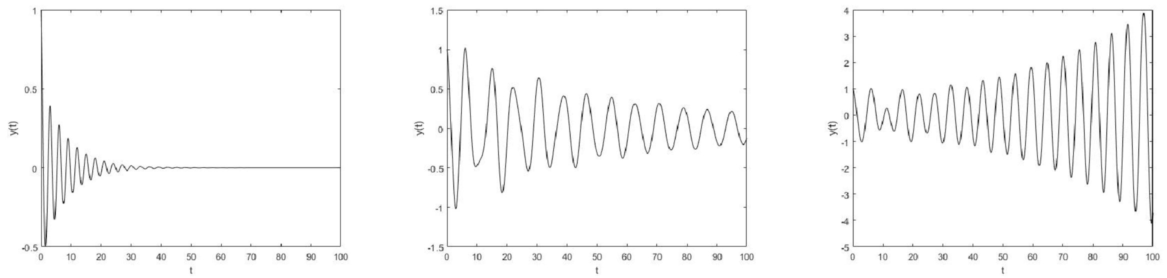
For any c satisfying the first condition, there exist a time delay such that (4.6) is stable. The interval of possible choices for τ gets smaller with decreasing c.

Taking (iii), the system (4.6) is asymptotically stable if and only if

$$0 > c > -\frac{\omega}{4}$$

$$0 < \tau < \frac{\pi}{2(\omega - c)} \quad \text{or} \quad \frac{(4(i-1)+3)\pi}{2(c+\omega)} < \tau < \frac{(4i+1)\pi}{2(\omega - c)}, \quad i = 1, 2, \dots, \left\lceil \frac{\omega}{-4c} \right\rceil.$$

Here for greater c, there are more possibilities for τ because more indexes i may be chosen. Anyway for any c satisfying the first condition, there are some τ such that (4.6)



(a) $\omega = 1; \tau = 0.7; c = -1$ (b) $\omega = 1; \tau = 6.1; c = -0.2$ (c) $\omega = 1; \tau = 6.8; c = -0.2$

Figure 4.1: Oscillation by (4.6) with a negative control constant and initial conditions $y(t) = 1, \dot{y}(t) = 0$ for $-\tau \leq t \leq 0$

is asymptotically stable. Together with conclusions from (ii), we may find a delay time τ, possibly very small, such that the system (4.6) is asymptotically stable for any negative $c < 0$.

For the positive control constant $c > 0$, there is just one possible statement from ($\star\star$), namely the statement (iv). This gives the conditions

$$0 < c < \frac{\omega}{2}$$
$$\frac{(4i+1)\pi}{2(\omega - c)} < \tau < \frac{(4i+3)\pi}{2(\omega + c)}, \quad i = 0, 1, 2, \ldots, \left\lceil \frac{\omega}{4c} - \frac{1}{2} \right\rceil.$$

There are always some τs for which the system (4.6) is asymptotically stable with any positive control constant $c > 0$ satisfying the first condition. For smaller c, there are more possibilities for τ stabilising the controlled system.

Putting together both negative and positive cases, the system (4.6) may be asymptotically stable for any $c < \frac{\omega}{2}$ except $c = 0$. The control's time delay must satisfy any condition derived above for a particular choice of c.

As a brief remark, the conditions ($\star$) applied on the system (4.6) give the same results. For a positive control constant $c > 0$, the statement (ii) from ($\star$) the conditions on τ and c. The case of a negative $c < 0$ would be more complicated, because one will have to compose results by statements (iii) (v) from ($\star$).

4.3 Two-dimensional system with Pyragas control

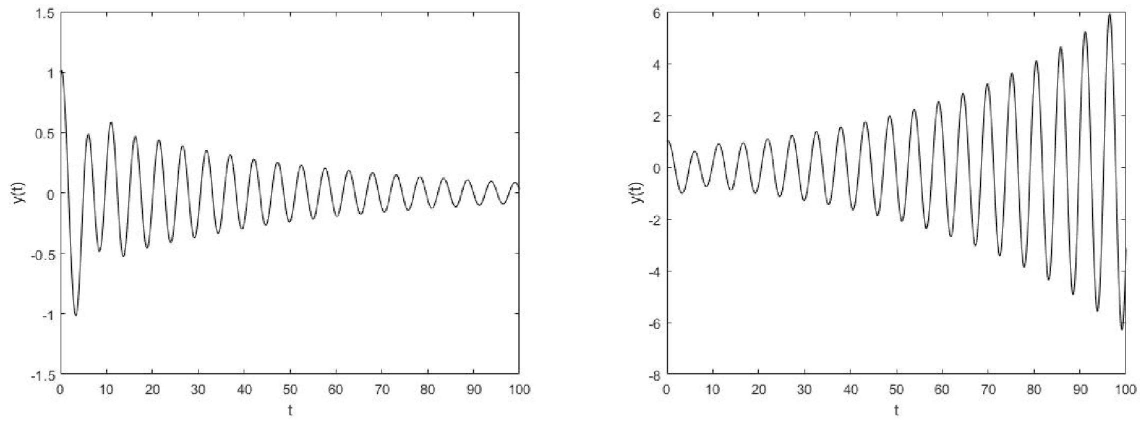
Recall, Pyragas control has the form

$$u(t) = c\,(y(t) - y(t - \tau))\,.$$

To make the procedure easier, the controller will be designed to be as similar as possible to the diagonal gain matrix from the previous section.

Consider the system

$$\begin{aligned} \dot{y_1}(t) &= y_2(t) + c(y_1(t) - y_1(t-\tau)) \\ \dot{y_2}(t) &= -\omega^2 y_1(t) + c(y_2(t) - y_2(t-\tau)). \end{aligned} \tag{4.7}$$



(a) $\omega = 1; \tau = 3.8; c = 0.2$ (b) $\omega = 1; \tau = 4.1; c = 0.2$

Figure 4.2: Oscillation by (4.6) with a positive control constant and initial conditions $y(t) = 1, \dot{y}(t) = 0$ for $-\tau \leq t \leq 0$

This may be equivalently rewritten as

$$\begin{aligned} \dot{y_1}(t) &= y_2(t) + cy_1(t) - cy_1(t-\tau) \\ \dot{y_2}(t) &= -\omega^2 y_1(t) + cy_2(t) - cy_2(t-\tau). \end{aligned}$$

From this formulation, it can be written in the matrix form as (4.1). Thus,

$$A = \begin{pmatrix} c & 1 \\ -\omega^2 & c \end{pmatrix}, \quad B = \begin{pmatrix} -c & 0 \\ 0 & -c \end{pmatrix}, \quad \mathbf{u}(t) = \begin{pmatrix} y_1(t-\tau) \\ y_2(t-\tau) \end{pmatrix}.$$

The conditions ($\star\star$) may not be used because it is not possible convert this system to the form of (4.5). However, the matrix B may be written as $B = -cI$. This means that the matrices A and B are commuting. Thus, the conditions ($\star$) may be used.

For doing this, the eigenvalues of both A and B must be computed. The eigenvalues of A are

$$\alpha = c \pm \mathrm{i}\omega.$$

The matrix B has the double eigenvalue

$$\beta = -c.$$

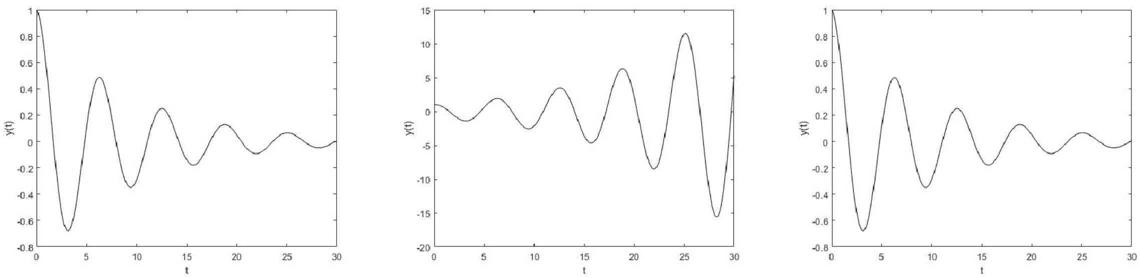
Surprisingly, the case of the positive control constant $c > 0$ does not fit to any partial conditions ($\star$). This case will be considered as unstable for any choice ω, τ. Also the case $c = 0$ will not be considered because there would be no control and the system (4.7) would become the harmonic oscillator.

Now set the control constant c negative. In this case,

$$\beta > 0, \quad \mathcal{R}(\alpha) + |\beta| = 0 \quad \text{and} \quad |\mathcal{R}(\alpha)| - \beta = 0.$$

Thus, the only possibility how to use the conditions ($\star$) is to take the statement (ii). This says that the equation (4.3) for the eigenvalues pair $(-\mathrm{i}\omega, -c)$ have all roots with a negative real part if and only if

$$\tau \neq -\frac{2l_-\pi}{\omega}, \quad l_- \in \mathbb{Z}. \tag{4.8}$$



(a) $\omega = 1; \tau = 1; c = -0.1$ (b) $\omega = 1; \tau = 1; c = 0.1$ (c) $\omega = -1; \tau = 1; c = -0.1$

Figure 4.3: Oscillation by (4.7) with initial conditions $y(t) = 1, \dot{y}(t) = 0$ for $-\tau \leq t \leq 0$

Similarly, it holds for the pair $(\mathrm{i}\omega, -c)$ if and only if

$$\tau \neq \frac{2l_+\pi}{\omega}, \quad l_+ \in \mathbb{Z}. \tag{4.9}$$

Clearly, if we set $l_- = -l_+$, the conditions (4.8) and (4.9) become identical as

$$\tau \neq \frac{2l\pi}{\omega}, \quad l \in \mathbb{Z}$$

for both eigenvalues pairs $(-\mathrm{i}\omega, \tau)$ and $(\mathrm{i}\omega, \tau)$.

There are no other restrictions on ω and c except the basic assumption $\omega > 0$. Moreover, the positiveness of ω is not necessary here. With $\omega < 0$, the system (4.7) would lose the character of a mechanical oscillator, but the conditions $(\star)$ set such a system as a asymptotically stable system. The only restriction on ω is $\omega \neq 0$ because no statements of the conditions $(\star)$ would be fulfilled for any τ. As the conclusion of this section, the system (4.7) is asymptotically stable if and only if

$$\omega \neq 0, \quad c < 0$$

and $\tau > 0$ except countably many values

$$\frac{2l\pi}{\omega}, \quad l \in \mathbb{Z}.$$

In the end of this section, consider a general system of n differential equations controlled by Pyragas controller, i.e.

$$\dot{\mathbf{y}}(t) = \mathbf{f}(\mathbf{y}(t)) + B(\mathbf{y}(t) - \mathbf{y}(t - \tau)) \tag{4.10}$$

The uncontrolled system has the form

$$\dot{\mathbf{y}}(t) = \mathbf{f}(\mathbf{y}(t)) = A(t)\mathbf{y}(t)$$

where $A(t)$ is a general matrix of continuous functions. This system is widely studied in Floquet theory. One can compute monodromy matrix's characteristic multipliers of the matrix $A(t)$. This also deals with a fundamental matrix of the system (see [12]). They are called Floquet multipliers.

Although the system (4.10) is too general and complicated to be solved, there exist a powerful theorem helping to determine a particular case. The theorem called "Odd

Number Limitation Theorem" has been stated in [13] by Japanese engineer Hiroyuki Nakajima. It says that if the uncontrolled system (4.10) with $K = 0$ has an odd number of real Floquet multipliers greater than unity, the unstable periodic orbit can never be stabilised by a time-delay feedback control of Pyragas form with any values of the gain matrix [13].

This statement gives at least a particular case when the system (4.10) can not be stabilised for any combinations c, τ. The described system has been studied in [14] as well. Here has been proved, that Odd Number Limitation Theorem does not hold in general. For states close to a bifurcation point, the system can by stabilised by a proper type of Pyragas control. Furthermore, it is possible to stabilise the system by other types of controllers.

5 Synchronisation by feedback delay controls

The system of oscillators from a mechanical point of view is a group of, in general, n connected oscillators. Every oscillator is connected with at least one other element of the system. This connection is called coupling. The coupled oscillator influences, and also is influenced, by other coupled oscillators of the system. Since this thesis focuses on a delay control, the coupling will be done by at least one delayed state. The controls will be shown for coupled harmonic oscillators.

The DDE of the general i-th element of the system can be written as

$$\ddot{y}_i(t) + \omega_i^2 y_i(t) + u_i(y_1(t-\tau_1), y_2(t-\tau_2), \dots, y_n(t-\tau_n)) = 0$$

where u_i is a delayed control function which does not have to depend on all elements of the system. In general, delays τ_j may be different. If a particular delay would be zero, the corresponding state will influence the system by its motion in actual time. We simplify the following problem by the assumption that the delays are equal, i.e. $\tau_j = \tau > 0$ for each $j = 1, 2, \dots, n$.

We say that elements are synchronised if there is a common behaviour of elements' characteristics. Mathematically, the goal of the synchronisation for mechanical oscillators by [15] is

$$\lim_{t\to\infty} |y_i(t) - y_j(t)| = 0, \quad \lim_{t\to\infty} |\dot{y}_i(t) - \dot{y}_j(t)| = 0$$

for all $i, j = 1, 2, \dots, n$. Thus, coupled oscillators are synchronised if the differences between positions and velocities of coupled elements become zero for time t goes to infinity.

In the following sections, three types of controllers will be studied. Two coupled oscillators will create the synchronising system. Note, the design of partial elements will be simplified in sense of constants. We will assume that the frequency ω, the control constant c and also the time delay τ are equal for both partials of the synchronised system. In general, the constants may be different but things become more complicated. Due to these simplifications, the results from the previous chapters can be used.

5.1 Control by a difference of delayed states

For a general control by a difference of delayed states in a system of n oscillators, the i-th controller u_i is designed as

$$u_i(t) = \sum_{\substack{k=1 \\ k\neq i}}^{n} c_{ik} \left(y_i(t-\tau) - y_k(t-\tau) \right).$$

It has been said that the case of two coupled oscillators will be considered, i.e.

$$\begin{aligned} \ddot{y}_1(t) + \omega_1^2 y_1(t) + c_1 \left(y_1(t-\tau) - y_2(t-\tau) \right) &= 0 \\ \ddot{y}_2(t) + \omega_2^2 y_2(t) + c_2 \left(y_2(t-\tau) - y_1(t-\tau) \right) &= 0 \end{aligned}$$

is the general studied system of this section. This system will be more simplified by the identity of the constants $\omega_1 = \omega_2 = \omega$ and $c_1 = c_2 = c$. Thus, the system being worked on is

$$\ddot{y_1}(t) + \omega^2 y_1(t) + c\,(y_1(t-\tau) - y_2(t-\tau)) = 0 \tag{5.1}$$
$$\ddot{y_2}(t) + \omega^2 y_2(t) + c\,(y_2(t-\tau) - y_1(t-\tau)) = 0. \tag{5.2}$$

Our way in this chapter is to find a smart substitution for some combinations of (5.1), (5.2) and convert the system to some known expressions.

First, the sum and the difference of (5.1) and (5.2) is done:

$$(5.1) + (5.2): \quad \ddot{y_1}(t) + \ddot{y_2}(t) + \omega^2\,(y_1(t) + y_2(t)) = 0 \tag{5.3}$$
$$(5.1) - (5.2): \quad \ddot{y_1}(t) - \ddot{y_2}(t) + \omega^2\,(y_1(t) - y_2(t)) + 2c\,(y_1(t-\tau) - y_2(t-\tau)) = 0. \tag{5.4}$$

Now substitute

$$z_1(t) = y_1(t) + y_2(t), \quad z_2(t) = y_1(t) - y_2(t), \quad K = 2c. \tag{5.5}$$

Under these substitutions, the equations (5.3), (5.4) become

$$\ddot{z_1}(t) + \omega^2 z_1(t) = 0 \tag{5.6}$$
$$\ddot{z_2}(t) + \omega^2 z_2(t) + K z_2(t-\tau) = 0. \tag{5.7}$$

Two autonomous equations are obtained. Equation (5.7) is clearly similar to (2.3) which has been solved in the section 3.1. Following the results from this section, the (c,τ) stability region, where the function $z_2(t)$ is asymptotically stable, is given by the inequalities

$$\frac{2k\pi}{\omega} < \tau < \frac{(2k+1)\pi}{\omega}, \quad k \in \mathbb{N} \cup \{0\},$$
$$0 > c > \max\left\{\left(\frac{2k\pi}{\sqrt{2}\tau}\right)^2 - \frac{\omega^2}{2}, \frac{\omega^2}{2} - \left(\frac{(2k+1)\pi}{\sqrt{2}\tau}\right)^2\right\}$$

or

$$\frac{(2k+1)\pi}{\omega} < \tau < \frac{2(k+1)\pi}{\omega}, \quad k \in \mathbb{N} \cup \{0\},$$
$$0 < c < \min\left\{\frac{\omega^2}{2} - \left(\frac{(2k+1)\pi}{\sqrt{2}\tau}\right)^2, \left(\frac{2(k+1)\pi}{\sqrt{2}\tau}\right)^2 - \frac{\omega^2}{2}\right\}.$$

If these conditions hold, the zero solution of the function $z_2(t)$ is asymptotically stable. Recall from (5.5), $z_2(t)$ is the difference between $y_1(t)$ and $y_2(t)$ which should be synchronised. If $z_2(t)$ goes to the zero for large t, also the difference between $y_1(t)$ and $y_2(t)$ decreases and so the oscillations $y_1(t), y_2(t)$ go closer to the other. Thus, the system of coupled oscillators (5.1), (5.2) with a given ω can be synchronised for c,τ such that the above conditions hold.

The equation (5.6) gives an interesting result as well. This equation has a form of a (uncontrolled) harmonic oscillator. Since it is an ordinary differential equation of order two, it is easy to obtain a general analytical solution. By [1], the solution is

$$z_1(t) = C_1\cos(\omega t) + C_2\sin(\omega t)$$

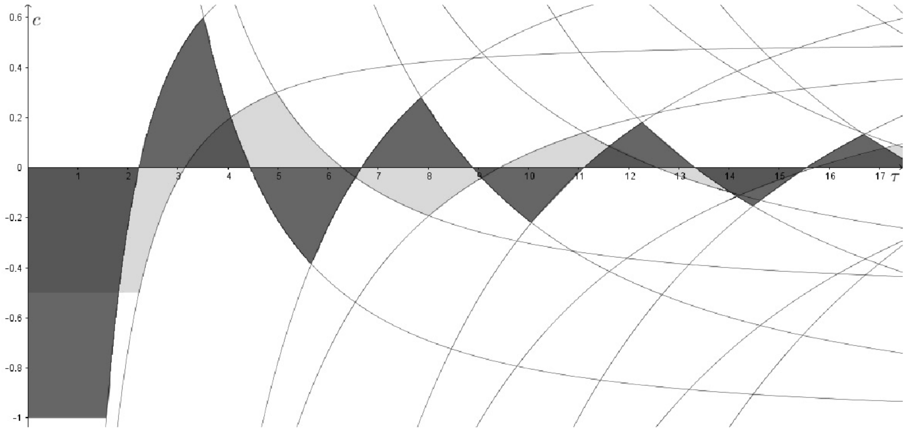


Figure 5.1: The synchronisation region of the equations (5.1), (5.2) with $\omega = 1$ (light grey) and $\omega = \sqrt{2}$ (dark grey)

where constants $C_1, C_2 \in \mathbb{R}$ are given by initial conditions.

Recall from (5.5), $z_1(t)$ is the sum of $y_1(t)$ and $y_2(t)$. The first important result of this fact it that the sum of $y_1(t), y_2(t)$ is predictable and periodic with zero in its codomain. This also means, there are infinitely many time moments $t_i, i \in \mathbb{N}$ such that $y_1(t_i) = -y_2(t_i)$ or even $y_1(t_i) = y_2(t_i) = 0$.

If the above results for $z_1(t)$ and $z_2(t)$ are put together, $y_1(t)$ and $y_2(t)$ are very close (approximately identical) for large t. Their sum is a periodic function with a constant amplitude. This holds for large t, too. Consequentially,

$$y_1(t) \approx y_2(t) \approx \frac{z_1(t)}{2}$$

for large time t. Since the analytic solution of $z_1(t)$ is known, the analytic solutions of both $y_1(t)$ and $y_2(t)$ can by derived. Namely

$$y_1(t) \approx y_2(t) \approx \frac{1}{2}\left(C_1 \cos(\omega t) + C_2 \sin(\omega t)\right)$$

for large t and C_1, C_2 constants given by initial conditions.

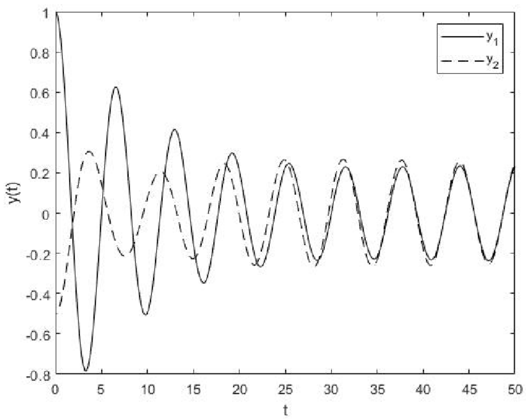
5.2 Control by a difference of a current state and other states with a delay

Consider now the controller

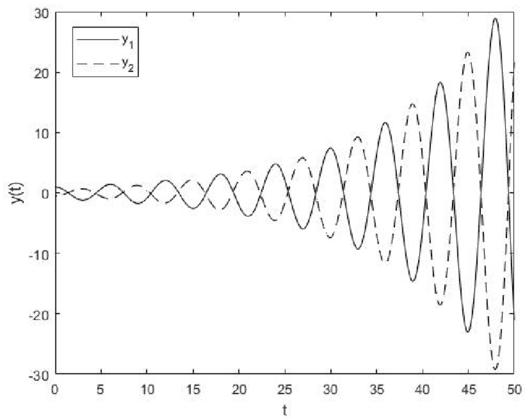
$$u_i(t) = \sum_{\substack{k=1 \\ k \neq i}}^{n} c_{ik}(y_i(t) - y_k(t - \tau))$$

of the i-th element in a system of n coupled oscillator. This control seems to be close to Pyragas control. For the case of two coupled oscillators, the whole system is given by

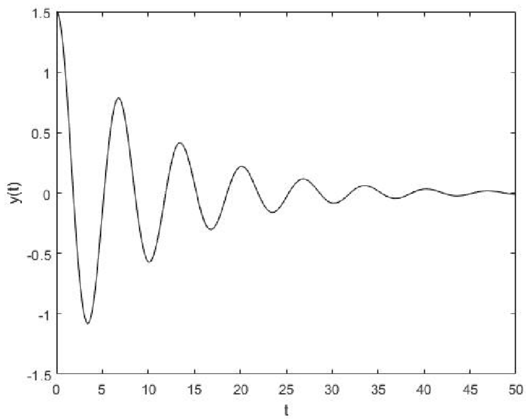
$$\begin{aligned}
\ddot{y}_1(t) + \omega_1^2 y_1(t) + c_1\left(y_1(t) - y_2(t - \tau)\right) &= 0 \\
\ddot{y}_2(t) + \omega_2^2 y_2(t) + c_2\left(y_2(t) - y_1(t - \tau)\right) &= 0.
\end{aligned}$$



(a) $\omega = 1; \tau = 1; c = -0.1$



(b) $\omega = 1; \tau = 1; c = 0.1$



(c) The difference of $y_1(t), y_2(t)$ with $\omega = 1$; $\tau = 1; c - 0.1$

Figure 5.2: Oscillation by (5.1), (5.2) with initial conditions $y_1(t) = 1, \dot{y_1}(t) = 0, y_2(t) = -0.5, \dot{y_2}(t) = 0$ for $-\tau \leq t \leq 0$

Under the simplifications mentioned in the beginning of the current chapter, the system becomes

$$\ddot{y_1}(t) + \omega^2 y_1(t) + c\,(y_1(t) - y_2(t-\tau)) = 0 \tag{5.8}$$
$$\ddot{y_2}(t) + \omega^2 y_2(t) + c\,(y_2(t) - y_1(t-\tau)) = 0, \tag{5.9}$$

The algorithm for solving this problem will be similar to the previous one. At first, the sum and the difference of (5.8) and (5.9):

$(5.8) + (5.9)$:
$$\ddot{y_1}(t) + \ddot{y_2}(t) + \omega^2(y_1(t) + y_2(t)) + c(y_1(t) + y_2(t) - y_1(t-\tau) - y_2(t-\tau)) = 0$$
$(5.8) - (5.9)$:
$$\ddot{y_1}(t) - \ddot{y_2}(t) + \omega^2(y_1(t) - y_2(t)) + c(y_1(t) - y_2(t) + y_1(t-\tau) - y_2(t-\tau)) = 0.$$

Now the substitutions are added:

$$z_1(t) = y_1(t) + y_2(t), \quad z_2(t) = y_1(t) - y_2(t).$$

With them, the system has the form

$$\ddot{z}_1(t) + \omega^2 z_1(t) + c(z_1(t) - z_1(t-\tau)) = 0 \tag{5.10}$$
$$\ddot{z}_2(t) + \omega^2 z_2(t) + c(z_2(t) + z_2(t-\tau)) = 0. \tag{5.11}$$

Two autonomous DDEs are obtained. They are similar to each other (not equal) and both of them have forms close to a system with the Pyragas control. The idea of using results, which have been already obtained, is destroyed by the sing inside the control part in (5.11). This is the equation which we have to manage at first. The function $z_2(t)$ is the difference of $y_1(t)$ and $y_2(t)$ and the goal of synchronisation is to make this difference zero, i.e. to determine parameters for which $z_2(t)$ is asymptotically stable.

Due to the sign inside the control part in (5.11), any results from the section 3.3 may not be used. Fortunately, the algorithm will be very similar. It begins by converting to a form similar to the harmonic system from the section 3.1.

The equations (5.11) may be equivalently written as

$$\ddot{z}_2(t) + (\omega^2 + c)z_2(t) + cz_2(t-\tau) = 0.$$

Considering the substitution $\bar{a} = \omega^2 + c$,

$$\ddot{z}_2(t) + \bar{a}z_2(t) + cz_2(t-\tau) = 0. \tag{5.12}$$

This form is similar to the equation (2.3). It is more useful to see this equation equivalently to (3.22) because from now, we will copy the algorithm shown in the section 3.3.

First, the condition $\Delta(0) > 0$ following from the Pontryagin's theorem turns into

$$\tau\bar{a}(\bar{a} + c) > 0.$$

Equivalently,

$$\tau(\omega^2 + c)(\omega^2 + 2c) > 0.$$

This inequality holds if both brackets are either positive or negative. This is true if

$$c > -\frac{1}{2}\omega^2 \quad \text{or} \quad c < -\omega^2. \tag{5.13}$$

By the final results from 3.1.1 and 3.1.2, the function $z_2(t)$ given by the equation (5.11) equivalent to (5.12) is asymptotically stable if and only if

$$\frac{2k\pi}{\sqrt{\bar{a}}} < \tau < \frac{(2k+1)\pi}{\sqrt{\bar{a}}}, \quad k \in \mathbb{N} \cup \{0\},$$
$$0 > c > \max\left\{\left(\frac{2k\pi}{\tau}\right)^2 - \bar{a}, \bar{a} - \left(\frac{(2k+1)\pi}{\tau}\right)^2\right\}$$

or

$$\frac{(2k+1)\pi}{\sqrt{\bar{a}}} < \tau < \frac{2(k+1)\pi}{\sqrt{\bar{a}}}, \quad k \in \mathbb{N} \cup \{0\},$$
$$0 < c < \min\left\{\bar{a} - \left(\frac{(2k+1)\pi}{\tau}\right)^2, \left(\frac{2(k+1)\pi}{\tau}\right)^2 - \bar{a}\right\}.$$

Using the backward substitution $\bar{a} = \omega^2 + c$, the results above are equivalent to

$$\frac{2k\pi}{\sqrt{\omega^2+c}} < \tau < \frac{(2k+1)\pi}{\sqrt{\omega^2+c}}, \quad k \in \mathbb{N} \cup \{0\},$$
$$0 > c > \max\left\{\left(\frac{2k\pi}{\tau}\right)^2 - \omega^2 - c, \omega^2 + c - \left(\frac{(2k+1)\pi}{\tau}\right)^2\right\}$$

or

$$\frac{(2k+1)\pi}{\sqrt{\omega^2+c}} < \tau < \frac{2(k+1)\pi}{\sqrt{\omega^2+c}}, \quad k \in \mathbb{N} \cup \{0\},$$
$$0 < c < \min\left\{\omega^2 + c - \left(\frac{(2k+1)\pi}{\tau}\right)^2, \left(\frac{2(k+1)\pi}{\tau}\right)^2 - \omega^2 - c\right\}.$$

Since we consider a real positive τ, a new condition $c > -\omega^2$ comes from the results. This mutes the second inequality in (5.13) but it is weaker then the first condition in (5.13).

As in the section 3.3, there appears a problem because the control constant c is on both left-hand and right-hand sides of some inequalities. To solve this problem, the two inequalities with max and min operators will be replaced by four triple inequalities so the results are now split into four parts. After that, the inequalities in each part will be simplified as much as possible with the goal to find out conditions on τ and c. The whole algorithm is fully described in the section 3.3. Some inequalities may be neglected which is evident after drawing regions in the (τ, c) plane.

Skipping a proper derivation, we write the final conditions on τ and c as

$$\frac{2k\pi}{\omega} < \tau < \frac{(2k+1)\pi}{\omega}, \quad k \in \mathbb{N} \cup \{0\},$$
$$0 > c > \frac{1}{2}\left[\left(\frac{2k\pi}{\tau}\right)^2 - \omega^2\right]$$

or

$$\frac{(2k+1)\pi}{\omega} < \tau < \frac{2(k+1)\pi}{\omega}, \quad k \in \mathbb{N} \cup \{0\},$$
$$0 < c < \frac{1}{2}\left[\left(\frac{2(k+1)\pi}{\tau}\right)^2 - \omega^2\right].$$

Under these conditions, the function $z_2(t)$ is asymptotically stable. In other words, the difference of $y_1(t)$ and $y_2(t)$ is converging, i.e. $y_1(t)$ and $y_2(t)$ are synchronised.

Also the function $z_1(t)$ must not be forgotten. As we have seen in the section 5.1, the sum of $y_1(t)$ and $y_2(t)$ decides the behaviour of the synchronised functions.

The equation (5.10) is a DDE with the Pyragas control, it is equivalent to the DDE (3.20), even no substitutions are needed here. This system has been fully solved in the section 3.3. Taking the results from this section, the function $z_1(t)$ is asymptotically stable if

$$\frac{2k\pi}{\omega} < \tau < \frac{(2k+1)\pi}{\omega}, \quad k \in \mathbb{N} \cup \{0\},$$
$$0 < c < \frac{1}{2}\left[\left(\frac{(2k+1)\pi}{\tau}\right)^2 - \omega^2\right]$$

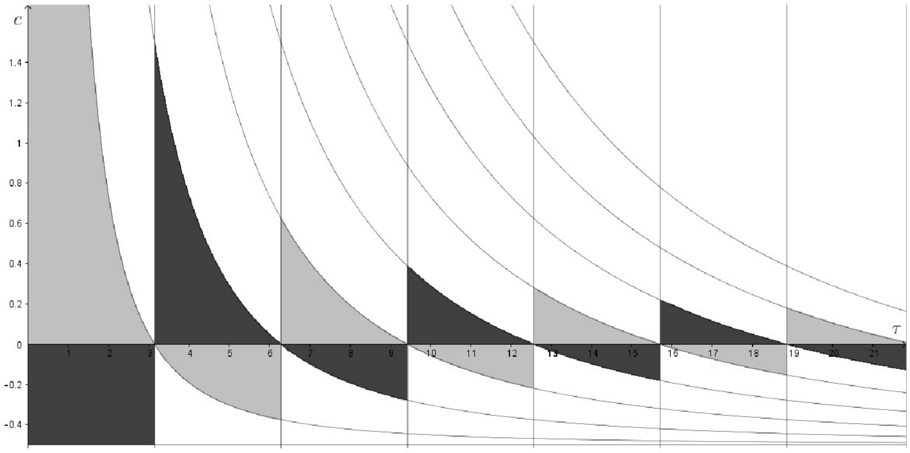


Figure 5.3: The stability regions of $z_1(t)$ (light grey) and $z_2(t)$ (dark grey) for $\omega = 1$

or

$$\frac{(2k+1)\pi}{\omega} < \tau < \frac{2(k+1)\pi}{\omega}, \quad k \in \mathbb{N} \cup \{0\},$$
$$0 > c > \frac{1}{2}\left[\left(\frac{(2k+1)\pi}{\tau}\right)^2 - \omega^2\right].$$

Under these conditions, the sum of the functions $y_1(t)$ and $y_2(t)$ converges. This happens when either both functions are converging to the zero function or the function are in antiphase.

If stability regions of $z_1(t)$ and $z_2(t)$ are drawn in the (τ, c) plane, one can see that there is no combination of τ and c where both $z_1(t)$ and $z_2(t)$ are asymptotically stable. See the figure 5.3. This fact goes from the sign of c in the conditions for each function. The first parts of the conditions set particular active intervals of τ. The intervals are equivalent for both functions. However, if c must be positive on a particular interval of τ for one function, c is negative on the same τ interval for the other and vice versa.

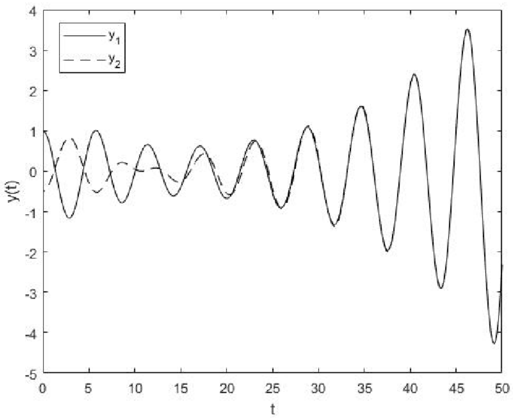
The conclusion of this section is that the system given by (5.8) and (5.9) may be synchronised. However, if the system is synchronised, then the sum of $y_1(t)$ and $y_2(t)$ is, up to some specific cases, unbounded. On the other hand, if the parameters are chosen such that the sum of $y_1(t)$ and $y_2(t)$ is tending to the zero, the difference of these functions is unbounded. The only possibility is that the functions $y_1(t)$ and $y_2(t)$ are in antiphase. If one would choose the parameters such that they would be neither in the stability region of $z_1(t)$ nor $z_2(t)$, the system will be strongly unbounded.

5.3 Control by a difference of a current state and all states with a delay

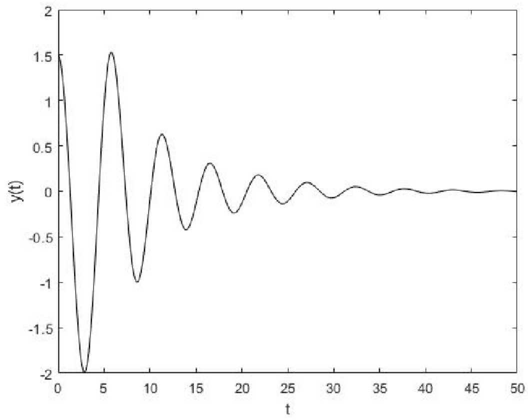
The last studied control has a general form

$$u_i(t) = \sum_{k=1}^{n} c_{ik} \left(y_i(t) - y_k(t-\tau)\right).$$

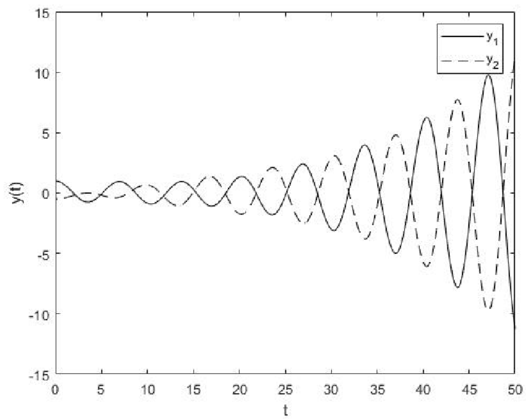
This control may be seen as a sum of differences between the current state and every delayed coupled states. Each difference is also multiplied by a constant c_{ik}.



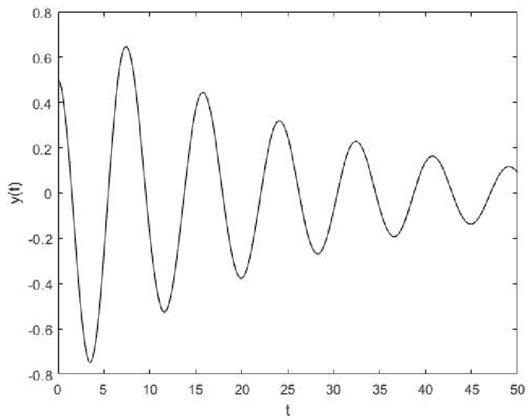
(a) $c = 0.2$



(b) The difference of $y_1(t), y_2(t)$ with $c = 0.2$



(c) $c = -0.2$



(d) The sum of $y_1(t), y_2(t)$ with $c = -0.2$

Figure 5.4: Oscillation by (5.8), (5.9) with $\omega = 1$, $\tau = 4.5$ and initial conditions $y_1(t) = 1, \dot{y_1}(t) = 0$, $y_2(t) = -0.5, \dot{y_2}(t) = 0$ for $-\tau \leq t \leq 0$

A system of two elements with this controller is given by

$$\begin{aligned}
\ddot{y_1}(t) + \omega_1^2 y_1(t) + c_{11}\left(y_1(t) - y_1(t-\tau)\right) + c_{12}\left(y_1(t) - y_2(t-\tau)\right) = 0 \\
\ddot{y_2}(t) + \omega_2^2 y_2(t) + c_{21}\left(y_2(t) - y_1(t-\tau)\right) + c_{22}\left(y_2(t) - y_2(t-\tau)\right) = 0.
\end{aligned}$$

As in the previous sections, the simplifications are considered. Namely,

$$\omega_1 = \omega_2 = \omega, \quad c_{11} = c_{12} = c_{21} = c_{22} = c.$$

The system is now represented as

$$\ddot{y_1}(t) + \omega^2 y_1(t) + c\left(2y_1(t) - y_1(t-\tau) - y_2(t-\tau)\right) = 0 \tag{5.14}$$

$$\ddot{y_2}(t) + \omega^2 y_2(t) + c\left(2y_2(t) - y_1(t-\tau) - y_2(t-\tau)\right) = 0. \tag{5.15}$$

Similarly as in the previous systems, the sum and the difference of (5.14) and (5.15) have to be done,

$$\begin{aligned}
&(5.14) + (5.15): \\
&\qquad \ddot{y_1}(t) + \ddot{y_2}(t) + \omega^2(y_1(t) + y_2(t)) + 2c(y_1(t) + y_2(t) - y_1(t-\tau) - y_2(t-\tau)) = 0 \\
&(5.14) - (5.15): \\
&\qquad \ddot{y_1}(t) - \ddot{y_2}(t) + \omega^2(y_1(t) - y_2(t)) + 2c(y_1(t) - y_2(t)) = 0.
\end{aligned}$$

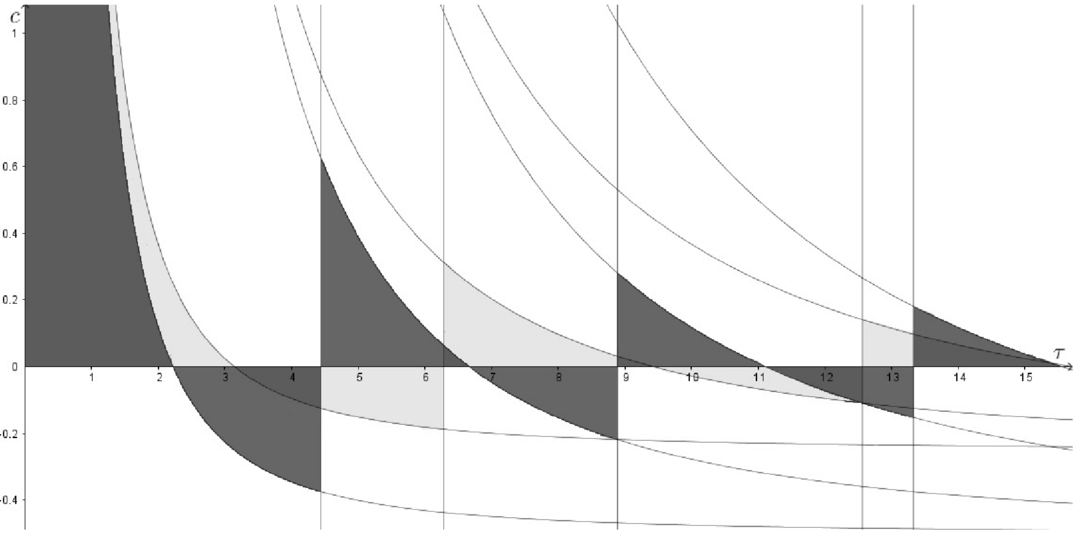


Figure 5.5: The synchronisation region of the equations (5.14), (5.15) with $\omega = 1$ (light grey) and $\omega = \sqrt{2}$ (dark grey)

Considering the known substitutions

$$z_1(t) = y_1(t) + y_2(t), \quad z_2(t) = y_1(t) - y_2(t), \quad K = 2c,$$

the system may be equivalently written as

$$\ddot{z}_1(t) + \omega^2 z_1(t) + K(z_1(t) - z_1(t-\tau)) = 0 \tag{5.16}$$
$$\ddot{z}_2(t) + \omega^2 z_2(t) + K z_2(t) = 0. \tag{5.17}$$

Now the equation (5.16) representing the sum of (5.14), (5.15) has the form of harmonic oscillation with Pyragas control. Stability anallysis of (5.16) implies the conditions

$$\begin{aligned} \frac{2k\pi}{\omega} < \tau < \frac{(2k+1)\pi}{\omega}, \quad k \in \mathbb{N} \cup \{0\}, \\ 0 < c < \frac{1}{4}\left[\left(\frac{(2k+1)\pi}{\tau}\right)^2 - \omega^2\right] \end{aligned}$$

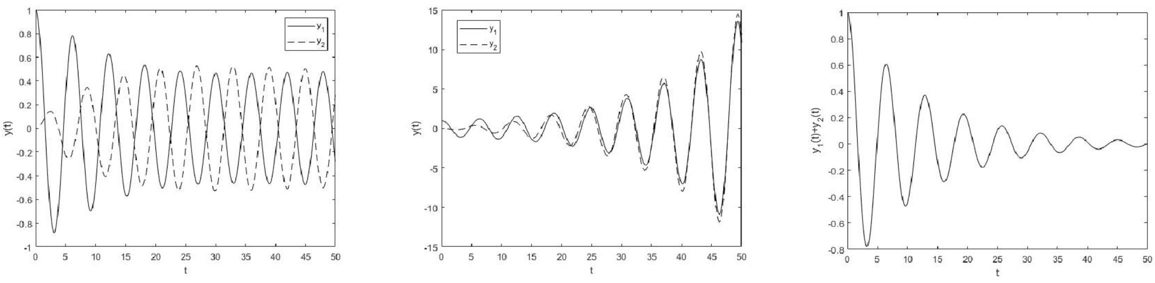
or

$$\begin{aligned} \frac{(2k+1)\pi}{\omega} < \tau < \frac{2(k+1)\pi}{\omega}, \quad k \in \mathbb{N} \cup \{0\}, \\ 0 > c > \frac{1}{4}\left[\left(\frac{(2k+1)\pi}{\tau}\right)^2 - \omega^2\right], \end{aligned}$$

and the necessary condition $2c > -\omega^2$ must be satisfied.

Under these conditions, the zero state of $z_1(t)$ is asymptotically stable and so the sum of $y_1(t)$ and $y_2(t)$ tends to the zero. This can be seen as $y_1(t) \approx -y_2(t)$ (this admits also the particular case $y_1(t) \approx y_2(t) \approx 0$) for large t.

What does the equation (5.17) mean? The difference of $y_1(t)$ and $y_2(t)$ is a harmonic function (we still assume that $2c > -\omega^2$). Moreover, there are infinitely many time moments $\hat{t}$ when $y_1(\hat{t}) - y_2(\hat{t}) = 0$. This implies that the above mentioned particular



(a) $\omega = 1; \tau = 0.8; c = 0.1$ (b) $\omega = 1; \tau = 0.8; c = -0.1$ (c) The sum of $y_1(t), y_2(t)$ with $\omega = 1; \tau = 0.8;\ c = 0.1$

Figure 5.6: Oscillations by (5.14), (5.15) with initial conditions $y_1(t) = 1, \dot{y_1}(t) = 0$, $y_2(t) = 0, \dot{y_2}(t) = 0$ for $-\tau \leq t \leq 0$

case $y_1(t) \approx y_2(t) \approx 0$ can not occur (except a system with a specific choice of initial functions).

The clarification above indicates that solutions of (5.14) and (5.15) oscillate and appear in antiphase, i.e. they oscillate (harmonically for large time t) but in opposite half-planes. So the oscillations are not synchronised. However, it gives an interesting result about behaviours of $y_1(t)$ and $y_2(t)$ interacting to each other.

6 Numerical methods for solving DDEs in MATLAB

Numerical methods create a powerful apparatus in many branches of mathematics. In particular, their use in ODEs and DDEs is necessary when no analytical methods of their solving are known. Clearly, using numerical methods in the industry is more suitable than computing solutions one by one.

The problem of numerical methods for ODEs is one of the most widely studied branches. There are many algorithms for solving ODEs or systems of ODEs. Each method can be described by its order, some types of errors and numerical stability. However, we are not going to deal with these characteristics, they are significant if the exact solution is needed. This thesis is based solely on the stability of differential equations.

Recall some of the well known numerical methods for ODEs from [16]. Euler's method is probably the easiest one. Further, it can be used in the form of the forward method or the backward method. Derivatives are substituted by the formula of forward or backward substitution with a chosen step size. The next method is the Runge - Kutta method. Here, some additional coefficients must be computed. This method has a higher order than Euler's methods. The last noted method is the multistep method. In particular, Adams – Bashforth methods and Adams – Moulton methods are usually presented.

In general, the input for numerical methods for ODEs consists of the size of the time step, the size of the space step, boundary or initial conditions, space limitations, the initial time, the final time and the right-hand side function itself, of course. During the method's computation, some coefficients are calculated if they are needed and the derivatives are substituted. By doing that, the next space step is computed. When the whole space is fulfilled, the computations are repeated for the next time step until the final time is reached. Now an important question of this chapter is coming. Is it possible to use a similar algorithm also in the case of DDEs?

The answer is yes! Most of the numerical methods for ODEs can be modified for the DDEs case. Clearly, something more has to be added and the methods are not as problem free as the methods for ODEs. One simply must be more careful with the inputs. For more details, see [17].

The first difference in the inputs consists in the form of the initial conditions. For the ODE case, the initial condition is the value of the function and its derivatives in the initial time t_0. Note that they are just values here. On the other hand, in the case of DDEs, the initial function is needed as well. It has also been stated in the section 2.2 that the initial function $\psi(t)$ must be defined on the interval $-\tau \leq t \leq 0$ and that it must also be smooth enough on this interval. By this function, all initial data are obtained and the numerical method can be used.

Another different part of the input is the delay factor which is in the numerical case often called lag. This is an unchanged constant value in the whole studied interval. In the previous chapters, systems have been stated with one delay only. But this is not the only possibility. In general, there can be a finite series of positive delays $\{\tau_i\}_i$. In the process of solving, the method takes the information from different past moments which have been given or already computed. However, the initial function is stated on the interval bounded by just one of the delays τ_i, call this the specific delay $\hat{\tau}$. Then the initial function $\psi(t)$ is defined on $-\hat{\tau} \leq \tau \leq 0$. The user must choose $\hat{\tau}$ so that all initial values are well defined, i.e. all needed values before the initial time must be well defined. Intuitively, this holds if

$$\hat{\tau} = \max_i \tau_i.$$

If $\psi(t)$ were defined on an interval bounded by a different value, i.e. $\hat{\tau} \neq \max_i \tau_i$, at minimum, the initial values for the delay $\tau = \max_i \tau_i$ would be missing.

6.1 The predefined MATLAB function for solving DDEs

In the beginning, it is useful to say something about the MATLAB function `ode23`. This is a one-step method for solving ODEs by explicit Runge-Kutta. Following [18], it can be called in MATLAB by

```
[t,y] = ode23(odefun,tspan,y0,options).
```

The function `odefun` is the differential system which shall be solved. If the original differential equation has a higher order, it must be rewritten to a system of ODEs of order one. The vector `tspan` usually given as $[t_0, t_f]$ is a time interval. On this interval, the equation is solved. The initial conditions are given by `y0`. This is a vector with the same length as the number of ODEs in `odefun`. By `options`, one can set some special modifications to the solution.

The outputs are arrays `t` and `y` representing the time mesh and values of the solution evaluated at time values of `t`, respectively. The array `y` has as many rows as there are ODEs of order one in the original system `odefun`. Each of the rows of `y` has the same length as the vector `t`. From these outputs, a plot of the solution can be generated.

The solver `dde23` is used with a similar syntax. Following [19], the predefined function in MATLAB is called by

```
sol = dde23(ddefun,lags,history,tspan,options).
```

The input `ddefun` keeps the same function as `odefun` in the solver `ode23`. In the case of a higher order equation, it must be written as a system of DDEs of order one. Furthermore, there is no change in `tspan` and `options` for the functions `ode23` and `dde23`. The inputs `lags` and `history` are more interesting. The constant vector `lags` represents the delays in the system. There can by finitely many delays and all of them must be positive. The input `history` gives the initial conditions of the system. It can be a constant or a function of time t. In the case of a constant, `history` is a column vector of the same number of columns as there are DDEs of order one in the system `ddefun`. The length of the initial function is taken by the solver automatically. As it has been said, it depends on the largest time delay, i.e. the highest value in `lags`.

The output `sol` includes both the time mesh and the evaluations of the solution $y(t)$. By using the command `sol.x`, the time mesh is obtained. The $y(t)$ values are called by `sol.y`. If `ddefun` is a system of DDEs, then `sol.y` has as many columns and, in particular, the column i can be obtained by the command `sol.y(i,:)`

To complete this classification, something more should be said about the function `ddefun`. It has been said already that the function is, in general, a system of DDEs of order one. One crucial problem is that of defining the delayed part of the system.

Probably the best way of doing this is to create the solved system by a function declaration, for example, as following,

```
yp = f(t,y,ylag,consts).
```

Here, the inputs of the function `f` are time `t`, the motion `y`, some constants shown by `consts`. The delayed part is represented by `ylag`. This is an array of dimension $n \times m$, where n is the number of DDEs in the system, i.e. number of rows of `yp`. Further, m is the number of different delays in the system. Clearly, it is also the dimension of the input `lags` from the solver `dde23`. The output is a system of n equations `yp` which will be solved by the solver.

As an example, if the user calls `ylag(i,j)` in the MATLAB function, the solver will work with the i-th variable. Its first derivative is described by the i-th equation of the system. This will be delayed by the j-th value of the vector `lags`.

In the following examples, this problem of defining delay will be reduced because the systems in this thesis only deal with problems of one delay. The following sections show in particular the use of the solver `dde23` in MATLAB.

6.2 Harmonic oscillator with a feedback delay control in MATLAB

The focus of this section is the numerical solution of the harmonic oscillator controlled by the standard feedback delay control. This system is described by (2.3), recall,

$$\begin{aligned} &\ddot{y}(t) + \omega^2 y(t) + cy(t-\tau) = 0, \quad t > 0 \\ &y(t) = \psi(t), \quad -\tau \leq t \leq 0, \end{aligned} \tag{2.3}$$

where $\psi(t) \in C(\langle -\tau; 0\rangle; \mathbb{R})$ is an initial function.

The solver `dde23` needs the problem in the form of a system of DDEs of order one. To do this, (2.3) becomes

$$\begin{aligned} \dot{y_1}(t) &= y_2(t) \\ \dot{y_2}(t) &= -\omega^2 y_1(t) - cy_1(t-\tau). \end{aligned}$$

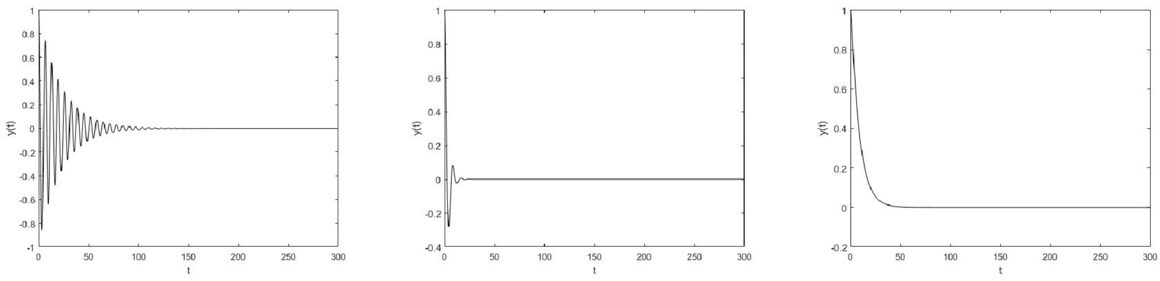
This shall be added to the MATLAB script as a function declaration. One way of doing this is

```
function yp = f(t,y,ylag,omega,c)
    yp = [y(2)
    -omega^2*y(1)-c*ylag(1,1)];
end
```
(6.1)

where `omega` and `c` are constants which can be declared in the script itself. An interesting feature of this declaration is that the delay τ is not yet present in the system.

The solver is called by

```
sol = dde23(@f, tau, y_initial, [0, 300], [], omega, c).
```



(a) $c = -0.1$ (b) $c = -0.5$ (c) $c = -0.9$

Figure 6.1: Oscillation by (2.3) with $\omega = 1$, $\tau = 1$ and initial conditions $y(t) = 1, \dot{y}(t) = 0$ for $-\tau \leq t \leq 0$

The input `@f` calls the function (6.1). The input `tau` represents the delays. In this case, the delay is one positive real value. The initial conditions are represented by `y_initial`. A smooth initial function does not influence the stability of the system. For simplicity, it is chosen as

```
y_initial=[1,0]
```

for all following cases. This means that the point mass has the constant position $y(t) = 1$ and velocity $\dot{y}(t) = 0$ for $-\tau \leq t \leq 0$. The system is solved for the time interval `[0, 300]`. The next `[]` means that no additional options are considered, just the same constants `omega, c` as in (6.1) must be added to the solver.

