PALACKÝ UNIVERSITY OLOMOUC
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Department of Optics

Nonlocalizable genuine multipartite entanglement

BACHELOR THESIS

Dominik Koutný

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Nelokalizovatelná úplná multipartitní provázanost

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Form of study:	Full – time
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Abstract

Quantum entanglement plays essential role in quantum information processing. Our task in this theoretically orientated thesis is to find a class of three-qubit fully-inseparable states and analyzed this class of states in the question of localizability of quantum entanglement between two parties of our state by performing measurement on the third party. The issue of localizability of quantum entanglement by POVM elements and utility of the state for quantum information processing is discussed.

Keywords

Discrete variable quantum information theory, quantum entanglement, multipartite entanglement, nonlocalizable entanglement

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I would like to thank my supervisor Mgr. Ladislav Mišta Ph.D. for his guidance, endless patience and advices. I am also grateful to the rest of the Department of Optics for stimulating enviroment. Last but not least, I would like to thank my dear family for support.

Declaration

I declare that I have written Bachelor Thesis "Nonlocalizable genuine multipartite entanglement" on my own under the guidance of Mgr. Ladislav Mišta Ph.D. by using resources, which are reffered to in the list of literature. I agree with the further usage of this document according to the requirements of the Department of Optics.

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Contents

1 Introduction

The thesis deals with genuine multipartite entanglement with finite dimensional Hilbert state space. This type of entanglement found and finds many applications in quantum information science.
The interesting such a kind of multipartite entanglement has arisen due to the presence of a new type of quantum nonlocality, the called Greenberger-Horne-Zeilinger (GHZ) nonlocality [1]. Genuine multipartite entanglement also found applications in some older quantum information protocols including assisted quantum teleportation [2], assisted dense coding [3], third man quantum cryptography [4] or quantum secret sharing schemes.
New applications of genuine entanglement are in cluster state quantum computing [5] or one-way quantum computing [6, 7], quantum simulations and topological quantum computing [8].
The performace of all the assisted protocols [3, 9] as well as the quantum secret sharing [10] rely on the possibility to localize entanglement between a pair of subsystems of a genuine multipartite entanglement obviously plays a vital role also in cluster-state implementation of entangling quantum gates. The term localizable entanglement was first introduced in references [11] and [12] in the context of a measure of entanglement and it is defined as a maximum amount of entanglement that can be localized between two subsystems of a multipartite system by performing local measurements on the other subsystems.
Besides being useful in applications, localizability of entanglement in genuine multipartite entangled state is distillable if one can distill by local operations and classical communication (LOCC) more entangled bipartite [13] or multipartite [14] states from only partially entangled states. Localization of entanglement between two initially separable subsystems of a multipartite quantum state by a measurement on the remaining subsystems of the state can therefore be seen as a kind of a simple distillation protocol. The absence of localizable entanglement in a given quantum state thus indicates that either more sophisticated protocols have to be used for its distillation or that the state contains nondistillable (bound) entanglement [15]. Therefore, nonlocalizability of entanglement in genuine multipartite entangled state is also a necessary prerequisite for the presence of bound entanglement in this kind of states. Note that so far, an example of a genuine multipartite bound entangled state has not been found.
In our thesis, we investigate genuine multipartite entanglement which does not posses localizable entanglement. First we construct an example of such a state for a simplest system consisting of three quantum bits (qubits). The next part of this thesis contains analysis of the experimental realization of the proposed state. Finally, in the last part of the thesis

we discuss the utility of the state for quantum information processing.

2 Theoretical background

In this section we briefly explain the basic quantum-mechanical concepts used throughout the thesis. The present thesis deals with quantum entanglement in multipartite systems and therefore we start with the description of states such systems in quantum mechanics.

2.1 Formalism of quantum states

In classical physics, a state of a system is fully described by a point in the phase space. In contrast, in quantum theory, a state is characterized by a normalized vector in a complex Hilbert space $\mathcal{H}$ [16]. In the present thesis we work only with systems with finite-dimensional Hilbert state space and therefore we restrict our attention to these systems in what follows. In the discrete variables, dimension of Hilbert space grows with the amount of particles as 2^N, where N in a number of particles. Representation via complex vectors also give us maximum affordable information about given quantum state. In the further text we will impose on theory for discrete variables, because my thesis deals with the discrete ones.

2.1.1 Pure states

States, which can be represented as a vectors in Hilbert space are called pure states. Mathematically, we can express these states as

$$|\psi\rangle = \frac{1}{\sqrt{\langle\tilde{\psi}|\tilde{\psi}\rangle}} |\tilde{\psi}\rangle \,, \tag{2.1}$$

where Dirac bra-ket notation is used and $\sqrt{\langle\tilde{\psi}|\tilde{\psi}\rangle}$ is called the norm of the state. Pure states are idealization. In practice, we do not have a complete information about state of a quantum system. Then we talk about a mixed state and we have to describe it by the so called density matrix.

2.1.2 Mixed states

Density matrix is used to describe a quantum system in a mixed state, a statistical ensemble of several quantum states. For a mixed state, a density matrix is made of system in the quantum state $|\psi_j\rangle$ with appropriate probability p_j and can be expressed as

$$\varrho = \sum_j p_j \, |\psi_j\rangle \, \langle\psi_j| \,, \tag{2.2}$$

where ϱ is a common symbol for density operator. An operator ϱ describes a density matrix if and only if ϱ is Hermitean, positive-semidefinite operator with a unit trace. The Hermicity means that

$$\varrho^\dagger = \varrho, \tag{2.3}$$

where $\dagger$ is a symbol for transposition and complex conjugate. Positive-semidefiniteness is defined as

$$\langle\psi|\,\varrho\,|\psi\rangle \geq 0, \tag{2.4}$$

for every $|\psi\rangle \neq 0$ and the unit trace condition reads as

$$\mathrm{Tr}[\varrho] = \sum_j p_j = 1. \tag{2.5}$$

Density operator for a pure state apparently satisfies relation

$$\varrho^2 = \varrho. \tag{2.6}$$

2.2 Quantum bits

Quantum information is an extension of classical information as the complex numbers are extension of the real ones. We define quantum bit [17], or qubit, as a quantum system with two-dimensional complex Hilbert state space, i.e. $\mathcal{H} = \mathbb{C}^2$. Unlike a classical bit, which can occur only in states 0 or 1 a quantum bit can exist in the superposition $\alpha\,|0\rangle + \beta\,|1\rangle$ where $|\alpha|^2 + |\beta|^2 = 1$. In laboratory, we can encode qubits for example to the polarization state of one photon. As a geometrical representation of two-level quantum system we use Bloch sphere.

