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Master of Science Mathematical Engineering Brno University of Technology (BUT)	Master of Science Mathematical Engineering University of L'Aquila (UAQ)

Master's Thesis

Discrete modeling of nonlinear beams under uniform external load

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ÚSTAV MATEMATIKY

DISCRETE MODELING OF NONLINEAR BEAMS UNDER UNIFORM EXTERNAL LOAD

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MASTER'S THESIS

DIPLOMOVÁ PRÁCE

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MASTER'S THESIS

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Assignment Master's Thesis

Institut:	Institute of Mathematics
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As provided for by the Act No. 111/98 Coll. on higher education institutions and the BUT Study and Examination Regulations, the director of the Institute hereby assigns the following topic of Master's Thesis:

Discrete modeling of nonlinear beams under uniform external load

Brief Description:

The Euler–Bernoulli beam theory and Timoshenko beam theory are the most used beam theories. The difference between the two theories is the consideration of the shear effect. The Euler–Bernoulli beam theory neglects the shear effect, while the Timoshenko beam theory does not. In practice, the Euler–Bernoulli beam theory is suitable for the analysis of thin beams. In contrast, the Timoshenko beam theory has a wider range of applications since it can be used for both thin and thick beams. However, numerical performances of Timoshenko beam formulations suffer from the shear–locking effect. Therefore, in Timoshenko beam formulations, special treatments are generally required to ensure accurate results for an analysis of thin beams. Unfortunately, these theories have inherently two drawbacks, namely:

1. The available discretization schemes used in solving these theories possess some inherent drawbacks when considering large deformation and displacement regimes.
2. The formulation of these theories are based as a mathematical object and they are void of inclusion of the mechanical phenomenon for describing their continuum models.

The present approach, the Hencky–type discrete model, tends to eliminate the inherent issue of discretization as the model is considered a discrete finite Lagrangian formulation. Similarly, adequate consideration is given to the model's mechanical phenomenon in its formulation, making it a richer model predicting real–world application better than previous models. Comparison of results between the theories is numerically proven, and possible areas of modification will also be reported.

Master's Thesis goals:

To study the behavior of a beam in the non–linear regime using an efficient discrete model.

Recommended bibliography:

BAROUDI, D., GIORGIO, I., BATTISTA, A., TURCO, E.,IGUMNOV, L.A. Non-linear dynamics of uniformly loaded Elastica: Experimental and numerical evidence of motion around curled stable equilibrium configurations. Z. Angew. Math. Mech. 99 (2019), Article ID e201800121, 20 pp.

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VO, D., NANAKORN, P. A total Lagrangian Timoshenko beam formulation for geometrically non-linear isogeometric analysis of planar, curved beams. Acta Mech. 231 (2020), 2827–2847.

Deadline for submission Master's Thesis is given by the Schedule of the Academic year 2022/23

In Brno,

L. S.

doc. Mgr. Petr Vašík, Ph.D.
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FME dean

Abstract

The concept of Beam theory is extensively studied in the fields of computational and structural mechanics, with widespread applications in both industry and academia. However, the existing body of knowledge lacks the derivation of important deformation equations due to the overly constrained assumptions made by early researchers in this area. This research aims to overcome these limitations by investigating beam deformation through the study of the centerline beam deformation theory, thus relaxing the previously adopted assumptions.

To achieve this goal, the energy functionals variational formulation was employed to derive a classical formulation that avoids the inherent assumptions of the Euler-Bernoulli and Timoshenko beam model equations. A discrete approach, known as Hencky-Type, was utilized to verify the inextensibility constraint of the nonlinear Euler-Bernoulli Beam. Furthermore, the linearized case was derived using variational methods applied to its nonlinear counterpart.

The derived models were then applied to two types of beams: the cantilever or clamped-Free (CF) beam and the simply supported beam (SS). A comparison was made to evaluate the superiority of these models. The nonlinear model formulation was solved using the weak formulation math model of COMSOL Multiphysics software.

This study aims to pave the way for more accurate model formulations and the development of novel numerical schemes that can effectively handle nonlinear models, which are often avoided due to their complexity. The findings from this work hold the potential to significantly advance the field and facilitate the exploration of various practical applications.

Keywords

Euler-Bernoulli, Timoshenko, Hencky-Type, Centerline Beam Theory, Variational, Weak & Classical Formulation, Energy Functional, Cantilever Beam, Simply Supported Beam, COMSOL Multiphysics

Folorunsho, S.S.: *Discrete modeling of nonlinear beams under uniform external load*, Brno University of Technology, Faculty of Mechanical Engineering, 2023. 68 pp. Supervisor: Professor Ivan Giorgio.

I declare that I wrote the diploma thesis *Discrete modeling of nonlinear beams under uniform external load* independently under the guidance of *Prof. Ivan Giorgio Ph.D.* using the literature included in the list of references.

Sodiq Sunday FOLORUNSHO

I begin by expressing my heartfelt gratitude and appreciation to Allah, the most Beneficent and Merciful, for bestowing upon me the knowledge, wisdom, and understanding necessary to complete this thesis. I acknowledge His blessings and guidance throughout my academic journey.

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1 Introduction

Beams, which are mathematical abstractions representing slender structural elements that are elongated in one dimension compared to their other dimensions, hold great significance in various engineering and structural applications. They serve as vital components in buildings, bridges, aircraft, automobiles, and numerous other constructions. The primary function of beams is to support and distribute loads to their supports, ensuring structural stability. Recent research has also investigated beam deformation in functionally graded materials and additive manufacturing, both on micro scales, such as Lab-on-a-chip technology, and macro scales, like riser platforms for oil and gas applications.

A fundamental aspect of beam analysis revolves around accurately predicting the load-bearing capacity and *deformation* [1] behavior of these structural elements. This knowledge is crucial for ensuring the expected structural integrity in diverse structural engineering applications. Over the years, extensive studies and innovations have been dedicated to comprehending the behavior of beams, optimizing their design, and exploring novel materials and construction techniques. These endeavors aim to enhance beam performance, efficiency, and safety in various practical scenarios.

1.1 Beam Theories

Amongst the several deformation theories, the *balance of forces and momentum*, and *centerline deformation beam theory CBDT* are predominantly the industrial and academic accepted formulation. The deformation beam theories are mathematical models used to analyze the behaviour of beam when subjected to external loading. The ideology is that the deformation of an entire body can be estimated exactly by considering how the *centerline* [2] of the body deforms.
Notable of the deformation beam theories are:

Euler-Bernoulli Beam Theory (EBT)

The mathematical statement of the Euler-Bernoulli beam theory is described by the Euler-Bernoulli beam equation, which relates the bending moment and the curvature of a beam subjected to transverse loading. The equation is as follows:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) = q \tag{1.1}$$

In this equation:

[1]Deformation is often described as a displacement field, representing the change in position of a point in the structure (for the purpose of this work – beam) relative to its original undeformed state. It is mostly referred also as changes in shape, size and position that occur when a load or force is applied to a structure.

[2]Centerline of a body is a continuous curve that represents the axis or midline of the body. It is a one-dimensional geometric representation that runs through the center or midpoint of the body, typically along its longest dimension or main axis.

1. EI represents the flexural rigidity of the beam, which is the product of the Young's modulus (E) and the area moment of inertia (I).
2. $\frac{d^2w}{dx^2}$ denotes the second derivative of the beam deflection (w) with respect to the axial coordinate x. It represents the curvature of the beam.
3. q is the distributed load on the beam.

The Euler-Bernoulli beam theory assumes several assumptions:

- Planar motion deformation of the beam centerline.
- The axial deformation of the beam is negligible compared to its bending deformation.
- The beam cross-section remain planar and perpendicular to the undeformed centerline.
- EBT is suitable for slender beams with a high aspect ratio (i.e. length to thickness ratio)

These assumptions resulted in the linearisation of the inherent nonlinear deformation measure in its derivation. This makes the model only befitting for small deformations, and in the large deformation regime, the model fails drastically.

Remark. The words deflection and displacement will be used interchangeably.

Timoshenko Beam Theory (TBT)

The Timoshenko beam theory provides an extension to the Euler-Bernoulli beam theory by considering the effect of shear deformation. The mathematical statement of the Timoshenko beam theory is described by the Timoshenko beam equation, which takes into account both bending and shear effects. The equation is as follows:

$$\begin{aligned} \frac{d^2}{dx^2}\left(EI\frac{d\varphi}{dx}\right) &= q(x) \\ \frac{dw}{dx} &= \varphi - \frac{1}{\kappa AG}\frac{d}{dx}\left(EI\frac{d\varphi}{dx}\right). \end{aligned} \tag{1.2}$$

The displacements of the beam are assumed to be given by:

$$u_x(x, y, z) = -z\varphi(x); \quad u_y(x, y, z) = 0; \quad u_z(x, y) = w(x)$$

Where (x, y, z) are the coordinates of a point in the beam, u_x, u_y, u_z are the components of the displacement vector in the three directions.

In this equation:

1. EI represents the flexural rigidity of the beam, similar to the Euler-Bernoulli beam theory.
2. φ is the angle of rotation of the normal to the mid-surface of the beam
3. w is the displacement of the mid-surface in the z-direction
4. κ is the correction factor based on the cross-section of the beam.
5. GA represents the shear rigidity of the beam, where G is the shear modulus and A is the cross-sectional area of the beam.

6. $q(x)$ denotes the transverse loading acting on the beam as a function of the axial position x.

The assumptions utilized in this formulation is as stated below:

- Plane motion deformation of the beam's centerline.
- Inclusion of both bending and shear deformations.
- The beam cross-section are not necessarily perpendicular to the deformed centerline due to the effect of shear and rotary inertia.
- TBT is applicable to beams with low aspect ratios and significant shear forces, such as thick beams or beams with open cross-sections.

Just like Euler-Bernoulli, Timoshenko also considers only the linearized case of the equation. The essence of this work is to now work with the nonlinear cases of these two models to ensure the superiority of the nonlinear model when compared to the linearised ones. The Centerline deformation beam theory is going to be used to define the nonlinear measures of deformation and the expression relating them are derived from the geometry given below:

Centerline Deformation Beam Theory (CDBT)

The centerline deformation beam theory is a theoretical framework used to describe the nonlinear deformation behavior of beams. Unlike the traditional Euler-Bernoulli and Timoshenko beam theories, which assume linearized deformations, the centerline deformation theory considers the full nonlinear deformation of the beam. It focuses on the changes in the position and orientation of the beam's centerline during deformation. In this theory, the centerline of the beam is treated as a continuous curve, and the displacements and rotations of points along this curve are used to define the beam's deformation. The theory takes into account the effect of *bending, shear, and axial* deformation on the overall behavior of the beam.

The centerline deformation beam theory provides a more accurate representation of beam deformations, especially for cases involving large deflections, nonlinear material behavior, or complex loading conditions. By considering the full nonlinear behavior, this theory allows for a better understanding and analysis of the structural response of beams in engineering and mechanical systems.

Expressions and equations are derived to relate the centerline deformation measures, such as displacements and rotations, to the applied loads, material properties, and geometric characteristics of the beam. These mathematical formulations provide a basis for numerical simulations, design optimization, and structural analysis of beams under various conditions.

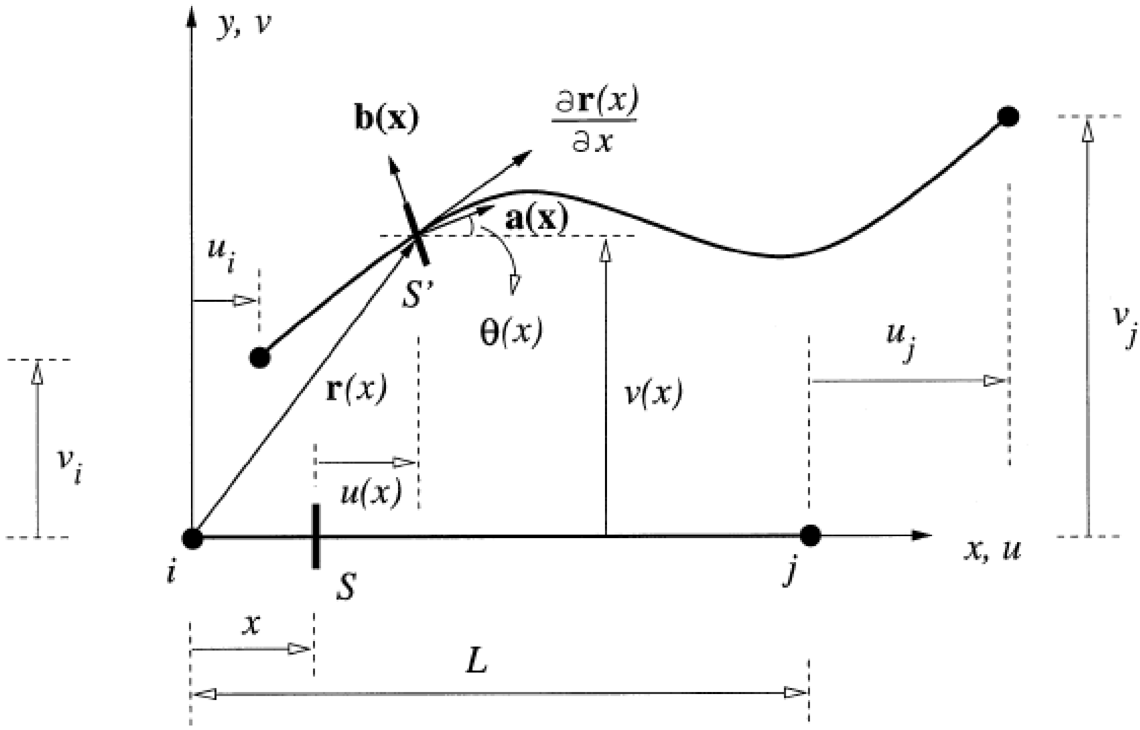

Figure 1.1: Plane beam: Initial and deformed configuration [10]

The crux of the centerline deformation beam theories is as explained in [10]. The beam is imagined as a one dimensional continuum. Its deformed configuration (Fig. 1.1) is described by a regular curve which is parameterised. The abscissa $x \in [0, L]$ is measured on the straight reference configuration of the beam, $u(x)$, $v(x)$ represents the *axial and transverse displacement components* and i and j are unit axis vectors. Each point on the beam axis is associated *cross-section* (S); and the angle $\theta(x)$ which defines the rotation of the cross section in the deformed configuration. The *deformation measures* ϵ, γ, κ which take values on the current deformed configuration alongside their constitutive relations are derived from the figure and material properties of the beam. These deformation measures allow for the proper definition of the total energy functional or Lagrangian on which analysis is performed to determine the minimum which coincides with the exact solution of primary deformation parameters.

1.2 Literature Review

Being one of the most exhaustively researched topic in engineering and structural application, it's practically impossible to give a thorough literature review on beams and its deformation. The review presented here is rather restrictive to conform to the relevant areas of review that's used in this current report. The categories of large deformation of elastic beams in the framework of nonlinear elasticity presented is taken from the works of [12].

Concerning the approach to the problem, we must state that two main lines may be identified based on the strategy employed to determine the governing equations: the variational method and the method of the balance of force and momentum. The variational approach, developed originally by Euler [3] and Lagrange [7, 8], allow the equations of motion and stability equilibrium conditions to be obtained in a relatively simple and efficient manner by minimizing the functional of energy (an example of the so-called principle of virtual work). The variational technique is evidently extremely commonly utilized while searching for mechanical system equilibria due to its ease of application and tremendous capabilities to examine even the most complex structures.

The review will be categorised in the following ways: large deformations of beam, different loads that can be applied to a beam, and deformation in beams.

Large Deflection of Beams: kinematic, Deformation and Action

To start out in the large deformations of beams, a recourse needs to be done first in analysing general problems in nonlinear mechanics and elasticity. Various aspects of nonlinear theories are entrenched in the books such as Antman [1], which focuses mainly on nonlinear problems in elasticity.

Antman [1] treated exhaustively the case of planar beam deformation and introduced theory of elastic rods , by assuming variational postulation of mechanics following the ground works of Euler [3] and Lagrange [7]. He gave a general theory for each kind of elastic body, by formulating specific problems and introduced the needed mathematical methods for strings, rods, shells and 3D bodies. Large part of his work is devoted to perturbation method, used in solving nonlinear differential equation, he likewise developed the bifurcation theory for problems of buckling, and in variational continuum mechanics, he introduced the classical multiplier rule and specifically the Lagrange multipliers, which are variables which are used predominantly in constrained optimization problems of beam theories.

Similarly, the works of Reissner [11] further amplified the deformation of nonlinear beams. His work was instrumental in the generalized form of constitutive relations in terms of axial force strain ϵ, shear force strain γ and bending strain κ. These are important measures of deformation used in almost every aspect of beam deformation theories. He looked into the classical Finite-Strain Beam Theory (FSBT) with application of the principle of virtual work, formulated a one-dimensional large strain theory for plane deformation of plane beams. This resulted in a system of non-linear strain displacement relations which is consistent with exact one-dimensional equilibrium equations for forces and moments. The article presented a novel way of incorporating transverse shear deformation into one-dimensional FSBT, this novelty rendered the earlier assumptions of Euler-Bernoulli classical theory assumption of absence of transverse shearing strain moot.

Distributed and Concentrated External Loads

In the problem of large deformation, while using the variational method approach, the energy due to external load plays a crucial role in obtaining the equation of motion as the final deformation shape experienced by a beam is largely dependent on the external applied load.
Following Reissner's variational principle [11], Humer [4], considered large deformations of

slender beams (Euler-Bernoulli) under a concentrated force. He gave the representation of solution in terms of elliptic integrals.

Also, Wang [14, 15] studied the case of deflection of a cantilever and simply supported beam. He presented the theoretical discussion on the nonlinear equation for the maximum deflection of the beam, likewise he also formulated a simple numerical method to analyse nonlinear bending of beams subjected to concentric load.

