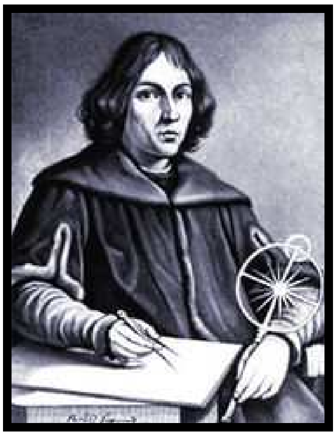

RELATIONSHIPS OF PARAMETERS OF PLANETARY ORBITS IN SOLAR-TYPE SYSTEMS

FACULTY OF SCIENCE,
PALACKÝ UNIVERSITY, OLOMOUC

RELATIONSHIPS OF PARAMETERS OF PLANETARY ORBITS IN SOLAR-TYPE SYSTEMS

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PAVEL PINTR

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„Relationships of parameters of planetary orbits in solar-type systems“

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Abstract

We deal with regularities of the distances in the solar system in chapter 2. On starting with the Titius--Bode law, these prescriptions include, as ``hidden parameters'', also the numbering of planets or moons. We reproduce views of mathematicians and physicists of the controversy between the opinions that the distances obey a law and that they are of a random origin. Hence, we pass to theories of the origin of the solar system and demonstrations of the chaotic dynamics and planetary migration, which at present lead to new theories of the origin of the solar system and exoplanets. We provide a review of the quantization on a cosmic scale and its application to derivations of some Bode-like rules.

We have utilized the fact in chapter 3 that the areal velocity of a planet is directly proportional to the appropriate number of the planet, while its distance is directly proportional to the square of this number. We have confirmed a previous proposal of the quantization of the planetary orbits, but with the first possible orbit of a planet in the solar system identical only to an order of magnitude. Using this method, we have treated moons of two planets and one extrasolar system. We have investigated a successive numbering and suggested a Schmidt-like formula in the planets and the Jovian moons.

We have introduced some new functions (called ``normalized parameters'') of usual parameters of extrasolar systems in chapter 4. One pair of these parameters exhibits areas, where the density of exoplanets is higher. One of these parameters along with the specific angular momentum indicate two groups of exoplanets with the Gaussian distributions. We have found that for five multi-planet extrasolar systems, the power function leads to the best determination of the product of the exoplanet distance and the stellar surface temperature by the specific angular momentum. We have revealed the role of the Schmidt law. We have also considered the spectral classes of the stars. We have also explored the data of 2321 exoplanet candidates from the Kepler mission.

We have determined the theoretical number of exoplanets using the statistical analysis of extrasolar systems for the spectral classes F, G, K and M in chapter 5. We have predicted many possible habitable exoplanets for the stellar spectral class G. The stellar spectral class F should have by 52% less possible habitable exoplanets than the class G, the stellar spectral class K should have by 67% less possible habitable exoplanets than the class G and the stellar spectral class M should have by 90 % less possible habitable exoplanets than the class G, i. e., the least possible habitable exoplanets.

We have also found the dependence of effective temperature of exoplanets on the orbital parameters of exoplanets. Using the model of planetary atmospheres, we have predicted habitable zones for the stellar spectral classes F, G, K and M.

In chapter 6 some brief conclusions are presented, concerning a comparison of the results from the chapter 2, chapter 3, chapter 4 and chapter 5.

Relationships of parameters of planetary orbits in solar-type systems

Pavel Pintr

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Chapter 1

Motivation

The 20th century is held as the golden age of astronomy and astrophysics, when many persistent questions were solved and the human view of the universe changed radically. In spite of this, at the beginning of the 21st century, one cannot find satisfactory answers to some questions our ancestors posed as early as in the 16th century. For instance, Kepler looked for a universal law, in his Mysterium cosmographicum, to explain the planetary distances in the solar system. Nowadays, when discoveries of other planetary systems occur, such a law could explain the distances of their planets.

The ongoing search of extrasolar planets is one of the most attractive fields of research in astrophysics and astronomy. Up to February 11, 2012, 759 exoplanets in 609 extrasolar systems have been discovered near stars with similar mass as the Sun. There is also discovery related to the so-called Earth-like planets. With regards to these discoveries, one intriguing question is whether there is relationship between orbit distance of the planets and their stars.

On our planet we find extreme conditions under which organisms are able to not only sustain metabolic processes, but thrive and grow. This understand- ing informs our precepts on how life formed in our solar system and also the possibility of similar processes in exoplanetary systems. The habitable zone is a key concept in our understanding of the conditions under which basic life can form and survive. In particular, the response of different atmospheres to varying amounts of stellar flux allows the determination of habitable zone boundaries for known exoplanetary systems.

Chapter 2

Regularities in systems of planets and recent works

2.1 Introduction

The solar system wakes admiration and an attempt at a reasonable argument for this feeling suggests regularities in the systems of planets and moons. In this chapter, we restrict ourselves to the regularities that the distances of secondaries from their primaries indicate. Other parameters leading to the concept of resonances are not treated (cf. [Murray and Dermott (1999)], pages 9, 321). A frequent argument in favor of a formal treatment of the regularities is the failure of the theories of the origin of the solar system, which should be essential at least from the materialist viewpoint. At present, rather new theories are spoken of the successful theories of a new generation and the weight of the usual argument is lesser. Some theories seem to confirm the regularity formulae. Therefore, we include also the theories of the origin of the solar system and the extrasolar planets.

Neglecting that both the major sciences, mathematics and physics, have undergone a historical development, we pay attention to the fact that as late as at the times of J. Kepler, astronomy (and astrology) was counted to the mathematics. Kepler's inventions have ushered the establishment of the astronomy as a physical field. Even though till 1781 only six planets of the solar system were known, their distances from the Sun were measured with an appropriate precision.

In 1766, J. Titius von Wittenberg formulated his famous note on planetary distances. J. Bode has published this note and readdressed it, see [Nieto (1972)]. A temporary success of the Bode law may consist also in the fact that it is not quite simple. A geometric progression is obvious only after a subtraction of 0.4 AU (astronomic units) from the distance from the planet to the Sun.

The objection that empirical formulae may be arbitrarily complicated has led to attempts at simple formulae. So, Armellini's law has the form, $r_{n\mathrm{A}} = 1.53^n$, where n assumes the values: -2 for Mercury, -1 for Venus, 0 for the Earth, 1 for Mars, 2 for the asteroid Vesta, 3 for the asteroid Camilla, 4 for Jupiter, 5 for Saturn, 6 for Centaur Chiron, 7 for Uranus, 8 for Neptune and 9 for Pluto [de Oliveira Neto (1996)].

Kant (1755) understood the origin of the solar system as a scientific problem and worded the nebular hypothesis. By his theory, the Sun and the planets became from the gas, which had been located in the volume of the present solar system and had had a high temperature. He assumed that the gravitation could bring about the origin of a proto-Sun and transform the irregular motion into a rotation. The planets originated from the rotating mass.

Independently, Laplace (1796) indicated that the solar system had become from gas and assumed not only a high temperature of the matter, but also its rotation. The nebula rotated as a solid body. His scenario of the evolution includes the cooling, contraction, enhancement of the rotation and flattening. The nebula shed a gaseous ring, which becomes a ball. It repeats as many times as many the planets are. Similarly, the moons of the planets have originated. The Sun has become from the remainder of the nebula. From a single ring more small planets could originate. P. S. Laplace confessed that he was not convinced by his hypothesis.

Maxwell (1859) has provided results confirmed by the flybys by the Voyager spacecraft in the 1980s. In application to the solar nebula, he has remarked that the gaseous ring itself cannot wrap into a spherical body, a planet. It has been stated that the present planetary system and the Sun do not have the total angular momentum that leads to an instability of the rotating nebula. The theories of the origin by the external causes are called catastrophic.

The Chamberlin–Moulton planetesimal hypothesis has been proposed in 1905 by geologist T. C. Chamberlin and astronomer F. R. Moulton [Chamberlin (1905), Moulton (1905)]. The external cause consists in that the star passed close enough to the Sun. Jeans (1914) assumed close encounter between the Sun and a second star. The difference from the previous hypothesis is in that Chamberlin and Moulton assumed separation of some mass on the adjacent and opposite sides of the Sun and the accretion of planetesimals. J. Jeans assumed the separation of the mass only on the adjacent side of the Sun and a direct origin of planets. This hypothesis has been assumed also by the mathematician and astronomer H. Jeffreys, who considered also a collision theory [Jeffreys (1924)]. In the 1920s, H. N. Russell was persuaded by the Jean-Jeffreys tidal hypothesis to affirm that planetary systems are "infrequent" and inhabited planets "matter of pure speculation." Two decades later, however, he gave up this opinion [Russell (1943)]. Russell (1935) measured spectra of binary stars and was interested in the origin of planetary systems. Lyttleton (1936) as an expert on the binary stars assumed that the Sun had been part of such a system.

In contrast, the nebular hypothesis has been resumed. E.g., Nölke (1930) did not derive the shedding of the rings from the assumption of the rotation, but the turbulence. von Weizsäcker (1943) elaborated a similar theory. Kuiper (1951) uses the concept of a protoplanet. The electromagnetic forces have been considered by Alfvén (1942), Dauvillier and Desguins (1942) and Schmidt (1944). In the 1960s, the massive-nebula model [Cameron (1962)] and low-mass-nebula model coexisted [Safronov (1960, 1969)]. The latter has evolved into a "standard" model [Lissauer (1993)].

In section 2.2, we mention a discussion of the mathematicians, who have not been satisfied with the statement that "the Bode law fits data well enough". They have constructed alternative hypotheses and have found that the likelihood ratio differs significantly from unity in one case [Good (1969)] and it does not differ from unity significantly in the other case [Efron (1971)]. Specialists may pay attention to a hypothesis competing with the statement that the Bode law fits the data well. They are not content with the repetition of a mathematician's idea, but they use the astronomical knowledge. They suggest a random origin of the regularities [Hayes and Tremaine (1998), Murray and Dermott (1999), p. 5]. These imposing analyses are not persuasive, on considering their model dependence [Lynch (2003)].

In section 2.3, we touch the resumed nebular hypothesis. The topic is estimates of the total mass of the solar nebula and the distribution of its mass [Weidenschilling (1977), Hayashi (1981)] and a modification for the extrasolar nebulae [Kuchner (2004)]. We mention next the 'standard" model of planet formation [Lissauer (1993)]. Finally, we touch dynamical theories of the Titius–Bode law [Graner and Dubrulle (1994), Dubrulle and Graner (1994)]. These theories for the restriction to the Titius–Bode law comprise well intended simplifications. In this framework, Christodoulou and Kazanas (2008) have been able to provide a theory of the dependence of the planetary distance on its ordinary number, which does not express this dependence by a closed formula, but fits the data well. This means even attempts at an application to extrasolar planets. We expand on a method of derivation of the rule involving squares of the ordinary numbers instead of the geometric progression [Krot (2009)]. We provide the dependence of the planetary distances on their ordinary numbers, which is no more based on the squares of the ordinary numbers, but it is satisfactory.

Already in the book [Murray and Dermott (1999), p. 409], a whole chapter is devoted to the chaos and long-term evolution along with appropriate references. In section 2.4, we pay attention to such reports on numerical integration [Laskar (1989), Sussman and Wisdom (1992)]. The theory of origin of the extrasolar planets meets a difficulty that the Jupiter-mass planets are present on small orbits. This has led to the theory of migrating planets [Murray, Hansen, Holman and Tremaine (1998), Murray, Paskowitz and Holman (2002)]. Long-time scales are not accepted by creationists [Spencer (2007)].

In section 2.5, we devote attention to the quantization on a cosmic scale. The observed deviations of the absorption lines from the Lyman-α frequency have led to a hypothesis of their origin, which includes the quantization of "megascopic" systems [Greenberger (1983)]. The quantization of microscopic systems has been simulated [de Oliveira Neto (1996), Nottale, Schumacher and Gay (1997), Agnese and Festa (1997), Rubčić and Rubčić (1998), Carneiro (1998), Agnese and Festa (1998, 1999), Nottale, Schumacher and Lefèvre (2000)]. We concentrate ourselves to the approaches, which replace a dynamical theory of the Titius–Bode law by the quantization of orbits and thus derive rather the use of the square of a planet's ordinary number instead of the geometric progression. Next we remember the indirect use of the quantization for the discretization of orbits. First, a wavefunction for a planet is chosen and then the expectation value of the distance from the particle to the central body is compared with the observed distance from the planet to the Sun [de Oliveira Neto, Maia and Carneiro (2004)]. Finally, we return to our publications. Pintr and Peřinová (2003–2004) have commented on the proposal of Mohorovičić (1938) positively and have modified it to the moons of giant planets and extrasolar planets. Peřinová, Lukš and Pintr (2007) intended to replace the close relationship of the paper [de Oliveira Neto, Maia and Carneiro (2004)] to the article [de Oliveira Neto (1996)] by a connection with the paper [Agnese and Festa (1997)]. This intention has been realized in part. Pintr, Peřinová and Lukš (2008) have derived a discrete system of orbits using mainly the classical physics. Like an incomplete dynamical theory of the Titius–Bode law, or the quantization of orbits using wavefunctions, this theory assigns the distances to the nodal lines of standing waves, even though indirectly, through a transformation.

The first five sections are based on the chapter [Peřinová, Lukš and Pintr (2012)]. In this contribution the introduction presents a history of the interest in distances from planets to the Sun and from moons to a central planet. In section 2.2, views of mathematicians and physicists of the contrast between the opinions that the distances obey a law and that they are of a random origin are reproduced. In sections 2.3 and 2.4, theories of the origin of the solar system and demonstrations of the chaotic dynamics and planetary migration are mentioned. In section 2.5, a review of the quantization on a cosmic scale and its application to derivations of some Bode-like rules is provided.

2.2 Statistical decision making

In the Titius–Bode law doubling occurs, which enables anybody to write down a mathematical formula for the planets Venus, Earth, Mars, Ceres, Jupiter, Saturn, Uranus, Neptune and Pluto. The powers of two may be linearly extrapolated,

$$a_{\mathrm{Mercury}} = a,\, a_{\mathrm{Venus}} = a + b,\, a_{\mathrm{Earth}} = a + 2b,\, a_{\mathrm{Mars}} = a + 4b,\, a_{\mathrm{Ceres}} = a + 8b,$$
$$a_{\mathrm{Jupiter}} = a + 16b,\, a_{\mathrm{Saturn}} = a + 32b,\, a_{\mathrm{Uranus}} = a + 64b,\, a_{\mathrm{Neptune}} = a + 96b,$$
$$a_{\mathrm{Pluto}} = a + 128b, \tag{2.1}$$

where $a = 0.4$, $b = 0.3$, cf. [Christodoulou and Kazanas (2008), Povolotsky (2007)]. The continuation in the formula up to Uranus is obvious.

The Titius–Bode law with slight irregularities provokes to improvement, but also to proposing formulas, whose "validity" is saved by the neglect of critique of numbering of the planets, cf. [Nieto (1972)]. Good (1969) has emended the Bode law to the form

$$b_{\mathrm{Mercury}} = a + b,\ b_{\mathrm{Venus}} = a + 2b,\ b_{\mathrm{Earth}} = a + 4b,\ b_{\mathrm{Mars}} = a + 8b,\ b_{\mathrm{Ceres}} = a + 16b,$$

$$b_{\mathrm{Jupiter}} = a + 32b,\ b_{\mathrm{Saturn}} = a + 64b,\ b_{\mathrm{Uranus}} = a + 128b,\ b_{\mathrm{Neptune}} = a + 256b,$$

$$b_{\mathrm{Pluto}} = a + 512b. \tag{2.2}$$

Here $a = 0.4$, $b = 0.075$, however.

Just as Efron (1971) has indicated in a footnote of a statistician, it can be expected that he or she practises the "numerology". It seems that the three purposes of his article are given in descending order of importance for the astronomy: (1) The validity of Bode's law, (2) testing whether or not the observed sequence of numbers follows some simple rule, (3) the logical basis of Fisherian significance testing.

The statistical decision making assumes:

(i) A statistical model describing what the statement means that Bode's law is real.

(ii) An alternative statistical model describing the statement that Bode's law is artifactual.

The question of the validity of Bode's law is transformed to a problem of hypothesis testing [Efron (1971)].

The statisticians apply Bode's law only to the planets Venus through Uranus. According to [Good (1969)], the statistical model (i) consists in a normal distribution of logarithms of planetary distances. These distances are independent random variables. The means are given by the Titius–Bode law. All the variances are σ^2. It is accepted that the three parameters a, b, and σ^2 are estimated. Model (ii) is a uniform distribution of the logarithms of the planetary distances on some interval $[\log \delta_{\mathrm{l}}, \log \delta_{\mathrm{u}}]$, where the subscript l (u) stands for lower (upper), respecting the observed order of the planets. It is accepted that the parameters $\log \delta_{\mathrm{l}}$, $\log \delta_{\mathrm{u}}$ are estimated. By the use of the Bayesian methods, it has been derived that the data witness for Bode's law. Efron (1971) admits that, in a non-Bayesian framework, the result would be the same, but he adopts the Fisherian methodology along with the model C instead of (ii).

A formulation of the model C demonstrates that, in the mathematical statistics, it is not necessary to specify a joint distribution of the planetary distances completely. Such a distribution, if any, is characterized in terms of the ratios of the planetary spacings to the difference between the distances of Uranus and Venus. For simplicity, we speak of a spacing instead of the difference between the distances from a planet and its successor to the Sun. The planets can be mapped on the interval $[0, 1]$, with 0 corresponding to Venus and 1 to Uranus and the points pertaining to other planets respecting not only the observed order, but also the "law of increasing differences." This way, the model C "impertinently" draws near Bode's law. A classical statistical analysis has utilized a distance statistics, Δ, and it has led to the conclusion that the data do not witness for Bode's law.

It can be expected that a physicist's viewpoint will differ from a matematician's method. Indeed, Hayes and Tremaine (1998) do not believe in the law of increasing differences. They approach the generation of random planetary systems rather in the sense of the simpler model (ii), with $\delta_{\mathrm{l}} = 0.2$ AU and $\delta_{\mathrm{u}} = 50$ AU. The generation is completed by the rejection of some obviously unstable planetary systems. Nine semi-major axes $r_0,\ldots,\ r_8$ are generated. Then, a nonlinear least-squares fit of the distances r_i, $i = 0, 1, \ldots, 8$, is performed to the "law" $a + bc^i$, with c being another parameter. Next, a fit is performed on leaving out j out of nine planets, $j = 0, 1, 2, 3$, and, for each j, the best reduction is chosen. The entire procedure is repeated, but, after the generation of semi-major axes, one gap is inserted between two neighbouring planets with the largest ratio of $\frac{r_{i+1}}{r_i}$. It means that such planets will have numbers i, $i + 2$, and the numbering ends with number nine.

Nine "reasons" of rejecting have been formulated, first of all, the rejection need not have been attempted at all. Further, such a reason has been a violation of the condition

$$r_{i+1} - r_i > r_{i+1}V_{i+1} + r_iV_i, \tag{2.3}$$

where, e.g., $V_i = H_{M_i}, 2H_{M_i}, 4H_{M_i}, 8H_{M_i}$, M_i being the mass of the planet i in the solar system, H_{M_i} is the fractional Hill radius,

$$H_{M_i} = \sqrt[3]{\frac{M_i}{3M_{\mathrm{Sun}}}}. \tag{2.4}$$

When the observed distances in the solar system are processed in the same way as the nine generated semi-major axes, the best fit is obtained in the case, where a gap is added between Mars and Jupiter, whereas Mercury, Neptune and Pluto are ignored.

The view of Lynch (2003) approaches the statement by Efron (1971) that the statistical decision, whether the observed patterns have a physical basis or can be ascribed to chance, depends on the model. He readdresses the geometric progression of orbital periods of five major satellites of Uranus. According to the literature, the model leaves the period of Miranda unchanged and the following satellites have the periods equal to the products of one (Ariel) to four (Oberon) random factors respecting observed ratios of successive periods. He presents a simpler procedure consisting just in random choice of orbital periods in bands covering the values produced by the formula. In the original model, the probability of random origin is about 80 per cent and, in the new model, it is about 20 per cent for the chosen bands.

Since the planetary radii and periods are related by Kepler's third law, Lynch (2003) investigates the solar system in a similar way. First of all, he simplifies the Titius–Bode law to a geometric progression. Further, he repeats the comparison of two models or procedures. The procedure, which is a continuation of the study of major Uranian satellites, indicates that the probability of random origin is only about 40 per cent. In the new model, it is 0.99 for the bands chosen in the similar way as in the new investigation of the Uranian satellites. Even though rigorous mathematical methods used by Efron (1971) may throw new light on these results, Lynch (2003) is right that the possibility of a physical explanation for the observed distributions remains open.

2.3 Theories of Bode-like laws

The rejected nebular hypothesis had the advantage that it assumed a mass in the region of the present solar system. After the hypothesis has been resumed, a search could begin for a shortcut of the road leading from the assumptions of the model to the Titius–Bode law. In this section, we first remember an estimate of the total mass of the solar nebula and distribution of its mass [Weidenschilling (1977)] and another one for extrasolar nebulae [Kuchner (2004)]. We mention the model of planet formation, which was standard till recent times [Lissauer (1993)]. We remember some dynamical theories of the Titius–Bode law [Graner and Dubrulle (1994), Dubrulle and Graner (1994), Krot (2009)]. These publications can be criticized, as any of them provides not a unique approach, but at least two different ones. Exceptionally, Christodoulou and Kazanas (2008) have been able to provide a unified approach.

2.3.1 Resumption of the nebular hypothesis

Weidenschilling (1977) remembers theories of cosmogony. Most such theories assume that the planetary system formed from a nebula. It is assumed that the mass fraction of Fe in the solar matter is 1.2×10^{-3}. The mass fractions are at disposal even for the terrestrial planets. To each such a planet, the mass is determined, which has the solar composition. For the planets Jupiter, Saturn, Uranus and Neptune, one proceeds differently, but also in such a way that the appropriate masses are determined.

Weidenschilling (1977) reconstructs the solar nebula by spreading the augmented planetary masses through zones surrounding their orbits. He determines the zones (AU) $(0.22, 0.56)$ for Mercury, $(0.56, 0.86)$ for Venus, $(0.86, 1.26)$ for the Earth, $(1.26, 2.0)$ for Mars, $(2.0, 3.3)$ for asteroids, $(3.3, 7.4)$ for Jupiter, $(7.4, 14.4)$ for Saturn, $(14.4, 24.7)$ for Uranus and $(24.7, 35.5)$ for Neptune in terms of the observed distances.

On the determination of the zones, surface densities can already be found. The surface density is preferred to the volume density, which can be determined only on other assumptions about the vertical structure of the nebula. The surface density proportional to $r^{-\frac{3}{2}}$ has been found, which continues the paper [Cameron and Pine (1973)]. Anomalies (deviations from the power law with the exponent $-\frac{3}{2}$) in Mercury, Mars and asteroids are stated. Processes exist for selective removal of matter from these regions.

So the nebular hypothesis includes the loss of light elements during the planetary formation. The mass of the nebula must be at least between 0.01 and 0.1 solar masses. We may calculate that the mass of the nebula is

$$\begin{aligned} M_{\text{nebula}} &= 2\pi \int_0^{r_{\text{nebula}}} r\sigma_{\text{Earth}} \left(\frac{r}{r_{\text{Earth}}}\right)^{-\frac{3}{2}} \mathrm{d}r \\ &= 4\pi r_{\text{Earth}}^2 \sigma_{\text{Earth}} \sqrt{\frac{r_{\text{nebula}}}{r_{\text{Earth}}}}, \end{aligned} \tag{2.5}$$

where $\sigma_{\text{Earth}} = 32000$ kgm^{-2}, $r_{\text{nebula}} = 35.5$ AU, outer limit of Neptune's zone.

Hayashi (1981) pays attention to the importance of magnetic effects on the origin of the solar system. Nevertheless, he begins with a model of the solar nebula without magnetic effects. He expounds the properties of the nebula, which entail the magnetic effects, and their significance. He takes into account the magnetic and turbulent viscosities. He attempts at an initial condition of a more ancient stage of the evolution.

The model is related to the stages, where according to the theories of planetary formation, the dust sedimented on the equatorial plane. The mass of the nebula is of the order of 0.01 M_{Sun}. It is assumed that $\frac{\mathrm{d}P}{\mathrm{d}r}$, where P is the pressure, has a negligible value. The half-thickness of the nebula, with the temperature dependent on r, is given. The nebula is heated by the Sun and the field temperature is

$$T = 280\sqrt{\frac{r_{\text{Earth}}}{r}} \tag{2.6}$$

for the luminosity of the Sun identified with the present value.

In the interval $[0.35, 36]$ AU, three kinds of the surface density are considered. Two components of the total density are related only to the dust and gas, but the density of rock increases from 2.7 AU (in the asteroid belt) due to the presence of ice. The gas prevails. The exponent $-\frac{3}{2}$ is utilized and the surface density $\rho_s(r_{\text{Earth}})$ $= 17000$ kgm^{-2}. As long as the magnetic effects are negligible, the volume density is

$$\rho(r, z) = \rho_0 \left(\frac{r}{r_{\text{Earth}}}\right)^{-\frac{11}{4}} \exp\left(-\frac{z^2}{z_0^2(r)}\right), \tag{2.7}$$

where $\rho_0 = 1.4\times10^{-6}$ kgm^{-3} and $z_0(r) = 0.0472 r_{\text{Earth}} \left(\frac{r}{r_{\text{Earth}}}\right)^{\frac{5}{4}}$. A simple vertical structure is assumed, so that the surface density is

$$\rho_s(r) = \sqrt{\pi}\rho_0 z_0(r) \left(\frac{r}{r_{\text{Earth}}}\right)^{-\frac{11}{4}}$$

$$= \quad \rho_s(r_{\mathrm{Earth}}) \left(\frac{r}{r_{\mathrm{Earth}}}\right)^{-\frac{3}{2}}. \tag{2.8}$$

Here $\sqrt{\pi}1.4 \times 10^{-6} \times 0.0472 r_{\mathrm{Earth}} = \sqrt{\pi}1.4 \times 10^{-6} \times 0.0472 \times 1.5 \times 10^{11} = 1.7 \times 10^4$ kgm^{-2}.