We can express any pure state of a qubit as

$$|\psi(\vartheta,\phi)\rangle = \cos\left(\frac{\vartheta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\vartheta}{2}\right)|1\rangle\,, \tag{2.7}$$

where $\vartheta \in [0,\pi]$ and $\phi \in [0, 2\pi]$. In further work we deal with multiqubit systems involving two or three qubits. States of such a system is described by vectors in $\mathcal{H}\otimes\mathcal{H}$ or $\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}$. The basis is a product basis $\{|00\rangle\,, |01\rangle\,, |10\rangle\,, |11\rangle\}$ or or $\{|ijk\rangle\}^1_{i,j,k=0}$. Composite systems of qubits may exhibit quantum entanglement which will be described in detail in Section 2.4.

2.3 Measurement in quantum mechanics

In our thesis we deal with localizability of quantum entanglement in multipartite entangled quantum systems by measurement on part of this states and thus is necessary to briefly introduce basic concepts from the theory of quantum measurement.

2.3.1 Projective measurement

Physical quantities called observables are in quantum mechanics described by Hermitean operators. Such the operators have real eigenvalues and orthogonal eigenvectors. Consider an observable described by a Hermitian operator A, which is assumed for simplicity to have non-degenerate eigenvalues a' and the corresponding eigenvectors $|a'\rangle$. Then we can associate with the observable the so called projective measurement described by the following set of projective operators $P_{a'\in\mathcal{A}}$, where

$$P_{a'} = |a'\rangle \langle a'| \,, \tag{2.8}$$

is the projector onto the eigenvector corresponding to the eigenvalue a' and the symbol $\mathcal{A}$ denotes the set of eigenvalues of the operator A. Operator P satisfies a conditions $P^2 = P$, $P \geq 0$ and $P^\dagger = P$. The probability of getting a measurement outcome a' when measuring the observable A on a pure state $|\psi\rangle$ reads as

$$p_{a'} = |\,\langle a'|\psi\rangle\,|^2, \tag{2.9}$$

and the corresponding state after the measurement is given by

$$|a'\rangle = \frac{P_{a'}\,|\psi\rangle}{\sqrt{\langle\psi|\,P_{a'}^\dagger P_{a'}\,|\psi\rangle}}. \tag{2.10}$$

On the other hand, when the measurement is carried out on a density matrix ϱ one gets with the probability

$$p_{a'} = \mathrm{Tr}[\varrho P_{a'}] \tag{2.11}$$

the post-measurement state of the form

$$\varrho' = \frac{P_{a'}\varrho P_{a'}}{\mathrm{Tr}[\varrho P_{a'}]}. \tag{2.12}$$

Suppose, that state ϱ of a system is composed of two subsystems, density operator ϱ is acting on Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. If we are interested in measurement on one of a subsystems, the unnormalized outcome will take a form

$$\varrho^{(A)} = \mathrm{Tr}_B[\varrho^{(AB)} P^{(B)}], \tag{2.13}$$

where $P^{(B)}$ is projective operator acting on the subsystem B. The symbol Tr_i in equation (2.13) stands for the partial trace over the subsystem B and is defined as
$\mathrm{Tr}_i[\cdot] = \sum_n {}_i\langle n| \cdot |n\rangle_i$, where $\{|n\rangle_i\}$ is the basis in the Hilbert space $\mathcal{H}_i$. The density matrix (2.13) is not normalized and is called a reduced density matrix.

2.3.2 Generalized measurement

Projective measurement is made of projectors projecting on orthogonal subspaces and projective operators satisfy $P^2 = P$. Positive operator value measurement (POVM) elements do not necessarily have this property. Let's have a set of non-orthonormal Hermitian positive semi-definite projective operators $\{\Pi_i\}$. From the condition that these operators are Hermitian we have that probabilities p_i are real numbers

$$\Pi_i = \Pi_i^\dagger \Rightarrow p_i \equiv \mathrm{Tr}(\varrho \Pi_i) \in \mathbb{R} \tag{2.14}$$

Semi-definite positivity tells us that these probabilities are equal or greater then zero

$$\Pi_i \geq 0 \Rightarrow p_i \geq 0. \tag{2.15}$$

POVM elements decompose identity operator, thus the probabilities p_i sum up to 1

$$\sum_i \Pi_i = \mathbb{1} \Rightarrow \sum_i p_i = 1. \tag{2.16}$$

POVM is defined by equations (2.14)-(2.16). Probability of outcome a' after measurement on $|\psi\rangle$ is given by

$$p_{a'} = \langle\psi| \Pi_{a'} |\psi\rangle . \tag{2.17}$$

Likewise, for a system made of two subsystems A and B, we can write the unnormalized density matrix of subsystem A after the measurement of the POVM element $\Pi_i^{(B)}$ on subsystem B as

$$\varrho^{(A)} = \mathrm{Tr}_B[\varrho^{(AB)} \Pi_i^{(B)}], \tag{2.18}$$

2.4 Quantum entanglement and its properties

Quantum entanglement plays a central role in quantum information processing and it is also a main advantage against classical information theory. The properties of quantum entanglement have been summarized by Horodecki's in a nice review paper [18]. If only two parties are entangled we talk about bipartite entanglement whereas if more parties a involved we talk about multipartite entanglement. In two-level systems, quantum states can be entangled or separable. A quantum system is called separable, if we can write its density matrix as a sum of local density matrices, written as

$$\varrho = \sum_i p_i \varrho_i^{(A)} \otimes \varrho_i^{(B)}, \tag{2.19}$$

where $\varrho_i^{(A)}$ ($\varrho_i^{(B)}$) is a local state of subsystem A (B).
Entangled states are states, which cannot be written in the form of (2.19). Trivial example of a separable state in two-qubit system is the state $|0\rangle_A \otimes |0\rangle_B$. When there is not a risk of confusion we omit the symbol of the tensor product $\otimes$ for the sake of brevity. As an example of entangled states of a two-qubit system can serve us Bell states
$|\Phi_\pm\rangle = (1/\sqrt{2})(|0\rangle_A |0\rangle_B \pm |1\rangle_A |1\rangle_B)$ and $|\Psi_\pm\rangle = (1/\sqrt{2})(|0\rangle_A |1\rangle_B \pm |1\rangle_A |0\rangle_B)$.

2.4.1 PPT separability criterion

While decision about entanglement of pure states is simple, situation is much more complicated for mixed states. For this purpose, various separability criteria have been created [18]. In this thesis we will use positive partial transpose (PPT) criterion. PPT criterion was firstly introduced by Peres [19]. This criterion is a very strong necessary condition for separability. It says, that if ϱ_{AB} is a separable density matrix, then its partial transpose $\varrho_{AB}^{T_B}$ is also a legitimate density matrix. The symbol T_B denotes the partial transpose operation with respect to subsystem B which is defined as

$$\langle ij| \varrho^{T_A} |lm\rangle \equiv \langle im| \varrho |lj\rangle \tag{2.20}$$

In this thesis, we will also work with density matrices of three qubits A, B and C and we will use the partial transpose operation with respect to individual qubits. For the qubit A the operation is defined as

$$\langle ijk| \varrho^{T_A} |lmn\rangle \equiv \langle ljk| \varrho |imn\rangle\,, \tag{2.21}$$

whereas for qubits B and C it is defined similarly just by interchanging the second and third indices, respectively, of the matrix element on the right-hand-side of equation (2.21).