After an extensive review of the existing literature, it becomes evident that the majority of beam models still heavily rely on the assumptions put forth by Euler-Bernoulli, which assume that the beam's cross-section remains rigid and planar to the neutral axis before and after deformation. Furthermore, both the Euler-Bernoulli and Timoshenko formulations exhibit linearity, which can be attributed to the limited computational capabilities available during the time of their development. These linear models represent simplified versions of their non-linear counterparts, potentially overlooking more sophisticated deformation behaviors.

Addressing these gaps, the present study takes a novel approach by deriving the deformation behavior of beams from first principles without resorting to linearization. This approach forms the foundation for formulating the total energy functional or Lagrangian, capturing the true deformation characteristics of the beam. By employing the Lagrangian formulation, the classical model is presented, which offers a more accurate representation of the beam's deformation behavior.

This gap identification and rectification in beam analysis research not only challenges the long-standing assumptions but also strives to improve the understanding and predictive capabilities of beam behavior. By departing from linear models and incorporating non-linear deformation behaviors derived from first principles, the study aims to advance the field and pave the way for more sophisticated computational analyses and design optimizations in the future.

1.3 Variational Formulation

In this section the variational formulation which is the center piece on which this work rested is described in a rather intuitive and succinct way as contained in the works of [6].

The variational formulation is a generalization of the *principle of minimum potential energy*, which states that a system reaches the minimum accessible energy. In this formulation, the potential energy of a state is described by a functional $\Phi(u)$ $(\Phi(u) : M \longrightarrow \mathbb{R})$. The problem formulation involves determining the set of admissible states M in which the minimum is sought. The minimization problem for the functional Φ on the set M can be expressed as:

Find $u \in M$ such that $\Phi(u) \leq \Phi(v) \quad \forall v \in M$.

"Variation problem" refers to finding the minimum u of the functional $\Phi(u)$ by solving the equation for the first variation (i.e., the first differential) $\partial_v \Phi(u) = 0$ for all v, which has the same form as the weak formulation. In our context, the variational formulation involves the minimization of integral functionals $\Phi(u)$.

Weak and Variational Formulation

Consider a Hilbert space W and its subspace V. Let $A(u,v)$ be a continuous bilinear form on $W \times W$, and $b(v)$ be a continuous linear functional on W. We assume that the form $Q(u,v)$ is elliptic on V, meaning there exists a constant $\alpha > 0$ such that $Q(u,u) \geq \alpha \|u\|^2$ holds for all $u \in V$. For the nonhomogeneous Dirichlet condition, let U be an element in W that determines the linear set $V_U = \{v + U \mid v \in V\}$, which is a subset of W. We aim to find the solution in the set V_U. The weak formulation is given as follows: Find a function $u \in W$ such that $u - U \in V$, or equivalently, $u \in V_U$, and,

$$a(u,v) = b(v) \quad \forall v \in V.$$

We assume that the form $A(u,v)$ is symmetric, satisfying:

$$a(v,u) = a(u,v) \quad \forall u,v \in W.$$

Definition 1.1 (Variational formulation Of The Problem). Problem (V) We are looking for $u \in V_U$ such that $\Phi(u) \leq \Phi(v) \quad \forall v \in V_U$.

Let us remark that in the definition the condition $\Phi(u) \leq \Phi(v)\, \forall v \in V_U$ can be replaced by any of the following conditions:

1. $\Phi(u) \leq \Phi(u+v) \quad \forall v \in V$.
2. $\Phi(u) = \inf\{\Phi(v) \mid v \in V_U\}$.
3. $\Phi(u) = \inf\{\Phi(u+v) \mid v \in V\}$.

Theorem 1.2. *(Weak and Variational formulation)*
Let W be the Hilbert space and $\boldsymbol{Q}(u,v)$ the symmetric continuous bilinear form on the space $W \times W$, which is elliptic on V, i.e. V-elliptic, $b \in V^*$ and $\Phi(v)$ the functional $\Phi(v) = \frac{1}{2}a(v,v) - b(v)$. Then both problems are equivalent: i. e. the function u is solution of the weak formulation problem (W), if and only if the function u is the solution of the variational problem (V).

Proof. ($\Rightarrow$) Let us prove, that the weak solution is also the variational solution. Let $u \in V_U$ be a solution to the problem (W) and v an element of the space V. Let us estimate the difference $\Phi(u+v) - \Phi(u)$. Using linearity of b, symmetry of $Q(v,u) = Q(u,v)$ and linearity of the form a in both variables

$$\begin{aligned}
\Phi(u+v) - \Phi(u) &= \frac{1}{2}a(u+v,u+v) - b(u+v) - \frac{1}{2}a(u,u) + b(u) = \\
&= \frac{1}{2}[\boldsymbol{A}(u,u) + \boldsymbol{Q}(u,v) + \boldsymbol{A}(v,u) + \boldsymbol{A}(v,v) - \boldsymbol{A}(u,u)] - b(v) = \\
&= \boldsymbol{Q}(u,v) - b(v) + \frac{1}{2}\boldsymbol{Q}(v,v).
\end{aligned}$$

Since u is the weak solution, $Q(u,v) = b(v)$ holds and the first two terms mutually subtracts. Due to ellipticity of the form $Q(v,v) \geq \alpha\|v\|^2$ on V we obtain

$$\Phi(u+v) - \Phi(u) \geq \alpha\|v\|^2.$$

which implies $\Phi(u) \leq \Phi(u+v)$. Since v was arbitrary, the function u is the variational solution. Moreover the estimate implies that $\Phi(u+v)$ attains its minimum in the element u only, the solution is unique.

($\Leftarrow$) Conversely, let us prove that the variational solution is also the weak solution. Let $u \in V_U$ be the solution to the problem (V) and $v \in V$ arbitrary. Let us consider the real function,

$$\psi(t) = \Phi(u+tv).$$

Since the functional Φ attains its minimum in u, the function $\psi(t)$ attains its minimum for $t = 0$. Since $\psi(t)$ is differentiable, in its minimum $\psi'(0) = 0$ holds. Let us compute the derivative

$$\begin{aligned}\psi'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\Phi(u+tv) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}a(u+tv,u+tv) - b(u+tv)\right] = \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}a(u,u) + \frac{1}{2}ta(u,v) + \frac{1}{2}ta(v,u) + \frac{1}{2}t^2Q(v,v) - b(u) - tb(v)\right] = \\ &= \frac{1}{2}Q(u,v) + \frac{1}{2}Q(v,u) + tQ(v,v) - b(v).\end{aligned}$$

Using the symmetry $Q(v,u) = Q(u,v)$, the equality $\psi'(0) = 0$ implies $Q(u,v) - b(v) = 0$. Since $v \in V$ was arbitrary, the function u is the weak solution. □

Theorem 1.3 (Existence and uniqueness of the variational solution). *Under the assumptions of the Theorem 1.2 the functional $\Phi(u)$ is bounded from below on V_U and attains its infimum m in the unique point u, i.e. the variational problem (V) admits unique solution.*

1.4 Goal of Thesis

We aim to derive from first principle the nonlinear deformation equation for both Euler-Bernoulli and Timoshenko beam models and compare them with models in literature.
Similarly, we aim to study the behavior of beam in the non-linear regime using an efficient discrete model (Hencky-Type)

1.5 Thesis Outline

As earlier presented until now, section 1 looked at the introduction and some literary text on Beam theories and an important area of gap identified in these literature paved way for this current study.

Chapter2 looked extensively at the first variation formulation of the strong form equation for an Euler-Bernoulli Beam catering for the non-linearities that has been oversimplified or downright not looked into in literature.

Following same spirit as Chapter 2 , Chapter 3 presented same light but in the case of Timoshenko-Beam.

Chapter 4 is dedicated to the discrete aspect of the project. As presented in Literature, Henkcy-Type model has been reported to predict exactly or near exact the behaviour of the non-linear Euler-Bernoulli beam subjected to pure bending (In-extensibility constraint). So, to affirm this claim, Section 4 looked extensively in the Discrete description of a beam.

Chapter 5 exhaustively looked into application of the above derived model to 2 separate types of boundary conditions for beams, the Simply Supported Beam (SS) and, Clamped Free (CF) or Cantilever Beam. Also, in the same section, the claim of Henkcy-type beam predicting an Euler-Bernoulli Beam is also presented.

Chapter 6 presents the conclusion of the study and outlines possible future research directions.

2 Euler- Bernoulli Beam

In the Euler-Bernoulli beam theory, it is assumed that plane cross sections perpendicular to the axis of the beam remain plane and perpendicular to the axis after deformation [13]

The following assumptions are required to be fulfilled in the formulation of Euler-Bernoulli Beam theory viz:

1. Assume only planar motion.
2. Cross-section is assumed to be rigid.
3. The Variables of interest - displacement components, and angle of cross-section are continuous (i.e. $u(s), v(s), \varphi(s) \in C^0[0, L]$).
4. The cross-section is parallel/collinear to the tangent of the deformation curve, i.e. $\eta_{cs} \parallel \tau_\gamma$ as shown in Figure (2.1). This implies that the *shear deformation* [3] = 0.

2.1 Kinematic Description and Deformation Energy

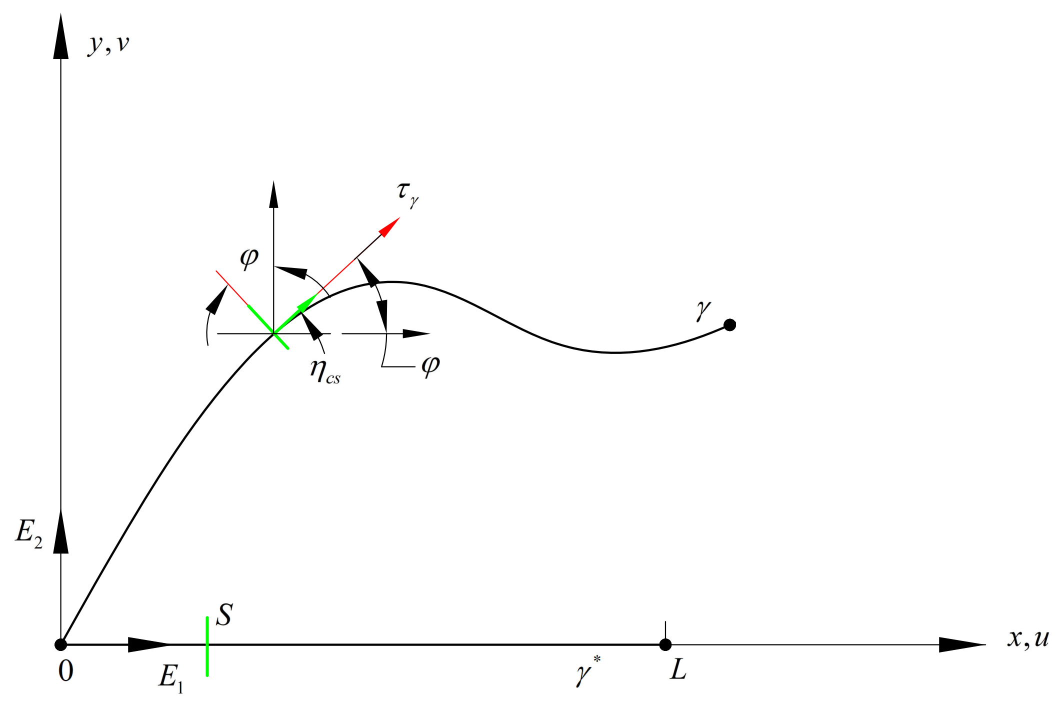

Figure 2.1: Centerline deformation for Euler-Bernoulli Beam

The beam is imagined as a one dimensional continuum (has high aspect ratio - length to cross-section area). Its deformed configuration (Fig. 2.1) is described by a regular curve which is

[3] Shear deformation is the difference between η_{cs} and τ_γ

parameterised. The abscissa $S \in [0, L]$ is measured on the *straight reference configuration* of the beam γ^*, $u(S)$, $v(S)$ represents the axial and transverse displacement components. Each point on the beam axis is associated cross-section (S); and the angle $\varphi(S)$ which defines the rotation of the cross section in the current or deformed configuration γ. The deformation measures of an Euler-Bernoulli beam is given as: *axial deformation* ϵ, and *bending deformation* κ which take values on the current deformed configuration.

Mathematical Models

In planar motion, there are 3 types of deformation mode that can be introduced to define the deformation of a beam. These modes are explained as:

- Axial deformation of the beam

Deformation curve of the current configuration γ can be parameterised as:

$$\gamma(S) = \begin{cases} S + u(S), \\ v(S). \end{cases} \tag{2.1}$$

The tangent vector to the curve γ can be expressed from the derivative as:

$$\tau_\gamma(S) = \begin{cases} 1 + u'(S), \\ v'(S). \end{cases} \tag{2.2}$$

The length of the deformed beam in the current configuration is expressed as:

$$L^* = \int_0^L \|\tau_\gamma\| dS = \int_0^L g dS.$$

where,

$$g = \sqrt{(1 + u')^2 + (v')^2}.$$

Change of Length

$$L^* - L = \int_0^L g dS - \int_0^L dS.$$

$$\varepsilon = \frac{d(L^* - L)}{ds} = (g - 1)\,.$$

The measure of elongation is expressed as

$$\varepsilon = \sqrt{(1 + u')^2 + (v')^2} - 1. \tag{2.3}$$

- Bending deformation of the beam

From Figure(2.1), the measure of the cross-section rotation angle φ is expressed as:

$$\varphi = tan^{-1}\left(\frac{v'}{1+u'}\right).$$

Then bending deformation is expressed as:

$$\kappa - \kappa^* = \frac{d\varphi}{dS}.$$

but, $\kappa^* = 0$ since the reference configuration for the beam is straight.

Performing the differentiation, the bending deformation is thus calculated as:

$$\kappa = \frac{v''\left(1+u'\right) - u''v'}{\left(1+u'\right)^2 + \left(v'\right)^2}. \tag{2.4}$$

2.1.1 Euler-Bernoulli (Non-linear Model)

Since we've established that the deformation of the beam can be measured by the quantities $(g-1)$ and κ, the associated deformation energy reads as:

$$\mathscr{E}_{\mathrm{EB}}^{\mathrm{Def}} = \int_0^L \frac{1}{2}k_e\left(g-1\right)^2 ds + \int_0^L \frac{1}{2}k_b\kappa^2 ds. \tag{2.5}$$

Substituting the expression of g and κ into Equ.(2.5), we have:

$$\mathscr{E}_{\mathrm{EB}}^{\mathrm{Def}} = \int_0^L \frac{1}{2}k_e\left(\sqrt{\left(1+u'\right)^2+\left(v'\right)^2}-1\right)^2 ds + \int_0^L \frac{1}{2}k_b\left[\frac{v''\left(1+u'\right)-u''v'}{\left(1+u'\right)^2+\left(v'\right)^2}\right]^2 ds. \tag{2.6}$$

The two addends in the above integrals are, respectively, called "*extensional energy,*" and "*flexural energy,*" and the positive material parameters k_e and k_b are the associated stiffness. The total energy or *Lagrangian* of Euler-Bernoulli of the system is given as below as:

$$\mathscr{E}^{\mathrm{Tot}} \equiv \mathscr{L} = \mathscr{E}_{\mathrm{EB}}^{\mathrm{Def}} + \mathscr{W}^{\mathrm{ext}}. \tag{2.7}$$

Where $\mathscr{W}$ is the potential energy contribution of the external loads applied to the beam. The first variation of the total energy above is expressed using Theroem 1.2

$$\delta\mathscr{E}^{\mathrm{Tot}} \equiv \delta\mathscr{L} = \delta\mathscr{E}_{\mathrm{EB}}^{\mathrm{Def}} + \delta\mathscr{W}^{\mathrm{ext}} = 0. \tag{2.8}$$

The first Variation of the deformation energy for the above deformation energy thus becomes

$$
\begin{aligned}
\delta\mathscr{E}_{\mathrm{EB}}^{\mathrm{Def}} = & \int_0^L \frac{1}{2}k_e \cdot 2\,(g-1) \times \frac{1}{2\cdot g} \times 2\,(1+u')\,\delta u' + \int_0^L \frac{1}{2}\cdot k_e \cdot 2\,(g-1) \times \frac{2v'}{2\cdot g}\delta v' \\
& + \int_0^L k_b \cdot \kappa \cdot \frac{(1+u')\,\delta v''}{g^2} + \int_0^L k_b \cdot \kappa \cdot \frac{(-v\prime)\,\delta u''}{g^2} \\
& + \int_0^L k_b \cdot \kappa \cdot \frac{\left(v''\left(g^2 - 2(1+u')^2\right) - 2u''(1+u')v'\right)\delta u'}{g^4} \\
& - \int_0^L k_b \cdot \kappa \cdot \frac{\left(u''\left(g^2 - 2(v')^2\right) + 2v''(1+u')v'\right)\delta v'}{g^4}.
\end{aligned}
$$

Collecting like terms and performing Integration by parts to each term of the above expression with respect to the test functions $\delta u'$, $\delta v'$, $\delta u''$, $\delta v''$, and combining them afterward gives:

$$
\begin{aligned}
& \delta\mathscr{E}_{\mathrm{EB}}^{\mathrm{Def}} = \\
& - \int_0^L \left(\left[k_e \left(1 - \frac{1}{g}\right)(1+u') \right]' + \left[\frac{k_b \kappa v'}{g^2} \right]'' + \left[\frac{k_b \kappa}{g^4} \left(v'' \left(g^2 - 2\,(1+u')^2 \right) - 2u''\,(1+u')^2\, v' \right) \right]' \right) \delta u ds \\
& - \int_0^L \left(\left[k_e \left(1 - \frac{1}{g}\right) v' \right]' - \left[\frac{k_b \kappa\,(1+u')}{g^2} \right]'' - \left[\frac{k_b \kappa}{g^4} \left(u'' \left(g^2 - 2\,(v')^2 \right) + 2v''\,(1+u')\, v' \right) \right]' \right) \delta v ds \\
& + \left(\left[k_e \left(1 - \frac{1}{g}\right)(1+u') \right] + \left[\frac{k_b \kappa v'}{g^2} \right]' + \left[\frac{k_b \kappa}{g^4} \left(v'' \left(g^2 - 2\,(1+u')^2 \right) - 2u''\,(1+u')^2\, v' \right) \right] \right) \delta u \Big|_0^L \\
& + \left(\left[k_e \left(1 - \frac{1}{g}\right) v' \right] - \left[\frac{k_b K\,(1+u')}{g^2} \right]' + \left[\frac{k_b \kappa}{g^4} \left(u'' \left(g^2 - 2\,(v')^2 \right) + 2v''\,(1+u')\, v' \right) \right] \right) \delta v \Big|_0^L \\
& - \left[\frac{k_b \kappa v'}{g^2} \right] \delta u' \Big|_0^L - \left[\frac{k_b \kappa\,(1+u')}{g^2} \right] \delta v' \Big|_0^L = 0 \quad \forall\, \delta u, \delta v, \delta u', \delta v'. \qquad (2.9)
\end{aligned}
$$

Considering a beam Fig.(2.2) subject to [complex loading [4]] of all possible external loads that can be sustained by a beam. This approach is to enable the non-linear model be robust to cater for all possible combinations of loading a beam can be subjected.
The external work potential for these complex loading subjected to the beam is expressed as.

$$
\delta\mathscr{W}^{\mathrm{ext}} = - \int_0^L b_1(s)\delta u - \int_0^L b_2(s)\delta v - \int_0^L \mu(s)\delta\varphi - F\delta v|_0^L - N\delta u|_0^L - M\delta\varphi|_0^L = 0. \qquad (2.10)
$$

Recall that $\varphi = f(u, v)$, hence the first variation of this parameter is given as:

$$
\delta\mathscr{W}^{\mathrm{ext}} = - \int_0^L b_1(s)\delta u - \int_0^L b_2(s)\delta v - \int_0^L \mu(s)\frac{\partial\varphi}{\partial u'}\delta u' ds
$$

[4] It's impossible to have a beam subjected to these hypothetical complex loading simultaneously in real life application

$$-\int_0^L \mu(s)\frac{\partial\varphi}{\partial v'}\delta v' ds - F\delta v|_0^L - N\delta u|_0^L - M\frac{\partial\varphi}{\partial u'}\delta u' \;|_0^L - M\frac{\partial\varphi}{\partial v'}\delta v' \;|_0^L = 0. \qquad (2.11)$$

where

1. $b_1(s)$ is the uniform axial load along the length of beam
2. $b_2(s)$ is the uniform transverse load along the length of the beam
3. $\mu(s)$ is the uniform couple load along the the length of the beam
4. F is the transverse force applied to the end of the beam
5. M is the bending moment applied to the free end of the beam
6. N is the axial load applied to the free end of the beam

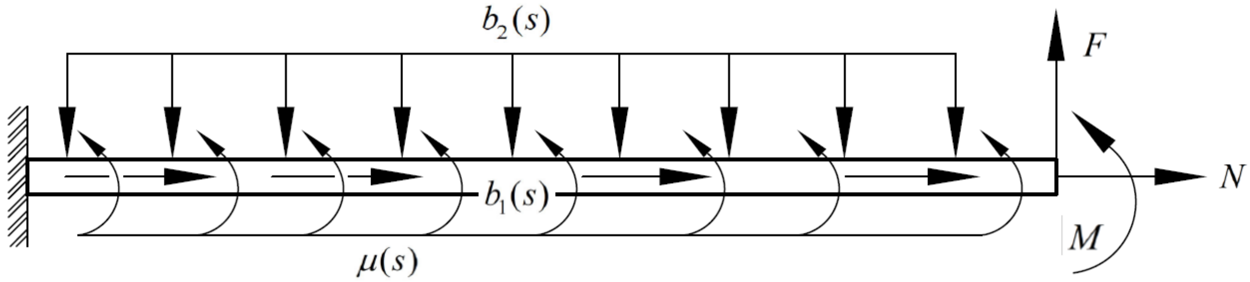

Figure 2.2: Complex Loading of a Clamped-Free Beam

Simplifying the above first variation by incorporating the dependence of u and v on φ, we have:

$$\begin{aligned}\delta\mathscr{W}^{\text{ext}} =& -\int_0^L b_1(s)\delta u - \int_0^L b_2(s)\delta v + \int_0^L \mu(s)\frac{v'}{g^2}\delta u' - \int_0^L \mu(s)\left(\frac{1+u'}{g^2}\right)\delta v' s \\ & - F\delta v|_0^L - N\delta u|_0^L + M\left(\frac{v'}{g^2}\right)\delta u'\Big|_0^L - M\left(\frac{1+u'}{g^2}\right)\delta v'\Big|_0^L = 0.\end{aligned}$$

Performing integration by part on the first variation of external potential to undifferentiate the test functions $\delta u'$ and $\delta v'$, we have:

$$\begin{aligned}\delta\mathscr{W}^{\text{ext}} =& -\int_0^L \left(b_1(s) + \left(\frac{\mu(s)v'}{g^2}\right)'\right)\delta u - \int_0^L \left(b_2(s) - \left(\frac{\mu(s)(1+u')}{g^2}\right)'\right)\delta v + \left(\frac{\mu(s)v'}{g^2}\right)\delta u\Big|_0^L \\ & - \left(\frac{\mu(S)(1+u')}{g^2}\right)\delta v\Big|_0^L + \left(\frac{Mv'}{g^2}\right)\delta u'\Big|_0^L - \left(\frac{M(1+u')}{g^2}\right)\delta v'\Big|_0^L = 0 \;\forall\, \delta u, \delta v, \delta u', \delta v'.\end{aligned} \qquad (2.12)$$

The strong formulation for the system is obtained by substitution Equ.(2.9) and Equ.(2.12) into Equ.(2.8). Using the test lemma defined in 7.7, performing necessary mathematical manipulation, the differential equation and the boundary conditions describing the deformation of Euler-Bernoulli beam is given below as:

$$\left[k_e\left(1-\frac{1}{g}\right)(1+u')\right]' + \left[\frac{\mu(s)v'}{g^2}\right]' + \left[\frac{k_b\kappa v'}{g^2}\right]'' + \left[\frac{k_b\kappa}{g^4}\left(v''\left(g^2-2\left(1+u'\right)^2\right)-2u''\left(1+u'\right)v'\right)\right]' + b_1(s) = 0. \tag{2.13}$$

$$\left[k_e\left(1-\frac{1}{g}\right)v'\right]' - \left[\frac{\mu(s)(1+u')}{g^2}\right]' - \left[\frac{k_b\kappa\left(1+u'\right)}{g^2}\right]'' + \left[\frac{k_b\kappa}{g^4}\left(u''\left(g^2-2\left(v'\right)^2\right)+2v''\left(1+u'\right)v'\right)\right]' + b_2(s) = 0. \tag{2.14}$$

Using the following definition,

$$\begin{aligned}
Bu &:= \left[k_e\left(1-\frac{1}{g}\right)(1+u')\right] + \left[\frac{\mu(s)v'}{g^2}\right] + \left[\frac{k_b\kappa v'}{g^2}\right]' + \left[\frac{k_b\kappa}{g^4}\left(v''\left(g^2-2\left(1+u'\right)^2\right)-2u''\left(1+u'\right)v'\right)\right]\\
Bv &:= \left[k_e\left(1-\frac{1}{g}\right)v'\right] - \left[\frac{\mu(s)(1+u')}{g^2}\right] - \left[\frac{k_b k\left(1+u'\right)}{g_2}\right]' + \left[\frac{k_b\kappa}{g^4}\left(u''\left(g^2-2\left(v'\right)^2\right)+2v''\left(1+u'\right)v'\right)\right]\\
Bu_p &:= \left[\frac{k_b\kappa+M}{g^2}\right]v'\\
Bv_p &:= \left[\frac{k_b\kappa-M}{g^2}\right](1+u')
\end{aligned}$$

The boundary condition for the system is thus given as:

$$Bu = N \quad \text{or} \quad \delta u = 0. \tag{2.15}$$

$$Bv = F \quad \text{or} \quad \delta v = 0. \tag{2.16}$$

$$Bu_p = 0 \quad \text{or} \quad \delta u' = 0. \tag{2.17}$$

$$Bv_p = 0 \quad \text{or} \quad \delta v' = 0. \tag{2.18}$$

2.1.2 Euler-Bernoulli (Linear Model)

The linear approximation of the Euler-Bernoulli model is achieved by linearising the elongation and bending measure of deformation of a beam as discussed earlier. The two measures of deformation are expanded using Taylor's series and truncated at the linear term. This procedure is as exemplified below:

Elongation Approximation

Recall that $\varepsilon = \sqrt{(1+u')^2+(v')^2}-1$.

$$\varepsilon\left(u', v'\right) = \varepsilon|_{\substack{u'=0 \\ v'=0}} + \left.\frac{\partial \varepsilon}{\partial u'}\right|_{\substack{u'=0 \\ v'=0}} u' + \left.\frac{\partial \varepsilon}{\partial v'}\right|_{\substack{u'=0 \\ v'=0}} v' + \cdots + h.o.t.$$

Each term of the expansion is as expressed below:

$$\varepsilon|_{\substack{u'=0 \\ v'=0}} = 0.$$

$$\frac{\partial \varepsilon}{\partial u'} = \left.\frac{(1+u')}{\sqrt{(1+u)')' + (v')^2}}\right|_{\substack{u'=0 \\ v'=0}} = 1.$$

$$\frac{\partial \varepsilon}{\partial v'} = \left.\frac{(v')}{\sqrt{(1+u)')' + (v')^2}}\right|_{\substack{u'=0 \\ v'=0}} = 0.$$

The linearised elongation term is therefore given as:

$$\varepsilon_{lin} = u'. \tag{2.19}$$

Bending Curvature Linear Approximation

Recall that the bending curvature was earlier derived as:

$$\kappa = \frac{v''\left(1+u'\right) - u''v'}{\left(1+u'\right)^2 + \left(v'\right)^2}.$$

Performing Taylor's series expansion of this parameter around the origin we have:

$$\kappa\left(u', v', u'', v''\right) = \kappa|_{\substack{u'=0 \\ v'=0 \\ u''=0 \\ v''=0}} + \left.\frac{\partial \kappa}{\partial u'}\right|_{\substack{u'=0 \\ v'=0 \\ u''=0 \\ v''=0}} u' + \left.\frac{\partial \kappa}{\partial v'}\right|_{\substack{u'=0 \\ v'=0 \\ u''=0 \\ v''=0}} v' + \left.\frac{\partial \kappa}{\partial u''}\right|_{\substack{u'=0 \\ v'=0 \\ u''=0 \\ v''=0}} u'' + \left.\frac{\partial \kappa}{\partial v''}\right|_{\substack{u'=0 \\ v'=0 \\ u''=0 \\ v''=0}} v'' + \cdots + h.o.t.$$

Performing the same procedure as the elongation case, the linearised bending curvature is evaluated as:

$$\kappa_{lin} = v''. \tag{2.20}$$

Substituting the evaluated linear approximations into the deformation energy functional, we have:

$$\mathscr{E}^{\text{Def}}_{\text{EB-lin}} = \int_0^L \frac{1}{2} k_e \,(u')^2\, ds + \int_0^L \frac{1}{2} k_b \,(v'')^2\, ds. \tag{2.21}$$

Remark. 1. The first term of the linearised energy functional corresponds to the case of elongation of the beam.
2. The second term caters for the pure bending and (or) the in-extensible characteristics

The first variation of the linearised deformation energy is expressed as:

$$\delta\mathscr{E}^{\text{Def}}_{\text{EB-lin}} = \int_0^L k_e u' \delta u' ds + \int_0^L k_b (v'') \delta v'' ds = 0 \quad \forall \delta u',\, \delta v''.$$

Performing integration by part to the expression above, we have the

$$\begin{aligned}\delta\mathscr{E}^{\text{Def}}_{\text{EB-lin}} = &- \int_0^L (k_e u')' \delta u ds \quad + (k_e u')\, \delta u|_0^L - \int_0^L (k_b v'')'' \delta v ds \quad - (k_b v'')'\, \delta v\Big|_0^L \\ &+ (k_b v'')\, \delta v'|_0^L \quad = 0 \quad \forall \delta u, \delta v, \delta v'. \end{aligned}\tag{2.22}$$

Utilizing the external work first variation expression derived in Equ.(2.10), and combining with the Equ.(2.22) and test lemma definition used earlier in the nonlinear case, yields the strong formulation equation for both the elongation and bending curvature measures of deformation respectively as:
Axial Elongation Deformation:

$$\begin{aligned} k_e u'' + b_1(s) = 0, \quad s \in [0, L] \quad &\text{if}, k_e = c \in \mathbb{R}. \\ &\text{Subject to the following boundary conditions} \\ [(k_e u') - N]\, \delta u|_0^L &= 0. \\ &\text{This is expressed as:} \\ (k_e u')\,|_0 = 0, \quad &\text{or} \quad \delta u = 0. \\ (k_e u')\,|_L = N, \quad &\text{or} \quad \delta u = 0. \end{aligned}\tag{2.23}$$

Similarly the Transverse Deformation is given as:

$$\begin{aligned} k_b v'''' + \mu(s) - b_2(s) = 0, \quad s \in [0, L] \quad &\text{if}, k_b = c \in \mathbb{R}. \\ &\text{Subject to the following boundary conditions} \\ [k_b v''' - F - \mu]\, \delta v|_0^L &= 0. \\ [(k_b v'') - M]\, \delta v'|_0^L &= 0. \end{aligned}\tag{2.24}$$

Careful examination of Equ. 2.24, we can see that it is the recovery of Euler-Bernoulli Beam theory defined in Equ. 1.1. This is achieved by setting $\mu(s) = 0$, $q(x) = b_2(s)$, and $k_b = EI$

2.2 Application to a Clamped-Free (CF) Beam

Applying the above equation to a Clamped-Free/Cantilever Beam, we have the equation above reduced to:

Figure 2.3: Cantilever Beam (CF) and its boundary conditions

$$\left[k_e\left(1-\frac{1}{g}\right)(1+u')\right]' + \left[\frac{k_b\kappa v'}{g^2}\right]'' + \left[\frac{k_b\kappa}{g^4}\left(v''\left(g^2-2(1+u')^2\right)-2u''(1+u')v'\right)\right]' = 0. \tag{2.25}$$

$$\left[k_e\left(1-\frac{1}{g}\right)v'\right]' - \left[\frac{k_b\kappa(1+u')}{g^2}\right]'' + \left[\frac{k_b\kappa}{g^4}\left(u''\left(g^2-2(v')^2\right)+2v''(1+u')v'\right)\right]' = 0. \tag{2.26}$$

$$\begin{aligned} Bu|_L &= 0, \quad Bv|_L = F. \\ Bu_p|_0 &= 0, \quad Bv_p|_0 = 0. \end{aligned} \tag{2.27}$$

2.3 Application to a Simply Supported (SS) Beam

Figure 2.4: Simply Supported Beam (SS) and its boundary conditions

The corresponding Euler-Bernoulli deflection equation and it's corresponding boundary condition for the Simply Supported (SS) Beam is expressed below as:

$$\left[k_e\left(1-\frac{1}{g}\right)(1+u')\right]' + \left[\frac{k_b\kappa v'}{g^2}\right]'' + \left[\frac{k_b\kappa}{g^4}\left(v''\left(g^2-2\,(1+u')^2\right)-2u''\,(1+u')\,v'\right)\right]' = 0. \tag{2.28}$$

$$\left[k_e\left(1-\frac{1}{g}\right)v'\right]' - \left[\frac{k_b\kappa\,(1+u')}{g^2}\right]'' + \left[\frac{k_b\kappa}{g^4}\left(u''\left(g^2-2\,(v')^2\right)+2v''\,(1+u')\,v'\right)\right]' = 0. \tag{2.29}$$

$$u(0) = u(L) = 0.$$
$$Bu_p|_0 = 0, \quad Bv_p|_L = M. \tag{2.30}$$

3 Timoshenko Beam

3.1 Kinematic Description and Deformation Energy

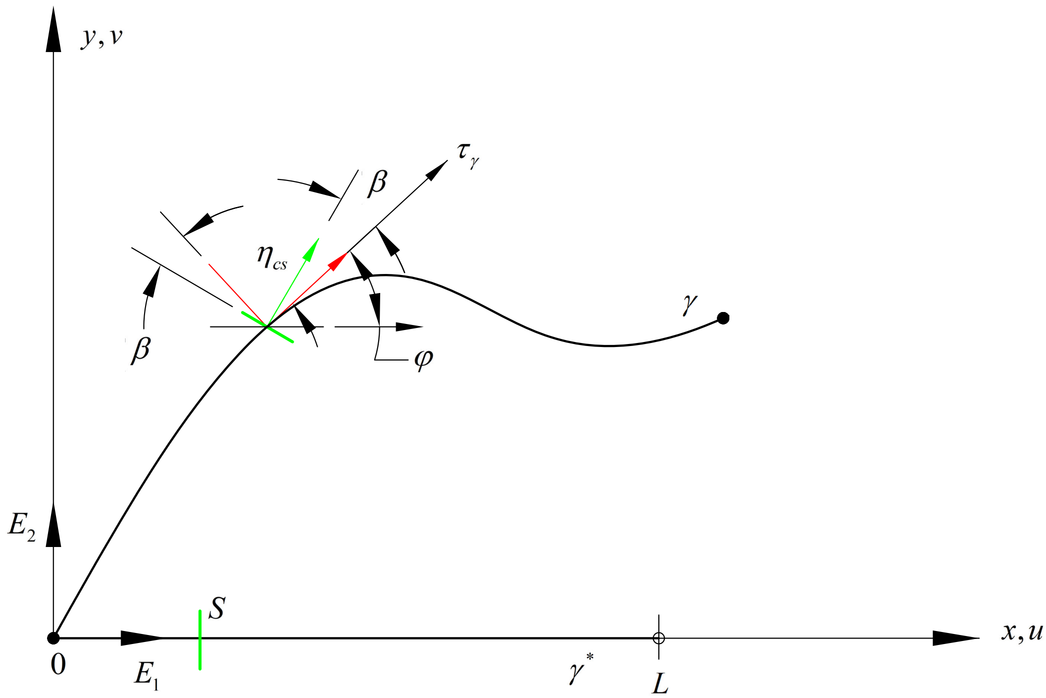

Figure 3.1: Centerline deformation for Timoshenko Beam

The beam is imagined as a one dimensional continuum (has high aspect ratio - length to cross-section area). Its deformed configuration (Fig. 3.1) is described by a regular curve which is parameterised. The abscissa $S \in [0, L]$ is measured on the straight reference configuration of the beam γ^*, and $u(S)$, $v(S)$, $\varphi(s)$ represents the axial and transverse displacement components and cross-section rotation angle respectively. Each point on the beam axis is associated cross-section (S); and the angle $\beta(S)$ which defines the shear angle between normal to the cross-section of the geometry to the tangent to deformed configuration in the current configuration γ. The deformation measures of an Timoshenko beam is given as: axial deformation ϵ, bending deformation κ and shear deformation β which take values on the current deformed configuration.