We note that the exponent $\frac{5}{4} > 1$. Magnitudes of the magnetic fields H_1 and H_2 are determined such that the vertical structure of the nebula is affected for $H > H_1$, with H being the magnitude of a field present in the solar nebula, and a deviation from the Keplerian velocity of rotation occurs for $H > H_2$. The field magnitudes H_1 and H_2 decrease with the increase of r. In a uniformly ionized gas, magnetic fields grow and decay according to a result of magnetohydrodynamics.

The turbulence of an "equilibrium" solar nebula leads to the origin of seed magnetic fields. A possibility of the redistribution of gas density in the solar nebula is studied, which is caused by angular momentum transport due to the presence of magnetic and mechanical turbulent viscosity. The effect of the mechanical viscosity is reduced to the diffusion in the radial direction.

This way the exponent $-\frac{3}{2}$ can be derived. For the isothermal case, the exponent -2 is given. It is admitted that, without further calculations, it is not certain whether the effect of mechanical viscosity alone suffices or magnetic viscosity must contribute. According to an accomplished theory of planetary formation, the planets have been formed except for Uranus and Neptune before the dissipation of the solar nebula, Saturn being formed in an intermediate stage [Hayashi (1981)].

The magnetic effects on the structure of the nebula are negligible in regions of the terrestrial planets, even though it is not valid for its outermost layers. Contrary to this, these effects are significant in regions of the giant planets. Hayashi (1981) recognizes a numerical simulation of the cloud that preceded the nebula. He discusses an initial condition of the collapse. He does not adopt the spherical Jeans condition for it, but properties of the fragment of a rotating isothermal disk.

Kuchner (2004) reviews the mental pictures of the minimum-mass solar nebula. He includes also the papers [Weidenschilling (1977), Hayashi (1981)]. He attempts to take into account the extrasolar planets. He introduces the concept of a minimum-mass extrasolar nebula. He concentrates himself to few-planet, i.e., two-planet and three-planet systems discovered by precise Doppler methods. The astronomer can infer planets with a suitable relation between the orbital period and the ratio of the mass of the planet to the mass of the star corrected by the angle of sight. They detect radial velocity variations of the planet.

Kuchner (2004) chooses 1000 M_{Earth} for the augmented masses of most of the extrasolar planets. The dependence of the surface density on the semi-major axis is obtained by mixing of data of different systems. It emerges that the surface density is proportional to r^{-2}. Separately fitted nebulae are not taken too seriously and their exponents are -2.42 through -1.50. The mixed data force one to admit even the solar exponent $-\frac{3}{2}$.

The minimum-mass solar nebula (exponent $-\frac{3}{2}$) and the uniform-α accretion disk model [Shakura and Sunyaev (1973)] suggest that giant planets do not form at the centre of the disk. Migration theories are quite acceptable, if the power law has the exponent $-\beta$, $\beta > 2$. The minimum-mass solar nebula was based on Laplace's concept of the solar nebula that broke up into rings that condensed into planets.

2.3.2 Protoplanetary disks

The origin of the solar system is a recognized problem of science [Lissauer (1993)]. Models of planetary formation are developed using the solar system and limited astrophysical observations of star-forming regions and circumstellar disks. Other planetary systems are detected around main sequence stars and pulsars.

A theory of the origin should explain the following facts:

1. Both the orbits of the planets and those of most of asteroids are nearly coplanar and this plane is close to that of the Sun's equator. The orbits of the planets are nearly circular and planets orbit the Sun in the same sense as the Sun rotates.

2. Spacing between the orbits of the planets increase with the distance from the Sun. The orbits of eight planets do not cross. Even though Pluto's orbit crosses that of Neptune, the dwarf planet avoids close encounters with the planet due to $3:2$ resonance.

3. Comets orbit the Sun.

4. Six of the eight planets rotate around their axis in the same direction, in which they revolve around the Sun (cf. point 1), and their obliquities (tilts of their axes) are less than $30°$.

5. Most planets have natural satellites.

6. Planetary masses account for less than 0.2 % of the mass of the solar system.

7. Over 98 % of the angular momentum in the solar system is contained in the orbital motions of the Jovian planets.

8. Planets and asteroids have compositions which are rather well known.

9. The size of asteroids and parameters of asteroidal orbits are rather well known.

10. Nearly all meteorites come from the asteroid belt.

11. Ages of meteorites are relatively well known.

12. Isotopic ratios are about the same in all solar system bodies.

13. Meteorites argue for rapid heating and cooling and for magnetic fields of order 1 Gauss (10^{-4} T).

14. Most solid planetary and satellite surfaces are heavily cratered.

Point 1 suggested the hypothesis of planetary formation in a flattened disk [Kant (1755), Laplace (1796)]. In the 1990s, the evidence for the presence of appropriately large disks around pre-main sequence stars increased. Protoplanetary disks contain a mixture of gas and condensed matter. A lower bound on the mass of the protoplanetary disk has been mentioned above [Weidenschilling (1977)]. The planetesimal hypothesis asserts that planets grow within circumstellar disks through pairwise accretion of small solid bodies, the so-called planetesimals. Sufficiently massive planetary bodies embedded in a gas-rich disk can gravitationally capture much gas and produce Jovian-type planets. The absence of a planet in the dynamically stable region inside Mercury's orbit can be attributed to two reasons. Close to the early Sun, nebula temperatures were such high that condensation of material did not take place. Possibly, solid planetesimals felt so strong a gas drag that their decay depleted the region considered of condensed matter [Lissauer (1993)].

Theories considering instabilities of the gas leading to giant gaseous protoplanets fail to explain just compositions of the Jovian planets. Models of planetary growth from small solid bodies do not suffer from such dificulties. Heating in the consequence of the collapse of a molecular cloud core to the solar system dimensions is admitted. Afterwards the disk can cool. Various compounds condense into microscopic grains. The motions of small grains in a protoplanetary disk are strongly coupled to the gas. The vertical component of the star's gravity causes sedimentation onto the midplane of the disk. Models suggest that the volume of the solid material was able to agglomerate into bodies of macroscopic size at least in the terrestrial planet region of the solar nebula [Weidenschilling and Cuzzi (1993)].

With respect to a possible pressure gradient in the radial direction, the gas circles the star less rapidly than the Keplerian rate. Large particles which move at nearly the Keplerian speed experience a headwind, and so the material that survives to form the planets, must accomplish the transition from cm size to km size rather quickly. Further forces upon planetesimals are remembered besides the star's gravity. They are gravitational interactions with other planetesimals and protoplanets. Further mutual inelastic collisions and gas drag.

It is stated that the simplest analytic approach to the evolution of planetesimal velocities is based on methods of the kinetic theory of gases. For the final stages of the evolution, the number of planetesimals becomes small enough that a direct treatment of individual planetesimal orbits is feasible. A numerical n-body integration has been replaced by the alteration of precessing elliptic orbits by close encounters with other planetesimals.

Physical collisions dissipate part or all of the relative kinetic energy of colliding bodies. Models comprise a Boltzmann collision operator for hard spheres modified to allow for inelastic collisions and gravitational interaction. Gas drag damps excentricities and inclinations of planetesimals, especially small ones.

It is accepted that 3-body effects may be neglected. After Safronov (1969), accretion zones and the protoplanet as the largest body in the given zone are introduced. The region, in which 3-body effects are significant, is limited with the radius of protoplanet's Hill Sphere

$$h_{\mathrm{S}} = \sqrt[3]{\frac{M}{3M_{\mathrm{s}}}}a, \tag{2.9}$$

where M and M_{s} are the masses of the protoplanet and the star, respectively, and a is the semi-major axis of protoplanet's orbit.

It is mentioned that the accretion rate of a protoplanet is enhanced by the squared ratio of the escape velocity from the point of contact to the relative velocity of the bodies. It is pointed out that during simulation of early stages, a simultaneous calculation of the velocity evolution and size evolution in the planetesimal swarm is necessary. It was found that the size evolution is of two kinds. The slower evolution exhibits a regular growth of all planetesimals. The more rapid evolution that is related to exceptional planetesimals shows a "runaway" accretion to the largest planetesimal in the local region. The discrete form of the coagulation equation has solutions of two kinds (bifurcation) [Safronov (1969)].

The condition of low velocities of planetesimals for the runaway accretion is remembered. When a protoplanet consumes most of the planetesimals within its gravitational reach, its mass is equal to the so-called isolation mass and the rapid runaway growth may cease. Mechanisms of further growth have been considered. Attention is drawn to the origin of semi-major axes of protoplanets in an arithmetic progression. Such a configuration is not dynamically stable for a long time. Crossing orbits, close gravitational encounters and violent collisions are predicted. It can be referred to, e.g., the paper [Wetherill (1990)].

The necessity of a high velocity, post-runaway growth phase is emphasized. Times closer to 10^8 are given. Next it is stated that the model of minimum-mass protoplanetary disk does not yield appropriate times for the cores of the giant planets. The exposition of giant planet atmospheres in [Lissauer (1993)] begins with the hypothesis that the atmospheres of the terrestrial planets and other small bodies come from material accreted as solid planetesimals. The view is adopted that first the core of a giant planet was formed. The masses of such cores are of order 10 M_{Earth}. The accretion of the gaseous envelopes of the giant planets long lacked explanation.

The origin of planetary rotation, point 4, is a question, which is difficult to answer. Although part of literature denies a random occurrence, part relies on stochastic factors. The angular momenta brought in by individual planetesimals and hydrodynamic accretion of gas provide planets with rotational angular momentum perpendicular to the midplane of the disk essentially, because the disk is flat.

The analogy of the moons and rings of the giant planets to miniature planetary systems is admitted. Inside Roche's limit, where tidal forces from the planet suffice to disrupt a moon held together only by its gravity, planetary rings dominate. In the outer regions of the satellite systems of Jupiter, Saturn and Neptune, small bodies on highly excentric and inclined orbits occur. This way planetary satellites can be classified as "regular" and "irregular". Regular satellites are related to a circumplanetary disk. Irregular moons may be captured planetoids from the solar nebula. The satellites of Earth and Mars and some other planetary satellites are harder to classify in this manner.

The analogy of the circumplanetary disk to the larger circumsolar disk is nearly imperfect. Satellite systems are much more compact than the planetary system and are evolved dynamically. Most satellite rotations have been locked by planetary tidal forces. Among several moons mean motion commensurabilities persist. Satellites of satellites have not been observed. For the explanation of this simplification, many reasons have been brought in.

The Earth's Moon is different and indicates a stochastic event. Complexity of the satellite systems of the four giant planets in our solar system suggests that stochastic processes participated in satellite formation. It is pointed out to, e.g., little total mass in the asteroid region and their large number. Also the excentricities and inclinations of the orbits of these bodies are higher than the planetary values. We remember also the diversity of the composition. Proximity to Jupiter is an explanation. For dynamical models of the formation of the asteroid belt, one refers to the papers [Wetherill (1989, 1991, 1992)].

The division of comets into two groups is mentioned: short-period comets, with the orbital period shorter than 200 years, and long-period comets, which return after more than 200 years. Hyperbolic orbits are admitted as consequences of perturbations by the planets. The long-period comets come from the Oort cloud. Some short-period comets come from the Oort cloud, but most of them populate the Kuiper belt. The origin of comets, especially the Oort cloud, is connected to the ejection of small planetesimals from the giant planet region. This excludes a restriction of the study to the minimum-mass protoplanetary disk. Also the outer limit of Neptune's zone cannot be the edge of the protoplanetary disk, on considering the formation of the Kuiper belt from planetesimals.

In conclusion, the planetesimal hypothesis is a viable theory of the growth of the terrestrial planets, the cores of the giant planets, and the smaller bodies present in the solar system. The formation of solid bodies of planetary size should be common around young stars, which do not have stellar companions at planetary distances. On certain conditions, planets could form within circumpulsar disks. A summary of the theory of planetary growth from planetesimal accretion within a circumstellar disk is provided in [Lissauer (1993)]. Several different ideas have been added, e.g., the rule that a more massive protoplanetary disk of the same radial extent will probably produce a smaller number of larger planets.

2.3.3 Dynamical theories of the Titius–Bode law

Graner and Dubrulle (1994) point out to the book [Nieto (1972)], in which the notion of a dynamical theory of the Titius–Bode law has been introduced. In spite of the diversity of these theories, in each of the models a Titius–Bode like law arises. Let us note that it means a geometric progression in planetary distances.

Essentially, a description using partial differential equations, which are symmetric, is assumed. The symmetries considered are:

(P1) the invariance with respect to the rotation around an axis z,

(P2) the invariance with respect to the dilatation (scale transformation) in the plane perpendicular to z,

(P3) the time independence.

On denoting $g_j(r,\theta)$ the physical quantities obeying these equations and γ_j the respective exponents, $\Lambda^{\gamma_j} g_j(\Lambda r,\theta)$, where $\Lambda > 0$, satisfy also these equations.

Unfortunately, the exposition comprises imperfections, which we do not evaluate, but must mention them. A physical quantity $g(r,\theta)$ with an exponent γ and its Fourier series decomposition

$$g(r,\theta) = \mathrm{Re}\left\{\sum_m a_m(r)\exp\left[i(m\theta+\theta_m)\right]\right\}, \tag{2.10}$$

where θ_m are real numbers, are considered. However, the closest usual form of this equation is

$$g(r,\theta) = \sum_m a_m(r)\exp(im\theta). \tag{2.11}$$

We see that both Re and θ_m are superfluous and that the usual statement that $a_0(r)$ is a real number, is substituted with a feeling of $a_0(r)$ being a complex number. But it cannot be defined uniquely.

It is assumed that the coefficient a_m of the decomposition fulfils a symmetric equation

$$h_m\left(\frac{a_m a_m^*}{r^{2\gamma}}, \frac{r}{a_m}\frac{\partial a_m}{\partial r}\right) = 0, \tag{2.12}$$

where h_m has arisen in a manipulation and the asterisk means the complex conjugation. As a first-order ordinary differential equation, it can be cast to the form

$$\frac{\partial a_m}{\partial r} = \frac{a_m}{r} H_m\left(\frac{|a_m|^2}{r^{2\gamma}}\right), \tag{2.13}$$

where H_m has arisen in another manipulation, if the dependence of the second argument on the first one can be made explicit. Using the substitution

$$b_m = \frac{a_m}{r^\gamma}, x = \log\left(\frac{r}{r_0}\right), \tag{2.14}$$

where r_0 is a normalizing radius, we obtain that

$$\frac{\partial b_m}{\partial x} = b_m G_m \left(|b_m|^2\right), \tag{2.15}$$

where $G_m = H_m - \gamma$. As long as $|b_m|^2$ is small, we may utilize the property

$$G_m(|b_m|^2) = \mu_m + ik_m + (\eta_m + i\kappa_m)|b_m|^2 + \mathrm{O}(|b_m|^4), \tag{2.16}$$

where

$$\mu_m + ik_m = G_m(|b_m|^2)\Big|_{|b_m|^2=0}, \eta_m + i\kappa_m = \left.\frac{\mathrm{d}G_m(|b_m|^2)}{\mathrm{d}|b_m|^2}\right|_{|b_m|^2=0}. \tag{2.17}$$

Let us remark that, in the paper [Graner and Dubrulle (1994)], the subscript m at μ, η, k and κ is omitted. Neglecting the dependence of G_m on $|b_m|^2$ at all, we obtain that

$$b_m = B_0 \left(\frac{r}{r_0}\right)^{\mu} \exp\left[ik \log\left(\frac{r}{r_0}\right)\right], \tag{2.18}$$

where $\mu \equiv \mu_m$, $k \equiv k_m$. A derivation of a Titius–Bode law is possible in the case of one mode, or in the case of independence of the modal index m. It is also necessary to assume $k \neq 0$. Equal phase cylinders are of the form $r = r_n$, where

$$r_n = r_0 K^n, \quad K = \exp\left(\frac{2\pi}{k}\right). \tag{2.19}$$

We have criticized that the derivation is only pertinent to $m \neq 0$, so it can hardly be applied to rotationally symmetric solutions, which are important. In spite of this, Graner and Dubrulle (1994) modify the solution (2.18) to include further the linear term of the Taylor series of the function G_m in $|b_m|^2$. The distances r_n change, do not form a geometric progression and only illustrate the so-called nonlinear Titius–Bode law.

Dubrulle and Graner (1994) utilize the theory from [Graner and Dubrulle (1994), part I] even though, in part I, the assumption (P3) of time independence was made too, which is not made here at first. Therefore, they mention the scale invariance of equations for the hydrodynamic description of the solar nebula. The scalar invariance need not always occur as a consequence of the polytropic gas law

$$P = P_0 \left(\frac{\rho}{\rho_0}\right)^{1+\frac{1}{n}}, \tag{2.20}$$

where P is the pressure, ρ the volume density, P_0 and ρ_0 are the normalizing pressure and the normalizing density, respectively, and n is the polytropic index, which constrains the validity of the scale invariance to $n = 3$. That is why, Dubrulle and Graner (1994) assume $P = 0$. Another difficulty presents the Poisson equation, in which the second-order partial derivatives occur, while the outline of a theory has included only the first-order derivatives. Therefore, an integral expression of the gravitational potential with the linear density is assumed.

On specifying the equilibrium solution, the assumption of the time independence is abandoned again and the stability of the equilibrium solution is studied by the method of linearization around this solution. The linearized equation is solved by the method of the Fourier series decomposition, which is appropriate to the surface density decreasing like r^{-2}. An option is typical of these problems and their adversity. We obtain

$$\tilde{v}^2 = \left(\frac{\partial}{\partial x} - 1\right)\tilde{\phi}, \tag{2.21}$$

where $\tilde{v}$ is the scale-invariant tangential component of the velocity and $\tilde{\phi}$ is the scale-invariant gravitational potential of the disk.

The question arises, which mode k (k is a wavenumber of the Fourier series) is suitable for a derivation of the Titius–Bode law. Dubrulle and Graner (1994) argue for $k = k_c$, where k_c separates, which wavenumbers are low and which are high. The low wavenumbers ($< k_c$) are linearly stable and the high wavenumbers ($> k_c$) are linearly unstable. The critical wavenumber k_c depends on $\frac{M_{\rm eg}}{M_C}$, where $M_{\rm eg}$ is an effective gravitating disk mass, substituting for its actual mass, M_D, in a rather complicated manner, and M_C is the mass of the central body.

Christodoulou and Kazanas (2008) have dealt with equilibrium structures of rotating fluids with cylindrical symmetry. They have derived exact results and are convinced that the results are relevant to the location of the planetary orbits. The famous Titius–Bode law also expresses this location, but it is actually opposed to the authors' effort.

The Lane–Emden equation is mentioned, which describes the equilibria of non-rotating fluids. This equation has a spherical symmetry. It can be modified to an equation with cylindrical symmetry easily, what have been utilized by physicists outside astrophysics. In the astrophysics, the paper [Jeans (1914)] may be typically quoted. The results for a finite polytropic index n are known, whereas Christodoulou and Kazanas (2008) deal with the case where $n \to \infty$.

The Titius–Bode "law" is held for the Titius–Bode algorithm. It is also referred to the critiques, which we consider in this review, or to papers, which somewhat underpin this rule. Typically, a pertinent paper expresses both cons and (at least essentially) pros.

Some methodologically related analyses are mentioned. The simple cylindrical formulation is defended [Jeans (1914)]. Rules are evaluated, which would approximate the observed distances. Only two rules are considered. Such rules rely on a consecutive numbering of planets, to which both asteroid Ceres is counted and the dwarf planet Pluto. The rules do not predict positions of the end bodies. The rule of arithmetic mean, $\frac{1}{2}(a_{i-1} + a_{i+1})$, has errors up to 23.5 and 27.9%. The rule of

Index i	Planet	$\mathcal{M}_i$	Error (%)
1	Mercury		
2	Venus	0.8	-9.0
3	Earth	1.9	8.7
4	Mars	2.4	16.3
5	Ceres	2.0	11.7
6	Jupiter	1.8	8.5
7	Saturn	2.2	16.3
8	Uranus	1.1	-4.3
9	Neptune	0.9	-5.5
10	Pluto		

Table 2.1: Magnification ratios $\mathcal{M}_i$ of planetary orbits and relative errors of the means with exponent $\frac{1}{2}$.

geometric mean, $(a_{i-1}a_{i+1})^{\frac{1}{2}}$, has errors up to -11.8 and -14.0 %.

In the same vein, we could try another possibility, the rule of the mean with the exponent of $\frac{1}{2}$, $[\frac{1}{2}(a_{i-1}^{\frac{1}{2}}+a_{i+1}^{\frac{1}{2}})]^2$. Having tried it, we obtain the error up to 16.3 % in two cases. A good fit to the rule of the arithmetic mean in the bodies that neighbour the end ones is stated. In the intermediate bodies, the rule of the geometric mean is suitable. The use of the rule, which better predicts the distance to the body, leads to errors up to 5.0 and 9.1 %. We have not improved it using the rule of the mean with exponent $\frac{1}{2}$! It is obvious that the Titius–Bode law begins with a three-term arithmetic progression, which is the shortest nontrivial progression of this kind.

Christodoulou and Kazanas (2008) then mention interesting connections or analogies. In optics, an analogue of the quantity

$$\mathcal{M}_i \equiv \frac{a_{i+1} - a_i}{a_i - a_{i-1}} \tag{2.22}$$

can be found. Obviously, it is a quantity reducing to the Titius–Bode base two and independent of their coefficients a and b.

We have calculated the ratios $\mathcal{M}_i$ according to equation (2.22) and obtained results can be found in Table 2.1 (we insert the error of the rule of the mean with exponent $\frac{1}{2}$). The ratios $\mathcal{M}_i$ can be further rounded to 1 and 2. So the three-term arithmetic progression leads to 1 for single Venus and the Titius–Bode law leads to 2 for the Earth to Saturn. In what preceded, we have presented a continuation of the Titius–Bode rule.

Nevertheless, Christodoulou and Kazanas (2008) have concentrated themselves to the Lane–Emden equation. They have not been attracted by the generous identification of the Titius–Bode sequence with the geometric progression, which has been done by Graner and Dubrulle (1994) and Dubrulle and Graner (1994).

Christodoulou and Kazanas (2008) distinguish these notions. They have evaluated also the approach of the cited authors.

The isothermal equilibrium is assumed, i.e., the pressure balances the general gravity. The cylindrical symmetry is assumed. It is assumed that the angular velocity has the form

$$\Omega(r) = \Omega_0 f_{\mathrm{CK}}\left(\frac{r}{r_0}\right), \tag{2.23}$$

where $\Omega_0 = \Omega(0)$ for centrally condensed models and $f_{\mathrm{CK}}(x)$ is an infinite-dimensional parameter such that $f_{\mathrm{CK}}(0) = 1$. It is assumed that an isothermal equation of state for the pressure P and the gas density ρ holds (it is a volume density, but it has much in common with the surface density with respect to the cylindrical symmetry)

$$P = c_0^2\rho, \tag{2.24}$$

where c_0 is the isothermal sound velocity. It is declared that for finite polytropic indices, significantly different results are not obtained.

Euler's equation for the unknown functions ρ, Ω, P, and ϕ is presented,

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} + \frac{\mathrm{d}\phi}{\mathrm{d}r} = \Omega^2 r, \tag{2.25}$$

but we already know that $\Omega \equiv \Omega(r)$ is rather a parameter. Poisson's equation for these functions is also given,

$$\frac{1}{r^{d-1}}\frac{\mathrm{d}}{\mathrm{d}r}r^{d-1}\frac{d\phi}{dr} = 4\pi G\rho, \tag{2.26}$$

where $d = 2$ with respect to the cylindrical symmetry and G is the gravitational constant. On the elimination of ϕ and with the substitution $x = \frac{r}{r_0}$, an equation is obtained, which may be called the Lane–Emden equation with rotation,

$$\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}x\frac{\mathrm{d}}{\mathrm{d}x}\log\tau + \tau = \frac{\beta_0^2}{2x}\frac{\mathrm{d}}{\mathrm{d}x}(x^2 f_{\mathrm{CK}}^2), \tag{2.27}$$

where $\tau = \frac{\rho}{\rho_0}$ and $\beta_0^2 = \frac{\Omega_0^2}{2\pi G\rho_0}$. Here ρ_0 is the maximum density.

For $\beta_0 f_{\mathrm{CK}} = 0$, equation (2.27) differs from the usual Lane–Emden equation by the assumed cylindrical symmetry of the matter described and the polytropic index $n \to \infty$. Closed solutions are presented. We ask for which next choice of $\beta_0 f_{\mathrm{CK}}$, the solution will be viable and physically meaningful. It is assumed that

$$\tau = \frac{\beta_0^2}{2x}\frac{\mathrm{d}}{\mathrm{d}x}(x^2 f_{\mathrm{CK}}^2), \tag{2.28}$$

then the term with the derivatives on the left-hand side of equation (2.27) equals zero,

$$\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}x\frac{\mathrm{d}}{\mathrm{d}x}\log\tau = 0. \tag{2.29}$$

Then the density depends on two integration constants, A and k,

$$\tau(x) = \frac{\beta_0^2}{2}\, A x^{k-1} \tag{2.30}$$

and $\beta_0 f_{\rm CK}$ also depends on the integration constant B,

$$\beta_0 f_{\rm CK}(x) = \frac{\sqrt{A g(x) + B}}{x}, \tag{2.31}$$

where

$$g(x) = \begin{cases} \frac{x^{k+1}}{k+1}, & \text{if } \ k \neq -1, \\ \log x, & \text{if } \ k = -1, \end{cases} \tag{2.32}$$

implying that $\frac{\mathrm{d}g}{\mathrm{d}x} = x^k$ for all values of k. Five points are added to this result. Point 4 on the composite profiles is important. These profiles can be utilized to the predictions we have mentioned above and we will also return to them below. The striking assumption (2.29) is connected with the enthalpy gradient being constant.

Even though the surface density corresponds to $k = -\frac{1}{2}$, also $k = 1$ is interesting, for which we obtain a finite value at the axis of the cylinder, $\tau(0) = \beta_0^2$, which cannot be changed on a choice of A, since necessarily $A = 2$ with respect to the boundary condition $f(0) = 1$. The boundary condition $\tau(0) = 1$ is mentioned, which contradicts the property $\tau(0) = \beta_0^2$, in general. The boundary conditions are "imposed" to the desired function at the axis of the cylinder. It is stated that it has been performed using a numerical integration.