Similar to the two-qubit case, if the qubit A is separable from the pair of qubits (BC), then $\rho^{T_A} \geq 0$. Thus if $\rho^{T_A} < 0$, then the qubit A is entangled with the remaining two qubits. However, in contrast to two qubits, in the three-qubit case the condition $\rho^{T_A} \geq 0$ is not sufficient for separability of the qubit A due to the existence of the so called PPT entangle states [20].

2.4.2 Separability for three-qubit systems

Following the classification in [14] the three qubits can be split into two groups in three ways, i.e. we have $A-(BC)$, $B-(AC)$ and $C-(AB)$ bipartite splittings. According to the separability of a given quantum state with respect to the splittings we distinguish the following five separability classes:
(1) Fully inseparable states are states, which are entangled with respect to all bipartite splittings $A-(BC)$, $B-(AC)$ and $C-(AB)$. An example is maximally entangled of three-qubit GHZ state [1], $|GHZ\rangle = (1/\sqrt{2})(|000\rangle_{ABC} + |111\rangle_{ABC})$.
(2) One-qubit biseparable states are entangled with respect to splitting of two bipartite splittings but separable with respect to the third one. An example can be a state $|\alpha\rangle = |\Psi_+\rangle_{AB}|0\rangle_C$.
(3) Two-qubit biseparable states which are entangled with respect to one bipartite splitting. But separable with respect to the remaining two bipartite splitting.
(4) Three-qubit biseparable states are separable across all three bipartite splittings but cannot be written in the form

$$\varrho = \sum_i |a_i\rangle_A \langle a_i| \otimes |b_i\rangle_B \langle b_i| \otimes |c_i\rangle_C \langle c_i| \,, \tag{2.22}$$

where $|a_i\rangle$, $|b_i\rangle$ and $|c_i\rangle$ are unnormalized states of subsystems A, B and C. (5) Fully separable states can be written in the form (2.22).
A vast majority of applications of multipartite entanglement relies on fully inseparable states and, in particular, on the GHZ state. An important property of the GHZ state for applications such as secret sharing or assisted dense quantum coding is, that we can by measuring one qubit in the basis $|\pm\rangle = (1/\sqrt{2})(|0\rangle \pm |1\rangle)$ and communicating the measurement result to the locations of the other qubits establish a maximally entangled Bell state $|\Phi_+\rangle$ between the qubits. In other worlds, we can localize maximally entangled state between two qubits by a measurement on the third qubit.
In our thesis, we deal with states, which belongs to the same class as GHZ state, but do not have this property.

2.5 Localizable entanglement

Localizable entanglement is a property of given state. Localizable entanglement is a maximal entanglement quantified by some entanglement measure, which can be created between two qubits of a three-qubit state by a measurement on the third qubit supplemented by some correcting operations on the two qubits, which depend on the measurement outcome.

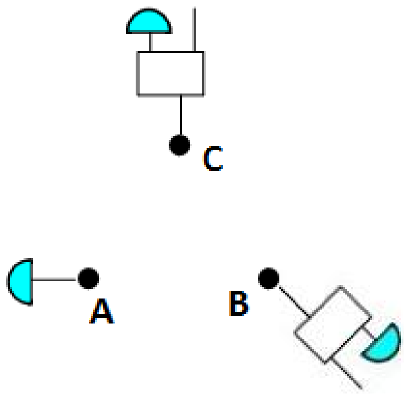

Figure 1: Localizable entanglement is a maximum entanglement which can be obtained by measuring, for example, a qubit A and communicating measurement result with other parties. Boxes in the figure denote correcting quantum operations on qubits B and C, which depend on the outcome of the measurement on qubit A. Let's have a GHZ state. By performing measurement on qubit A in the basis $|\pm\rangle_A$ we get $|\Phi_\pm\rangle_{BC}$. Communicating measurement result and applying Pauli matrix $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$, where $\{|0\rangle, |1\rangle\}$ represent normalized basis vectors in two-dimensional vector space, we obtain $|\Phi_+\rangle$. This is the basic principle underlying the performance of the quantum secret sharing or assisted dense coding.

3 Original results

Entanglement localizability in multipartite systems is crucial in many applications in quantum information processing such as quantum computation [6] or assisted quantum teleportation [2]. Another important property of multipartite mixed entangled quantum states is a distillation of entanglement to the maximally entangled multipartite state such as the GHZ state [1] or to Bell's states, maximally entangled states in bipartite systems. In conjuction with distillation there are still open questions in the sense of so-called bound entangled states. This term is used for states which are entangled but no pure entangled states can be obtained from them by using local operations and classical communication. Existence of fully inseparable multipartite bound entangled states is still an open question. In this part of my thesis I construct a three-qubit state with nonlocalizable entanglement which is also a nontrivial necessary prerequisite for the presence of multipartite bound entanglement.

3.1 Our state

We analyze the concept of nonlocalizable entanglement on a two-parametric family of three-qubit states.
Our work is mainly based on the paper published by Dür, Cirac and Tarrach [14]. They consider a set of three-qubit density operators in the following form:

$$\varrho_3 = \sum_{\sigma=\pm} \lambda_0^\sigma \left|\Psi_0^\sigma\right\rangle \left\langle\Psi_0^\sigma\right| + \sum_{j=1}^{3} \lambda_j(\left|\Psi_j^+\right\rangle \left\langle\Psi_j^+\right| + (\left|\Psi_j^-\right\rangle \left\langle\Psi_j^-\right|) \tag{3.1}$$

with

$$\left|\Psi_j^\pm\right\rangle = \frac{1}{\sqrt{2}}(\left|j\right\rangle_{AB} \left|0\right\rangle_C \pm \left|(3-j)\right\rangle_{AB} \left|1\right\rangle_C), \tag{3.2}$$

where $\left|j\right\rangle_{AB} \equiv \left|j_1\right\rangle_A \left|j_2\right\rangle_B$ with $j = j_1 j_2$ is understood in binary notation. For example, $\left|\Psi_0^\pm\right\rangle = \left|GHZ_\pm\right\rangle = (1/\sqrt{2})(\left|000\right\rangle \pm \left|111\right\rangle)$ are standard GHZ states. The state (3.1) describes a quantum state only if

$$\lambda_i \geq 0 \quad \text{for} \quad i = 1, 2, 3, +, -, \tag{3.3}$$

and λs are restricted by condition $\mathrm{Tr}(\varrho_3) = 1$. The state (3.1) has negative partial transpose over given qubit if

$$\begin{aligned} \varrho_3^{T_A} &< 0 \quad \text{if} \quad \Delta > 2\lambda_2, \\ \varrho_3^{T_B} &< 0 \quad \text{if} \quad \Delta > 2\lambda_1, \\ \varrho_3^{T_C} &< 0 \quad \text{if} \quad \Delta > 2\lambda_3, \end{aligned} \tag{3.4}$$

where $\Delta \equiv \lambda_0^+ - \lambda_0^- \geq 0$. In the following text we will instead of $|GHZ^+\rangle$ use the expression $|GHZ\rangle$, because $|GHZ^+\rangle$ state is part of a state which will be analyzed.
We start the construction of a fully inseparable state with nonlocalizable entanglement on a simple example of a state given by a convex mixture of the GHZ state and the maximally mixed state described by the identity operator. The latter state has the form It has form

$$\varrho_p = p\,|GHZ\rangle\,\langle GHZ| + \frac{1-p}{8}\mathbb{1}. \tag{3.5}$$

where $p \in [0,1]$ is a parameter controlling the ratio between the GHZ state and the maximally mixed state. This state is in the form of state (3.1) if $\lambda_0^- = 0$, $\lambda_0^+ = p + (1-p)/8$ and $\lambda_1 = \lambda_2 = \lambda_3 = (1-p)/8$. Moreover, from the inequalities (3.4) it follows that the state (3.5) is fully inseparable for $p > 1/5$. It has been shown in [13] that states in the form (3.5) are distillable using purification protocol described in [13] if $p > 0.32$. For values of parameter p greater then 0.32 one may still be able to distill GHZ state. In the reference [21] it was also shown that all fully inseparable states of the form (3.1) are also distillable which implies the distillability of the state (3.5) for all $p > 1/5$. These states can be also easily prepared in the laboratory. This is important because of experimental confirmation of our results.
In order to construct a fully inseparable state which can be also a candidate for a bound entangled state, we have to go beyond the structure (3.1). For simplicity, we construct a state which is fully symmetric, i.e., it is invariant under the exchange of any two qubits.

$$\varrho_{p,\mu} = p\varrho_\mu + \frac{1-p}{8}\mathbb{1}, \tag{3.6}$$

where ϱ_μ is a density matrix defined as

$$\varrho_\mu = \mu\,|GHZ\rangle\,\langle GHZ| + \frac{1-\mu}{3}(|001\rangle\,\langle 001| + |010\rangle\,\langle 010| + |100\rangle\,\langle 100|), \tag{3.7}$$

where the parameter $\mu \in [0,1]$ controls the ratio between the GHZ and the correlated noise given by the second term on the right-hand side of equation (3.7). The generalized state (3.6) belongs to the class of states (3.1) only in special cases when $\mu = 1$ or $p = 1$. If $\mu = 1$ then state (3.6) reduces to state (3.5). If $p = 1$ then we get only the mixture of the GHZ state with correlated noise (3.7). Due to the fact that these states have some eigenvalues equal to zero as can be seen from the state conditions (3.3), states lie on the boundary of the set of quantum states and therefore are not suitable for experimental proof of our results. For this reason we add to the state (3.7) also a noise term represented by the identity operator, which shifts the state from the boundary and thus makes it more friendly to experimental demonstration. In the following section we analyze entanglement as well as its localizability for the density matrix (3.6).

3.2 Nonlocalizable entanglement

We consider a fully symmetric three-qubit state (3.6), which is a mixture of a fully inseparable state, a separable correlated noise and a completely depolarized state. Now, we will answer the question when entanglement is present in this class of states and when it can be localized. In particular, we would like to know whether there exists a non-empty set of the states which are fully inseparable but the entanglement is nonlocalizable.

In order to find out, when our class of states is fully inseparable we will use the PPT criterion. Because the state is fully symmetric, it is sufficient to investigate the inseparability of the states only with respect to one bipartite splitting, say $A-(BC)$. The criterion states, that if the state (3.6) has a negative partial transpose with respect to the qubit A then the state is entangled with respect to the $A-(BC)$ splitting.

Actually, we want to find a condition on parameters p and μ due to PPT criterion so that $\rho_{p,\mu}^{T_A}$ has some eigenvalue negative. After some algebra we arrive at the condition telling us when the state (3.6) is fully inseparable in the form

$$p_{\mathrm{PPT}}(\mu) \equiv \frac{1-4\mu+4\sqrt{1-2\mu+10\mu^2}}{5-8\mu+48\mu^2} \leq p \leq 1 \tag{3.8}$$

The set of fully inseparable states in the plane of parameters p and μ is depicted by a gray area in Figure 2. where the presence of entanglement in the state (3.6) is show as a two-parametric set of states above for parameters p and μ above blue line.

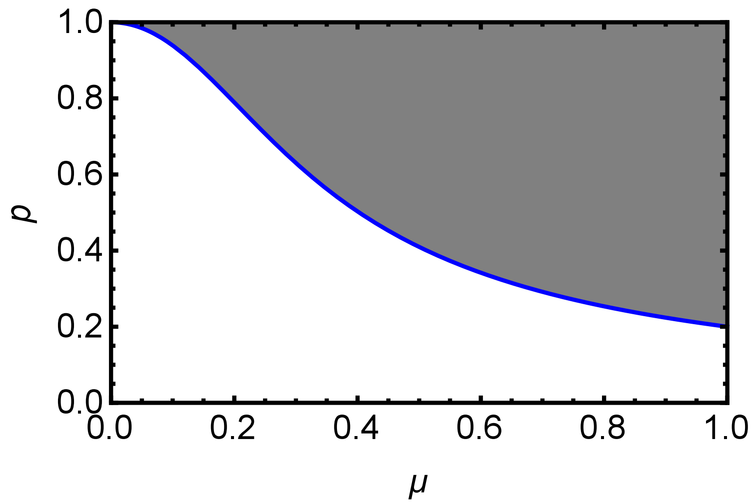

Figure 2: Gray area defined by the condition (3.8) depicts the set of parameters p and μ for which the state (3.6) is NPT and therefore fully inseparable. The threshold value $p_{\mathrm{NPT}}(\mu)$ above which the state (3.6) is fully inseparable is depicted by the blue curve. Note, that in principle there can be also entangled states in the white area below the blue curve due to the fact that in the case of entangled states between one qubit and two qubits (2×4 system) there can exist PPT entangled states [20].

From the inequality (3.8) we see that at given μ there exists a bound $p_{\mathrm{NPT}}(\mu)$ above which the state (3.4) is NPT and therefore fully inseparable. The Figure 2 further reveals, that the threshold value $p_{\mathrm{NPT}}(\mu)$ is a monotonously decreasing function of μ which attains a minimum of $p_{\mathrm{NPT,min}} = 0.2$ at $\mu = 1$. Therefore, if we decrease p below 0.2 the influence of maximally mixed state turns our state into a PPT state. On the other hand, at sufficiently large fixed value of the parameter p NPT property and therefore entanglement appears when we increase the influence of the GHZ state by increasing the value of the parameter μ.

Now, we would like to address a question, whether there is a subset of parameters in the gray area in Figure 2 which contains states the entanglement of which cannot be localized. Such states then lack an important property of fully inseparable states which is used in several quantum information applications.

In what follows, we try to find conditions for p and μ such that no entanglement can be localized between qubits A and B of the state (3.6) by any projective measurement on qubit C. Let us therefore consider a generic projective measurement on qubit C projecting onto the pure states $|\psi(\vartheta, \phi)\rangle$ and $|\psi_{\perp}(\vartheta, \phi)\rangle$ (2.8).

In order to decide, whether there is a state (3.6) from which entanglement cannot be

localized by any such measurement it is sufficient to investigate separability of the conditional state of qubits A and B after the projection onto the state $|\psi(\vartheta,\phi)\rangle$. If the conditional state is PPT for any choice of the angles ϑ and ϕ, then the state is separable. Then, also the conditional state after the projection onto the orthogonal state $|\psi_\perp(\vartheta,\phi)\rangle$ will be inevitably separable. As mixing of such states, which might be also transformed by some local operations dependent on the measurement outcome, cannot turn the separable states into entangled states, the resulting state will also be separable and thus no entanglement can be localized as required.
Now we will do measurement covering the whole Bloch sphere on qubit C (2.8) on our state (3.6), then we will find the least eigenvaule of a reduced density matrix after partial transpose and we will find angles ϑ and ϕ such as that least eigenvalue will be minimal. We need to have least eigenvalue minimal because due to PPT criterion we want to localize such entanglement as we can.
After measuring on qubit C (2.13) with (2.8) we get reduced unnormalized state

$$\tilde{\varrho}_{AB}(\vartheta,\phi) = \mathrm{Tr}_C(\rho_p \,|\psi(\vartheta,\phi)\rangle_C \,\langle\psi(\vartheta,\phi)|). \tag{3.9}$$

Using the definition of partial trace we can easily obtain elements of matrix (3.9) as follows

$$\begin{aligned} \tilde{\varrho}_{AB}(\vartheta,\phi) = \mathrm{Tr}_C(\varrho_{p,\mu}\,|\psi(\vartheta,\phi)\rangle_C\,\langle\psi(\vartheta,\phi)|) = \\ \mathrm{Tr}_C\left\{\varrho_{p,\mu}\left[\cos\left(\frac{\vartheta}{2}\right)|0\rangle_C + e^{i\phi}\sin\left(\frac{\vartheta}{2}\right)|1\rangle_C\right]\left[\cos\left(\frac{\vartheta}{2}\right)\langle 0|_C + e^{-i\phi}\sin\left(\frac{\vartheta}{2}\right)\langle 1|_C\right]\right\}. \end{aligned} \tag{3.10}$$

After multiplying brakets in (3.10) and using property of trace we obtain

$$\begin{aligned} \tilde{\varrho}_{AB}(\vartheta,\phi) = \frac{1+\cos(\vartheta)}{2}\;{}_C\langle 0|\,\varrho_{p,\mu}\,|0\rangle_C + \frac{1-\cos(\vartheta)}{2}\;{}_C\langle 1|\,\varrho_{p,\mu}\,|1\rangle_C + \\ \frac{\sin(\vartheta)}{2}\left(e^{i\phi}\;{}_C\langle 0|\,\varrho_{p,\mu}\,|1\rangle_C + e^{-i\phi}\;{}_C\langle 1|\,\varrho_{p,\mu}\,|0\rangle_C\right) = \cdots = \\ \frac{1}{2}\left[1+\cos(\vartheta)(\varrho_{00}-\varrho_{11}) + 2\sin(\vartheta)\,\Re\left(e^{i\phi}\varrho_{01}\right)\right], \end{aligned} \tag{3.11}$$

where

$$\varrho_{ij} = \sum_{k,l,m,n=0}^{1} {}_{ABC}\langle kli|\,\varrho_{p,\mu}\,|mnj\rangle_{ABC}\,|kl\rangle_{AB}\,\langle mn|\,. \tag{3.12}$$

Normalized state (3.9) is a state in a form

$$\varrho_{AB}(\vartheta,\phi) = \frac{\tilde{\varrho}_{AB}(\vartheta,\phi)}{\mathrm{Tr}_{AB}\tilde{\varrho}_{AB}(\vartheta,\phi)}. \tag{3.13}$$

If we use results recieved from (3.11) together with (3.12) and when we applied them on our state (3.6), result can be written in a matrix form

$$\varrho_{AB}(\vartheta,\phi) = \frac{1}{\mathcal{N}}\begin{pmatrix} c & 0 & 0 & b \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ b^* & 0 & 0 & d \end{pmatrix} \tag{3.14}$$

where $\mathcal{N}$ is a norm of (3.9) and a, b, b^*, c and d have the following expressions

$$a = \frac{1}{24}\left[3 + p - 4p\mu - 4p(-1+\mu)\cos(\vartheta)\right] \tag{3.15}$$

$$b = \frac{1}{4}e^{i\phi}p\mu\sin(\vartheta), \tag{3.16}$$

$$b^* = \frac{1}{4}e^{-i\phi}p\mu\sin(\vartheta), \tag{3.17}$$

$$c = \frac{1}{24}\left[3 + p + 2p\mu + 2p(-2+5\mu)\cos(\vartheta)\right] \tag{3.18}$$

$$d = \frac{1}{8}\left[1 - p + 2p\mu - 2p\mu\cos(\vartheta)\right]. \tag{3.19}$$

The norm of the matrix (3.14) $\mathcal{N}$ is equal to $(1/6)\left[3 - p(-1+\mu)\cos(\vartheta)\right]$ and it is positive for every $p \in [0,1]$, $\mu \in [0,1]$ and $\vartheta \in [0,\pi]$. Thus for finding the condition when matrix (3.13) is entangled according to PPT criterion we can do computation without norm of that matrix.

To find out under what conditions matrix (3.9) is entangled, we need to carry out partial transpose on (3.9), find the smallest eigenvalue and discuss its sign how PPT criterion says with respect to ϑ and ϕ. If the smallest eigenvalue after partial transpose of (3.9) will be negative then (3.9) is entangled.

After the partial transpose over qubit A (2.21) of matrix (3.9) we get

$$\tilde{\varrho}_{AB}^{T_A}(\vartheta,\phi) = \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b^* & a & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \tag{3.20}$$

We can reduce a problem of finding the smallest eigenvalue of (3.20) to finding the smallest eigenvalue of the matrix

$$\Lambda(\vartheta,\phi) = \begin{pmatrix} a & b \\ b^* & a \end{pmatrix}, \tag{3.21}$$

because eigenvalues corresponding to matrix element c and d will be positive for every p and μ from interval $[0,1]$.

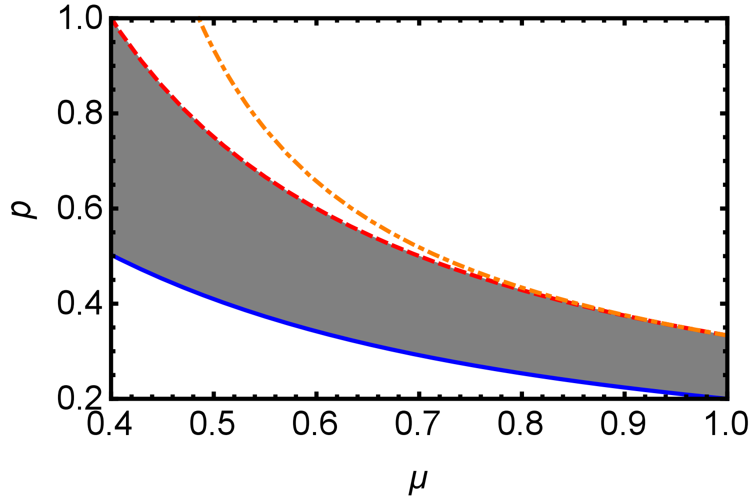

Figure 3: Red dashed line is curve of optimal measurement on $|\pm\rangle$. Under this line is a reduced density matrix (3.9) due to PPT criterion separable. Orange dot-dashed line is also a a curve of optimal measurement, but for $\vartheta \in [0, \pi/2]$. Above blue line is a density matrix (3.6) entangled, below that line is separable. Two-parametric set of states from gray area have entanglement which we can not localize with any projective measurement and are not usable for application in quantum information

The smallest eigenvalue of a matrix (3.21) will be after some easy computing in the form $\lambda_1 = a - |b|$. If we take a look on (3.16) we will find, that the smallest eigenvalue of (3.21) does not depend on ϕ. After substituting (3.15) and (3.16) into λ_1 we get three-parametric equation

$$\lambda_1 = \frac{1}{24}\left[3 + p - 4p\mu - 4p(-1+\mu)\cos(\vartheta) - \frac{1}{4}p\mu|\sin(\vartheta)|\right]. \tag{3.22}$$

We want to find the minimum of λ_1 with respect to ϑ so mathematically we are solving equation $\frac{d\lambda_1}{d\vartheta} = 0$. Solution of our optimization leads to the condition on ϑ which looks as follows

$$\tan(\vartheta) = \frac{3}{2}\frac{\mu}{\mu - 1}. \tag{3.23}$$

What does the equation (3.23) tell us? If we mention that $\vartheta \in [0, \pi]$ and function tangent is discontinuous at $\pi/2$ we have to split interval where ϑ is defined onto two subintervals $[0, \pi/2]$ and $[\pi/2, \pi]$.

Tangent for $\vartheta \in [0, \pi/2]$ is positive so the optimal measurement will be done under the angle corresponding to $\vartheta = \arctan(\frac{3}{2}\frac{\mu}{\mu-1})$. On the other hand, for $\vartheta \in [\pi/2, \pi]$ tangent is negative, but the right hand side of equation (3.23) is always positive, so extreme have to lie on the border of the interval $\vartheta \in [\pi/2, \pi]$. If we consider a sign of derivation, mathematical analysis tells us, that the eigenvalue λ_1 is increasing function of $\vartheta \in [\pi/2, \pi]$. Again according to the PPT criterion, we want to have this eigenvalue minimal so the optimal measurement will be reached for $\vartheta = \pi/2$. If we substitute this result into equation (2.8) we will find that minimize the lowest value of the smallest eigenvalue after partial transpose of the state (3.6) will be obtained by performing measurement on $|+\rangle$. Because optimal measurement is not phase-depended, measuring on $|-\rangle$ will give us the same result. We can localize the most of entanglement of the conditional state (3.13) by measuring the qubit C in the $|\pm\rangle$ basis as it can be seen in Figure 3. We will not localize more if we will be measuring in basis where $\vartheta = \arctan(\frac{3}{2}\frac{\mu}{\mu-1})$. This result was expected because of symmetry of our state (3.6).

3.3 Experimental data processing

For verification of our results, we collaborated with Quantum Optics Laboratory Olomouc, in particular with dr. M. Mičuda. We reconstructed density matrix using maximum likehood method [23] for $\mu = 1$ and $p = 0.25$. Values of parameters μ and p were chosen such that we can easily reconstruct our state in laboratory and so that we can verify if the entanglement is nonlocalizable . We reconstructed state

$$\varrho_{1/4} = \frac{1}{4} |GHZ\rangle \langle GHZ| + \frac{3}{32} \mathbb{1}. \tag{3.24}$$

State (3.24) is a mixture of maximally entangled GHZ state with weight 1/4 and noise with weight 3/4. Dr. M. Mičuda reconstructed one thousand density matrix corresponding to state (3.24). Now is our task to verify our theoretical results. First of all, we tested if state (3.24) is fully-inseparable due to splitting considered in 2.4.2.

j	A	B	C
$10^2 \times \mathrm{Min}[\mathrm{eig}(\varrho^{T_j})]$	-3.00 ± 0.05	-1.98 ± 0.05	-3.37 ± 0.05

Table 1: Values of eigenvalues of (3.6) after partial transpose over given qubit. All of the smallest eigenvalues are negative. So according to PPT criterion is (3.24) nonseparable with respect to splitting $A - (BC), B - (AC)$ and $C - (AB)$.

Results shown in Table 3.3 tells us that state (3.24) is nonseparable with respect to all splittings. This is in conformity with our theoretical result as can be seen in Figure 3.
We verified nonseparability and now we want to verify nonlocalizability. We have to do measurement on experimentally reconstructed matrices using (3.12) and do partial transpose over qubit A and find minimal eigenvalue. Finding minimal eigenvalue analytic way

is quite problematic, because experimental density matrix have every element non-zero. After performing measurement on the experimentally reconstructed state (3.24) and partial transpose, we decided to find the least eigenvalue numerically.
We divide both intervals for $\vartheta \in [0, \pi]$ and $\phi \in [0, 2\pi]$ to 63 values equally distributed on these intervals. Then we calculated value of the smallest eigenvalue numerically for every measurement (2.7) and corresponding values of ϑ and ϕ.