Mathematical Model

The axial and bending measure of deformation described in Euler-Bernoulli framework is also valid in the case of Timoshenko beam. The additional measure of deformation that's considered in Timoshenko's model is that of shear deformation. Also, it's pertinent to emphasise that the rotation angle $\varphi(s)$ in the case of the present model is independent of the axial and transverse displacement $u(s), v(s)$. The shear deformation is expressed from Figure(3.1) as:

$$\beta = tan^{-1}\left(\frac{v'}{1+u'}\right) - \varphi. \tag{3.1}$$

3.1.1 Timoshenko Beam (Non-Linear Model)

Since we've established that the deformation of the beam can be measured by the quantities $(g-1)$, κ, and β the associated energy reads as:

$$\mathscr{E}_{\text{TIM}} = \int_0^L \frac{1}{2}k_e\left(g-1\right)^2 ds + \int_0^L \frac{1}{2}k_b\left(\varphi'\right)^2 ds + \int_0^L \frac{1}{2}k_s\beta^2 ds. \tag{3.2}$$

where,

$$g = \sqrt{(1+u')^2 + (v')^2}.$$

$$\beta = \tan^{-1}\left(\frac{v'}{1+u'}\right) - \varphi.$$

Substituting the expression of g, and β into Equ.(3.2), we have:

$$\mathscr{E}^{\text{Def}}_{\text{TIM}} = \int_0^L \frac{1}{2}k_{\text{e}}\left(\sqrt{(1+u')^2+(v')^2}-1\right)^2 ds + \int_0^L \frac{1}{2}k_b\left(\varphi'\right)^2 ds + \int_0^L \frac{1}{2}k_s\left[\tan^{-1}\left(\frac{v'}{1+u'}\right)-\varphi\right]^2 ds. \tag{3.3}$$

The three addends in the above integrals are, respectively, called "*extensional energy*", "*flexural energy*", and "*shear energy*" and the positive material parameters k_e, k_b and k_s are the associated stiffness.

$$\delta\mathscr{E}^{\text{Def}}_{\text{TIM}} =$$

$$\int_0^L \frac{1}{2}k_e\cdot 2\left(\sqrt{(1+u')^2+(v')^2}-1\right)\cdot\frac{1}{2\sqrt{(1+u')^2+(v')^2}}\cdot 2\left(1+u'\right)\delta u' ds$$

$$+\frac{1}{2}\cdot k_e\cdot 2\left(\sqrt{(1+u')^2+(v')^2}-1\right)\cdot\frac{2v'}{2\sqrt{(1+u')^2+(v')^2}}\delta v' ds + \frac{1}{2}k_b\cdot 2\left(\varphi'\right)\delta\varphi' ds$$

$$+\frac{1}{2}k_s\cdot 2\left(\tan^{-1}\left(\frac{v'}{1+u'}\right)-\varphi\right)\cdot\frac{1}{1+\left(\frac{v'}{1+u'}\right)^2}\cdot v'(-1)\left(1+u'\right)^{-2}\delta u' ds$$

$$+\frac{1}{2}k_s\cdot 2\left(\tan^{-1}\left(\frac{v'}{1+u'}\right)-\varphi\right)\cdot\frac{1}{1+\left(\frac{v'}{1+u'}\right)^2}\cdot\frac{\delta v'}{(1+u')}+\frac{1}{2}k_s\cdot 2\left(\tan^{-1}\left(\frac{v'}{1+u'}\right)-\varphi\right)(-1)\delta\varphi ds.$$

Performing some necessary housekeeping, using the definition defined above. The above expression is simplified as:

$$\delta\mathscr{E}^{\text{Def}}_{\text{TIM}} = \int_0^L \left[k_e\left(1-\frac{1}{g}\right)(1+u') - \frac{k_s\beta v'}{g^2}\right]\delta u' ds + \int_0^L \left[k_e\left(1-\frac{1}{g}\right)v' + \frac{k_s\beta}{g^2}(1+u')\right]\delta v' ds$$
$$+ \int_0^L k_b\varphi'\delta\varphi' ds - \int_0^L k_s\beta\delta\varphi ds. \tag{3.4}$$

Performing integration by parts to undifferentiate the test functions in each variable, yields the expression below:

$$\delta\mathscr{E}^{\text{Def}}_{\text{TIM}} = -\int_0^L \left[k_e\left(1-\frac{1}{g}\right)(1+u') - \frac{k_s\beta v'}{g^2}\right]' \delta u ds + \left[k_e\left(1-\frac{1}{g}\right)(1+u') - \frac{k_s\beta v'}{g^2}\right]\Bigg|_0^L \delta u$$
$$-\int_0^L \left[k_e\left(1-\frac{1}{g}\right)v' + \frac{k_s\beta}{g^2}(1+u')\right]' \delta v ds + \left[k_e\left(1-\frac{1}{g}\right)v' + \frac{k_s\beta}{g^2}(1+u')\right]\Bigg|_0^L \delta v$$
$$-\int_0^L \left[(k_b\varphi')' + k_s\beta\right]\delta\varphi ds + [k_b\varphi']|_0^L \delta\varphi = 0 \quad \forall \delta u,\ \delta v\ \delta\varphi. \tag{3.5}$$

Recall that

$$\delta\mathscr{E}^{Tot} \equiv \delta\mathscr{L} = \delta\mathscr{E}^{\text{Def}} + \delta\mathscr{W}^{\text{ext}} = 0. \tag{3.6}$$

The general work potential of external loading subjected to a beam is expressed as:

$$\delta\mathscr{W}^{\text{ext}} = -\int_0^L b_1(s)\delta u ds - \int_0^L \mu(s)\delta\varphi ds - \int_0^L b_2(s)\delta v ds - F\delta v|_0^L - N\delta u|_0^L - M\delta\varphi|_0^L = 0.$$

Substitute the above work potential into the Lagrangian expression above we have the following systems of the differential equation.

$$\left[k_e\left(1-\frac{1}{g}\right)(1+u') - \frac{k_s\beta v'}{g^2}\right]' + b_1(s) = 0. \tag{3.7}$$

$$\left[k_e\left(1-\frac{1}{g}\right)v' + \frac{k_s\beta}{g^2}(1+u')\right]' + b_2(s) = 0. \tag{3.8}$$

$$[k_b\varphi']' + k_s\beta + \mu(s) = 0. \tag{3.9}$$

Using the following definition,

$$Cu := \left[k_e\left(1-\frac{1}{g}\right)(1+u') - \frac{k_s\beta v'}{g^2}\right].$$
$$Cv := \left[k_e\left(1-\frac{1}{g}\right)v' + \frac{k_s\beta}{g^2}(1+u')\right].$$
$$C\varphi := [k_b\varphi'].$$

The boundary condition for the system is thus given as:

$$Cu = N \quad \text{or} \quad \delta u = 0. \tag{3.10}$$

$$Cv = F \quad \text{or} \quad \delta v = 0. \tag{3.11}$$

$$C\varphi = M \quad \text{or} \quad \delta \varphi = 0. \tag{3.12}$$

3.1.2 Timoshenko Beam (Linear Model)

The linear elongation contribution approximation is as derived in the case of Euler-Bernoulli above. Timoshenko has its bending curvature case not simplified as it depends only on φ (i.e. $\kappa_{lin} = \varphi'$) and not u and v like that of Euler-Bernoulli. The only term that needs further linearization is the shear term β.
Recall that the shear angle is earlier derived as:

$$\beta = \tan^{-1}\left(\frac{v'}{1+u'}\right) - \varphi.$$

Linearising this term gives:

$$\beta_{lin} = v' - \varphi. \tag{3.13}$$

Substituting the evaluated linear approximation into the deformation energy functional, we have:

$$\mathscr{E}^{\text{Def}}_{\text{TM-lin}} = \int_0^L \frac{1}{2}k_e\,(u')^2\,ds + \int_0^L \frac{1}{2}k_b\,(\varphi')^2\,ds + \int_0^L \frac{1}{2}k_s\,(v'-\varphi)^2\,ds. \tag{3.14}$$

Remark. 1. The first term of the linearised energy functional corresponds to the case of elongation of the beam.
2. The second term caters for the pure bending and (or) the in-extensible characteristics
3. The third term predicts the shear effects of the beam when subjected to external load.

For the first term of the energy functional, the analysis follows similarly as that carried out in Euler-Bernoulli. Attention is placed on the coupled bending and shear contribution:

$$\mathscr{E}^{\text{Def}}_{\text{bs-lin}} = \int_0^L \frac{1}{2}k_b\,(\varphi')^2\,ds + \int_0^L \frac{1}{2}k_s\,(v'-\varphi)^2\,ds. \tag{3.15}$$

The first variation of Equ.(3.15) above becomes:

$$\delta\mathscr{E}^{\text{Def}}_{\text{bs-lin}} = \int_0^L k_b\,(\varphi')\,\delta\varphi' ds - \int_0^L k_s\,(v'-\varphi)\,\delta\varphi ds + \int_0^L k_s\,(v'-\varphi)\,\delta v' ds = 0.$$

Performing integration by part to the expression above, we have the

$$\begin{aligned}\delta\mathscr{E}^{\text{Def}}_{\text{bs-lin}} = &-\int_0^L (k_b\varphi')'\delta\varphi ds \quad + (k_b\varphi')\,\delta\varphi|_0^L - \int_0^L k_s(v'-\varphi)\delta\varphi ds \\ &-\int_0^L (k_s(v'-\varphi))'\,\delta v ds \quad + (k_s(v'-\varphi))\,\delta v|_0^L = 0 \quad \forall \delta v, \delta\varphi. \end{aligned} \tag{3.16}$$

Utilizing the external work first variation expression derived in Equ.(2.10), and combining with the Equ.(3.16) yields the classical formulation of the coupled bending and shear contribution given below as:

Bending Deformation:

$$\begin{aligned} k_s\,((v'-\varphi))' + b_2(s) = 0, \quad s \in [0, L] \quad \text{if}, k_s = c \in \mathbb{R}. \\ \text{Subject to the following boundary conditions} \\ [k_s\,((v'-\varphi)) - F]\,\delta v|_0^L = 0. \\ \text{This is expressed as:} \\ k_s\,((v'-\varphi))\,|_0 = 0, \quad \text{or} \quad \delta v = 0. \\ k_s\,((v'-\varphi))\,|_L = F, \quad \text{or} \quad \delta v = 0. \end{aligned} \tag{3.17}$$

The Shear Deformation is also expressed as:

$$\begin{aligned} k_b\,(\varphi)' + k_s\,((v'-\varphi)) + \mu(s) = 0, \quad s \in [0, L] \quad \text{if}, k_s, k_b = c \in \mathbb{R}. \\ \text{Subject to the following boundary conditions :} \\ [k_b\varphi' - M]\,\delta\varphi|_0^L = 0. \\ \text{This is expressed as:} \\ (k_b\varphi' - M)\,|_0 = 0, \quad \text{or} \quad \delta\varphi = 0. \\ (k_b\varphi' - M)\,|_L = M, \quad \text{or} \quad \delta\varphi = 0. \end{aligned} \tag{3.18}$$

3.1.3 Saint-Venants Micro-Macro Identification (Linearised)

The relation between the elongation, bending and shear stiffness k_e, k_b, and k_s and its constitutive properties is given by Saint-Venant and it is as described below:

$$k_e = Y \cdot A. \tag{3.19}$$

$$k_b = Y \cdot J. \tag{3.20}$$

$$k_s = q \cdot G \cdot A. \tag{3.21}$$

where,

1. A is the cross-section area of the beam.
2. J is the second moment of area of the cross-section of the beam.
3. Y is the Young's modulus of the beam.
4. G is the section modulus of the beam expressed as:

$$G = \frac{Y}{2(1+\nu)}.$$

5. q is the corrective factor. According to [5], the numerical value for q is given as:

$$q = \frac{5}{6} \quad \text{and} \quad \frac{6}{7}$$

for rectangular and circular cross-section respectively.

6. ν is the poisson's ratio of the material of the beam.

3.2 Application to Clamped-Free (CF)beam

The differential equation as well as its boundary condition is expressed as:

$$\begin{aligned}
&\left[k_e\left(1-\frac{1}{g}\right)(1+u') - \frac{k_s\beta v'}{g^2}\right]' = 0.\\
&\left[k_e\left(1-\frac{1}{g}\right)v' + \frac{k_s\beta}{g^2}(1+u')\right]' = 0.\\
&[k_b\varphi']' + k_s\beta = 0.
\end{aligned}$$

Subject to the following BCs.:

$$\begin{aligned}
&\left[K_e\left(1-\frac{1}{g}\right)(1+u') - \frac{k_s\beta v'}{g^2}\right]\Big|_L = 0.\\
&\left[K_e\left(1-\frac{1}{g}\right)v' + \frac{k_s\beta}{g^2}(1+u')\right]\Big|_L = F.\\
&[K_b\varphi']|_L = 0.
\end{aligned}$$

3.3 Application to Simply Supported (SS) Beam

$$\begin{aligned}
&\left[k_e\left(1-\frac{1}{g}\right)(1+u') - \frac{k_s\beta v'}{g^2}\right]' = 0.\\
&\left[k_l\left(1-\frac{1}{g}\right)v' + \frac{k_s\beta}{g^2}(1+u')\right]' = 0.\\
&[k_b\varphi']' + k_s\beta = 0.
\end{aligned}$$

Subject to the following BCs.:

$$\left[k_e\left(1-\frac{1}{g}\right)(1+u')-\frac{k_s\beta'}{g^2}\right]\bigg|_L = 0.$$
$$\left[k_e\left(1-\frac{1}{g}\right)v'+\frac{k_s\beta}{g^2}(1+u')\right|_L = 0$$
$$[k_b\varphi']|_L = M.$$

4 Hencky-Type Discrete Model

4.1 Kinematic Description

To formulate the Hencky-Type model, the following assumptions which are peculiar to continuous Euler-Bernoulli beam model are used [2]:

1. The conservative part of deformation energy (i.e. elastic energy) depends upon the relative Hencky-bars rotations (i.e. discrete).
2. The Hencky bars are rigid (i.e. the axis of the Euler beam is inextensible).
3. As a result of the simplicity of the Hencky model then the concept of shear deformation cannot even be formulated when considering the discrete model, this situation corresponds to the assumption that the the cross-section with respect to the axis of the beams is negligible; and that this cross-section must be regarded to be not deformable.