The solution for $\tau(0) = \beta_0^2$ and $\frac{\mathrm{d}\tau}{\mathrm{d}x}\Big|_{x=0} = 0$ represents a "baseline" solution, about which the solution oscillates that fulfils the proper initial condition. The oscillatory behaviour of the density profiles is utilized for fitting the observed planetary distances to the density peaks. A construction of the density profile and rotation law in three regions, $x \leq x_1$, $x_1 < x < x_2$, $x \geq x_2$ is described. The model depends on at least three parameters, $x_1 > 0$, $x_2 > x_1$, and $1 - k \equiv \delta > 2$, with k being related to the region $x_1 < x < x_2$. The fourth parameter of the model is β_0. The mentioned four parameters have been chosen such that they predict the observed planetary distances. In the discussions, many further interesting connections with research of planet formation are presented. The model has been applied to the solar nebula. An application to the satellites of Jupiter and the five planets of 55 Cancri is possible.

In the paper [Christodoulou and Kazanas (2008)], solutions have been found successfully, which exhibit a pronounced oscillatory behaviour. The adaptable slope of the profile of the differential rotation leads to arithmetic partial or geometric partial progressions of the mass density peaks. It is a clear explanation of the Titius–Bode "law" of planetary distances. A criticism of the explanations of the order invoking new dynamical laws or "universal" constants and solar-system "quantizations" is indicated. At this place, we could object that the choices of slopes of the profile of the differential rotation have only been verified using a numerical integration. It is admitted that the currently available observations have not supported the idea that our solar system is a representant of the planetary systems around stars. The most similar systems such as 55 Cancri and HD37124 are mentioned.

Krot (2009) mentions his statistical theory of gravity. He promises to expound the gravitational condensation. He derives an antidiffusion equation. To our opinion, this derivation is faulty. His equation (48) is the beginning, but it already expresses what one has intended to derive. After a certain time, the maximum value of the density from the reference time t_0 is present in a point at a distance Δr from the origin, where the maximum was attained at the time t_0. Thus, the density is closer to the origin than it was at the time t_0. It does not seem to be an error, but equation (48) is not so evident. Then the antidiffusion equation is combined with the Euler equation. The uniform rotation around the axis z is assumed. For example, a distribution with the shape of oblate ellipsoid, but still the Laplacian–Gaussian one, is derived.

The derivation of the distribution of specific angular momentum λ, $\lambda = \Omega h^2$, where Ω is the angular velocity of the uniform rotation around the axis z and h is the distance from the axis z, is promised. Let us note that it means the double of the areal velocity. This distribution is

$$f(\lambda) = \frac{\alpha(1-\varepsilon_0^2)}{2\Omega} \exp\left(-\frac{\alpha(1-\varepsilon_0^2)}{2\Omega}\lambda\right), \tag{2.33}$$

where $\alpha \equiv \alpha(t)$ is a positively defined monotonically increasing function and ε_0 is a constant, ε_0^2 is a squared eccentricity of ellipse [Krot (2009)].

Schmidt's hypothesis is a picture that particles or planetesimals, which have sufficiently close values of the specific angular momentum or those of the areal velocity, condense or accrete [Schmidt 1944]. The hypothesis may include just cutting of the distribution of the areal velocities to equal parts.

A solution of this problem is described. We look for specific angular momenta μ_n, λ_n, such that

$$\mu_n = \frac{\lambda_n + \lambda_{n+1}}{2}, \tag{2.34}$$

$$\lambda_n = \frac{\int_{\mu_{n-1}}^{\mu_n} \lambda f(\lambda)\mathrm{d}\lambda}{\int_{\mu_{n-1}}^{\mu_n} f(\lambda)\mathrm{d}\lambda}. \tag{2.35}$$

Using the approximation

$$\lambda_n \approx \frac{\lambda_{n+1} + \lambda_{n-1}}{2}, \tag{2.36}$$

we obtain λ_n in the form of an arithmetic progression. This way, Schmidt (1944) has derived his law for planetary distances, r_n. Then he has encountered the fact that it is not accurate enough for all the planets together. He has recommended to use it separately for the terrestrial planets and the Jovian planets.

The approximation (2.36) is based on the possibility of considering $f(\lambda)$ a constant in the interval $[\mu_{n-1}, \mu_n]$. Krot (2009) performs at least the linear approximation of the function $f(\lambda)$ (his equation (101)),

$$f(\lambda) \approx \frac{\alpha(1-\epsilon_0^2)}{2\Omega}[1 - \frac{\alpha(1-\epsilon_0^2)}{2\Omega}\lambda]. \tag{2.37}$$

He adds a generalization of the Schmidt law using the inverted equation

$$\lambda_n = \sqrt{2GM_{\rm Sun}r_n}. \tag{2.38}$$

We will return to the derivation below.

The proposed theory is applied to the formation of the solar system. As usual, the whole enterprise with the planets is based on merely eight planets and on the dwarf planet Pluto. We criticize the fitting of data to the theoretic dependence. An excessive number of parameters, namely seven, only the number of planets minus one, have been proposed. So we are very close to a mere data transformation, which is not recommended. Every further comparison then results in favor of this *incorrect* procedure, whose inappropriateness may not be conceded by the author.

Now, the planets are successively numbered, although the asteroid belt might deserve its number. The asteroid belt is equated one planet usually. Krot (2009) simply neglects this belt. We criticize a large number of parameters for expressing a relatively small number of observed distances. Let us see, how the author has arrived at this solution. The author has processed various assumptions to the "hypothesis" that the cloud of particles has an exponential distribution of specific angular momentum λ with the probability density (2.33). Later he has sacrificed this assumption to the approximation (2.37). On substituting into equation (2.35), he has obtained that

$$\lambda_n \approx \frac{\frac{\mu_n+\mu_{n-1}}{2} - \frac{\alpha(1-\varepsilon_0^2)}{6\Omega}(\mu_n^2 + \mu_n\mu_{n-1} + \mu_{n-1}^2)}{1 - \frac{\alpha(1-\varepsilon_0^2)}{4\Omega}(\mu_n + \mu_{n-1})}, \tag{2.39}$$

where μ_n is defined by equation (2.34). On substituting (2.34) into (2.39) and using the property

$$\bar{\lambda} = \frac{2\Omega}{\alpha(1-\varepsilon_0^2)}, \tag{2.40}$$

the author obtains the finite difference equation

$$\lambda_n = \frac{\bar{\lambda}(C_n + D_n) - \frac{1}{3}\left(C_n^2 + C_nD_n + D_n^2\right)}{4\bar{\lambda} - (C_n + D_n)}, \tag{2.41}$$

where $C_n = \lambda_{n+1} + \lambda_n$, $D_n = \lambda_n + \lambda_{n-1}$ and where, for readability, we have already written the equality sign instead of the more correct $\approx$.

We mind that this finite difference equation is not completed with initial or boundary conditions. Before we complete it, we will return to the original formulation. The sequence to be found is not denoted by a single letter, but it has the form

$$(\lambda_1, \mu_1, \lambda_2, \mu_2, \lambda_3, \mu_3, \ldots). \tag{2.42}$$

So the finite difference equations (2.34) and (2.35) are equations for a single sequence, even though it is denoted by two letters. But it is justified by the alternation of the equations. We begin with equation (2.35) for $n = 1$, where we put $\mu_0 = 0$, hopefully. The author has not explained it. We proceed with equation (2.34) for $n = 1$, etc. Another assumption to be considered is the hypothesis that the sequence is finite. In the model of formation of the solar system, it follows from the finite number of the planets.

Krot (2009) does not recommend a length of the sequence. The author demonstrates that the longer sequence that includes Pluto does not fit well. Although we have not made numerical calculations, we could also exclude Pluto. Then we obtain

$$(\lambda_1, \mu_1, \lambda_2, \mu_2, \ldots, \lambda_7, \mu_7, \lambda_8). \tag{2.43}$$

To each term of this sequence, an equation exists. We simplify the last equation with the assumption $\mu_8 = +\infty$. This assumption is reasonable. In combination with the approximation (2.37) it should rather be assumed that $\mu_8 = \bar{\lambda}$. So we have arrived at fifteen equations for fifteen unknowns, which depend only on the parameter of the distribution $f(\lambda)$. It is unique for the time being. It is far from the result by Krot (2009), who asserts something else.

To assess the equations, which are solved in an obviously incorrect way, we note that somewhere a "usual" mistake occurs and the numbering is shifted by one. Certainly, $\lambda_0 = a_0$ for $n = 0$ in equation

$$\lambda_n = a_0 + dn, \tag{2.44}$$

where d is the difference and a_0 is the first (the zeroth) term of the arithmetic progression. Later in the misleading substitution,

$$\lambda_n = Z^n, \quad n = 1, 2, 3, \ldots, \tag{2.45}$$

$n \geq 1$ already.

Since, in equation (2.41), the variable n is comprised only as the subscript, we can utilize our note. In this equation, we put $n = 2, \ldots, 7$, considering the boundary conditions

$$\lambda_1 = \frac{\bar{\lambda}(\lambda_2 + \lambda_1) - \frac{1}{3}(\lambda_2 + \lambda_1)^2}{4\bar{\lambda} - (\lambda_2 + \lambda_1)}, \tag{2.46}$$

$$\lambda_8 = \frac{\bar{\lambda}(2\bar{\lambda} + \lambda_8 + \lambda_7) - \frac{1}{3}\left[(2\bar{\lambda} + \lambda_8 + \lambda_7)^2 - 2\bar{\lambda}(\lambda_8 + \lambda_7)\right]}{4\bar{\lambda} - (2\bar{\lambda} + \lambda_8 + \lambda_7)}. \tag{2.47}$$

Although the problem is formulated, it is meaningless to expound its solution. One of many reasons is that Krot (2009) proceeds in a completely different way. We will explain the essence of our method using a different class of distributions of the specific angular momentum than (2.33).

Let us assume that the random variable $\underline{\lambda}$ has the log uniform distribution, i.e., that $\log \underline{\lambda}$ has the uniform distribution on the interval $[\log \lambda_{\min}, \log \lambda_{\max}]$. Then

$$f(\lambda) = \frac{1}{\lambda \log\left(\frac{\lambda_{\max}}{\lambda_{\min}}\right)}. \tag{2.48}$$

Then the finite difference equation (2.35) becomes

$$\lambda_n = \frac{\mu_n - \mu_{n-1}}{\log\left(\frac{\mu_n}{\mu_{n-1}}\right)}, n = 1, \ldots, 8, \tag{2.49}$$

where $\mu_0 = \lambda_{\min}$, $\mu_8 = \lambda_{\max}$. Given the differential rotation of the form $\Omega_0 \left(\frac{h}{h_0}\right)^{(k-1)/2}$, with $h_0 \equiv r_0$, the specific angular momentum will be

$$\lambda = \lambda_0 \left(\frac{h}{h_0}\right)^{\frac{k+3}{2}}, \; -3 < k < 1, \tag{2.50}$$

where λ_0 is the specific angular momentum for $h = h_0$. Our distribution of distances has the probability density

$$l(h) = \frac{k+1}{h_{\max}^{k+1} - h_{\min}^{k+1}} h^k, k \neq -1, \tag{2.51}$$

where $h_{\min}$ and $h_{\max}$ are the least and greatest possible distances, respectively. For $k = -1$, a limiting formula with the logarithm holds. Since

$$\frac{h}{h_0} = \left(\frac{\lambda}{\lambda_0}\right)^{\frac{2}{k+3}}, \tag{2.52}$$

it holds that

$$\begin{aligned} f(\lambda) &= l(h) \left|\frac{\mathrm{d}h}{\mathrm{d}\lambda}\right| \\ &= \frac{k+1}{\lambda_{\max}^{\frac{2(k+1)}{k+3}} - \lambda_{\min}^{\frac{2(k+1)}{k+3}}} \lambda^{\frac{k-1}{k+3}}, k \neq -1. \end{aligned} \tag{2.53}$$

For $k = -1$, a limiting formula with the logarithm holds, see above.

We have solved this problem with a small change, on leaving out the case $n = 1$, namely Mercury. Contrary to [Krot (2009)], we include the asteroid Ceres with number $n = 5$. Here $n = 8$ means Uranus. But the optimal choice of μ_1 and μ_8

leads to relative errors up to forty per cent. Then the difference equation (2.35) has also the form

$$\lambda_n = \frac{2k+2}{3k+5} \frac{\mu_n^{\frac{3k+5}{k+3}} - \mu_{n-1}^{\frac{3k+5}{k+3}}}{\mu_n^{\frac{2k+2}{k+3}} - \mu_{n-1}^{\frac{2k+2}{k+3}}}, n = 2, 3, 4, 5, 6, 7, 8, k \neq -\frac{5}{3}, -1. \tag{2.54}$$

For $k = -\frac{5}{3}, -1$, a limiting formula with the logarithm holds. We search, moreover, for the optimum k. We have found that the best fit is achieved for $k = -2.228$, min=0.017387.

For the numerical illustration, we assume that

$$\begin{aligned} a_1 &= 0.3871, a_2 = 0.7233, a_3 = 1.0000, a_4 = 1.5237, \\ a_5 &= 2.765, a_6 = 5.2028, a_7 = 9.580, a_8 = 19.141, \qquad (2.55) \\ r_{\text{Earth}} &= 1.49597870691 \times 10^{11}, M_{\text{Sun}} = 1.9891 \times 10^{30}, G = 6.67428 \times 10^{-11}, \qquad (2.56) \end{aligned}$$

where a_j, $j = 1, \ldots, 8$, are the main half-axes of planet orbits beginning the Mercury and ending Uranus in the astronomical units, r_{Earth} is the astronomical unit, M_{Sun} is the mass of the Sun and G is the gravitational constant. The values a_j for $j \neq 5$ are according to [Krot (2009)], a_5 corresponds to the asteroid Ceres and it has been involved according to [Christodoulou and Kazanas (2008)].

We choose $k \neq -1$, but in the surroundings of the value $k = -1$. We search for 15 unknowns creating the sequence

$$(\mu_1, \lambda_2, \mu_2, \lambda_3, \mu_3, \lambda_4, \mu_4, \lambda_5, \mu_5, \lambda_6, \mu_6, \lambda_7, \mu_7, \lambda_8, \mu_8), \tag{2.57}$$

which obey the equation

$$\sum_{n=2}^{8} \left(\frac{d_n - a_n}{a_n} \right)^2 = \min, \tag{2.58}$$

where

$$\begin{aligned} d_n &= \frac{\lambda_n^2}{2GM_{\text{Sun}}} \frac{1}{r_{\text{Earth}}}, \qquad (2.59) \\ \mu_n &= \frac{\lambda_n + \lambda_{n+1}}{2}, n = 2, 3, 4, 5, 6, 7, \qquad (2.60) \\ \lambda_n &= \frac{2k+2}{3k+5} \frac{\mu_n^{\frac{3k+5}{k+3}} - \mu_{n-1}^{\frac{3k+5}{k+3}}}{\mu_n^{\frac{2k+2}{k+3}} - \mu_{n-1}^{\frac{2k+2}{k+3}}}, n = 2, 3, 4, 5, 6, 7, 8. \qquad (2.61) \end{aligned}$$

Probability density function $f(\lambda)$ of specific angular momentum λ is proportional to $\lambda^{\frac{k-1}{k+3}}$, $k = -2.228$. The nebula is divided formally into rings suitable for assignment of the angular momentum to circular loops of the radii d_n. The values of d_n and errors $\frac{d_n - a_n}{a_n} 100\%$ can be found in Table 2.2.

Index n	Planet	d_n	Error (%)
2	Venus	0.702972	-2.81
3	Earth	0.994763	-0.52
4	Mars	1.578346	3.59
5	Ceres	2.745513	-0.70
6	Jupiter	5.079846	-2.36
7	Saturn	9.748512	1.76
8	Uranus	19.085846	-0.29

Table 2.2: The radii d_n and their relative errors $\frac{d_n - a_n}{a_n}$ 100 %.

2.4 Chaotic behaviour and migration

Laskar (1989) has reported on an extensive analytic system of averaged differential equations describing secular evolution of the orbits of eight main planets, accurate to the second order in the planetary masses and to the fifth order in eccentricity and inclination. His effort reminds one of the numerical integration of the long-term evolution of the solar system, e.g., [Sussman and Wisdom (1988)]. Through the analysis of results, it has been decided, whether the initial conditions of the solar system lead to a quasiperiodic or a chaotic solution. In the time of the publication, direct numerical integration was not yet able to take into account the inner planets and their orbital motion was held for too rapid. Laskar (1989) has estimated the maximum Lyapunov exponent to be about $\frac{1}{5}$ Myr^{-1}. It indicates a chaotic motion. This conclusion has been left for next evaluation.

Sussman and Wisdom (1992) have numerically integrated the evolution of the whole planetary system for a time span of nearly 100 million years. They have estimated the Lyapunov exponent to be about $\frac{1}{4}$ Myr^{-1}. They have confirmed the characteristics of Pluto's motion [Sussman and Wisdom (1988)]. They have encountered the complication that the subsystem of the Jovian planets is chaotic or quasiperiodic in the dependence on the initial values. The method of integration resembles an approach intended for a provocation of the chaos, but it can be used in a regime, which excludes that the exponential divergence is a numerical artifact.

Murray, Hansen, Holman and Tremaine (1998) have elaborated an explanation of the presence of Jupiter-mass planets in small orbits. E.g., a planet orbits τ Bootis at a distance of 0.0462 AU (Internet gives a more recent value of 0.0481 AU). They have assumed that the giant planets may form at orbital radii of several AU and then migrate inwards. Such a planet originates in a planetesimal disk and it decreases the angular momentum of the planetesimals with resonant interactions. The planetesimals either collide with or are ejected by the planet. The ejection process removes orbital energy from the planet, which moves closer to the star. Other consequences of a close encounter may be a collision of the planetesimal with the star or a long-term capture into a mean-motion resonance.

Murray, Paskowitz and Holman (2002) have concentrated themselves to the resonant migration of planets, which produces large eccentricities. In their study, they have distinguished the migration due to ejection of planetesimals and that by tidal torques. These possibilities rely respectively on the assumptions of planetesimal and gas disks. The resonant migration in the gas disk is related to two bodies of roughly Jupiter mass.

Spencer (2007) reviews the migrating planets from a creationist's standpoint. He remembers the Nebular Hypothesis, which comprised the idea that all the planets in the our solar system formed in the regions, where they are now located. Extrasolar planetary systems require rather pictures of a migration. As the long accepted "naturalistic" origins explanations for our solar system do not operate for extrasolar planetary systems, planetary scientists modify past theories for our solar system. It is welcome, when the nebular models have had problems with the formation of Uranus and Neptune. The Nice model is much appealing. The theory has been published as the paper [Tsiganis, Gomes, Morbidelli and Levinson (2005)]. The initial order of the giant planets JSNU has changed to the actual order JSUN after 6.6 Myr. It can be described also as a "rapid inclination of the orbits of Uranus and Neptune and exchange of their orders in distance from the Sun". The time axis is drawn in the length of over 80 Myr. Spencer (2007) does not accept naturalistic origins scenarios that conflict with the biblical time scale.

It can be expected that long time spans, not only 10^9 yr, but also 10^{18} yr, lead to considerations of the inclusion of the quantum gravity and the uncertainty relations. For them, we refer to the essay [Page (2010)].

2.5 Quantization on a cosmic scale

In this section, we provide an account of our publications. Pintr and Peřinová (2003–2004) have reported on the proposal of Mohorovičić (1938). The then popular Bohr–Sommerfeld quantum theory has convinced him of the occurrence of "allowed" orbits in planetary science. His "magnification ratios" cf., [Christodoulou and Kazanas (2008)]) are lesser than one (reduction ratios). Another peculiar hypothesis is the connection of the allowed orbits with moons, which have "subjected" themselves to the primaries later. Peřinová, Lukš and Pintr (2007) have tried to use the concept of wavefunction instead of or together with the notion of an allowed orbit. Pintr, Peřinová and Lukš (2008) have returned to the allowed orbits using mainly the classical physics. Some modern physics concepts are applied as well.

Before this account, we review ideas which influenced us. Greenberger (1983) admitted that quantization of "megascopic" systems exists, which has a macroscopic limit in common with the familiar quantization of microscopic systems. de Oliveira Neto (1996) and Agnese and Festa (1998) have constructed Bode-like laws for our solar system. Later, de Oliveira Neto, Maia and Carneiro (2004) have provided a continuation of the paper [de Oliveira Neto (1996)], using the concept of a wavefunction. Interesting coincidences can be found even in the application of these ideas to extrasolar planets.

2.5.1 Quantization of megascopic systems

Greenberger (1983) has concerned himself with absorption lines in quasars with a high red-shift parameter z. He assumes that a hydrogen atom is attracted gravitationally to a quasar. The world is quantized on a microscopic scale and is quantized on a cosmic scale as well by the model theory. He restricts himself to the absorption of the Lyman-α frequency of ultraviolate radiation by an atom, i.e., the transition from the level $n = 1$ to $n = 2$. When the atom also passes over into a higher gravitational state, the absorption lines different from the Lyman-α line will be present in the analogy with the Raman effect.

Greenberger (1983) has expressed the quantized gravitational energy as

$$E_n = -\frac{E_0}{n^{\frac{2}{3}}}, \tag{2.62}$$

where E_0 can only be derived theoretically in a complicated way and it is not as yet observable. In observing the absorption lines with the wavelengths λ_k, $k = 1, 2$, we must consider jumps from the gravitational levels n_{k1} to n_{k2} and the equation

$$\frac{hc}{\lambda_k} - \frac{hc}{\lambda_\alpha} = \frac{E_0}{1+z}\left(n_{k1}^{-\frac{2}{3}} - n_{k2}^{-\frac{2}{3}}\right), \; k = 1, 2, \tag{2.63}$$

where h is Planck's constant and λ_α is the wavelength of the Lyman-α line seen on the Earth. From this

$$\frac{\frac{\lambda_\alpha}{\lambda_1} - 1}{\frac{\lambda_\alpha}{\lambda_2} - 1} = \frac{n_{11}^{-\frac{2}{3}} - n_{12}^{-\frac{2}{3}}}{n_{21}^{-\frac{2}{3}} - n_{22}^{-\frac{2}{3}}}, \tag{2.64}$$

and E_0 is not needed.

An analysis of absorption spectra of four quasars has provided good fit for two of them and the third comparison has been also meaningful. The fourth case has provided an illustration of the method at least. This complicated procedure is not similar to the Titius–Bode law. Considerations of elliptical rings around the normal elliptical galaxies in [Malin and Carter (1980)] remind us of the Titius–Bode law. The major half-axes very closely fit the formula

$$r_n = r_0 n^{\frac{2}{3}}. \tag{2.65}$$

Greenberger (1983) distinguishes the "kinetic" momentum from the canonical momentum and remembers that

$$[x, p_c] = i\hbar, \tag{2.66}$$

where x is a position coordinate and p_c the canonical momentum. We assume that the physical system is described by a Hamiltonian H. Then

$$\dot{x} \equiv \frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{i}{\hbar}[x, H] = \frac{\partial H}{\partial p_c} \tag{2.67}$$

and the kinetic momentum is $p = m\dot{x}$, when the "system" means a particle of the mass m. We have understood that the author considers equations for $\dot{x}$ of the form

$$[x, \dot{x}] = C(\dot{x}), \tag{2.68}$$

where the commutator C may depend on $\dot{x}$ again. We obtain an equation for the Hamiltonian H of the form

$$\frac{\partial^2 H}{\partial p_c^2} = -\frac{i}{\hbar} C\left(\frac{\partial H}{\partial p_c}\right). \tag{2.69}$$

For the original treatment, we refer to [Greenberger (1983)].

de Oliveira Neto (1996) starts with an imaginary observer, who is compelled to apply the quantum theory to our solar system by his/her cosmic size. He represents the mean planetary distances by the formula

$$r_{nm} = \frac{n^2 + m^2}{2} r_0, \, 0 \leq m \leq n, \tag{2.70}$$

where n and m are integers and r_0 is the mean distance of Mercury to the Sun. In the dependence on a planet, we note that

$$n(\mathrm{Mercury}) = 1, \;\; n(\mathrm{Venus}) = n(\mathrm{Earth}) = n(\mathrm{Mars}) = 2, \;\; n(\mathrm{Jupiter}) = 4,$$

$$n(\mathrm{Saturn}) = 5, \;\; n(\mathrm{Uranus}) = 7, \;\; n(\mathrm{Neptune}) = 9, \;\; n(\mathrm{Pluto}) = 10. \tag{2.71}$$

Next, $m(p) = n(p)$ for $p =$ Mercury, Mars, Jupiter, Saturn, Uranus, Neptune, Pluto. Three of the inner planets are described only by a variation of m: $m(\mathrm{Venus}) = 0$, $m(\mathrm{Earth}) = 1$, $m(\mathrm{Mars}) = 2$. Besides the planets and Pluto, the dwarf planet, some asteroids have been represented.

Nottale, Schumacher and Gay (1997) continue the approach in the book [Nottale (1993)]. They assume that at very large time-scales, the solar system can be described in terms of fractal trajectories governed by a Schrödinger-like equation. They let the fractal motion depend on a universal constant w_0 having the dimension of a velocity. According to an illustrative example by Greenberger (1983), a connection of w_0 with the fundamental length λ may be

$$w_0 \lambda = \frac{GM}{c}, \tag{2.72}$$

where $M = M_{\mathrm{Sun}}$. The average distance to the Sun is given in the form

$$\langle r_{n\ell} \rangle = \frac{1}{2}[3n^2 - \ell(\ell+1)]\frac{GM}{w_0^2}, \tag{2.73}$$

where n is the principal number of the quantized orbit and ℓ is the number of quanta of the angular momentum, $\ell = 0, 1, \ldots, n-1$. It has been found that the appropriate value for the inner system is $w_0 = (144.3 \pm 1.2)$ km s^{-1} and $w_0 \to w_{\mathrm{out}}$ for the outer system, $w_{\mathrm{out}} = \frac{w_0}{5}$, where $w_0 = (140 \pm 3)$ km s^{-1}.

Agnese and Festa (1997) remind the paper [Greenberger (1983)], especially his equation (8) where, without loss of generality, they write $\bar{\lambda} f$ instead of λf, $\bar{\lambda} = \frac{\lambda}{2\pi}$. We have understood that these authors propose the commutator

$$[x, m\dot{x}] = i(\hbar + mc\bar{\lambda}), \tag{2.74}$$

where $\hbar = \frac{h}{2\pi}$ is the reduced Planck constant and

$$\dot{x} = -\frac{i}{\hbar}[x, H]. \tag{2.75}$$

As above, we obtain that

$$\frac{\partial^2 H}{\partial p_c^2} = \frac{1}{m} + \frac{c\bar{\lambda}}{\hbar}, \tag{2.76}$$

or

$$H = \left(\frac{1}{m} + \frac{c\bar{\lambda}}{\hbar}\right)\frac{p_c^2}{2} + \text{potential energy}. \tag{2.77}$$

It is just the direction, in which Greenberger (1983) did not intend to proceed. It is obvious that, sometimes, an alternative expression for the kinetic energy may be useful.

Agnese and Festa (1997) do not utilize the full quantum mechanics, but the Bohr–Sommerfeld discretization rules for (multiply) periodic motions

$$\oint p_j \mathrm{d}q_j = n_j 2\pi\hbar, \tag{2.78}$$

where q_j, p_j are generalized coordinates and canonically conjugate momenta, respectively, and n_j are integers. Using the equation

$$\hbar = \frac{\tilde{e}^2}{\alpha_c c}, \tag{2.79}$$

where c is the velocity of light and the charge of the electron $\tilde{e}$,

$$\tilde{e} = \frac{e}{\sqrt{4\pi}}\epsilon_0, \tag{2.80}$$

with the elementary charge $e = 1.602 \times 10^{-19}$C, the permittivity of the vacuum ϵ_0, the constant of fine structure α_c, we obtain that the periodic plane motions of a particle should fulfil the equations

$$\oint p_r \mathrm{d}r = k2\pi \frac{\tilde{e}^2}{\alpha_c c}, \tag{2.81}$$

$$\oint p_\varphi \mathrm{d}\varphi = l2\pi \frac{\tilde{e}^2}{\alpha_c c}, \tag{2.82}$$

where k and l are the radial and azimuthal numbers, respectively. This way the particle-mass independence of the orbits is not obtained. Hypothetically, the authors replace

$$\frac{\tilde{e}^2}{\alpha_c} \to \frac{GMm}{\alpha_g}, \tag{2.83}$$

which gives

$$\oint p_r \mathrm{d}r = k2\pi \frac{GMm}{\alpha_g c}, \tag{2.84}$$

$$\oint p_\varphi \mathrm{d}\varphi = l2\pi \frac{GMm}{\alpha_g c}, \tag{2.85}$$

where α_g is the gravitational constant. This assumption entails the particle-mass, m, the independence of orbits. Nevertheless, a new quantization rule is not needed, when one correctly writes p_c instead of p and uses the new kinetic energy. In application to the solar system, we put $M = M_{\mathrm{Sun}}$. The principal number $n = k + l$ is introduced.

The major half-axes of the elliptic orbits have the form

$$a_n = a_1^{\mathrm{Sun}} n^2, \tag{2.86}$$

where

$$a_1^{\mathrm{Sun}} = \frac{1}{\alpha_g^2} \frac{GM_{\mathrm{Sun}}}{c^2}. \tag{2.87}$$

For circular orbits, the canonical momentum $p_r = 0$, $k = 0$. The estimation of a_1^{Sun} has been carried out and, simultaneously, the numbers $n(p)$ for the planets have been searched for. It has been found that $a_1^{\mathrm{Sun}} = (0.0439 \pm 0.0004)$ AU and

$$n(\mathrm{Mercury}) = 3, \quad n(\mathrm{Venus}) = 4, \quad n(\mathrm{Earth}) = 5, \quad n(\mathrm{Mars}) = 6, \quad n(\mathrm{Ceres}) = 8,$$
$$n(\mathrm{Jupiter}) = 11, \quad n(\mathrm{Saturn}) = 15, \quad n(\mathrm{Uranus}) = 21, \quad n(\mathrm{Neptune}) = 26,$$
$$n(\mathrm{Pluto}) = 30. \tag{2.88}$$

From this

$$\frac{1}{\alpha_g} = 2113 \pm 15. \tag{2.89}$$

It can be objected that the number of parameters is greater by one than the number of observations, so that the solution of the problem is too easy. But it would be an oversimplified opinion, since most of the unknowns are integers. Such parameters cannot be counted as "full-blown" degrees of freedom. Intuitively, a quasicontinuum of consecutive integers (here $3, 4, 5, 6$) would be counted as a value of a single parameter.

The assumption that α_g is a universal constant is tested by satellites of the planets. The first allowed orbit, which lies out of the planet, has a relatively high (≥ 19) principal number for Mercury, Venus, the Earth, and Mars. For the giant planets, such orbits have lesser numbers. But the "universality" of the constant α_g has not been confirmed.

Agnese and Festa (1997) indicate that the spin of a purely gravitationally bounded celestial body may be proportional to the square of its mass,

$$J = \frac{1}{2}\frac{Gm^2}{\alpha_g c}. \tag{2.90}$$

In fact, the law $J = pM^2$, where p is a proportionality constant, has been proposed, with $p = 8 \times 10^{-17}$ m^{2}kg^{-1}s^{-1} [Wesson (1981)]. In continuation, these authors have calculated $G/(2\alpha_g c) = 2.35 \times 10^{-16}$ m^{2}kg^{-1}s^{-1}.

Rubčić and Rubčić (1998) have criticized the vacant orbits in the quadratic law in [Agnese and Festa (1997)]. They have been aware that it is advantageous to analyze the terrestrial planets and the Jovian planets as separate systems. The quadratic law governs even the satellites of Jupiter, Saturn and Uranus [Rubčić and Rubčić (1995, 1996)].

They relate the quantization in the cosmic world to the effect of chaos [Nottale (1993, 1996)]. Rubčić and Rubčić (1998) begin with the "quantization" of the specific angular momentum $\frac{J_n}{m_n}$, where J_n is the angular momentum of the planet and m_n is its mass. Then $\frac{J_n}{m_n} = nH'$, with $H' = fAM$, where M is the mass of the central body,

$$A = 2\pi\frac{G}{\alpha_c c}, \tag{2.91}$$

and f indicates the impossibility of an invariable quantization of the systems considered. Particularly, $f = 2.41 \pm 0.03$ for the terrestrial planets and $f = 12.61 \pm 0.16$ for the Jovian ones. The principle of consecutive numeration is respected. Especially, Mercury, Venus, the Earth, Mars, Ceres are numbered by 3, 4, 5, 6, 8 and, e.g., Jupiter, Saturn and Uranus are labeled by 2, 3 and 4, respectively. The asteroid Ceres only represents the main belt, which is related to numbers 7 to 9.

As an approximation, the radius of the nth orbit is

$$r_n = \frac{1}{G}(fA)^2 Mn^2. \tag{2.92}$$

Through an analysis of the behaviour of f^{-1}, a rougher approximation (cf., equation (13) in [Rubčić and Rubčić (1998)]) has been derived,

$$r_n = \frac{GM}{v_0^2 k^2} n^2, \tag{2.93}$$

where $v_0 = (25.0 \pm 0.7)$ kms^{-1} and $k = 1$ for the Jovian planets and the satellites of Uranus, $k = 2, 4$ for the satellites of Jupiter and Saturn, respectively, and $k = 6$ for the terrestrial planets. The value $v_0 \approx 24$ kms^{-1} is one of increments of galactic redshifts [Arp and Sulentic (1985), Rubčić and Rubčić (1998)], cf., also [Arp (1998)].

The universal constant A can be written in the form

$$A = \frac{h}{m_0^2}, \tag{2.94}$$

where $m_0^2 = \alpha_c m_{\mathrm{P}}^2$, m_{P} is Planck's mass.

After the observation of quantized redshifts, Dersakissian (1984) formulated a cosmic form of quantum theory. The studies of the redshift of galaxies reverberate also in [Carvalho (1985)]. In that paper, the quantized gravitational energy (2.62) has been obtained using a model, which we understand on paying attention to an illustrative example in [Greenberger (1983)]. Indeed, the equation for p, $[x, p] = i\hbar g(x)$, where $g(x)$ is a potential function, can be written as an equation for H,

$$\frac{\partial^2 H}{\partial p_c^2} = \frac{1}{m} g(x), \tag{2.95}$$

which leads to the Hamiltonian by Carvalho (1985) on generalization to three dimensions, where, e.g., $g(\mathbf{r})$ must replace $g(x)$ and be proportional to $\frac{1}{|\mathbf{r}|^2}$.

2.5.2 Tentative "universal" constants

Carneiro (1998) remembers Dirac's hypothesis that cosmological large numbers, as mass M, radius R and age T of our Universe can be related to the typical values of mass m, size r and life time t appearing in particle physics, by a scale factor $\Lambda \sim 10^{38}$–10^{41} [Dirac (1937, 1938)]

$$\frac{T}{t} = \frac{R}{r} = \left(\frac{M}{m}\right)^{\frac{1}{2}} = \Lambda. \tag{2.96}$$

A simple one-dimensional analysis is utilized for introducing new concepts as a scaled quantum of action H,

$$\frac{H}{h} = \Lambda^3. \tag{2.97}$$

If Universe rotates, its spin must be of the order $\frac{H}{2\pi} \sim 10^8$ Js. Another, intermediate, scale of quantization related to the angular momenta of stars is proposed, whose values concentrate around $\frac{H'}{2\pi} \sim 10^{42}$ Js,

$$\frac{T'}{t} = \frac{R'}{r} = \left(\frac{M'}{m}\right)^{\frac{1}{3}} = \lambda. \tag{2.98}$$

Here $\lambda = \sqrt{\Lambda}$, $T' \sim 10^4$ s, $R' \sim 10^4$ m and $M' \sim 10^{30}$ kg. For the astrophysical meaning of these values, we refer to the paper [Carneiro (1998)]. Gravitation constants of three kinds are considered, $G = G_2$ on the Λ scale, $G' = G_3$ on the λ scale and g on the microscopic scale,

$$G_n = g\left(\frac{M_n}{m}\right)^{\frac{1}{n}-1}, n = 2, 3, \tag{2.99}$$

where $M_2 = M$ and $M_3 = M'$. Here g can be named the strong gravity constant [Recami, Raciti, Rodrigues, Jr., and Zanchin (1994)]. It has been guaranteed that $G_3 = G_2$, or $G' = G$.

In a continuation of [Agnese and Festa (1997)], it has been chosen

$$H' = 2\pi\sqrt{GMm^2 r_1}, \tag{2.100}$$

where m is the average mass of planets of the solar system and $r_1 \equiv a_1^{\mathrm{Sun}}$, and calculated that $H' = 1.2 \times 10^{42}$ Js for $m = 2.10 \times 10^{26}$ kg [Carneiro (1998)].

Agnese and Festa (1998) have considered the universal constant of the form

$$v_* = \alpha_g c \tag{2.101}$$

and have found that $v_* \sim 143.7$ km s^{-1}. An elliptical orbit has been tried for Pluto, $(n, l) = (30, 29)$, and for 1996TL$_{66}$, $(n, l) = (44, 36)$, which is not much valued even by themselves. They have analyzed about twenty star–planet pairs. The extrasolar planets can be discovered near the star. The mass of these stars was not known, but it can be at least roughly estimated from the star type. From the theory, it follows that the ratio of the orbital periods of planets orbiting different stars at the level $n = 1$ equals the ratio between the masses of the respective stars. The predictions have agreed fairly for (stars) HD187123, τ Bootis and HD75289 and their companions.

Agnese and Festa (1999) have refined the analysis in the paper [Agnese and Festa (1997)] with the assumption that the distance of Jupiter from the Sun has least changed since the formation. They have obtained that

$$a_1^{\mathrm{Sun}} = 0.04297 \text{ AU}, \quad \frac{1}{a_g^{\mathrm{Sun}}} = 2086 \pm 14. \tag{2.102}$$

The planetary system of υ Andromedae with three planets was paid attention to. These planets are called b, c, d and the planet c is the most massive. From the orbital period and the principal number of the planet, the mass of the star can be found. As the mass of υ Andromedae is known with some accuracy, it has been feasible to number the planet c with $n = 4$. The authors then have corrected the mass of the star. Similarly, the numbers $n = 1$ for the planet b and $n = 7$ for the planet d have been determined, but the value of the stellar mass have no more been changed. The authors are aware that their scheme contradicts the theories of the planet migration. Since the numbers $n = 1, 4, 7$ are not consecutive, it is natural to assume the existence of smaller, not yet observed, planets.

Nottale, Schumacher and Lefèvre (2000) have continued the study in [Nottale (1996)]. They have accepted the gravitation coupling (not the "structure") constant $\alpha_g = \frac{w_0}{c}$. Planets in the inner solar system and exoplanets have been treated altogether. The mean velocity has been calculated from the parent star mass, M, and the planet period, P, as $v = \sqrt[3]{\frac{M}{P}}$. The histogram of the values of $|\delta n| = \left|\frac{144}{v} - n\right|$, with n being the nearest integer of $\frac{144}{v}$, has been plotted. The distribution has differed from the uniform one significantly. Rather a condensation of δn about $\delta n = 0$ has been observed.

de Oliveira Neto, Maia and Carneiro (2004) apply full quantum mechanics in contrast to [Agnese and Festa (1997)]. Such an approach is peculiar in that the indeterminism of the complete quantum mechanics does not permit to find the planetary distances with certainty. As in many similar cases, the predictions are formulated in terms of the planetary mean distances. In the application of the full quantum mechanics, wave functions are associated with the planets. For the calculation of the planetary mean distances, squares of moduli of these wave functions are used. Next comments may resemble the philosophy of quantum mechanics. A superposition of planetary wave functions is a wave function again. But this function does not describe all the planets simultaneously. At least in quantum mechanics, Schrödinger's cat paradox is not related to a description of two cats, but to that of two *states* of a single cat. In application to the planetary formation, we need a nonstandard interpretation of quantum mechanics indeed. Let us consider a weighted average of squares of moduli of wave functions, which differs from a square of a modulus of an appropriate superposition of these functions in the neglect of the interference terms. The weighted average respecting the masses of protoplanets much resembles the solar nebula according to the paper [Christodoulou and Kazanas (2008)].

de Oliveira Neto, Maia and Carneiro (2004) remember the paper [Nelson (1966)] based on the idea from the paper [Fényes (1952)]. A cosmic Brownian motion could lead to the validity of a Schrödinger-type diffusion equation. In the framework of this model, we can hardly imagine a Brownian motion, whose complexity would depend on the number of the planets to be formed. A theory of planet formation must then content with the superposition of wave functions, which "paradoxically" do not correspond to states of a single protoplanet, but diretly to the

Body	k	ℓ	$r_{k\ell}$ [AU]	Error [%]
—	$\frac{1}{2}$	0	0.055	—
Mercury	$\frac{3}{2}$	1	0.332	-15
Mercury	$\frac{3}{2}$	0	0.387	-1
Venus	$\frac{5}{2}$	2	0.829	15
Earth	$\frac{5}{2}$	1	0.995	-0.5
Earth	$\frac{5}{2}$	0	1.050	5
Mars	$\frac{7}{2}$	3	1.548	2
Hungaria	$\frac{7}{2}$	2	1.824	-6
Hungaria	$\frac{7}{2}$	1	1.990	2.5
Hungaria	$\frac{7}{2}$	0	2.046	5.5
Vesta	$\frac{9}{2}$	4	2.488	5.5
Ceres	$\frac{9}{2}$	3	2.875	9
Hygeia	$\frac{9}{2}$	2	3.151	-0.5
Camilla	$\frac{9}{2}$	1	3.317	-4.5
Camilla	$\frac{9}{2}$	0	3.372	-3
Jupiter	$\frac{11}{2}$	0	5.031	-3
—	$\frac{13}{2}$	0	7.021	—
Saturn	$\frac{15}{2}$	0	9.343	-2
Chiron	$\frac{17}{2}$	0	11.997	-12.5
Chiron	$\frac{19}{2}$	0	14.982	9.5
Uranus	$\frac{21}{2}$	0	18.300	-4.5
—	$\frac{23}{2}$	0	21.948	—
HA2 (1992), DW2 (1995)	$\frac{25}{2}$	0	25.929	4.5
Neptune	$\frac{27}{2}$	0	30.241	0.5
—	$\frac{29}{2}$	0	34.885	—
Pluto	$\frac{31}{2}$	0	39.861	1

Table 2.3: Predicted distances of bodies from the Sun.

protoplanets.

It is proper to remember the book [Nottale (1993)], where the inner and outer planetary systems are treated separately. de Oliveira Neto, Maia and Carneiro (2004) solve the time-independent Schrödinger equation with Newton's potential in two dimensions. In order a relationship to the planetary system to be established, also the component of the Hamiltonian, which is the kinetic energy, must be changed, as we have mentioned above. As Peřinová, Lukš and Pintr (2007) proceed almost in the same way, deviating only in detail, we do not give the theory twice, we present it only in what follows. Using the numbers k, $k = n - \frac{1}{2}$, where n is the principal quantum number and ℓ, $\ell = l - 1$, the following correspondences have been established in Table 2.3.

It is obvious that, only in several cases, the assumption of circular orbits [Agnese and Festa (1997)] $l = n$, or $\ell + 1 = k + \frac{1}{2}$, or $\ell = k - \frac{1}{2}$ is used.

2.5.3 Membrane model

Mohorovičić (1938) returned to the matter of distances of the bodies of the solar system. He established a law, which can be expressed as follows: The distance of every body of the solar system from the Sun is given in the astronomic units by the formula

$$r_k^{\mp} = 3.363(1 \mp 0.88638^k), \tag{2.103}$$

where the minus sign corresponds to inner parts of the solar system and the plus sign describes the more remote regions of the solar system. The number k can assume positive integer values for the minus sign, it can take both positive and negative integer values for the plus sign in the formula (2.103). If other real numbers are substituted, the result corresponds to unstable orbits. This formula encompasses all inner planets and asteroids, giant planets, and even the distances of comets. It cannot be recommended without tables, in which the numeration of planets is striking. In comparison with the Titius–Bode series, we have $k = 1, 2, 3, 5, 14$ for the bodies that are respected by the Titius–Bode series. But the Mohorovičić formula is double similar as proposals of some other authors. Perhaps, the split is peculiar. So far we have mentioned the terrestrial planets and the asteroid Ceres, the formula with the upper sign. We have $k = 5, -5, -13, -17, -20$ for farther bodies, which are described by the Titius–Bode series well or wrong. The Titius–Bode series would give good predictions to the bodies with $k = 5, -5, -13$ and the lower sign. These chosen values evidence the attention paid to the asteroids and comets. Typical is the condensation of orbits in the asteroid belt, where at 3.363 AU there is also the interface.

Pintr and Peřinová (2003–2004) generalize the Mohorovičić formula for stars of different masses, noting that the formula (2.103) can be rewritten in the form

$$a_k = a_{\text{lim}}(1 \mp 0.88638^k), \tag{2.104}$$

where $a_{\text{lim}} = 3.363$ AU for our solar system. Substituting

$$a_{\text{lim}} = \frac{GM_{\text{s}}}{\mu_g^2 c^2 1.5 \times 10^{11}}, \tag{2.105}$$

where M_{s} is the stellar mass, c is the speed of light and μ_g is a Mohorovičić constant with the property

$$\frac{1}{\mu_g} = 18448.1, \tag{2.106}$$

they obtain the calculated distances of the bodies in AU in the form

$$a_k = \frac{GM_{\text{s}}(1 \mp 0.88638^k)}{\mu_g^2 c^2 1.5 \times 10^{11}}. \tag{2.107}$$

In a comparison with the paper [Agnese and Festa (1997)], it can be stated that the interface, $a_{\rm lim}$, is obtained for $n = 10$, but the correspondence is very rough.

The relation (2.107) has been modified and for distances of moons of the giant planets, a similar relation has been obtained

$$a_k = \frac{GM_{\rm p}(1 \mp 0.88638^k)}{\mu_g^2 c^2}, \qquad (2.108)$$

where the distance is measured in meters. Tables for moons and rings of Jupiter, Saturn, Uranus and Neptune are presented, in which they observe that:
(i) In the case of Jupiter, $k = 18$ for an inner moon and $k = 8$ for an outer moon is utilized.
(ii) In the case of Saturn, $k = 33$ for an inner ring and $k = 14$ for an outer ring is utilized.
(iii) In the case of Uranus, the interface is not utilized, the ring and moons are outer.
(iv) In the case of Neptune, the interface is not utilized, the moons are outer.
When the interface is utilized, the tables lack indication of the sign in the formula (2.108).

The formula (2.107) has been applied to the systems of υ Andromedae and 47 Ursae Majoris. The interface has not been utilized. The planets are inner.

Peřinová, Lukš and Pintr (2007) have solved the Schrödinger equation with appropriately modified Hamiltonian deviating in detail from the paper [de Oliveira Neto, Maia and Carneiro (2004)]. They formulate the problem as follows. They consider a body of the mass $M_{\rm p}$, which orbits a central body of the mass $M_{\rm s}$ and has the potential energy $V(x, y, z)$ in its gravitational field. As planets and moons of the giant planets revolve approximately in the same plane, they consider $z = 0$. As they revolve in the same direction, they choose directions of the axes x, y and z such that the planets or moons of giant planets revolve anticlockwise. Then they write the modified Schrödinger equation for the wave function $\psi = \psi(x, y)$ from the part of the Hilbert space $L_2(R^2) \cap C^2(R^2)$ and the eigenvalue $0 > E \in R$ in the form

$$-\frac{\hbar_M^2}{2M_{\rm p}}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi + V(x, y)\psi = E\psi, \qquad (2.109)$$

where $\hbar_M \approx 1.48 \times 10^{15} M_{\rm p}$, $V(x, y) = V(x, y, z)$ and E is the total energy. Negative values of E classically correspond to the elliptic Kepler orbits and the localization property (bound state) is also conserved in the quantum mechanics for such total energies E. The factor 1.48×10^{15} is not a dimensionless number, but the unit of its measurement is $\rm m^2 s^{-1}$. With respect to the unusual unit, they do not wonder that Agnese and Festa (1997) consider this factor in the form of a product, such that $\hbar_M = \bar{\lambda}_M c M_{\rm p}$, where $\bar{\lambda}_M \approx 4.94 \times 10^6$ m.

They transform equation (2.109) into the polar coordinates,

$$-\frac{\hbar_M^2}{2M_{\rm p}}\left(\frac{\partial^2\tilde{\psi}}{\partial r^2} + \frac{1}{r}\frac{\partial\tilde{\psi}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\tilde{\psi}}{\partial\theta^2}\right) + \tilde{V}(r)\tilde{\psi} = E\tilde{\psi}, \qquad (2.110)$$

where $\tilde{\psi} \equiv \tilde{\psi}(r,\theta)=\psi(r\cos\theta, r\sin\theta)$, $\tilde{V}(r) = V(r\cos\theta, r\sin\theta)$ does not depend on θ. Particularly, they choose

$$\tilde{V}(r) = -\frac{GM_\mathrm{p}M_\mathrm{s}}{r}. \tag{2.111}$$

With respect to the Fourier method, they assume a solution of equation (2.111) in the form

$$\tilde{\psi}(r,\theta) = R(r)\Theta(\theta). \tag{2.112}$$

The original eigenvalue problem is transformed equivalently to two eigenvalue problems

$$\Theta''(\theta) = -\Lambda\Theta(\theta), \tag{2.113}$$

$$\Theta(0) = \Theta(2\pi) \tag{2.114}$$

and

$$R''(r) + \frac{1}{r}R'(r) + \left\{-\frac{\Lambda}{r^2} + \left[E - \tilde{V}(r)\frac{2M_\mathrm{p}}{\hbar_M^2}\right]\right\} R(r) = 0, \tag{2.115}$$

$$\lim_{r\to 0+}[\sqrt{r}R(r)] = 0, \quad \sqrt{r}R(r) \in L_2((0,\infty)). \tag{2.116}$$

The solution of the problem (2.113)–(2.114) has the form

$$\Theta_\ell(\theta) = \frac{1}{\sqrt{2\pi}}\exp(i\ell\theta) \tag{2.117}$$

for $\ell = \pm\sqrt{\Lambda} \in Z$.

Here $\ell = 0$ should mean a body, which does not revolve at all. In the classical mechanics, such a body moves close to a line segment ending at the central body, and it spends a short time in the vicinity of this body. Peřinová, Lukš and Pintr (2007) utilize some – not all – of the concepts of quantum mechanics and will not avoid the case $\ell = 0$ [de Oliveira Neto, Maia and Carneiro (2004)]. In the formula (2.117), $\ell = 1, 2, \ldots, \infty$ corresponds to the anticlockwise revolution.

On respecting (2.111), equation (2.115) becomes

$$R''(r) + \frac{1}{r}R'(r) + \left\{-\frac{\ell^2}{r^2} - B - \frac{2M_\mathrm{p}}{\hbar_M^2}\left(-\frac{GM_\mathrm{p}M_\mathrm{s}}{r}\right)\right\} R(r) = 0, \tag{2.118}$$

where

$$B = -\frac{2M_\mathrm{p}E}{\hbar_M^2} = -\frac{2}{(\bar{\lambda}_M c)^2}\frac{E}{M_\mathrm{p}}. \tag{2.119}$$

It holds that

$$\frac{M_\mathrm{p}GM_\mathrm{p}M_\mathrm{s}}{\hbar_M^2} = \frac{GM_\mathrm{s}}{(\bar{\lambda}_M c)^2}. \tag{2.120}$$

On substituting $r = \frac{\rho}{2\sqrt{B}}$ and introducing

$$\tilde{R}(\rho) = R\left(\frac{\rho}{2\sqrt{B}}\right), \tag{2.121}$$

equation (2.118) becomes

$$\tilde{R}''(\rho) + \frac{1}{\rho}\tilde{R}'(\rho) + \left(-\frac{1}{4} + \frac{k}{\rho} - \frac{\ell^2}{\rho^2}\right)\tilde{R}(\rho) = 0, \tag{2.122}$$

where

$$k = \frac{GM_\mathrm{s}}{(\bar{\lambda}_M c)^2\sqrt{B}}. \tag{2.123}$$

For later reference, it holds inversely that

$$\sqrt{B} = \frac{GM_\mathrm{s}}{(\bar{\lambda}_M c)^2 k}, \tag{2.124}$$

$$\frac{-E}{M_\mathrm{p}} = \frac{(\bar{\lambda}_M c)^2}{2}B = \frac{(GM_\mathrm{s})^2}{2(\bar{\lambda}_M c)^2 k^2}. \tag{2.125}$$

Expressing $\tilde{R}(\rho)$ in the form

$$\tilde{R}(\rho) = \frac{1}{\sqrt{\rho}}u\left(\rho\right), \tag{2.126}$$

they obtain an equation for $u(\rho)$,

$$u''(\rho) + \left[-\frac{1}{4} + \frac{k}{\rho} - \left(\ell'^2 - \frac{1}{4}\right)\frac{1}{\rho^2}\right]u\left(\rho\right) = 0, \tag{2.127}$$

where $\ell' = \ell$. It is familiar that this equation has two linear independent solutions $M_{k,\ell'}(\rho)$, $M_{k,-\ell'}(\rho)$, if ℓ' is not an integer number. When ℓ' is integer, the solution $M_{k,-\ell'}(\rho)$ must be replaced with a more complicated solution. It can be proven that the other solution is not regular for $\rho = 0$ (it diverges as $\ln\rho$ for $\rho \to 0$). The remaining solution $M_{k,\ell}(\rho)$ can be transformed to a wave function from the space $L_2((0,\infty))$ if and only if $k - \ell - \frac{1}{2} = n_\mathrm{r}$ is any nonnegative integer number. They choose this function to be

$$u_{k\ell}(\rho) = C_{k\ell}M_{k,\ell}(\rho), \tag{2.128}$$

where $C_{k\ell}$ is an appropriate normalization constant and $M_{k,\ell}(\rho)$ is a Whittaker function, namely

$$M_{k,\ell}(\rho) = \rho^{\ell+\frac{1}{2}}\exp\left(-\frac{\rho}{2}\right)\Phi\left(\ell - k + \frac{1}{2}, 2\ell + 1; \rho\right), \tag{2.129}$$

where Φ is the confluent (or degenerate) hypergeometric function. In the formula (2.128), the constant $C_{k\ell}$ has the property

$$\int_0^\infty r[R_{k\ell}(r)]^2\,\mathrm{d}r = 1, \tag{2.130}$$

or it is

$$C_{k\ell} = 2\sqrt{B}\frac{1}{(2\ell)!}\sqrt{\frac{(n+\ell-1)!}{2k(n-\ell-1)!}}. \tag{2.131}$$

Then

$$R_{k\ell}(r) = 2\sqrt{B}\sqrt{\frac{(n-\ell-1)!}{2k\Gamma(n+\ell)}}\exp(-r\sqrt{B})(2r\sqrt{B})^\ell L^{2\ell}_{n-\ell-1}(2r\sqrt{B}), \tag{2.132}$$

where $n = k + \frac{1}{2}$, $L^{2\ell}_{n-\ell-1}(2r\sqrt{B})$ is a Laguerre polynomial and the relation (2.124) holds.

Having solved the modified Schrödinger equation, they address an interpretation of the formulae derived. The probability density $P_{k\ell}(r)$ of the revolving body occurring at the distance r from the central body is

$$P_{k\ell}(r) = r[R_{k\ell}(r)]^2,\, r \in [0,\infty). \tag{2.133}$$

Mean distances of the planets are given by the relation

$$r_{k\ell} = \int_0^\infty rP_{k\ell}(r)\,\mathrm{d}r \tag{2.134}$$

$$= \frac{(\bar{\lambda}_M c)^2}{4GM_\mathrm{s}}\left[(2k-n_\mathrm{r})(2k-n_\mathrm{r}+1) + 4n_\mathrm{r}(2k-n_\mathrm{r}) + n_\mathrm{r}(n_\mathrm{r}-1)\right], \tag{2.135}$$

where $n_\mathrm{r} = n - \ell - 1$, $k = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, \infty$ and $\ell = 0, 1, 2, \ldots, n$. In fact, it is a particular case $d = 2$ of a formula depending on the dimension d. It reduces to the familiar formula for $d = 3$. The derivations in the framework of the results of [Nouri (1999)] are easy.

Peřinová, Lukš and Pintr (2007) recall that de Oliveira Neto, Maia and Carneiro (2004) have defined the Bohr radius of the solar system $r_{\frac{1}{2}0} = 0.055$ AU. As Agnese and Festa (1997) have preferred circular orbits, they expect an approach with $\ell = k - \frac{1}{2}$, which has not been adopted by de Oliveira Neto, Maia and Carneiro (2004). As Agnese and Festa (1997) have demonstrated, such an approach may require a different Bohr radius.

The orbits, on which big bodies – planets – may originate, are listed in Table 2.4. It emerges that for every number k, there exists only one stable orbit, on which a big body – a planet – may originate. Then they can interpret the number k as the principal quantum number and ℓ as the orbital quantum number equal to the

Body	k	ℓ	$r_{k\ell}$ [AU]	Error [%]
—	$\frac{1}{2}$	0	0.055	—
Mercury	$\frac{3}{2}$	1	0.332	-15
Venus	$\frac{5}{2}$	2	0.83	15.5
Mars	$\frac{7}{2}$	3	1.54	1.5
Vesta	$\frac{9}{2}$	4	2.49	5.5
Fayet comet	$\frac{11}{2}$	5	3.64	-3.5
Jupiter	$\frac{13}{2}$	6	5.03	-3.5
Neujmin comet	$\frac{15}{2}$	7	6.636	-2.5
—	$\frac{17}{2}$	8	8.46	—
Saturn	$\frac{19}{2}$	9	10.5	10
—	$\frac{21}{2}$	10	12.77	—
Westphal comet	$\frac{23}{2}$	11	15.26	-2.5
Pons–Brooks comet	$\frac{25}{2}$	12	17.97	4
Uranus	$\frac{27}{2}$	13	20.9	9
—	$\frac{29}{2}$	14	24.055	—
—	$\frac{31}{2}$	15	27.43	—
Neptune	$\frac{33}{2}$	16	31.02	3
—	$\frac{35}{2}$	17	34.84	—
Pluto	$\frac{37}{2}$	18	38.88	-1.5
—	$\frac{39}{2}$	19	43.134	—

Table 2.4: Bodies with stable circular orbits.

number of possible orbits, but only for the greatest ℓ there exists a stable orbit of a future body. A planet which does not confirm this theory, is the Earth. Since the description based on the modified Schrödinger equation for the planetary system is not fundamental, it could not fit all the stable orbits. Other deviations are likely to be incurred by collisions of the bodies in early stages of the origin of the planets, thus nowadays it is already possible to observe elliptical orbits, which are very close to circular orbits.

Using the graphs of the probability densities that they have plotted for every predicted orbit of this system, they have obtained expected results. The graphs of the probability densities for each orbit with $k \leq \frac{9}{2}$ and with $\frac{11}{2} \leq k \leq \frac{39}{2}$ are contained in figure 3.1 and in figure 3.2, respectively. The vertical axis denotes the probability density $P_{k\ell}(r)$ and the longitudinal axis designates the planetary distance r from the Sun. In figure 3.1, the graph for $n = 2$ is interpreted such that the highest probability density is assigned to the orbit of the radius of 0.332 AU and from the calm shape of the graph they infer that an ideal circular orbit is tested.

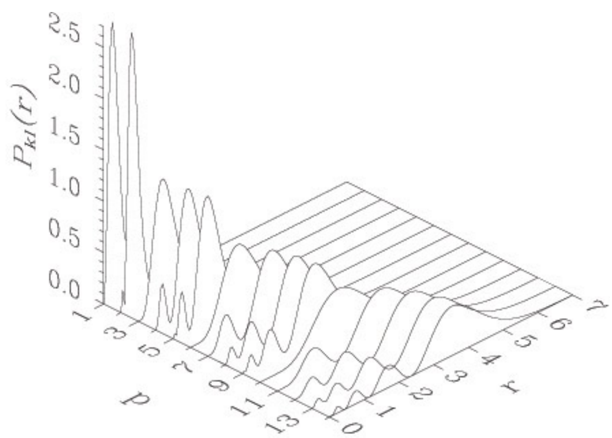

Figure 2.1: Probability densities for a particle in states with quantum numbers k, ℓ, which correspond, respectively ($n = k + \frac{1}{2}$), to Mercury ($n = 2$, $\ell = 1$), Venus ($n = 3$, $\ell = 2$), Mars ($n = 4$, $\ell = 3$) and asteroid Vesta ($n = 5$, $\ell = 4$). Here $k \in \{\frac{3}{2}, \frac{5}{2}, \ldots, \frac{9}{2}\}$ and r is measured in AU.

The previous procedure has been applied to moons of giant planets [Peřinová, Lukš and Pintr (2007)]. It emerges that the moons of giant planets are also fitted by the modified Schrödinger equation and appropriate expectation values. Especially, the predicted stable circular orbits of Jupiter's moons are presented in Table 2.5. For Jupiter it holds that $M_{\mathrm{s}} = M^{(\mathrm{Jupiter})}$ and the Bohr radius of this system is $r_1 = 6287$ km. It emerges that the predicted lunar orbits fit the measured orbits of the moons orbiting Jupiter.

Pintr, Peřinová and Lukš (2008) have paid attention to the following event. In the year 2004, a new theory emerged, which assumes basing on a study of chemical compounds in meteorites and a study of astronomical objects of the distant universe

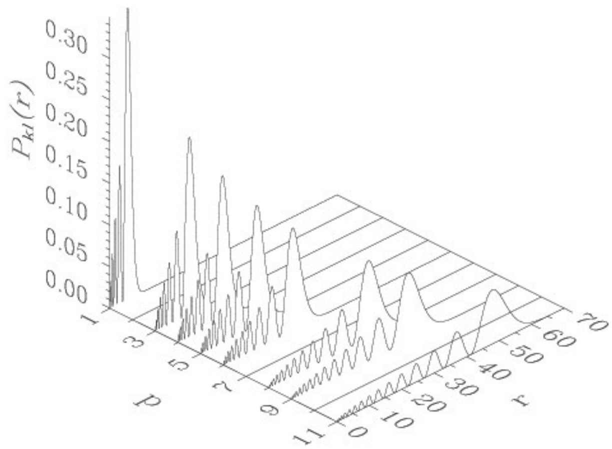

Figure 2.2: Probability densities for a particle in states with quantum numbers k, ℓ, which correspond, respectively ($n = k + \frac{1}{2}$), to Fayet comet ($n = 6$), Jupiter ($n = 7$), Neujmin comet ($n = 8$), Saturn ($n = 10$), Westphal comet ($n = 12$), Pons-Brooks comet ($n = 13$), Uranus ($n = 14$), Neptune ($n = 17$) and Pluto ($n = 19$), $k \in \{\frac{11}{2}, \frac{13}{2}, \ldots, \frac{39}{2}\}$, $\ell = n - 1$. Here r is measured in AU.

that, like most low-mass stars, the Sun formed in a high-mass star-forming region, where some stars went supernova [Desch, Healy and Leshin (2004)]. The radiation from massive stars carves out ionized cavities in the dense clouds, within which the stars formed. Such regions are called H II (ionized hydrogen regions). Examples of these regions of star formations are the Orion Nebula, the Eagle Nebula, and many other nebulae. The decisive argument of this new theory is the presence of isotope ^{60}Fe in meteorites. This isotope is unstable and it may originate only in cores of high-mass stars. The presence of the isotope ^{60}Fe in our solar system favours the hypothesis that the Sun was born near high-mass stars in ionized clouds of gas and dust. The protagonist of the new theory is Hester and the group of astronomers in the Arizona State University [Hester, Desch, Healy and Leshin (2004)]. The origin of the solar system happened probably as follows: A shock wave, which compresses molecular gas into dense cores, is driven in advance of an ionization front. After the cores emerge into the H II region interior, they evaporate. Some of the cores contain a star and a circumstellar disk. The disk evaporates. The massive stars pelt the low-mass young stellar objects and the protoplanetary disks, which surround them, with ejecta.

Assuming this to be a probable mechanism of the origin of stellar systems, Pintr, Peřinová and Lukš (2008) attempt to find answers to some basic questions about the traits of the solar system. They consider our solar system on some simplifications:

(i) The orbits of the bodies about the Sun are considered to be circular.

(ii) These orbits lay in one plane with respect to the Sun.

Body	k	ℓ	$r_{k\ell}$ [km]	Error [%]
—	$\frac{1}{2}$	0	6287	—
—	$\frac{3}{2}$	1	37722	—
Halo ring	$\frac{5}{2}$	2	94305	5.5–-23.5
Outer ring	$\frac{7}{2}$	3	176036	43–-27.5
—	$\frac{9}{2}$	4	282915	—
Io	$\frac{11}{2}$	5	414942	-1.5
Europa	$\frac{13}{2}$	6	572117	-14.5
—	$\frac{15}{2}$	7	754440	—
—	$\frac{17}{2}$	8	961911	—
Ganymede	$\frac{19}{2}$	9	1.19×10^6	11
—	$\frac{21}{2}$	10	1.452×10^6	—
Callisto	$\frac{23}{2}$	11	1.735×10^6	-8

Table 2.5: Moons of Jupiter with stable circular orbits.

They do not explain the elliptical orbits and the inclinations of these orbits. Then they can define a two-dimensional planetary model of a radius r_0.

Pintr, Peřinová and Lukš (2008) argue that an ionized nebula is the medium, where hydromagnetic waves may propagate at a velocity, v_{A} (Alfvén's velocity). For understanding possible evolution of the solar system, they introduce the circular membrane model (two-dimensional model) of the radius r_0. It describes a perfectly flexible circular membrane of a constant thickness [Brepta, Půst and Turek (1994)]. It is pulled with a force F_l over a unit of length acting at the distance r_0 from the centre of the membrane. In formulating the equation of motion, they take into account only the transverse displacements w, assuming that they are small in comparison with the size of the circular membrane. They let μ_{A} denote the mass of a unit of area.

Introducing polar coordinates r and φ with the relations $x = r\cos\varphi$ and $y = r\sin\varphi$, they solve the appropriate problem for the transverse displacement $\tilde{w}$ using the method of separation of variables. This method is based on solutions in the form

$$\tilde{w}(r,\varphi,t) = R(r,\varphi)T(t), \tag{2.136}$$

where $T(t)$ is a solution of a simpler problem and

$$R(r,\varphi) = R_1(r)\Phi(\varphi), \tag{2.137}$$

with $R_1(r)$ and $\Phi(\varphi)$ solutions of simpler problems. Below, $J_n(z)$ will be the Bessel function of the order n, $\mu_m^{(n)}$ will be positive roots of the equation

$$J_n(\mu) = 0, \tag{2.138}$$

and

$$\lambda_m^{(n)} = \pm\frac{\mu_m^{(n)}}{r_0}, \quad n = 0, 1, \dots, \infty, \quad m = 1, 2, \dots, \infty. \tag{2.139}$$

Omitting the standard developments, they state that

$$\tilde{w}_{nm}(r, \varphi, t) = R_{1nm}(r)\Phi_n(\varphi)T_{nm}(t), n = 0, 1, \dots, \infty, m = 1, 2, \dots, \infty, \tag{2.140}$$

where they have taken into account the multiplicity of the solutions and

$$R_{1nm}(r) = C_n J_n\left(\mu_m^{(n)}\frac{r}{r_0}\right), \tag{2.141}$$

with $\lambda_m^{(n)} \geq 0$,

$$\Phi_n(\varphi) = A_n \cos(n\varphi) + B_n \sin(n\varphi), \tag{2.142}$$

$$T_{nm}(t) = E_{nm} \exp(i\Omega_{nm}t), \tag{2.143}$$

where

$$\Omega_{nm} = \lambda_m^{(n)}\sqrt{\frac{F_l}{\mu_{\mathrm{A}}}}. \tag{2.144}$$

The superposition form of a solution suggests that, without a loss of generality, they may specify

$$A_{nm} = 1, B_{nm} = -i, C_{nm} = 1, E_{nm} = 1. \tag{2.145}$$

The superposition solution (2.140) is valid for any circular membrane. For $n = 0$, the membrane has a nonzero amplitude at the point with $r_0 = 0$. For other n, $J_n(0) = 0$. The subscripts n and m control the number of nodal lines. The membrane has $2n$ radial nodal segments and m nodal circles including the circumference.

The membrane model considered must take into account the mass M_{s} of a future central body and the membrane radius r_0 must also be connected with this mass. A multiplicity has been conceded. A distance a_r is defined in the form

$$a_r = \frac{GM_{\mathrm{s}}}{c^2}, \tag{2.146}$$

which is known as half the Schwarzschild radius. On using the fine structure constant α_e, the radii $r_0^{(a)}$ are defined in the form

$$r_0^{(a)} = \frac{a_r}{\alpha_e^a}, \tag{2.147}$$

where α_e has the property $\frac{1}{\alpha_e} \simeq 137.0$ and a is an exponent, $a = 0, 1, 2, ..., \infty$.

Substituting M_{s} = M_{Sun}, Pintr, Peřinová and Lukš (2008) have observed the necessity of the following radii: $r_0^{(0)} = 1.48$ km, half the Schwarzschild gravitational radius of the Sun; $r_0^{(4)} = 3.481$ AU, the membrane for terrestrial planets (the centre of the asteroid belt); $r_0^{(5)} = 476.897$ AU, the membrane for giant planets and $r_0^{(6)} = 65334.9$ AU, the membrane for formation of the Oort cloud of comets.

Pintr, Peřinová and Lukš (2008) replace a mechanical velocity v with the Alfvén velocity $v_{\rm A}$. They rewrite the angular frequency of the membrane (2.144) in the form

$$\Omega_{nm}^{(a)} = \frac{\mu_m^{(n)}}{r_0^{(a)}} v_{\rm A}. \tag{2.148}$$

They may calculate the time of revolution of the ionized parts,

$$T_{nm}^{(a)} = 2\pi \frac{r_0^{(a)}}{\mu_m^{(n)}} \frac{1}{v_{\rm A}}. \tag{2.149}$$

This period depends on the physical properties of the membrane. It is assumed that a future body revolving on a circular orbit about born central body at a distance $r_{nm}^{(a)}$ inherits this period from the wave. The velocity of such a body can be written in two forms,

$$v_{nm}^{(a)} = \frac{2\pi r_{nm}^{(a)}}{T_{nm}^{(a)}} = \sqrt{\frac{GM_{\rm s}}{r_{nm}^{(a)}}}, \tag{2.150}$$

or (cf., [Murray and Dermott (1999)])

$$(2\pi)^2 r_{nm}^{(a)3} = GM_{\rm s} T_{nm}^{(a)2}. \tag{2.151}$$

This equation for $r_{nm}^{(a)}$ has the solution

$$r_{nm}^{(a)} = \sqrt[3]{\frac{GM_{\rm s} T_{nm}^{(a)2}}{(2\pi)^2}} = \sqrt[3]{\frac{GM_{\rm s} r_0^{(a)2}}{\mu_m^{(n)2} v_{\rm A}^2}}, \tag{2.152}$$

where (2.149) have been utilized.

The membrane model admits that, under conditions present at the origin of stellar systems, a hydromagnetic wave originates, whose nodal lines provide stable orbits of future planets. For the calculation of possible orbits of planets, it has been assumed that $n = 1$ and the Alfvén velocity for the inner parts of the solar system was $v_{\rm A}^{(4)} = 16500$ ms^{-1} (Table 2.6) and for the formation of the outer planets, it was $v_{\rm A}^{(5)} = 7000$ ms^{-1}, while $n = 0$ (Table 2.7).

In application to giant planets, it holds that $M_{\rm s} = M_{\rm p}$. A membrane model of Jupiter's system has been formulated with $r_0^{(0)} = 0.0014$ km, half the Schwarzschild gravitational radius of Jupiter, $r_0^{(4)} = 496043$ km, the membrane for small moons (rings), $r_0^{(5)} = 6.796 \times 10^7$ km, the membrane for natural moons, and $r_0^{(6)} = 9.310 \times 10^9$ km, the membrane for the remotest parts of the system.

m	$\mu_m^{(1)}$	r [AU]	Body
1	3.832	1.390	Mars
2	7.016	0.930	Earth
3	10.173	0.725	Venus
4	13.323	0.606	asteroid 1999 MN
5	16.471	0.526	—
6	19.616	0.468	—
7	22.760	0.424	—
8	25.903	0.389	Mercury

Table 2.6: Distances of formation of inner bodies of the solar system up to 0.389 AU.

m	$\mu_m^{(0)}$	r [AU]	Body
1	2.405	89.364	planet X
2	5.520	51.359	—
3	8.654	38.050	Pluto – Charon
4	11.792	30.963	Neptune
5	14.931	26.455	—
6	18.071	23.294	—
7	21.212	20.934	—
8	24.353	19.090	Uranus
9	27.494	17.610	Halley comet
10	30.635	16.384	—
11	33.776	15.352	—
12	36.917	14.468	—
13	40.058	13.701	—
14	43.200	13.029	—
15	46.341	12.434	—
16	49.483	11.901	—
17	52.624	11.423	—
18	55.766	10.989	—
19	58.907	10.596	—
20	62.049	10.234	—
21	65.190	9.903	—
22	68.331	9.590	Saturn
..	$\dots$	$\dots$	$\dots$

Table 2.7: Distances of formation of outer bodies of the solar system up to 9.59 AU.

A similar model of Saturn's system has been reported with $r_0^{(0)} = 0.00042$ km, half the Schwarzschild gravitational radius of Saturn, $r_0^{(4)} = 14803$ km, the membrane for small moons (centre of rings), $r_0^{(5)} = 2.028 \times 10^7$ km, the membrane for natural moons, and $r_0^{(6)} = 2.778 \times 10^9$ km, the membrane for the remotest parts.

In application to planets of υ Andromedae, it holds that $M_{\rm s} = M^{(\upsilon\ \rm And)}$. A membrane model of the υ-Andromedae system has been formulated with $r_0^{(0)} = 1.705$ km, half the Schwarzschild gravitational radius of υ Andromedae, $r_0^{(4)} = 4.003$ AU, the membrane for terrestrial planets (probable centre of planetoids), $r_0^{(5)} = 548.432$ AU, the membrane for Jovian planets, and $r_0^{(6)} = 75135.2$ AU, the membrane for the remotest parts of the system.

The last calculations have been related to the star GJ876. Such a model has been presented with $r_0^{(0)} = 0.474$ km, half the Schwarzschild gravitational radius of GJ876, $r_0^{(4)} = 1.114$ AU, the membrane for terrestrial planets (probable centre of planetoids), $r_0^{(5)} = 152.607$ AU, the membrane for Jovian planets, and $r_0^{(6)} = 20907.2$ AU, the membrane for the remotest parts of the system.

Chapter 3

Areal velocities of planets and their comparison

3.1 Introduction

The formation of the solar system and its development are well described in [Montmerle, Augereau and Chaussidon (2006)]. The existing theories presume the age of the solar system as 4.5 milliard years and that the entire system was created approximately 100 million years after the formation of the Sun. Despite of this, some of the chronological events of the formation of the system still remain unknown to us.

Comparing with the young T Tauri stars, we can say that the Sun formed in the centre of a protoplanetary disk with the dimensions of approximately 1000 AU. The planets formed in the first 10 million years after the formation of the protoplanetary disk. The development of the solar system was terminated approximately 90 million years after the formation of the protoplanetary disk.

Very interesting papers have been devoted to the mechanism in protoplanetary disks [von Weizsäcker (1943, 1947)]. Turbulent processes have been described in nascent protoplanetary nebulae. These topics have been the subject of many papers. The beginning of modern theories dates since Kuiper [Kuiper (1951)], who has shown that the protoplanetary nebulae would have to be more massive than the algebraic sum of masses of all planets.

We can divide papers describing the distribution of distances into several categories. Many empirical formulae describe the distances on the condition of suitable numbering of planets [Pintr, Peřinová and Lukš (2008)], [Nottale, Schumacher and Lefèvre (2000)]. Graner and Dubrulle have shown that using the rotational symmetry and the scale invariance, we can derive a geometric progression for any model system in the form

$$r_n = r_0 K^n, \tag{3.1}$$

where n is an integer number, r_0 is an initial distance, and K is a constant that determines the distribution of distances in the system [Graner and Dubrulle (1994)], [Dubrulle and Graner (1994)]. A successive numbering of planets is assumed as is respectable to such an impressive formula. Krot has created an evolutionary model of the rotating and gravitating spherical body [Krot (2009)]. He has remembered that with the aid of specific angular momentum of protoplanets, Schmidt derived the square root of radius R_n of the orbit for the nth protoplanet [Schmidt (1944)],

$$\sqrt{R_n} = a + bn, \tag{3.2}$$

where a and b are constants. Then he has generalized the Schmidt law for the solar system, leaving (3.2) a mere linear approximation.

The first quantum formulas are comparable in complexity with (3.2). Agnese and Festa [Agnese and Festa (1997, 1999)] have described the distances of planets in the solar system as a gravitational atom [Greenberger (1983)] using the famous Bohr-Sommerfeld rules. The successive numbering is possible only for the terrestrial planets. The other planets are numbered “suitably”, i.e., so that a best fit is achieved. They have shown that this description can be applied to extrasolar systems. They have proposed a gravitational constant in conformity with the clue to the unification of gravitation and particle physics [Wesson (1981)]. The hypothesis of a fundamental orbital distance 0.055 AU has been a very interesting result [Agnese and Festa (1997, 1999), Nottale (1993), Nottale, El Naschie, Al-Athel, Ord, editors (1996)]. A derivation of the Schrödinger equation [Nelson (1966, 1985)] from the Newton mechanics has inspired many variations of quantum description [Greenberger (1983), Carvalho (1985), Chechelnitsky (2000), Neto, Maia and Carneiro (2004)]. We can find solutions of the Schrödinger equation in [de Oliveira Neto, Maia and Carneiro (2004), Peřinová, Lukš and Pintr (2007)], which lead to possible discrete orbits by means of the quantum averaging. The distances of planets obtained in such a way exhibit a dependence on the main and orbital quantum numbers. The probability densities have been derived for each orbit and the number of possible orbits in the solar system has been reduced [Peřinová, Lukš and Pintr (2007)].

Till now 360 extrasolar planets have been discovered near stars with similar mass as the Sun. Every day we observe new extrasolar planets or protoplanetary disks. Theories of migrating planets suppose that, if two high mass planets form near each other, both the planets will change orbits around the star and also the collisions with next big bodies will change orbits of planets in a young planetary system [Murray et al. (1998), Murray, Paskowitz and Holman (2002), Spencer (2007)]. According to these theories, the predictions of orbits are very problematic.

In section 3.2, we will expound the method under application and use it for the planets in the outer part of the solar system. In section 3.3, we will apply it to the systems of moons around Jupiter and Uranus. In section 3.4, we will consider the extrasolar system HD10180 [Lovis et al. (2010)].

3.2 Correlation of areal velocities

Agnese and Festa (1997, 1999) have invented allowed planetary orbits with the major semi-axes and excentricities

$$\bar{a}_n = \bar{a}_1 n^2, \bar{\varepsilon}_{nl} = \sqrt{1 - \frac{l^2}{n^2}}, \tag{3.3}$$

respectively, where $\bar{a}_1$ means a possible first orbit of a planet, n is a principal number and l is an azimuthal number, $l = 1, \ldots, n$, $n = 1, 2, 3, \ldots, \infty$. Assuming $l = n$ (circular orbits), they describe the distribution of planetary distances in words that we formalize as

$$r(p) = \bar{a}_{n(p)}, \tag{3.4}$$

where $p =$ Mercury, Venus, Earth, Mars, (Ceres,) Jupiter, Saturn, Uranus, Neptune, Pluto and $n(\text{Mercury}) = 3$, $n(\text{Venus}) = 4$, $n(\text{Earth}) = 5$, $n(\text{Mars}) = 6$, $n(\text{Ceres}) = 8$, $n(\text{Jupiter}) = 11$, $n(\text{Saturn}) = 15$, $n(\text{Uranus}) = 21$, $n(\text{Neptune}) = 26$, $n(\text{Pluto}) = 30$. It can be seen that the inner planets are successively numbered and the outer planets are rather numbered with a step of 5. For the first orbit it holds that

$$\bar{a}_1 = \frac{GM_{\text{Sun}}}{\alpha_g^2 c^2}, \tag{3.5}$$

where G is the gravitational constant, c is the speed of light, α_g is a gravitational structure constant, $1/\alpha_g = 2113 \pm 15$ [Agnese and Festa (1997)], and M_{Sun} is the mass of the Sun in application to the first possible orbit in the solar system. We remark that the gravitational structure constant, whose value was calculated from data of the solar system, has been tested against extrasolar planets and provided an orbit $\bar{a}_1 = 0.055$ AU [Agnese and Festa (1997, 1999), Nottale (1993), Nottale et al. (1996)]. This description has shown a very interesting connection between a model of the solar system and the hydrogen atom. We will show that it is also possible to use quantum physics for the determination of the distribution of orbits in planetary systems.

Let us consider the solar system, where we implement a simplification that the orbits of planets are circular and the positions of planetary orbits are in one plane. We can define a circular planetary model of the solar system, which looks like the model of hydrogen atom from the "old quantum theory". In the old quantum theory, it was only possible to explain the structure of the hydrogen atom or an ionized atom with a single electron. The absorption or emission lines for spectroscopy were obtained in terms of the energy differences of the electron on various orbits. For these orbits it holds that (cf. Bohr's model of hydrogen atom)

$$m_{\text{e}} \bar{v}_n \bar{r}_n = n\hbar, \tag{3.6}$$

where m_{e} is the mass of electron, $\bar{v}_n$ is the velocity of electron revolving around the nucleus, $\bar{r}_n$ is the distance of electron to the nucleus, and $\hbar$ is the reduced Planck

p	$v(\mathrm{p})r(\mathrm{p})$	$n(\mathrm{p})$	$K(\mathrm{p})$	$K^{(\mathrm{approx})}$	$n(\mathrm{p})K^{(\mathrm{approx})}$
Jupiter	1.02×10^{16}	10	1.02×10^{15}	1.00×10^{15}	1.00×10^{16}
Saturn	1.38×10^{16}	14	9.87×10^{15}	1.00×10^{15}	1.40×10^{16}
Uranus	1.95×10^{16}	20	9.75×10^{15}	1.00×10^{15}	2.00×10^{16}
Neptune	2.45×10^{16}	25	9.80×10^{15}	1.00×10^{15}	2.50×10^{16}

Table 3.1: Parameters $K(\mathrm{p})$, $K^{(\mathrm{approx})}$ and $n(\mathrm{p})$ for the outer part of the solar system.

constant. These orbits may not be occupied simply by more and more electrons, because the Coulombic interaction between these particles is not negligible. This quantization of orbits can be formally generalized to macroscopic bodies and the velocities related to the gravity. Agnese and Festa (1997) have modified the relation (3.6) to the form

$$m_{\mathrm{p}}\bar{v}_{\mathrm{n}}\bar{r}_{\mathrm{n}} = n(\hbar + m_{\mathrm{p}}c\bar{\lambda}), \tag{3.7}$$

where m_{p} is the mass of planet, $\bar{v}_{\mathrm{n}}$ is the orbital velocity of planet, $\bar{r}_{\mathrm{n}}$ is its distance to the central star, c is the speed of light and $\bar{\lambda}$ is a fundamental length.

With respect to the weak equivalence principle in the case of circular orbits, the velocities will only depend on the gravitational potential in the distance r to the Sun (the central body). On neglecting $\hbar$ on the right-hand side of the relation (3.7), the independence of allowed orbits of the mass m_{p} is obtained. In contrast to the electrons in the atom, the orbits around the Sun can be occupied by more than one macroscopic body as far as the gravitational interactions between them can be neglected. This ad hoc hypothesis explains the regularity of the planetary orbits dependent on the given, maybe too generous assumption.

Let us study the simplified model of the solar system and take into account the following consideration, which comes out of Kepler's second law: Areas which are swept out by the radius vector of planet in equal time intervals are equal, so the elementary area swept out by the radius vector of planet in the aphelium in the time dt is the same as the elementary area swept out by the radius vector of planet in the perihelium in the time dt. For the area which is swept out by the radius vector of planet in the circular model, Kepler's second law is valid as well. For the areal velocities of planets, $w(\mathrm{p})$, it holds that

$$2w(\mathrm{p}) = v(\mathrm{p})r(\mathrm{p}). \tag{3.8}$$

Let us compare the areal velocities of planets for the outer part of the solar system, with the allowed areal velocities $\bar{w}_n$, $2\bar{w}_n = \bar{v}_n\bar{r}_n$, which will be appropriately defined.

From Table 3.1, we can substitute a formula $v(\mathrm{p})r(\mathrm{p}) = n(\mathrm{p})K(\mathrm{p})$ by a new formula $\bar{v}_n\bar{r}_n = nK^{(\mathrm{approx})}$, where $n = n(\mathrm{p})$ and $K^{(\mathrm{approx})}$ is a constant, viz., an

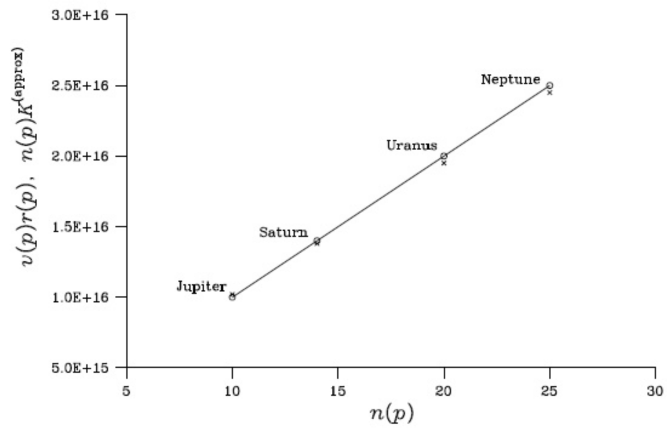

Figure 3.1: Comparison of real data $v(\mathrm{p})r(\mathrm{p})$ ($\times$) with the formula $n(\mathrm{p})K^{(\mathrm{approx})}$ ($\circ$) for the outer parts of the solar system.

approximate value of $K(\mathrm{p})$, or

$$v(\mathrm{p})r(\mathrm{p}) \approx n(\mathrm{p})K^{(\mathrm{approx})}, \tag{3.9}$$

where $\mathrm{p} =$ Jupiter, Saturn, Uranus, Neptune. A comparison of real data $v(\mathrm{p})r(\mathrm{p})$ with the approximate formula $n(\mathrm{p})K^{(\mathrm{approx})}$ is illustrated in Figure 3.1.

We find $\bar{v}_n$, $\bar{r}_n$ such that they fulfil the relation

$$\bar{v}_n\bar{r}_n = nK^{(\mathrm{approx})} \tag{3.10}$$

and Newton's gravitational law

$$\bar{v}_n = \sqrt{\frac{GM_{\mathrm{Sun}}}{\bar{r}_n}}. \tag{3.11}$$

We arrive at

$$\bar{r}_n = \bar{a}_1 n^2, \tag{3.12}$$

where

$$\bar{a}_1 = \frac{[K^{(\mathrm{approx})}]^2}{GM_{\mathrm{Sun}}} = 0.052\ \mathrm{AU}. \tag{3.13}$$

We will show that the planets can be numbered successively using $n(\mathrm{J})$ to $n(\mathrm{J}) + 3$, where J stands for the planet Jupiter, unlike 10, 14, 20, 25 in Table 3.1. Traditionally, we use the least squares method. We can determine the number $n(\mathrm{J})$ and a constant K such that

$$[1.02 - n(\mathrm{J})K]^2 + [1.38 - (n(\mathrm{J}) + 1)K]^2 + [1.95 - (n(\mathrm{J}) + 2)K]^2$$

p	$v(\mathrm{p})r(\mathrm{p})$	$n(\mathrm{p})$	$K(\mathrm{p})$	$K^{(\mathrm{approx})}$	$n(\mathrm{p})K^{(\mathrm{approx})}$
Io	7.31×10^{12}	7	1.04×10^{12}	1.00×10^{12}	7.00×10^{12}
Europa	9.21×10^{12}	9	1.02×10^{12}	1.00×10^{12}	9.00×10^{12}
Ganymedes	1.16×10^{13}	12	9.67×10^{11}	1.00×10^{12}	1.20×10^{13}
Callisto	1.50×10^{13}	15	1.00×10^{12}	1.00×10^{12}	1.50×10^{13}

Table 3.2: Parameters $K(\mathrm{p})$, $K^{(\mathrm{approx})}$ and $n(\mathrm{p})$ for the Jovian system of moons.

$$+\left[2.45 - (n(\mathrm{J}) + 3)K\right]^2 = \min. \tag{3.14}$$

This happens for $n(\mathrm{J}) = 2$, $K = 0.4857 \times 10^{16}$. We have arrived at a Schmidt-like formula.

The formula (3.12) is in accordance with the papers of Agnese and Festa [Agnese and Festa (1997), (1999)], but we do not use the gravitational structure constant for the definition of a possible first orbit. This is the main point in our considerations.

Now we can introduce a new parameter ρ_l of the system in the formula (3.13), which we call the length density of orbits,

$$\rho_l = \frac{G}{[K^{(\mathrm{approx})}]^2} \tag{3.15}$$

in units kgm^{-1}. This new parameter can be used for the classification of extrasolar systems.

3.3 Systems of moons around planets

The procedure which we derived above, is valid also for systems of moons around planets. For a system of moons, the formula (3.12) is valid, where for the first orbit it holds that

$$\bar{a}_1 = \frac{1}{\rho_l M_\mathrm{p}}, \tag{3.16}$$

where M_p is a mass of a planet that moons revolve around.

For the Jovian system, it holds that $M_{\mathrm{Jupiter}} = 1.9 \times 10^{27}$ kg, $\bar{a}_1 = 7890.79$ km, $K^{(\mathrm{approx})} = 1.00 \times 10^{12}\ \mathrm{m^2s^{-1}}$. In Table 3.2, the parameters for the Jovian system of moons can be found. A comparison of real data $v(\mathrm{p})r(\mathrm{p})$ with the approximate formula $n(\mathrm{p})K^{(\mathrm{approx})}$ is illustrated in Figure 3.2.

We will show that the moons can be labelled with successive numbers $n(\mathrm{I})$ to $n(\mathrm{I})+3$, where I stands for the moon Io, unlike 7, 9, 12, 15 in Table 3.2. Again, we use the least squares method. We can find the number $n(\mathrm{I})$ and a constant K such that

$$\left[7.31 - n(\mathrm{I})K\right]^2 + \left[9.21 - (n(\mathrm{I}) + 1)K\right]^2 + \left[11.6 - (n(\mathrm{I}) + 2)K\right]^2$$

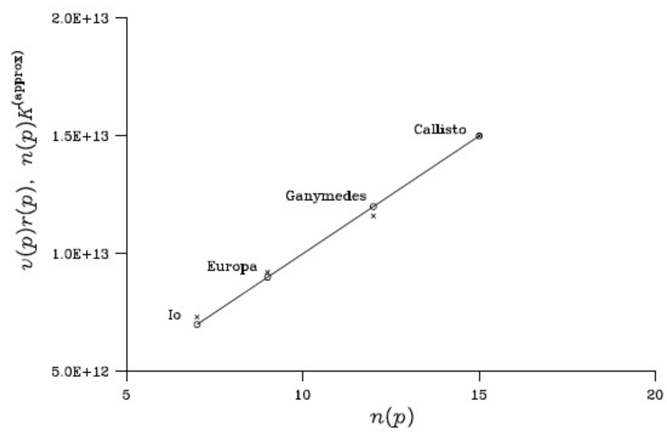

Figure 3.2: Comparison of real data $v(\mathrm{p})r(\mathrm{p})$ ($\times$) with the formula $n(\mathrm{p})K^{(\mathrm{approx})}$ ($\circ$) for the Jovian system of moons.

$$+\left[15.0-(n(\mathrm{I})+3)K\right]^2=\min. \tag{3.17}$$

This takes place for $n(\mathrm{I})=3$, $K=2.40\times10^{12}$. We have indicated a Schmidt-like formula.

For the Uranian system it is valid that $M_{\mathrm{Uranus}}=8.7\times10^{25}$ kg, $\bar{a}_1=1723.28$ km, $K^{(\mathrm{approx})}=1.00\times10^{11}$ $\mathrm{m^2s^{-1}}$. In Table 3.3, the parameters for the Uranian system of moons can be found. A comparison of real data $v(\mathrm{p})r(\mathrm{p})$ with the approximate formula $n(\mathrm{p})K^{(\mathrm{approx})}$ is illustrated in Figure 3.3.

p	$v(\mathrm{p})r(\mathrm{p})$	$n(\mathrm{p})$	$K(\mathrm{p})$	$K^{(\mathrm{approx})}$	$n(\mathrm{p})K^{(\mathrm{approx})}$
Miranda	8.68×10^{11}	9	9.64×10^{11}	1.00×10^{11}	9.00×10^{11}
Ariel	1.05×10^{12}	11	9.57×10^{11}	1.00×10^{11}	1.10×10^{12}
Umbriel	1.24×10^{12}	12	1.04×10^{11}	1.00×10^{11}	1.20×10^{12}
Titania	1.59×10^{12}	16	9.94×10^{11}	1.00×10^{11}	1.60×10^{12}
Oberon	1.84×10^{12}	18	1.02×10^{11}	1.00×10^{11}	1.80×10^{12}

Table 3.3: Parameters $K(\mathrm{p})$, $K^{(\mathrm{approx})}$ and $n(\mathrm{p})$ for the Uranian system of moons.

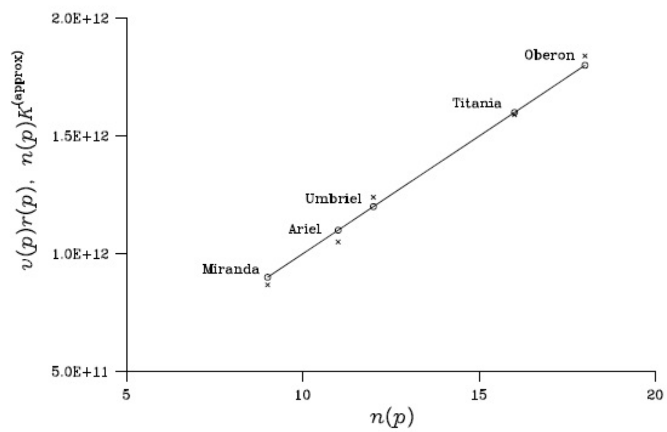

Figure 3.3: Comparison of real data $v(\mathrm{p})r(\mathrm{p})$ ($\times$) with the formula $n(\mathrm{p})K^{(\mathrm{approx})}$ ($\circ$) for the Uranian system of moons.

3.4 Extrasolar system HD 10180

We can also apply our consideration to extrasolar systems. For such systems it holds that

$$\bar{a}_1 = \frac{[K^{(\mathrm{approx})}]^2}{GM_{\mathrm{s}}}, \tag{3.18}$$

where M_{s} is a mass of a central star. Here M_{s}=$1.06 \pm 0.05 M_{\mathrm{Sun}}$. We calculate further orbits according to the formula (3.12).

The planetary system HD10180 was introduced in [Lovis et al. (2010)]. It is the most explored extrasolar system with 7 planets. This extrasolar system was examined with the aid of measurements of the radial velocities of the system HARPS and that is why we selected it for our considerations.

The planet b is at the distance 0.02226 AU to the central star with the mass 1.4 times more than the mass of the Earth, the planet c is at the distance 0.0641 AU to the central star with the mass 13.16 times more than the mass of the Earth, the planet d is at the distance 0.1286 AU to the central star with the mass 11.91 of the mass of the Earth, the planet e is at the distance 0.2695 AU to the central star with the mass 25.3 of the mass of the Earth, the planet f is at the distance 0.4923 AU to the central star with the mass 23.5 of the mass of the Earth, the planet g is at the distance 1.422 AU to the central star with the mass 21.3 of the mass of the Earth, the planet h is at the distance 3.4 AU to the central star with the mass 65.2 of the mass of the Earth.

The formulae (3.3) and (3.12), with $\bar{a}_1 = 0.055$ AU do not provide an appropriate allowed orbit, because the distance of the planet b to the central star r(b), is

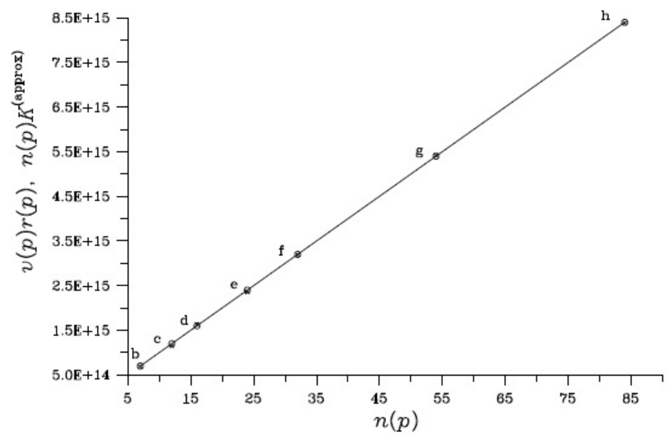

Figure 3.4: Comparison of real data $v(\mathrm{p})r(\mathrm{p})$ ($\times$) with the formula $n(\mathrm{p})K^{(\mathrm{approx})}$ ($\circ$) for the system HD10180.

much nearer than the radius of a possible first orbit $\bar{a}_1$. For the system HD10180, it holds that $K^{(\mathrm{approx})} = 1.00 \times 10^{14}$ m^{2}s^{-1}, with the first orbit $\bar{a}_1 = 0.000484$ AU and the length density of orbits $\rho_l = 6.67 \times 10^{-39}$ kgm^{-1}. If we compare the length density of orbits with the solar system, the system HD10180 has 100 times denser orbits than the solar system. Therefore, the architecture HD10180 is much nearer than the solar system. The extrasolar system HD10180 meets the formula (3.12), if we apply the correct first distance $\bar{a}_1$. In Table 3.4, the parameters for the system HD10180 are arranged. A comparison of real data $v(\mathrm{p})r(\mathrm{p})$ with the approximate formula $n(\mathrm{p})K^{(\mathrm{approx})}$ is illustrated in Figure 3.4.

p	$v(\mathrm{p})r(\mathrm{p})$	$n(\mathrm{p})$	$K(\mathrm{p})$	$K^{(\mathrm{approx})}$	$n(\mathrm{p})K^{(\mathrm{approx})}$
b	6.86×10^{14}	7	9.80×10^{13}	1.00×10^{14}	7.00×10^{14}
c	1.15×10^{15}	12	9.59×10^{13}	1.00×10^{14}	1.20×10^{15}
d	1.63×10^{15}	16	1.02×10^{14}	1.00×10^{14}	1.60×10^{15}
e	2.36×10^{15}	24	9.84×10^{13}	1.00×10^{14}	2.40×10^{15}
f	3.19×10^{15}	32	9.98×10^{13}	1.00×10^{14}	3.20×10^{15}
g	5.42×10^{15}	54	1.00×10^{14}	1.00×10^{14}	5.40×10^{15}
h	8.38×10^{15}	84	9.98×10^{13}	1.00×10^{14}	8.40×10^{15}

Table 3.4: Parameters $K(\mathrm{p})$, $K^{(\mathrm{approx})}$ and $n(\mathrm{p})$ for the system HD10180.

Chapter 4

Statistical and regression analyses of detected extrasolar systems

4.1 Introduction

A total of 759 planets in 609 extrasolar systems have been identified as of February 11, 2012 [Schneider (2012)]. We know also 99 multi-planet extrasolar systems with more than 2 planets in one system. In [Cassan et al. (2012)], it has been estimated that each of the 100 billion stars in our Galaxy hosts on average at least 1.6 planets and 160 billion stars are alone in our Galaxy. Many planets could also be in the habitable zones, where the liquid water could be on the exoplanet surface and also potential conditions for the life. For determining the habitable zones, bulk composition, atmosphere, and potential chemical interactions are very important. Stellar characteristics that are also very important include mass and luminosity, stable variability, high metallicity and at least orbital properties. The orbital properties have been studied by Laskar [Laskar (2000)]. In [Braun (2011)], it is argued that the physical quantities of the exoplanets are functions of astrophysical parameters of the central (host) stars, such as the stellar luminosity. Forms of such parametric dependence are investigated here by using statistical methods. Earlier some efforts have been made to study the statistical properties of exoplanets ([Lineweaver and Grether (2003), Grether (2006)] and references therein), but not many extrasolar systems were known at that time. Now we know 490 host stars.

Considerations of scaling have led us to two normalized parameters. As we know many extrasolar systems, we can compare them and prepare a statistical study of the position of exoplanets around host stars. In section 4.2, we present the joint distribution of normalized distance and normalized mass and that of the specific angular momentum and the normalized mass. In section 4.3, we look for the characteristics shared by six multi-planet systems such as solar system and five extrasolar systems. We state supremacy of the power function and assess the prediction of further proposed parameters by the specific angular momentum. In section 4.4, we continue this analysis within the four spectral classes of the stars. We illustrate the possible prediction of the product of the exoplanet distance and the stellar surface temperature by the specific angular momentum. Topically, we perform this study for the Kepler exoplanet candidates in section 4.5.

4.2 Normalization of extrasolar systems

For the statistical study, we need all extrasolar planets in one scale. For a stable orbit of planet, it is valid that

$$F_{\mathrm{g}} = \frac{GM_{\mathrm{s}}M_{\mathrm{p}}}{r_{\mathrm{p}}^2} = F_{\mathrm{o}} = M_{\mathrm{p}}\frac{v_{\mathrm{p}}^2}{r_{\mathrm{p}}}, \tag{4.1}$$

where F_{g} is the gravitational force, F_{o} is the centrifugal force, G is the gravitational constant and r_{p} is the distance of planet from the central star, M_{s} is the mass of the star, M_{p} is the mass of the planet, and v_{p} is the velocity of the planet. We can write

$$r_{\mathrm{p}}^2 = \frac{GM_{\mathrm{s}}M_{\mathrm{p}}}{F_{\mathrm{o}}} \tag{4.2}$$

and we get after a modification

$$\frac{r_{\mathrm{p}}}{\sqrt{M_{\mathrm{s}}}} = \sqrt{\frac{GM_{\mathrm{p}}}{F_{\mathrm{o}}}}. \tag{4.3}$$

We understand the left-hand side of equation (4.3) as the first normalized parameter for all extrasolar systems. We will use this normalized parameter for the creation of graphs of extrasolar systems on the x axis. As the second normalized parameter, we use equation (4.1) and we get after a modification

$$\frac{M_{\mathrm{p}}}{M_{\mathrm{s}}} = \frac{GM_{\mathrm{p}}}{v_{\mathrm{p}}^2 r_{\mathrm{p}}}. \tag{4.4}$$

We understand the left-hand side of equation (4.4) as the second normalized parameter for all extrasolar systems and we will use this parameter for the creation of graphs of extrasolar systems on the axis y.

The radial velocity detection method detects the exoplanets according to the displacements in the star spectral lines due to the Doppler effect. This method is most productive for the discovering of most exoplanets and it has the advantage for an application to stars with a wide range of characteristics. Nevertheless, using this method, we cannot determine planet's true mass, this method can only set a lower limit on the mass. Most exoplanets were detected by this detection method.

With the transiting method (the Kepler mission), we measure star's dimensions in time, which depend on the size of exoplanet. This method determines planet's true radius, but not the mass of exoplanets and the exoplanets should be confirmed with other methods for the detection of exoplanets.

It is very difficult to make a correct interpretation of the exoplanets data, because the selective factors for each detection method play the key role. In Fig. 4.1, we can see a distribution of extrasolar systems after the normalization. We have used data for exoplanets, which were confirmed by two and more detection methods and according to this we have decreased the effect of the selective factor to the minimum value.

In Fig. 4.1, we can see areas, where the density of extrasolar planets is higher than elsewhere. The reason could be the selective factor according to the detection methods or a possible probability that the formation of exoplanets around stars is higher in specific distances from the central stars. In this figure, we can see the distribution of extrasolar systems according to the parameter $v_{\mathrm{p}}r_{\mathrm{p}}$. The exoplanets form two groups with the Gaussian distributions. The explanation of higher probabilities for the formation of exoplanets around stars can be found in [Pintr, Peřinová and Lukš (2012)]. There the same idea has been applied to our equation (4.1). On using the Bohr-Sommerfeld hypothesis of quantization of angular momentum, a new constant g has been introduced in the form

$$M_{\mathrm{p}}v_{\mathrm{p}}r_{\mathrm{p}} = \frac{n_{\mathrm{p}}g}{2\pi}, \tag{4.5}$$

where n_{p} is a planetary quantum number and the distances of planets could be rewritten in the known form of gravitational Bohr-type radius,

$$r_{\mathrm{p}} = \frac{n_{\mathrm{p}}^2 G M_{\mathrm{s}}}{v_0^2}, \tag{4.6}$$

where $v_0 = 144$ km/sec for planetary systems in accordance with [Nottale, Schumacher and Lefèvre (2000)].

4.3 Regression method for the solar system and five multi-planet extrasolar systems

In [Pintr, Peřinová and Lukš (2012)], we have shown that the celestial systems can be described with the parameter $v_{\mathrm{p}}r_{\mathrm{p}}$. Now we will focus in the study of the solar system and the multi-planet extrasolar systems HD10180 [Lovis et al. (2011)], Kepler - 20 [Fressin et al. (2011)], 55Cnc [Marcy et al. (2002)], Gliese 876 [Marcy et al. (2001)] and Upsilon Andromedae [Butler et al. (1999)]. We would like to find physical parameters for the solar system and five multi-planet extrasolar systems,

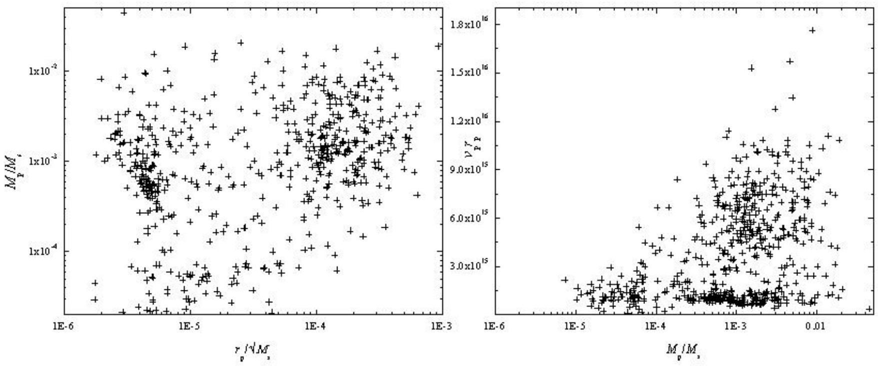

Figure 4.1: Normalization of extrasolar systems detected by the radial velocities. On the left-hand side, the density of exoplanets is higher in the specific areas. On the right-hand side, the exoplanets form two groups with the Gaussian distributions.

which are normalized (collective) in the dependence on distances of planets from the central stars. We can find these characteristics in Table 4.1.

In this paper, we use the statistical regression methods. A regression model relates Y to a function of X and β in the form

$$Y = f(X, \beta), \tag{4.7}$$

where β are unknown parameters, X are independent variables, and Y is a dependent variable. The observed data (x_i, y_i), $i = 1, 2, ..., n$, with n the number of observations, satisfy equations

$$y_i = f(x_i, \beta) + \epsilon_i, \tag{4.8}$$

where ϵ_i is an error term.

For a linear regression, we have two parameters β_0 and β_1 and it is valid that

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i. \tag{4.9}$$

The residual

$$\epsilon_i = y_i - f_i \tag{4.10}$$

is the difference between the true value of the dependent variable y_i and the value of the dependent variable predicted by the model f_i. We measure the goodness of fit by the coefficient of determination R^2,

$$R^2 = \frac{SS_{\text{reg}}}{SS_{\text{tot}}}, \tag{4.11}$$

where SS_{reg} is the regression sum of squares,

$$SS_{\text{reg}} = \sum_{i=1}^{n}(f_i - y_{\text{avg}})^2, \tag{4.12}$$

and SS_{tot} is the total sum of squares,

$$SS_{\text{tot}} = \sum_{i=1}^{n}(y_i - y_{\text{avg}})^2. \tag{4.13}$$

In (4.12) and (4.13), y_{avg} is the mean of observed values y_i. When the model function is not linear in the parameters, the sum $\sum_{i=1}^{n}\epsilon_i^2$ must be minimized by an iterative procedure.

We have also tried to interpolate our real data for the multi-planet extrasolar systems using these interpolations:

- Linear interpolation: $Y = A + Bx$.
- Logarithmic interpolation: $Y = \ln(A + Bx)$.
- Exponential interpolation: $Y = AB^x$.
- Power interpolation: $Y = Ax^B$.
- Polynomial interpolation: $Y = A_0 + A_1x + A_2x^2 + ...A_{n-1}x^{n-1}$.

First we have studied the dependence of the distance of planets r_{p} from the central stars on the parameter $v_{\text{p}}r_{\text{p}}$ for the multi-planet extrasolar systems. At the same time, we have found the best interpolation method for real data.

We have applied the regression analysis to other physical parameters of planets, i. e., $r_{\text{p}}T_{\text{eff}}$, $r_{\text{p}}L$, and $r_{\text{p}}J$, where $T_{\text{eff}} \equiv T_{\text{eff,s}}$ is the effective temperature of stellar surface, $L \equiv L_{\text{s}}$ is the luminosity of a star, $J \equiv J_{\text{s}}$ is the stellar irradiance. Using the best interpolation method, we have found such parameters that are normalized for the solar system and five multi-planet extrasolar systems.

We have calculated the luminosity of a star in the form [Luminosity of stars (2004)]

$$L \equiv L_{\text{s}} = 4\pi R_{\text{s}}^2\sigma T_{\text{eff}}^4, \tag{4.14}$$

where R_{s} is the stellar radius, which is given in Table 1 in radii of the Sun, σ is the Stefan-Boltzmann constant. We have used the Stefan-Boltzmann law for the calculation of the stellar irradiance,

$$J \equiv J_{\text{s}} = \sigma T_{\text{eff}}^4. \tag{4.15}$$

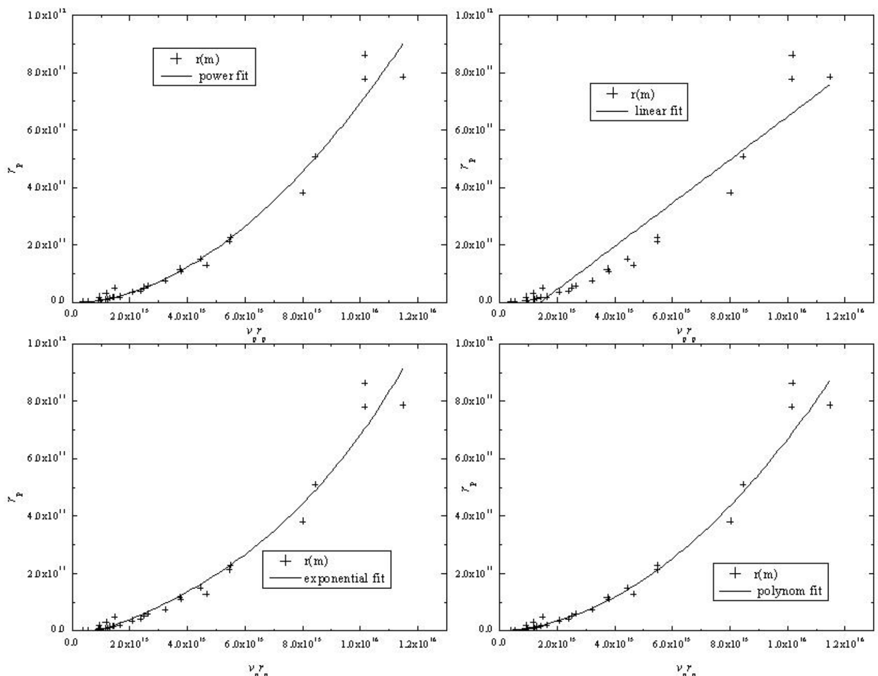

Figure 4.2: Finding the best interpolation for real data $r_{\rm p}$ on $v_{\rm p} r_{\rm p}$ for the solar system and five multi-planet extrasolar systems. The best is the power interpolation with the coefficient of determination $R^2 = 0.957$.

According to Fig. 4.2, we can describe the dependence of r_p on the parameter $v_\mathrm{p} r_\mathrm{p}$ by the power interpolation with the coefficient of determination $R^2 = 0.957$. The regression relation is in the form

$$y \approx K x^{1.769}, \tag{4.16}$$

where $y = r_\mathrm{p}$, $K = 3 \times 10^{-17}$, and $x = v_\mathrm{p} r_\mathrm{p}$.

Krot (2009) has shown that if we apply a statistical theory to a formation of protoplanets around a spheroidal body, we can derive the original Schmidt law in the form

$$\sqrt{r_n} = a + bn, \tag{4.17}$$

where r_n is the distance of planets from the central star, n is a natural number, and a, b are independent constants, and generalize it. Graner and Dubrulle (1994) have shown that the empirical laws are implicitly based on the assumption of the rotation and the scale invariance in the form

$$r_n = r_0 K^n, \tag{4.18}$$

where K is a constant.

Laskar has presented a simplified model of planetary accreation based on the conservation of mass and momentum [Laskar (2000)]. He has derived the initial mass distribution in the form

$$m(a) = \left(\frac{\tilde{C}}{k}\right)^{\frac{1}{3}} a^{\frac{1}{2}} (\rho(a))^{\frac{2}{3}}, \tag{4.19}$$

where a is a semi-major distance of planetesimals, $\rho(a) = \zeta a^p$, $\tilde{C} = C\sqrt{\mu}$, C is the angular momentum deficit and $\mu = GM_\mathrm{s}$. Using this relation, we get the planetary distributions corresponding to different initial mass distributions in the forms

- For $p = -\frac{1}{2}$, we get $a^{\frac{1}{3}} = a_0^{\frac{1}{3}} + \left(\frac{\tilde{C}}{k\zeta}\right)^{\frac{1}{3}} \frac{n_\mathrm{i}}{3}$,
  we have $\frac{1}{4} < k < \frac{1}{2}$.
- For $p = 0$, we get $\sqrt{a} = \sqrt{a_0} + \left(\frac{\tilde{C}}{2\zeta}\right)^{\frac{1}{3}} n_\mathrm{i}$,
  we have the power n_i^2 distribution for $a(n_\mathrm{i})$.
- For $p = -\frac{3}{2}$, we get $\log a = \log(a_0) + \left(2\frac{\tilde{C}}{\zeta}\right)^{\frac{1}{3}} n_\mathrm{i}$,
  we have the Bode-like power law.

Here n_i is the order increment.

We can modify our regression equation (4.16) to the form

$$\sqrt[1.769]{r_\mathrm{p}} \approx \sqrt[1.769]{3 \times 10^{-17}}\, x, \tag{4.20}$$

and arrive at the Schmidt-like law similar to laws (4.17) and (4.18) for five multi-planet extrasolar systems. Our regression result is also very similar to the power $n_{\rm i}^2$ law in [Laskar (2000)], with the constant initial distribution of planetesimals for $p = 0$.

Let us discuss the dependence of physical parameters $r_{\rm p}L$, $r_{\rm p}T_{\rm eff}$, and $r_{\rm p}J$ on the parameter $v_{\rm p}r_{\rm p}$. The luminosity of stars $L \equiv L_{\rm s}$ is one of many physical parameters for the determination of habitable zones [Turnbull and Tarter (2003)]. According to Fig. 4.3, we can find that the coefficient of determination for the parameter $r_{\rm p}L$ is $R^2 = 0.837$ and this parameter is not the collective normalized parameter for five multi-planet extrasolar systems. Better collective normalized parameter for such systems is the dependence including the irradiance of a star $J \equiv J_{\rm s}$, $r_{\rm p}J$, with the coefficient of determination $R^2 = 0.962$. The best collective normalized parameter for such systems is $r_{\rm p}T_{\rm eff}$ with the coefficient of determination $R^2 = 0.991$. The regression relation for the parameter $r_{\rm p}T_{\rm eff}$ is in the form

$$y \approx Kx^{1.892}, \tag{4.21}$$

where $y = r_{\rm p}T_{\rm eff}$, $K = 2 \times 10^{-15}$ and $x = v_{\rm p}r_{\rm p}$. We have a similar result, with the power $n_{\rm i}^2$ distribution as in [Laskar (2000)].

Why is the parameter $r_{\rm p}T_{\rm eff}$ a collective normalized parameter for five multi-planet extrasolar systems? As this parameter is directly proportional to the mass of central star $M_{\rm s}$ and according to (4.21), it is valid that

$$r_{\rm p}T_{\rm eff} \approx K(v_{\rm p}r_{\rm p})^{1.892} \tag{4.22}$$

and after the substitution $v_{\rm p} = \sqrt{\frac{GM_{\rm s}}{r_{\rm p}}}$, we get

$$T_{\rm eff} \approx K(GM_{\rm s})^{0.946} r_{\rm p}^{-0.054}. \tag{4.23}$$

4.4 Regression method for all detected exoplanets

Similarly as in the previous section, we will study the dependence of the parameter $r_{\rm p}T_{\rm eff}$ on the parameter $v_{\rm p}r_{\rm p}$ for all 759 detected extrasolar systems. As the distances of exoplanets from the central star could depend on $T_{\rm eff}$, we will pay attention to the spectral classes of stars F, G, K and M. This is justified since the life times of spectral classes of stars O, B and A are so small that the complex life will never form on the planets associated with them. According to [Schneider (2012)], spectral classes of our interest can be characterized as:

- spectral class F with $T_{\rm eff}$ between $6000 - 7500$ K,

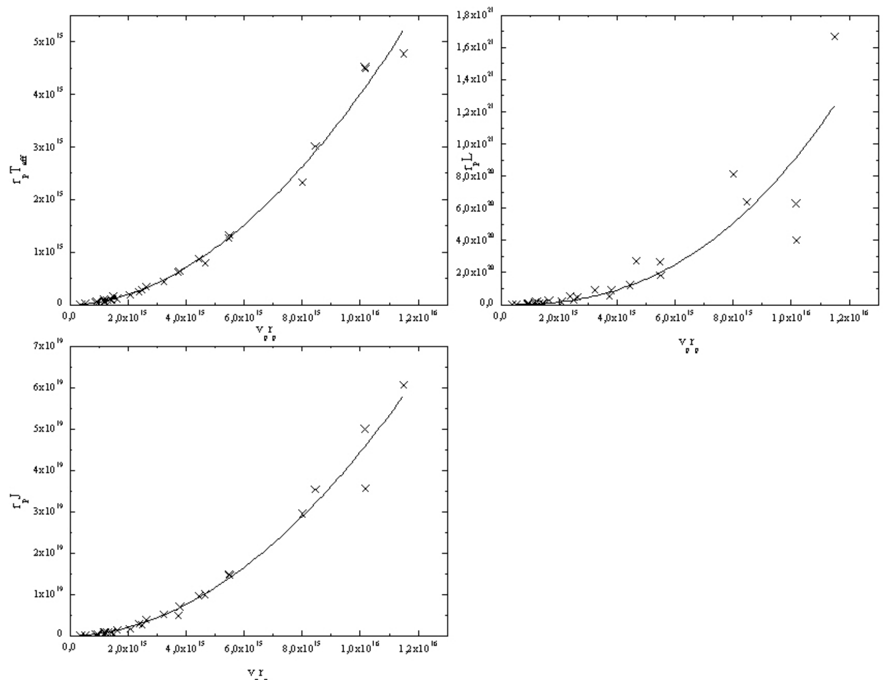

Figure 4.3: Dependence of the parameters $r_{\mathrm{p}}T_{\mathrm{eff}}$, $r_{\mathrm{p}}L$, and $r_{\mathrm{p}}J$ on $v_{\mathrm{p}}r_{\mathrm{p}}$ for the solar system and five multi-planet extrasolar systems. According to the regression analysis, the best power interpolation is with the coefficient of determination $R^2 = 0.991$ for the parameter $r_{\mathrm{p}}T_{\mathrm{eff}}$.

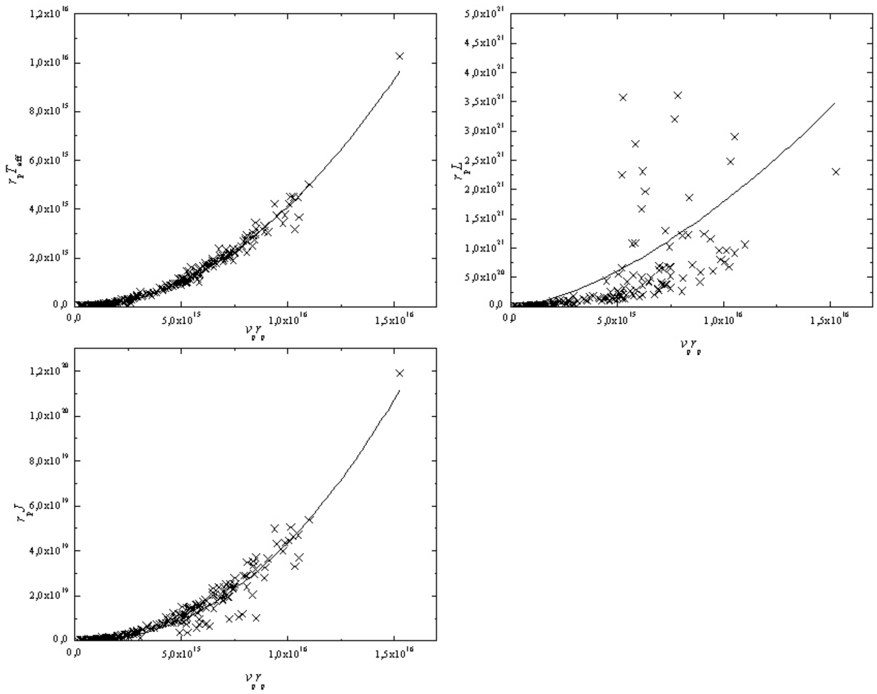

Figure 4.4: Dependence of the parameters $r_\mathrm{p}T_\mathrm{eff}$, $r_\mathrm{p}L$, and $r_\mathrm{p}J$ on the parameter $v_\mathrm{p}r_\mathrm{p}$ for the stellar spectral class G. According to the regression analysis, the best is the power interpolation with the coefficient of determination $R^2 = 0.983$ for the parameter $r_\mathrm{p}T_\mathrm{eff}$.

- spectral class G with T_eff between $5200 - 6000$ K,
- spectral class K with T_eff between $3700 - 5200$ K,
- spectral class M with T_eff less than 3700 K.

For stars belonging to the spectral type G, we can find from Fig. 4.4 that the parameter $r_\mathrm{p}T_\mathrm{eff}$ depends on the parameter $v_\mathrm{p}r_\mathrm{p}$ with the coefficient of determination $R^2 = 0.995$. The obtained equation of regression is in the form

$$r_\mathrm{p}T_\mathrm{eff} \approx Kx^{1.976}, \tag{4.24}$$

where $K = 1 \times 10^{-16}$ and $x = v_\mathrm{p}r_\mathrm{p}$.

For the stellar spectral type G, we observe that the parameter $r_\mathrm{p}L$, $L \equiv L_\mathrm{s}$, has big scattering with the coefficient of determination $R^2 = 0.895$. For the parameter

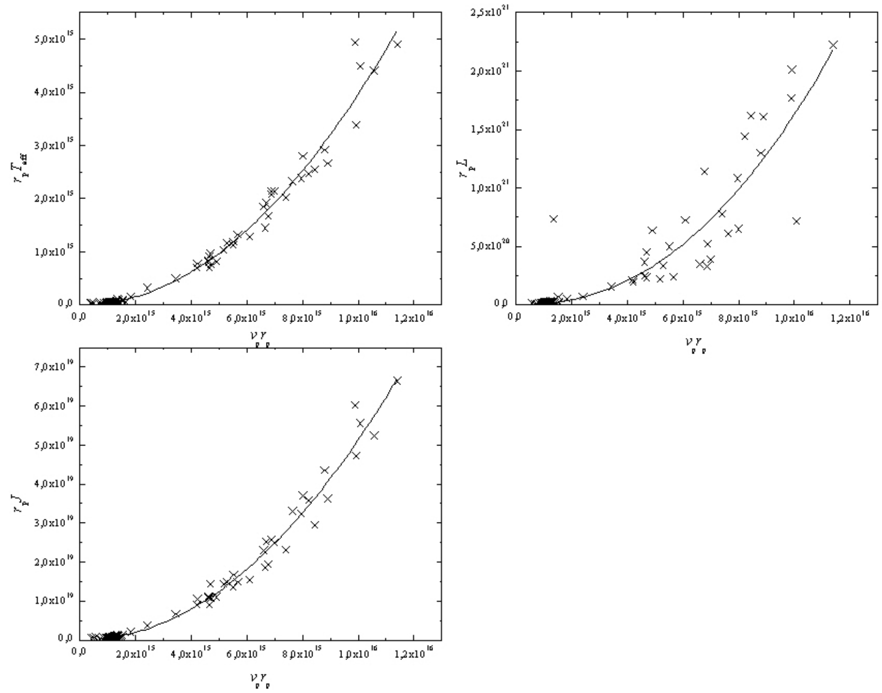

Figure 4.5: Dependence of the parameters $r_\mathrm{p} T_\mathrm{eff}$, $r_\mathrm{p} L$, and $r_\mathrm{p} J$ on $v_\mathrm{p} r_\mathrm{p}$ for the stellar spectral class F. According to the regression analysis, the best is the power interpolation with the coefficient of determination $R^2 = 0.997$ for the parameter $r_\mathrm{p} T_\mathrm{eff}$.

$r_\mathrm{p} J$, we get better results with the coefficient of determination $R^2 = 0.979$. From this it follows that the parameter $r_\mathrm{p} T_\mathrm{eff}$, $T_\mathrm{eff} \equiv T_\mathrm{eff,s}$, affects mainly the orbits of exoplanets. We get also the power n_i^2 distribution, with the constant initial distribution of planetesimals for $p = 0$, as in [Laskar (2000)].

For stars of the spectral type F, we can find from Fig. 4.5 that the parameter $r_\mathrm{p} T_\mathrm{eff}$ depends on the parameter $v_\mathrm{p} r_\mathrm{p}$ with the coefficient of determination $R^2 = 0.997$. The predicted equation of regression is in the form

$$r_\mathrm{p} T_\mathrm{eff} \approx K x^{1.9963}, \tag{4.25}$$

where $K = 4 \times 10^{-17}$ and $x = v_\mathrm{p} r_\mathrm{p}$.

For the stellar spectral type F, it is valid that the parameter $r_\mathrm{p} L$ has big scattering with the coefficient of determination $R^2 = 0.923$. For the parameter $r_\mathrm{p} J$, we get better results with the value of reliability $R^2 = 0.995$. From Fig. 4.5, it is

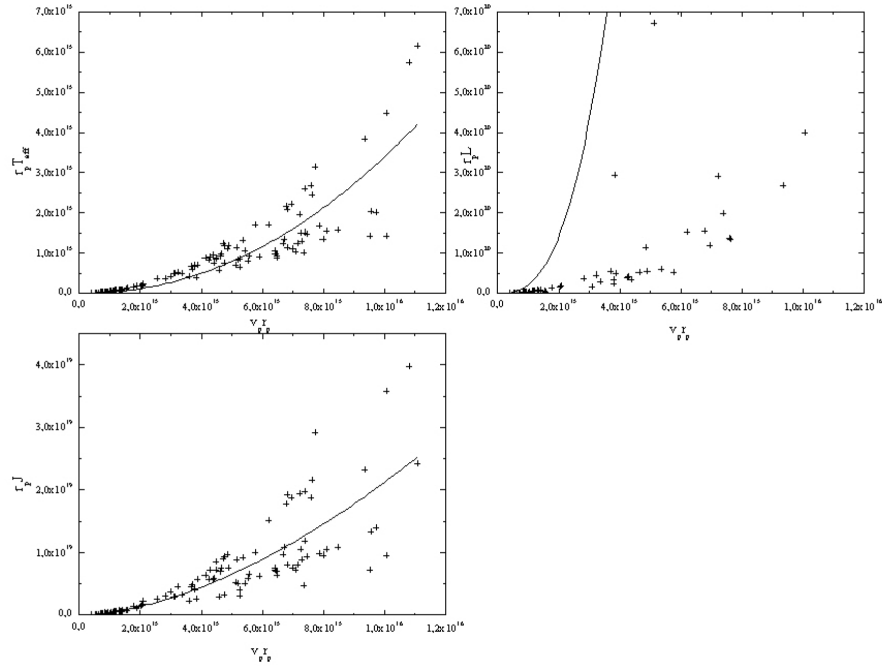

Figure 4.6: Dependence of the parameters $r_\mathrm{p}T_\mathrm{eff}$, $r_\mathrm{p}L$, and $r_\mathrm{p}J$ on $v_\mathrm{p}r_\mathrm{p}$ for the stellar spectral class K. According to the regression analysis, the best is the power interpolation with the coefficient of determination $R^2 = 0.776$ for the parameter $r_\mathrm{p}T_\mathrm{eff}$.

obvious that for the stellar spectral type F, the parameter $r_\mathrm{p}T_\mathrm{eff}$ affects mainly the orbits of exoplanets. We get also the power n_i^2 distribution, with the constant initial distribution of planetesimals for $p = 0$, as in [Laskar (2000)].

For stars of the spectral type K, we can find from Fig. 4.6 that the parameter $r_\mathrm{p}T_\mathrm{eff}$ depends on the parameter $v_\mathrm{p}r_\mathrm{p}$ with the coefficient of determination $R^2 = 0.776$. The predicted equation of regression is in the form

$$r_\mathrm{p}T_\mathrm{eff} \approx Kx^{1.807}, \tag{4.26}$$

where $K = 4 \times 10^{-14}$ and $x = v_\mathrm{p}r_\mathrm{p}$.

For stars of the stellar spectral type K, it is valid that the parameters $r_\mathrm{p}T_\mathrm{eff}$ and $r_\mathrm{p}L$ have the same regressions with the coefficient of determination $R^2 = 0.974$. This is the main difference from stars of the spectral type G. For stars of the spectral type K, the parameter $r_\mathrm{p}T_\mathrm{eff}$ affects mainly the orbits of exoplanets. In Fig. 4.6, we can see two types of distributions. Many exoplanets are described by the power

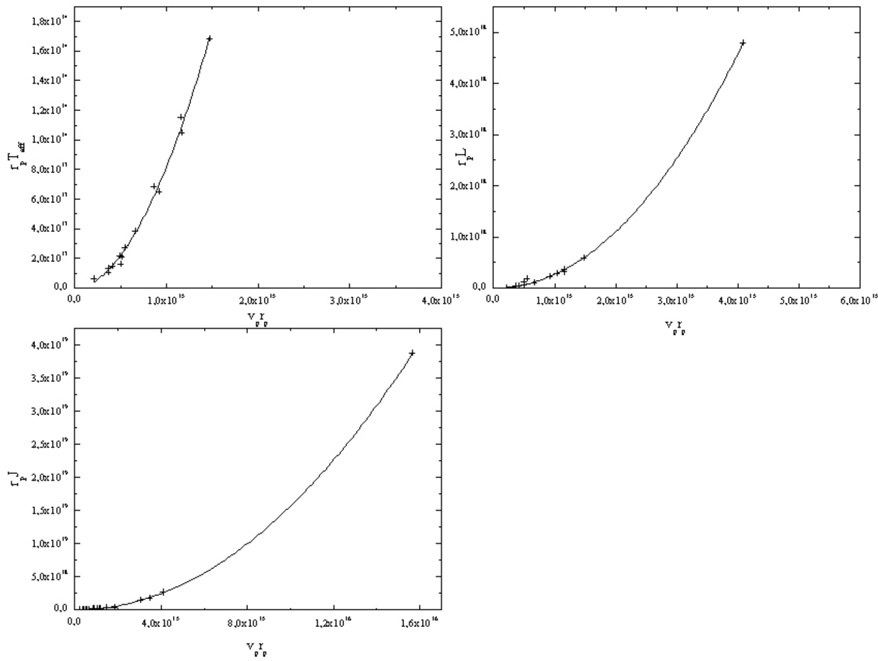

Figure 4.7: Dependence of the parameters $r_\mathrm{p}T_\mathrm{eff}$, $r_\mathrm{p}L$, and $r_\mathrm{p}J$ on $v_\mathrm{p}r_\mathrm{p}$ for the stellar spectral class M. According to the regression analysis, the best is the power interpolation with the coefficient of determination $R^2 = 0.974$ for the parameter $r_\mathrm{p}T_\mathrm{eff}$.

regression, but some exoplanets deviate from this function. We get also the power n_i^2 distribution, with the constant initial distribution of planetesimals for $p = 0$, as in [Laskar (2000)].

For stars of the spectral type M, we can find from Fig. 4.7 that the parameter $r_\mathrm{p}T_\mathrm{eff}$ depends on the parameter $v_\mathrm{p}r_\mathrm{p}$ with the coefficient of determination $R^2 = 0.974$. The predicted relation of regression is in the form

$$r_\mathrm{p}T_\mathrm{eff} \approx Kx^{1.814}, \tag{4.27}$$

where $K = 5 \times 10^{-14}$ and $x = v_\mathrm{p}r_\mathrm{p}$.

For stars of the stellar spectral type M, it is valid that the parameter $r_\mathrm{p}L$ has the coefficient of determination $R^2 = 0.950$. For the parameter $r_\mathrm{p}J$, we get better results with the coefficient of determination $R^2 = 0.967$. From Fig. 4.7, it is obvious that the parameter $r_\mathrm{p}T_\mathrm{eff}$ affects mainly the orbits of exoplanets. This result is similar to that for the stars of spectral types G and F. We get also the power n_i^2

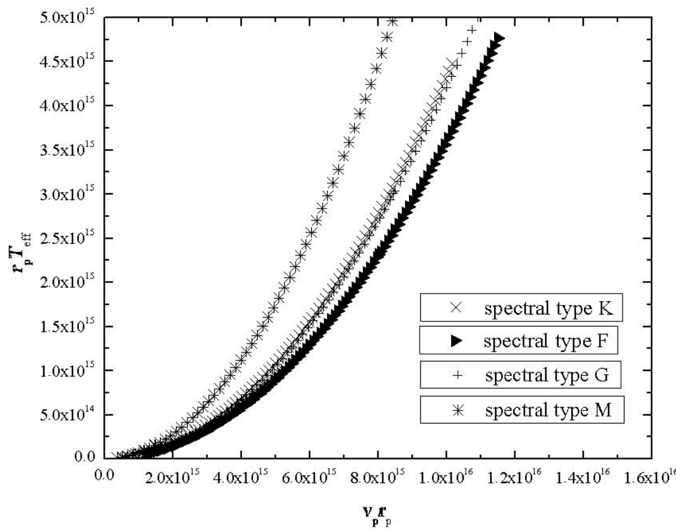

Figure 4.8: Dependence of the parameter $r_{\rm p}T_{\rm eff}$ on $v_{\rm p}r_{\rm p}$ for different stellar spectral classes.

distribution, with the constant initial distribution of planetesimals for $p = 0$, as in [Laskar (2000)].

For 759 exoplanets detected by various methods, we have found that the distances of exoplanets from the central star obey, in general, the Schmidt law and these distances $r_{\rm p}$ depend on the stellar surface temperature $T_{\rm eff}$. Each stellar spectral class has a different regression of the parameter $r_{\rm p}T_{\rm eff}$ on $v_{\rm p}r_{\rm p}$. The stellar spectral class M has a steeper regression of the parameter $r_{\rm p}T_{\rm eff}$ than the stellar spectral classes K and G. To the contrary, the stellar spectral class F has a smaller slope of regression for the parameter $r_{\rm p}T_{\rm eff}$ as is obvious in Fig. 4.8.

4.5 Kepler exoplanet candidates

The Kepler mission is designed to discover Earth-size planets orbiting another stars in our Milky Way galaxy. The Kepler mission is now in a full operation. First results were published on February 2010 [Borucki et al. (2010), Borucki et al. (2011a)], with initial discoveries of short period exoplanets. On December 2011, there were total of 2321 candidates [Borucki et al. (2011b), Koch and Gould (2012)]. 207 candidates are similar in size to the Earth, 680 candidates are super-Earths, 1181 candidates are Neptune-size exoplanets, 203 candidates are Jupiter-size exoplanets and 55 candidates are larger than Jupiter.

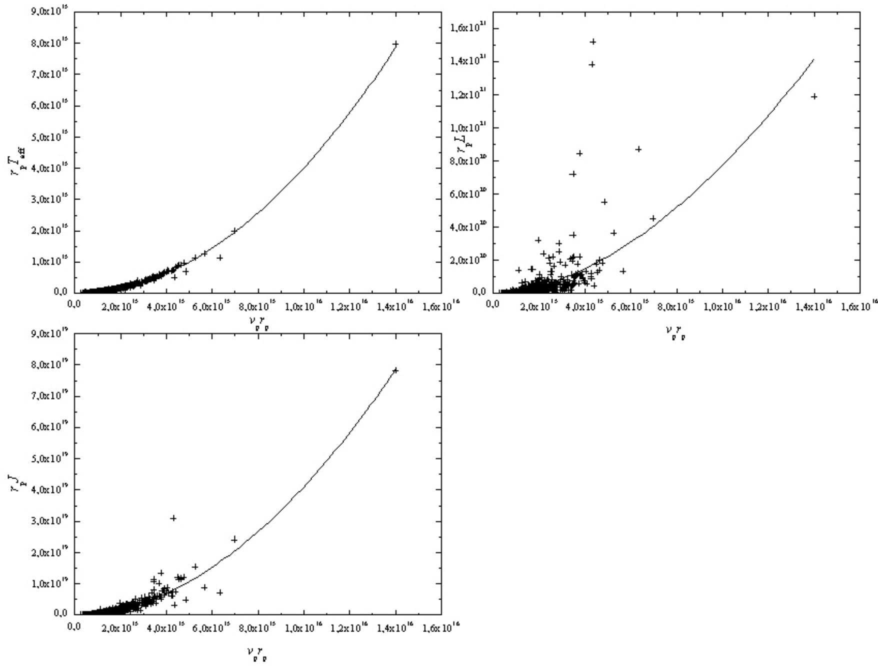

Figure 4.9: Dependence of the parameters $r_{\rm p}T_{\rm eff}$, $r_{\rm p}L$, and $r_{\rm p}J$ on $v_{\rm p}r_{\rm p}$ for the Kepler candidates. According to the regression analysis, the best is the power interpolation with the coefficient of determination $R^2 = 0.993$ for the parameter $r_{\rm p}T_{\rm eff}$.

Exoplanet data obtained from the Kepler mission are carefully analyzed in [Borucki et al. (2011b), Batalha et al. (2012)]. Initially, in [Borucki et al. (2011b)] first four months data were analyzed. In [Borucki et al. (2011b)], the Kepler candidates were divided into four groups according to the vetting flags. The flag no. 1 marks out a published and confirmed planet, the flag no. 2 marks out a strong probability candidate, which confirms all tests, the flag no. 3 marks out a moderate probability candidate, which does not confirm all tests and the flag no. 4 marks out an insufficient follow-up to perform full suite of vetting tests. Recently in [Batalha et al. (2012)], the Kepler data of first 16 months are analyzed in detail. Interestingly they confirm considerable increase in the fraction of smaller planets with longer orbital periods. The present study complements the analysis in [Batalha et al. (2012)], because we have investigated the statistical distribution of the Kepler candidates which have not been explored in [Batalha et al. (2012)]. To be precise, we have tested our previous considerations on the Kepler exoplanet candidates again, cf. Fig. 4.9.

The star luminosity L and the star irradiance J have no effect on the regression of distances of exoplanets. We have also applied these characteristics to 2321 exoplanet candidates from the Kepler mission. We have approved all regressions of exoplanets for $r_{\mathrm{p}}T_{\mathrm{eff}}$, $r_{\mathrm{p}}L$, and $r_{\mathrm{p}}J$.

System	Planet	$r_{\rm p}$ [m]	$v_{\rm p} r_{\rm p}$	Class	$M_{\rm s}$	$R_{\rm s}$	$T_{\rm eff}$ [K]
HD10180	c	9.59×10^{9}	1.16×10^{15}	G1V	1.06	1.20	5911
HD10180	d	1.92×10^{10}	1.65×10^{15}	G1V	1.06	1.20	5911
HD10180	e	4.04×10^{10}	2.38×10^{15}	G1V	1.06	1.20	5911
HD10180	f	7.37×10^{10}	3.22×10^{15}	G1V	1.06	1.20	5911
HD10180	g	2.13×10^{11}	5.47×10^{15}	G1V	1.06	1.20	5911
HD10180	h	5.09×10^{11}	8.46×10^{15}	G1V	1.06	1.20	5911
Kepler-20	b	6.79×10^{9}	9.06×10^{14}	G8	0.91	0.94	5466
Kepler-20	e	7.58×10^{9}	9.58×10^{14}	G8	0.91	0.94	5466
Kepler-20	c	1.39×10^{10}	1.30×10^{15}	G8	0.91	0.94	5466
Kepler-20	f	1.65×10^{10}	1.41×10^{15}	G8	0.91	0.94	5466
Kepler-20	d	5.17×10^{10}	2.50×10^{15}	G8	0.91	0.94	5466
55 Cnc	e	2.33×10^{9}	5.29×10^{14}	K0IV-V	0.91	0.94	5196
55 Cnc	b	1.72×10^{10}	1.44×10^{15}	K0IV-V	0.91	0.94	5196
55 Cnc	c	3.59×10^{10}	2.08×10^{15}	K0IV-V	0.91	0.94	5196
55 Cnc	f	1.17×10^{11}	3.75×10^{15}	K0IV-V	0.91	0.94	5196
55 Cnc	d	8.62×10^{11}	1.02×10^{16}	K0IV-V	0.91	0.94	5196
Solar	Mercury	5.79×10^{10}	2.64×10^{15}	G2V	1.00	1.00	5800
Solar	Venus	1.08×10^{11}	3.79×10^{15}	G2V	1.00	1.00	5800
Solar	Earth	1.50×10^{11}	4.46×10^{15}	G2V	1.00	1.00	5800
Solar	Mars	2.27×10^{11}	5.49×10^{15}	G2V	1.00	1.00	5800
Solar	Jupiter	7.78×10^{11}	1.02×10^{16}	G2V	1.00	1.00	5800
Gliese 876	b	3.12×10^{10}	1.18×10^{15}	M4V	0.33	0.36	3100
Gliese 876	c	1.94×10^{10}	9.27×10^{14}	M4V	0.33	0.36	3100
Gliese 876	d	3.11×10^{9}	3.71×10^{14}	M4V	0.33	0.36	3100
Gliese 876	e	5.00×10^{10}	1.49×10^{15}	M4V	0.33	0.36	3100
Ups And	b	8.83×10^{9}	1.22×10^{15}	F8V	1.27	1.48	6074
Ups And	c	1.29×10^{11}	4.66×10^{15}	F8V	1.27	1.48	6074
Ups And	d	3.81×10^{11}	8.02×10^{15}	F8V	1.27	1.48	6074
Ups And	e	7.85×10^{11}	1.15×10^{16}	F8V	1.27	1.48	6074

Table 4.1: Characteristics of the solar system and multi-planet extrasolar systems.

Chapter 5

Exoplanet habitability for stellar spectral classes F, G, K and M

5.1 Introduction

The initiation of life and the progress of biological evolution are probably extremely rare things in the Universe. In astrobiology, the habitable zone is the scientific term for the region around a star within which it is theoretically possible for a planet with sufficient atmospheric pressure to maintain liquid water on its surface. This concept is in accordance with the conditions favorable for the life on the Earth, because the liquid water is essential for all known forms of life. The idea of habitable zone originates from the fact that the liquid water is essential condition for the terrestrial life. Extrasolar planets in this zone are considered the most promising sites to host an extraterrestrial life.

Human interest in the habitable extrasolar planets is centuries-old. However, the initial interest and approaches were limited to the speculation and science fiction [Dole (1964), Gilster (2004)] and references therein. To be precise, the evidence of extrasolar planets was not available till mid-1990s. Consequently, any statistical analysis of extrasolar planets was not possible. Such investigations become possible only after the access of data from the missions dedicated to extrasolar planets. For example, the information about 800+ extrasolar planets and 2700+ Kepler candidates is now available [Schneider (2013)]. The availability of these data has led, in general, to several statistical analyses of extrasolar planets in the recent past [Kane et al. (2012), Borucki et al. (2011b), Pintr, Peřinová, Lukš and Pathak (2013)] and references therein. It has also initiated investigations specifically focused in the detection of habitable exoplanets. To be precise, the Kepler mission is specifically designed to survey a part of the Milky Way galaxy to discover the Earth-size planets in or near the habitable zone [Kepler mission (2013)]. Further, several resources have been developed for the analysis of habitable planets [Kane and Gelino (2012), Morris (2010), Circumstellar habitable zone simulator (2009)] and the habitable exoplanets have been studied from various perspectives [Jones, Sleep and Underwood (2008), Cuntz (2012), Wetherill (1996), Lineweaver, Fenner and Gibson (2004)]. This recent interest in habitable exoplanets has been further amplified with the recent observations of HARPS search, which have shown the existence of very low-mass habitable exoplanets around the stars HD20794, HD85512 and HD192310 [Pepe et al. (2011)]. All these recent developments have led to many interesting questions. One of them is: Is there any particular spectral class, where habitable exoplanets are more probable? This question is extremely relevant to the future searches.

Water is held for one of the most important species in exoplanetary atmospheres. Space-based transit observations have indicated that water is present in significant quantities in the exoplanets for which such observations have been possible. But observations of astronomical water spectra from the ground and spaceborne telescopes are so far difficult [Tinetti et al. (2012)].

Two strategies of the search have been identified for the life in other worlds [Schulze-Makuch et al. (2011)]. Two different indices have been proposed for them, the Earth Similarity Index (ESI) and the Planetary Habitability Index (PHI). The authors are convinced that the index ESI is related only to one type of the habitability. In general, the index PHI is appropriate. The authors hold their proposals for cumulative, the values can be updated and refined when new information is obtained. The proposals have been applied to the solar system, giant planets and their moons and other bodies. There is the hope that in the future a more accurate assessment of the distribution of life in the Universe will be possible.

In this chapter, we will deal with the habitability of exoplanets around the stars of spectral classes F, G, K and M. In Section 5.2, we explain the concept of the habitability of exoplanets. In Section 5.3, we present the statistical analysis of extrasolar systems and establish the theoretical number of exoplanets in the habitable zones. In Section 5.4, we discuss the effective temperature of exoplanets. In Section 5.5, we continue in this analysis. We determine the planetary surface temperature using the model of planetary atmospheres. In Section 5.6, we present new calculation of habitable zones in the framework of this model for the stellar spectral classes F, G, K and M. We illustrate the possible prediction of the product of the number of habitable zones and the total infrared optical thickness.

5.2 Exoplanet habitability

Optically thin atmospheres have been used and 600 million habitable planets in our Milky Way Galaxy have been estimated in 1964 [Dole (1964)]. Gilster has popularized these ideas by capturing the imagination exploring the possibilities of space colonization of other planetary systems [Gilster (2004)]. With the discovery of large Super-Earth type planets, the concept of habitable zones has been adopted.

The estimation of a habitable zone is difficult due to a number of factors. The inner edge r_{inner} and the outer edge r_{outer} of the habitable zone have been determined in the form [Kane and Gelino (2012)],

$$r_{\mathrm{inner}} = \sqrt{\frac{L_{\mathrm{s}}}{S_{\mathrm{inner}}}}, \quad r_{\mathrm{outer}} = \sqrt{\frac{L_{\mathrm{s}}}{S_{\mathrm{outer}}}}, \tag{5.1}$$

where L_{s} is the absolute luminosity of the star, S_{inner} and S_{outer} are stellar fluxes for the inner edge and the outer edge of the habitable zone, respectively,

$$\begin{aligned} S_{\mathrm{inner}} &= 4.190 \times 10^{-8} T_{\mathrm{eff}}^2 - 2.139 \times 10^{-4} T_{\mathrm{eff}} + 1.268, \\ S_{\mathrm{outer}} &= 6.190 \times 10^{-9} T_{\mathrm{eff}}^2 - 1.319 \times 10^{-5} T_{\mathrm{eff}} + 0.2341, \end{aligned} \tag{5.2}$$

where T_{eff} is the effective stellar surface temperature. New service has been established for the exoplanet community, which provides the habitable zone information for each exoplanetary system with known planetary orbital parameters [Kane and Gelino (2012)]. The service includes a table with the information of the percentage of orbital phase spent within the habitable zone, the effective planetary temperatures and of other basic planetary properties.

The habitable zone boundaries can be also computed according to empirical formulae in the form

$$r_{\mathrm{inner}} = \sqrt{\frac{L_{\mathrm{s}}}{1.1}}, \quad r_{\mathrm{outer}} = \sqrt{\frac{L_{\mathrm{s}}}{0.53}}, \tag{5.3}$$

where the value 1.1 is the magnitude of stellar flux at the inner boundary of the habitable zone and the value 0.53 is the magnitude of stellar flux at the outer boundary of the habitable zone [Morris (2010)]. The evolution of habitable zones has been also studied, because stars become gradually more luminous during their main sequence lives and the habitable zones must migrate outward. Very important factors have been used for the calculation of habitable zones such as the influence of tidal braking and orbital period of planets. The habitability of known extrasolar systems based on measured stellar properties has been discussed in [Jones, Sleep and Underwood (2008)]. Computer models have been applied to 152 exoplanetary systems that are sufficiently well characterized. The proportion of the systems that could contain habitable Earth-mass planets has been determined.

Many authors have determined positions of habitable planets from empirical laws. Cuntz has identified four new possible hypothetical planetary positions in 0.081, 0.41, 1.51 and 2.95 AU from the star in 55 Cancri System [Cuntz (2012)]. Possible habitability conditions for these planets have been also discussed.

Some authors have focused in the formation and habitability of extrasolar planets [Wetherill (1996)]. 500 new simulations of planetary formation have been used. These simulations show that from 5% to 15% of the simulated planetary systems associated with stars as small as 0.5 of the Sun mass M_{Sun} and as large as 1.5 of M_{Sun} involve habitable planets.

Atmospheric conditions and their influence on habitable planets have been discussed in [Selsis et al. (2007)]. Atmospheric models for the evolution of Venus and Mars have been used and both the theoretical and empirical distances of habitable planets for the stars of spectral types F, G, K and M have been derived. These results have been applied to the extrasolar system Gliese 581. The antigreenhouse effect has been discussed in [McKay, Lorenz and Lunine (1999)]. The antigreenhouse equation has been applied to the Titan moon. A simple expression for the vertical convective fluxes in planetary atmospheres can be found in [Lorenz and McKay (2003)]. The grey radiative model of an atmosphere has been used for the calculation of greenhouse effect and the derivation of an analytic expression for the convective flux.

The galactic habitable zone and the age distribution of complex life in the Milky Way have been discussed in [Lineweaver, Fenner and Gibson (2004)]. The Galactic habitable zone between 7 and 9 kiloparsecs from the Galactic centre has been identified that widens with time and is composed of stars that formed between 4 and 8 billion years ago.

The correct eccentricity of exoplanet is one of the main parameters for the habitability. The eccentricity distributions for the Kepler planet candidates have been compared and it has been shown that the mean eccentricity of the Kepler planet candidates decreases with the decreasing planet size [Kane et al. (2012)]. Smaller planets are preferentially found in low-eccentricity orbits. We will use the Kepler planet candidates [Borucki et al. (2011b)] for the study of habitable zones for certain stellar spectral classes.

5.3 Statistical analysis of exoplanet habitability

We use the normalized parameter $r_{\mathrm{p}}^{\mathrm{norm}}$ in the form [Pintr, Peřinová, Lukš and Pathak (2013)]

$$r_{\mathrm{p}}^{\mathrm{norm}} = \frac{r_{\mathrm{p}}}{\sqrt{M_{\mathrm{s}}}}, \tag{5.4}$$

where r_{p} is the distance of exoplanet from the central star and M_{s} is the mass of central star. For the statistical study, we need all extrasolar planets in one scale. We use the first 50 neighbour exoplanets for each stellar spectral class. We can calculate the difference of normalized distances of two neighbour exoplanets from the central star, $\Delta_{n,n-1}^{\mathrm{norm}}$,

$$\Delta_{n,n-1}^{\mathrm{norm}} = r_{\mathrm{p}_n}^{\mathrm{norm}} - r_{\mathrm{p}_{n-1}}^{\mathrm{norm}}, \tag{5.5}$$

where n is the order of exoplanet.

We get the average of differences of normalized distances of neighbour exoplanets from the central star, namely, the average normalized difference $\Delta_n^{\mathrm{norm,avg}}$

for n exoplanets from the stellar spectral classes F, G, K and M in the form

$$\Delta_n^{\mathrm{norm,avg}} = \frac{1}{n}\sum_{i=2}^{n}\Delta_{i,i-1}^{\mathrm{norm}}. \tag{5.6}$$

The average of differences of unnormalized distances of neighbour exoplanets from the central star, namely, the average unnormalized difference $\Delta_n^{\mathrm{unnorm,avg}}$ is of the form

$$\Delta_n^{\mathrm{unnorm,avg}} = \Delta_n^{\mathrm{norm,avg}}\sqrt{M_{\mathrm{s}}^{\mathrm{avg}}}. \tag{5.7}$$

Using the data for the stellar spectral class F from [Borucki et al. (2011b)], we get the average mass of central star $M_{\mathrm{s}}^{\mathrm{avg}} = 2.195 \times 10^{30}$, the average normalized difference $\Delta_n^{\mathrm{norm,avg}}$ for the 50 neighbour exoplanets,

$$\Delta_{50}^{\mathrm{norm,avg}} = 8.353 \times 10^{-8}, \tag{5.8}$$

and the average unnormalized difference in AU

$$\Delta_{50}^{\mathrm{unnorm,avg}} = 8.251 \times 10^{-4}. \tag{5.9}$$

For the stellar spectral class G, we get the average mass of central star $M_{\mathrm{s}}^{\mathrm{avg}} = 1.972 \times 10^{30}$, the average normalized difference $\Delta_{50}^{\mathrm{norm,avg}}$

$$\Delta_{50}^{\mathrm{norm,avg}} = 3.371 \times 10^{-8}, \tag{5.10}$$

and the average unnormalized difference in AU

$$\Delta_{50}^{\mathrm{unnorm,avg}} = 3.156 \times 10^{-4}. \tag{5.11}$$

For the stellar spectral class K, we get the average mass of central star $M_{\mathrm{s}}^{\mathrm{avg}} = 1.214 \times 10^{30}$, the average normalized difference $\Delta_{50}^{\mathrm{norm,avg}}$,

$$\Delta_{50}^{\mathrm{norm,avg}} = 4.291 \times 10^{-8}, \tag{5.12}$$

and the average unnormalized difference in AU

$$\Delta_{50}^{\mathrm{unnorm,avg}} = 3.152 \times 10^{-4}. \tag{5.13}$$

Thus, the average unnormalized difference for the first 50 neighbour exoplanets is 3.15×10^{-4} AU for the stellar spectral class K.

For the stellar spectral class M, we get the average mass of central star $M_{\mathrm{s}}^{\mathrm{avg}} = 8.101 \times 10^{29}$, the average normalized difference $\Delta_{12}^{\mathrm{norm,avg}}$,

$$\Delta_{12}^{\mathrm{norm,avg}} = 1.466 \times 10^{-6}, \tag{5.14}$$

and the average unnormalized difference in AU

$$\Delta_{12}^{\mathrm{unnorm,avg}} = 8.796 \times 10^{-3}. \tag{5.15}$$

Thus, the average unnormalized difference for the first 12 neighbour exoplanets is 8.796×10^{-3} AU for the stellar spectral class M. We are not able to get all 50 neighbour exoplanets from [Borucki et al. (2011b)], because more Kepler candidates are under measurements for this stellar spectral class.

Morris has shown that the luminosity of star gradually increases during the main sequence [Morris (2010)]. As the star gets more luminous, the habitable zone moves outward the star. According to this, we can define three stages of habitable zones during the stellar life:

- habitable zone at zero age of main sequence (ZAMS),
- habitable zone transit during the life of star on main sequence (HZT),
- habitable zone at the end of main sequence (MSE).

We know the width w of habitable zone during the life of star [Morris (2010)]. We determine the average unnormalized difference for exoplanets $\Delta^{\mathrm{unnorm,avg}}$. The theoretical number of exoplanets N_{p} in the habitable zone for a stellar spectral class is obtained in the form

$$N_{\mathrm{p}} = \frac{w}{\Delta^{\mathrm{unnorm,avg}}}. \tag{5.16}$$

We can find these characteristics in Table 5.1.

stellar spectral class	w ZAMS	w HZT	w MSE	N_{p} ZAMS	N_{p} HZT	N_{p} MSE
F	0.444	0.640	0.718	538.182	775.758	870.303
G	0.354	0.510	0.573	1120.253	1613.924	1813.291
K	0.116	0.167	0.187	368.254	530.159	593.651
M	0.067	0.097	0.109	7.617	11.028	12.392

Table 5.1: Widths of habitable zones in AU and the theoretical numbers of habitable exoplanets.

We can see two trends in Table 5.1. