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VYSOKÉ UČENÍ TECHNICKÉ V BRNĚ

FACULTY OF ELECTRICAL ENGINEERING AND COMMUNICATION
DEPARTMENT OF MATHEMATICS

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ÚSTAV MATEMATIKY

REPRESENTATION OF SOLUTIONS OF LINEAR DISCRETE SYSTEMS WITH DELAY

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DIZERTAČNÍ PRÁCE

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REPREZENTACE ŘEŠENÍ LINEÁRNÍCH DISKRÉTNÍCH SYSTÉMŮ SE ZPOŽDĚNÍM

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ABSTRACT

The dissertation thesis is concerned with linear discrete systems with constant matrices of linear terms with a single or two delays. The main objective is to obtain formulas analytically describing exact solutions of initial Cauchy problems. To this end, some matrix special functions called discrete matrix delayed exponentials are defined and used. Their basic properties are proved. Such special matrix functions are used to derive analytical formulas representing the solutions of initial Cauchy problems.

First the initial problem

$$\Delta x(k) = Bx(k-m) + f(k), \quad k \in \mathbb{N} \cup \{0\}, \tag{a}$$

$$x(k) = \varphi(k), \quad k = -m, -m+1, \ldots, 0 \tag{b}$$

is discussed where B is a constant square matrix, f is a given nonhomogeneity, φ is an initial function and m is a positive integer. It is assumed that impulses are acting at some prescribed points and formulas describing the solutions of problem (a) and (b) are derived. Then, instead of the problem (a), (b), a generalized problem

$$x(k+1) = Ax(k) + Bx(k-m) + f(k), \quad k \in \mathbb{N} \cup \{0\},$$

$$x(k) = \varphi(k), \quad k = -m, -m+1, \ldots, 0$$

with impulses acting at each point is considered where A is a constant square matrix and B, f, φ and m are as above.

In the next part of the dissertation, two definitions of discrete matrix delayed exponentials for two delays are given and their basic properties are proved. Such discrete special matrix functions make it possible to find representations of solutions of linear systems with two delays. This is done in the last part of dissertation thesis. The below problem

$$\Delta x(k) = Bx(k-m) + Cx(k-n) + f(k), \quad k \in \mathbb{N} \cup \{0\}, \tag{c}$$

$$x(k) = \varphi(k), \quad k = -\max\{m,n\}, -\max\{m,n\}+1, \ldots, 0 \tag{d}$$

is considered where C is a constant square matrix, n is a positive integer and B, f, φ and m are as above. Two different formulas giving the analytical solution of problem (c), (d) are derived.

KEYWORDS

difference equation, systems of difference equations, linear systems, delay, representation of solution

ABSTRAKT

Dizertační práce se zabývá lineárními diskrétními systémy s konstantními maticemi a s jedním nebo dvěma zpožděními. Hlavním cílem je odvodit vzorce analyticky popisující řešení počátečních úloh. K tomu jsou definovány speciální maticové funkce zvané diskrétní maticové zpožděné exponenciály a je dokázána jejich základní vlastnost. Tyto speciální maticové funkce jsou základem analytických vzorců reprezentujících řešení počáteční úlohy.

Nejprve je uvažována počáteční úloha

$$\Delta x(k) = Bx(k-m) + f(k), \quad k \in \mathbb{N} \cup \{0\}, \tag{a}$$

$$x(k) = \varphi(k), \quad k = -m, -m+1, \dots, 0, \tag{b}$$

kde B je konstantní čtvercová matice, f je daná nehomogenita, φ je počáteční funkce a m je přirozené číslo. Dále předpokládáme, že v některých předepsaných bodech působí na řešení úlohy (a), (b) impulsy. Poté, kromě úlohy (a), (b), uvažujeme zobecněnou úlohu

$$x(k+1) = Ax(k) + Bx(k-m) + f(k), \quad k \in \mathbb{N} \cup \{0\},$$
$$x(k) = \varphi(k), \quad k = -m, -m+1, \dots, 0$$

s impulsy působícími v každém bodě, kde A je konstantní čtvercová matice (B, f, φ a m byly definovány výše).

V další části dizertační práce jsou definovány dvě různé diskrétní maticové zpožděné exponenciály pro dvě zpoždění a jsou dokázány jejich základní vlastnosti. Tyto diskrétní maticové zpožděné exponenciály nám dávají možnost najít reprezentaci řešení lineárních systémů se dvěma zpožděními. Tato řešení jsou konstruována v poslední kapitole dizertační práce, kde je uvažován problém

$$\Delta x(k) = Bx(k-m) + Cx(k-n) + f(k), \quad k \in \mathbb{N} \cup \{0\}, \tag{c}$$

$$x(k) = \varphi(k), \quad k = -\max\{m,n\}, -\max\{m,n\}+1, \dots, 0 \tag{d}$$

(C je konstantní čtvercová matice, n je přirozené číslo a B, f, φ, m byly definovány výše). Řešení problému (c), (d) je dáno pomocí dvou různých vzorců.

KLÍČOVÁ SLOVA

diferenční rovnice, systémy diferenčních rovnic, lineární systémy, zpoždění, reprezentace řešení

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DECLARATION

I declare that I have written my doctoral thesis on the theme of "Representation of Solutions of Linear Discrete Systems with Delay" independently, under the guidance of the doctoral thesis supervisor and using the technical literature and other sources of information which are all quoted in the thesis and detailed in the list of literature at the end of the thesis.

As the author of the doctoral thesis I furthermore declare that, as regards the creation of this doctoral thesis, I have not infringed any copyright. In particular, I have not unlawfully encroached on anyone's personal and/or ownership rights and I am fully aware of the consequences in the case of breaking Regulation § 11 and the following of the Copyright Act No 121/2000 Sb., and of the rights related to intellectual property right and changes in some Acts (Intellectual Property Act) and formulated in later regulations, inclusive of the possible consequences resulting from the provisions of Criminal Act No 40/2009 Sb., Section 2, Head VI, Part 4.

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CONTENTS

1 INTRODUCTION

The dissertation is concerned with the representation of solutions of systems of discrete equations with delays. In this field, many valuable results have been achieved recently, which are also useful in aplications of discrete systems, e.g., for solving problems in the control theory. The motivation for writing this dissertation was the results of the papers [6, 7], which the dissertation extends.

The dissertation is devoted to problems of representation of solutions of linear discrete systems containing delays as partial cases of a general system

$$x(k+1) = Ax(k) + Bx(k-m) + Cx(k-n) + f(k), \quad k \in \mathbb{N} \cup \{0\} \tag{1.1}$$

where A, B and C are constant square matrices and f is a given nonhomogeneity. The main tool to derive appropriate formulas is a so-called discrete matrix delayed exponential (and its generalizations). Along with the system (1.1), the dissertation considers some of its special cases and discusses the influence of impulses at given points on the solution.

Detailed summary of the current state of this problem is given in Chapter 2. In addition to the papers [6, 7], other papers (devoted to discrete equations as well as differential equations) are close to the issues considered by the dissertation. We refer, e.g., to the papers [1–5, 9–11, 17–24, 26] and to the references therein. Among them, the papers most related to the dissertation's topic include [20, 24].

Chapter 3 considers an initial Cauchy problem

$$x(k+1) = Ax(k) + Bx(k-m) + f(k), \quad k \in \mathbb{N} \cup \{0\}, \tag{1.2}$$

$$x(k) = \varphi(k), \quad k = -m, -m+1, \dots, 0 \tag{1.3}$$

where A, B are constant square matrices. The problem of representation of the solution of (1.2), (1.3) is solved under the assumption that impulses are acting on solution at prescribed points. The main result is given in Theorem 3.10.

In Chapter 4 two generalizations of discrete matrix delayed exponential for two delays are given (in Definition 4.1 and Definition 4.5). For both generalized discrete matrix delayed exponentials, their main properties are proved (in Theorem 4.4 and Theorem 4.8). Differences between two definitions of such exponentials naturally lead to different formulas for representation of initial Cauchy problem in Chapter 5 (Theorems 5.1, 5.2, 5.4, 5.5). The exponential given by Definition 4.1 corresponds the definition of the discrete matrix delayed exponential for a single delay, but its application needs the existence of an inverse $(B+C)^{-1}$. From this point of view, the second definition of discrete matrix delayed exponential is better, as no assumption on the existence of an inverse is necessary.

Throughout the dissertation, we use some special notation and known facts. This is the reason why their formulations are given below.

For integers s, t, $s \le t$, we define a set $\mathbb{Z}_s^t := \{s, s+1, \dots, t-1, t\}$. Similarly, we define sets $\mathbb{Z}_{-\infty}^t := \{\dots, t-1, t\}$ and $\mathbb{Z}_s^\infty := \{s, s+1, \dots\}$.

The function $\lfloor \cdot \rfloor$ is the floor integer function. We will employ the following property of the floor integer function:

$$x - 1 < \lfloor x \rfloor \le x \tag{1.4}$$

where $x \in \mathbb{R}$.

Define binomial coefficients as usual, i.e., for $n \in \mathbb{Z}$ and $k \in \mathbb{Z}$,

$$\binom{n}{k} := \begin{cases} \dfrac{n!}{k!(n-k)!} & \text{if} \quad n \ge k \ge 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.5}$$

We will also use the well-known identities

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \tag{1.6}$$

where $n, k \in \mathbb{N}$ and

$$\binom{i}{i} = \binom{i-1}{i-1}, \quad \binom{j}{0} = \binom{j-1}{0}, \quad \binom{i+j}{i} = \binom{i+j-1}{i-1} + \binom{i+j-1}{i} \tag{1.7}$$

where $i, j \in \mathbb{N}$.

We recall that, for a well-defined discrete function $\omega(k)$, the forward difference operator Δ is defined as $\Delta\omega(k) = \omega(k+1) - \omega(k)$. In the dissertation, we also adopt the customary notation $\sum\limits_{i=i_1}^{i_2} g_i = 0$, $\prod\limits_{i=i_1}^{i_2} g_i = 1$ if $i_2 < i_1$. In the case of double sums, we set

$$\sum_{i=i_1, j=j_1}^{i_2, j_2} g_{ij} = 0 \tag{1.8}$$

if at least one of inequalities $i_2 < i_1$, $j_2 < j_1$ holds.

Let a function $F\colon \mathbb{Z}_{k_0}^\infty \times \mathbb{Z}_{n_0}^\infty \to \mathbb{R}^n$ be given. Define a partial difference operator Δ_k acting by the formula

$$\Delta_k F(k, n) := F(k+1, n) - F(k, n), \quad k \ge k_0, \ n \ge n_0.$$

It is easy to see [6, Lemma 3.3] that

$$\Delta_k \left[\sum_{j=1}^{k} F(k, j) \right] = F(k+1, k+1) + \sum_{j=1}^{k} \Delta_k F(k, j), \quad k \ge k_0, \ n \ge n_0. \tag{1.9}$$

2 CURRENT STATE

A "by-steps" method or "method of steps" is one of the basic methods of the theory of differential equations with delay to find a solution to the initial problem. It is effective especially for linear equations and their systems. In [18], method of steps is formalized for linear systems with a constant matrix and with a single delay. This formalization was achieved by utilizing a delayed matrix exponential.

Later, delayed matrix exponential was used in many papers. In [1–3] it is used for boundary-value problems of linear differential equations with delay. A modification of delayed matrix exponential was given in [17] where delayed matrix sine and delayed matrix cosine are introduced to find solutions of oscillating systems. Papers [10, 19] deal with applications of the delayed matrix exponential, delayed matrix sine, and delayed matrix cosine in the control theory. Paper [9] is concerned with systems of linear partial differential equations of a parabolic type with a single delay. The stability of nonlinear differential systems with delay through the delayed matrix exponential is discussed in [23]. Paper [21] extends the definition of delayed matrix exponential to several delays and with applications to analysis of stability. In [4, 5] definitions are given of delayed matrix sine and delayed matrix cosine for several matrices and representations of solutions are derived. The results of [21] are used in [22] to study the exponential stability of fractional differential equations. Paper [26] describes the construction of a matrix exponential for equations with functional delays.

A discrete version of delayed matrix exponential was defined in [6, 7]. In addition to the definition of a discrete matrix exponential its application is considered to solutions of initial-value problems for linear discrete systems with a single delay and representations of solutions are obtained. It also served as a useful tool for solving problems of control theory in [11]. A generalization of discrete delayed matrix exponential to several delays can be found in [20]. In [24], discrete delayed matrix exponential is used to investigate the stability of delay difference equations.

In the following sections, we give a detailed overview of the known results, which are then used or generalized in Chapters 3, 4 and 5. The proofs are also shown of the results formulated (the original proofs are simplified) because some parts of the proofs are referred to in these chapters.

2.1 Discrete Matrix Delayed Exponential

In the present dissertation, we use a special matrix function called a discrete matrix delayed exponential. Such a discrete matrix function was first defined in [6, 7].

Definition 2.1. *For an $n \times n$ constant matrix B, $k \in \mathbb{Z}$ and fixed $m \in \mathbb{N}$, we define a discrete matrix delayed exponential e_m^{Bk} as follows:*

$$\mathrm{e}_m^{Bk} := \begin{cases} \Theta \quad \text{if} \quad k \in \mathbb{Z}_{-\infty}^{-m-1}, \\ I \quad \text{if} \quad k \in \mathbb{Z}_{-m}^{0}, \\ I + B\binom{k}{1} \quad \text{if} \quad k \in \mathbb{Z}_{1}^{m+1}, \\ I + B\binom{k}{1} + B^2\binom{k-m}{2} \quad \text{if} \quad k \in \mathbb{Z}_{(m+1)+1}^{2(m+1)}, \\ I + B\binom{k}{1} + B^2\binom{k-m}{2} + B^3\binom{k-2m}{3} \quad \text{if} \quad k \in \mathbb{Z}_{2(m+1)+1}^{3(m+1)}, \\ \dots \\ I + B\binom{k}{1} + B^2\binom{k-m}{2} + \dots + B^\ell\binom{k-m(\ell-1)}{\ell} \\ \qquad \text{if} \quad \ell = 0, 1, 2, \dots\ , \ k \in \mathbb{Z}_{(\ell-1)(m+1)+1}^{\ell(m+1)} \end{cases}$$

where Θ is the $n \times n$ null matrix and I is the $n \times n$ unit matrix.

Remark 2.2. Definition 2.1 can be shortened to

$$\mathrm{e}_m^{Bk} := \begin{cases} \Theta & \text{if} \quad k \in \mathbb{Z}_{-\infty}^{-m-1}, \\ I + \sum_{i=1}^{\ell} B^i\binom{k-m(i-1)}{i} & \text{if} \quad \ell = 0, 1, 2, \dots\ , \ k \in \mathbb{Z}_{(\ell-1)(m+1)+1}^{\ell(m+1)}\ . \end{cases}$$

Next, Theorem 2.3 is taken from [7, Theorem 2.1].

Theorem 2.3. *Let B be a constant $n \times n$ matrix. Then, for $k \in \mathbb{Z}_{-m}^{\infty}$,*

$$\Delta\, \mathrm{e}_m^{Bk} = B\mathrm{e}_m^{B(k-m)}. \tag{2.1}$$

Proof. Let a matrix B and a positive integer m be fixed. Then, for an integer k satisfying

$$(\ell-1)(m+1) + 1 \le k \le \ell(m+1),$$

in accordance with the definition of e_m^{Bk},

$$\Delta\, \mathrm{e}_m^{Bk} = \Delta\left[I + \sum_{i=1}^{\ell} B^i\binom{k-m(i-1)}{i}\right].$$

Since $\Delta I = \Theta$, we have

$$\Delta \, \mathrm{e}_m^{Bk} = \Delta \left[\sum_{i=1}^{\ell} B^i \binom{k - m(i-1)}{i} \right]. \tag{2.2}$$

By the definition of the forward difference, i.e.,

$$\Delta \mathrm{e}_m^{Bk} = \mathrm{e}_m^{B(k+1)} - \mathrm{e}_m^{Bk}, \tag{2.3}$$

we conclude that it is reasonable to divide the proof into two parts with respect to the value of integer k. We will consider two cases, one with k such that

$$(\ell - 1)(m+1) + 1 \le k < k+1 \le \ell(m+1),$$

and one with

$$k = \ell(m+1).$$

I. The case $(\ell - 1)(m+1) + 1 \le k < k+1 \le \ell(m+1)$.

In this case,

$$k - m \in [(\ell - 2)(m+1) + 1, (\ell - 1)(m+1)]$$

and, by Definition 2.1,

$$\mathrm{e}_m^{B(k-m)} = I + \sum_{i=1}^{\ell-1} B^i \binom{k - m - m(i-1)}{i} = I + \sum_{i=1}^{\ell-1} B^i \binom{k - mi}{i}. \tag{2.4}$$

We prove that

$$\Delta \mathrm{e}_m^{Bk} = B \mathrm{e}_m^{B(k-m)} = B \left[I + \sum_{i=1}^{\ell-1} B^i \binom{k - mj}{i} \right]. \tag{2.5}$$

With the aid of (2.2), (2.3), and (1.6), we get

$$\begin{aligned}
\Delta \mathrm{e}_m^{Bk} &= \mathrm{e}_m^{B(k+1)} - \mathrm{e}_m^{Bk} \\
&= \sum_{i=1}^{\ell} B^i \binom{k+1 - m(i-1)}{i} - \sum_{i=1}^{\ell} B^i \binom{k - m(i-1)}{i} \\
&= \sum_{i=1}^{\ell} B^i \left[\binom{k+1 - m(i-1)}{i} - \binom{k - m(i-1)}{i} \right] \\
&= \sum_{i=1}^{\ell} B^i \binom{k - m(i-1)}{i-1} \\
&= B \binom{k - m(1-1)}{1-1} + \sum_{i=2}^{\ell} B^i \binom{k - m(i-1)}{i-1} \\
&= B \left[I + \sum_{i=2}^{\ell} B^{i-1} \binom{k - m(i-1)}{i-1} \right].
\end{aligned}$$

Now we change the summation index i to $i+1$. Then,

$$\Delta \mathrm{e}_m^{Bk} = B\left[I + \sum_{i=1}^{\ell-1} B^i \binom{k-mi}{i}\right] = B\mathrm{e}_m^{B(k-m)}$$

and, by (2.4), we conclude that formula (2.5) is valid.

II. The case $\boldsymbol{k = \ell(m+1)}$.

In this case we have by Definition 2.1

$$\mathrm{e}_m^{B(k+1)} = I + \sum_{i=1}^{\ell+1} B^i \binom{k+1-m(i-1)}{i}.$$

and

$$\mathrm{e}_m^{B(k-m)} = I + \sum_{i=1}^{\ell} B^i \binom{k-m-m(i-1)}{i} = I + \sum_{i=1}^{\ell} B^i \binom{k-mi}{i}. \tag{2.6}$$

We prove that

$$\Delta \mathrm{e}_m^{Bk} = B\mathrm{e}_m^{B(k-m)} = B\left[I + \sum_{i=1}^{\ell} B^i \binom{k-mi}{i}\right]. \tag{2.7}$$

Therefore,

$$\begin{aligned}
\Delta\, \mathrm{e}_m^{Bk} &= \mathrm{e}_m^{B(k+1)} - \mathrm{e}_m^{Bk} \\
&= \sum_{i=1}^{\ell+1} B^i \binom{k+1-m(i-1)}{i} - \sum_{i=1}^{\ell} B^i \binom{k-m(i-1)}{i} \\
&= B^{\ell+1} \binom{k+1-m(\ell+1-1)}{\ell+1} \\
&\quad + \sum_{i=1}^{\ell} B^i \left[\binom{k+1-m(i-1)}{i} - \binom{k-m(i-1)}{i}\right].
\end{aligned}$$

With the aid of $k = \ell(m+1)$, we get

$$\binom{k+1-m(\ell+1-1)}{\ell+1} = \binom{\ell+1}{\ell+1} = 1$$

and, by (1.6) and (1.7), we have

$$\begin{aligned}
\Delta\, \mathrm{e}_m^{Bk} &= B^{\ell+1} + \sum_{i=1}^{\ell} B^i \binom{k-m(i-1)}{i-1} \\
&= B^{\ell+1} + B\binom{k-m(1-1)}{1-1} + \sum_{i=2}^{\ell} B^i \binom{k-m(i-1)}{i-1}
\end{aligned}$$

$$= B\left[B^\ell + I + \sum_{i=2}^{\ell} B^{i-1}\binom{k-m(i-1)}{i-1}\right].$$

Now we change the summation index i to $i+1$. Then,

$$\Delta \mathrm{e}_m^{Bk} = B\left[I + B^\ell + \sum_{i=1}^{\ell-1} B^{i}\binom{k-mi}{i}\right].$$

For $k = \ell(m+1)$, we have

$$\binom{k-m\ell}{\ell} = \binom{\ell(m+1)-m\ell}{\ell} = \binom{\ell}{\ell} = 1$$

and

$$\begin{aligned}\Delta \mathrm{e}_m^{Bk} &= B\left[I + B^\ell\binom{k-m\ell}{\ell} + \sum_{i=1}^{\ell-1} B^{i}\binom{k-mi}{i}\right]\\ &= B\left[I + \sum_{i=1}^{\ell} B^{i}\binom{k-mi}{i}\right]\\ &= B\mathrm{e}_m^{B(k-m)}.\end{aligned}$$

By (2.6), formula (2.7) is proved. □

2.2 Solutions of Linear Discrete Systems with Single Delay

Consider an initial Cauchy problem

$$\Delta x(k) = Bx(k-m) + f(k), \quad k \in \mathbb{Z}_0^\infty, \tag{2.8}$$

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^0 \tag{2.9}$$

where $m \geq 1$ is a fixed integer, $B = (b_{ij})$ is a constant $n \times n$ matrix, $x\colon \mathbb{Z}_{-m}^\infty \to \mathbb{R}^n$, $f\colon \mathbb{Z}_0^\infty \to \mathbb{R}^n$, $\varphi\colon \mathbb{Z}_{-m}^0 \to \mathbb{R}^n$, and $\Delta x(k) = x(k+1) - x(k)$.

With the aid of discrete matrix delayed exponential, we will derive formulas for solutions of the homogeneous and nonhomogeneous initial problem (2.8), (2.9).

2.2.1 Homogeneous Initial Problem

Consider first a homogeneous initial problem (2.8), (2.9), i.e., the problem

$$\Delta x(k) = Bx(k-m), \quad k \in \mathbb{Z}_0^\infty, \tag{2.10}$$

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^0. \tag{2.11}$$

Theorem 2.4 (Theorem 3.1 in [7]). *Let B be a constant $n \times n$ matrix. Then, a solution of the problem* (2.10), (2.11) *can be expressed as*

$$x(k) = \mathrm{e}_m^{Bk} \varphi(-m) + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B(k-m-j)} \Delta\varphi(j-1), \tag{2.12}$$

where $k \in \mathbb{Z}_{-m}^{\infty}$.

Proof. We are going to find a solution to the problem (2.10), (2.11) in the form

$$x(k) = \mathrm{e}_m^{Bk}\, v + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B(k-m-j)} \Delta\psi(j-1), \quad k \in \mathbb{Z}_{-m}^{\infty} \tag{2.13}$$

with an unknown constant vector v and a discrete function $\psi : \mathbb{Z}_{-m}^{0} \to \mathbb{R}^n$. Because of linearity (taking into account that k varies), we have

$$\begin{aligned}
\Delta x(k) &= \Delta \left[\mathrm{e}_m^{Bk}\, v + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B(k-m-j)} \Delta\psi(j-1) \right] \\
&= \Delta \mathrm{e}_m^{Bk}\, v + \sum_{j=-m+1}^{0} \Delta \left[\mathrm{e}_m^{B(k-m-j)} \Delta\psi(j-1) \right] \\
&= \Delta \left[\mathrm{e}_m^{Bk} \right] v + \sum_{j=-m+1}^{0} \Delta \left[\mathrm{e}_m^{B(k-m-j)} \right] \Delta\psi(j-1).
\end{aligned}$$

We use formula (2.1)

$$\begin{aligned}
\Delta x(k) &= B\mathrm{e}_m^{B(k-m)}\, v + \sum_{j=-m+1}^{0} B\mathrm{e}_m^{B(k-2m-j)} \Delta\psi(j-1) \\
&= B \left[\mathrm{e}_m^{B(k-m)}\, v + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B(k-2m-j)} \Delta\psi(j-1) \right].
\end{aligned}$$

Now we conclude that, for any v and ψ, the equation $\Delta x(k) = Bx(k-m)$ holds. We will try to satisfy the initial conditions (2.11). By (2.10), we have

$$\mathrm{e}_m^{B(k-m)}\, v + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B(k-2m-j)} \Delta\psi(j-1) = x(k-m).$$

We consider values k such that $k - m \in \mathbb{Z}_{-m}^{0}$. Simultaneously, we change the argument k to $k + m$. We get

$$\mathrm{e}_m^{Bk}\, v + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B(k-m-j)} \Delta\psi(j-1) = \varphi(k),$$

for $k \in \mathbb{Z}_{-m}^{0}$. We rewrite the last formula as

$$\mathrm{e}_m^{Bk}\, v + \sum_{j=-m+1}^{k} \mathrm{e}_m^{B(k-m-j)} \Delta\psi(j-1) + \sum_{j=k+1}^{0} \mathrm{e}_m^{B(k-m-j)} \Delta\psi(j-1) = \varphi(k). \tag{2.14}$$

By Definition 2.1, the first sum becomes

$$\sum_{j=-m+1}^{k} \mathrm{e}_m^{B(k-m-j)} \Delta\psi(j-1) = \sum_{j=-m+1}^{k} \Delta\psi(j-1) = \psi(k) - \psi(-m)$$

and the second one turns into the zero vector. Finally, since

$$\mathrm{e}_m^{Bk} \equiv I, \quad k \in \mathbb{Z}_{-m}^{0},$$

equation (2.14) becomes

$$v + \psi(k) - \psi(-m) = \varphi(k)$$

and one can define

$$\psi(k) := \varphi(k), \quad k \in \mathbb{Z}_{-m}^{0};$$
$$v := \psi(-m) = \varphi(-m).$$

In order to get formula (2.12) it remains to substitute v and ψ into (2.13). □

2.2.2 Nonhomogeneous Initial Problem

We consider a nonhomogeneous initial Cauchy problem (2.8), (2.9), i.e. the problem

$$\Delta x(k) = Bx(k-m) + f(k), \quad k \in \mathbb{Z}_0^{\infty}, \tag{2.15}$$
$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^{0}. \tag{2.16}$$

We get this solution, in accordance with the theory of linear equations, as the sum of a solution of the adjoint homogeneous problem (2.10), (2.11) (satisfying the same initial data) and a particular solution of (2.15) being zero on the initial interval. Therefore, we are going to find such a particular solution $x_p(k)$, $k \in \mathbb{Z}_{-m}^{\infty}$ of the initial Cauchy problem

$$\Delta x(k) = Bx(k-m) + f(k), \quad k \in \mathbb{Z}_0^{\infty}, \tag{2.17}$$
$$x(k) = 0, \quad k \in \mathbb{Z}_{-m}^{0}. \tag{2.18}$$

Theorem 2.5 (Theorem 3.5 in [7]). *The solution $x = x_p(k)$ of the initial Cauchy problem* (2.17)*,* (2.18) *can be represented on $\mathbb{Z}_{-m}^{\infty}$ in the form*

$$x_p(k) = \sum_{i=1}^{k} \mathrm{e}_m^{B(k-m-j)} f(j-1).$$

Proof. We are going to find a particular solution $x_p(k)$ of the problem (2.17), (2.18) using the idea of the method of variation of arbitrary constants in the form

$$x_p(k) = \sum_{i=1}^{k} \mathrm{e}_m^{B(k-m-j)}\omega(j), \tag{2.19}$$

where $\omega : \mathbb{Z}_1^\infty \to \mathbb{R}^n$ is a discrete function. We put (2.19) in (2.17). Then, with the aid of (1.9) and (2.1), we obtain

$$\begin{aligned}
\Delta x(k) &= \Delta\left[\sum_{i=1}^{k} \mathrm{e}_m^{B(k-m-j)}\omega(j)\right] \\
&= \mathrm{e}_m^{B((k+1)-m-(k+1))}\omega(k+1) + \sum_{j=1}^{k}\Delta\left[\mathrm{e}_m^{B(k-m-j)}\omega(j)\right] \\
&= \mathrm{e}_m^{B(-m)}\omega(k+1) + B\sum_{j=1}^{k}\mathrm{e}_m^{B(k-2m-j)}\omega(j) \\
&= \mathrm{e}_m^{B(-m)}\omega(k+1) + B\left[\sum_{j=1}^{k-m}\mathrm{e}_m^{B(k-2m-j)}\omega(j) + \sum_{j=k-m+1}^{k}\mathrm{e}_m^{B(k-2m-j)}\omega(j)\right].
\end{aligned}$$

By Definition 2.1, $\mathrm{e}_m^{B(-m)} = I$ and, for $j \in \mathbb{Z}_{k-m+1}^k$, $\mathrm{e}_m^{B(k-2m-j)} = \Theta$. Then,

$$\Delta x(k) = \omega(k+1) + B\sum_{j=1}^{k-m}\mathrm{e}_m^{B(k-2m-j)}\omega(j).$$

We define

$$\omega(k) := f(k-1), \quad k \in \mathbb{Z}_0^\infty$$

and get

$$\begin{aligned}
\Delta x(k) &= f(k) + B\sum_{j=1}^{k-m}\mathrm{e}_m^{B(k-2m-j)}f(j-1) \\
&= f(k) + Bx(k-m).
\end{aligned}$$

This ends the proof. □

Combining the results of Theorem 2.4 and Theorem 2.5, we get immediately

Theorem 2.6 (Theorem 3.6 in [7]). *On $\mathbb{Z}_{-m}^\infty$, the solution $x = x(k)$ of the initial Cauchy problem* (2.15), (2.16) *can be represented in the form*

$$x(k) = \mathrm{e}_m^{Bk}\,\varphi(-m) + \sum_{j=-m+1}^{0}\mathrm{e}_m^{B(k-m-j)}\Delta\varphi(j-1) + \sum_{j=1}^{k}\mathrm{e}_m^{B(k-m-j)}f(j-1).$$

2.3 Solutions of Linear Discrete Systems with Single Delay – Generalization

Consider a linear discrete system

$$x(k+1) = Ax(k) + Bx(k-m) + f(k) \tag{2.20}$$

where $m \geq 1$ is a fixed integer, $k \in \mathbb{Z}_0^\infty$, $A = (a_{ij})$, $\det A \neq 0$ and $B = (b_{ij})$ are constant $n \times n$ matrices with the commutative property

$$AB = BA, \tag{2.21}$$

$f\colon \mathbb{Z}_0^\infty \to \mathbb{R}^n$, $x\colon \mathbb{Z}_{-m}^\infty \to \mathbb{R}^n$. Together with equation (2.20), we consider an initial Cauchy problem

$$x(k) = \varphi(k) \tag{2.22}$$

with a given $\varphi\colon \mathbb{Z}_{-m}^0 \to \mathbb{R}^n$.