The constant inputs of the script are `omega, tau` and `c`. They will be chosen with respect to the figures 3.1 and 3.3. By doing this, the correctness of the results from section 3.1 will be verified.

Take, for example, $\omega = 1$, $\tau = 1$ and observe the influence of constant c. It is clear from the figure 3.3 that the system is asymptotically stable for $c \in (-1, 0)$. This also holds for the numerical solution (see figure 6.1). The speed of the stabilisation depends on the value of the constant c. It can be seen that the power of the control increases with a decreasing value of c.

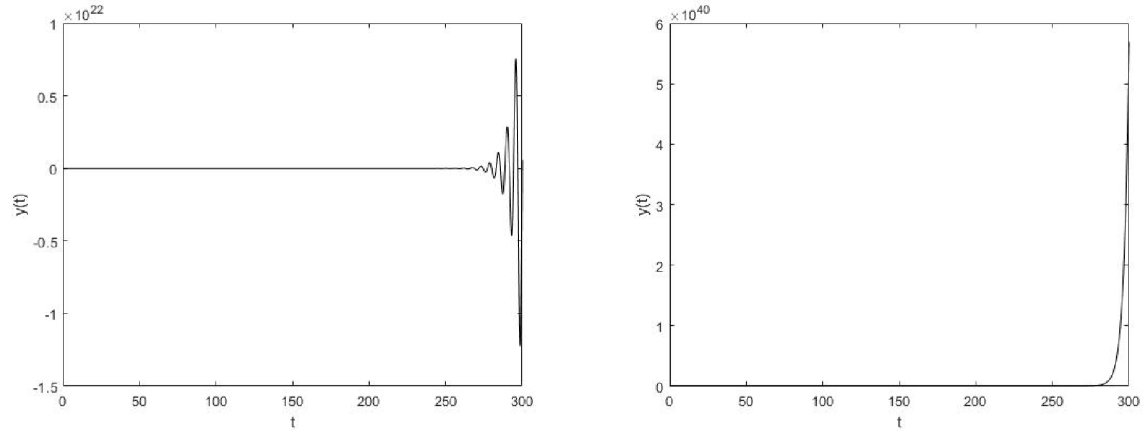
With the same values of ω and τ, the system becomes unstable if $c > 0$ or $c < -1$. Graphs with such a choice can be seen in 6.2. Even if the graphs seem to be constant for "small t", they oscillate in the whole interval. The reason for this almost constant curve is the huge scale of y-axis in comparison with the x-axis.

For $c = -1$, the solution is an "approximately constant" curve. This constant curve is influenced by the initial conditions. Anyway, if c were chosen to be slightly greater, the solution $y = 0$ would become asymptotically stable. On the other, if c were slightly smaller, the system would be unstable. Both of these cases can be seen in 6.3. So the system with $c = -1$ is not stable.

To be illustrative, choose $\omega = 1$, $c = -0.25$ and change values of τ around the boundary of the stability region. For such ω and c, the system should be asymptotically stable for $\tau \in (0, \pi/\sqrt{1.25})$. If τ is chosen close to 0, the curve approaches the stable position slowly. The same situation is for τ close to the boundary curve. The stabilisation is the fastest somewhere in the middle of the interval. All graphs may be seen in 6.4.

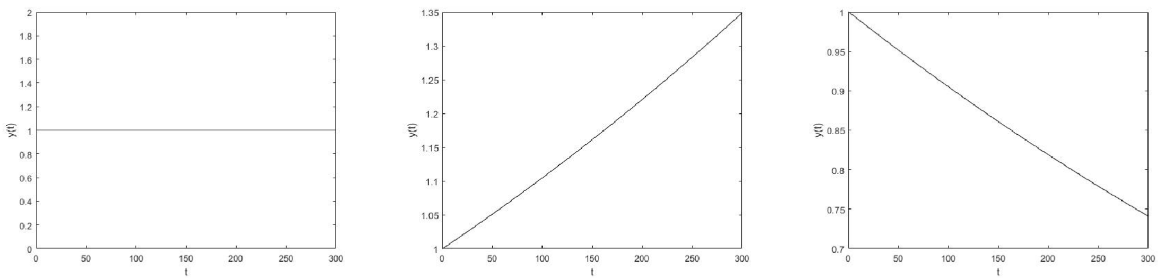
If values around the boundary

$$\tau = \frac{\pi}{\sqrt{\omega^2 - c}}$$



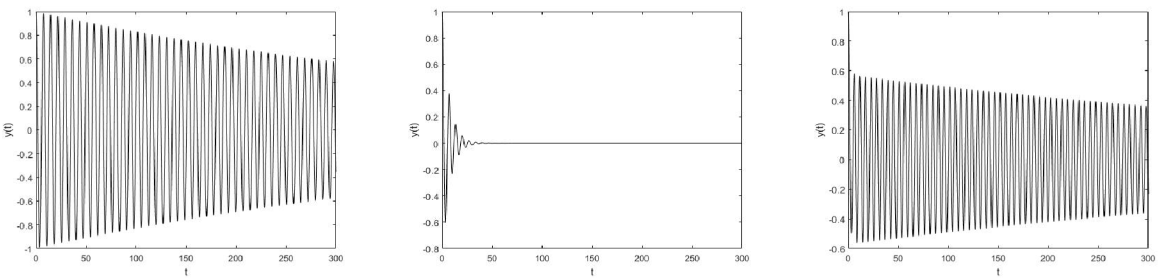
(a) $c = 0.5$ (b) $c = -1.5$

Figure 6.2: Oscillation by (2.3) with $\omega = 1$, $\tau = 1$ and initial conditions $y(t) = 1, \dot{y}(t) = 0$ for $-\tau \leq t \leq 0$



(a) $c = -1$ (b) $c = -1.001$ (c) $c = -0.999$

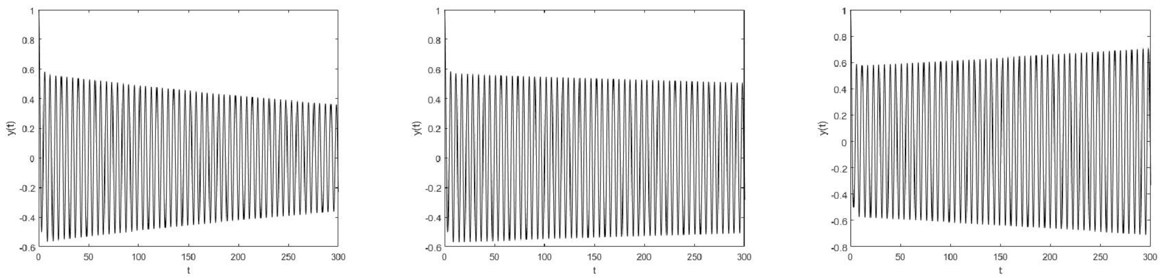
Figure 6.3: Oscillation by (2.3) with $\omega = 1$, $\tau = 1$ and initial conditions $y(t) = 1, \dot{y}(t) = 0$ for $-\tau \leq t \leq 0$



(a) $\tau = 0.01$ (b) $\tau = 1.4$ (c) $\tau = 2.8$

Figure 6.4: Oscillation by (2.3) with $\omega = 1$, $c = -0.25$ and initial conditions $y(t) = 1, \dot{y}(t) = 0$ for $-\tau \leq t \leq 0$

are considered, the numerical solution's behaviour is not like it has been predicted. In the case of $\tau = \pi/\sqrt{1.25}$, the curve goes asymptotically to the state $y = 0$. On the other hand, if τ is slightly greater, the system becomes unstable. In the opposite direction, the prediction holds, so the system is asymptotically stable. The reason for the behaviour for $\tau = \pi/\sqrt{1.25}$ could be some relative errors in the MATLAB computations.



(a) $\tau = \frac{\pi}{\sqrt{1.25}} - 0.01$ (b) $\tau = \frac{\pi}{\sqrt{1.25}}$ (c) $\tau = \frac{\pi}{\sqrt{1.25}} + 0.01$

Figure 6.5: Oscillation by (2.3) with $\omega = 1$, $c = -0.25$ and initial conditions $y(t) = 1, \dot{y}(t) = 0$ for $-\tau \leq t \leq 0$

For this choice of ω and c, the system is also asymptotically stable if $\tau \in (2\pi/\sqrt{0.75}, 3\pi/\sqrt{1.25})$. The behaviour of the system is similar to the interval above. Around the boundaries

$$\tau = \frac{2\pi}{\sqrt{\omega^2 + c}} \quad \text{and} \quad \tau = \frac{3\pi}{\sqrt{\omega^2 - c}},$$

the curve does not have the predicted shape either.

Following the algorithm above, the remaining problems from the chapter 3 can be considered. The MATLAB scripts would be easily rewritten to the case of a damped oscillator from the section 3.2 or the cases of oscillators controlled by the Pyragas control from 3.3. The next discussion would also be about possible different choices of the constants in the system.

6.3 Synchronisation of a system by a difference of delayed states in MATLAB

The results from the section 5.1 will be studied now. Namely, the equations

$$\ddot{y}_1(t) + \omega^2 y_1(t) + c\,(y_1(t-\tau) - y_2(t-\tau)) = 0 \tag{5.1}$$
$$\ddot{y}_2(t) + \omega^2 y_2(t) + c\,(y_2(t-\tau) - y_1(t-\tau)) = 0 \tag{5.2}$$

are considered. Since the system shall be solved numerically, initial functions must be added. The system is fully defined with initial functions

$$y_1(t) = \psi_1(t) \quad \text{and} \quad y_2(t) = \psi_2(t)$$

for $-\tau \leq t \leq 0$. Clearly, these functions have to satisfy $\psi_1(t), \psi_2(t) \in C(\langle -\tau; 0\rangle; \mathbb{R})$

For using the MATLAB solver `dde23`, the system must be rewritten into the form of a system of DDEs of the order one:

$$\begin{aligned}
\dot{y}_1(t) &= v_1(t)\\
\dot{v}_1(t) &= -\omega^2 y_1(t) - c(y_1(t-\tau) - y_2(t-\tau))\\
\dot{y}_2(t) &= v_2(t)\\
\dot{v}_2(t) &= -\omega^2 y_2(t) - c(y_2(t-\tau) - y_1(t-\tau)).
\end{aligned}$$

This system is typed into the MATLAB script as a fuction declaration

```
function yp = f(t,y,ylag,omega,c)
    yp = [y(2)
    -omega^2*y(1)-c*(ylag(1,1)-ylag(3,1)
    y(4)
    -omega^2*y(3)-c*(ylag(3,1)-ylag(1,1)];
end
```
(6.2)

The solver is called by

```
sol = dde23(@f, tau, y_initial, [0, 300], [], omega, c).
```

The input `@f` calls the function (6.2). As before, the input `tau` represents the delays. The initial conditions are represented by `y_initial`. Smooth initial functions do not influence the stability of the system. For simplicity, they are chosen as

```
y_initial=[1,0,-0.5,0]
```

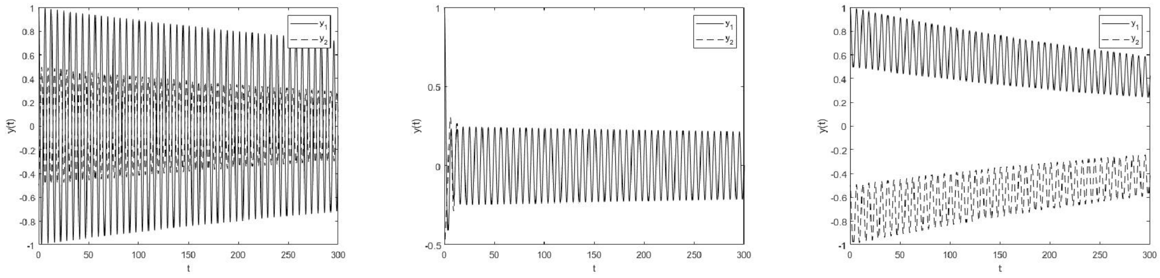
for all following cases. This means that the oscillation $y_1(t)$ has a constant position $y_1(t) = 1$ and a velocity $\dot{y_1}(t) = 0$ for $-\tau \leq t \leq 0$. The oscillation $y_2(t)$ has a constant position $y_2(t) = -0.5$ and a velocity $\dot{y_2}(t) = 0$ for $-\tau \leq t \leq 0$. Recall that the controller acts if $t > 0$, so the partial systems do not influence each other for $-\tau \leq t \leq 0$ and also the initial functions are independent. The system is solved for the time interval `[0, 300]`. No additional options are considered, which is represented by the next `[]`. Nevertheless, the constants `omega, c` must be added to the solver just like in (6.2).

Now, the results from the section 5.1 will be checked. The stability region is shown in the figure 5.1. As in the section before, the goal is to take different values of the constants ω, τ and c, especially the boundary cases, and to observe the behaviour of the system.

In the first part, fix $\omega = 1$ and $\tau = 1$ and study the system with different values of c. It shall be synchronised for $c \in (-0.5, 0)$. This hypothesis holds but the behaviour inside the interval is different than in the section 6.2. For the system (6.2), the synchronisation is the fastest for a value in the middle of the interval. Near the boundary values of the interval, the synchronisation is significantly slower. See the figure 6.7. Also note that the behaviour is different for a chosen c near the left or the right boundary of the interval. If $c \approx 0$, the maximal amplitude of both partial system is synchronised in time. If $c \approx -0.5$, the amplitudes are set immediately and the positions of the partial systems are synchronised in time. See the figure 6.6.

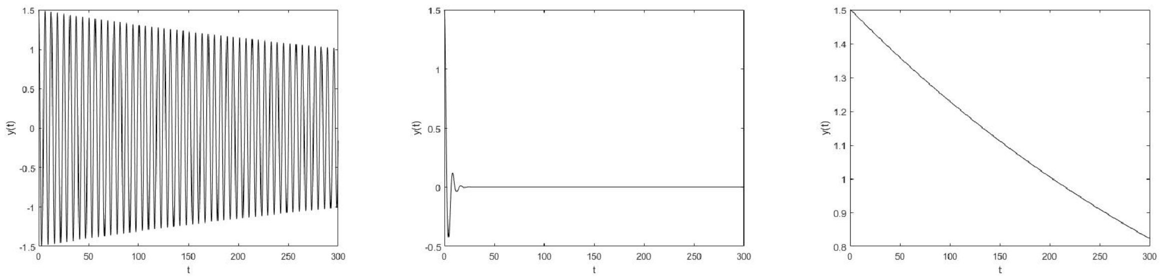
On the boundaries of the interval, the graphs more or less copy the results from the chapter 6.2. For $c = -0.5$, i.e. on the lower boundary, no synchronisation is going on, both the partial systems oscillate with a constant difference. The system is asymptotically stable for c slightly greater than -0.5 (see figures 6.6c, 6.7c) and it is unstable for c slightly smaller than -0.5. The system is also unstable for any $c > 0$.

Now fix $\omega = 1$ and $c = -0.25$. The system should be synchronised for $\tau \in (0, \pi/\sqrt{1.5})$. After doing the numerical tests, the computations are confirmed. The slowest synchronisation comes with a positive τ close to zero. A bit faster case is obtained for τ slightly smaller than $\pi/\sqrt{1.5}$. With τ in the middle of the interval, the synchronisation is the fastest.



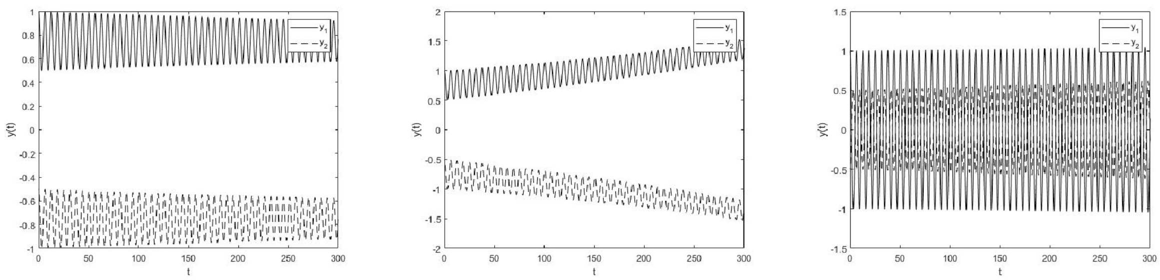
(a) $c = -0.001$ (b) $c = -0.25$ (c) $c = -0.499$

Figure 6.6: Oscillation by (5.1), (5.2) with $\omega = 1$, $\tau = 1$ and initial conditions $y_1(t) = 1, \dot{y_1}(t) = 0, y_2(t) = -0.5, \dot{y_2}(t) = 0$ for $-\tau \leq t \leq 0$



(a) $c = -0.001$ (b) $c = -0.25$ (c) $c = -0.449$

Figure 6.7: The difference of $y_1(t), y_2(t)$ from oscillation by (5.1), (5.2) with $\omega = 1$, $\tau = 1$ and initial conditions $y_1(t) = 1, \dot{y_1}(t) = 0, y_2(t) = -0.5, \dot{y_2}(t) = 0$ for $-\tau \leq t \leq 0$

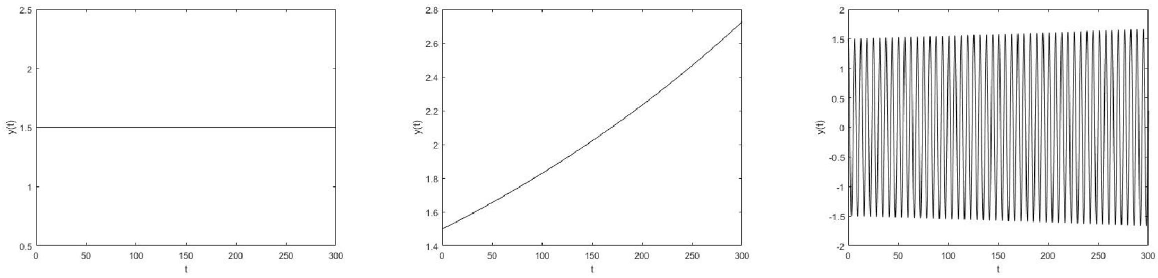


(a) $c = -0.5$ (b) $c = -0.501$ (c) $c = 0.001$

Figure 6.8: Oscillation by (5.1), (5.2) with $\omega = 1$, $\tau = 1$ and initial conditions $y_1(t) = 1, \dot{y_1}(t) = 0, y_2(t) = -0.5, \dot{y_2}(t) = 0$ for $-\tau \leq t \leq 0$

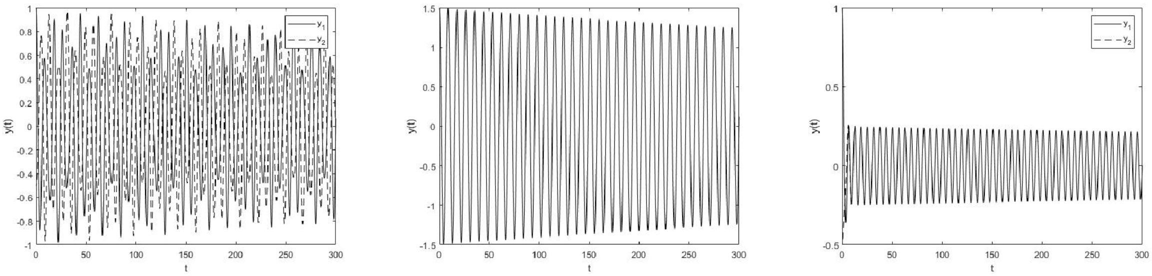
The situation near the boundary $\tau = \pi/\sqrt{1.5}$ is also similar to the analysis in the section before. From the theory in the section 5.1, the system should be stable, and for greater τ, the system should be unstable. By doing a numerical test, the system is still stable for τ on the boundary. Moreover, it is stable for some values greater than $\pi/\sqrt{1.5}$, too. The instability comes somewhere between $\pi/\sqrt{1.5} + 0.001$ and $\pi/\sqrt{1.5} + 0.002$. For higher values of τ, the system is unstable. There is no other interval of τ where the system is stable with the constants $\omega = 1$ and $c = -0.25$.

The reason for how the numerical results near the boundary turned out may be a numerical inaccuracy of the MATLAB solver. Similar results are also obtained around the boundaries for a different choice of ω and c.



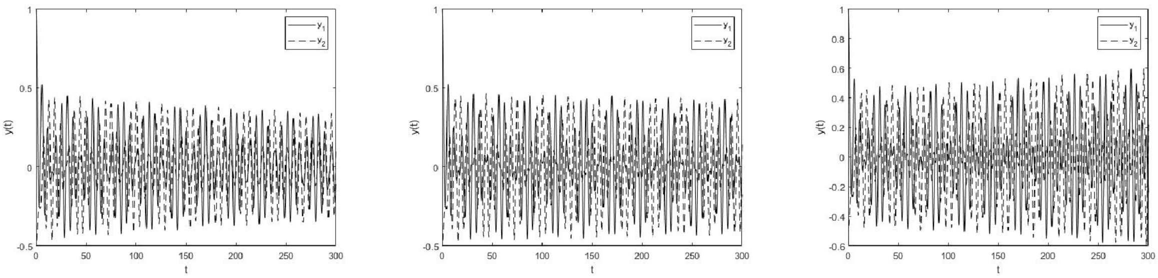
(a) $c = -0.5$ (b) $c = -0.501$ (c) $c = 0.001$

Figure 6.9: The difference of $y_1(t), y_2(t)$ from oscillation by (5.1), (5.2) with $\omega = 1$, $\tau = 1$ and initial conditions $y_1(t) = 1, \dot{y}_1(t) = 0, y_2(t) = -0.5, \dot{y}_2(t) = 0$ for $-\tau \leq t \leq 0$



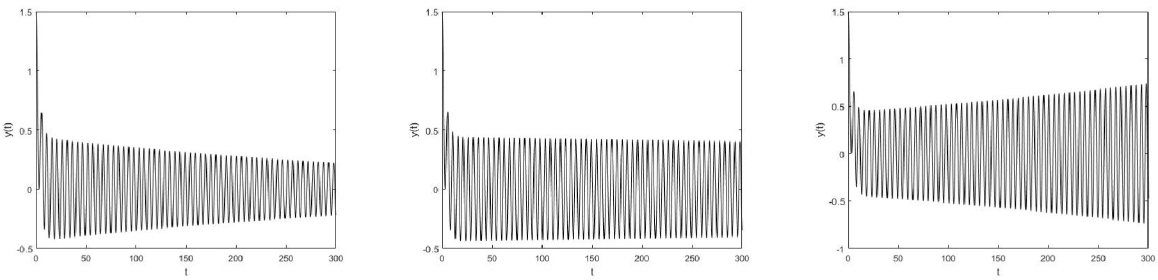
(a) $\tau = 0.001$ (b) $y_1(t) - y_2(t)$ for $\tau = 0.001$ (c) $\tau = 1.2$

Figure 6.10: Oscillation by (5.1), (5.2) with $\omega = 1$, $c = -0.25$ and initial conditions $y_1(t) = 1, \dot{y}_1(t) = 0, y_2(t) = -0.5, \dot{y}_2(t) = 0$ for $-\tau \leq t \leq 0$



(a) $\tau = \frac{\pi}{\sqrt{1.5}} - 0.001$ (b) $\tau = \frac{\pi}{\sqrt{1.5}}$ (c) $\tau = \frac{\pi}{\sqrt{1.5}} + 0.01$

Figure 6.11: Oscillation by (5.1), (5.2) with $\omega = 1$, $c = -0.25$ and initial conditions $y_1(t) = 1, \dot{y}_1(t) = 0, y_2(t) = -0.5, \dot{y}_2(t) = 0$ for $-\tau \leq t \leq 0$



(a) $\tau = \frac{\pi}{\sqrt{1.5}} - 0.001$ (b) $\tau = \frac{\pi}{\sqrt{1.5}}$ (c) $\tau = \frac{\pi}{\sqrt{1.5}} + 0.01$

Figure 6.12: The difference of $y_1(t), y_2(t)$ from oscillation by (5.1), (5.2) with $\omega = 1$, $c = -0.25$ and initial conditions $y_1(t) = 1, \dot{y}_1(t) = 0, y_2(t) = -0.5, \dot{y}_2(t) = 0$ for $-\tau \leq t \leq 0$

7 Conclusion

In this thesis, dynamical systems with feedback delay controls were studied. As a representative of dynamical systems, oscillators have been chosen. The main aim was to discussed stability and controllability of the chosen systems. Since we have used DDEs, it was not possible to simply check the real parts of all roots of the system. Recall that the characteristic equation was an exponential polynomial. This equation has, in general, infinitely many roots.

The thesis has been divided into seven chapters. In the chapter 2, we have recalled some basics of mechanical oscillators. The background of controls has also been introduced, especially the feedback delay control. We have also discussed stability of ODEs and DDEs.

Stability of DDEs has been studied in the chapter 3. The exponential polynomial has been introduced as a consequence of the characteristic polynomial from the theory of ODEs. This is necessary for determining the systems' stability. We have stated Pontryagin's theorem. The theorem gives conditions under which all the roots of the exponential polynomial have a negative real part. The theorem has been used in the following sections with different combinations of oscillators and controls.

First, harmonic oscillation has been controlled by a simple control $u(y(t)) = cy(t-\tau)$. This resulted into inequalities defining the stability region. This region has also been visualised in diagrams. The process has been split into two parts, one with negative constant c, and the other with positive constant c. In fact, the lower and the upper half-plane of the diagrams have been studied separately. Two options of diagrams have been shown. In the first, time delay τ is fixed. The diagram shows a dependency of the frequency ω on the control constant c. The stability region consists of areas similar to triangles. The areas are exactly triangles in the case of dependency of ω^2 on c. These triangles become taller with increasing ω. By Pontryagin's theorem, the system is asymptotically stable for the parameters lying inside the triangles. For the constants on the boundary of the region, the system is stable. If the values from the curves creating the boundary of the region but not exactly the boundary values of the region are taken, the stability has not been determined. There has been no proof confirming any hypothesis until now. Finally, the system is unstable for any other combination of ω and c. The second option of the diagram was the case with a fixed ω. This diagram showing the dependency of τ and c was significantly different. Here, the partial regions became smaller with increasing values on the x-axis, i.e. τ values. The discussion about chosen parts of the diagram, especially the case of the boundary curves, would be similar to the one done for the diagram with the fixed delay τ.

A damped oscillator is obtained by adding friction to the system. Its control by the basic control has been studied in the section 3.2. As before, Pontryagin's theorem has been used for this mechanical system with two forms of the exponential polynomial. We converted the damped system to the harmonic system by a substitution. Hence, the results from the section of the harmonic oscillator might be used. By this algorithm, sufficient conditions have been obtained. These conditions have been projected to a stability diagram. Since our aim was to state necessary and sufficient conditions, the original exponential polynomial has been recomputed. Pontryagin's theorem has been used as well but the results have been taken as already derived statements. It follows from the resource that if the mechanical system is under some parameters' assumptions, the necessary and sufficient conditions have been stated. Unfortunately, the method requests the computation of roots and signs of a function for particular values and so it was not possible to make stability diagrams.

The next part of the chapter 3 deals with the Pyragas control $u(y(t)) = c(y(t) - y(t-\tau))$. The stability problem has been solved by converting it to already studied problems. For harmonic oscillations with this non-invasive controller, we substituted some constants. Thus, the system was similar to the harmonic oscillator with the basic control and also the result might be taken from that section. Due to the union of the particular stability regions, the final boundary curves could be simplified. The Pyragas control has been applied to the damped oscillator, too. By a substitution, we obtained a DDE similar to the damped system with the basic control. The next work has been reduced to the question of necessary and sufficient conditions. The system's constants had to be limited by some assumptions. For such a system, necessary and sufficient conditions have been obtained. These conditions depend on the roots of a function following from the exponential polynomial and so the stability diagrams could not have been drawn.

The last part of the chapter 3 shows comparisons of the basic control and Pyragas control. For harmonic oscillators, there are no combinations of τ and c stabilising the systems with both controls. Considering the control $u(y(t)) = cy(t-\tau)$ and Pyragas control, we have shown and compared the effects of these controls. This discussion followed mostly from the stability diagrams. Similar diagrams could not be stated for damped problems.

The next chapter was dedicated to systems in a matrix form. Such systems are described by two matrices and the stability of the system depends on their eigenvalues. We have stated general conditions for stabilising the systems which required systems' matrices to be commutative. This theorem might be simplified with an assumption of gain matrices' real roots. Moreover, another simplification of the theorem has been stated for a particular version of the matrix system. In this form, the theorem does not work with matrices' eigenvalues, but the stability conditions follow right from the systems' parameters.

A possible use of the theorem has been shown for a system which was derived by conversion of the model of harmonic oscillations into a system of ODEs of the first order. To both of the equations, the simple control $u(y(t)) = cy(t-\tau)$ was added. Due to the theorem, the problem had to be split into parts with a positive control constant and a negative control constant. As before, a set of conditions for the constants in the system has been obtained. The conditions for τ were similar in both parts of the algorithm.

The theorem has been used for the next system, too. Here, Pyragas control has been applied. It has been stated that there is no asymptotically stable solution for a positive control constant. On the other hand, the system has, say, a lot of solutions for a negative control constant. For a fixed frequency, there exist only countably many values of the time delay for which the system is unstable.

The chapter 5 shows some ways of synchronising two mechanical systems by delayed controls. The algorithms are based on converting the systems into two autonomous differential equations. These equations are either ODEs which can be solved analytically by well known methods, or DDEs already solved in the chapter 3.

The first system has been controlled by the difference of delayed states. By converting it, an ODE for the harmonic oscillator and a DDE for the harmonic oscillator with the basic control have been obtained. The stability region has been created by the already obtained results. The system is synchronised if the DDE is asymptotically stable, i.e. constants are taken from the stability region. The synchronised partial systems oscillations are approximated by the harmonic motion.

In the next example, the systems have been controlled by the difference of a current state and other states with a delay. Using the same trick as before, two independent DDEs have been obtained with Pyragas controls. We have derived stability regions for both the equations, but in fact these regions do not intersect. Taking parameters from one of the regions, the coupled system was synchronised, but the oscillations became unbounded. On the other hand, if parameters were taken from the second stability region, the oscillations became unbounded and they appeared in antiphase.

The last system in the chapter 5 presents a hybrid form of the controller. With this control, the synchronisation in the usual meaning has not been achieved. Anyway, one can use this control to set the motions of the partial system dependently on each other. For large time t, the systems are in antiphase.

Since DDEs cannot be solved analytically, numerical methods play an essential part in the feedback delay control. The solver `dde23` is a predefined function in MATLAB designed for these problems. In the last part of this thesis, the solver `dde23` has been introduced. Its background and use have been shown. Further, the solver has been used for two systems theoretically derived before, namely the harmonic oscillator with the basic control, and the synchronisation controlled by the difference of delayed states. The theoretical results for these systems have been checked numerically. They have been mostly confirmed, moreover, some additional information about the power of the control has been given. There were cases which did not precisely fit the theory. Since the non-fitting part was in order of a thousandth, it has been accounted as a numerical inaccuracy.

All graphs in the thesis have been created in MATLAB by `dde23`. The diagram have been created by the software Geogebra.

The thesis can be extended by considering more complex problems. In chapter 3, some other types of control could be added to the systems (for example a combination of states from different time delays or some other non-invasive controls). Also, the control of the damped oscillator with negative friction is still an open problem. An extension of the results of chapter 4 could be done by considering more general matrix systems, especially those where gain matrices may also have imaginary eigenvalues. When studying the synchronisation of a system, a natural extension consists of the discussion of systems with different frequencies or control constants. Another possibility is to create controllers with states in different delay times.

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8 List of Symbols

$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{Z}$	set of integers
$\mathbb{N}$	set of natural numbers
C	set of continuous functions
$\mathcal{R}(\alpha)$	real part of a complex number α
$\mathcal{I}(\alpha)$	imaginary part of a complex number α
$\arg(\alpha)$	argument of a complex number α
$\lceil \alpha \rceil$	ceiling function of a real number α; the least integer greater than or equal to α
c	control constant
I	identity matrix
k	stiffness of a spring
l	viscous damping coefficient
m	mass
T	period
$u(t)$	control function
$u(y(t))$	feedback control function
τ	time delay
$\psi(t)$	initial function
ω	frequency
ODE	ordinary differential equation
DDE	delay differential equation