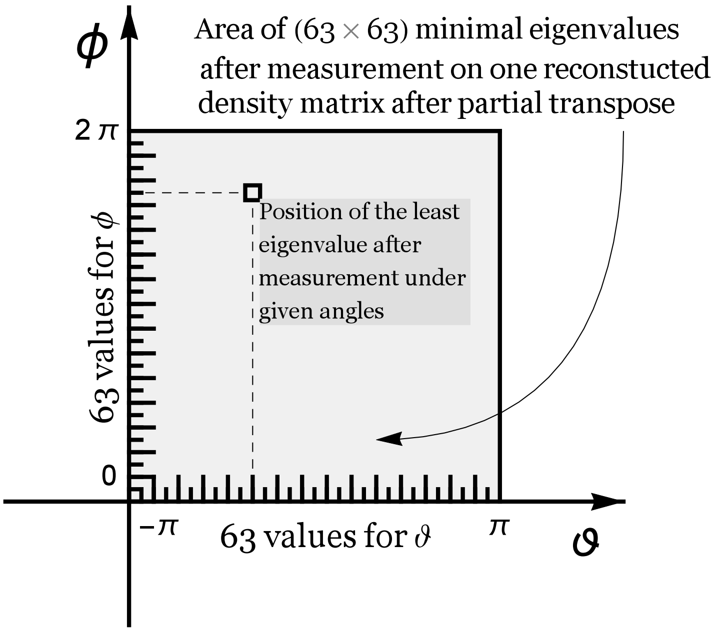

Figure 4: An example of computing the least eigenvalue after measurement on one reconstructed density matrix (3.24) and partial transpose over qubit A.

In the Figure 4 we can see how this works for one matrix. After measuring, doing partial transpose and dividing intervals for ϑ and μ into 63 discrete values, our reduced density matrix will split onto 63^2 matrices, which contains only numbers. Then using numerical algorithms we can easily find the smallest eigenvalue. We obtain 63×63 matrix, which elements are minimal values of eigenvalues after measurement and partial transpose. So we can see, for which ϑ and ϕ we can do the best measurement.
We do that on every reconstructed density matrix. Mean value of given smallest eigenvalue and standard deviation is $\text{Min}[\text{eig}(\varrho_{red}^{T_A})] = 10^{-2} \times (6.14 \pm 0.08)$ and their corresponding values ϑ and ϕ are equal with small deviation to $\vartheta = \pi/4$ and $\phi = 0$ or $\pi/2$. Because the minimal eigenvalue is positive, there is no entanglement which we can localize. We also check that optimal measurement is performed in $|\pm\rangle$ basis. These results are with perfect agreement with theoretical results.

3.4 Nonlocalizability of entanglement by POVM

In Section 3.2 we showed that there exist two-parametric set of states, which contain entanglement nonlocalizable by any projective measurement. But is there a non-projective measurement described in theory 2.3.2 which can localize more then projective measurement? Answer of this question is no, we cannot localize more of entanglement then by projective measurement. Prove of this statement is based on fact, that we can express every POVM element in the orthogonal basis using spectral decomposition. And because we cannot localize entanglement by projective measurement, we also cannot localize entanglement by POVM elements. That is because mixing of results of projective measurement is LOCC operation and entanglement cannot be created using these operations. These can be mathematically expressed as

$$\varrho_K^{(AB)} = \mathrm{Tr}_C[\varrho_{ABC}\Pi_K^{(C)}] = \sum_i \lambda_i \,\mathrm{Tr}_C[\varrho_{ABC} \left|\lambda_i\right\rangle_C \left\langle\lambda_i\right|] = \sum_i \lambda_i \varrho_{K,i}^{AB}, \tag{3.25}$$

where we spread POVM elements $\Pi_K^{(C)}$ acting on qubit C into diagonal basis using spectral decomposition $\Pi_K^{(C)} = \sum_i \lambda_i \left|\lambda_i\right\rangle_C \left\langle\lambda_i\right|$. If for every $\varrho_{K,i}^{AB} = \sum_l p_l \varrho_l^{(A)} \otimes \varrho_l^{(B)}$, e. g. $\varrho_{K,i}^{AB}$ is separable, then ϱ_K^{AB} is also separable.

3.5 Entanglement localization by unitary operation

3.5.1 Probabilistic localization

Even though entanglement in (3.24) is nonlocalizable, there exist techniques how to extract him. Lets have CNOT operation on qubits B and C, where B is control qubit and C target qubit. CNOT gate flips target qubit if and only if control qubit is $|1\rangle$. Let's have our reconstructed (3.24) state but with no exact p value. After CNOT operation on (3.6) we get

$$\begin{gathered} U_{CNOT}^{\dagger}\,\varrho_p\, U_{CNOT} = \\ p(\left|\Phi_+\right\rangle_{AB} \left\langle\Phi_+\right|) \otimes \left|0\right\rangle_C + \frac{1-p}{8}\mathbb{1}_{AB} \otimes \left|0\right\rangle_C, \end{gathered} \tag{3.26}$$

where U_{CNOT} is an unitary operation acting on qubits B and C and where we used the formula $U_{CNOT}^{BC} \left|GHZ\right\rangle_{ABC} = \left|\Phi_+\right\rangle_{AB} \left|0\right\rangle_C$. $\left|\Phi_+\right\rangle = (1/\sqrt{2})(|00\rangle + |11\rangle)$ is one of the fully inseparable Bell basis. If we will perform measurement $|0\rangle$ on C we get unnormalized Werner state [22]

$$\varrho_W = p\left|\Phi_+\right\rangle \left\langle\Phi_+\right| + \frac{1-p}{8}\mathbb{1}. \tag{3.27}$$

Norm of this state is equal to $(1+p)/2$. So normalized output looks as follows

$$\varrho_W = \frac{2p}{1+p} |\Phi_+\rangle \langle\Phi_+| + \frac{1-p}{4(1+p)}\mathbb{1}. \tag{3.28}$$

PPT criterion tells us, that the state (3.28) is entangled if coefficient which stands before $|\Phi_+\rangle \langle\Phi_+|$ is greater then 1/3. In our reconstructed state (3.24) we used $p = 1/4$. If we substitute this into (3.28), normalized Werner state will be in a form

$$\varrho_W = \frac{2}{5} |\Phi_+\rangle \langle\Phi_+| + \frac{3}{20}\mathbb{1}. \tag{3.29}$$

Because $2/5 > 1/3$, is state (3.29) entangled. Although we cannot localize entanglement contained in (3.24) we can obtain entangled state (3.29) using local CNOT operation propabilistically.

3.5.2 Deterministic localization

In Section 3.5 we showed that we can extract entanglement from class of states ϱ_p (3.5) into entangled class of states 3.28 for $1/5 < p \leq 1/3$ using CNOT operation on qubits BC. But this extraction of quantum entanglement is only probabilistic. Alternatively, we can extract a maximally entangled state deterministically applying local completely positive map (CPM) [24] to qubits B and C defined by $\mathcal{E}_{BC}(\varrho) = \sum_j \mathcal{O}_{BC}^{(j)} \varrho \mathcal{O}_{BC}^{(j)\dagger}$ with Kraus operators $\mathcal{O}_{BC}^{(1)} = \mathbb{1}_B \otimes |0\rangle_C \langle 0|$, $\mathcal{O}_{BC}^{(2)} = |0\rangle_B \langle 0| \otimes |1\rangle_C \langle 1|$ and $\mathcal{O}_{BC}^{(3)} = |0\rangle_B \langle 1| \otimes |1\rangle_C \langle 1|$. Kraus operators satisfy $\sum_j \mathcal{O}^{(j)\dagger}\mathcal{O}^{(j)} = \mathbb{1}$. Throwing away qubit C leaves the state

$$\begin{aligned}\varrho_p^{(AB)} = \mathrm{Tr}_C(\mathcal{E}_{BC}(\varrho_p^{(ABC)})) = p\,|\Phi_+\rangle \langle\Phi_+| + \frac{3(1-p)}{8}&(|00\rangle \langle 00| + |10\rangle \langle 10|) + \\ \frac{1-p}{8}&(|11\rangle \langle 11| + |01\rangle \langle 01|).\end{aligned} \tag{3.30}$$

A state $\varrho_p^{(AB)}$ is entangled (distillable) for $p > 3.02169$ due to PPT criterion. Earlier we showed, that class of states ϱ_p (3.5) is entangled for $1/5 < p \leq 1$ and entanglement is localizable for $1/3 < p \leq 1$. This result tells us that the deterministic way we can not extract entanglement from class of states ϱ_p (3.5) everywhere where is entangled.

4 Conclusion

We constructed three-bubit state with nonlocalizable entanglement. In subsection 3.2 we showed that there exist a two-parametric set of states containing entanglement which we cannot localize with a projective measurement. We also showed, that nonlocalizability with projective measurement implies nonlocalizability with general POVM elements. We also prove our results experimentally. In subsection 3.5 we showed that we can extract probabilistically maximally entangled state $|\Phi_+\rangle$ after CNOT operations on BC measuring $|0\rangle$ on qubit C. Deterministic extraction of entanglement in Section 3.5.2 showed that we can extract bipartite entangled state (3.30) for $p > 3.02169$.
We spread our analysis of CPM introduced in Section 3.5.2 on two parametric set of states $\varrho_{p,\mu}$ (3.6) and tried to use the output for dense coding. But analysis showed that the local von Neumann entropy ($S(\varrho) = -\operatorname{Tr}(\varrho \log \varrho)$) $S(\varrho_{p,\mu}^{(B)})$ [25] is smaller then a entropy $S(\varrho_{p,\mu}^{(AB)})$, where $\varrho_{p,\mu}^{(B)} = \operatorname{Tr}_A[\varrho_{p,\mu}^{(AB)}]$ and $\varrho_{p,\mu}^{(AB)}$ is a state $\varrho_{p,\mu}$ (3.6) after CNOT operation, CPM and throwing away qubit C. For dense coding is necessary to given state to be more disordered locally then globally. The question of utility of more sophisticated dense coding skills will be addressed elsewhere.
The question of distillability of the states (3.6) using methods of [21] is left for future work.

References

[1] D. M. Greenberger, M. Horne, and A. Zeilinger, in *Bell's Theorem, Quantum Theory, and Conceptions of the universe*, edited by M. Kafatos (Kluwer, Dordrecht, 1989), p. 69.

[2] C. H. Benett *et. al.* *Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels*. Phys. Rev. Lett. **70**, 1895 (1993)

[3] J. C. Hao, C. F. Li, and G. C. Guo. *Controlled dense coding using the Greenberger-Horne-Zeilinger state*. Phys. Rev. A **63**, 054301 (2001)

[4] Y. A. Chen, A. N. Zhang, Z. Zhao, X. Q. Zhou C. Y. Lu, C. Z. Peng, T. Yang, and J. W. Pan. *Experimental quantum secret sharing and third-man quantum cryptography.* Phys. Rev. Lett. **95**, 200502 (2005)

[5] W. B. Gao *et. al.* *Experimental measurement-based quantum computing beyond the cluster-state model.* NATURE PHOTONICS **5**, 117 (2011)

[6] P. Walther, K. J. Resch, T. Rudolph, E. Schenck, H. Weinfurter, V. Vedral, M. Aspelmeyer, and A. Zeilinger. *Experimental one-way quantum computing.* NATURE **434**, 169 (2005)

[7] R. Raussendorf and H. J. Briegel. *A one-way quantum computer*. Phys. Rev. Lett. **86**, 5188 (2001)

[8] X. C. Yao, *et. al.*, *Experimental demonstration of topological error correction.* NATURE **482**, 459 (2012)

[9] M. Hillery, V. Bužek, and A. Berthiaume. *Quantum secret sharing.* Phys. Rev. A **59**, 1829 (1999)

[10] M. Ziman and V. Bužek. *Correlation-assisted quantum communication.* Phys. Rev. A **67**, 042321 (2003).

[11] F. Verstraete, M. Popp, and J. I. Cirac. *Entanglement versus Correlations in Spin Systems.* Phys. Rev. Lett. **92**, 027901 (2004)

[12] M. Popp, F. Verstraete, M. A. Martín-Delgado, and J. I. Cirac. *Localizable entanglement.* Phys. Rev. A **71**, 042306 (2005)

[13] M. Murao, M. B. Plenio, S. Popescu, V. Vedral, and P. L. Knight. *Multiparticle entanglement purification protocols*. Phys. Rev. A **57**, 4075 (1998)

[14] W. Dür, J. I. Cirac, and R. Tarrach. *Separability and Distillability of Multipartite Quantum Systems*. Phys. Rev. Lett. **83**, 3562 (1999).

[15] W. Dür and J. I. Cirac. *Activating bound entanglement in multiparticle systems*. Phys. Rev. A **62**, 022302 (2000)

[16] J. J. Sakuraii. *Modern quantum mechanics*. Reading: Addison-Wesley, 1994.

[17] M. A. Nielsen and I. L. Chuang. *Quantum computation and quantum information*. Cambridge: Cambridge University Press, 2005.

[18] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki. *Quantum entanglement*. Rev. Mod. Phys. **81**, 865 (2009).

[19] A. Peres. *Separability Criterion for Density Matrices*. Phys. Rev. Lett. **77**, 1413 (1996)

[20] P. Horodecki. *Separability criterion and inseparable mixed states with positive partial transposition*. Phys. Lett. A **232**, 333 (1997)

[21] H. Aschauer, W. Dür, and H.-J. Briegel. *Multiparticle entanglement purification for two-colorable graph states*. Phys. Rev. A **71**, 012319 (2005)

[22] R. F. Werner. *Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model*. Phys. Rev. A **40**, 4277 (1989)

[23] M. Paris, J. Rehacek. *Quantum state estimation*. Springer-Verlag Berlin Heidelberg, 2004.

[24] T. S. Cubitt, F. Verstraete, W. Dür, and J. I. Cirac. *Separable states can be used to distribute entanglement*. Phys. Rev. Lett. **86**, 5184 (2001)

[25] M. A. Nielsen and J. Kempe. *Separable states are more disordered globally then locally*. Phys. Rev. Lett. **86**, 5184 (2001)