Substituting in (2.20)

$$x(k) = A^k y(k)$$

with $k \in \mathbb{Z}_{-m}^\infty$ we get

$$y(k+1) = y(k) + B_1 y(k-m) + A^{-k-1} f(k)$$

with $B_1 = A^{-k-1} B A^{k-m}$. By the property (2.21), we obtain $B_1 = A^{-1} B A^{-m}$ and matrix B_1 becomes a constant matrix. Using the difference operator, we write an equivalent form to (2.20) as

$$\Delta y(k) = B_1 y(k-m) + A^{-k-1} f(k), \quad k \in \mathbb{Z}_0^\infty. \tag{2.23}$$

The corresponding equivalent initial data with respect to (2.23) are

$$y(k) = A^{-k} \varphi(k), \quad k \in \mathbb{Z}_{-m}^0.$$

We consider an initial Cauchy problem for a homogeneous linear matrix equation:

$$X(k+1) = AX(k) + BX(k-m), \quad k \in \mathbb{Z}_0^\infty, \tag{2.24}$$

$$X(k) = A^k, \quad k \in \mathbb{Z}_{-m}^0 \tag{2.25}$$

with $n \times n$ matrices A and B satisfying conditions $AB = BA$ and $\det A \neq 0$. Here $X\colon \mathbb{Z}_{-m}^\infty \to \mathbb{R}^{n \times n}$ is an unknown matrix.

Theorem 2.7 (Theorem 2.2 in [6])**.** *The matrix*

$$X = X_0(k) := A^k \, \mathrm{e}_m^{B_1 k}, \quad k \in \mathbb{Z}_{-m}^\infty \tag{2.26}$$

solves the problem (2.24)*,* (2.25)*.*

Proof. We put

$$X(k) = X_0(k), \quad k \in \mathbb{Z}_{-m}^{\infty}$$

in matrix equation (2.24). Then,

$$A^{k+1} \operatorname{e}_m^{B_1(k+1)} = A^{k+1} \operatorname{e}_m^{B_1 k} + BA^{k-m} \operatorname{e}_m^{B_1(k-m)}$$

and

$$\Delta \operatorname{e}_m^{B_1 k} = B_1 \operatorname{e}_m^{B_1(k-m)}, \quad k \in \mathbb{Z}_{-m}^{\infty}.$$

This equality is valid, by (2.1), for every $k \in \mathbb{Z}_{-m}^{\infty}$. □

In the following parts, we derive a matrix form of the solution of a homogeneous and nonhomogeneous initial problem (2.20), (2.22). We use the matrix function $X_0(k)$ defined as a discrete matrix delayed exponential by formula (2.26).

2.3.1 Homogeneous Initial Problem

Consider a homogeneous initial problem (2.20), (2.22), i.e., the problem

$$x(k+1) = Ax(k) + Bx(k-m), \quad k \in \mathbb{Z}_0^{\infty}, \tag{2.27}$$

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^{0}. \tag{2.28}$$

Theorem 2.8 (Theorem 3.1 in [6]). *Let A, B be constant $n \times n$ matrices, $AB = BA$ and* $\det A \neq 0$. *Then, the solution of* (2.27), (2.28) *can be expressed as*

$$x(k) = X_0(k)A^{-m}\varphi(-m) + A^m \sum_{j=-m+1}^{0} X_0(k-m-j)\left[\varphi(j) - A\varphi(j-1)\right] \tag{2.29}$$

where $k \in \mathbb{Z}_{-m}^{\infty}$.

Proof. We put $x(k) = A^k y(k)$, $k \in \mathbb{Z}_{-m}^{\infty}$. Then, the problem (2.27), (2.28) is changed to

$$\Delta y(k) = B_1 y(k-m), \quad k \in \mathbb{Z}_0^{\infty}, \tag{2.30}$$

$$y(k) = A^{-k}\varphi(k), \quad k \in \mathbb{Z}_{-m}^{0}. \tag{2.31}$$

We will try to find a solution of (2.30), (2.31) for $k \in \mathbb{Z}_{-m}^{\infty}$ in the form

$$y(k) = \operatorname{e}_m^{B_1 k} C + \sum_{j=-m+1}^{0} \operatorname{e}_m^{B_1(k-m-j)} \Delta\pi(j-1), \tag{2.32}$$

where C is an unknown constant vector and $\pi : \mathbb{Z}_{-m}^{0} \to \mathbb{R}^n$ is a discrete function.

First we show that expression (2.32) is a solution of homogeneous system (2.30) for arbitrary C and π and for $k \in \mathbb{Z}_0^\infty$. We compute $\Delta y(k)$, $k \in \mathbb{Z}_{-m}^\infty$. The linearity yields

$$\begin{aligned}\Delta y(k) &= \Delta \left[\mathrm{e}_m^{B_1 k} C + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B_1(k-m-j)} \Delta\pi(j-1) \right] \\ &= \Delta \mathrm{e}_m^{B_1 k} C + \sum_{j=-m+1}^{0} \Delta \left[\mathrm{e}_m^{B_1(k-m-j)} \Delta\pi(j-1) \right] \\ &= \Delta \left[\mathrm{e}_m^{B_1 k} \right] C + \sum_{j=-m+1}^{0} \Delta \left[\mathrm{e}_m^{B_1(k-m-j)} \right] \Delta\pi(j-1).\end{aligned}$$

We apply formula (2.1) relative to the increments of discrete exponential:

$$\begin{aligned}\Delta y(k) &= B_1 \mathrm{e}_m^{B_1(k-m)} C + \sum_{j=-m+1}^{0} B_1 \mathrm{e}_m^{B_1(k-2m-j)} \Delta\pi(j-1) \\ &= B_1 \left[\mathrm{e}_m^{B_1(k-m)} C + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B_1(k-2m-j)} \Delta\pi(j-1) \right].\end{aligned} \tag{2.33}$$

Consequently,

$$\Delta y(k) = B_1 y(k-m)$$

for $k \in \mathbb{Z}_0^\infty$ and expression (2.32) really solves homogeneous system (2.30) for arbitrary C and π.

Now we try to fix C and π in order to satisfy initial condition (2.31) for $k \in \mathbb{Z}_{-m}^0$. We use the representation of increment (2.33) substituting it into (2.30). Easy simplification leads to

$$\mathrm{e}_m^{B_1(k-m)} C + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B_1(k-2m-j)} \Delta\pi(j-1) = y(k-m), \quad k \in \mathbb{Z}_0^m.$$

We choose the vector C and function π in such a manner that initial conditions (2.31) holds. We change the index k to $k+m$. Then, the last equation can be written as

$$\mathrm{e}_m^{B_1 k} C + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B_1(k-m-j)} \Delta\pi(j-1) = y(k) = A^{-k}\varphi(k),$$

where $k \in \mathbb{Z}_{-m}^0$. Moreover, let us rewrite the last formula again. We get

$$\mathrm{e}_m^{B_1 k} C + \sum_{j=-m+1}^{k} \mathrm{e}_m^{B_1(k-m-j)} \Delta\pi(j-1) + \sum_{j=k+1}^{0} \mathrm{e}_m^{B_1(k-m-j)} \Delta\pi(j-1) = A^{-k}\varphi(k). \tag{2.34}$$

Consider the first sum. If $j > k$, then, by definition, the first sum equals zero. Therefore, we consider only the case $j \leq k$ so that

$$k - m - j \geq k - m - k = -m$$

and, moreover since $j \geq -m + 1$ and $k \leq 0$

$$k - m - j \leq -m - j \leq -m + m - 1 = -1.$$

By Definition 2.1,

$$\mathrm{e}_m^{B_1(k-m-j)} \equiv I$$

and the first sum is equivalent to

$$\sum_{j=-m+1,\, j\leq k}^{k} \mathrm{e}_m^{B_1(k-m-j)} \Delta\pi(j-1) = \sum_{j=-m+1,\, j\leq k}^{k} \Delta\pi(j-1) = \pi(k) - \pi(-m).$$

Now we consider the second sum. If $j > 0$, then, by definition, the second sum equals zero. This holds for $k = 0$, i.e., it is sufficient to consider $k \in \mathbb{Z}_{-m}^{-1}$ only. Since $j \geq k + 1$, we have

$$k - m - j \leq k - m - k - 1 = -m - 1 < -m$$

and, by Definition 2.1,

$$\mathrm{e}_m^{B_1(k-m-j)} \equiv \Theta.$$

Finally, since $\mathrm{e}_m^{B_1 k} \equiv I$ if $k \in \mathbb{Z}_{-m}^{0}$, (2.34) becomes

$$C + \pi(k) - \pi(-m) = A^{-k}\varphi(k)$$

and one can put

$$\pi(k) := A^{-k}\varphi(k), \quad k \in \mathbb{Z}_{-m}^{0} \quad \text{and} \quad C := \pi(-m) = A^{-m}\varphi(-m).$$

In order to get formula (2.29) it remains to substitute C and π in (2.32). We take into account that, from the definition of discrete exponential, it follows that matrices A and discrete exponential commute. Now

$$\begin{aligned} x(k) = A^k y(k) &= A^k \left[\mathrm{e}_m^{B_1 k} A^{-m}\varphi(-m) + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B_1(k-m-j)} \Delta \left[A^{-(j-1)}\varphi(j-1) \right] \right] \\ &= X_0(k) A^{-m}\varphi(-m) + \sum_{j=-m+1}^{0} A^{m+j} X_0(k-m-j) \left[A^{-j}\varphi(j) - A^{-(j-1)}\varphi(j-1) \right] \\ &= X_0(k) A^{-m}\varphi(-m) + A^m \sum_{j=-m+1}^{0} X_0(k-m-j) \left[\varphi(j) - A\varphi(j-1) \right]. \end{aligned}$$

□

2.3.2 Nonhomogeneous Initial Problem

We consider the nonhomogeneous initial Cauchy problem (2.20), (2.22), i.e.,

$$x(k+1) = Ax(k) + Bx(k-m) + f(k), \quad k \in \mathbb{Z}_0^\infty, \tag{2.35}$$

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^0. \tag{2.36}$$

We derive a solution of this problem as the sum of a solution of the adjoint homogeneous problem (2.27), (2.28) (satisfying the same initial data) and a particular solution of (2.35) being zero on the initial interval. Therefore, we will try to find such a particular solution.

Now we find a solution $x = x_p(k)$, $k \in \mathbb{Z}_{-m}^\infty$ of the problem

$$x(k+1) = Ax(k) + Bx(k-m) + f(k), \quad k \in \mathbb{Z}_0^\infty, \tag{2.37}$$

$$x(k) = 0, \quad k \in \mathbb{Z}_{-m}^0. \tag{2.38}$$

Theorem 2.9 (Theorem 3.4 in [6]). *Let A, B be constant $n\times n$ matrices, $AB = BA$ and $\det A \neq 0$. Then, a solution $x = x_p(k)$ of the initial Cauchy problem* (2.37), (2.38) *can be represented on $\mathbb{Z}_{-m}^\infty$ in the form*

$$x_p(k) = A^m \sum_{i=1}^{k} X_0(k-m-j) f(j-1).$$

Proof. We put $x(k) = A^k y(k)$, $k \in \mathbb{Z}_{-m}^\infty$. Then, problem (2.37), (2.38) can be written as

$$\Delta y(k) = B_1 y(k-m) + A^{-k-1} f(k), \quad k \in \mathbb{Z}_0^\infty, \tag{2.39}$$

$$y(k) = 0, \quad k \in \mathbb{Z}_{-m}^0. \tag{2.40}$$

We will try to find a particular solution $y_p(k)$ of (2.39), (2.40) on $\mathbb{Z}_1^\infty$ in the form

$$y_p(k) = \sum_{i=1}^{k} \mathrm{e}_m^{B_1(k-m-j)} \omega(j), \tag{2.41}$$

where $\omega : \mathbb{Z}_1^\infty \to \mathbb{R}^n$ is a discrete function. We put $y_p(k)$ into (2.39). Then,

$$\Delta \left[\sum_{i=1}^{k} \mathrm{e}_m^{B_1(k-m-j)} \omega(j) \right] = B_1 \left[\sum_{i=1}^{k-m} \mathrm{e}_m^{B_1(k-2m-j)} \omega(j) \right] + A^{-k-1} f(k).$$

Considering Δ as Δ_k, we obtain, with the aid of formula (1.9),

$$\mathrm{e}_m^{B_1((k+1)-m-(k+1))} \omega(k+1) + \sum_{i=1}^{k} \Delta \left[\mathrm{e}_m^{B_1(k-m-j)} \omega(j) \right]$$
$$= B_1 \left[\sum_{i=1}^{k-m} \mathrm{e}_m^{B_1(k-2m-j)} \omega(j) \right] + A^{-k-1} f(k).$$

Using formula (2.1), we have

$$\Delta \mathrm{e}_m^{B_1(k-m-j)} = B_1 \mathrm{e}_m^{B_1(k-2m-j)}$$

and the last equation becomes

$$\begin{aligned} &\mathrm{e}_m^{B_1(-m)} \omega(k+1) + B_1 \sum_{i=1}^{k} \mathrm{e}_m^{B_1(k-2m-j)} \omega(j) \\ &\qquad\qquad = B_1 \sum_{i=1}^{k-m} \mathrm{e}_m^{B_1(k-2m-j)} \omega(j) + A^{-k-1} f(k). \end{aligned} \tag{2.42}$$

Since $\mathrm{e}_m^{B_1(-m)} \equiv I$ and

$$\sum_{i=1}^{k} \mathrm{e}_m^{B_1(k-2m-j)} \omega(j) = \sum_{i=1}^{k-m} \mathrm{e}_m^{B_1(k-2m-j)} \omega(j) + \sum_{j=k-m+1}^{k} \mathrm{e}_m^{B_1(k-2m-j)} \omega(j),$$

where by Definition 2.1

$$\mathrm{e}_m^{B_1(k-2m-j)} \equiv \Theta \quad \text{if} \quad j \in \mathbb{Z}_{k-m+1}^{k},$$

(2.42) turns into

$$\omega(k+1) + B_1 \sum_{i=1}^{k-m} \mathrm{e}_m^{B_1(k-2m-j)} \omega(j) = B_1 \sum_{i=1}^{k-m} \mathrm{e}_m^{B_1(k-2m-j)} \omega(j) + A^{-k-1} f(k).$$

Both sides will be equivalent if we define

$$\omega(k) := A^{-k} f(k-1), \quad k \in \mathbb{Z}_1^{\infty}$$

and substitute this function into (2.41). This ends the proof since

$$\begin{aligned} x_p(k) = A^k y_p(k) &= A^k \sum_{i=1}^{k} \mathrm{e}_m^{B_1(k-m-j)} \omega(j) = A^k \sum_{i=1}^{k} \mathrm{e}_m^{B_1(k-m-j)} A^{-j} f(j-1) \\ &= \sum_{i=1}^{k} A^{m+j} A^{k-m-j} \mathrm{e}_m^{B_1(k-m-j)} A^{-j} f(j-1) = A^m \sum_{i=1}^{k} X_0(k-m-j) f(j-1). \end{aligned}$$

□

Combining the results of Theorem 2.8 and Theorem 2.9, we get immediately

Theorem 2.10 (Theorem 3.5 in [6])**.** *On $\mathbb{Z}_{-m}^{\infty}$, a solution $x = x(k)$ of the initial Cauchy problem* (2.35), (2.36) *can be represented in the form*

$$\begin{aligned} x(k) = X_0(k) A^{-m} \varphi(-m) &+ A^m \sum_{j=-m+1}^{0} X_0(k-m-j) \left[\varphi(j) - A\varphi(j-1)\right] \\ &+ A^m \sum_{i=1}^{k} X_0(k-m-j) f(j-1). \end{aligned}$$

2.4 Discrete Matrix Multi-Delayed Exponential

In [20], a problem is considered of the representation of solutions to linear discrete systems with several delays and constant matrices. Below, we reproduce the main results of [20]. In Section 5.4, we compare these results with ours.

Theorem 2.11. *Let $m_1, m_2 \geq 1$, B_1, B_2 be $N \times N$ matrices such that $B_1 B_2 = B_2 B_1$, and $X(k) = \mathrm{e}_{m_1}^{B_1(k-m_1)}$. Then, the matrix solution of the equation*

$$\Delta Y(k) = B_1 Y(k-m_1) + B_2 Y(k-m_2), \qquad k \geq 0, \tag{2.43}$$

satisfying

$$Y(k) = \begin{cases} \Theta, & \text{if} \quad k \in \mathbb{Z}_{-\infty}^{-1}, \\ E, & \text{if} \quad k = 0, \end{cases} \tag{2.44}$$

has the form $Y(k) = \mathrm{e}_{m_1,m_2}^{B_1,B_2(k-m_2)}$ where

$$\mathrm{e}_{m_1,m_2}^{B_1,B_2 k} = \begin{cases} \Theta, & \text{if} \quad k \in \mathbb{Z}_{-\infty}^{-m_2-1}, \\ X(k+m_2) + B_2 \sum\limits_{j_1=1}^{k} X(k-j_1)X(j_1-1) + \cdots \\ \quad \cdots + B_2^l \sum\limits_{\substack{j_1=(l-1) \\ \times(m_2+1)+1}}^{k} \sum\limits_{\substack{j_2=(l-1) \\ \times(m_2+1)+1}}^{j_1} \cdots \sum\limits_{\substack{j_l=(l-1) \\ \times(m_2+1)+1}}^{j_{l-1}} X(k-j_1) \\ \quad \times \prod\limits_{i=1}^{l-1} X(j_i - j_{i+1}) X(j_l - (l-1)(m_2+1)-1), \\ & \text{if} \quad k \in \mathbb{Z}_{(l-1)(m_2+1)+1}^{l(m_2+1)},\ l \in \mathbb{Z}_0^{\infty} \end{cases} \tag{2.45}$$

Theorem 2.12. *Let $1 \leq m_1 \leq m_2$, $\varphi\colon \mathbb{Z}_{-m_2}^{0} \to \mathbb{R}^N$ be a given function and B_1, B_2 $N \times N$ permutable matrices, i.e., $B_1 B_2 = B_2 B_1$. Then, the solution of the initial Cauchy problem consisting of equation*

$$\Delta x(k) = B_1 x(k-m_1) + B_2 x(k-m_2), \qquad k \geq 0$$

and initial condition

$$x(k) = \varphi(k), \qquad k \in \mathbb{Z}_{-m_2}^{0}$$

has the form

$$\begin{aligned} x(k) = Y(k+m_2)\varphi(-m_2) &+ \sum_{j=-m_2+1}^{0} Y(k-j)\Delta\varphi(j-1) \\ &- B_1 \sum_{j=-m_2}^{-1-m_1} Y(k-1-m_1-j)\varphi(j) \end{aligned} \tag{2.46}$$

for $k \in \mathbb{Z}_{-m_2}^{\infty}$, where $Y(k) = \mathrm{e}_{m_1,m_2}^{B1,B2(k-m_2)}$.

Definition 2.13. *Let $n > 1$, $m_1, \dots, m_n \geq 1$ and $B_1, \dots, B_n$ be pairwise permutable $N \times N$ matrices, i.e., $B_iB_j = B_jB_i$ for each $i, j \in \{1, \dots, n\}$. For each $j = 2, \dots, n$, the discrete multi-delayed matrix exponential corresponding to delays $m_1, \dots, m_j$ and matrices $B_1, \dots, B_j$ is defined as follows*

$$\mathrm{e}_{m_1,\dots,m_j}^{B_1,\dots,B_j k} = \begin{cases} \Theta, & \text{if} \quad k \in \mathbb{Z}_{-\infty}^{-m_j-1}, \\ X_{j-1}(k+m_j) + B_j \sum\limits_{i_1=1}^{k} X_{j-1}(k-i_1)X_{j-1}(i_1-1) + \cdots \\ \quad \cdots + B_j^l \sum\limits_{\substack{i_1=(l-1)\\ \times(m_j+1)+1}}^{k} \sum\limits_{\substack{i_2=(l-1)\\ \times(m_j+1)+1}}^{i_1} \cdots \sum\limits_{\substack{i_l=(l-1)\\ \times(m_j+1)+1}}^{i_{l-1}} X_{j-1}(k-i_1) \\ \quad \times \prod\limits_{s=1}^{l-1} X_{j-1}(i_s - i_{s+1}) X_{j-1}(i_l - (l-1)(m_j+1) - 1), \\ & \text{if} \quad k \in \mathbb{Z}_{(l-1)(m_j+1)+1}^{l(m_j+1)}, \ l \in \mathbb{Z}_0^{\infty} \end{cases} \tag{2.47}$$

where $X_{j-1}(k) = \mathrm{e}_{m_1,\dots,m_j-1}^{B_1,\dots,B_{j-1}(k-m_{j-1})}$.

Theorem 2.14. *Let $n > 0$, $m_1, \dots, m_n \geq 1$, and $B_1, \dots, B_n$ be pairwise permutable $N \times N$ matrices. Then, the matrix solution of the equation*

$$\Delta Y(k) = B_1 Y(k-m_1) + \cdots + B_n Y(k-m_n), \quad k \geq 0,$$

satisfying condition

$$Y(k) = \begin{cases} \Theta, & \text{if} \quad k \in \mathbb{Z}_{-\infty}^{-1}, \\ E, & \text{if} \quad k = 0, \end{cases}$$

has the form

$$Y(k) = \begin{cases} \mathrm{e}_{m_1}^{B_1(k-m_1)}, & \text{if } n = 1, \\ \mathrm{e}_{m_1,\dots,m_n}^{B_1,\dots,B_n(k-m_n)}, & \text{if } n > 1. \end{cases}$$

where $k \in \mathbb{Z}_0^{\infty}$

Theorem 2.15. *Let $n > 1$, $m_1, \dots, m_n \geq 1$, $m = \max\{m_1, \dots, m_n\}$, $\varphi\colon \mathbb{Z}_{-m}^0 \to \mathbb{R}^N$ be a given function and $B_1, \dots, B_n$ pairwise permutable $N \times N$ matrices. Then, the solution of the initial Cauchy problem consisting of the equation*

$$\Delta x(k) = B_1 x(k-m_1) + \cdots + B_n x(k-m_n) \tag{2.48}$$

and initial condition

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^0, \tag{2.49}$$

has the form

$$\begin{aligned} x(k) = Y(k+m)\varphi(-m) + \sum_{j=-m+1}^{0} Y(k-j)\Delta\varphi(j-1) \\ - \sum_{i=1}^{n} B_i \sum_{j=-m}^{-1-m_i} Y(k-1-m_i-j)\varphi(j) \end{aligned} \tag{2.50}$$

for $k \in \mathbb{Z}_{-m}^{\infty}$*, where* $Y(k) = \mathrm{e}_{m_1,\ldots,m_n}^{B_1,\ldots,B_n(k-m_n)}$.

Theorem 2.16. *Let* $n > 1$*,* $m_1, \ldots, m_n \geq 1$*,* $m = \max\{m_1, \ldots, m_n\}$*,* $B_1, \ldots, B_n$ *be pairwise permutable* $N \times N$ *matrices,* $\varphi\colon \mathbb{Z}_{-m}^{0} \to \mathbb{R}^N$ *and* $f\colon \mathbb{Z}_0^{\infty} \to \mathbb{R}^N$ *be given functions. Then, the solution of the nonhomogeneous initial Cauchy problem consisting of the equation*

$$\Delta x(k) = B_1 x(k-m_1) + \cdots + B_n x(k-m_n) + f(k), \qquad k \geq 0, \tag{2.51}$$

and initial condition

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^{0}, \tag{2.52}$$

has the form

$$\begin{aligned} x(k) = {} & Y(k+m)\varphi(-m) + \sum_{j=-m+1}^{0} Y(k-j)\Delta\varphi(j-1) \\ & - \sum_{i=1}^{n} B_i \sum_{j=-m}^{-1-m_i} Y(k-1-m_i-j)\varphi(j) + \sum_{j=1}^{k} Y(k-j) f(j-1) \end{aligned} \tag{2.53}$$

for $k \in \mathbb{Z}_{-m}^{\infty}$*, where* $Y(k) = \mathrm{e}_{m_1,\ldots,m_n}^{B_1,\ldots,B_n(k-m_n)}$.

Corollary 2.17. *Let* $n > 1$*,* $m_1, \ldots, m_n \geq 1$*,* $m = \max\{m_1, \ldots, m_n\}$*,* $A, B_1, \ldots, B_n$ *be pairwise permutable* $N \times N$ *matrices,* $\det A \neq 0$*,* $\varphi\colon \mathbb{Z}_{-m}^{0} \to \mathbb{R}^N$ *and* $f\colon \mathbb{Z}_0^{\infty} \to \mathbb{R}^N$ *be given functions. Then, the solution of the nonhomogeneous initial problem consisting of the equation*

$$x(k+1) = Ax(k) + B_1 x(k-m_1) + \cdots + B_n x(k-m_n) + f(k), \quad k \geq 0, \tag{2.54}$$

and initial condition

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^{0}, \tag{2.55}$$

has the form

$$\begin{aligned} x(k) = {} & \tilde{Y}(k+m)\varphi(-m) + \sum_{j=-m+1}^{0} \tilde{Y}(k-j)[\varphi(j) - A\varphi(j-1)] \\ & - \sum_{i=1}^{n} B_i \sum_{j=-m}^{-1-m_i} \tilde{Y}(k-1-m_i-j)\varphi(j) + \sum_{j=1}^{k} \tilde{Y}(k-j) f(j-1) \end{aligned} \tag{2.56}$$

for $k \in \mathbb{Z}_{-m}^{\infty}$*, where* $\tilde{Y}(k) = A^k \mathrm{e}_{m_1,\ldots,m_n}^{\tilde{B}_1,\ldots,\tilde{B}_n(k-m_n)}$ *and* $\tilde{B}_i = B_i A^{-1-m_i}$ *,* $i = 1, \ldots, n$.

3 SOLUTIONS OF LINEAR DISCRETE SYSTEMS WITH IMPULSES

In this chapter we present results on representation of solutions of linear discrete systems with impulses. The results of Sections 3.1 – 3.3 are published in [8, 13–15, 25] while the results in Section 3.4 are new.

Consider an initial Cauchy problem

$$\Delta x(k) = Bx(k-m) + f(k), \quad k \in \mathbb{Z}_0^\infty, \tag{3.1}$$

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^0 \tag{3.2}$$

where $m \geq 1$ is a fixed integer, $B = (b_{ij})$ is a constant $n \times n$ matrix, $x\colon \mathbb{Z}_{-m}^\infty \to \mathbb{R}^n$, $f\colon \mathbb{Z}_0^\infty \to \mathbb{R}^n$, $\varphi\colon \mathbb{Z}_{-m}^0 \to \mathbb{R}^n$ and $\Delta x(k) = x(k+1) - x(k)$.

We assume that impulses are acting on x at some prescribed points. Particularly, the problem (3.1), (3.2) is considered if impulses are focused on the first point of every interval $\mathbb{Z}_{(\ell-1)(m+1)+1}^{\ell(m+1)}$:

$$x((\ell-1)(m+1)+1) = x((\ell-1)(m+1)+1-0) + J_\ell,$$

$\ell \geq 1$, $\ell = \left\lfloor \frac{k+m}{m+1} \right\rfloor$, $k \in \mathbb{Z}_0^\infty$, $J_\ell \in \mathbb{R}^n$ (resuts are given in Theorems 3.2, 3.4), or on the p-th point of such intervals:

$$x((\ell-1)(m+1)+p) = x((\ell-1)(m+1)+p-0) + J_\ell,$$

$p \in \{1, 2, 3, \ldots, m+1\}$, $\ell \geq 1$, $\ell = \left\lfloor \frac{k+m}{m+1} \right\rfloor$, $k \in \mathbb{Z}_0^\infty$, $J_\ell \in \mathbb{R}^n$ (resuts are given in Theorems 3.5, 3.6), or, in a general case, impulses are added to each point k (Theorems 3.7, 3.8).

In Section 3.4 further generalization is given of the results from Section 3.3. The problem considered has a more general form

$$x(k+1) = Ax(k) + Bx(k-m) + f(k), \quad k \in \mathbb{Z}_0^\infty, \tag{3.3}$$

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^0, \tag{3.4}$$

$$x(k+1) = Cx(k+1-0) + J_{k+1}, \quad k \in \mathbb{Z}_0^\infty \tag{3.5}$$

and the main assumption is that matrices A and B commute ($AB = BA$) and $ACB = BCA$ (Theorem 3.10).

3.1 Problem (3.1), (3.2) with Impulses at Points Having the Form $(\ell-1)(m+1)+1$

We will consider problem (3.1), (3.2) with impulses $J_\ell \in \mathbb{R}^n$ added to x at points having the form $(\ell-1)(m+1)+1$ where the index $\ell \geq 1$ is defined as $\ell = \left\lfloor \frac{k+m}{m+1} \right\rfloor$

for every $k \in \mathbb{Z}_0^\infty$, i.e., we set

$$x((\ell-1)(m+1)+1) = x((\ell-1)(m+1)+1-0) + J_\ell \tag{3.6}$$

and investigate the solution of the problem (3.1), (3.2), (3.6).

Before we deal with the solution of (3.1), (3.2), (3.6), we will consider an example to get a better understanding of the problem. The example illustrates the influence of impulses on the solution and serves as a motivation for the related results.