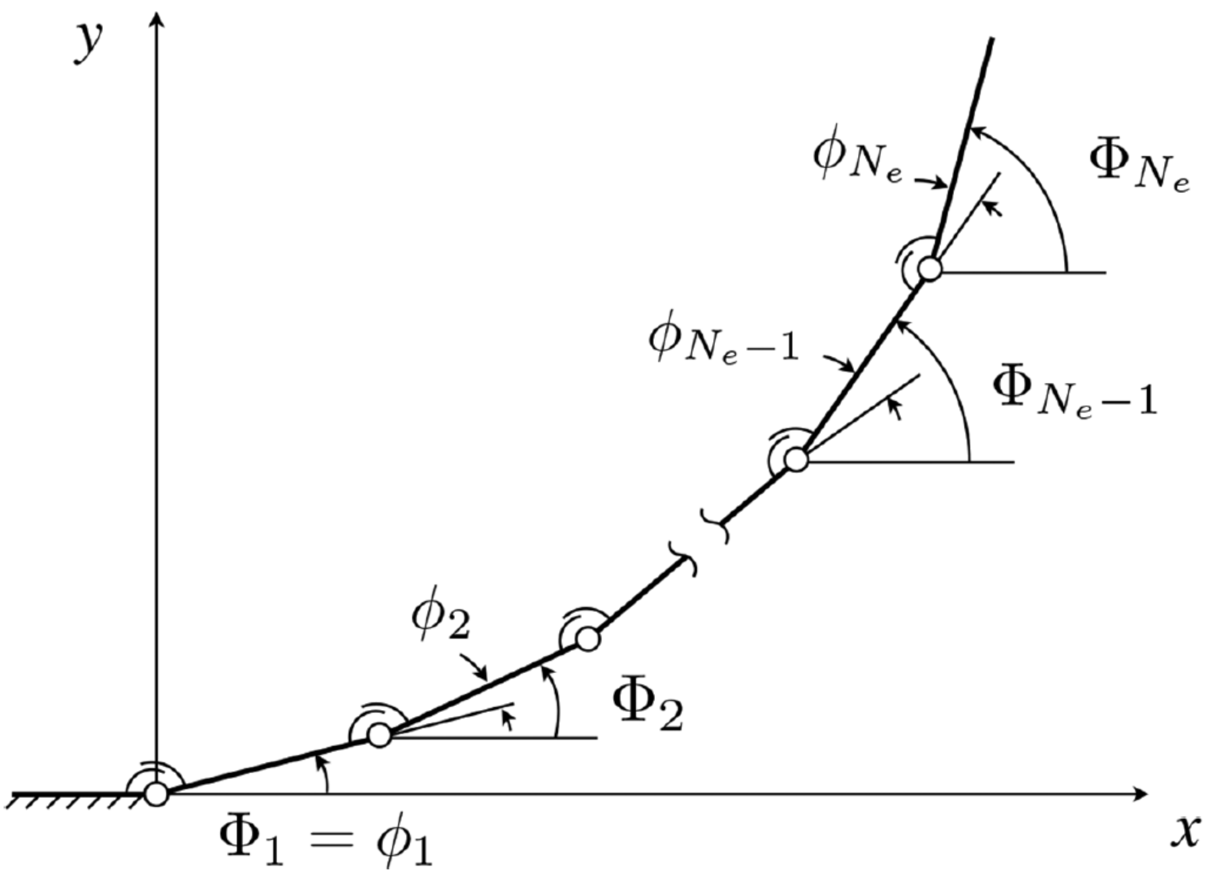

Figure 4.1: Hencky-Type lumped mass-spring model for high flexible beam.

Adapting the Hencky technique, we consider a discrete system made of an 'articulated' sequence of N_e (rigid, as the beam is assumed inextensible) rods of length η which are constrained at their terminal points by perfect hinges, *i.e.* hinges which are exerting vanishing couples. Modelling the elasticity effects which are resistant to the bending of a mechanical system, we assume a rotational spring is applied at each joint connecting adjacent rods. See Fig. 4.1.

To fully describe each Lagrangian configuration of the Hencky system, it suffices only to specify the Lagrangian angular coordinates Φ_i.These coordinates determine the orientation of each of the considered rigid rods relative to the x-axis which is chosen to be horizontal (the applied loads will be assumed to be vertical). The direction along which the applied loads are assumed to be directed coincides with the y-axis, which is oriented vertically upwards.
The set of bars, before the deformation, is assumed to be straight along the y-axis. The springs will have their rest length such that this configuration will have the minimum of deformation energy.

4.2 Mathematical Model

4.2.1 Hencky-Type Approximation

The bending deformation energy of the Hencky discrete model is postulated in [2] as:

$$\Psi_{el} = \sum_{i=1}^{Ne} k_{b_i} \frac{\phi_i^2}{2}. \tag{4.1}$$

Where,

- k_{b_i} is the bending stiffness for each section of the rod.
- ϕ_i is the relative angles between adjacent rods.
- Ne is the total number of adjacent rods that makes the beam.

Because of clamping constraint, we set $\phi_1 = \Phi_1$, while

$$\begin{aligned} \phi_i &= \Phi_{i+1} - \Phi_i \quad \text{for} \quad i \geq 2 \\ \phi_i &= \frac{\Phi_{i+1} - \Phi_i}{\eta} \eta. \end{aligned} \tag{4.2}$$

Substitute the expression above into the quadratic bending deformation energy given in Equ.(4.1), and evaluating it as $\eta \longrightarrow 0$, we have:

$$\Psi_{el} = \sum_{i=1}^{Ne} k_{b_i} \left(\frac{\Phi_{i+1} - \Phi_i}{\eta_i} \right)^2 \frac{\eta_i^2}{2}. \tag{4.3}$$

$$\Psi_{el} = \lim_{\eta_i \to 0} \left(\sum_{i=1}^{Ne} k_{b_i} \left(\frac{\Phi_{i+1} - \Phi_i}{\eta_i} \right)^2 \frac{\eta_i^2}{2} \right). \tag{4.4}$$

Using the definition that:

$$\varphi' = \lim_{\eta_i \to 0} \left(\frac{\Phi_{i+1} - \Phi_i}{\eta_i} \right).$$

The Hencky-discrete model represented by Equ.(4.4), after using the definition above becomes:

$$\Psi_{el} = \int_0^L \frac{1}{2} k_b (\varphi')^2. \tag{4.5}$$

Where φ is the rotation angle as defined in the Euler-Bernoulli continuous framework, and it retains its earlier definition of

$$\varphi = tan^{-1}\left(\frac{v'}{1+u'}\right).$$

As analyzed above, the limit case of the Hencky-type approach gives the inextensibility constraint of the Non-linear Euler-Bernoulli model which validates or affirms the claim of Hencky model as reported in [2].

4.2.2 Inextensibility Constraint $(g-1)=0$ of Euler-Bernoulli

Let's recall that the Non-linear Euler-Bernoulli energy functional is given as:

$$\mathscr{E}_{\text{EB}}^{\text{Def}} = \int_0^L \frac{1}{2} k_e (g-1)^2 ds + \int_0^L \frac{1}{2} k_b \left[\frac{v''(1+u') - u''v'}{(1+u')^2 + (v')^2}\right]^2 ds.$$

Imposing the inextensibility constraint: $(g-1)=0$ and g is recalled to be

$$g = \sqrt{(1+u')^2 + (v')^2}.$$

The energy functional after imposing the inextensibility constraint thus becomes:

$$\mathscr{E}_{\text{EB}}^{\text{Def}} = \int_0^L N(g-1) ds + \int_0^L \frac{1}{2} k_b \left[\frac{v''(1+u') - u''v'}{(1+u')^2 + (v')^2}\right]^2 ds. \tag{4.6}$$

Where,
N is the lagrange mutliplier which interpretes as the axial force per-unit line that has to be imposed on the system, in order to satisfy the inextensibility constraint. The first variation of the above energy thus becomes:

$$\delta\mathscr{E}_{\text{EB}}^{\text{Def}} = \int_0^L N\delta(g-1) ds + \int_0^L \frac{1}{2} k_b \delta \left[\frac{v''(1+u') - u''v'}{(1+u')^2 + (v')^2}\right]^2 ds. \tag{4.7}$$

The second addend's first variation has been derived earlier in the Euler-Bernoulli case. The

First addend needs our attention and it's expressed as:

$$\int_0^L N\delta(g-1)ds = \int_0^L \frac{N\,(1+u')}{g}\delta u' ds + \int_0^L \frac{Nv'}{g}\delta v' ds.$$

Performing Integration by part we have:

$$= -\int_0^L \left[\frac{N\,(1+u')}{g}\right]' \delta u ds + \left[\frac{N\,(1+u')}{g}\right] \delta u|_0^L - \int_0^L \left[\frac{Nv'}{g}\right]' \delta v ds + \left[\frac{Nv'}{g}\right] \delta v|_0^L.$$

The first variation of the work potential of the external load the beam can be subjected is expressed as:

$$\delta \mathscr{W}^{ext} = -\int_0^L b_2(s)\delta v ds - \int_0^L \mu(s)\delta\varphi ds - M\delta\varphi\;|_0^L - F\delta v|_0^L = 0. \tag{4.8}$$

Where b_2, μ, $M\,and\,F$ retain their definition from the earlier analysed Timoshenko and Euler-Bernoulli beams

The rotation angle φ is recalled as:

$$\varphi = tan^{-1}\left(\frac{v'}{1+u'}\right).$$

The first variation of the Equ.(4.9) using the first variation of φ, we have:

$$\begin{aligned}\delta \mathscr{W}^{ext} = &-\int_0^L b_2(s)\delta v ds - \int_0^L \mu(s)\frac{\partial\varphi}{\partial u'}\delta u' ds - \int_0^L \mu(s)\frac{\partial\varphi}{\partial v'}\delta v' ds \\ &- M\frac{\partial\varphi}{\partial u'}\delta u'\;|_0^L - M\frac{\partial\varphi}{\partial v'}\delta v'\;|_0^L - F\delta v|_0^L = 0.\end{aligned} \tag{4.9}$$

Using the first variation analysis of the complex loading previously analyzed. Combining the simplification of the first and second addend of Equ.(4.7) alongside the first variation of Equ.(4.9). The strong formulation recovered after the first variation of the total energy and imploying the test function lemma, we have:

$$\begin{aligned}&\left(\frac{N(s)\,(1+u')}{g}\right)' + \left(\frac{k_b\kappa v'}{g^2}\right)'' + \left(\frac{k_b\kappa}{g^4}\left(v''\left(g^2-2(1+u')^2\right) - 2u''(1+u')v'\right)\right)' \\ &\qquad + \left(\frac{\mu(s)v}{g^2}\right)' = 0.\end{aligned} \tag{4.10}$$

$$b_2(s) + \left(\frac{N(s)v'}{g}\right)' - \left(\frac{k_b\kappa(1+u')}{g^2}\right)'' - \left(\frac{k_b\kappa}{g^4}\left(u''\left(g^2-2(v')^2\right) + 2v''(1+u')\right)\right)'$$

$$-\left(\frac{\mu(s)(1+u')}{g^2}\right)' = 0. \tag{4.11}$$

$$g - 1 = 0. \tag{4.12}$$

Substituting Equ.(4.12) into Equations (4.10 and 4.11), and defining the inextensible bending curvature as $\bar{\kappa} = v''(1+u')^2 + (v')^2$. The resulting equations and their corresponding boundary conditions is expressed as:

$$(N(s)\,(1+u'))' + (k_b\bar{\kappa}v')'' + (\mu(s)v)' + \left(k_b\bar{\kappa}\left(v''\left(1-2(1+u')^2\right) - 2u''(1+u')v'\right)\right)' = 0. \tag{4.13}$$

$$b_2(s) + (N(s)v')' - (k_b\bar{\kappa}(1+u'))'' - (\mu(s)(1+u'))' - \left(k_b\bar{\kappa}\left(u''\left(1-2(v')^2\right) + 2v''(1+u')\right)\right)' = 0. \tag{4.14}$$

Subject to the following boundary conditions:

$$\begin{aligned}
\left[N(s)(1+u') + (k_b\bar{\kappa}v')' + \mu(s)v + k_b\bar{\kappa}(v''(1-2(1+u')^2) - 2u''(1+u')v')\right]\delta u|_0^L &= 0. \\
\left[N(s)v' - (k_b\bar{\kappa}(1+u'))' - \mu(s)(1+u') - k_b\bar{\kappa}(v''(1-2(v')^2) - 2v''(1+u')) - F\right]\delta v|_0^L &= 0. \\
\left[(M - k_b\bar{\kappa})\,v'\right]\delta u'|_0^L &= 0. \\
\left[(k_b\bar{\kappa} - M)\,(1+u')\right]\delta v'|_0^L &= 0.
\end{aligned} \tag{4.15}$$

5 Numerical Solution and Results

5.1 Finite Element Methods

The Finite Element Method (FEM) is a numerical technique used to solve differential equations (DEs) by dividing the problem domain into smaller elements and approximating the solution within each element using basis functions. The general formulation of FEM involves the following steps:

1. Discretization: The problem domain is discretized into a collection of elements (e.g., triangles, quadrilaterals, tetrahedra, hexahedra) that together form a mesh. Each element has a set of nodes, which serve as the interpolation points for the solution.
2. Basis Functions: The basis functions are chosen to approximate the solution within each element. Two commonly used types of basis functions in FEM are Lagrange polynomials and Hermite polynomials.
   (a) Lagrange polynomials are simple functions that interpolate the solution values at the element nodes. They ensure that the approximated solution passes exactly through the nodal values. The Lagrange polynomials can be of different orders (e.g., linear, quadratic, cubic), allowing for different levels of accuracy in the approximation.
   (b) Hermite polynomials not only interpolate the function values at the nodes but also account for the derivatives. They accurately represent both the function and its derivatives at the nodes. Hermite polynomials are particularly useful when high accuracy or gradient-dependent quantities are involved.
3. Ritz Approximation: The Ritz approximation technique, a specific implementation of the Galerkin approximation, is employed to determine the coefficients of the basis functions in order to minimize the error between the actual solution and the approximated solution. The Ritz method involves choosing trial functions, which are typically a linear combination of the basis functions, and finding the coefficients that minimize the residual functional. This leads to a set of algebraic equations that can be solved to obtain the coefficients. The steps taken in this case is as highlighted below:
   (a) Define the Trial Functions: Let's denote the trial functions as $\phi_i(x)$, where i ranges from 1 to the total number of unknowns in the problem. The trial functions are typically chosen as a linear combination of the basis functions.

$$\phi_i(x) = \sum_{j=1}^{N} c_{ij}\psi_j(x).$$

   Here, N represents the total number of basis functions, c_{ij} are the coefficients to be determined, and $\psi_j(x)$ are the basis functions.
   (b) Substitute Trial Functions into the Governing Equation: Replace the unknown function in the governing equation with the trial functions

$$F(\phi_i(x)) = 0.$$

This equation represents the original governing equation (DE) in terms of the trial functions.

(c) Minimize the Residual Functional: The goal is to find the coefficients c_{ij} that minimize the residual functional, defined as the square of the residual between the actual solution and the trial solution,

$$R = \int_{\Omega} [F(\phi_i(x))]^2 dx.$$

Here, Ω represents the problem domain.

(d) Variation of the Residual Functional: Take the variation of the residual functional with respect to the coefficients c_{ij} and set it to zero to find the minimum.

$$\frac{\partial R}{\partial c_{ij}} = 0.$$

This variation gives rise to a set of algebraic equations that can be solved to determine the coefficients c_{ij}.

(e) Solve the Algebraic Equations: Solve the resulting algebraic equations to obtain the values of the coefficients c_{ij}. These coefficients determine the approximation of the solution within each element.

By combining the discretization, basis functions (such as Lagrange or Hermite polynomials), and Ritz approximation, the FEM allows for the formulation of a system of algebraic equations that can be solved numerically to obtain the approximate solution to the original DE problem. This process provides a versatile and efficient approach for analyzing and solving a wide range of engineering and scientific problems.

5.1.1 What is COMSOL Multiphysics

COMSOL Multiphysics is a modelling and simulation software that allows scientist and engineers to analyze complex physical systems. It allows the use of in-built Physics package that caters to the need of engineering and scientific problems worldwide.

For the purpose of this thesis, we'll be making use of the mathematical package, particularly the weak formulation capabilities of this software. This is one of the advantage of this software over other modelling and simulation software. This allows for the proper assignment of regularities of the field variables to suit the mathematical model's first variation being considered.

It follows the same usual practices of modelling and simulating process viz: Definition of geometry (3D, 2D, or 1D), specifying physics (Structural Mechanics, Acoutics, Fluid Flow, Heat transer, AC/DC, Mathematics etc), meshing, solving(Using sophisticated in-built numerical algorithms that's constantly updated) and post processing of result (visualization of results and also capabilities of exporting the data of plots to be used in other software for a more refined visualization goals one prefers). [9]

5.1.2 First Variation and Weak Formulation of COMSOL Multiphysics

The weak formulation, derived through the utilization of the first variation of the total energy or action functional of the system, allows for a more flexible approach to solving deformation problems. It relaxes the requirements on the differentiability of the solution and permits the use of piecewise continuous functions, such as finite element shape functions, for approximation. When utilizing the weak form module of COMSOL Multiphysics, it is important to pay careful attention to the regularities of the variables or parameters in the energy functional.

COMSOL Multiphysics provides a user-friendly interface and computational environment for defining the physics, specifying boundary conditions, and setting up the weak form equations using appropriate variational forms. In the case of axial elongation, the Lagrange polynomial is employed as the interpolating polynomial, ensuring continuity up to the continuous function. However, for handling bending deformation, the Lagrange polynomial breaks down in maintaining continuity to the first derivative. Therefore, the Hermite polynomial is used to arrive at a solution that effectively handles bending deformation.

5.2 Comparison of Results For Euler-Bernoulli and Timoshenko Beams

To assess the competitive advantage of the non-linear beam models, a comparison between the linear and non-linear models was conducted. A cantilever beam (Clamped-Free) was subjected to a transverse force of $10kN$, and the parameters used in COMSOL Multiphysics for this comparison are presented in Table (5.1) and Table (5.2).

The comparison revealed significant differences in the predictions of the linear and non-linear models. Fig. 5.1 shows that the linear models predicted zero axial deflection for the beam, neglecting the inextensibility constraint. In contrast, the non-linear models accurately predicted a compressive behavior when the beam was loaded. This discrepancy highlights the necessity for the present study.

Furthermore, Fig. 5.2 focuses on the transverse deflection predicted by both the linear and non-linear models. The figure clearly demonstrates that the linear case over-predicted the transverse deflection at the tip of the beam. Consequently, the true measure of deformation at the tip aligns with the predictions of the non-linear models.

Subsequent figures provide further comparisons of deformation measures and variations in parameters of interest for the non-linear models, specifically the Timoshenko and Euler-Bernoulli formulations.

Table 5.1: The parameters and methods used in COMSOL Simulation

Model	Mathematics (Weak Form PDE)
Space Dimension	1D
Study	Stationary
Mesh Size	Extremely Fine
Number of Elements	550
Iterations	100

Table 5.2: Definition of parameters for weak formulation simulation

Name	Expression	Value	Description
L	100[cm]	1 m	Length of Rod
b	5[cm]	0.05 m	Cross-section Length
a	2.5[cm]	0.025 m	Cross-section Width
Yb	70[GPa]	7E10 Pa	Young's Modulus
Ac	a*b	0.00125 m^2	Area of Cross-Section
Ke	Yb*Ac	8.75E7 N	Elastic Stiffness
J	$a^3b/12$	$6.5104E-8\,m^4$	Second Moment of Area
Kb	Yb*J	4.5573E6 kg·m^3/s^2	Bending Stiffness
ML	10[kN*m]	10000 N·m	Bending Moment
FL	10[kN]	10000 N	Transverse Force

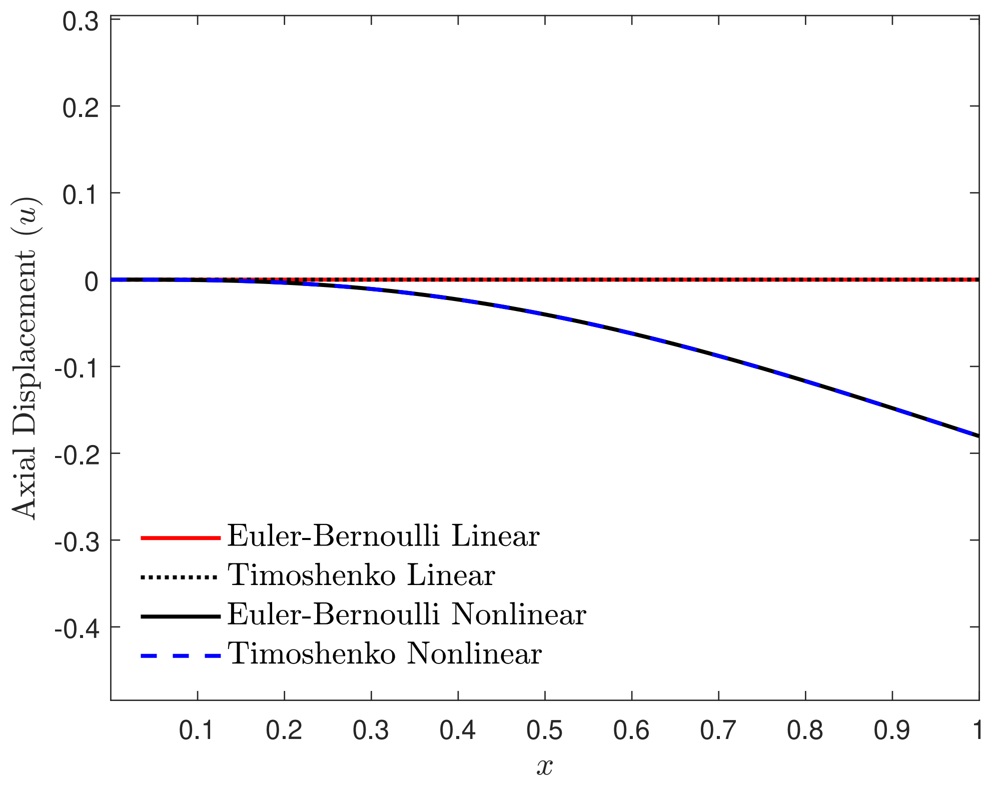

Figure 5.1: Axial displacement comparison across model

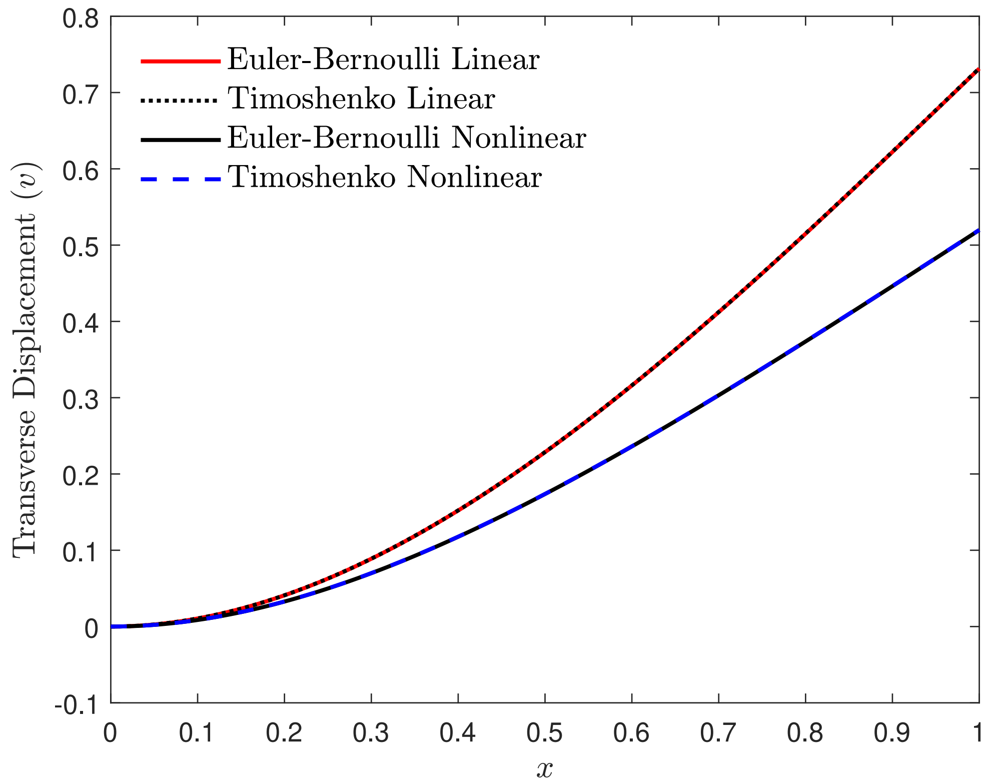

Figure 5.2: Transverse displacement comparison across model

5.2.1 Clamped-Free (Cantilever) Beam

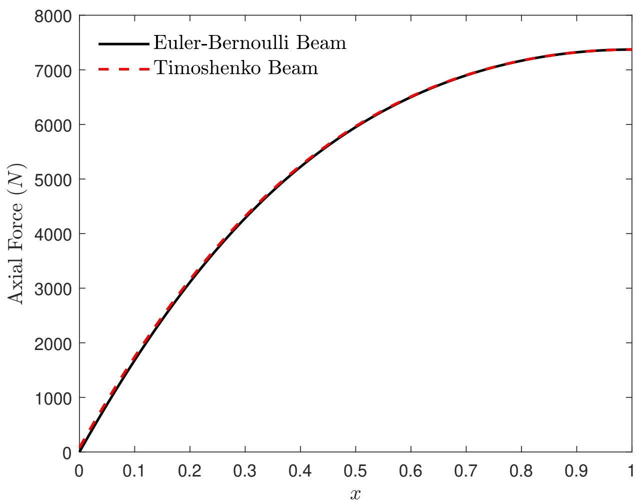

Figure 5.3: Axial Force Comparison- Euler-Bernoulli Vs Timoshenko (CF)

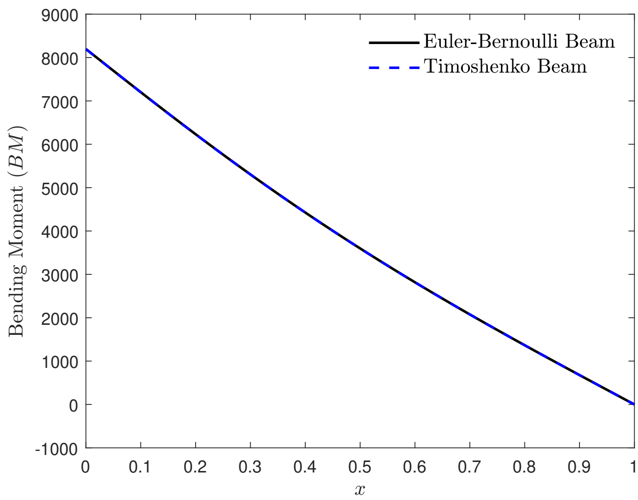

Figure 5.4: Bending Moment Comparison- Euler-Bernoulli Vs Timoshenko (CF)

The results obtained from the analysis of the non-linear models, specifically the axial deflection, transverse deflection, axial force, bending moment, and shear force diagram, demonstrate consistent and comparable predictions. Both the Timoshenko and Euler-Bernoulli models exhibit the same trends and behavior, as illustrated in the figures.

A noteworthy finding can be observed in the variation of shear angle shown in Fig. 5.7.

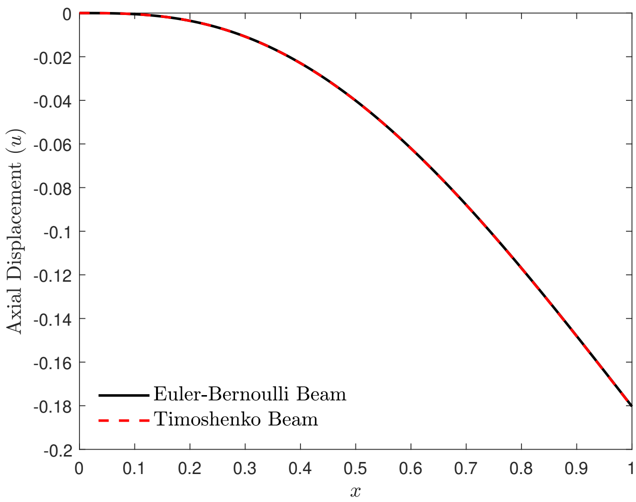

Figure 5.5: Axial Displacement Comparison- Euler-Bernoulli Vs Timoshenko (CF)

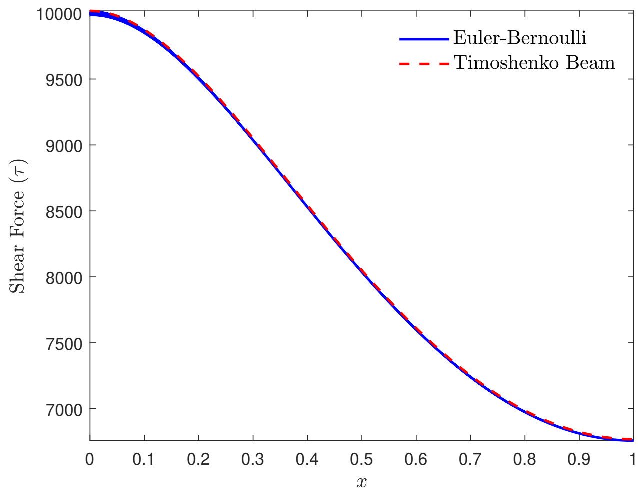

Figure 5.6: Shear Force Comparison- Euler-Bernoulli Vs Timoshenko (CF)

It reveals an inverse relationship between the shear stiffness and shear angle. As the shear stiffness decreases by an order of magnitude, the shear angle increases. Consequently, a significant difference in the deformation parameters, specifically the axial and transverse deflections (u and v), is expected when the shear stiffness is further reduced.

This expectation is confirmed in Fig. (5.8, 5.9, and 5.10), where noticeable changes in the plots are observed, deviating from the exact predictions of both models. The behavior can be attributed to the increased shear deformation resulting from the reduced shear stiffness. When the shear stiffness is high, the beam exhibits high resistance to shear, making the shear deformation negligible and aligning with the exact predictions. However, reducing the shear stiffness by an order of magnitude introduces noticeable differences in the recent plots.

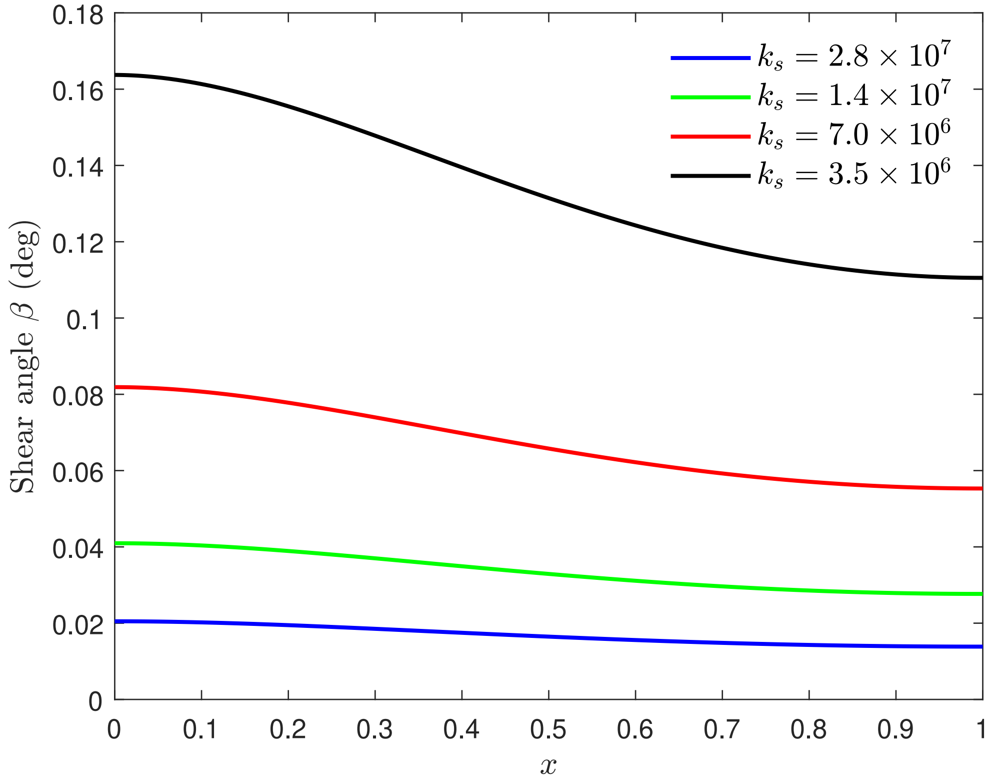

Figure 5.7: Shear angle variation for Timoshenko Beam (CF)

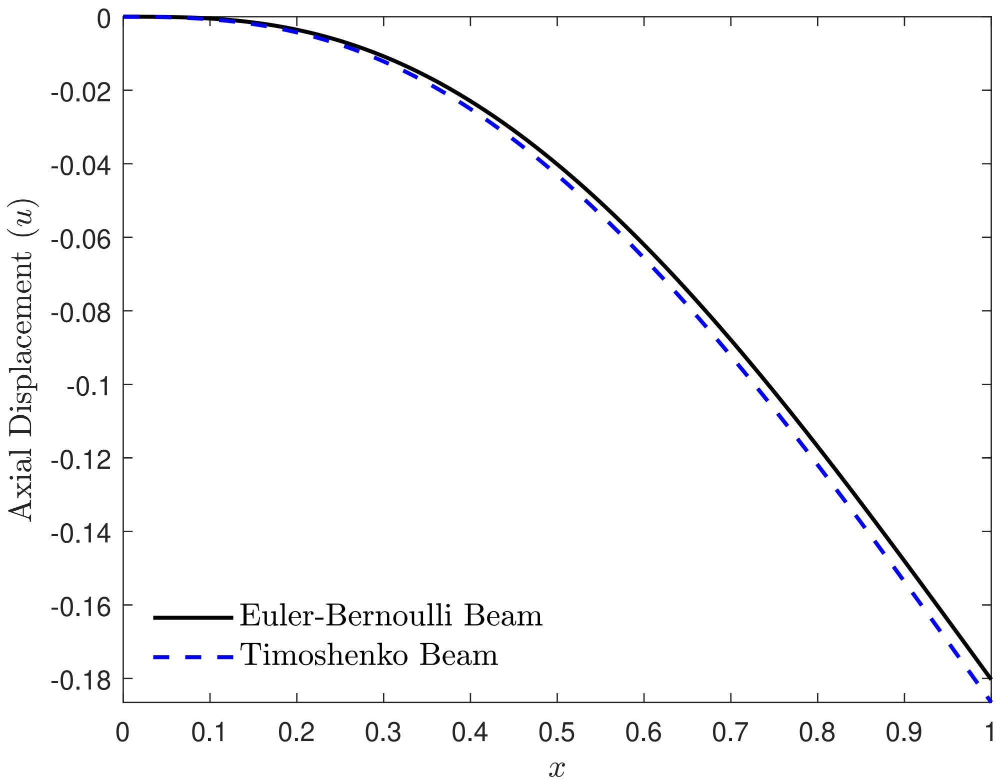

Figure 5.8: Axial deflection at $k_s = 2.1875 \times 10^5$

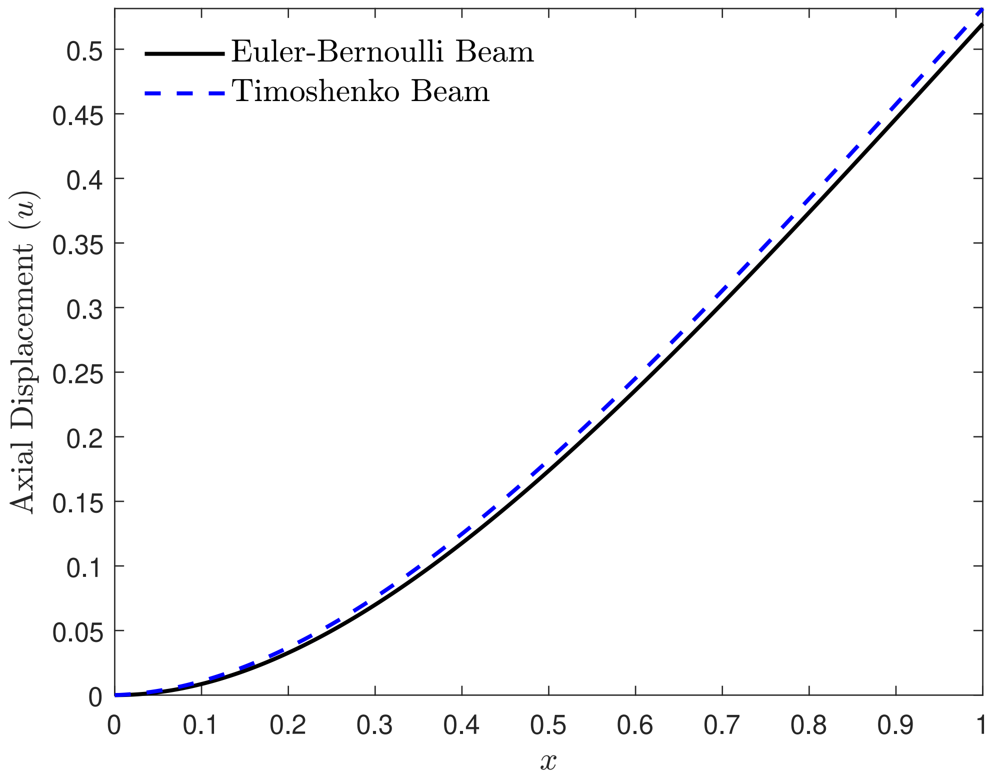

Figure 5.9: Transverse deflection at $k_s = 2.1875 \times 10^5$

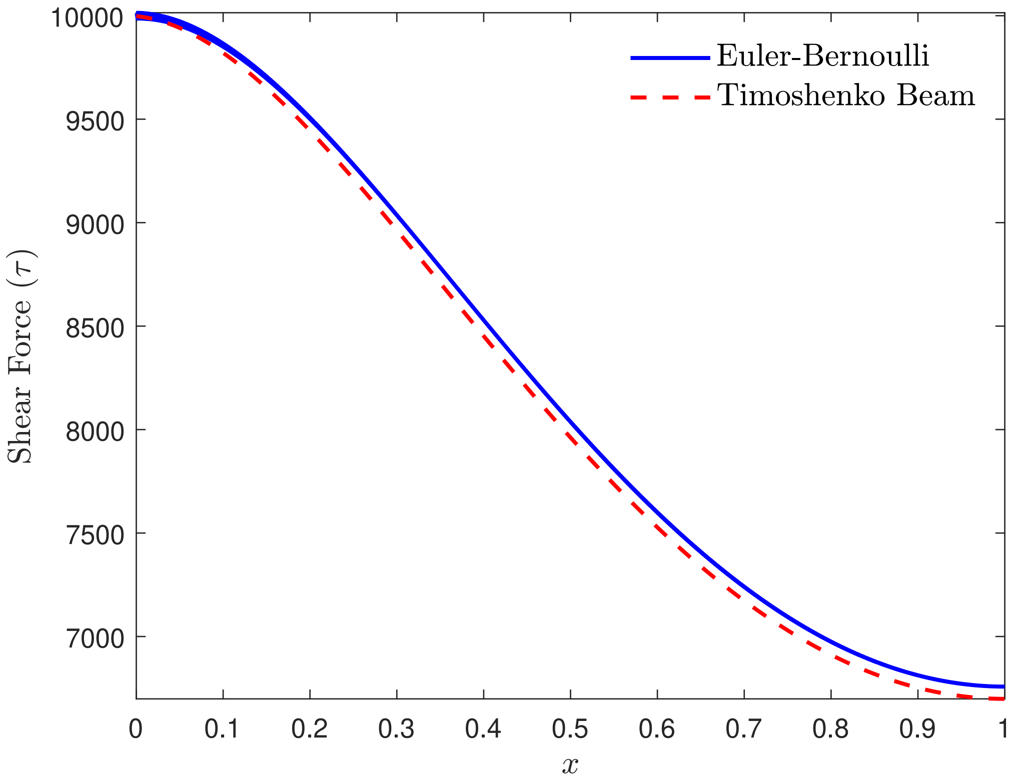

Figure 5.10: Shear Force at $k_s = 2.1875 \times 10^5$

5.2.2 Simply-Supported Beam

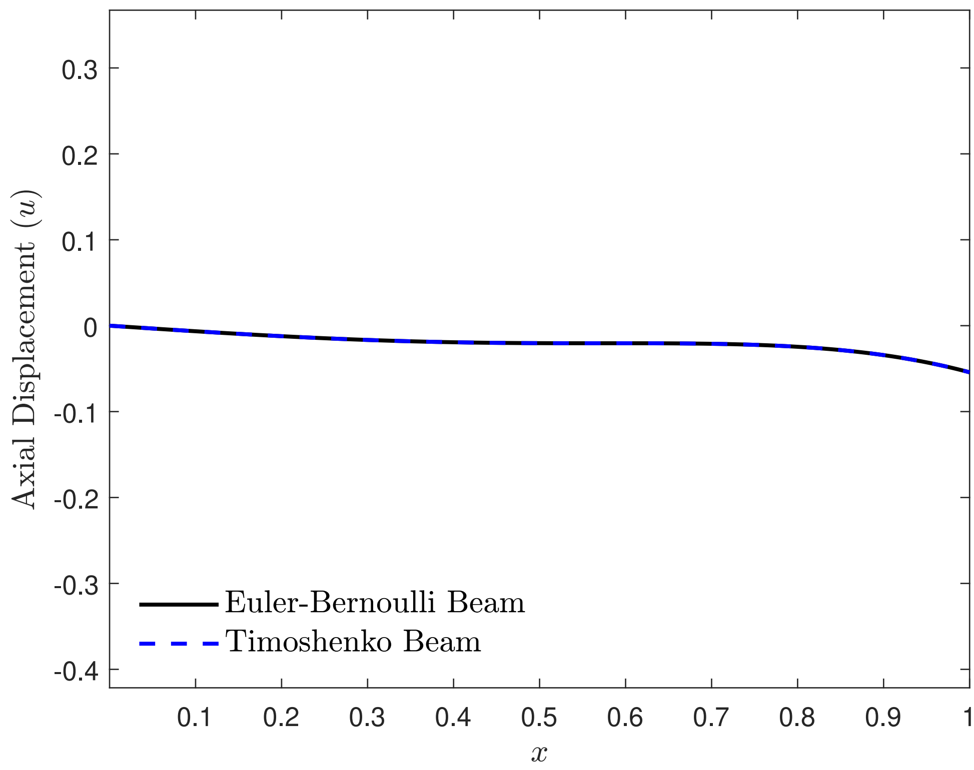

Figure 5.11: Axial Displacement Comparison- Euler-Bernoulli Vs Timoshenko (SS)

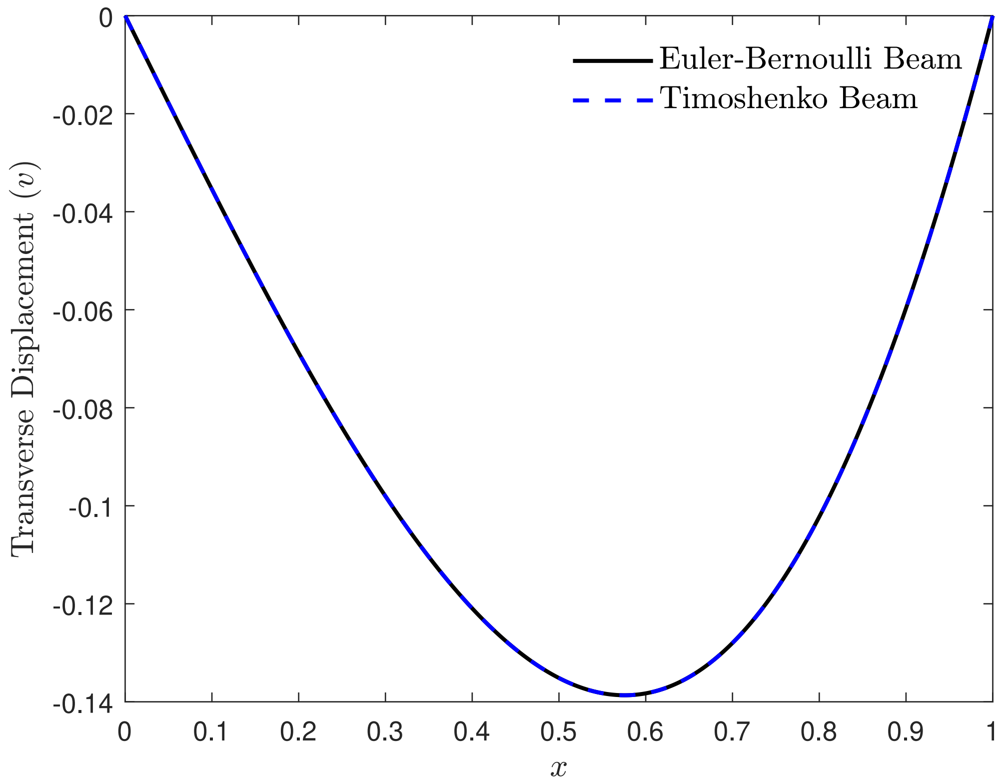

Figure 5.12: Transverse Displacement Comparison- Euler-Bernoulli Vs Timoshenko (SS)

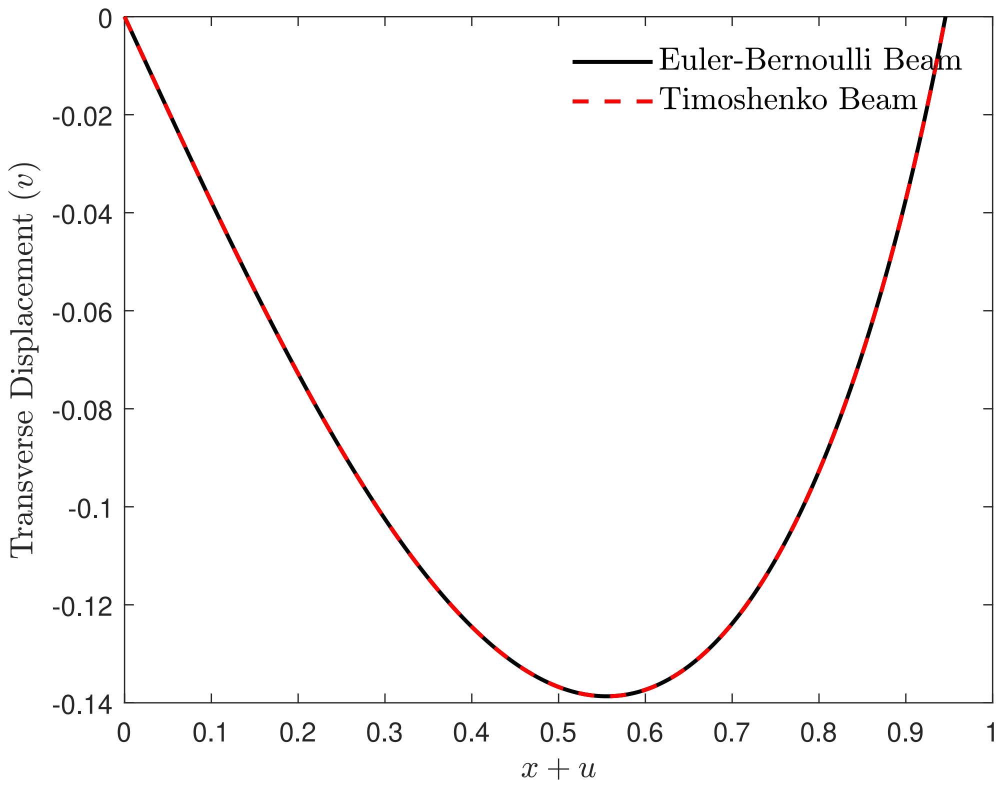

Figure 5.13: Transverse Displacement and Axial Displacement Comparison- Euler-Bernoulli Vs Timoshenko (SS)

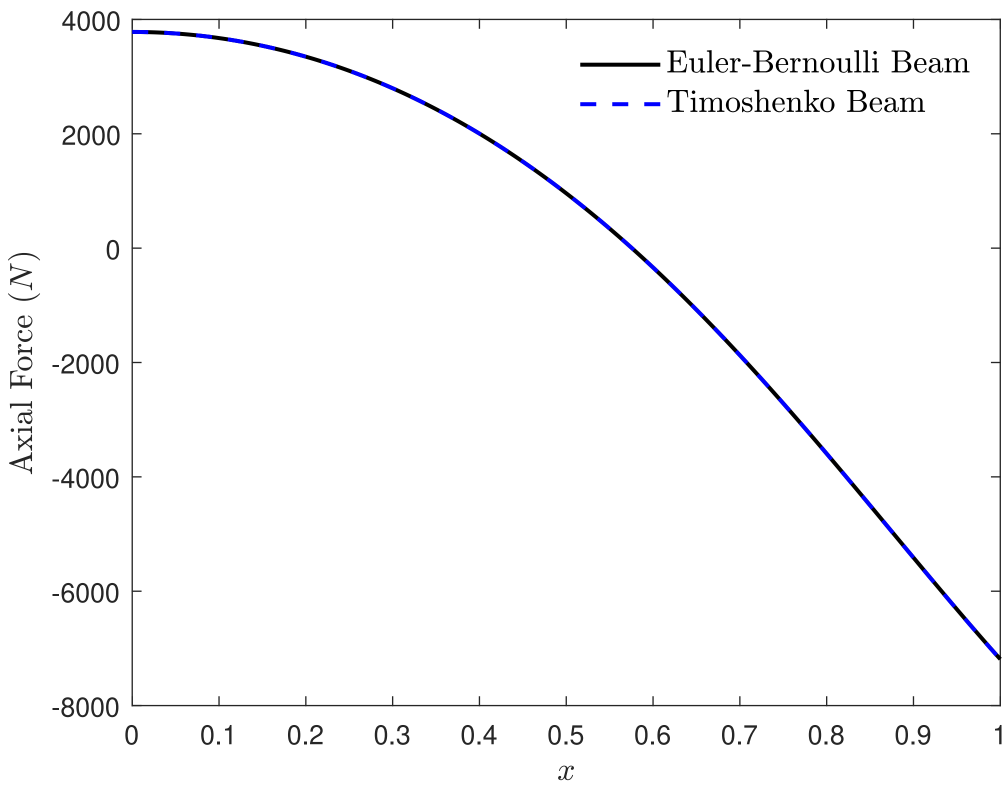

Figure 5.14: Axial Force Comparison- Euler-Bernoulli Vs Timoshenko (SS)

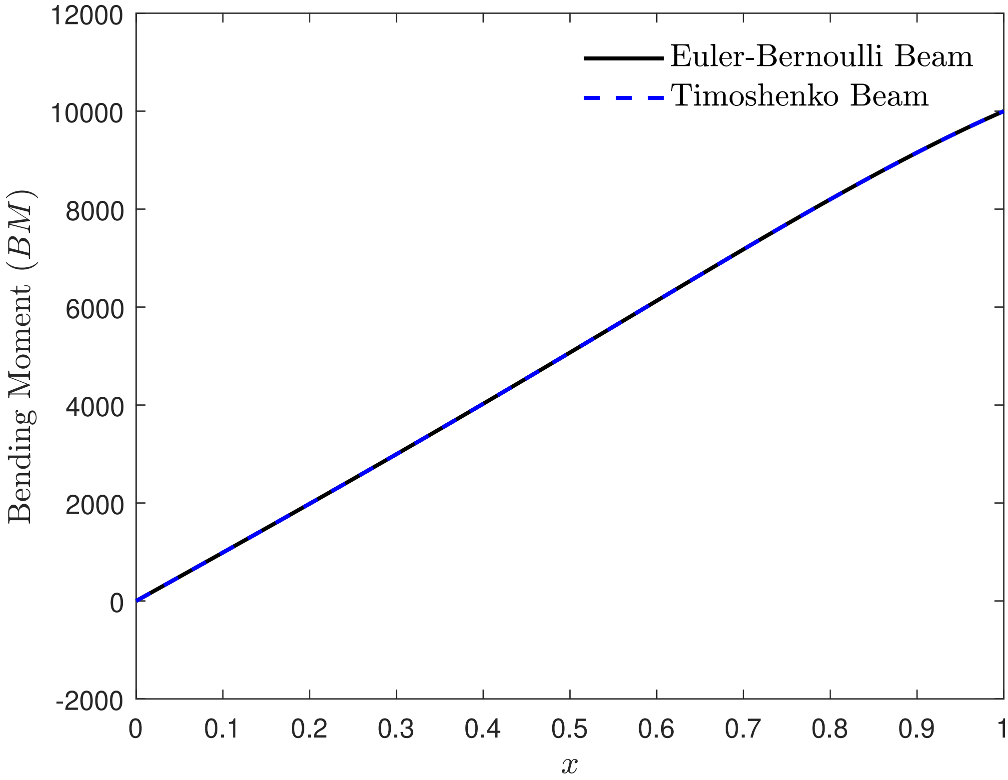

Figure 5.15: Bending Moment Comparison- Euler-Bernoulli Vs Timoshenko (SS)

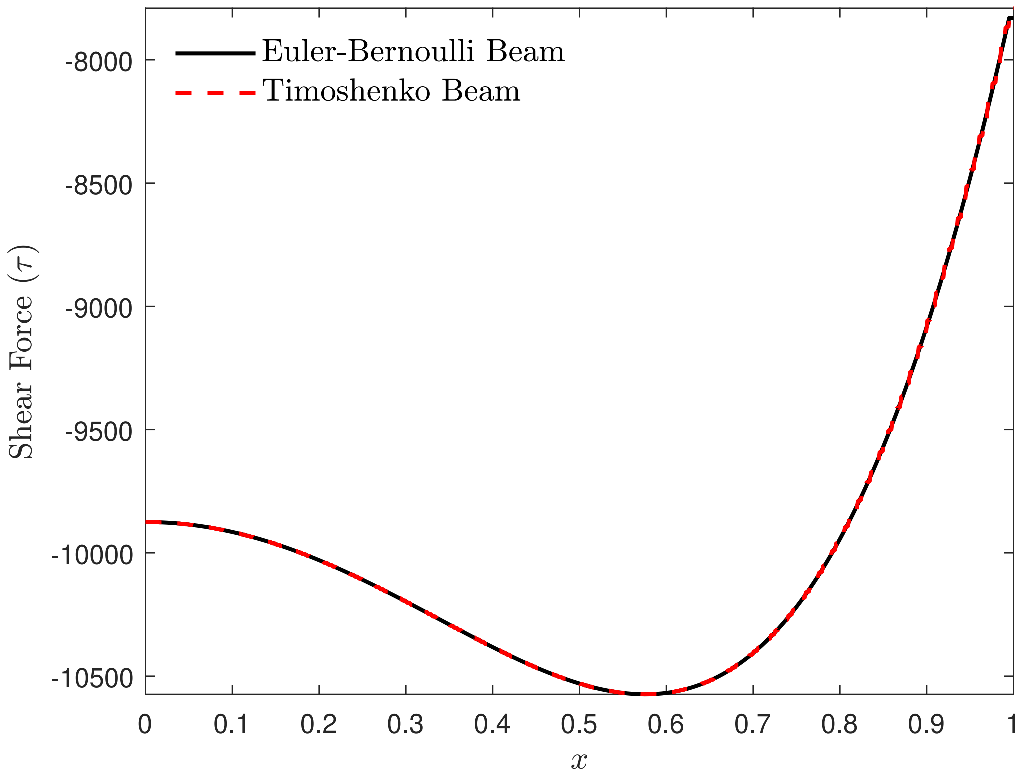

Figure 5.16: Shear Force Comparison- Euler-Bernoulli Vs Timoshenko (SS)

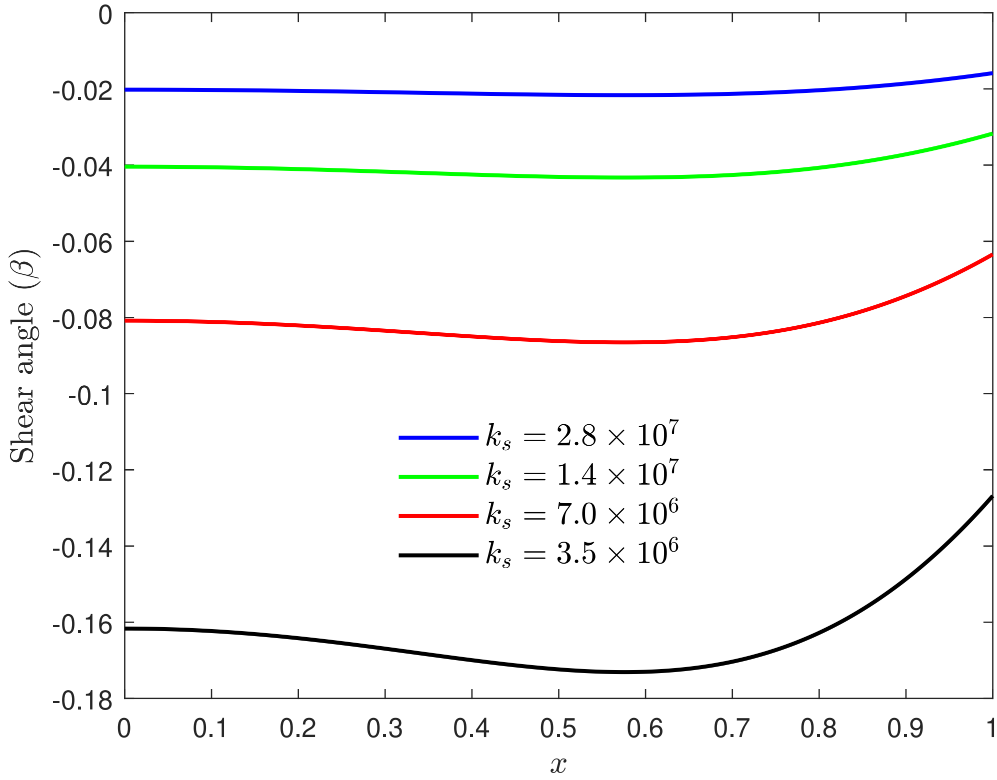

Figure 5.17: Shear angle variation for Timoshenko Beam (SS)

The analysis of the non-linear models for the simply supported beam provides insightful findings. In Fig. 5.11, a slight axial displacement is observed for both non-linear models, which is unexpected since the beam is subjected to pure bending.

For the transverse deformation of the beam, both Fig. 5.12 and Fig. 5.13 predict the same trend. This deformation behavior aligns with the expected physics of a simply supported beam under pure bending.

The axial force and bending moment distribution on the beam are depicted in Fig. 5.14, and Fig. 5.15, respectively. The reaction force on the beam is approximately $4kN$, and the highest bending moment occurs at the end of the beam, as anticipated. The shear force distribution throughout the beam remains almost constant, as shown in Fig. 5.16.

Furthermore, Fig. 5.17, demonstrates an inverse relationship between the shear angle and the shear stiffness. Notably, a significant change in the plots is observed when reducing the order of magnitude of the shear stiffness k_s. This observation corresponds to the findings observed in the analysis of the cantilever beam.

Overall, the results from the analysis of the simply supported beam provide valuable insights into the axial displacement, transverse deformation, axial force, bending moment, and shear force behavior, further highlighting the influence of shear stiffness on the shear angle.

6 Conclusion

In conclusion, this thesis introduces an innovative and exact nonlinear classical formulation for the deformation equations of Euler-Bernoulli and Timoshenko Beam models. Departing from the prevalent linearized mode descriptions found in literature, this formulation directly incorporates the nonlinear deformation mode kinematics. By utilizing the first variational formulation derived from the total energy or Lagrangian functional, a more accurate and comprehensive representation of beam behavior is achieved.

Furthermore, the thesis validates the assertion that the Hencky-Type discrete model converges to the inextensibility constraint applied on the nonlinear Euler-Bernoulli beam model. This affirmation reinforces the reliability and applicability of the proposed formulation.

To investigate the practical implementation of the models, the weak formulation of the nonlinear problem is solved using the Finite Element Method (FEM) in the math module of COMSOL Multiphysics software. Comparative simulations between the linear and nonlinear cases are performed, and the results are thoroughly analyzed and discussed. The plots demonstrate the precise prediction of deformation measures and parameters by both nonlinear models. Notably, when the shear stiffness of a beam is high, the Euler-Bernoulli model precisely aligns with the Timoshenko model. Conversely, a reduced shear stiffness reveals noticeable differences in the deformation behavior, aligning with the expected physical characteristics of the problem. This analysis confirms the superiority of the nonlinear models in accurately estimating the true deformation experienced by a loaded beam.

Moving forward, a promising avenue for future research involves applying the presented classical formulation to explore buckling, modal, and harmonic analysis of beams. These analyses are essential for determining critical loading thresholds until beam failure occurs. Additionally, the application of these models in areas such as Functionally Graded Materials (FGM), Additive Manufacturing (3D Printing), Biomedical Engineering (bone structures), and the structural, automotive, and aviation sectors holds great potential for further advancements. By incorporating the proposed models, these areas can benefit from improved accuracy and reliability in their respective analyses and designs.

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7 Appendix

7.1 Mathematical Background and Defintion of Terms

Definition 7.1. A normed space is a pair $(V, \|\cdot\|)$, where V is a vector space over a field of scalars (typically the real numbers or complex numbers), and $\|\cdot\|$ is a norm on V, i.e., a function that maps each vector x in V to a non-negative real number $\|x\|$ such that:

1. $\|x\| = 0$ if and only if $x = 0$ (the zero vector).
2. $\|\alpha x\| = |\alpha|\|x\|$ for all scalars α and all vectors x in V.
3. $\|x + y\| \leq \|x\| + \|y\|$ for all vectors x and y in V (the triangle inequality).

A normed space provides a natural way to measure the size or magnitude of vectors, and it allows one to define concepts such as convergence, continuity, and completeness, which are important in the study of analysis and topology. Examples of normed spaces include Euclidean spaces, function spaces, and sequence spaces.

Definition 7.2. A metric space X is *complete* if every Cauchy sequence in X converges to a point in X. A *Banach Space* is a complete normed space

Definition 7.3. A Hilbert space is a complete inner product space (complete with respect to the norm $||x|| = \sqrt{\langle x, x\rangle}$)

Definition 7.4. The set of measurable functions on a domain Ω with respect to the Lebesgue measure $m(\cdot)$ will be denoted by $\Lambda(\Omega)$. Let Ω be a domain in $\mathbb{R}^N$. For an exponent $p \in (1, \infty)$ we define a functional $\|\cdot\|_p$ by the relation

$$\|u\|_p \equiv \|u\|_{p;\Omega} \overset{\text{def}}{=} \left[\int \Omega |u(x)|^p \,\mathrm{d}x\right]^{1/p}$$

and we consider a subset of the set of measurable functions $\Lambda(\Omega)$

$$\mathscr{L}^p(\Omega) = \left\{u \in \Lambda(\Omega) | \|u\|_p < \infty\right\}.$$

The functional $\|\cdot\|_p$ on the linear space $\mathscr{L}^p(\Omega)$ satisfies the first two axioms of the norm: the homogeneity and the triangle inequality, see Minkowski inequality, but it does not satisfy the third axiom $\|u\|_p = 0 \Rightarrow u = 0$, since $\|u\|_p = 0$ is true for each function u, which is zero on $\Omega - \mathcal{N}$, where $\mathcal{N}$ is a zero measure set.

Using almost everywhere equality (briefly we shall write $\overset{\text{c.e.}}{=}$) in the space $\mathscr{L}^p(\Omega)$, we identify functions, which mutually differ on a zero measure subset. In this way we obtain the Lebesgue space $L^P(\Omega)$

$$L^p(\Omega) = \mathscr{L}^p(\Omega)|_{\overset{\text{c.e.}}{=}}.$$

Its elements are classes of functions, which differ at most on a zero measure set.

Definition 7.5 (Sobolev Spaces). [6] Sobolev spaces are the basic mean in the theory of generalized formulation of the boundary value problems. Sobolev spaces are spaces of functions with integrable generalized derivatives. The Sobolev spaces are usually denoted by $W^{k,p}(\Omega)$ or $H^{k,p}(\Omega)$, where
k - positive integer yielding the highest order of the derivatives, $k = 1, 2, 3, \ldots$.
p - exponent, as in the case of Lebesgue spaces $p \in (1, \infty)$.

To define the Norm on a Sobolev space we introduce (k, p)-norm, which is equivalent to the norm:

$$\|u\|k, p = \left[\sum |a| \leq k \, \|D^{\alpha}u\| \, p^{p}\right]^{1/p} = \left[\int \Omega|\alpha| \leq k \, |D^{\alpha}u|^{p} \ \mathrm{d}x\right]^{1/p},$$

The definition of Sobolev space is therefore defined in three different ways namely:

1. Sobolev space $W^{k,p}(\Omega)$ is the completion of the space $\mathscr{E}(\Omega)$ in $\|\cdot\|_{k,p}$.

$$W^{k,p}(\Omega) \overset{\text{def}}{=} \overline{\mathscr{E}(\Omega)}^{\|\cdot\|_{k,p}}.$$

and $W_0^{k,p}(\Omega)$ is the completion of the space $C_0^{\infty}(\Omega)$ in the same (k, p)-norm.

$$W_0^{k,p}(\Omega) \overset{\text{def}}{=} \overline{C_0^{\infty}(\Omega)}^{\|\cdot\|_{k,p}}.$$

2. The space $H^{k,p}(\Omega)$ is the space of functions, which have integrable derivatives in the sense of distributions:

$$H^{k,p}(\Omega) = \left\{u \in \Lambda(\Omega) \, \middle| \partial^{\alpha}u \in L^{p}(\Omega) \quad \forall \middle| \, \alpha \, |\leq k\right\}.$$

3. Beppo-Levi space $BL^{1p}(\Omega)$ is a space of functions Absolutely Continuous $AC(\Omega)$, which have integrable derivative:

$$BL^{1,p}(\Omega) = \left\{u \in AC(\Omega) \mid \frac{\partial u}{\partial x_1} \ldots, \frac{\partial u}{\partial x_N} \in L^{p}(\Omega)\right\}.$$

Definition 7.6 (Functionals). 1. A function $u(x)$ is a rule of correspondence such that for all x in D there is assigned a unique element $u(x)$ in R. A functional $F[u(x)]$ is a rule of correspondence such that for all $u(x)$ in R there is assigned a unique element $F[u(x)]$ in Ω in other words, a functional is a function of a function.
2. Linear functional $l\left(\alpha_1, u + u_2 v\right) = \alpha_1 l(u) + \alpha_2 l(v)$.
3. Bilinear functiunal $B\left(\alpha_1 u_1 + \alpha_2 u_2, \mathrm{v}\right) = \alpha_1 B\left(u_1, \mathrm{v}\right) + \alpha_2 B\left(u_2, v\right)$. $B\left(u, \alpha_1 v_1 + \alpha_2 \mathrm{v}_2\right) = \alpha_1 B\left(u, v_1\right) + \alpha_2 B\left(u, v_2\right)$.
4. Symmetric bilinear functional: $B(u, \mathrm{v}) = B(\mathrm{v}, u)$.
5. u and v can either be scalars or vectors.

Theorem 7.7 (Test Lemma). *[6] Let f be a continuous function on a domain Ω and let the following* *integral identity hold:*

$$\int_{\Omega} f(x)\varphi(x)\mathrm{d}x = 0 \quad \forall \varphi \in C_0^{\infty}(\Omega)$$

then $f = 0$ in Ω.

Proof. We prove the lemma by a contradiction. Let us assume that function f is not zero on Ω. Let e.g. $f(x_0) = k > 0$ for a $x_0 \in \Omega$. Since f is continuous at x_0, f is nonzero also in some neighborhood. Let us take a smaller neighborhood $\mathrm{B}(x_0, \delta)$, where f is bigger than $k/2$: i.e. $f(x) \geq k/2$ for $x \in \mathrm{B}(x_0, \delta)$. Let us insert into the test equality for φ the function $\varphi(x) = \varphi_\delta(x - x_0)$ from the previous proof and we arrive to a contradiction:

$$0 = \int_{\Omega} f(x)\varphi(x)\mathrm{d}x = \int_{B(x_0,\delta)} f(x)\varphi(x)\mathrm{d}x \geq \frac{k}{2}\int_{B(0,\delta)} \varphi_\delta(x)\mathrm{d}x = \frac{k}{2} > 0.$$

□

Theorem 7.8 (On Solvability of Abstract Variational Problem). *[6] Let V be a reflexive Banach space, M its nonempty closed convex subset and $\Phi(u)$ a real functional on M, i.e. $\Phi : M \longrightarrow \mathbb{R}$, which is*

1. *coercive in case when the set M is not bounded:*

$$\lim_{\|u\| \to \infty} \Phi(u) = \infty.$$

2. *weakly lower semi-continuous (w.l.s.c.):*

$$u_n \rightharpoonup u \quad \Longrightarrow \quad \liminf_{n \to \infty} \Phi(u_n) \geq \Phi(u).$$

Then the functional $\Phi(u)$ is bounded from below and attains its minimum, i. e. the variational problem admits its solution.

Theorem 7.9 (On uniqueness of the solution). *[6] Let the functional $\Phi : V \longrightarrow \mathbb{R}$ be strictly convex, i. e. for each two $u_0 \neq u_1 \in V$ and $\lambda \in (0,1)$ the following inequality holds*

$$\Phi((1-\lambda)u_0 + \lambda u_1) < (1-\lambda)\Phi(u_0) + \lambda\Phi(u_1).$$

Then there exists at most one $u^\star \in V$, where the functional $\Phi(u)$ attains its minimum, thus the variational problem admits at most one solution.

If the functional $\Phi(u)$ is only convex, then the points where the functional $\Phi(u)$ attains its minimum, is a convex closed set, which can also be one-point or empty.

Proof. Let $\Phi(u)$ attains its minimum m in two different functions $u_0 \neq u_1$. Then the value of the strictly convex functional $\Phi(u)$ in the middle $(u_0 + u_1)/2$ is smaller. Indeed, for $\lambda = 1/2$ the coercivity condition yields

$$\Phi\left(\frac{1}{2}(u_0+u_1)\right) < \frac{1}{2}\left[(\Phi(u_0)+\Phi(u_1)\right] = m$$

which is a contradiction to the assumption that the minimum $\Phi(u)$ is m. □

Lemma 7.10 (Properties of Coercive Functionals). *[6]*

1. *The coercive functionals form a cone: for coercive functionals $\Phi_i(u)$ and positive reals α_i the "positive" linear combination $\sum_i \alpha_i \Phi_i(u)$ is also the coercive functional.*
2. *Let $\Phi_0(u)$ be a coercive functional and $\Phi_1(u)$ nonnegative functional $\Phi_1(u) \geq 0$, or bounded from below, i.e. $\Phi(u) \geq -K$ on the subspace V or the subset M. Then their sum $\Phi_0 + \Phi_1$ is also coercive.*
3. *Let the coercive functional $\Phi_0(u)$ satisfy $\Phi_0(u) \geq \alpha_0\|u\|^p - K_0$ and $\Phi_1(u)$ decreases "slower" i.e. $\Phi_1(u) \geq -\alpha_1\|u\|^q - K_1$, where $0 < q < p$.*

Then their sum $\Phi_0(u) + \Phi_1(u)$ is also coercive, i. e. $\Phi_1(u)$ "does not spoilt" coerciveness of the functional $\Phi_0(u)$.

Lemma 7.11 (Properties of W.L.S.C Functionals). *[6]*

1. *If the functional $\Phi(u)$ is weakly lower semi-continuous and $\alpha > 0$ positive, then $\alpha\Phi(u)$ is weakly lower semi-continuous too.*
2. *Each continuous linear functional is also weakly continuous and thus also weakly lower semi-continuous and weakly upper semi-continuous.*
3. *Each continuous convex functional is also weakly lower semi-continuous.*