Example 3.1. We consider a particular case of (3.1) if $n = 1$, $B = b \neq 0$, $b \in \mathbb{R}$, $m = 3$ and $f(k) = 0$, $k \in \mathbb{Z}_0^\infty$ together with an initial problem (3.2) for $\varphi(k) = 1$, $k \in \mathbb{Z}_{-3}^0$ and with impulses $J_i \in \mathbb{R}$ at points $(\ell-1)(m+1)+1 = 4(\ell-1)+1$ where $\ell \geq 1$, $\ell = \left\lfloor \frac{k+m}{m+1} \right\rfloor = \left\lfloor \frac{k+3}{4} \right\rfloor$:

$$\Delta x(k) = bx(k-3), \tag{3.7}$$

$$x(-3) = x(-2) = x(-1) = x(0) = 1, \tag{3.8}$$

$$x(4(\ell-1)+1) = x(4(\ell-1)+1-0) + J_\ell. \tag{3.9}$$

Rewriting equation (3.7) as

$$x(k+1) = x(k) + bx(k-3)$$

and solving it by the method of steps, we conclude that the solution of the problem can be written in the form:

$$x(k) = b^0\binom{k+3}{0} \qquad \text{if} \quad k \in \mathbb{Z}_{-3}^0,$$

$$x(k) = b^0\binom{k+3}{0} + b^1\binom{k}{1} + J_1\, b^0\binom{k-1}{0} \qquad \text{if} \quad k \in \mathbb{Z}_1^4,$$

$$x(k) = b^0\binom{k+3}{0} + b^1\binom{k}{1} + b^2\binom{k-3}{2} + J_1\left[b^0\binom{k-1}{0} + b^1\binom{k-4}{1}\right] + J_2\, b^0\binom{k-5}{0} \qquad \text{if} \quad k \in \mathbb{Z}_5^8,$$

$$x(k) = b^0\binom{k+3}{0} + b^1\binom{k}{1} + b^2\binom{k-3}{2} + b^3\binom{k-6}{3} + J_1\left[b^0\binom{k-1}{0} + b^1\binom{k-4}{1} + b^2\binom{k-7}{2}\right] + J_2\left[b^0\binom{k-5}{0} + b^1\binom{k-8}{1}\right] + J_3\, b^0\binom{k-9}{0} \qquad \text{if} \quad k \in \mathbb{Z}_9^{12},$$

$$
\begin{aligned}
x(k) = {} & b^0\binom{k+3}{0} + b^1\binom{k}{1} + b^2\binom{k-3}{2} + b^3\binom{k-6}{3} + b^4\binom{k-9}{4} \\
& + J_1\left[b^0\binom{k-1}{0} + b^1\binom{k-4}{1} + b^2\binom{k-7}{2} + b^3\binom{k-10}{3}\right] \\
& + J_2\left[b^0\binom{k-5}{0} + b^1\binom{k-8}{1} + b^2\binom{k-11}{2}\right] \\
& + J_3\left[b^0\binom{k-9}{0} + b^1\binom{k-12}{1}\right] + J_4\, b^0\binom{k-13}{0} \qquad \text{if} \quad k \in \mathbb{Z}_{13}^{16},
\end{aligned}
$$

$$\vdots$$

$$
\begin{aligned}
x(k) = {} & b^0\binom{k+3}{0} + b^1\binom{k}{1} + b^2\binom{k-3}{2} + b^3\binom{k-6}{3} + \cdots + b^\ell\binom{k-3(\ell-1)}{\ell} \\
& + J_1\left[b^0\binom{k-1}{0} + b^1\binom{k-4}{1} + b^2\binom{k-7}{2} + b^3\binom{k-10}{3}\right. \\
& \qquad \left. + \cdots + b^{\ell-1}\binom{k-4-3(\ell-2)}{\ell-1}\right] \\
& + J_2\left[b^0\binom{k-5}{0} + b^1\binom{k-8}{1} + b^2\binom{k-11}{2} + b^3\binom{k-14}{3}\right. \\
& \qquad \left. + \cdots + b^{\ell-2}\binom{k-8-3(\ell-3)}{\ell-2}\right] \\
& + J_3\left[b^0\binom{k-9}{0} + b^1\binom{k-12}{1} + b^2\binom{k-15}{2} + b^3\binom{k-18}{3}\right. \\
& \qquad \left. + \cdots + b^{\ell-3}\binom{k-12-3(\ell-4)}{\ell-3}\right] \\
& + J_4\left[b^0\binom{k-13}{0} + b^1\binom{k-16}{1} + b^2\binom{k-19}{2} + b^3\binom{k-22}{3}\right. \\
& \qquad \left. + \cdots + b^{\ell-4}\binom{k-16-3(\ell-5)}{\ell-4}\right] \\
& + \cdots \\
& + J_i\left[b^0\binom{k-4i+3}{0} + b^1\binom{k-4i}{1} + b^2\binom{k-4i-3}{2} + b^3\binom{k-4i-6}{3}\right. \\
& \qquad \left. + \cdots + b^{\ell-i}\binom{k-4i-3(\ell-(i+1))}{\ell-i}\right] \\
& + \cdots \\
& + J_{\ell-2}\left[b^0\binom{k-4(\ell-2)+3}{0} + b^1\binom{k-4(\ell-2)}{1} + b^2\binom{k-4(\ell-2)-3}{2}\right]
\end{aligned}
$$

$$+ J_{\ell-1}\left[b^0\binom{k-4(\ell-1)+3}{0}+b^1\binom{k-4(\ell-1)}{1}\right]$$
$$+ J_\ell\, b^0\binom{k-4\ell+3}{0} \qquad \text{if} \quad \ell=\left\lfloor\frac{k+3}{4}\right\rfloor \quad \text{and} \quad k\in\mathbb{Z}_{-m}^{\infty}.$$

The solution of (3.7) – (3.9) can be shortened to

$$x(k)=\sum_{j=0}^{\ell} b^j\binom{k-3(j-1)}{j}+\sum_{q=1}^{\ell} J_q\sum_{j=0}^{\ell-q} b^j\binom{k-4q-3(j-1)}{j}, \tag{3.10}$$

for $\ell=\left\lfloor\frac{k+m}{m+1}\right\rfloor$ and $k\in\mathbb{Z}_{-m}^{\infty}$.

3.1.1 Homogeneous Initial Problem

Theorem 3.2. *Let B be a constant $n\times n$ matrix, m be a fixed integer, $J_\ell\in\mathbb{R}^n$, $\ell\geq 1$, $\ell=\left\lfloor\frac{k+m}{m+1}\right\rfloor$. Then, the solution of the initial Cauchy problem with impulses*

$$\Delta x(k)=Bx(k-m), \quad k\in\mathbb{Z}_0^{\infty}, \tag{3.11}$$
$$x(k)=\varphi(k), \quad k\in\mathbb{Z}_{-m}^{0}, \tag{3.12}$$
$$x((\ell-1)(m+1)+1)=x((\ell-1)(m+1)+1-0)+J_\ell \tag{3.13}$$

can be expressed in the form:

$$x(k)=\mathrm{e}_m^{Bk}\varphi(-m)+\sum_{j=-m+1}^{0}\mathrm{e}_m^{B(k-m-j)}\Delta\varphi(j-1)+\sum_{q=1}^{\ell}J_q\mathrm{e}_m^{B(k-q(m+1))} \tag{3.14}$$

where $k\in\mathbb{Z}_{-m}^{\infty}$.

Proof. We prove (3.14) for $k\geq 0$ at first. We substitute (3.14) into the left-hand side $\mathcal{L}$ of the equation (3.11):

$$\begin{aligned}
\mathcal{L}&=\Delta x(k)\\
&=\Delta\left[\mathrm{e}_m^{Bk}\varphi(-m)+\sum_{j=-m+1}^{0}\mathrm{e}_m^{B(k-m-j)}\Delta\varphi(j-1)+\sum_{q=1}^{\ell}J_q\mathrm{e}_m^{B(k-q(m+1))}\right]\\
&=\Delta\mathrm{e}_m^{Bk}\varphi(-m)+\sum_{j=-m+1}^{0}\Delta\mathrm{e}_m^{B(k-m-j)}\Delta\varphi(j-1)+\sum_{q=1}^{\ell}J_q\Delta\mathrm{e}_m^{B(k-q(m+1))}\\
&=[\text{according to the Theorem 2.3}]\\
&=B\mathrm{e}_m^{B(k-m)}\varphi(-m)+\sum_{j=-m+1}^{0}B\mathrm{e}_m^{B(k-2m-j)}\Delta\varphi(j-1)+\sum_{q=1}^{\ell}J_qB\mathrm{e}_m^{B(k-m-q(m+1))}\\
&=B\left[\mathrm{e}_m^{B(k-m)}\varphi(-m)+\sum_{j=-m+1}^{0}\mathrm{e}_m^{B(k-2m-j)}\Delta\varphi(j-1)+\sum_{q=1}^{\ell}J_q\mathrm{e}_m^{B(k-m-q(m+1))}\right].
\end{aligned}$$

Now we substitute (3.14) into the right-hand side $\mathcal{R}$ of the equation (3.11):

$$\begin{aligned}\mathcal{R} &= Bx(k-m)\\ &= B\left[\mathrm{e}_m^{B(k-m)}\varphi(-m) + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B(k-2m-j)}\Delta\varphi(j-1) + \sum_{q=1}^{\ell} J_q \mathrm{e}_m^{B(k-m-q(m+1))}\right].\end{aligned}$$

Since $\mathcal{L} = \mathcal{R}$, (3.14) is a solution of (3.11).

Now we have to prove that (3.13) holds, too. We substitute (3.14) into the left-hand side $\mathcal{L}^*$ and right-hand side $\mathcal{R}^*$ of (3.13):

$$\begin{aligned}\mathcal{L}^* &= x((\ell-1)(m+1)+1)\\ &= \mathrm{e}_m^{B((\ell-1)(m+1)+1)}\varphi(-m) + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B((\ell-1)(m+1)+1-m-j)}\Delta\varphi(j-1)\\ &\quad + \sum_{q=1}^{\ell} J_q \mathrm{e}_m^{B((\ell-1)(m+1)+1-q(m+1))},\end{aligned}$$

$$\begin{aligned}\mathcal{R}^* &= x((\ell-1)(m+1)+1-0) + J_\ell\\ &= \mathrm{e}_m^{B((\ell-1)(m+1)+1)}\varphi(-m) + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B((\ell-1)(m+1)+1-m-j)}\Delta\varphi(j-1)\\ &\quad + \sum_{q=1}^{\ell-1} J_q \mathrm{e}_m^{B((\ell-1)(m+1)+1-q(m+1))} + J_\ell.\end{aligned}$$

Since

$$\begin{aligned}\sum_{q=1}^{\ell} J_q \mathrm{e}_m^{B((\ell-1)(m+1)+1-q(m+1))} &= \sum_{q=1}^{\ell-1} J_q \mathrm{e}_m^{B((\ell-1)(m+1)+1-q(m+1))}\\ &\quad + J_\ell \mathrm{e}_m^{B((\ell-1)(m+1)+1-\ell(m+1))}\\ &= \sum_{q=1}^{\ell-1} J_q \mathrm{e}_m^{B((\ell-1)(m+1)+1-q(m+1))} + J_\ell \mathrm{e}_m^{B(-m)}\\ &= \left[\text{according to the Definition 2.1, } \mathrm{e}_m^{B(-m)} = I\right]\\ &= \sum_{q=1}^{\ell-1} J_q \mathrm{e}_m^{B((\ell-1)(m+1)+1-q(m+1))} + J_\ell\end{aligned}$$

it is obvious that $\mathcal{L}^* = \mathcal{R}^*$ and (3.13) holds.

We prove that (3.12) holds as well. For $k \in \mathbb{Z}_{-m}^{0}$ is $\ell = 0$ and

$$\sum_{q=1}^{\ell} J_q \mathrm{e}_m^{B(k-q(m+1))} = \sum_{q=1}^{0} J_q \mathrm{e}_m^{B(k-q(m+1))} = 0.$$

Therefore, (3.14) becomes (2.12). This formula was proved in the proof of Theorem 2.4. □

Example 3.3. We consider the problem (3.11) – (3.13) where $n = 1$, $B = b$, $m = 3$, $\varphi(k) = 1$ for $k \in \mathbb{Z}^0_{-3}$. Then (3.14) takes the form:

$$x(k) = \mathrm{e}_3^{bk}\varphi(-3) + \sum_{j=-3+1}^{0} \mathrm{e}_3^{b(k-3-j)}\Delta\varphi(j-1) + \sum_{q=1}^{\ell} J_q \mathrm{e}_3^{b(k-q(3+1))}. \tag{3.15}$$

This problem was also solved in Example 3.1. We will show that the representations (3.15) and (3.10) are equivalent.

We write all the addition terms of (3.15):

$$\begin{aligned}
\mathrm{e}_3^{bk}\varphi(-3) &= \text{[according to Definition 2.1]}\\
&= 1 + b\binom{k}{1} + b^2\binom{k-3}{2} + b^3\binom{k-6}{3} + \dots + b^\ell\binom{k-3(\ell-1)}{\ell}\\
&= b^0\binom{k+3}{0} + b^1\binom{k}{1} + b^2\binom{k-3}{2} + b^3\binom{k-6}{3} + \dots + b^\ell\binom{k-3(\ell-1)}{\ell}\\
&= \sum_{j=0}^{\ell} b^j\binom{k-3(j-1)}{j},
\end{aligned}$$

$$\begin{aligned}
\sum_{j=-3+1}^{0} \mathrm{e}_3^{b(k-3-j)}\Delta\varphi(j-1) &= \sum_{j=-2}^{0} \mathrm{e}_3^{b(k-3-j)}\Delta\varphi(j-1)\\
&= \mathrm{e}_3^{b(k-1)}\Delta\varphi(-3) + \mathrm{e}_3^{b(k-2)}\Delta\varphi(-2) + \mathrm{e}_3^{b(k-3)}\Delta\varphi(-1)\\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\sum_{q=1}^{\ell} J_q \mathrm{e}_3^{b(k-q(3+1))} &= \sum_{q=1}^{\ell} J_q \mathrm{e}_3^{b(k-4q)}\\
&= \text{[according to Definition 2.1]}\\
&= \sum_{q=1}^{\ell} J_q \sum_{j=0}^{\ell} b^j \binom{k-4q-3(j-1)}{j}\\
&= \sum_{q=1}^{\ell} J_q \Bigg[b^0\binom{k-4q+3}{0} + b^1\binom{k-4q}{1} + b^2\binom{k-4q-3}{2}\\
&\quad + \dots + b^{\ell-q}\binom{k-4q-3(\ell-q-1)}{\ell-q}\\
&\quad + b^{\ell-q+1}\binom{k-4q-3(\ell-q)}{\ell-q+1}\\
&\quad + \dots + b^\ell\binom{k-4q-3(\ell-1)}{\ell}\Bigg].
\end{aligned}$$

We prove that the binomical coeficients

$$\binom{k-4q-3(\ell-q)}{\ell-q+1}, \dots, \binom{k-4q-3(\ell-1)}{\ell}$$

are equal to zero. These coeficients can be writen as

$$\binom{k-4q-3(\ell-q+p-1)}{\ell-q+p} \quad \text{where } p=1,2,\ldots,q.$$

Since $k\in\mathbb{Z}^{4\ell}_{4(\ell-1)+1}$, we can write $k=4(\ell-1)+h$ where $h=1,2,3,4$. Thus,

$$\begin{aligned}\binom{k-4q-3(\ell-q+p-1)}{\ell-q+p} &= \binom{4(\ell-1)+h-4q-3(\ell-q+p-1)}{\ell-q+p}\\ &= \binom{\ell-q-3p-1+h}{\ell-q+p}\\ &= [\text{because } -3p-1+h<p]\\ &= 0.\end{aligned}$$

Hence,

$$\sum_{q=1}^{\ell} J_q e_3^{b(k-4q)} = \sum_{q=1}^{\ell} J_q \sum_{j=0}^{\ell-q} b^j \binom{k-4q-3(j-1)}{j}.$$

Then, the solution (3.15) is in the form

$$x(k)=\sum_{j=0}^{\ell} b^j\binom{k-3(j-1)}{j} + \sum_{q=1}^{\ell} J_q \sum_{j=0}^{\ell-q} b^j\binom{k-4q-3(j-1)}{j},$$

which is the solution of Example 3.1.

3.1.2 Nonhomogeneous Initial Problem

Theorem 3.4. *Let B be a constant $n\times n$ matrix, m be a fixed integer, $J_\ell\in\mathbb{R}^n$, $\ell\geq 1$, $\ell=\left\lfloor\frac{k+m}{m+1}\right\rfloor$. Then, the solution of the initial Cauchy problem with impulses*

$$\Delta x(k) = Bx(k-m)+f(k), \quad k\in\mathbb{Z}_0^\infty, \tag{3.16}$$

$$x(k)=\varphi(k), \quad k\in\mathbb{Z}^0_{-m}, \tag{3.17}$$

$$x((\ell-1)(m+1)+1) = x((\ell-1)(m+1)+1-0)+J_\ell, \tag{3.18}$$

can be expressed in the form:

$$\begin{aligned}x(k) &= e_m^{Bk}\varphi(-m) + \sum_{j=-m+1}^{0} e_m^{B(k-m-j)}\Delta\varphi(j-1)\\ &\quad + \sum_{j=1}^{k} e_m^{B(k-m-j)} f(j-1) + \sum_{q=1}^{\ell} J_q e_m^{B(k-q(m+1))}\end{aligned}$$

where $k\in\mathbb{Z}^\infty_{-m}$.

Proof. The problem (3.16) – (3.18) with $f(k) = 0$, $k \in \mathbb{Z}_{-m}^{\infty}$ is considered in Theorem 3.2. So, it is suifficient to prove only the following:

$$\Delta \sum_{j=1}^{k} \mathrm{e}_m^{B(k-m-j)} f(j-1) = B \sum_{j=1}^{k-m} \mathrm{e}_m^{B(k-2m-j)} f(j-1) + f(k).$$

This formula was proved in the proof of Theorem 2.5 and so we refer to it for a detailed explanation. □

3.2 Problem (3.1), (3.2) with Impulses at Points Having the Form $(\ell - 1)(m + 1) + p$

We will consider the problem (3.1), (3.2) with impulses $J_\ell \in \mathbb{R}^n$ added to x at points having the form $(\ell-1)(m+1)+p$ where p is a fixed integer from the set $\{1, 2, 3, \dots, m+1\}$ and the index $\ell \geq 1$ is defined as $\ell = \left\lfloor \frac{k+m}{m+1} \right\rfloor$ for every $k \in \mathbb{Z}_0^{\infty}$, i.e., we set

$$x((\ell-1)(m+1)+p) = x((\ell-1)(m+1)+p-0) + J_\ell \tag{3.19}$$

and investigate the solutions of both the homogeneous and nonhomogeneos problems (3.1), (3.2), (3.19).

The following theorems generalize the results from Section 3.1 where a particular case of this problem (if $p = 1$) was solved.

Theorem 3.5. *Let B be a constant $n \times n$ matrix, m be a fixed integer, p be a fixed integer from the set $\{1, 2, 3, \dots, m+1\}$, $J_\ell \in \mathbb{R}^n$, $\ell \geq 1$, $\ell = \left\lfloor \frac{k+m}{m+1} \right\rfloor$. Then, the solution of the homogeneous initial Cauchy problem with impulses*

$$\Delta x(k) = Bx(k-m), \quad k \in \mathbb{Z}_0^{\infty}, \tag{3.20}$$

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^{0}, \tag{3.21}$$

$$x((\ell-1)(m+1)+p) = x((\ell-1)(m+1)+p-0) + J_\ell, \tag{3.22}$$

can be expressed in the form:

$$x(k) = \mathrm{e}_m^{Bk}\varphi(-m) + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B(k-m-j)} \Delta\varphi(j-1) + \sum_{q=1}^{\ell} J_q \mathrm{e}_m^{B(k-(p-1)-q(m+1))} \tag{3.23}$$

where $k \in \mathbb{Z}_{-m}^{\infty}$.

Proof. First we prove (3.23) for $k \geq 0$. We substitute (3.23) into the left-hand side $\mathcal{L}$ of the equation (3.20):

$$\begin{aligned}
\mathcal{L} &= \Delta x(k) \\
&= \Delta \left[\mathrm{e}_m^{Bk}\varphi(-m) + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B(k-m-j)}\Delta\varphi(j-1) + \sum_{q=1}^{\ell} J_q \mathrm{e}_m^{B(k-(p-1)-q(m+1))} \right] \\
&= \Delta\mathrm{e}_m^{Bk}\varphi(-m) + \sum_{j=-m+1}^{0} \Delta\mathrm{e}_m^{B(k-m-j)}\Delta\varphi(j-1) + \sum_{q=1}^{\ell} J_q \Delta\mathrm{e}_m^{B(k-(p-1)-q(m+1))} \\
&= [\text{according to Theorem 2.3}] \\
&= B\mathrm{e}_m^{B(k-m)}\varphi(-m) + \sum_{j=-m+1}^{0} B\mathrm{e}_m^{B(k-2m-j)}\Delta\varphi(j-1) + \sum_{q=1}^{\ell} J_q B\mathrm{e}_m^{B(k-m-(p-1)-q(m+1))} \\
&= B \left[\mathrm{e}_m^{B(k-m)}\varphi(-m) + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B(k-2m-j)}\Delta\varphi(j-1) + \sum_{q=1}^{\ell} J_q \mathrm{e}_m^{B(k-m-(p-1)-q(m+1))} \right].
\end{aligned}$$

Now we substitute (3.23) into the right-hand side $\mathcal{R}$ of equation (3.20):

$$\begin{aligned}
\mathcal{R} &= Bx(k-m) \\
&= B \left[\mathrm{e}_m^{B(k-m)}\varphi(-m) + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B(k-2m-j)}\Delta\varphi(j-1) + \sum_{q=1}^{\ell} J_q \mathrm{e}_m^{B(k-m-(p-1)-q(m+1))} \right].
\end{aligned}$$

Since $\mathcal{L} = \mathcal{R}$, (3.23) is a solution of (3.20).

Now we have to prove that (3.22) holds, too. We substitute (3.23) into the left-hand side $\mathcal{L}^*$ and right-hand side $\mathcal{R}^*$ of (3.22):

$$\begin{aligned}
\mathcal{L}^* &= x((\ell-1)(m+1)+p) \\
&= \mathrm{e}_m^{B((\ell-1)(m+1)+p)}\varphi(-m) + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B((\ell-1)(m+1)+p-m-j)}\Delta\varphi(j-1) \\
&\quad + \sum_{q=1}^{\ell} J_q \mathrm{e}_m^{B((\ell-1)(m+1)+p-(p-1)-q(m+1))}, \\
\mathcal{R}^* &= x((\ell-1)(m+1)+p-0) + J_\ell \\
&= \mathrm{e}_m^{B((\ell-1)(m+1)+p)}\varphi(-m) + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B((\ell-1)(m+1)+p-m-j)}\Delta\varphi(j-1) \\
&\quad + \sum_{q=1}^{\ell-1} J_q \mathrm{e}_m^{B((\ell-1)(m+1)+p-(p-1)-q(m+1))} + J_\ell.
\end{aligned}$$

Since

$$\sum_{q=1}^{\ell} J_q \mathrm{e}_m^{B((\ell-1)(m+1)+p-(p-1)-q(m+1))} = \sum_{q=1}^{\ell-1} J_q \mathrm{e}_m^{B((\ell-1)(m+1)+p-(p-1)-q(m+1))}$$

$$
\begin{aligned}
&+ J_\ell \mathrm{e}_m^{B((\ell-1)(m+1)+p-(p-1)-\ell(m+1))}\\
&= \sum_{q=1}^{\ell-1} J_q \mathrm{e}_m^{B((\ell-1)(m+1)+p-(p-1)-q(m+1))} + J_\ell \mathrm{e}_m^{B(-m)}\\
&= \left[\text{according to the Definition 2.1 , } \mathrm{e}_m^{B(-m)} = I\right]\\
&= \sum_{q=1}^{\ell-1} J_q \mathrm{e}_m^{B((\ell-1)(m+1)+p-(p-1)-q(m+1))} + J_\ell,
\end{aligned}
$$

it is obvious that $\mathcal{L}^* = \mathcal{R}^*$ and (3.22) holds.

We prove that (3.21) holds as well. For $k \in \mathbb{Z}_{-m}^0$, we have $\ell = 0$ and

$$
\sum_{q=1}^{\ell} J_q \mathrm{e}_m^{B(k-(p-1)-q(m+1))} = \sum_{q=1}^{0} J_q \mathrm{e}_m^{B(k-(p-1)-q(m+1))} = 0.
$$

Therefore, (3.23) chenges to (2.12). This formula was proved in the proof of Theorem 2.4. □

Theorem 3.6. *Let B be a constant $n \times n$ matrix, m be a fixed integer, p be a fixed integer from the set $\{1, 2, 3, \ldots, m+1\}$, $J_\ell \in \mathbb{R}^n$, $\ell \geq 1$, $\ell = \left\lfloor \frac{k+m}{m+1} \right\rfloor$. Then, the solution of the nonhomogeneous initial Cauchy problem with impulses*

$$
\Delta x(k) = Bx(k-m) + f(k), \quad k \in \mathbb{Z}_0^\infty, \tag{3.24}
$$

$$
x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^0, \tag{3.25}
$$

$$
x((\ell-1)(m+1)+p) = x((\ell-1)(m+1)+p-0) + J_\ell, \tag{3.26}
$$

can be expressed in the form:

$$
\begin{aligned}
x(k) = \mathrm{e}_m^{Bk}\varphi(-m) &+ \sum_{j=-m+1}^{0} \mathrm{e}_m^{B(k-m-j)} \Delta\varphi(j-1)\\
&+ \sum_{j=1}^{k} \mathrm{e}_m^{B(k-m-j)} f(j-1) + \sum_{q=1}^{\ell} J_q \mathrm{e}_m^{B(k-(p-1)-q(m+1))}
\end{aligned}
$$

where $k \in \mathbb{Z}_{-m}^\infty$.

Proof. The problem (3.24) – (3.26) with $f(k) = 0$, $k \in \mathbb{Z}_{-m}^\infty$ is considered in Theorem 3.5. So, it is sufficient to prove only the following:

$$
\Delta \sum_{j=1}^{k} \mathrm{e}_m^{B(k-m-j)} f(j-1) = B \sum_{j=1}^{k-m} \mathrm{e}_m^{B(k-2m-j)} f(j-1) + f(k).
$$

This formula was proved in the proof of Theorem 2.5 and, therefore, we refer to it for a detailed explanation. □

3.3 Problem (3.1), (3.2) with Impulses at Each Point

We will consider the problem (3.1), (3.2) with impulses $J_{k+1} \in \mathbb{R}^n$ added to x in every $k \in \mathbb{Z}_1^\infty$, i.e., we set

$$x(k+1) = x(k+1-0) + J_{k+1} \tag{3.27}$$

and investigate the solutions of both the homogeneous and nonhomogeneous problems (3.1), (3.2), (3.27).

Theorem 3.7. *Let B be a constant $n \times n$ matrix, m be a fixed integer, $J_i \in \mathbb{R}^n$. Then, the solution of the homogeneous initial Cauchy problem with impulses*

$$\Delta x(k) = Bx(k-m), \quad k \in \mathbb{Z}_0^\infty, \tag{3.28}$$

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^0, \tag{3.29}$$

$$x(k+1) = x(k+1-0) + J_{k+1}, \quad k \in \mathbb{Z}_0^\infty \tag{3.30}$$

can be expressed in the form:

$$x(k) = \mathrm{e}_m^{Bk}\varphi(-m) + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B(k-m-j)}\Delta\varphi(j-1) + \sum_{i=1}^{k} J_i \mathrm{e}_m^{B(k-(i+m))} \tag{3.31}$$

where $k \in \mathbb{Z}_{-m}^\infty$.

Proof. The problem (3.28) – (3.30) is equivalent to the problem

$$\Delta x(k) = Bx(k-m) + J_{k+1}, \quad k \in \mathbb{Z}_0^\infty, \tag{3.32}$$

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^0 \tag{3.33}$$

We prove (3.31) for $k \geq 0$ first. We substitute (3.31) into the left-hand side $\mathcal{L}$ of the equation (3.32):

$$\begin{aligned}
\mathcal{L} &= \Delta x(k) \\
&= \Delta \left[\mathrm{e}_m^{Bk}\varphi(-m) + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B(k-m-j)}\Delta\varphi(j-1) + \sum_{i=1}^{k} J_i \mathrm{e}_m^{B(k-(i+m))} \right] \\
&= \Delta \mathrm{e}_m^{Bk}\varphi(-m) + \sum_{j=-m+1}^{0} \Delta \mathrm{e}_m^{B(k-m-j)}\Delta\varphi(j-1) + \Delta \sum_{i=1}^{k} J_i \mathrm{e}_m^{B(k-(i+m))} \\
&= [\text{according to } 1.9] \\
&= \Delta \mathrm{e}_m^{Bk}\varphi(-m) + \sum_{j=-m+1}^{0} \Delta \mathrm{e}_m^{B(k-m-j)}\Delta\varphi(j-1) + \sum_{i=1}^{k} J_i \Delta \mathrm{e}_m^{B(k-(i+m))} + J_{k+1}\mathrm{e}_m^{B(-m)}
\end{aligned}$$

$$= [\text{according to Theorem 2.3 and Definition 2.1}]$$

$$= B\mathrm{e}_m^{B(k-m)}\varphi(-m) + \sum_{j=-m+1}^{0} B\mathrm{e}_m^{B(k-2m-j)}\Delta\varphi(j-1) + \sum_{i=1}^{k} J_i B\mathrm{e}_m^{B(k-2m-i)} + J_{k+1}$$

$$= B\Bigg[\mathrm{e}_m^{B(k-m)}\varphi(-m) + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B(k-2m-j)}\Delta\varphi(j-1) + \sum_{i=1}^{k-m} J_i\mathrm{e}_m^{B(k-2m-i)}$$

$$+ \sum_{i=k-m+1}^{k} J_i\mathrm{e}_m^{B(k-2m-i)}\Bigg] + J_{k+1}$$

$$= \Big[\text{according to Definition 2.1 is } \mathrm{e}_m^{B(k-2m-i)} = \Theta \text{ for } i \in \mathbb{Z}_{k-m+1}^{k}\Big]$$

$$= B\left[\mathrm{e}_m^{B(k-m)}\varphi(-m) + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B(k-2m-j)}\Delta\varphi(j-1) + \sum_{i=1}^{k-m} J_i\mathrm{e}_m^{B(k-2m-i)}\right] + J_{k+1}.$$

Now we substitute (3.31) into the right-hand side $\mathcal{R}$ of the equation (3.32):

$$\mathcal{R} = Bx(k-m) + J_{k+1}$$

$$= B\left[\mathrm{e}_m^{B(k-m)}\varphi(-m) + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B(k-2m-j)}\Delta\varphi(j-1) + \sum_{i=1}^{k-m} J_i\mathrm{e}_m^{B(k-2m-i)}\right] + J_{k+1}.$$

Since $\mathcal{L} = \mathcal{R}$, (3.31) is a solution of (3.32).

For $k \in \mathbb{Z}_{-m}^{0}$, no impulses exist. The problem (3.32), (3.33) becomes the problem (2.10), (2.11) solved in Theorem 2.4. □

Theorem 3.8. *Let B be a constant $n \times n$ matrix, m be a fixed integer, $J_i \in \mathbb{R}^n$. Then, the solution of the nonhomogeneous initial Cauchy problem with impulses*

$$\Delta x(k) = Bx(k-m) + f(k), \quad k \in \mathbb{Z}_0^{\infty}, \tag{3.34}$$

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^{0}, \tag{3.35}$$

$$x(k+1) = x(k+1-0) + J_{k+1}, \quad k \in \mathbb{Z}_0^{\infty} \tag{3.36}$$

can be expressed in the form:

$$x(k) = \mathrm{e}_m^{Bk}\varphi(-m) + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B(k-m-j)}\Delta\varphi(j-1)$$

$$+ \sum_{j=1}^{k} \mathrm{e}_m^{B(k-m-j)} f(j-1) + \sum_{i=1}^{k} J_i\mathrm{e}_m^{B(k-(i+m))}$$

where $k \in \mathbb{Z}_{-m}^{\infty}$.

Proof. The problem (3.34) – (3.36) with $f(k) = 0$, $k \in \mathbb{Z}_{-m}^{\infty}$ is considered in Theorem 3.7. So, it is sufficient to prove only the following:

$$\Delta \sum_{j=1}^{k} \mathrm{e}_m^{B(k-m-j)} f(j-1) = B\sum_{j=1}^{k-m} \mathrm{e}_m^{B(k-2m-j)} f(j-1) + f(k).$$

As mentioned above, this formula was proved in the proof of Theorem 2.5 and, therefore, we refer to it for a detailed explanation. □

3.4 Problem (3.3) – (3.5) with Impulses at Each Point

Consider the discrete system

$$x(k+1) = Ax(k) + Bx(k-m) + f(k), \quad k \in \mathbb{Z}_0^\infty, \tag{3.37}$$

where $m \geq 1$ is a fixed integer, $A = (a_{ij})$ and $B = (b_{ij})$ are regular constant $n \times n$ matrices with the commutative property

$$AB = BA,$$

$f\colon \mathbb{Z}_0^\infty \to \mathbb{R}^n$ and $x\colon \mathbb{Z}_{-m}^\infty \to \mathbb{R}^n$.

Together with equation (3.37), we will consider the initial Cauchy problem

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^0, \tag{3.38}$$

with a given $\varphi\colon \mathbb{Z}_{-m}^0 \to \mathbb{R}^n$.

We will consider problem (3.37), (3.38) together with the condition

$$x(k+1) = Cx(k+1-0) + J_{k+1} \tag{3.39}$$

where $k \in \mathbb{Z}_0^\infty$, $C = (c_{ij})$ is a regular constant $n \times n$ matrix and $J_i \in \mathbb{R}^n$ are impulses.

We assume that, for matrices A, B and C, equality

$$ACB = BCA$$

holds.

Remark 3.9. For $A = C = I$, the problem (3.37) – (3.39) turns into (3.34) – (3.36) considered in Theorem 3.8.

Theorem 3.10. *Let A, B, C be constant $n \times n$ matrices with the proterty $ACB = BCA$, m be a fixed integer and $J_i \in \mathbb{R}^n$. Then, the solution of the initial Cauchy problem with impulses*

$$x(k+1) = Ax(k) + Bx(k-m) + f(k), \quad k \in \mathbb{Z}_0^\infty, \tag{3.40}$$

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^0, \tag{3.41}$$

$$x(k+1) = Cx(k+1-0) + J_{k+1}, \quad k \in \mathbb{Z}_0^\infty \tag{3.42}$$

can be expressed in the form:

$$\begin{aligned} x(k) = {} & X_0(k)(CA)^{-m}\varphi(-m) \\ & + (CA)^m \sum_{j=-m+1}^{0} X_0(k-m-j)\left[\varphi(j) - (CA)\varphi(j-1)\right] \\ & + (CA)^m \sum_{i=1}^{k} X_0(k-m-i)\left[Cf(i-1) + J_i\right] \end{aligned} \tag{3.43}$$

where $k \in \mathbb{Z}_1^\infty$, $X_0(k) = (CA)^k \mathrm{e}_m^{B_1 k}$, $B_1 = (CA)^{-1}CB(CA)^{-m}$.

Proof. The problem (3.40) – (3.42) is equivalent to the problem without impulses

$$x(k+1) = CAx(k) + CBx(k-m) + Cf(k) + J_{k+1}, \quad k \in \mathbb{Z}_0^{\infty}, \tag{3.44}$$
$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^{0}$$

We prove (3.43) for $k \geq 0$ first. We substitute (3.43) into the left-hand side $\mathcal{L}$ and right-hand side $\mathcal{R}$ of the equation (3.44):

$$\begin{aligned}
\mathcal{L} &= x(k+1) \\
&= X_0(k+1)(CA)^{-m}\varphi(-m) \\
&\qquad + (CA)^m \sum_{j=-m+1}^{0} X_0(k+1-m-j)\left[\varphi(j) - (CA)\varphi(j-1)\right] \\
&\qquad + (CA)^m \sum_{i=1}^{k+1} X_0(k+1-m-i)\left[Cf(i-1) + J_i\right]
\end{aligned}$$

$$\begin{aligned}
\mathcal{R} &= CAx(k) + CBx(k-m) + Cf(k) + I_{k+1} \\
&= CAX_0(k)(CA)^{-m}\varphi(-m) \\
&\qquad + CA(CA)^m \sum_{j=-m+1}^{0} X_0(k-m-j)\left[\varphi(j) - (CA)\varphi(j-1)\right] \\
&\qquad + CA(CA)^m \sum_{i=1}^{k} X_0(k-m-i)\left[Cf(i-1) + J_i\right] \\
&\qquad + CBX_0(k-m)(CA)^{-m}\varphi(-m) \\
&\qquad + CB(CA)^m \sum_{j=-m+1}^{0} X_0(k-2m-j)\left[\varphi(j) - (CA)\varphi(j-1)\right] \\
&\qquad + CB(CA)^m \sum_{i=1}^{k-m} X_0(k-2m-i)\left[Cf(i-1) + J_i\right] + Cf(k) + J_{k+1}
\end{aligned}$$

We will divide the proof into the following three parts:
1) We prove:

$$\begin{aligned}
X_0(k+1)(CA)^{-m}\varphi(-m) &= CAX_0(k)(CA)^{-m}\varphi(-m) \\
&\qquad + CBX_0(k-m)(CA)^{-m}\varphi(-m)
\end{aligned} \tag{3.45}$$

$$\begin{aligned}
\mathcal{L}_1 &= X_0(k+1) \\
&= (CA)^{k+1}\, \mathrm{e}_m^{B_1(k+1)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}_1 &= CAX_0(k) + CBX_0(k-m) \\
&= CA(CA)^k\, \mathrm{e}_m^{B_1 k} + CB(CA)^{k-m}\mathrm{e}_m^{B_1(k-m)}
\end{aligned}$$

$$
\begin{aligned}
&= (CA)^{k+1}\, \mathrm{e}_m^{B_1 k} + CB(CA)^{k-m} B_1^{-1} B_1 \mathrm{e}_m^{B_1(k-m)} \\
&= (CA)^{k+1}\, \mathrm{e}_m^{B_1 k} + CB(CA)^{k-m}(CA)^m (CB)^{-1} CA\Delta \mathrm{e}_m^{B_1 k} \\
&= (CA)^{k+1}\, \mathrm{e}_m^{B_1 k} + (CA)^{k+1} \left(\mathrm{e}_m^{B_1(k+1)} - \mathrm{e}_m^{B_1 k}\right) \\
&= (CA)^{k+1}\, \mathrm{e}_m^{B_1(k+1)}
\end{aligned}
$$

$\mathcal{L}_1 = \mathcal{R}_1$ and (3.45) holds.

2) We prove:

$$
\begin{aligned}
&(CA)^m \sum_{j=-m+1}^{0} X_0(k+1-m-j)\left[\varphi(j) - (CA)\varphi(j-1)\right] \\
&\qquad = CA(CA)^m \sum_{j=-m+1}^{0} X_0(k-m-j)\left[\varphi(j) - (CA)\varphi(j-1)\right] + \\
&\qquad\quad + CB(CA)^m \sum_{j=-m+1}^{0} X_0(k-2m-j)\left[\varphi(j) - (CA)\varphi(j-1)\right],
\end{aligned} \tag{3.46}
$$

e.i.,

$$
\begin{aligned}
&\sum_{j=-m+1}^{0} X_0(k+1-m-j)\left[\varphi(j) - (CA)\varphi(j-1)\right] = \\
&\qquad = \sum_{j=-m+1}^{0} \left[CAX_0(k-m-j) + CBX_0(k-2m-j)\right]\left[\varphi(j) - (CA)\varphi(j-1)\right]
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_2 &= X_0(k+1-m-j) \\
&= (CA)^{k+1-m-j} \mathrm{e}_m^{B_1(k+1-m-j)} \\
\mathcal{R}_2 &= CAX_0(k-m-j) + CBX_0(k-2m-j) \\
&= CA(CA)^{k-m-j} \mathrm{e}_m^{B_1(k-m-j)} + CB(CA)^{k-2m-j} \mathrm{e}_m^{B_1(k-2m-j)} \\
&= (CA)^{k+1-m-j} \mathrm{e}_m^{B_1(k-m-j)} + CB(CA)^{k-2m-j} B_1^{-1} B_1 \mathrm{e}_m^{B_1(k-2m-j)} \\
&= (CA)^{k+1-m-j} \mathrm{e}_m^{B_1(k-m-j)} + CB(CA)^{k-2m-j}(CA)^m (CB)^{-1} CA\Delta \mathrm{e}_m^{B_1(k-m-j)} \\
&= (CA)^{k+1-m-j} \mathrm{e}_m^{B_1(k-m-j)} + (CA)^{k+1-m-j} \left(\mathrm{e}_m^{B_1(k+1-m-j)} - \mathrm{e}_m^{B_1(k-m-j)}\right) \\
&= (CA)^{k+1-m-j} \mathrm{e}_m^{B_1(k+1-m-j)}
\end{aligned}
$$

$\mathcal{L}_2 = \mathcal{R}_2$ and (3.46) holds.

3) We prove:

$$\begin{aligned}(CA)^m \sum_{i=1}^{k+1} X_0(k+1-m-i)\,[Cf(i-1)+J_i] \\ = CA(CA)^m \sum_{i=1}^{k} X_0(k-m-i)\,[Cf(i-1)+J_i] + \\ + CB(CA)^m \sum_{i=1}^{k-m} X_0(k-2m-i)\,[Cf(i-1)+J_i] \\ + Cf(k) + J_{k+1}\end{aligned} \tag{3.47}$$

Since

$$\begin{aligned}(CA)^m \sum_{i=1}^{k+1} X_0(k+1-m-i)\,[Cf(i-1)+J_i] \\ &= (CA)^m \sum_{i=1}^{k} X_0(k+1-m-i)\,[Cf(i-1)+J_i] \\ &\quad + (CA)^m X_0(-m)\,[Cf(k)+J_{k+1}] \\ &= (CA)^m \sum_{i=1}^{k} X_0(k+1-m-i)\,[Cf(i-1)+J_i] \\ &\quad + (CA)^m (CA)^{-m}\, \mathrm{e}_m^{B_1(-m)}\,[Cf(k)+J_{k+1}] \\ &= (CA)^m \sum_{i=1}^{k} X_0(k+1-m-i)\,[Cf(i-1)+J_i] + Cf(k) + J_{k+1},\end{aligned}$$

it is enough to prove

$$\begin{aligned}\sum_{i=1}^{k} X_0(k+1-m-i)\,[Cf(i-1)+J_i] \\ = CA \sum_{i=1}^{k} X_0(k-m-i)\,[Cf(i-1)+J_i] \\ + CB \sum_{i=1}^{k-m} X_0(k-2m-i)\,[Cf(i-1)+J_i]\end{aligned}$$

$$\begin{aligned}\mathcal{L}_3 &= \sum_{i=1}^{k} X_0(k+1-m-i)\,[Cf(i-1)+J_i] \\ &= \sum_{i=1}^{k} (CA)^{k+1-m-i} \mathrm{e}_m^{B_1(k+1-m-i)}\,[Cf(i-1)+J_i]\end{aligned}$$

$$\begin{aligned}\mathcal{R}_3 &= CA \sum_{i=1}^{k} X_0(k-m-i)\,[Cf(i-1)+J_i] \\ &\quad + CB \sum_{i=1}^{k-m} X_0(k-2m-i)\,[Cf(i-1)+J_i]\end{aligned}$$

$$
\begin{aligned}
&= CA\sum_{j=1}^{k} X_0(k-m-i)\,[Cf(i-1)+J_i] \\
&\quad + CB\left(\sum_{i=1}^{k} X_0(k-2m-i)\,[Cf(i-1)+J_i]\right. \\
&\qquad\quad \left. - \sum_{i=k-m+1}^{k} X_0(k-2m-i)\,[Cf(i-1)+J_i]\right) \\
&= CA\sum_{i=1}^{k}(CA)^{k-m-i}\,\mathrm{e}_m^{B_1(k-m-i)}\,[Cf(i-1)+J_i] \\
&\quad + CB\sum_{i=1}^{k}(CA)^{k-2m-i}\,\mathrm{e}_m^{B_1(k-2m-i)}\,[Cf(i-1)+J_i] \\
&\quad - CB\sum_{i=k-m+1}^{k}(CA)^{k-2m-i}\,\mathrm{e}_m^{B_1(k-2m-i)}\,[Cf(i-1)+J_i] \\
&= \left[\text{for } i\in\mathbb{Z}_{k-m+1}^{k} \text{ is } \mathrm{e}_m^{B_1(k-2m-i)}=\Theta\right] \\
&= \sum_{i=1}^{k}\left[CA(CA)^{k-m-i}\,\mathrm{e}_m^{B_1(k-m-i)} + CB(CA)^{k-2m-i}\,\mathrm{e}_m^{B_1(k-2m-i)}\right][Cf(i-1)+J_i] \\
&= \sum_{i=1}^{k}\left[(CA)^{k+1-m-i}\,\mathrm{e}_m^{B_1(k-m-i)} + CB(CA)^{k-2m-i}B_1^{-1}B_1\,\mathrm{e}_m^{B_1(k-2m-i)}\right] \\
&\quad \cdot [Cf(i-1)+J_i] \\
&= \sum_{i=1}^{k}\left[(CA)^{k+1-m-i}\,\mathrm{e}_m^{B_1(k-m-i)} + CB(CA)^{k-2m-i}(CA)^{m}(CB)^{-1}CA\Delta\,\mathrm{e}_m^{B_1(k-m-i)}\right] \\
&\quad \cdot [Cf(i-1)+J_i] \\
&= \sum_{i=1}^{k}\left[(CA)^{k+1-m-i}\,\mathrm{e}_m^{B_1(k-m-i)} + (CA)^{k+1-m-i}\left(\mathrm{e}_m^{B_1(k+1-m-i)} - \mathrm{e}_m^{B_1(k-m-i)}\right)\right] \\
&\quad \cdot [Cf(i-1)+J_i] \\
&= \sum_{i=1}^{k}(CA)^{k+1-m-i}\,\mathrm{e}_m^{B_1(k+1-m-i)}\,[Cf(i-1)+J_i]
\end{aligned}
$$

$\mathcal{L}_3 = \mathcal{R}_3$ and (3.47) holds.

Since (3.45), (3.46), (3.47) holds, it is obvious that $\mathcal{L} = \mathcal{R}$ and (3.43) is a solution of (3.44). □

Remark 3.11. For $k \in \mathbb{Z}_{-m}^{0}$, the problem (3.40) – (3.42) is equivalent to the problem (2.27), (2.28) and, by Theorem 2.8, formula (3.43) is equal to formula (2.29),i.e.,

$$
x(k) = X_0(k)A^{-m}\varphi(-m) + A^{m}\sum_{j=-m+1}^{0} X_0(k-m-j)\,[\varphi(j) - A\varphi(j-1)]
$$

where $X_0(k) = A^k\,\mathrm{e}_m^{B_1 k}$, $B_1 = A^{-1}BA^{-m}$.

4 DISCRETE MATRIX DELAYED EXPONENTIALS FOR TWO DELAYS

In this chapter we define two discrete matrix delayed exponentials for two delays and we prove their main properties. Results of this chapter were already published in [12, 16].

4.1 Discrete Matrix Delayed Exponential e_{mn}^{BCk}

We define a discrete matrix function e_{mn}^{BCk} called the discrete matrix delayed exponential for two delays $m, n \in \mathbb{N}$, $m \neq n$ and for two $r \times r$ commuting constant matrices B, C.

Definition 4.1. *Let B, C be constant $r \times r$ matrices with the property $BC = CB$ and let $m, n \in \mathbb{N}$, $m \neq n$ be fixed integers. We define a discrete $r \times r$ matrix function e_{mn}^{BCk} called the discrete matrix delayed exponential for two delays m, n and for two $r \times r$ constant matrices B, C as follows:*

$$\mathrm{e}_{mn}^{BCk} := \begin{cases} \Theta \quad \text{if} \quad k \in \mathbb{Z}_{-\infty}^{-\max\{m,n\}-1}, \\ I \quad \text{if} \quad k \in \mathbb{Z}_{-\max\{m,n\}}^{0}, \\ I + (B+C) \sum\limits_{i=0,j=0}^{p_{(k)}-1, q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-mi-nj}{i+j+1} \quad \text{if} \quad k \in \mathbb{Z}_1^{\infty} \end{cases}$$

where

$$p_{(k)} := \left\lfloor \frac{k+m}{m+1} \right\rfloor, \quad q_{(k)} := \left\lfloor \frac{k+n}{n+1} \right\rfloor \tag{4.1}$$

and Θ is the $r \times r$ null matrix, I is the $r \times r$ unit matrix.

Remark 4.2. Because

$$\binom{i+j}{i} = \frac{(i+j)!}{i!j!} = \binom{i+j}{j},$$

we can replace $\binom{i+j}{i}$ by $\binom{i+j}{j}$ in Definition 4.1 and in the computations below if deemed necessary.

Let us show an example illustrating this special exponential function.

Example 4.3. For $k \in \mathbb{Z}_0^{12}$ we will construct the matrix e_{mn}^{BCk} if $m = 2$ and $n = 3$. Computing particular matrices generating $\mathrm{e}_{2,3}^{BCk}$ for $k \in \mathbb{Z}_0^{12}$, we get

$\mathrm{e}_{2,3}^{BC0} = I,$

$\mathrm{e}_{2,3}^{BC1} = I + B + C,$

$$
\begin{aligned}
\mathrm{e}_{2,3}^{BC2} &= I + (B + C)2,\\
\mathrm{e}_{2,3}^{BC3} &= I + (B + C)3,\\
\mathrm{e}_{2,3}^{BC4} &= I + (B + C)(4 + B),\\
\mathrm{e}_{2,3}^{BC5} &= I + (B + C)(5 + 3B + C),\\
\mathrm{e}_{2,3}^{BC6} &= I + (B + C)(6 + 6B + 3C),\\
\mathrm{e}_{2,3}^{BC7} &= I + (B + C)(7 + 10B + 6C + B^2),\\
\mathrm{e}_{2,3}^{BC8} &= I + (B + C)(8 + 15B + 10C + 4B^2 + 2BC),\\
\mathrm{e}_{2,3}^{BC9} &= I + (B + C)(9 + 21B + 15C + 10B^2 + 8BC + C^2),\\
\mathrm{e}_{2,3}^{BC10} &= I + (B + C)(10 + 28B + 21C + 20B^2 + 20BC + 4C^2 + B^3),\\
\mathrm{e}_{2,3}^{BC11} &= I + (B + C)(11 + 36B + 28C + 35B^2 + 40BC + 10C^2 + 5B^3 + 3B^2C),\\
\mathrm{e}_{2,3}^{BC12} &= I + (B + C)(12 + 45B + 36C + 56B^2 + 70BC + 20C^2 + 15B^3 + 15B^2C\\
&\qquad + 3BC^2).
\end{aligned}
$$

The main property of e_{mn}^{BCk} is given by the following theorem.

Theorem 4.4. *Let B, C be constant $r \times r$ matrices with the property $BC = CB$ and let $m, n \in \mathbb{N}$, $m \neq n$ be fixed integers. Then,*

$$
\Delta \mathrm{e}_{mn}^{BCk} = B\mathrm{e}_{mn}^{BC(k-m)} + C\mathrm{e}_{mn}^{BC(k-n)} \tag{4.2}
$$

holds for $k \in \mathbb{Z}_0^{\infty}$.

Proof. Let $k \geq 1$. From (1.4) and (4.1), we can see easily that, for an integer k satisfying

$$
(p_{(k)} - 1)(m + 1) + 1 \leq k \leq p_{(k)}(m + 1) \wedge (q_{(k)} - 1)(n + 1) + 1 \leq k \leq q_{(k)}(n + 1),
$$

the relation

$$
\Delta \mathrm{e}_{mn}^{BCk} = \Delta \left[I + (B + C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k - mi - nj}{i + j + 1} \right]
$$

holds in accordance with Definition 4.1 of e_{mn}^{BCk}. Since $\Delta I = \Theta$, we have

$$
\Delta \mathrm{e}_{mn}^{BCk} = \Delta \left[(B + C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k - mi - nj}{i + j + 1} \right]. \tag{4.3}
$$

By the definition of the forward difference, i.e.,

$$
\Delta \mathrm{e}_{mn}^{BCk} = \mathrm{e}_{mn}^{BC(k+1)} - \mathrm{e}_{mn}^{BCk}, \tag{4.4}
$$

we conclude that it is reasonable to divide the proof into four parts with respect to the value of integer k. In case one, k is such that

$$
(p_{(k)} - 1)(m + 1) + 1 \leq k < p_{(k)}(m + 1) \wedge (q_{(k)} - 1)(n + 1) + 1 \leq k < q_{(k)}(n + 1),
$$

in case two

$$k = p_{(k)}(m+1) \wedge (q_{(k)}-1)(n+1)+1 \le k < q_{(k)}(n+1),$$

in case three

$$(p_{(k)}-1)(m+1)+1 \le k < p_{(k)}(m+1) \wedge k = q_{(k)}(n+1),$$

and in case four

$$k = p_{(k)}(m+1) \wedge k = q_{(k)}(n+1).$$

We see that the above cases cover all the possible relations between k, $p_{(k)}$ and $q_{(k)}$.

I. $(p_{(k)}-1)(m+1)+1 \le k < p_{(k)}(m+1)$ $\wedge(q_{(k)}-1)(n+1)+1 \le k < q_{(k)}(n+1)$

From (1.4) and (4.1), we get

$$p_{(k-m)} = \left\lfloor \frac{k-m+m}{m+1} \right\rfloor \le \frac{k}{m+1} < p_{(k)},$$
$$p_{(k-m)} = \left\lfloor \frac{k-m+m}{m+1} \right\rfloor > \frac{k}{m+1} - 1 = \frac{k-m-1}{m+1} > p_{(k)} - 2.$$

Therefore, $p_{(k-m)} = p_{(k)} - 1$ and, by Definition 4.1,

$$\mathrm{e}_{mn}^{BC(k-m)} = I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k-m)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-mi-nj}{i+j+1}. \tag{4.5}$$

Similarly, omitting details, we get (using (1.4), and (4.1)) $q_{(k-n)} = q_{(k)} - 1$ and

$$\mathrm{e}_{mn}^{BC(k-n)} = I + (B+C) \sum_{i=0,j=0}^{p_{(k-n)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-n-mi-nj}{i+j+1}. \tag{4.6}$$

Let $q_{(k-m)} \ge 1$. We show that

$$\binom{k-m-mi-nj}{i+j+1} = 0 \quad \text{if} \quad i \ge 0, \quad j \ge q_{(k-m)}. \tag{4.7}$$

In accordance with (1.4),

$$q_{(k-m)} = \left\lfloor \frac{k-m+n}{n+1} \right\rfloor > \frac{k-m+n}{n+1} - 1 = \frac{k-m-1}{n+1}$$

or

$$k-m < (n+1)q_{(k-m)} + 1 \le (m+1)i + (n+1)j + 1 \quad \text{if} \quad i \ge 0, \quad j \ge q_{(k-m)}.$$

From the last inequality, we get

$$k - m - mi - nj < i + j + 1 \quad \text{if} \quad i \geq 0, \quad j \geq q_{(k-m)}$$

and (4.7) holds by (1.5). For that reason and since $q_{(k-m)} \leq q_{(k)}$, we can replace $q_{(k-m)}$ by $q_{(k)}$ in (4.5). Thus, we have

$$\mathrm{e}_{mn}^{BC(k-m)} = I + (B + C) \sum_{i=0,j=0}^{p_{(k)}-2, q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k - m(i+1) - nj}{i+j+1}. \tag{4.8}$$

It is easy to see that, by (1.8), formula (4.8) can be used instead of (4.5) if $q_{(k-m)} < 1$ also.

Let $p_{(k-n)} \geq 1$. Similarly, we can show that

$$\binom{k - n - mi - nj}{i+j+1} = 0 \quad \text{if} \quad i \geq p_{(k-n)}, \quad j \geq 0$$

and, since $p_{(k-n)} \leq p_{(k)}$, we can replace $p_{(k-n)}$ by $p_{(k)}$ in (4.6). Thus, we have

$$\mathrm{e}_{mn}^{BC(k-n)} = I + (B + C) \sum_{i=0,j=0}^{p_{(k)}-1, q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k - mi - n(j+1)}{i+j+1}. \tag{4.9}$$

It is easy to see that, by (1.8), formula (4.9) can be used instead of (4.6) if $p_{(k-n)} < 1$, too.

By (1.4), we also conclude that

$$p_{(k+1)} = p_{(k)}, \quad q_{(k+1)} = q_{(k)} \tag{4.10}$$

because

$$p_{(k+1)} = \left\lfloor \frac{k+1+m}{m+1} \right\rfloor \leq \frac{k}{m+1} + 1 < p_{(k)} + 1,$$

and

$$p_{(k+1)} = \left\lfloor \frac{k+1+m}{m+1} \right\rfloor > \frac{k+1+m}{m+1} - 1 = \frac{k}{m+1} \geq p_{(k)} - 1 + \frac{1}{m+1}.$$

The second formula can be proved similarly.

Now we are able to prove that

$$\begin{aligned} \Delta \mathrm{e}_{mn}^{BCk} &= B\mathrm{e}_{mn}^{BC(k-m)} + C\mathrm{e}_{mn}^{BC(k-n)} \\ &= B \left[I + (B + C) \sum_{i=0,j=0}^{p_{(k)}-2, q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k - m(i+1) - nj}{i+j+1} \right] \\ &\quad + C \left[I + (B + C) \sum_{i=0,j=0}^{p_{(k)}-1, q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k - mi - n(j+1)}{i+j+1} \right]. \end{aligned} \tag{4.11}$$

With the aid of (4.3), (4.4), (1.6), and (4.10), we get

$$\begin{aligned}
\Delta \mathrm{e}_{mn}^{BCk} &= \mathrm{e}_{mn}^{BC(k+1)} - \mathrm{e}_{mn}^{BCk} \\
&= I + (B+C) \sum_{i=0,j=0}^{p_{(k+1)}-1, q_{(k+1)}-1} B^i C^j \binom{i+j}{i} \binom{k+1-mi-nj}{i+j+1} \\
&\quad - I - (B+C) \sum_{i=0,j=0}^{p_{(k)}-1, q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-mi-nj}{i+j+1} \\
&= I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-1, q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k+1-mi-nj}{i+j+1} \\
&\quad - I - (B+C) \sum_{i=0,j=0}^{p_{(k)}-1, q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-mi-nj}{i+j+1} \\
&= (B+C) \sum_{i=0,j=0}^{p_{(k)}-1, q_{(k)}-1} B^i C^j \binom{i+j}{i} \left[\binom{k+1-mi-nj}{i+j+1} - \binom{k-mi-nj}{i+j+1} \right] \\
&= (B+C) \sum_{i=0,j=0}^{p_{(k)}-1, q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-mi-nj}{i+j} \\
&= (B+C) \left[I + \sum_{i=1}^{p_{(k)}-1} B^i C^0 \binom{i}{i} \binom{k-mi}{i} + \sum_{j=1}^{q_{(k)}-1} B^0 C^j \binom{j}{0} \binom{k-nj}{j} \right. \\
&\quad \left. + \sum_{i=1,j=1}^{p_{(k)}-1, q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-mi-nj}{i+j} \right].
\end{aligned}$$

By (1.7), we have

$$\begin{aligned}
\Delta \mathrm{e}_{mn}^{BCk} &= (B+C) \left[I + \sum_{i=1}^{p_{(k)}-1} B^i C^0 \binom{i-1}{i-1} \binom{k-mi}{i} + \sum_{j=1}^{q_{(k)}-1} B^0 C^j \binom{j-1}{0} \binom{k-nj}{j} \right. \\
&\quad + \sum_{i=1,j=1}^{p_{(k)}-1, q_{(k)}-1} B^i C^j \binom{i+j-1}{i-1} \binom{k-mi-nj}{i+j} \\
&\quad \left. + \sum_{i=1,j=1}^{p_{(k)}-1, q_{(k)}-1} B^i C^j \binom{i+j-1}{i} \binom{k-mi-nj}{i+j} \right] \\
&= (B+C) \left[I + \sum_{i=1,j=0}^{p_{(k)}-1, q_{(k)}-1} B^i C^j \binom{i+j-1}{i-1} \binom{k-mi-nj}{i+j} \right. \\
&\quad \left. + \sum_{i=0,j=1}^{p_{(k)}-1, q_{(k)}-1} B^i C^j \binom{i+j-1}{i} \binom{k-mi-nj}{i+j} \right].
\end{aligned}$$

Now, in the first sum, we replace the summation index i by $i+1$ and, in the second sum, we replace the summation index j by $j+1$. Then,

$$\Delta \mathrm{e}_{mn}^{BCk} = (B+C) \left[I + \sum_{i=0,j=0}^{p_{(k)}-2, q_{(k)}-1} B^{i+1} C^j \binom{i+j}{i} \binom{k-m(i+1)-nj}{i+j+1} \right.$$

$$+\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^iC^{j+1}\binom{i+j}{i}\binom{k-mi-n(j+1)}{i+j+1}\Bigg]$$

$$= B + B(B+C)\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-m(i+1)-nj}{i+j+1}$$

$$+ C + C(B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^iC^j\binom{i+j}{i}\binom{k-mi-n(j+1)}{i+j+1}$$

$$= B\left[I + (B+C)\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-m(i+1)-nj}{i+j+1}\right]$$

$$+ C\left[I + C(B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^iC^j\binom{i+j}{i}\binom{k-mi-n(j+1)}{i+j+1}\right]$$

$$= B\mathrm{e}_{mn}^{BC(k-m)} + C\mathrm{e}_{mn}^{BC(k-n)}.$$

By (4.8) and (4.9), we conclude that formula (4.11) is valid.

II. $k = p_{(k)}(m+1) \wedge (q_{(k)}-1)(n+1)+1 \le k < q_{(k)}(n+1)$

In this case,

$$p_{(k-m)} = \left\lfloor \frac{k-m+m}{m+1} \right\rfloor = \left\lfloor \frac{k}{m+1} \right\rfloor = p_{(k)},$$

$$p_{(k+1)} = \left\lfloor \frac{k+1+m}{m+1} \right\rfloor \le \frac{k+1+m}{m+1} = \frac{k}{m+1} + 1 = p_{(k)} + 1,$$

$$p_{(k+1)} = \left\lfloor \frac{k+1+m}{m+1} \right\rfloor > \frac{k+1+m}{m+1} - 1 = \frac{k}{m+1} = p_{(k)}$$

and $p_{(k+1)} = p_{(k)} + 1$. In addition to this (see relevant computations performed in the case I.), we have $q_{(k-n)} = q_{(k)} - 1$ and $q_{(k+1)} = q_{(k)}$. Then

$$\mathrm{e}_{mn}^{BCk} = I + (B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-mi-nj}{i+j+1},$$

$$\mathrm{e}_{mn}^{BC(k+1)} = I + (B+C)\sum_{i=0,j=0}^{p_{(k)},q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k+1-mi-nj}{i+j+1}.$$

and

$$\mathrm{e}_{mn}^{BC(k-m)} = I + (B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k-m)}-1} B^iC^j\binom{i+j}{i}\binom{k-m-mi-nj}{i+j+1}, \quad (4.12)$$

$$\mathrm{e}_{mn}^{BC(k-n)} = I + (B+C)\sum_{i=0,j=0}^{p_{(k-n)}-1,q_{(k)}-2} B^iC^j\binom{i+j}{i}\binom{k-n-mi-nj}{i+j+1}. \quad (4.13)$$

As with the computations performed in the previous part of the proof, we get

$$\binom{k-m-mi-nj}{i+j+1} = 0 \quad \text{if} \quad i \geq 0, \quad j \geq q_{(k-m)}$$

and

$$\binom{k-n-mi-nj}{i+j+1} = 0 \quad \text{if} \quad i \geq p_{(k-n)}, \quad j \geq 0.$$

So we can substitute $q_{(k)}$ for $q_{(k-m)}$ in (4.12) and $p_{(k)}$ for $p_{(k-n)}$ in (4.13). Accordingly, we have

$$\mathrm{e}_{mn}^{BC(k-m)} = I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m(i+1)-nj}{i+j+1}, \tag{4.14}$$

$$\mathrm{e}_{mn}^{BC(k-n)} = I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-mi-n(j+1)}{i+j+1}. \tag{4.15}$$

It is easy to see that, by (1.8), formula (4.14) can also be used instead of (4.12) if $q_{(k-m)} < 1$ and formula (4.15) can also be used instead of (4.13) if $p_{(k-n)} < 1$.

We have to prove

$$\begin{aligned}
\Delta\mathrm{e}_{mn}^{BCk} &= B\mathrm{e}_{mn}^{BC(k-m)} + C\mathrm{e}_{mn}^{BC(k-n)} \\
&= B\left[I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m(i+1)-nj}{i+j+1}\right] \\
&\quad + C\left[I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-mi-n(j+1)}{i+j+1}\right].
\end{aligned} \tag{4.16}$$

Therefore,

$$\begin{aligned}
\Delta\mathrm{e}_{mn}^{BCk} &= \mathrm{e}_{mn}^{BC(k+1)} - \mathrm{e}_{mn}^{BCk} \\
&= I + (B+C) \sum_{i=0,j=0}^{p_{(k)},q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k+1-mi-nj}{i+j+1} \\
&\quad - I - (B+C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-mi-nj}{i+j+1} \\
&= (B+C)\left[\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \left[\binom{k+1-mi-nj}{i+j+1} - \binom{k-mi-nj}{i+j+1}\right]\right. \\
&\quad \left. + \sum_{j=0}^{q_{(k)}-1} B^{p_{(k)}} C^j \binom{p_{(k)}+j}{p_{(k)}} \binom{k+1-mp_{(k)}-nj}{p_{(k)}+j+1}\right].
\end{aligned}$$

With the aid of the equation $k = p_{(k)}(m+1)$, we get

$$\binom{k+1-mp_{(k)}-nj}{p_{(k)}+j+1} = \binom{p_{(k)}+1-nj}{p_{(k)}+1+j} = 0 \quad \text{if} \quad j > 0$$

and, by (1.6), we have

$$\begin{aligned}
\Delta \mathrm{e}_{mn}^{BCk} &= (B+C)\left[\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-mi-nj}{i+j} + B^{p_{(k)}}\right]\\
&= (B+C)\left[I + \sum_{i=1}^{p_{(k)}-1} B^iC^0\binom{i}{i}\binom{k-mi}{i} + \sum_{j=1}^{q_{(k)}-1} B^0C^j\binom{j}{0}\binom{k-nj}{j}\right.\\
&\quad\left. + \sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-mi-nj}{i+j} + B^{p_{(k)}}\right].
\end{aligned}$$

By (1.7), we have

$$\begin{aligned}
\Delta \mathrm{e}_{mn}^{BCk} &= (B+C)\left[I + \sum_{i=1}^{p_{(k)}-1} B^iC^0\binom{i-1}{i-1}\binom{k-mi}{i} + \sum_{j=1}^{q_{(k)}-1} B^0C^j\binom{j-1}{0}\binom{k-nj}{j}\right.\\
&\quad + \sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j-1}{i-1}\binom{k-mi-nj}{i+j}\\
&\quad\left. + \sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j-1}{i}\binom{k-mi-nj}{i+j} + B^{p_{(k)}}\right]\\
&= (B+C)\left[I + \sum_{i=1,j=0}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j-1}{i-1}\binom{k-mi-nj}{i+j}\right.\\
&\quad\left. + \sum_{i=0,j=1}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j-1}{i}\binom{k-mi-nj}{i+j} + B^{p_{(k)}}\right].
\end{aligned}$$

Now we replace in the first sum the summation index i by $i+1$ and, in the second sum, we replace the summation index j by $j+1$. Then,

$$\begin{aligned}
\Delta \mathrm{e}_{mn}^{BCk} &= (B+C)\left[I + \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^{i+1}C^j\binom{i+j}{i}\binom{k-m(i+1)-nj}{i+j+1}\right.\\
&\quad\left. + \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^iC^{j+1}\binom{i+j}{i}\binom{k-mi-n(j+1)}{i+j+1} + B^{p_{(k)}}\right]\\
&= B + B(B+C)\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-m(i+1)-nj}{i+j+1} + B^{p_{(k)}}(B+C)\\
&\quad + C + C(B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^iC^j\binom{i+j}{i}\binom{k-mi-n(j+1)}{i+j+1}\\
&= B\left[I + (B+C)\left(\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-m(i+1)-nj}{i+j+1} + B^{p_{(k)}-1}\right)\right]\\
&\quad + C\left[I + (B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^iC^j\binom{i+j}{i}\binom{k-mi-n(j+1)}{i+j+1}\right].
\end{aligned}$$

For $k = p_{(k)}(m+1)$, we have

$$B^{p_{(k)}-1} = \sum_{j=0}^{q_{(k)}-1} B^{p_{(k)}-1} C^j \binom{p_{(k)}-1+j}{p_{(k)}-1}\binom{k-m(p_{(k)}-1+1)-nj}{p_{(k)}-1+j+1}$$

where

$$\binom{k-m(p_{(k)}-1+1)-nj}{p_{(k)}-1+j+1} = \binom{k-mp_{(k)}-nj}{p_{(k)}+j} = 0 \quad \text{if} \quad j > 0.$$

Thus,

$$\begin{aligned}
\Delta \mathrm{e}_{mn}^{BCk} &= B\left[I + (B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i}\binom{k-m(i+1)-nj}{i+j+1}\right] \\
&\quad + C\left[I + (B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i}\binom{k-mi-n(j+1)}{i+j+1}\right] \\
&= B\mathrm{e}_{mn}^{BC(k-m)} + C\mathrm{e}_{mn}^{BC(k-n)}
\end{aligned}$$

and formula (4.16) is proved.

III. $\boldsymbol{(p_{(k)}-1)(m+1)+1 \leq k < p_{(k)}(m+1) \wedge k = q_{(k)}(n+1)}$

In this case, we have (see the relevant computations in the cases I. and II.)

$$p_{(k-m)} = p_{(k)} - 1, \quad p_{(k+1)} = p_{(k)}$$

and

$$q_{(k-n)} = q_{(k)}, \quad q_{(k+1)} = q_{(k)} + 1.$$

Then,

$$\begin{aligned}
\mathrm{e}_{mn}^{BCk} &= I + (B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i}\binom{k-mi-nj}{i+j+1}, \\
\mathrm{e}_{mn}^{BC(k+1)} &= I + (B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}} B^i C^j \binom{i+j}{i}\binom{k+1-mi-nj}{i+j+1}
\end{aligned}$$

and

$$\mathrm{e}_{mn}^{BC(k-m)} = I + (B+C)\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k-m)}-1} B^i C^j \binom{i+j}{i}\binom{k-m-mi-nj}{i+j+1}, \tag{4.17}$$

$$\mathrm{e}_{mn}^{BC(k-n)} = I + (B+C)\sum_{i=0,j=0}^{p_{(k-n)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i}\binom{k-n-mi-nj}{i+j+1}. \tag{4.18}$$

Like with the computations performed in case I., we can get

$$\binom{k-m-mi-nj}{i+j+1} = 0 \quad \text{if} \quad i \geq 0, \quad j \geq q_{(k-m)}$$

and

$$\binom{k-n-mi-nj}{i+j+1} = 0 \quad \text{if} \quad i \geq p_{(k-n)}, \quad j \geq 0.$$

So we can substitute $q_{(k)}$ for $q_{(k-m)}$ in (4.17) and $p_{(k)}$ for $p_{(k-n)}$ in (4.18). Thus, we have

$$\mathrm{e}_{mn}^{BC(k-m)} = I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m(i+1)-nj}{i+j+1}, \tag{4.19}$$

$$\mathrm{e}_{mn}^{BC(k-n)} = I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-mi-n(j+1)}{i+j+1}. \tag{4.20}$$

It is easy to see that, by (1.8), formula (4.19) can also be used instead of (4.17) if $q_{(k-m)} < 1$ and formula (4.20) can also be used instead of (4.18) if $p_{(k-n)} < 1$.

Now we have to prove

$$\begin{aligned} \Delta \mathrm{e}_{mn}^{BCk} &= B\mathrm{e}_{mn}^{BC(k-m)} + C\mathrm{e}_{mn}^{BC(k-n)} \\ &= B\left[I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m(i+1)-nj}{i+j+1}\right] \\ &\quad + C\left[I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-mi-n(j+1)}{i+j+1}\right]. \end{aligned} \tag{4.21}$$

Considering the difference by its definition, we get

$$\begin{aligned} \Delta \mathrm{e}_{mn}^{BCk} &= \mathrm{e}_{mn}^{BC(k+1)} - \mathrm{e}_{mn}^{BCk} \\ &= I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}} B^i C^j \binom{i+j}{i} \binom{k+1-mi-nj}{i+j+1} \\ &\quad - I - (B+C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-mi-nj}{i+j+1} \\ &= (B+C) \left[\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \left[\binom{k+1-mi-nj}{i+j+1} - \binom{k-mi-nj}{i+j+1}\right] \right. \\ &\quad \left. + \sum_{i=0}^{p_{(k)}-1} B^i C^{q_{(k)}} \binom{i+q_{(k)}}{i} \binom{k+1-mi-nq_{(k)}}{i+q_{(k)}+1}\right]. \end{aligned}$$

With the aid of the equation $k = q_{(k)}(n+1)$, we get

$$\binom{k+1-mi-nq_{(k)}}{i+q_{(k)}+1} = \binom{q_{(k)}+1-mi}{q_{(k)}+1+i} = 0 \quad \text{if} \quad i > 0,$$

and

$$\Delta \mathrm{e}_{mn}^{BCk} = (B+C) \left[\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-mi-nj}{i+j} + C^{q_{(k)}}\right]$$

$$= (B+C)\left[I + \sum_{i=1}^{p_{(k)}-1} B^i C^0 \binom{i}{i}\binom{k-mi}{i} + \sum_{j=1}^{q_{(k)}-1} B^0 C^j \binom{j}{0}\binom{k-nj}{j} + \sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i}\binom{k-mi-nj}{i+j} + C^{q_{(k)}}\right].$$

By (1.7), we have

$$\begin{aligned}
\Delta \mathrm{e}_{mn}^{BCk} &= (B+C)\left[I + \sum_{i=1}^{p_{(k)}-1} B^i C^0 \binom{i-1}{i-1}\binom{k-mi}{i} + \sum_{j=1}^{q_{(k)}-1} B^0 C^j \binom{j-1}{0}\binom{k-nj}{j}\right. \\
&\quad + \sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j-1}{i-1}\binom{k-mi-nj}{i+j} \\
&\quad \left. + \sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j-1}{i}\binom{k-mi-nj}{i+j} + C^{q_{(k)}}\right] \\
&= (B+C)\left[I + \sum_{i=1,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j-1}{i-1}\binom{k-mi-nj}{i+j}\right. \\
&\quad \left. + \sum_{i=0,j=1}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j-1}{i}\binom{k-mi-nj}{i+j} + C^{q_{(k)}}\right].
\end{aligned}$$

Now we replace in the first sum the summation index i by $i+1$ and, in the second sum, we replace the summation index j by $j+1$. Then,

$$\begin{aligned}
\Delta \mathrm{e}_{mn}^{BCk} &= (B+C)\left[I + \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^{i+1} C^j \binom{i+j}{i}\binom{k-m(i+1)-nj}{i+j+1}\right. \\
&\quad \left. + \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^i C^{j+1} \binom{i+j}{i}\binom{k-mi-n(j+1)}{i+j+1} + C^{q_{(k)}}\right] \\
&= B + B(B+C)\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^i C^j \binom{i+j}{i}\binom{k-m(i+1)-nj}{i+j+1} \\
&\quad + C + C(B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i}\binom{k-mi-n(j+1)}{i+j+1} \\
&\quad + C^{q_{(k)}}(B+C) \\
&= B\left[I + (B+C)\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^i C^j \binom{i+j}{i}\binom{k-m(i+1)-nj}{i+j+1}\right] \\
&\quad + C\left[I + (B+C)\left(\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i}\binom{k-mi-n(j+1)}{i+j+1}\right.\right. \\
&\qquad\qquad \left.\left. + C^{q_{(k)}-1}\right)\right].
\end{aligned}$$

For $k = q_{(k)}(n+1)$, we have

$$C^{q_{(k)}-1} = \sum_{i=0}^{p_{(k)}-1} B^i C^{q_{(k)}-1} \binom{i+q_{(k)}-1}{i} \binom{k-mi-n(q_{(k)}-1+1)}{i+q_{(k)}-1+1}$$

where

$$\binom{k-mi-n(q_{(k)}-1+1)}{i+q_{(k)}-1+1} = \binom{k-mi-nq_{(k)}}{i+q_{(k)}} = 0 \quad \text{if} \quad i > 0.$$

Thus,

$$\begin{aligned}
\Delta \mathrm{e}_{mn}^{BCk} &= B \left[I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-2, q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m(i+1)-nj}{i+j+1} \right] \\
&\quad + C \left[I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-1, q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-mi-n(j+1)}{i+j+1} \right] \\
&= B\mathrm{e}_{mn}^{BC(k-m)} + C\mathrm{e}_{mn}^{BC(k-n)}
\end{aligned}$$

and formula (4.21) is proved.

IV. $\boldsymbol{k = p_{(k)}(m+1) \wedge k = q_{(k)}(n+1)}$

In this case, we have (see similar combinations in the cases II. and III.)

$$p_{(k-m)} = p_{(k)}, \quad p_{(k+1)} = p_{(k)} + 1$$

and

$$q_{(k-n)} = q_{(k)}, \quad q_{(k+1)} = q_{(k)} + 1.$$

Then,

$$\begin{aligned}
\mathrm{e}_{mn}^{BCk} &= I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-1, q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-mi-nj}{i+j+1}, \\
\mathrm{e}_{mn}^{BC(k+1)} &= I + (B+C) \sum_{i=0,j=0}^{p_{(k)}, q_{(k)}} B^i C^j \binom{i+j}{i} \binom{k+1-mi-nj}{i+j+1}.
\end{aligned}$$

and

$$\mathrm{e}_{mn}^{BC(k-m)} = I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-1, q_{(k-m)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-mi-nj}{i+j+1}, \tag{4.22}$$

$$\mathrm{e}_{mn}^{BC(k-n)} = I + (B+C) \sum_{i=0,j=0}^{p_{(k-n)}-1, q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-n-mi-nj}{i+j+1}. \tag{4.23}$$

As before,

$$\binom{k-m-mi-nj}{i+j+1} = 0 \quad \text{if} \quad i \geq 0, \quad j \geq q_{(k-m)}$$

and

$$\binom{k-n-mi-nj}{i+j+1} = 0 \quad \text{if} \quad i \geq p_{(k-n)}, \quad j \geq 0.$$

So we can substitute $q_{(k)}$ for $q_{(k-m)}$ in (4.22) and $p_{(k)}$ for $p_{(k-n)}$ in (4.23) and

$$\mathrm{e}_{mn}^{BC(k-m)} = I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m(i+1)-nj}{i+j+1}, \tag{4.24}$$

$$\mathrm{e}_{mn}^{BC(k-n)} = I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-mi-n(j+1)}{i+j+1}. \tag{4.25}$$

It is easy to see that, by (1.8), formula (4.24) can also be used instead of (4.22) if $q_{(k-m)} < 1$ and formula (4.25) can also be used instead of (4.23) if $p_{(k-n)} < 1$.

Now it is possible to prove the formula

$$\begin{aligned} \Delta \mathrm{e}_{mn}^{BCk} &= B\mathrm{e}_{mn}^{BC(k-m)} + C\mathrm{e}_{mn}^{BC(k-n)} \\ &= B\left[I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m(i+1)-nj}{i+j+1}\right] \\ &\quad + C\left[I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-mi-n(j+1)}{i+j+1}\right]. \end{aligned} \tag{4.26}$$

By (4.4), we get

$$\begin{aligned} \Delta \mathrm{e}_{mn}^{BCk} &= \mathrm{e}_{mn}^{BC(k+1)} - \mathrm{e}_{mn}^{BCk} \\ &= I + (B+C) \sum_{i=0,j=0}^{p_{(k)},q_{(k)}} B^i C^j \binom{i+j}{i} \binom{k+1-mi-nj}{i+j+1} \\ &\quad - I - (B+C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-mi-nj}{i+j+1} \\ &= (B+C) \left[\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \left[\binom{k+1-mi-nj}{i+j+1} - \binom{k-mi-nj}{i+j+1}\right]\right. \\ &\quad + \sum_{j=0}^{q_{(k)}} B^{p_{(k)}} C^j \binom{p_{(k)}+j}{p_{(k)}} \binom{k+1-mp_{(k)}-nj}{p_{(k)}+j+1} \\ &\quad \left. + \sum_{i=0}^{p_{(k)}} B^i C^{q_{(k)}} \binom{i+q_{(k)}}{i} \binom{k+1-mi-nq_{(k)}}{i+q_{(k)}+1}\right]. \end{aligned}$$

With the aid of equations $k = p_{(k)}(m+1)$, $k = q_{(k)}(n+1)$, we get

$$\binom{k+1-mp_{(k)}-nj}{p_{(k)}+j+1} = \binom{p_{(k)}+1-nj}{p_{(k)}+1+j} = 0 \quad \text{if} \quad j > 0,$$

$$\binom{k+1-mi-nq_{(k)}}{i+q_{(k)}+1} = \binom{q_{(k)}+1-mi}{q_{(k)}+1+i} = 0 \quad \text{if} \quad i > 0,$$

and

$$\Delta \mathrm{e}_{mn}^{BCk} = (B+C)\left[\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-mi-nj}{i+j} + B^{p_{(k)}} + C^{q_{(k)}}\right]$$
$$= (B+C)\left[I + \sum_{i=1}^{p_{(k)}-1} B^iC^0\binom{i}{i}\binom{k-mi}{i} + \sum_{j=1}^{q_{(k)}-1} B^0C^j\binom{j}{0}\binom{k-nj}{j}\right.$$
$$\left. + \sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-mi-nj}{i+j} + B^{p_{(k)}} + C^{q_{(k)}}\right].$$

By (1.7), we have

$$\Delta \mathrm{e}_{mn}^{BCk} = (B+C)\left[I + \sum_{i=1}^{p_{(k)}-1} B^iC^0\binom{i-1}{i-1}\binom{k-mi}{i} + \sum_{j=1}^{q_{(k)}-1} B^0C^j\binom{j-1}{0}\binom{k-nj}{j}\right.$$
$$+ \sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j-1}{i-1}\binom{k-mi-nj}{i+j}$$
$$\left. + \sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j-1}{i}\binom{k-mi-nj}{i+j} + B^{p_{(k)}} + C^{q_{(k)}}\right]$$
$$= (B+C)\left[I + \sum_{i=1,j=0}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j-1}{i-1}\binom{k-mi-nj}{i+j}\right.$$
$$\left. + \sum_{i=0,j=1}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j-1}{i}\binom{k-mi-nj}{i+j} + B^{p_{(k)}} + C^{q_{(k)}}\right].$$

We replace in the first sum the summation index i by $i+1$ and, in the second sum, we substitute the summation index j by $j+1$. Then,

$$\Delta \mathrm{e}_{mn}^{BCk} = (B+C)\left[I + \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^{i+1}C^j\binom{i+j}{i}\binom{k-m(i+1)-nj}{i+j+1}\right.$$
$$\left. + \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^iC^{j+1}\binom{i+j}{i}\binom{k-mi-n(j+1)}{i+j+1} + B^{p_{(k)}} + C^{q_{(k)}}\right]$$
$$= B + B(B+C)\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-m(i+1)-nj}{i+j+1}$$
$$+ C + C(B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^iC^j\binom{i+j}{i}\binom{k-mi-n(j+1)}{i+j+1}$$
$$+ B^{p_{(k)}}(B+C) + C^{q_{(k)}}(B+C)$$
$$= B\left[I + (B+C)\left(\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-m(i+1)-nj}{i+j+1} + B^{p_{(k)}-1}\right)\right]$$

$$
+ C\left[I + (B+C)\left(\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^iC^j\binom{i+j}{i}\binom{k-mi-n(j+1)}{i+j+1}\right.\right.
$$
$$
\left.\left. + C^{q_{(k)}-1}\right)\right].
$$

Because $k = p_{(k)}(m+1) = q_{(k)}(n+1)$, we can $B^{p_{(k)}-1}$ and $C^{q_{(k)}-1}$ express in the form

$$
\begin{aligned}
B^{p_{(k)}-1} &= \sum_{j=0}^{q_{(k)}-1} B^{p_{(k)}-1}C^j\binom{p_{(k)}-1+j}{p_{(k)}-1}\binom{k-m(p_{(k)}-1+1)-nj}{p_{(k)}-1+j+1},\\
C^{q_{(k)}-1} &= \sum_{i=0}^{p_{(k)}-1} B^iC^{q_{(k)}-1}\binom{i+q_{(k)}-1}{i}\binom{k-mi-n(q_{(k)}-1+1)}{i+q_{(k)}-1+1}
\end{aligned}
$$

where

$$
\begin{aligned}
\binom{k-m(p_{(k)}-1+1)-nj}{p_{(k)}-1+j+1} &= \binom{k-mp_{(k)}-nj}{p_{(k)}+j} = 0 \quad \text{if} \quad j>0,\\
\binom{k-mi-n(q_{(k)}-1+1)}{i+q_{(k)}-1+1} &= \binom{k-mi-nq_{(k)}}{i+q_{(k)}} = 0 \quad \text{if} \quad i>0.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\Delta \mathrm{e}_{mn}^{BCk} &= B\left[I + (B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-m(i+1)-nj}{i+j+1}\right]\\
&\quad + C\left[I + (B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-mi-n(j+1)}{i+j+1}\right]\\
&= B\mathrm{e}_{mn}^{BC(k-m)} + C\mathrm{e}_{mn}^{BC(k-n)}.
\end{aligned}
$$

Therefore, formula (4.26) is valid.

We have proved that formula (4.2) holds in each of the above cases I., II., III. and IV. for $k \geq 1$. If $k = 0$, the proof can be done directly because $p_{(0)} = q_{(0)} = 0$, $p_{(1)} = q_{(1)} = 1$,

$$
\begin{aligned}
\Delta \mathrm{e}_{mn}^{BC0} &= \mathrm{e}_{mn}^{BC1} - \mathrm{e}_{mn}^{BC0}\\
&= I + (B+C)\sum_{i=0,j=0}^{0,0} B^iC^j\binom{i+j}{i}\binom{1-mi-nj}{i+j+1}\\
&\quad - I - (B+C)\sum_{i=0,j=0}^{-1,-1} B^iC^j\binom{i+j}{i}\binom{-mi-nj}{i+j+1}\\
&= I + B + C - I = B + C,
\end{aligned}
$$

and

$$
B\mathrm{e}_{mn}^{BC(-m)} + C\mathrm{e}_{mn}^{BC(-n)} = BI + CI = B + C.
$$

Formula (4.2) holds again. Theorem 4.4 is proved. □

4.2 Discrete Matrix Delayed Exponential $\tilde{\mathrm{e}}_{mn}^{BCk}$

Analyzing the applicability of e_{mn}^{BCk} to a representation of the solution to initial problem (5.1), (5.2) we see that, unfortunately, this does not lead to satisfactory results because, as we will see below, an additional condition $\det(B + C) \neq 0$ is necessary. A small difference in the definition results in representations of solutions of initial problems without this assumption.

Now we give a second definition of a discrete matrix delayed exponential for two delays $\tilde{\mathrm{e}}_{mn}^{BCk}$.

Definition 4.5. *Let B, C be constant $r \times r$ matrices with the property $BC = CB$ and let $m, n \in \mathbb{N}$, $m < n$ be fixed integers. We define a discrete $r \times r$ matrix function $\tilde{\mathrm{e}}_{mn}^{BCk}$ called the discrete matrix delayed exponential for two delays m, n and for two $r \times r$ constant matrices B, C as follows*

$$\tilde{\mathrm{e}}_{mn}^{BCk} := \begin{cases} \Theta \quad \text{if} \quad k \in \mathbb{Z}_{-\infty}^{-1}, \\ I \quad \text{if} \quad k \in \mathbb{Z}_0^m, \\ I + B \displaystyle\sum_{i=0,j=0}^{p_{(k)}-1, q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-mi-nj}{i+j+1} \\ \quad + C \displaystyle\sum_{i=0,j=0}^{p_{(k)}-1, q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-n-mi-nj}{i+j+1} \quad \text{if} \quad k \in \mathbb{Z}_{m+1}^{\infty} \end{cases}$$

where

$$p_{(k)} := \left\lfloor \frac{k+m}{m+1} \right\rfloor, \quad q_{(k)} := \left\lfloor \frac{k+n}{n+1} \right\rfloor \tag{4.27}$$

and Θ is the $r \times r$ null matrix, I is the $r \times r$ unit matrix.

Remark 4.6. For $k \in \mathbb{Z}_0^n$, it is easy to deduce that $\tilde{\mathrm{e}}_{mn}^{BCk} = \mathrm{e}_m^{B(k-m)}$.

In order to compare both types of discrete delayed matrices for two delays and see the difference between both definitions, we consider the following example where delays are the same as in Example 4.3.

Example 4.7. For $k \in \mathbb{Z}_0^{12}$, we will construct the matrix $\tilde{\mathrm{e}}_{mn}^{BCk}$ if $m = 2$ and $n = 3$. Computing particular matrices generating $\tilde{\mathrm{e}}_{2,3}^{BCk}$ for $k \in \mathbb{Z}_0^{12}$, we get

$\tilde{\mathrm{e}}_{2,3}^{BC0} = I,$
$\tilde{\mathrm{e}}_{2,3}^{BC1} = I,$
$\tilde{\mathrm{e}}_{2,3}^{BC2} = I,$
$\tilde{\mathrm{e}}_{2,3}^{BC3} = I + B,$
$\tilde{\mathrm{e}}_{2,3}^{BC4} = I + 2B + C,$

$$
\begin{aligned}
\tilde{\mathrm{e}}_{2,3}^{BC5} &= I + 3B + 2C,\\
\tilde{\mathrm{e}}_{2,3}^{BC6} &= I + 4B + 3C + B^2,\\
\tilde{\mathrm{e}}_{2,3}^{BC7} &= I + 5B + 4C + 3B^2 + 2BC,\\
\tilde{\mathrm{e}}_{2,3}^{BC8} &= I + 6B + 5C + 6B^2 + 6BC + C^2,\\
\tilde{\mathrm{e}}_{2,3}^{BC9} &= I + 7B + 6C + 10B^2 + 12BC + 3C^2 + B^3,\\
\tilde{\mathrm{e}}_{2,3}^{BC10} &= I + 8B + 7C + 15B^2 + 20BC + 6C^2 + 4B^3 + 3B^2C,\\
\tilde{\mathrm{e}}_{2,3}^{BC11} &= I + 9B + 8C + 21B^2 + 30BC + 10C^2 + 10B^3 + 12B^2C + 3BC^2,\\
\tilde{\mathrm{e}}_{2,3}^{BC12} &= I + 10B + 9C + 28B^2 + 42BC + 15C^2 + 20B^3 + 30B^2C + 12BC^2 + C^3\\
&\quad + B^4.
\end{aligned}
$$

The main property of $\tilde{\mathrm{e}}_{mn}^{BCk}$ is given by the following theorem.

Theorem 4.8. *Let B, C be constant $r \times r$ matrices with the property $BC = CB$ and let $m, n \in \mathbb{N}$, $m < n$ be fixed integers. Then,*

$$
\Delta \tilde{\mathrm{e}}_{mn}^{BCk} = B\tilde{\mathrm{e}}_{mn}^{BC(k-m)} + C\tilde{\mathrm{e}}_{mn}^{BC(k-n)} \tag{4.28}
$$

holds for $k \in \mathbb{Z}_0^\infty$.

Proof. Let $k \geq 1$. From (1.4) and (4.27), we can see easily that, for an integer k satisfying

$$
(p_{(k)} - 1)(m+1) + 1 \leq k \leq p_{(k)}(m+1) \wedge (q_{(k)} - 1)(n+1) + 1 \leq k \leq q_{(k)}(n+1),
$$

the equation

$$
\begin{aligned}
\Delta \tilde{\mathrm{e}}_{mn}^{BCk} = \Delta \Bigg[I + B \sum_{i=0,j=0}^{p_{(k)}-1, q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-mi-nj}{i+j+1} \\
+ C \sum_{i=0,j=0}^{p_{(k)}-1, q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-n-mi-nj}{i+j+1} \Bigg]
\end{aligned}
$$

holds by Definition 4.5 of $\tilde{\mathrm{e}}_{mn}^{BCk}$. Since $\Delta I = \Theta$, we have

$$
\begin{aligned}
\Delta \tilde{\mathrm{e}}_{mn}^{BCk} = \Delta \Bigg[B \sum_{i=0,j=0}^{p_{(k)}-1, q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-mi-nj}{i+j+1} \\
+ C \sum_{i=0,j=0}^{p_{(k)}-1, q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-n-mi-nj}{i+j+1} \Bigg].
\end{aligned} \tag{4.29}
$$

By the definition of the forward difference, i.e.,

$$
\Delta \tilde{\mathrm{e}}_{mn}^{BCk} = \tilde{\mathrm{e}}_{mn}^{BC(k+1)} - \tilde{\mathrm{e}}_{mn}^{BCk}, \tag{4.30}
$$

we conclude that it is reasonable to divide the proof into four parts given by the four values of integer k. In the first case, k is such that

$$(p_{(k)}-1)(m+1)+1 \le k < p_{(k)}(m+1) \wedge (q_{(k)}-1)(n+1)+1 \le k < q_{(k)}(n+1),$$

in the second case,

$$k = p_{(k)}(m+1) \wedge (q_{(k)}-1)(n+1)+1 \le k < q_{(k)}(n+1),$$

in the third case,

$$(p_{(k)}-1)(m+1)+1 \le k < p_{(k)}(m+1) \wedge k = q_{(k)}(n+1),$$

and, in the fourth case,

$$k = p_{(k)}(m+1) \wedge k = q_{(k)}(n+1).$$

We see that the above four cases cover all the possible relations between k, $p_{(k)}$ and $q_{(k)}$.

Now we consider (in parts I.–IV. below) all four cases and perform auxiliary computations. The proof will be finished in part V.

I. $\boldsymbol{(p_{(k)}-1)(m+1)+1 \le k < p_{(k)}(m+1)}$ $\boldsymbol{\wedge (q_{(k)}-1)(n+1)+1 \le k < q_{(k)}(n+1)}$

From (1.4) and (4.27), we get

$$\begin{aligned} p_{(k-m)} &= \left\lfloor \frac{k-m+m}{m+1} \right\rfloor \le \frac{k}{m+1} < p_{(k)}, \\ p_{(k-m)} &= \left\lfloor \frac{k-m+m}{m+1} \right\rfloor > \frac{k}{m+1} - 1 = \frac{k-m-1}{m+1} > p_{(k)} - 2. \end{aligned}$$

Therefore, $p_{(k-m)} = p_{(k)} - 1$ and, by Definition 4.5,

$$\begin{aligned} \tilde{e}_{mn}^{BC(k-m)} = I + B &\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k-m)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-m-mi-nj}{i+j+1} \\ + C &\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k-m)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-n-mi-nj}{i+j+1}. \end{aligned} \tag{4.31}$$

Similarly, omitting details, using (1.4) and (4.27), we get $q_{(k-n)} = q_{(k)} - 1$ and

$$\begin{aligned} \tilde{e}_{mn}^{BC(k-n)} = I + B &\sum_{i=0,j=0}^{p_{(k-n)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-n-m-mi-nj}{i+j+1} \\ + C &\sum_{i=0,j=0}^{p_{(k-n)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-n-n-mi-nj}{i+j+1}. \end{aligned} \tag{4.32}$$

Let $q_{(k-m)} \geq 1$. We show that

$$\begin{aligned} \binom{k-m-m-mi-nj}{i+j+1} &= 0 \quad \text{if} \quad i \geq 0, \quad j \geq q_{(k-m)}, \\ \binom{k-m-n-mi-nj}{i+j+1} &= 0 \quad \text{if} \quad i \geq 0, \quad j \geq q_{(k-m)}. \end{aligned} \tag{4.33}$$

By (1.4),

$$q_{(k-m)} = \left\lfloor \frac{k-m+n}{n+1} \right\rfloor > \frac{k-m+n}{n+1} - 1 = \frac{k-m-1}{n+1}$$

or

$$k-m-m < (n+1)q_{(k-m)} + 1 \leq (m+1)i + (n+1)j + 1 \quad \text{if} \quad i \geq 0, \quad j \geq q_{(k-m)}$$

and

$$k-m-n < (n+1)q_{(k-m)} + 1 \leq (m+1)i + (n+1)j + 1 \quad \text{if} \quad i \geq 0, \quad j \geq q_{(k-m)}.$$

From the last inequalities, we get

$$\begin{aligned} k-m-m-mi-nj &< i+j+1 \quad \text{if} \quad i \geq 0, \quad j \geq q_{(k-m)}, \\ k-m-n-mi-nj &< i+j+1 \quad \text{if} \quad i \geq 0, \quad j \geq q_{(k-m)}, \end{aligned}$$

and (4.33) holds by (1.5). For that reason and since $q_{(k-m)} \leq q_{(k)}$, we can replace $q_{(k-m)}$ by $q_{(k)}$ in (4.31). Thus, we have

$$\begin{aligned} \tilde{\mathrm{e}}_{mn}^{BC(k-m)} = I &+ B \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-m(i+1)-nj}{i+j+1} \\ &+ C \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-n-m(i+1)-nj}{i+j+1}. \end{aligned} \tag{4.34}$$

It is easy to see that, by (1.8), formula (4.34) can be used instead of (4.31) if $q_{(k-m)} < 1$, too.

Let $p_{(k-n)} \geq 1$. Similarly, we can show that

$$\begin{aligned} \binom{k-n-m-mi-nj}{i+j+1} &= 0 \quad \text{if} \quad i \geq p_{(k-n)}, \quad j \geq 0, \\ \binom{k-n-n-mi-nj}{i+j+1} &= 0 \quad \text{if} \quad i \geq p_{(k-n)}, \quad j \geq 0. \end{aligned} \tag{4.35}$$

and, since $p_{(k-n)} \leq p_{(k)}$, we can replace $p_{(k-n)}$ by $p_{(k)}$ in (4.32). Thus, we have

$$\begin{aligned} \tilde{\mathrm{e}}_{mn}^{BC(k-n)} = I &+ B \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-m-mi-n(j+1)}{i+j+1} \\ &+ C \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-n-mi-n(j+1)}{i+j+1}. \end{aligned} \tag{4.36}$$

It is easy to see that, by (1.8), formula (4.36) can be used instead of (4.32) if $p_{(k-n)} < 1$, too.

By Definition 4.5,

$$\tilde{e}_{mn}^{BC(k+1)} = I + B \sum_{i=0,j=0}^{p_{(k+1)}-1,q_{(k+1)}-1} B^i C^j \binom{i+j}{i} \binom{k+1-m-mi-nj}{i+j+1} + C \sum_{i=0,j=0}^{p_{(k+1)}-1,q_{(k+1)}-1} B^i C^j \binom{i+j}{i} \binom{k+1-n-mi-nj}{i+j+1}.$$

By (1.4), we also conclude that

$$p_{(k+1)} = p_{(k)}, \quad q_{(k+1)} = q_{(k)} \tag{4.37}$$

because

$$p_{(k+1)} = \left\lfloor \frac{k+1+m}{m+1} \right\rfloor \le \frac{k}{m+1} + 1 < p_{(k)} + 1,$$

and

$$p_{(k+1)} = \left\lfloor \frac{k+1+m}{m+1} \right\rfloor > \frac{k+1+m}{m+1} - 1 = \frac{k}{m+1} \ge p_{(k)} - 1 + \frac{1}{m+1}.$$

The second formula can be proved similarly. Then

$$\tilde{e}_{mn}^{BC(k+1)} = I + B \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k+1-m-mi-nj}{i+j+1} + C \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k+1-n-mi-nj}{i+j+1}.$$

Now we are able to prove that

$$\begin{aligned}
\Delta\tilde{e}_{mn}^{BCk} &= B\tilde{e}_{mn}^{BC(k-m)} + C\tilde{e}_{mn}^{BC(k-n)} \\
&= B \Bigg[I + B \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-m(i+1)-nj}{i+j+1} \\
&\qquad + C \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-n-m(i+1)-nj}{i+j+1} \Bigg] \\
&\quad + C \Bigg[I + B \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-m-mi-n(j+1)}{i+j+1} \\
&\qquad + C \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-n-mi-n(j+1)}{i+j+1} \Bigg].
\end{aligned} \tag{4.38}$$

II. $\boldsymbol{k = p_{(k)}(m+1) \wedge (q_{(k)}-1)(n+1)+1 \le k < q_{(k)}(n+1)}$

In this case,

$$p_{(k-m)} = \left\lfloor \frac{k-m+m}{m+1} \right\rfloor = \left\lfloor \frac{k}{m+1} \right\rfloor = p_{(k)},$$
$$p_{(k+1)} = \left\lfloor \frac{k+1+m}{m+1} \right\rfloor \le \frac{k+1+m}{m+1} = \frac{k}{m+1} + 1 = p_{(k)} + 1,$$
$$p_{(k+1)} = \left\lfloor \frac{k+1+m}{m+1} \right\rfloor > \frac{k+1+m}{m+1} - 1 = \frac{k}{m+1} = p_{(k)}$$

and $p_{(k+1)} = p_{(k)} + 1$. In addition to this (see the relevant computations performed in case I.), we have $q_{(k-n)} = q_{(k)} - 1$ and $q_{(k+1)} = q_{(k)}$. Then,

$$\begin{aligned} \tilde{\mathrm{e}}_{mn}^{BC(k+1)} = I + B &\sum_{i=0,j=0}^{p_{(k)},q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k+1-m-mi-nj}{i+j+1} \\ + C &\sum_{i=0,j=0}^{p_{(k)},q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k+1-n-mi-nj}{i+j+1} \end{aligned} \tag{4.39}$$

and

$$\begin{aligned} \tilde{\mathrm{e}}_{mn}^{BC(k-m)} = I + B &\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k-m)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-m-mi-nj}{i+j+1} \\ + C &\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k-m)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-n-mi-nj}{i+j+1}, \end{aligned} \tag{4.40}$$

$$\begin{aligned} \tilde{\mathrm{e}}_{mn}^{BC(k-n)} = I + B &\sum_{i=0,j=0}^{p_{(k-n)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-n-m-mi-nj}{i+j+1} \\ + C &\sum_{i=0,j=0}^{p_{(k-n)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-n-n-mi-nj}{i+j+1}. \end{aligned} \tag{4.41}$$

For $k = p_{(k)}(m+1)$, $i = p_{(k)}$ and $j \ge 0$, we have

$$\begin{aligned} \binom{k+1-m-mi-nj}{i+j+1} &= \binom{p_{(k)}+1-m-nj}{p_{(k)}+1+j} = 0, \\ \binom{k+1-n-mi-nj}{i+j+1} &= \binom{p_{(k)}+1-n-nj}{p_{(k)}+1+j} = 0 \end{aligned} \tag{4.42}$$

and, for $k = p_{(k)}(m+1)$, $i = p_{(k)} - 1$ and $j \ge 0$, we have

$$\begin{aligned} \binom{k-m-m-mi-nj}{i+j+1} &= \binom{p_{(k)}-m-nj}{p_{(k)}+j} = 0, \\ \binom{k-m-n-mi-nj}{i+j+1} &= \binom{p_{(k)}-n-nj}{p_{(k)}+j} = 0. \end{aligned} \tag{4.43}$$

Thus, we can substitute $p_{(k)}-1$ for $p_{(k)}$ in (4.39) and $p_{(k)}-2$ for $p_{(k)}-1$ in (4.40).

As with the computations performed in the previous part of the proof, (4.33), (4.35) hold. So we can substitute $q_{(k)}$ for $q_{(k-m)}$ in (4.40) and $p_{(k)}$ for $p_{(k-n)}$ in (4.41).

Accordingly, we have

$$\tilde{e}_{mn}^{BC(k+1)} = I + B\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k+1-m-mi-nj}{i+j+1} + C\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k+1-n-mi-nj}{i+j+1},$$

$$\begin{aligned}\tilde{e}_{mn}^{BC(k-m)} = I &+ B\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-m-m(i+1)-nj}{i+j+1} \\ &+ C\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-n-m(i+1)-nj}{i+j+1},\end{aligned} \tag{4.44}$$

$$\begin{aligned}\tilde{e}_{mn}^{BC(k-n)} = I &+ B\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^iC^j\binom{i+j}{i}\binom{k-m-mi-n(j+1)}{i+j+1} \\ &+ C\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^iC^j\binom{i+j}{i}\binom{k-n-mi-n(j+1)}{i+j+1}.\end{aligned} \tag{4.45}$$

It is easy to see that, by (1.8), formula (4.44) can also be used instead of (4.40) if $q_{(k-m)} < 1$ and formula (4.45) can also be used instead of (4.41) if $p_{(k-n)} < 1$. Therefore, we see that (like in part I.) the relation (4.38) must be proved.

III. $\boldsymbol{(p_{(k)}-1)(m+1)+1 \leq k < p_{(k)}(m+1) \wedge k = q_{(k)}(n+1)}$

In this case, we have (see the relevant computations in cases I. and II.)

$$p_{(k-m)} = p_{(k)}-1, \quad p_{(k+1)} = p_{(k)}$$

and

$$q_{(k-n)} = q_{(k)}, \quad q_{(k+1)} = q_{(k)}+1.$$

Then,

$$\begin{aligned}\tilde{e}_{mn}^{BC(k+1)} = I &+ B\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}} B^iC^j\binom{i+j}{i}\binom{k+1-m-mi-nj}{i+j+1} \\ &+ C\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}} B^iC^j\binom{i+j}{i}\binom{k+1-n-mi-nj}{i+j+1}\end{aligned} \tag{4.46}$$

and

$$\begin{aligned}\tilde{e}_{mn}^{BC(k-m)} &= I + B\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k-m)}-1} B^iC^j\binom{i+j}{i}\binom{k-m-m-mi-nj}{i+j+1}\\ &\quad + C\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k-m)}-1} B^iC^j\binom{i+j}{i}\binom{k-m-n-mi-nj}{i+j+1},\end{aligned} \tag{4.47}$$

$$\begin{aligned}\tilde{e}_{mn}^{BC(k-n)} &= I + B\sum_{i=0,j=0}^{p_{(k-n)}-1,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-n-m-mi-nj}{i+j+1}\\ &\quad + C\sum_{i=0,j=0}^{p_{(k-n)}-1,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-n-n-mi-nj}{i+j+1}.\end{aligned} \tag{4.48}$$

For $k = q_{(k)}(n+1)$, $j = q_{(k)}$ and $i \geq 0$, we have

$$\begin{aligned}\binom{k+1-m-mi-nj}{i+j+1} &= \binom{q_{(k)}+1-m-mi}{i+q_{(k)}+1} = 0,\\ \binom{k+1-n-mi-nj}{i+j+1} &= \binom{q_{(k)}+1-n-mi}{i+q_{(k)}+1} = 0\end{aligned} \tag{4.49}$$

and, for $k = q_{(k)}(m+1)$, $j = q_{(k)} - 1$ and $i \geq 0$, we get

$$\begin{aligned}\binom{k-n-m-mi-nj}{i+j+1} &= \binom{q_{(k)}-m-mi}{i+q_{(k)}} = 0,\\ \binom{k-n-n-mi-nj}{i+j+1} &= \binom{q_{(k)}-n-mi}{i+q_{(k)}} = 0.\end{aligned} \tag{4.50}$$

Thus, we can replace $q_{(k)}$ by $q_{(k)} - 1$ in (4.46) and $q_{(k)} - 1$ by $q_{(k)} - 2$ in (4.48).

Like with the computations performed in cases I. and II., formulas (4.33), (4.35) hold and we can substitute $q_{(k)}$ for $q_{(k-m)}$ in (4.47) and $p_{(k)}$ for $p_{(k-n)}$ in (4.48). This means that

$$\begin{aligned}\tilde{e}_{mn}^{BC(k+1)} &= I + B\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k+1-m-mi-nj}{i+j+1}\\ &\quad + C\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k+1-n-mi-nj}{i+j+1},\end{aligned}$$

$$\begin{aligned}\tilde{e}_{mn}^{BC(k-m)} &= I + B\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-m-m-mi-nj}{i+j+1}\\ &\quad + C\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-m-n-mi-nj}{i+j+1},\end{aligned} \tag{4.51}$$

$$\begin{aligned}\tilde{e}_{mn}^{BC(k-n)} &= I + B\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^iC^j\binom{i+j}{i}\binom{k-n-m-mi-nj}{i+j+1}\\ &\quad + C\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^iC^j\binom{i+j}{i}\binom{k-n-n-mi-nj}{i+j+1}.\end{aligned} \tag{4.52}$$

It is easy to see that, by (1.8), formula (4.51) can also be used instead of (4.47) if $q_{(k-m)} < 1$ and formula (4.52) can also be used instead of (4.48) if $p_{(k-n)} < 1$. Therefore, we see that (as in parts I., II.) the equation (4.38) must be proved.

IV. $\boldsymbol{k = p_{(k)}(m+1) \wedge k = q_{(k)}(n+1)}$

In this case, we have (see similar combinations in the cases II. and III.)

$$p_{(k-m)} = p_{(k)}, \quad p_{(k+1)} = p_{(k)} + 1$$

and

$$q_{(k-n)} = q_{(k)}, \quad q_{(k+1)} = q_{(k)} + 1.$$

Then,

$$\begin{aligned} \tilde{\mathrm{e}}_{mn}^{BC(k+1)} = I + B \sum_{i=0,j=0}^{p_{(k)},q_{(k)}} B^i C^j \binom{i+j}{i} \binom{k+1-m-mi-nj}{i+j+1} \\ + C \sum_{i=0,j=0}^{p_{(k)},q_{(k)}} B^i C^j \binom{i+j}{i} \binom{k+1-n-mi-nj}{i+j+1} \end{aligned} \tag{4.53}$$

and

$$\begin{aligned} \tilde{\mathrm{e}}_{mn}^{BC(k-m)} = I + B \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k-m)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-m-mi-nj}{i+j+1} \\ + C \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k-m)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-n-mi-nj}{i+j+1}, \end{aligned} \tag{4.54}$$

$$\begin{aligned} \tilde{\mathrm{e}}_{mn}^{BC(k-n)} = I + B \sum_{i=0,j=0}^{p_{(k-n)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-n-m-mi-nj}{i+j+1} \\ + C \sum_{i=0,j=0}^{p_{(k-n)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-n-n-mi-nj}{i+j+1}. \end{aligned} \tag{4.55}$$

As in part II., for $k = p_{(k)}(m+1)$, $i = p_{(k)}$, and $j \geq 0$, formulas (4.42) hold and, for $k = p_{(k)}(m+1)$, $i = p_{(k)} - 1$, and $j \geq 0$, formulas (4.43) hold. Thus, we can substitute $p_{(k)} - 1$ for $p_{(k)}$ in (4.53) and $p_{(k)} - 2$ for $p_{(k)} - 1$ in (4.54).

As in part III., for $k = q_{(k)}(n+1)$, $j = q_{(k)}$, and $i \geq 0$, formulas (4.49) hold and, for $k = q_{(k)}(m+1)$, $j = q_{(k)} - 1$, and $i \geq 0$, formulas (4.50) hold. Thus, we can replace $q_{(k)}$ by $q_{(k)} - 1$ in (4.53) and $q_{(k)} - 1$ by $q_{(k)} - 2$ in (4.55).

As before, (4.33), (4.35) hold and we can substitute $q_{(k)}$ for $q_{(k-m)}$ in (4.54) and $p_{(k)}$ for $p_{(k-n)}$ in (4.55).

Thus, we have

$$\begin{aligned}
\tilde{\mathrm{e}}_{mn}^{BC(k+1)} &= I + B \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k+1-m-mi-nj}{i+j+1} \\
&\quad + C \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k+1-n-mi-nj}{i+j+1},
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathrm{e}}_{mn}^{BC(k-m)} &= I + B \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-m-mi-nj}{i+j+1} \\
&\quad + C \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-n-mi-nj}{i+j+1},
\end{aligned} \tag{4.56}$$

$$\begin{aligned}
\tilde{\mathrm{e}}_{mn}^{BC(k-n)} &= I + B \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-n-m-mi-nj}{i+j+1} \\
&\quad + C \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-n-n-mi-nj}{i+j+1}.
\end{aligned} \tag{4.57}$$

It is easy to see that, by (1.8), formula (4.56) can also be used instead of (4.54) if $q_{(k-m)} < 1$ and formula (4.57) can also be used instead of (4.55) if $p_{(k-n)} < 1$. Therefore, we see that (as in all the previous parts) the equation (4.38) must be proved.

V. The proof of formula (4.38)

Now we prove the equation (4.38). With the aid of (4.29), (4.30), (1.6), and (4.37), we get

$$\begin{aligned}
\Delta\tilde{\mathrm{e}}_{mn}^{BCk} &= \tilde{\mathrm{e}}_{mn}^{BC(k+1)} - \tilde{\mathrm{e}}_{mn}^{BCk} \\
&= I + B \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k+1-m-mi-nj}{i+j+1} \\
&\quad + C \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k+1-n-mi-nj}{i+j+1} \\
&\quad - I - B \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-mi-nj}{i+j+1} \\
&\quad - C \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-n-mi-nj}{i+j+1} \\
&= B \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \left[\binom{k+1-m-mi-nj}{i+j+1} \right. \\
&\qquad \left. - \binom{k-m-mi-nj}{i+j+1} \right]
\end{aligned}$$

$$
\begin{aligned}
&+ C \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \left[\binom{k+1-n-mi-nj}{i+j+1} \right. \\
&\qquad\qquad\qquad\qquad\qquad \left. - \binom{k-n-mi-nj}{i+j+1} \right] \\
&= B \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-mi-nj}{i+j} \\
&\quad + C \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-n-mi-nj}{i+j} \\
&= B \left[I + \sum_{i=1}^{p_{(k)}-1} B^i C^0 \binom{i}{i} \binom{k-m-mi}{i} + \sum_{j=1}^{q_{(k)}-1} B^0 C^j \binom{j}{0} \binom{k-m-nj}{j} \right. \\
&\qquad \left. + \sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-mi-nj}{i+j} \right] \\
&\quad + C \left[I + \sum_{i=1}^{p_{(k)}-1} B^i C^0 \binom{i}{i} \binom{k-n-mi}{i} + \sum_{j=1}^{q_{(k)}-1} B^0 C^j \binom{j}{0} \binom{k-n-nj}{j} \right. \\
&\qquad \left. + \sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-n-mi-nj}{i+j} \right].
\end{aligned}
$$

By (1.7), we have

$$
\begin{aligned}
\Delta \tilde{e}_{mn}^{BCk} = B &\left[I + \sum_{i=1}^{p_{(k)}-1} B^i C^0 \binom{i-1}{i-1} \binom{k-m-mi}{i} \right. \\
&\quad + \sum_{j=1}^{q_{(k)}-1} B^0 C^j \binom{j-1}{0} \binom{k-m-nj}{j} \\
&\quad + \sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j-1}{i-1} \binom{k-m-mi-nj}{i+j} \\
&\quad \left. + \sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j-1}{i} \binom{k-m-mi-nj}{i+j} \right] \\
+ C &\left[I + \sum_{i=1}^{p_{(k)}-1} B^i C^0 \binom{i-1}{i-1} \binom{k-n-mi}{i} \right. \\
&\quad + \sum_{j=1}^{q_{(k)}-1} B^0 C^j \binom{j-1}{0} \binom{k-n-nj}{j} \\
&\quad + \sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j-1}{i-1} \binom{k-n-mi-nj}{i+j} \\
&\quad \left. + \sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j-1}{i} \binom{k-n-mi-nj}{i+j} \right]
\end{aligned}
$$

$$= B\left[I + \sum_{i=1,j=0}^{p_{(k)}-1,q_{(k)}-1} B^iC^j \binom{i+j-1}{i-1}\binom{k-m-mi-nj}{i+j}\right.$$
$$\left. + \sum_{i=0,j=1}^{p_{(k)}-1,q_{(k)}-1} B^iC^j \binom{i+j-1}{i}\binom{k-m-mi-nj}{i+j}\right]$$
$$+ C\left[I + \sum_{i=1,j=0}^{p_{(k)}-1,q_{(k)}-1} B^iC^j \binom{i+j-1}{i-1}\binom{k-n-mi-nj}{i+j}\right.$$
$$\left. + \sum_{i=0,j=1}^{p_{(k)}-1,q_{(k)}-1} B^iC^j \binom{i+j-1}{i}\binom{k-n-mi-nj}{i+j}\right].$$

Now, in the first and third sums, we replace the summation index i by $i+1$ and, in the second and fourth sums, we replace the summation index j by $j+1$. Then

$$\Delta\tilde{\mathrm{e}}_{mn}^{BCk} = B\left[I + \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^{i+1}C^j \binom{i+j}{i}\binom{k-m-m(i+1)-nj}{i+j+1}\right.$$
$$\left. + \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^iC^{j+1} \binom{i+j}{i}\binom{k-m-mi-n(j+1)}{i+j+1}\right]$$
$$+ C\left[I + \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^{i+1}C^j \binom{i+j}{i}\binom{k-n-m(i+1)-nj}{i+j+1}\right.$$
$$\left. + \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^iC^{j+1} \binom{i+j}{i}\binom{k-n-mi-n(j+1)}{i+j+1}\right]$$
$$= B + B^2 \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^iC^j \binom{i+j}{i}\binom{k-m-m(i+1)-nj}{i+j+1}$$
$$+ BC \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^iC^j \binom{i+j}{i}\binom{k-m-mi-n(j+1)}{i+j+1}$$
$$+ C + BC \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^iC^j \binom{i+j}{i}\binom{k-n-m(i+1)-nj}{i+j+1}$$
$$+ C^2 \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^iC^j \binom{i+j}{i}\binom{k-n-mi-n(j+1)}{i+j+1}$$
$$= B\left[I + B \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^iC^j \binom{i+j}{i}\binom{k-m-m(i+1)-nj}{i+j+1}\right.$$
$$\left. + C \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^iC^j \binom{i+j}{i}\binom{k-n-m(i+1)-nj}{i+j+1}\right]$$
$$+ C\left[I + B \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^iC^j \binom{i+j}{i}\binom{k-m-mi-n(j+1)}{i+j+1}\right.$$
$$\left. + C \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^iC^j \binom{i+j}{i}\binom{k-n-mi-n(j+1)}{i+j+1}\right]$$

$$= B\tilde{e}_{mn}^{BC(k-m)} + C\tilde{e}_{mn}^{BC(k-n)}.$$

By (4.34) and (4.36), we conclude that formula (4.38) is valid.

We have proved that formula (4.28) holds in each of the considered cases I., II., III. and IV. for $k \geq 1$. If $k = 0$, the proof can be done directly because $p_{(0)} = q_{(0)} = 0$, $p_{(1)} = q_{(1)} = 1$,

$$\begin{aligned}
\Delta\tilde{e}_{mn}^{BC0} &= \tilde{e}_{mn}^{BC1} - \tilde{e}_{mn}^{BC0} \\
&= I + B \sum_{i=0,j=0}^{0,0} B^i C^j \binom{i+j}{i} \binom{1-m-mi-nj}{i+j+1} \\
&\quad + C \sum_{i=0,j=0}^{0,0} B^i C^j \binom{i+j}{i} \binom{1-n-mi-nj}{i+j+1} \\
&\quad - I - B \sum_{i=0,j=0}^{-1,-1} B^i C^j \binom{i+j}{i} \binom{-m-mi-nj}{i+j+1} \\
&\quad - C \sum_{i=0,j=0}^{-1,-1} B^i C^j \binom{i+j}{i} \binom{-n-mi-nj}{i+j+1} \\
&= I + B\Theta + C\Theta - I - B\Theta - C\Theta = \Theta,
\end{aligned}$$

and

$$B\tilde{e}_{mn}^{BC(-m)} + C\tilde{e}_{mn}^{BC(-n)} = B\Theta + C\Theta = \Theta.$$

Formula (4.28) holds again. Theorem 4.8 is proved. $\square$

5 SOLUTIONS OF LINEAR DISCRETE SYSTEMS WITH TWO DELAYS

In this chapter, we deal with the discrete system

$$\Delta x(k) = Bx(k-m) + Cx(k-n) + f(k) \tag{5.1}$$

where m, $n \in \mathbb{N}$, $m < n$ are fixed, $k \in \mathbb{Z}_0^\infty$, $B = (b_{ij})$, $C = (c_{ij})$ are constant $r \times r$ matrices, $f\colon \mathbb{Z}_0^\infty \to \mathbb{R}^r$ is a given $r \times 1$ vector, and $x\colon \mathbb{Z}_{-n}^\infty \to \mathbb{R}^r$.

Together with equation (5.1), we consider an initial Cauchy problem

$$x(k) = \varphi(k) \tag{5.2}$$

with a given $\varphi\colon \mathbb{Z}_{-n}^0 \to \mathbb{R}^r$.

With the aid of both discrete matrix delayed exponentials, we give formulas for the solutions of homogeneous and nonhomogeneous initial problems (5.1), (5.2). The results of this chapter are published in [16].

5.1 Homogeneous Initial Problem

Consider a homogeneous initial Cauchy problem

$$\Delta x(k) = Bx(k-m) + Cx(k-n), \quad k \in \mathbb{Z}_0^\infty, \tag{5.3}$$

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-n}^0. \tag{5.4}$$

First, we derive formulas for the solution of (5.3), (5.4) with the aid of the discrete matrix delayed exponential e_{mn}^{BCk} and then with the aid of the discrete matrix delayed exponential $\tilde{\mathrm{e}}_{mn}^{BCk}$.

Theorem 5.1. *Let B, C be constant $r \times r$ matrices such that*

$$BC = CB, \quad \det(B + C) \neq 0,$$

and let $m, n \in \mathbb{N}$, $m < n$ be fixed integers. Then, the solution of the initial Cauchy problem (5.3), (5.4) *can be expressed in the form*

$$x(k) = \sum_{j=0}^{n} \mathrm{e}_{mn}^{BC(k+j)} v_j, \tag{5.5}$$

where $k \in \mathbb{Z}_{-n}^\infty$ and

$$v_0 = \varphi(-n) - \sum_{s=1}^{n} v_s,$$

$$v_\ell = (B+C)^{-1} \left[\Delta\varphi(-\ell) - \sum_{t=1}^{n-\ell} \Delta \mathrm{e}_{mn}^{BCt} v_{t+\ell} \right], \quad \ell \in \mathbb{Z}_1^n.$$

Proof. We are going to find the solution of the problem (5.3), (5.4) in the form

$$x(k) = \sum_{j=0}^{n} \mathrm{e}_{mn}^{BC(k+j)} v_j, \quad k \in \mathbb{Z}_{-n}^{\infty} \tag{5.6}$$

with unknown constant vectors v_j. Because of linearity (taking into account that k varies), we have, for $k \geq 0$,

$$\Delta x(k) = \Delta \sum_{j=0}^{n} \mathrm{e}_{mn}^{BC(k+j)} v_j = \sum_{j=0}^{n} \Delta \left[\mathrm{e}_{mn}^{BC(k+j)} v_j \right] = \sum_{j=0}^{n} \Delta \left[\mathrm{e}_{mn}^{BC(k+j)} \right] v_j.$$

Using formula (4.2):

$$\begin{aligned} \Delta x(k) &= \sum_{j=0}^{n} \left(B \mathrm{e}_{mn}^{BC(k-m+j)} + C \mathrm{e}_{mn}^{BC(k-n+j)} \right) v_j \\ &= B \sum_{j=0}^{n} \mathrm{e}_{mn}^{BC(k-m+j)} v_j + C \sum_{j=0}^{n} \mathrm{e}_{mn}^{BC(k-n+j)} v_j \\ &= Bx(k-m) + Cx(k-n). \end{aligned}$$

Now we conclude that, for any v_j and $k \in \mathbb{Z}_0^{\infty}$, the equation $\Delta x(k) = Bx(k-m) + Cx(k-n)$ holds. We will try to satisfy initial conditions (5.4). By (5.6), we have, for $k \in \mathbb{Z}_{-n}^{0}$,

$$\begin{aligned} &\mathrm{e}_{mn}^{BC0} v_0 + \mathrm{e}_{mn}^{BC1} v_1 + \mathrm{e}_{mn}^{BC2} v_2 + \cdots \\ &\qquad + \mathrm{e}_{mn}^{BC(n-2)} v_{n-2} + \mathrm{e}_{mn}^{BC(n-1)} v_{n-1} + \mathrm{e}_{mn}^{BCn} v_n = \varphi(0), \end{aligned}$$

$$\begin{aligned} &\mathrm{e}_{mn}^{BC(-1)} v_0 + \mathrm{e}_{mn}^{BC0} v_1 + \mathrm{e}_{mn}^{BC1} v_2 + \cdots \\ &\qquad + \mathrm{e}_{mn}^{BC(n-3)} v_{n-2} + \mathrm{e}_{mn}^{BC(n-2)} v_{n-1} + \mathrm{e}_{mn}^{BC(n-1)} v_n = \varphi(-1), \end{aligned}$$

$$\begin{aligned} &\mathrm{e}_{mn}^{BC(-2)} v_0 + \mathrm{e}_{mn}^{BC(-1)} v_1 + \mathrm{e}_{mn}^{BC0} v_2 + \cdots \\ &\qquad + \mathrm{e}_{mn}^{BC(n-4)} v_{n-2} + \mathrm{e}_{mn}^{BC(n-3)} v_{n-1} + \mathrm{e}_{mn}^{BC(n-2)} v_n = \varphi(-2), \end{aligned}$$

$$\begin{aligned} &\mathrm{e}_{mn}^{BC(-3)} v_0 + \mathrm{e}_{mn}^{BC(-2)} v_1 + \mathrm{e}_{mn}^{BC(-1)} v_2 + \cdots \\ &\qquad + \mathrm{e}_{mn}^{BC(n-5)} v_{n-2} + \mathrm{e}_{mn}^{BC(n-4)} v_{n-1} + \mathrm{e}_{mn}^{BC(n-3)} v_n = \varphi(-3), \\ &\qquad\qquad \vdots \end{aligned}$$

$$\begin{aligned} &\mathrm{e}_{mn}^{BC(-n+3)} v_0 + \mathrm{e}_{mn}^{BC(-n+4)} v_1 + \mathrm{e}_{mn}^{BC(-n+5)} v_2 + \cdots \\ &\qquad + \mathrm{e}_{mn}^{BC1} v_{n-2} + \mathrm{e}_{mn}^{BC2} v_{n-1} + \mathrm{e}_{mn}^{BC3} v_n = \varphi(-n+3), \end{aligned}$$

$$\begin{aligned} &\mathrm{e}_{mn}^{BC(-n+2)} v_0 + \mathrm{e}_{mn}^{BC(-n+3)} v_1 + \mathrm{e}_{mn}^{BC(-n+4)} v_2 + \cdots \\ &\qquad + \mathrm{e}_{mn}^{BC0} v_{n-2} + \mathrm{e}_{mn}^{BC1} v_{n-1} + \mathrm{e}_{mn}^{BC2} v_n = \varphi(-n+2), \end{aligned}$$

$$\mathrm{e}_{mn}^{BC(-n+1)}v_0+\mathrm{e}_{mn}^{BC(-n+2)}v_1+\mathrm{e}_{mn}^{BC(-n+3)}v_2+\cdots$$
$$+\mathrm{e}_{mn}^{BC(-1)}v_{n-2}+\mathrm{e}_{mn}^{BC0}v_{n-1}+\mathrm{e}_{mn}^{BC1}v_n=\varphi(-n+1),$$

$$\mathrm{e}_{mn}^{BC(-n)}v_0+\mathrm{e}_{mn}^{BC(-n+1)}v_1+\mathrm{e}_{mn}^{BC(-n+2)}v_2+\cdots$$
$$+\mathrm{e}_{mn}^{BC(-2)}v_{n-2}+\mathrm{e}_{mn}^{BC(-1)}v_{n-1}+\mathrm{e}_{mn}^{BC0}v_n=\varphi(-n).$$

By Definition 4.1, $\mathrm{e}_{mn}^{BCk}=I$ for $k\in\mathbb{Z}_{-n}^0$. So we have

$$v_0+\mathrm{e}_{mn}^{BC1}v_1+\mathrm{e}_{mn}^{BC2}v_2+\cdots+\mathrm{e}_{mn}^{BC(n-2)}v_{n-2}+\mathrm{e}_{mn}^{BC(n-1)}v_{n-1}+\mathrm{e}_{mn}^{BCn}v_n=\varphi(0),\quad (E_0)$$

$$v_0+v_1+\mathrm{e}_{mn}^{BC1}v_2+\cdots+\mathrm{e}_{mn}^{BC(n-3)}v_{n-2}+\mathrm{e}_{mn}^{BC(n-2)}v_{n-1}+\mathrm{e}_{mn}^{BC(n-1)}v_n=\varphi(-1),\quad (E_1)$$

$$v_0+v_1+v_2+\cdots+\mathrm{e}_{mn}^{BC(n-4)}v_{n-2}+\mathrm{e}_{mn}^{BC(n-3)}v_{n-1}+\mathrm{e}_{mn}^{BC(n-2)}v_n=\varphi(-2),\quad (E_2)$$

$$v_0+v_1+v_2+\cdots+\mathrm{e}_{mn}^{BC(n-5)}v_{n-2}+\mathrm{e}_{mn}^{BC(n-4)}v_{n-1}+\mathrm{e}_{mn}^{BC(n-3)}v_n=\varphi(-3),\quad (E_3)$$

$$\vdots$$

$$v_0+v_1+v_2+\cdots+\mathrm{e}_{mn}^{BC1}v_{n-2}+\mathrm{e}_{mn}^{BC2}v_{n-1}+\mathrm{e}_{mn}^{BC3}v_n=\varphi(-n+3),\qquad (E_{n-3})$$

$$v_0+v_1+v_2+\cdots+v_{n-2}+\mathrm{e}_{mn}^{BC1}v_{n-1}+\mathrm{e}_{mn}^{BC2}v_n=\varphi(-n+2),\qquad (E_{n-2})$$

$$v_0+v_1+v_2+\cdots+v_{n-2}+v_{n-1}+\mathrm{e}_{mn}^{BC1}v_n=\varphi(-n+1),\qquad (E_{n-1})$$

$$v_0+v_1+v_2+\cdots+v_{n-2}+v_{n-1}+v_n=\varphi(-n).\qquad (E_n)$$

Subtracting the neighbouring equations ($E_{n-1}-E_n, E_{n-2}-E_{n-1},\dots,E_0-E_1$), we get

$$(\mathrm{e}_{mn}^{BC1}-I)v_n=\varphi(-n+1)-\varphi(-n),\qquad (E_{n-1}-E_n)$$

$$(\mathrm{e}_{mn}^{BC1}-I)v_{n-1}+(\mathrm{e}_{mn}^{BC2}-\mathrm{e}_{mn}^{BC1})v_n=\varphi(-n+2)-\varphi(-n+1),\quad (E_{n-2}-E_{n-1})$$

$$(\mathrm{e}_{mn}^{BC1}-I)v_{n-2}+(\mathrm{e}_{mn}^{BC2}-\mathrm{e}_{mn}^{BC1})v_{n-1}$$
$$+(\mathrm{e}_{mn}^{BC3}-\mathrm{e}_{mn}^{BC2})v_n=\varphi(-n+3)-\varphi(-n+2),\quad (E_{n-3}-E_{n-2})$$

$$\vdots$$

$$(\mathrm{e}_{mn}^{BC1}-I)v_3+(\mathrm{e}_{mn}^{BC2}-\mathrm{e}_{mn}^{BC1})v_4+\cdots$$
$$+(\mathrm{e}_{mn}^{BC(n-4)}-\mathrm{e}_{mn}^{BC(n-5)})v_{n-2}+(\mathrm{e}_{mn}^{BC(n-3)}-\mathrm{e}_{mn}^{BC(n-4)})v_{n-1}\qquad (E_2-E_3)$$
$$+(\mathrm{e}_{mn}^{BC(n-2)}-\mathrm{e}_{mn}^{BC(n-3)})v_n=\varphi(-2)-\varphi(-3),$$

$$(\mathrm{e}_{mn}^{BC1}-I)v_2+(\mathrm{e}_{mn}^{BC2}-\mathrm{e}_{mn}^{BC1})v_3+\cdots$$
$$+(\mathrm{e}_{mn}^{BC(n-3)}-\mathrm{e}_{mn}^{BC(n-4)})v_{n-2}+(\mathrm{e}_{mn}^{BC(n-2)}-\mathrm{e}_{mn}^{BC(n-3)})v_{n-1}\qquad (E_1-E_2)$$
$$+(\mathrm{e}_{mn}^{BC(n-1)}-\mathrm{e}_{mn}^{BC(n-2)})v_n=\varphi(-1)-\varphi(-2),$$

$$
\begin{aligned}
&(\mathrm{e}_{mn}^{BC1} - I)v_1 + (\mathrm{e}_{mn}^{BC2} - \mathrm{e}_{mn}^{BC1})v_2 + \cdots \\
&\quad + (\mathrm{e}_{mn}^{BC(n-2)} - \mathrm{e}_{mn}^{BC(n-3)})v_{n-2} + (\mathrm{e}_{mn}^{BC(n-1)} - \mathrm{e}_{mn}^{BC(n-2)})v_{n-1} \\
&\qquad + (\mathrm{e}_{mn}^{BCn} - \mathrm{e}_{mn}^{BC(n-1)})v_n = \varphi(0) - \varphi(-1).
\end{aligned} \tag{$E_0 - E_1$}
$$

By Definition 4.1, we have

$$
\mathrm{e}_{mn}^{BC1} - I = I + B + C - I = B + C,
$$

and, from the foregoing equations, we get

$$
\begin{aligned}
v_n &= (B+C)^{-1}\Delta\varphi(-n), \\
v_{n-1} &= (B+C)^{-1}\left[\Delta\varphi(-n+1) - (\mathrm{e}_{mn}^{BC2} - \mathrm{e}_{mn}^{BC1})v_n\right], \\
v_{n-2} &= (B+C)^{-1}\left[\Delta\varphi(-n+2) - (\mathrm{e}_{mn}^{BC2} - \mathrm{e}_{mn}^{BC1})v_{n-1} - (\mathrm{e}_{mn}^{BC3} - \mathrm{e}_{mn}^{BC2})v_n\right], \\
&\ \vdots \\
v_3 &= (B+C)^{-1}\Big[\Delta\varphi(-3) - (\mathrm{e}_{mn}^{BC2} - \mathrm{e}_{mn}^{BC1})v_4 - \cdots - (\mathrm{e}_{mn}^{BC(n-4)} - \mathrm{e}_{mn}^{BC(n-5)})v_{n-2} \\
&\quad - (\mathrm{e}_{mn}^{BC(n-3)} - \mathrm{e}_{mn}^{BC(n-4)})v_{n-1} - (\mathrm{e}_{mn}^{BC(n-2)} - \mathrm{e}_{mn}^{BC(n-3)})v_n\Big], \\
v_2 &= (B+C)^{-1}\Big[\Delta\varphi(-2) - (\mathrm{e}_{mn}^{BC2} - \mathrm{e}_{mn}^{BC1})v_3 - \cdots - (\mathrm{e}_{mn}^{BC(n-3)} - \mathrm{e}_{mn}^{BC(n-4)})v_{n-2} \\
&\quad - (\mathrm{e}_{mn}^{BC(n-2)} - \mathrm{e}_{mn}^{BC(n-3)})v_{n-1} - (\mathrm{e}_{mn}^{BC(n-1)} - \mathrm{e}_{mn}^{BC(n-2)})v_n\Big], \\
v_1 &= (B+C)^{-1}\Big[\Delta\varphi(-1) - (\mathrm{e}_{mn}^{BC2} - \mathrm{e}_{mn}^{BC1})v_2 - \cdots - (\mathrm{e}_{mn}^{BC(n-2)} - \mathrm{e}_{mn}^{BC(n-3)})v_{n-2} \\
&\quad - (\mathrm{e}_{mn}^{BC(n-1)} - \mathrm{e}_{mn}^{BC(n-2)})v_{n-1} - (\mathrm{e}_{mn}^{BCn} - \mathrm{e}_{mn}^{BC(n-1)})v_n\Big].
\end{aligned}
$$

The above formulas can be shortened to

$$
\begin{aligned}
v_\ell &= (B+C)^{-1}\left[\Delta\varphi(-\ell) - \sum_{t=1}^{n-\ell}(\mathrm{e}_{mn}^{BC(t+1)} - \mathrm{e}_{mn}^{BCt})v_{t+\ell}\right] \\
&= (B+C)^{-1}\left[\Delta\varphi(-\ell) - \sum_{t=1}^{n-\ell}\Delta\mathrm{e}_{mn}^{BCt}v_{t+\ell}\right]
\end{aligned}
$$

where $\ell \in \mathbb{Z}_1^n$. Finally, from equation (E_n), we get

$$
v_0 = \varphi(-n) - \sum_{s=1}^{n} v_s.
$$

Theorem 5.1 is proved. □

Now we express the solution of the homogeneous Cauchy problem by $\tilde{\mathrm{e}}_{mn}^{BC(k)}$. In this case, the condition $\det(B+C) \neq 0$ is not necessary.

Theorem 5.2. *Let B, C be constant $r\times r$ matrices with $BC = CB$ and let $m, n \in \mathbb{N}$, $m < n$ be fixed integers. Then, the solution of the initial Cauchy problem* (5.3), (5.4) *can be expressed in the form*

$$x(k) = \sum_{j=0}^{n} \tilde{\mathrm{e}}_{mn}^{BC(k+j)} w_j, \tag{5.7}$$

where $k \in \mathbb{Z}_{-n}^{\infty}$ and

$$\begin{aligned} w_\ell &= \Delta\varphi(-\ell-1) - \Delta\tilde{\mathrm{e}}_{mn}^{BC(-\ell+n-1)}\varphi(-n) \\ &\quad - \sum_{s=-n}^{-\ell-m-2} \Delta\tilde{\mathrm{e}}_{mn}^{BC(-\ell-s-2)}\Delta\varphi(s), \quad \ell \in \mathbb{Z}_0^{n-m-1}, \\ w_\ell &= \Delta\varphi(-\ell-1), \quad \ell \in \mathbb{Z}_{n-m}^{n-1}, \\ w_n &= \varphi(-n). \end{aligned}$$

Proof. We are going to find the solution of problem (5.3), (5.4) in the form

$$x(k) = \sum_{j=0}^{n} \tilde{\mathrm{e}}_{mn}^{BC(k+j)} w_j, \quad k \geq 0 \tag{5.8}$$

with unknown constant vectors w_j. Because of linearity (taking into account that k varies), we have

$$\Delta x(k) = \Delta \sum_{j=0}^{n} \tilde{\mathrm{e}}_{mn}^{BC(k+j)} w_j = \sum_{j=0}^{n} \Delta \left[\tilde{\mathrm{e}}_{mn}^{BC(k+j)} w_j\right] = \sum_{j=0}^{n} \Delta \left[\tilde{\mathrm{e}}_{mn}^{BC(k+j)}\right] w_j.$$

We use formula (4.28):

$$\begin{aligned} \Delta x(k) &= \sum_{j=0}^{n} \left(B\tilde{\mathrm{e}}_{mn}^{BC(k-m+j)} + C\tilde{\mathrm{e}}_{mn}^{BC(k-n+j)}\right) w_j \\ &= B\sum_{j=0}^{n} \tilde{\mathrm{e}}_{mn}^{BC(k-m+j)} w_j + C\sum_{j=0}^{n} \tilde{\mathrm{e}}_{mn}^{BC(k-n+j)} w_j \\ &= Bx(k-m) + Cx(k-n). \end{aligned}$$

Now we conclude that, for any w_j and $k \in \mathbb{Z}_0^{\infty}$, the equation $\Delta x(k) = Bx(k-m) + Cx(k-n)$ holds. We will try to satisfy initial conditions (5.4). By (5.8), we have, for $k \in \mathbb{Z}_{-n}^{0}$,

$$\begin{aligned} &\tilde{\mathrm{e}}_{mn}^{BC0} w_0 + \tilde{\mathrm{e}}_{mn}^{BC1} w_1 + \tilde{\mathrm{e}}_{mn}^{BC2} w_2 + \cdots \\ &\qquad + \tilde{\mathrm{e}}_{mn}^{BC(n-2)} w_{n-2} + \tilde{\mathrm{e}}_{mn}^{BC(n-1)} w_{n-1} + \tilde{\mathrm{e}}_{mn}^{BCn} w_n = \varphi(0), \end{aligned}$$

$$\tilde{\mathrm{e}}_{mn}^{BC(-1)}w_0 + \tilde{\mathrm{e}}_{mn}^{BC0}w_1 + \tilde{\mathrm{e}}_{mn}^{BC1}w_2 + \cdots$$
$$+\tilde{\mathrm{e}}_{mn}^{BC(n-3)}w_{n-2} + \tilde{\mathrm{e}}_{mn}^{BC(n-2)}w_{n-1} + \tilde{\mathrm{e}}_{mn}^{BC(n-1)}w_n = \varphi(-1),$$

$$\tilde{\mathrm{e}}_{mn}^{BC(-2)}w_0 + \tilde{\mathrm{e}}_{mn}^{BC(-1)}w_1 + \tilde{\mathrm{e}}_{mn}^{BC0}w_2 + \cdots$$
$$+\tilde{\mathrm{e}}_{mn}^{BC(n-4)}w_{n-2} + \tilde{\mathrm{e}}_{mn}^{BC(n-3)}w_{n-1} + \tilde{\mathrm{e}}_{mn}^{BC(n-2)}w_n = \varphi(-2),$$

$$\tilde{\mathrm{e}}_{mn}^{BC(-3)}w_0 + \tilde{\mathrm{e}}_{mn}^{BC(-2)}w_1 + \tilde{\mathrm{e}}_{mn}^{BC(-1)}w_2 + \cdots$$
$$+\tilde{\mathrm{e}}_{mn}^{BC(n-5)}w_{n-2} + \tilde{\mathrm{e}}_{mn}^{BC(n-4)}w_{n-1} + \tilde{\mathrm{e}}_{mn}^{BC(n-3)}w_n = \varphi(-3),$$
$$\vdots$$
$$\tilde{\mathrm{e}}_{mn}^{BC(-n+3)}w_0 + \tilde{\mathrm{e}}_{mn}^{BC(-n+4)}w_1 + \tilde{\mathrm{e}}_{mn}^{BC(-n+5)}w_2 + \cdots$$
$$+\tilde{\mathrm{e}}_{mn}^{BC1}w_{n-2} + \tilde{\mathrm{e}}_{mn}^{BC2}w_{n-1} + \tilde{\mathrm{e}}_{mn}^{BC3}w_n = \varphi(-n+3),$$

$$\tilde{\mathrm{e}}_{mn}^{BC(-n+2)}w_0 + \tilde{\mathrm{e}}_{mn}^{BC(-n+3)}w_1 + \tilde{\mathrm{e}}_{mn}^{BC(-n+4)}w_2 + \cdots$$
$$+\tilde{\mathrm{e}}_{mn}^{BC0}w_{n-2} + \tilde{\mathrm{e}}_{mn}^{BC1}w_{n-1} + \tilde{\mathrm{e}}_{mn}^{BC2}w_n = \varphi(-n+2),$$

$$\tilde{\mathrm{e}}_{mn}^{BC(-n+1)}w_0 + \tilde{\mathrm{e}}_{mn}^{BC(-n+2)}w_1 + \tilde{\mathrm{e}}_{mn}^{BC(-n+3)}w_2 + \cdots$$
$$+\tilde{\mathrm{e}}_{mn}^{BC(-1)}w_{n-2} + \tilde{\mathrm{e}}_{mn}^{BC0}w_{n-1} + \tilde{\mathrm{e}}_{mn}^{BC1}w_n = \varphi(-n+1),$$

$$\tilde{\mathrm{e}}_{mn}^{BC(-n)}w_0 + \tilde{\mathrm{e}}_{mn}^{BC(-n+1)}w_1 + \tilde{\mathrm{e}}_{mn}^{BC(-n+2)}w_2 + \cdots$$
$$+\tilde{\mathrm{e}}_{mn}^{BC(-2)}w_{n-2} + \tilde{\mathrm{e}}_{mn}^{BC(-1)}w_{n-1} + \tilde{\mathrm{e}}_{mn}^{BC0}w_n = \varphi(-n).$$

By Definition 4.5, we have $\tilde{\mathrm{e}}_{mn}^{BCk} = \Theta$ for $k \in \mathbb{Z}_{-\infty}^{-1}$ and $\tilde{\mathrm{e}}_{mn}^{BCk} = I$ for $k \in \mathbb{Z}_0^m$. Thus, we have

$$w_0 + w_1 + w_2 + \cdots + w_m + \tilde{\mathrm{e}}_{mn}^{BC(m+1)}w_{m+1} + \tilde{\mathrm{e}}_{mn}^{BC(m+2)}w_{m+2} + \cdots$$
$$+\tilde{\mathrm{e}}_{mn}^{BC(n-2)}w_{n-2} + \tilde{\mathrm{e}}_{mn}^{BC(n-1)}w_{n-1} + \tilde{\mathrm{e}}_{mn}^{BCn}w_n = \varphi(0), \qquad (\tilde{E}_0)$$

$$w_1 + w_2 + w_3 + \cdots + w_{m+1} + \tilde{\mathrm{e}}_{mn}^{BC(m+1)}w_{m+2} + \tilde{\mathrm{e}}_{mn}^{BC(m+2)}w_{m+3} + \cdots$$
$$+\tilde{\mathrm{e}}_{mn}^{BC(n-3)}w_{n-2} + \tilde{\mathrm{e}}_{mn}^{BC(n-2)}w_{n-1} + \tilde{\mathrm{e}}_{mn}^{BC(n-1)}w_n = \varphi(-1), \qquad (\tilde{E}_1)$$

$$w_2 + w_3 + w_4 + \cdots + w_{m+2} + \tilde{\mathrm{e}}_{mn}^{BC(m+1)}w_{m+3} + \tilde{\mathrm{e}}_{mn}^{BC(m+2)}w_{m+4} + \cdots$$
$$+\tilde{\mathrm{e}}_{mn}^{BC(n-4)}w_{n-2} + \tilde{\mathrm{e}}_{mn}^{BC(n-3)}w_{n-1} + \tilde{\mathrm{e}}_{mn}^{BC(n-2)}w_n = \varphi(-2), \qquad (\tilde{E}_2)$$
$$\vdots$$
$$w_{n-m-2} + w_{n-m-1} + w_{n-m} + \cdots$$
$$+w_{n-2} + \tilde{\mathrm{e}}_{mn}^{BC(m+1)}w_{n-1} + \tilde{\mathrm{e}}_{mn}^{BC(m+2)}w_n = \varphi(-n+m+2), \qquad (\tilde{E}_{n-m-2})$$

$$w_{n-m-1} + w_{n-m} + w_{n-m+1} + \cdots + w_{n-2} + w_{n-1} + \tilde{\mathrm{e}}_{mn}^{BC(m+1)} w_n = \varphi(-n+m+1), \qquad (\tilde{E}_{n-m-1})$$

$$w_{n-m} + w_{n-m+1} + w_{n-m+2} + \cdots + w_{n-2} + w_{n-1} + w_n = \varphi(-n+m), \qquad (\tilde{E}_{n-m})$$

$$\vdots$$

$$w_{n-2} + w_{n-1} + w_n \varphi(-n+2), \qquad (\tilde{E}_{n-2})$$

$$w_{n-1} + w_n = \varphi(-n+1), \qquad (\tilde{E}_{n-1})$$

$$w_n = \varphi(-n). \qquad (\tilde{E}_n)$$

We see directly that $w_n = \varphi(-n)$. Substracting the neighbouring equations ($\tilde{E}_{n-1} - \tilde{E}_n$, $\tilde{E}_{n-2} - \tilde{E}_{n-1}, \dots, \tilde{E}_{n-m} - \tilde{E}_{n-m+1}$), we immediately get the formulas for w_{n-1}, $w_{n-2}, \dots, w_{n-m}$:

$$w_{n-1} = \varphi(-n+1) - \varphi(-n) = \Delta\varphi(-n), \qquad (\tilde{E}_{n-1} - \tilde{E}_n)$$

$$w_{n-2} = \varphi(-n+2) - \varphi(-n+1) = \Delta\varphi(-n+1), \qquad (\tilde{E}_{n-2} - \tilde{E}_{n-1})$$

$$\vdots$$

$$w_{n-m+1} = \varphi(-n+m-1) - \varphi(-n+m-2) = \Delta\varphi(-n+m-2), \qquad (\tilde{E}_{n-m+1} - \tilde{E}_{n-m+2})$$

$$w_{n-m} = \varphi(-n+m) - \varphi(-n+m-1) = \Delta\varphi(-n+m-1), \quad (\tilde{E}_{n-m} - \tilde{E}_{n-m+1})$$

Further, substracting the neighbouring equations ($\tilde{E}_{n-m-1} - \tilde{E}_{n-m}$, $\tilde{E}_{n-m-2} - \tilde{E}_{n-m-1}$, $\dots, \tilde{E}_0 - \tilde{E}_1$), we get

$$w_{n-m-1} + \left[\tilde{\mathrm{e}}_{mn}^{BC(m+1)} - I\right] w_n = \varphi(-n+m+1) - \varphi(-n+m) \quad (\tilde{E}_{n-m-1} - \tilde{E}_{n-m})$$

$$\Rightarrow \quad w_{n-m-1} = \Delta\varphi(-n+m) - \left[\tilde{\mathrm{e}}_{mn}^{BC(m+1)} - I\right] \varphi(-n),$$

$$w_{n-m-2} + \left[\tilde{\mathrm{e}}_{mn}^{BC(m+1)} - I\right] w_{n-1} + \left[\tilde{\mathrm{e}}_{mn}^{BC(m+2)} - \tilde{\mathrm{e}}_{mn}^{BC(m+1)}\right] w_n = \varphi(-n+m+2) - \varphi(-n+m+1) \qquad (\tilde{E}_{n-m-2} - \tilde{E}_{n-m-1})$$

$$\Rightarrow \quad w_{n-m-2} = \Delta\varphi(-n+m+1) - \left[\tilde{\mathrm{e}}_{mn}^{BC(m+2)} - \tilde{\mathrm{e}}_{mn}^{BC(m+1)}\right] \varphi(-n) - \left[\tilde{\mathrm{e}}_{mn}^{BC(m+1)} - I\right] \Delta\varphi(-n),$$

$$\vdots$$

$$w_2 + \left[\tilde{\mathrm{e}}_{mn}^{BC(m+1)} - I\right] w_{m+3} + \left[\tilde{\mathrm{e}}_{mn}^{BC(m+2)} - \tilde{\mathrm{e}}_{mn}^{BC(m+1)}\right] w_{m+4} + \cdots + \left[\tilde{\mathrm{e}}_{mn}^{BC(n-4)} - \tilde{\mathrm{e}}_{mn}^{BC(n-5)}\right] w_{n-2} + \left[\tilde{\mathrm{e}}_{mn}^{BC(n-3)} - \tilde{\mathrm{e}}_{mn}^{BC(n-4)}\right] w_{n-1} + \left[\tilde{\mathrm{e}}_{mn}^{BC(n-2)} - \tilde{\mathrm{e}}_{mn}^{BC(n-3)}\right] w_n = \varphi(-2) - \varphi(-3) \qquad (\tilde{E}_2 - \tilde{E}_3)$$

$$
\begin{aligned}
\Rightarrow \quad w_2 = \Delta\varphi(-3) &- \left[\tilde{e}_{mn}^{BC(n-2)} - \tilde{e}_{mn}^{BC(n-3)}\right]\varphi(-n)\\
&- \left[\tilde{e}_{mn}^{BC(n-3)} - \tilde{e}_{mn}^{BC(n-4)}\right]\Delta\varphi(-n) - \left[\tilde{e}_{mn}^{BC(n-4)} - \tilde{e}_{mn}^{BC(n-5)}\right]\Delta\varphi(-n+1)\\
&- \cdots - \left[\tilde{e}_{mn}^{BC(m+2)} - \tilde{e}_{mn}^{BC(m+1)}\right]\Delta\varphi(-m-5)\\
&- \left[\tilde{e}_{mn}^{BC(m+1)} - I\right]\Delta\varphi(-m-4),
\end{aligned}
$$

$$
\begin{aligned}
w_1 &+ \left[\tilde{e}_{mn}^{BC(m+1)} - I\right] w_{m+2} + \left[\tilde{e}_{mn}^{BC(m+2)} - \tilde{e}_{mn}^{BC(m+1)}\right] w_{m+3} + \cdots\\
&+ \left[\tilde{e}_{mn}^{BC(n-3)} - \tilde{e}_{mn}^{BC(n-4)}\right] w_{n-2} + \left[\tilde{e}_{mn}^{BC(n-2)} - \tilde{e}_{mn}^{BC(n-3)}\right] w_{n-1} \qquad (\tilde{E}_1 - \tilde{E}_2)\\
&+ \left[\tilde{e}_{mn}^{BC(n-1)} - \tilde{e}_{mn}^{BC(n-2)}\right] w_n = \varphi(-1) - \varphi(-2)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad w_1 = \Delta\varphi(-2) &- \left[\tilde{e}_{mn}^{BC(n-1)} - \tilde{e}_{mn}^{BC(n-2)}\right]\varphi(-n)\\
&- \left[\tilde{e}_{mn}^{BC(n-2)} - \tilde{e}_{mn}^{BC(n-3)}\right]\Delta\varphi(-n) - \left[\tilde{e}_{mn}^{BC(n-3)} - \tilde{e}_{mn}^{BC(n-4)}\right]\Delta\varphi(-n+1)\\
&- \cdots - \left[\tilde{e}_{mn}^{BC(m+2)} - \tilde{e}_{mn}^{BC(m+1)}\right]\Delta\varphi(-m-4)\\
&- \left[\tilde{e}_{mn}^{BC(m+1)} - I\right]\Delta\varphi(-m-3),
\end{aligned}
$$

$$
\begin{aligned}
w_0 &+ \left[\tilde{e}_{mn}^{BC(m+1)} - I\right] w_{m+1} + \left[\tilde{e}_{mn}^{BC(m+2)} - \tilde{e}_{mn}^{BC(m+1)}\right] w_{m+2} + \cdots\\
&+ \left[\tilde{e}_{mn}^{BC(n-2)} - \tilde{e}_{mn}^{BC(n-3)}\right] w_{n-2} + \left[\tilde{e}_{mn}^{BC(n-1)} - \tilde{e}_{mn}^{BC(n-2)}\right] w_{n-1} \qquad (\tilde{E}_0 - \tilde{E}_1)\\
&+ \left[\tilde{e}_{mn}^{BCn} - \tilde{e}_{mn}^{BC(n-1)}\right] w_n = \varphi(0) - \varphi(-1)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad w_0 = \Delta\varphi(-1) &- \left[\tilde{e}_{mn}^{BCn} - \tilde{e}_{mn}^{BC(n-1)}\right]\varphi(-n)\\
&- \left[\tilde{e}_{mn}^{BC(n-1)} - \tilde{e}_{mn}^{BC(n-2)}\right]\Delta\varphi(-n) - \left[\tilde{e}_{mn}^{BC(n-2)} - \tilde{e}_{mn}^{BC(n-3)}\right]\Delta\varphi(-n+1)\\
&- \cdots - \left[\tilde{e}_{mn}^{BC(m+2)} - \tilde{e}_{mn}^{BC(m+1)}\right]\Delta\varphi(-m-3)\\
&- \left[\tilde{e}_{mn}^{BC(m+1)} - I\right]\Delta\varphi(-m-2).
\end{aligned}
$$

The previous formulas can be written as

$$
\begin{aligned}
w_\ell &= \Delta\varphi(-\ell-1) - \left[\tilde{e}_{mn}^{BC(-\ell+n)} - \tilde{e}_{mn}^{BC(-\ell+n-1)}\right]\varphi(-n)\\
&\qquad - \sum_{s=-n}^{-\ell-m-2} \left[\tilde{e}_{mn}^{BC(-\ell-s-1)} - \tilde{e}_{mn}^{BC(-\ell-s-2)}\right]\Delta\varphi(s),\\
&= \Delta\varphi(-\ell-1) - \Delta\tilde{e}_{mn}^{BC(-\ell+n-1)}\varphi(-n) - \sum_{s=-n}^{-\ell-m-2} \Delta\tilde{e}_{mn}^{BC(-\ell-s-2)}\Delta\varphi(s),\\
&\qquad \ell \in \mathbb{Z}_0^{n-m-1},\\
w_\ell &= \Delta\varphi(-\ell-1), \quad \ell \in \mathbb{Z}_{n-m}^{n-1},\\
w_n &= \varphi(-n).
\end{aligned}
$$

Theorem 5.2 is proved. □

5.2 Nonhomogeneous Initial Problem

We consider a nonhomogeneous initial Cauchy problem

$$\Delta x(k) = Bx(k-m) + Cx(k-n) + f(k), \quad k \in \mathbb{Z}_0^\infty, \tag{5.9}$$

$$x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-n}^0, \tag{5.10}$$

By the theory of linear equations, we can obtain its solution as the sum of a solution of adjoint homogeneous problem (5.3), (5.4) (satisfying the same initial data) and a particular solution of (5.9) being zero on an initial interval.

Let us, therefore, find such a particular solution $x_p(k)$, $k \in \mathbb{Z}_{-n}^\infty$ of the initial Cauchy problem

$$\Delta x(k) = Bx(k-m) + Cx(k-n) + f(k), \quad k \in \mathbb{Z}_0^\infty, \tag{5.11}$$

$$x(k) = 0, \quad k \in \mathbb{Z}_{-n}^0. \tag{5.12}$$

Theorem 5.3. *The solution $x = x_p(k)$ of the initial Cauchy problem* (5.11), (5.12) *can be represented on $\mathbb{Z}_{-n}^\infty$ in the form*

$$x_p(k) = \sum_{\ell=1}^{k} \tilde{\mathrm{e}}_{mn}^{BC(k-\ell)} f(\ell - 1), \quad k \in \mathbb{Z}_0^\infty. \tag{5.13}$$

Proof. We are going to find a particular solution $x_p(k)$ of problem (5.11), (5.12) in the form (5.13). We substitute (5.13) into (5.11). Then, we get

$$\begin{aligned} \Delta \left[\sum_{\ell=1}^{k} \tilde{\mathrm{e}}_{mn}^{BC(k-\ell)} f(\ell-1) \right] &= B \sum_{\ell=1}^{k-m} \tilde{\mathrm{e}}_{mn}^{BC(k-m-\ell)} f(\ell-1) \\ &\quad + C \sum_{\ell=1}^{k-n} \tilde{\mathrm{e}}_{mn}^{BC(k-n-\ell)} f(j-1) + f(k). \end{aligned} \tag{5.14}$$

We modify the left-hand side of equation (5.14). With the aid of (1.9), we obtain

$$\Delta \left[\sum_{\ell=1}^{k} \tilde{\mathrm{e}}_{mn}^{BC(k-\ell)} f(j-1) \right] = \tilde{\mathrm{e}}_{mn}^{BC((k+1)-(k+1))} f(k+1-1) + \sum_{\ell=1}^{k} \Delta \left[\tilde{\mathrm{e}}_{mn}^{BC(k-\ell)} f(j-1) \right],$$

and, applying Theorem 4.8, we get

$$\begin{aligned} \Delta \left[\sum_{\ell=1}^{k} \tilde{\mathrm{e}}_{mn}^{BC(k-\ell)} f(j-1) \right] &= \tilde{\mathrm{e}}_{mn}^{BC0} f(k) + \sum_{\ell=1}^{k} \left[B\tilde{\mathrm{e}}_{mn}^{BC(k-m-\ell)} + C\tilde{\mathrm{e}}_{mn}^{BC(k-n-\ell)} \right] f(j-1) \\ &= \tilde{\mathrm{e}}_{mn}^{BC0} f(k) + B \left[\sum_{\ell=1}^{k-m} \tilde{\mathrm{e}}_{mn}^{BC(k-m-\ell)} f(j-1) \right. \\ &\quad \left. + \sum_{\ell=k-m+1}^{k} \tilde{\mathrm{e}}_{mn}^{BC(k-m-\ell)} f(j-1) \right] \end{aligned}$$

$$+ C\left[\sum_{\ell=1}^{k-n}\tilde{\mathrm{e}}_{mn}^{BC(k-n-\ell)}f(j-1)+\sum_{\ell=k-n+1}^{k}\tilde{\mathrm{e}}_{mn}^{BC(k-n-\ell)}f(j-1)\right].$$

By Definition 4.5, we have $\tilde{\mathrm{e}}_{mn}^{BC0} = I$, $\tilde{\mathrm{e}}_{mn}^{BC(k-m-\ell)} = \Theta$ for $\ell \in \mathbb{Z}_{k-m+1}^{k}$, and $\tilde{\mathrm{e}}_{mn}^{BC(k-n-\ell)} = \Theta$ for $\ell \in \mathbb{Z}_{k-n+1}^{k}$. Thus, we get

$$\Delta\left[\sum_{\ell=1}^{k}\tilde{\mathrm{e}}_{mn}^{BC(k-\ell)}f(j-1)\right] = f(k)+B\sum_{\ell=1}^{k-m}\tilde{\mathrm{e}}_{mn}^{BC(k-m-\ell)}f(j-1)+C\sum_{\ell=1}^{k-n}\tilde{\mathrm{e}}_{mn}^{BC(k-n-\ell)}f(j-1)$$

and (5.14) holds. □

Combining the results of Theorems 5.1, 5.2 and 5.3, we get immediately the following two theorems, which describe the solution of (5.9), (5.10). The first theorem uses both discrete matrix delayed exponentials and the second one uses only the discrete matrix delayed exponential $\tilde{\mathrm{e}}_{mn}^{BCk}$.

Theorem 5.4. *Let B, C be constant $r \times r$ matrices such that*

$$BC = CB, \quad \det(B + C) \neq 0,$$

and let $m, n \in \mathbb{N}$, $m < n$ be fixed integers. Then, the solution of the initial Cauchy problem (5.9), (5.10) *can be expressed in the form:*

$$x(k) = \sum_{j=0}^{n}\mathrm{e}_{mn}^{BC(k+j)}v_j + \sum_{\ell=1}^{k}\tilde{\mathrm{e}}_{mn}^{BC(k-\ell)}f(\ell-1)$$

where $k \in \mathbb{Z}_{-n}^{\infty}$ and

$$v_0 = \varphi(-n) - \sum_{s=1}^{n} v_s,$$

$$v_\ell = (B + C)^{-1}\left[\Delta\varphi(-\ell) - \sum_{t=1}^{n-\ell}\Delta\mathrm{e}_{mn}^{BCt}v_{t+\ell}\right], \quad \ell \in \mathbb{Z}_1^n.$$

Theorem 5.5. *Let B, C be constant $r\times r$ matrices with $BC = CB$ and let $m, n \in \mathbb{N}$, $m < n$ be fixed integers. Then, the solution of the initial Cauchy problem* (5.9), (5.10) *can be expressed in the form:*

$$x(k) = \sum_{j=0}^{n}\tilde{\mathrm{e}}_{mn}^{BC(k+j)}w_j + \sum_{\ell=1}^{k}\tilde{\mathrm{e}}_{mn}^{BC(k-\ell)}f(\ell-1)$$

where $k \in \mathbb{Z}_{-n}^{\infty}$ and

$$w_\ell = \Delta\varphi(-\ell-1) - \Delta\tilde{\mathrm{e}}_{mn}^{BC(-\ell+n-1)}\varphi(-n) - \sum_{s=-n}^{-\ell-m-2}\Delta\tilde{\mathrm{e}}_{mn}^{BC(-\ell-s-2)}\Delta\varphi(s),$$
$$\ell \in \mathbb{Z}_0^{n-m-1},$$

$$w_\ell = \Delta\varphi(-\ell-1), \quad \ell \in \mathbb{Z}_{n-m}^{n-1},$$

$$w_n = \varphi(-n).$$

5.3 Examples

Below, we show four examples to demonstrate the results achieved.

Example 5.6. Let us represent the solution of the scalar ($r = 1$) problem (5.3), (5.4) where we put $m = 2$, $n = 3$, $B = b$, $C = c$, $\varphi(-3) = 1$, $\varphi(-2) = 2$, $\varphi(-1) = 3$, $\varphi(0) = 4$, using Theorem 5.1. We get

$$\Delta x(k) = bx(k-2) + cx(k-3), \quad k \in \mathbb{Z}_0^\infty, \tag{5.15}$$

$$\begin{aligned} x(-3) &= \varphi(-3) = 1, \\ x(-2) &= \varphi(-2) = 2, \\ x(-1) &= \varphi(-1) = 3, \\ x(0) &= \varphi(0) = 4. \end{aligned} \tag{5.16}$$

By Theorem 5.1, the solution of problem (5.15), (5.16) is

$$x(k) = \sum_{j=0}^{3} \mathrm{e}_{2,3}^{bc(k+j)} v_j, \quad k \in \mathbb{Z}_{-3}^\infty$$

where

$$\begin{aligned} v_3 &= (b+c)^{-1}\left[\Delta\varphi(-3) - \sum_{t=1}^{0} \Delta \mathrm{e}_{2,3}^{bct} v_{t+3}\right] = (b+c)^{-1}, \\ v_2 &= (b+c)^{-1}\left[\Delta\varphi(-2) - \sum_{t=1}^{1} \Delta \mathrm{e}_{2,3}^{bct} v_{t+2}\right] = (b+c)^{-1}\left[\Delta\varphi(-2) - \Delta \mathrm{e}_{2,3}^{bc1} v_3\right] \\ &= (b+c)^{-1}\left[1 - \left(\mathrm{e}_{2,3}^{bc2} - \mathrm{e}_{2,3}^{bc1}\right)(b+c)^{-1}\right] = (b+c)^{-1}\left[1 - (b+c)(b+c)^{-1}\right] = 0, \\ v_1 &= (b+c)^{-1}\left[\Delta\varphi(-1) - \sum_{t=1}^{2} \Delta \mathrm{e}_{2,3}^{bct} v_{t+1}\right] = (b+c)^{-1}\left[\Delta\varphi(-2) - \Delta \mathrm{e}_{2,3}^{bc1} v_2 - \Delta \mathrm{e}_{2,3}^{bc2} v_3\right] \\ &= (b+c)^{-1}\left[1 - \left(\mathrm{e}_{2,3}^{bc3} - \mathrm{e}_{2,3}^{bc2}\right)(b+c)^{-1}\right] = (b+c)^{-1}\left[1 - (b+c)(b+c)^{-1}\right] = 0, \\ v_0 &= \varphi(-3) - \sum_{s=1}^{3} v_s = 1 - (b+c)^{-1}. \end{aligned}$$

Thus, we get

$$x(k) = \mathrm{e}_{2,3}^{bck}\left[1 - (b+c)^{-1}\right] + \mathrm{e}_{2,3}^{bc(k+3)}(b+c)^{-1}.$$

We give values of $x(k)$ for $k \in \mathbb{Z}_1^8$:

$$\begin{aligned} x(1) &= 4 + 2b + c, \\ x(2) &= 4 + 5b + 3c, \\ x(3) &= 4 + 9b + 6c, \end{aligned}$$

$$
\begin{aligned}
x(4) &= 4 + 13b + 10c + \ \ 2b^2 + \ \ \ bc,\\
x(5) &= 4 + 17b + 14c + \ \ 7b^2 + \ \ 6bc + \ \ c^2,\\
x(6) &= 4 + 21b + 18c + 16b^2 + 17bc + \ 4c^2,\\
x(7) &= 4 + 25b + 22c + 29b^2 + 36bc + 10c^2 + 2b^3 + \ b^2c,\\
x(8) &= 4 + 29b + 26c + 46b^2 + 63bc + 20c^2 + 9b^3 + 9b^2c + 2bc^2.
\end{aligned}
$$

Example 5.7. Let us represent the solution of the scalar ($r = 1$) problem (5.9), (5.10) where we put $m = 2$, $n = 3$, $B = b$, $C = c$, $\varphi(-3) = 1$, $\varphi(-2) = 2$, $\varphi(-1) = 3$, $\varphi(0) = 4$, $f(k) = k + 1$, using Theorem 5.5. Thus, we have

$$\Delta x(k) = bx(k-2) + cx(k-3) + k + 1, \quad k \in \mathbb{Z}_0^\infty, \tag{5.17}$$

$$
\begin{aligned}
x(-3) &= \varphi(-3) = 1,\\
x(-2) &= \varphi(-2) = 2,\\
x(-1) &= \varphi(-1) = 3,\\
x(0) &= \varphi(0) = 4.
\end{aligned} \tag{5.18}
$$

By Theorem 5.5, the solution of problem (5.17), (5.18) is

$$x(k) = \sum_{j=0}^{3} \tilde{\text{e}}_{2,3}^{bc(k+j)} w_j + \sum_{\ell=1}^{k} \tilde{\text{e}}_{2,3}^{bc(k-\ell)} \ell, \quad k \in \mathbb{Z}_{-3}^\infty$$

where

$$
\begin{aligned}
w_0 &= \Delta\varphi(-1) - \Delta\tilde{\text{e}}_{2,3}^{bc2}\varphi(-3) - \sum_{s=-3}^{-4} \Delta\tilde{\text{e}}_{2,3}^{bc(-s-2)}\Delta\varphi(s) = 1 - \left(\tilde{\text{e}}_{2,3}^{bc3} - \tilde{\text{e}}_{2,3}^{bc2}\right)\cdot 1\\
&= 1 - (1 + b - 1) = 1 - b,\\
w_1 &= \Delta\varphi(-2) = 1,\\
w_2 &= \Delta\varphi(-3) = 1,\\
w_3 &= \varphi(-3) = 1.
\end{aligned}
$$

Thus, we get

$$x(k) = \tilde{\text{e}}_{2,3}^{bck}(1-b) + \tilde{\text{e}}_{2,3}^{bc(k+1)} + \tilde{\text{e}}_{2,3}^{bc(k+2)} + \tilde{\text{e}}_{2,3}^{bc(k+3)} + \sum_{\ell=1}^{k} \tilde{\text{e}}_{2,3}^{bc(k-\ell)} \ell.$$

The first eight values of the homogeneous problem are given in the previous Example 5.6. Now, we compute the first eight values of a particular solution $x_p(k) = \sum_{\ell=1}^{k} \tilde{\text{e}}_{2,3}^{bc(k-\ell)}\ell$:

$$
\begin{aligned}
x_p(1) &= \ 1,\\
x_p(2) &= \ 3,
\end{aligned}
$$

$$
\begin{aligned}
x_p(3) &= 6,\\
x_p(4) &= 10 + b,\\
x_p(5) &= 15 + 4b + c,\\
x_p(6) &= 21 + 10b + 4c,\\
x_p(7) &= 28 + 20b + 10c + b^2,\\
x_p(8) &= 36 + 35b + 20c + 5b^2 + 2bc.
\end{aligned}
$$

Together, we get

$$
\begin{aligned}
x(1) &= 5 + 2b + c,\\
x(2) &= 7 + 5b + 3c,\\
x(3) &= 10 + 9b + 6c,\\
x(4) &= 14 + 14b + 10c + 2b^2 + bc,\\
x(5) &= 19 + 21b + 15c + 7b^2 + 6bc + c^2,\\
x(6) &= 25 + 31b + 22c + 16b^2 + 17bc + 4c^2,\\
x(7) &= 32 + 45b + 32c + 30b^2 + 36bc + 10c^2 + 2b^3 + b^2c,\\
x(8) &= 40 + 64b + 46c + 51b^2 + 65bc + 20c^2 + 9b^3 + 9b^2c + 2bc^2.
\end{aligned}
$$

Example 5.8. Let us represent the solution of the scalar ($r = 1$) problem (5.3), (5.4) where we put $m = 2$, $n = 3$, $B = b = 4$, $C = c = -1$, $\varphi(-3) = 1$, $\varphi(-2) = 2$, $\varphi(-1) = 3$, $\varphi(0) = 4$, using Theorem 5.1. Thus, we have

$$
\Delta x(k) = 4x(k-2) - x(k-3), \quad k \in \mathbb{Z}_0^\infty, \tag{5.19}
$$

$$
\begin{aligned}
x(-3) &= \varphi(-3) = 1,\\
x(-2) &= \varphi(-2) = 2,\\
x(-1) &= \varphi(-1) = 3,\\
x(0) &= \varphi(0) = 4.
\end{aligned} \tag{5.20}
$$

By Theorem 5.1, the solution of problem (5.19), (5.20) is

$$
x(k) = \sum_{j=0}^{3} \mathrm{e}_{2,3}^{bc(k+j)} v_j, \quad k \in \mathbb{Z}_{-3}^\infty
$$

where

$$
v_3 = (b+c)^{-1}\left[\Delta\varphi(-3) - \sum_{t=1}^{0} \Delta \mathrm{e}_{2,3}^{bct} v_{t+3}\right] = (b+c)^{-1} = \frac{1}{3},
$$

$$
v_2 = (b+c)^{-1}\left[\Delta\varphi(-2) - \sum_{t=1}^{1} \Delta \mathrm{e}_{2,3}^{bct} v_{t+2}\right] = (b+c)^{-1}\left[\Delta\varphi(-2) - \Delta \mathrm{e}_{2,3}^{bc1} v_3\right]
$$

$$= (b+c)^{-1}\left[1-\left(e_{2,3}^{bc2}-e_{2,3}^{bc1}\right)(b+c)^{-1}\right] = (b+c)^{-1}\left[1-(b+c)(b+c)^{-1}\right] = 0,$$

$$v_1 = (b+c)^{-1}\left[\Delta\varphi(-1) - \sum_{t=1}^{2} \Delta e_{2,3}^{bct} v_{t+1}\right] = (b+c)^{-1}\left[\Delta\varphi(-2) - \Delta e_{2,3}^{bc1} v_2 - \Delta e_{2,3}^{bc2} v_3\right]$$

$$= (b+c)^{-1}\left[1-\left(e_{2,3}^{bc3}-e_{2,3}^{bc2}\right)(b+c)^{-1}\right] = (b+c)^{-1}\left[1-(b+c)(b+c)^{-1}\right] = 0,$$

$$v_0 = \varphi(-3) - \sum_{s=1}^{3} v_s = 1-(b+c)^{-1} = \frac{2}{3}.$$

Thus, we get

$$x(k) = e_{2,3}^{bck}\cdot\frac{2}{3} + e_{2,3}^{bc(k+3)}\cdot\frac{1}{3},$$

and

$$x(1) = e_{2,3}^{bc1}\cdot\frac{2}{3} + e_{2,3}^{bc4}\cdot\frac{1}{3} = 4\cdot\frac{2}{3} + 25\cdot\frac{1}{3} = 11,$$

$$x(2) = e_{2,3}^{bc2}\cdot\frac{2}{3} + e_{2,3}^{bc5}\cdot\frac{1}{3} = 7\cdot\frac{2}{3} + 49\cdot\frac{1}{3} = 21,$$

$$x(3) = e_{2,3}^{bc3}\cdot\frac{2}{3} + e_{2,3}^{bc6}\cdot\frac{1}{3} = 10\cdot\frac{2}{3} + 82\cdot\frac{1}{3} = 34,$$

$$x(4) = e_{2,3}^{bc4}\cdot\frac{2}{3} + e_{2,3}^{bc7}\cdot\frac{1}{3} = 25\cdot\frac{2}{3} + 172\cdot\frac{1}{3} = 74,$$

$$x(5) = e_{2,3}^{bc5}\cdot\frac{2}{3} + e_{2,3}^{bc8}\cdot\frac{1}{3} = 49\cdot\frac{2}{3} + 343\cdot\frac{1}{3} = 147,$$

$$x(6) = e_{2,3}^{bc6}\cdot\frac{2}{3} + e_{2,3}^{bc9}\cdot\frac{1}{3} = 82\cdot\frac{2}{3} + 622\cdot\frac{1}{3} = 262,$$

$$x(7) = e_{2,3}^{bc7}\cdot\frac{2}{3} + e_{2,3}^{bc10}\cdot\frac{1}{3} = 172\cdot\frac{2}{3} + 1228\cdot\frac{1}{3} = 524,$$

$$x(8) = e_{2,3}^{bc8}\cdot\frac{2}{3} + e_{2,3}^{bc11}\cdot\frac{1}{3} = 343\cdot\frac{2}{3} + 2428\cdot\frac{1}{3} = 1038.$$

Example 5.9. Let us represent the solution of the the scalar ($r = 1$) problem (5.9), (5.10) where we put $m = 2$, $n = 3$, $B = b = 4$, $C = c = -1$, $\varphi(-3) = 1$, $\varphi(-2) = 2$, $\varphi(-1) = 3$, $\varphi(0) = 4$, $f(k) = k+1$, using Theorem 5.5. Thus, we have

$$\Delta x(k) = 4x(k-2) - x(k-3) + k + 1, \quad k \in \mathbb{Z}_0^{\infty}, \tag{5.21}$$

$$\begin{aligned} x(-3) &= \varphi(-3) = 1,\\ x(-2) &= \varphi(-2) = 2,\\ x(-1) &= \varphi(-1) = 3,\\ x(0) &= \varphi(0) = 4. \end{aligned} \tag{5.22}$$

By Theorem 5.5, the solution of the problem (5.21), (5.22) is

$$x(k) = \sum_{j=0}^{3} \tilde{e}_{2,3}^{bc(k+j)} w_j + \sum_{\ell=1}^{k} \tilde{e}_{2,3}^{bc(k-\ell)}\ell, \quad k \in \mathbb{Z}_{-3}^{\infty}$$

where

$$w_0 = \Delta\varphi(-1) - \Delta\tilde{\mathrm{e}}_{2,3}^{bc2}\varphi(-3) - \sum_{s=-3}^{-4} \Delta\tilde{\mathrm{e}}_{2,3}^{bc(-s-2)}\Delta\varphi(s) = 1 - \left(\tilde{\mathrm{e}}_{2,3}^{bc3} - \tilde{\mathrm{e}}_{2,3}^{bc2}\right) \cdot 1$$
$$= 1 - (1 + b - 1) = 1 - b = -3,$$
$$w_1 = \Delta\varphi(-2) = 1,$$
$$w_2 = \Delta\varphi(-3) = 1,$$
$$w_3 = \varphi(-3) = 1.$$

Thus, we get

$$x(k) = \tilde{\mathrm{e}}_{2,3}^{bck}(-3) + \tilde{\mathrm{e}}_{2,3}^{bc(k+1)} + \tilde{\mathrm{e}}_{2,3}^{bc(k+2)} + \tilde{\mathrm{e}}_{2,3}^{bc(k+3)} + \sum_{\ell=1}^{k} \tilde{\mathrm{e}}_{2,3}^{bc(k-\ell)}\ell,$$

and

$$x(1) = \tilde{\mathrm{e}}_{2,3}^{bc1}(-3) + \tilde{\mathrm{e}}_{2,3}^{bc2} + \tilde{\mathrm{e}}_{2,3}^{bc3} + \tilde{\mathrm{e}}_{2,3}^{bc4} + \sum_{\ell=1}^{1} \tilde{\mathrm{e}}_{2,3}^{bc(1-\ell)}\ell = -3 + 1 + 5 + 8 + 1 = 12,$$
$$x(2) = \tilde{\mathrm{e}}_{2,3}^{bc2}(-3) + \tilde{\mathrm{e}}_{2,3}^{bc3} + \tilde{\mathrm{e}}_{2,3}^{bc4} + \tilde{\mathrm{e}}_{2,3}^{bc5} + \sum_{\ell=1}^{2} \tilde{\mathrm{e}}_{2,3}^{bc(2-\ell)}\ell = -3 + 5 + 8 + 11 + 3 = 24,$$
$$x(3) = \tilde{\mathrm{e}}_{2,3}^{bc3}(-3) + \tilde{\mathrm{e}}_{2,3}^{bc4} + \tilde{\mathrm{e}}_{2,3}^{bc5} + \tilde{\mathrm{e}}_{2,3}^{bc6} + \sum_{\ell=1}^{3} \tilde{\mathrm{e}}_{2,3}^{bc(3-\ell)}\ell = -15 + 8 + 11 + 30 + 6 = 40,$$
$$x(4) = \tilde{\mathrm{e}}_{2,3}^{bc4}(-3) + \tilde{\mathrm{e}}_{2,3}^{bc5} + \tilde{\mathrm{e}}_{2,3}^{bc6} + \tilde{\mathrm{e}}_{2,3}^{bc7} + \sum_{\ell=1}^{4} \tilde{\mathrm{e}}_{2,3}^{bc(4-\ell)}\ell = -24 + 11 + 30 + 57 + 14 = 88,$$
$$x(5) = \tilde{\mathrm{e}}_{2,3}^{bc5}(-3) + \tilde{\mathrm{e}}_{2,3}^{bc6} + \tilde{\mathrm{e}}_{2,3}^{bc7} + \tilde{\mathrm{e}}_{2,3}^{bc8} + \sum_{\ell=1}^{5} \tilde{\mathrm{e}}_{2,3}^{bc(5-\ell)}\ell = -33 + 30 + 57 + 93 + 30 = 177,$$
$$x(6) = \tilde{\mathrm{e}}_{2,3}^{bc6}(-3) + \tilde{\mathrm{e}}_{2,3}^{bc7} + \tilde{\mathrm{e}}_{2,3}^{bc8} + \tilde{\mathrm{e}}_{2,3}^{bc9} + \sum_{\ell=1}^{6} \tilde{\mathrm{e}}_{2,3}^{bc(6-\ell)}\ell = -90 + 57 + 93 + 202 + 57 = 319,$$
$$x(7) = \tilde{\mathrm{e}}_{2,3}^{bc7}(-3) + \tilde{\mathrm{e}}_{2,3}^{bc8} + \tilde{\mathrm{e}}_{2,3}^{bc9} + \tilde{\mathrm{e}}_{2,3}^{bc10} + \sum_{\ell=1}^{7} \tilde{\mathrm{e}}_{2,3}^{bc(7-\ell)}\ell = -171 + 93 + 202 + 400 + 114$$
$$= 638,$$
$$x(8) = \tilde{\mathrm{e}}_{2,3}^{bc8}(-3) + \tilde{\mathrm{e}}_{2,3}^{bc9} + \tilde{\mathrm{e}}_{2,3}^{bc10} + \tilde{\mathrm{e}}_{2,3}^{bc11} + \sum_{\ell=1}^{8} \tilde{\mathrm{e}}_{2,3}^{bc(8-\ell)}\ell = -279 + 202 + 400 + 715 + 228$$
$$= 1266.$$

5.4 Comparison with Known Results

During the preparation of this thesis, several papers on similar problems were published. In our opinion paper [20], overviewed in section 2.4, includes the most related results. Now we will compare our results with those given in [20].

For matrix systems with two delays (2.43), a solution of the problem (2.44) is described by using discrete matrix delayed exponential for two delays (2.45). This definition is rather complicated and its practical utilization is troublesome. These problems do not arise in our definitions (in Definition 4.1 and Definition 4.5).

Let us compare the formulas representing solutions. Formula (2.46) uses matrix solution of the problem (2.43), (2.44) where discrete matrix delayed exponential for two delays (2.45) is utilized. Therefore, the final formula (2.46) has all above disadvantages. Our formulas (5.5) and (5.7) to representat solutions of the problem (5.3), (5.4) are perhaps more suitable for possible applications.

Paper [20] considers homogeneous problem (2.48), (2.49) and nonhomogeneous problems (2.51), (2.52) and (2.54), (2.55). The derived formulas, (2.50), (2.53) and (2.56) representing solutions are based on the definition of discrete multi-delayed matrix exponential (2.47) which is even more complicated that the above discrete matrix delayed exponential for two delays (2.46). In the thesis, such problems are not considered. It is an open question how to extend our definitions of discrete matrix delayed exponentials (i.e. how to modify Definitions 4.1, 4.5) in order to get results similar to Theorems 5.1, 5.2, 5.4, 5.5.

6 CONCLUSION

Results presented in the thesis are important in two aspects. First, the known notion of discrete matrix delayed exponential function is used to get analytical representations of solutions to systems of linear dicscrete equations with a single delay and with impulses. Second, the definition of discrete matrix delayed exponential function is generalized to the case when linear systems contain two delays. A generalization is derived that copies, in a sense, the original definition of discrete matrix delayed exponential function. In addition to this, another definition of discrete matrix delayed exponential function is suggested. For both discrete matrix delayed exponentials, their main properties are proved and they are used in formulas describing analytical solutions of linear discrete systems with two delays.

The future progress can be achieved by further generalizations of discrete matrix delayed exponential functions to the case when problems with multiple delays are considered. The results known in this field seem to be too cumbersome and so it may be expected that new results on the representation of solutions of linear problems with multiple delays will be very useful for applications.

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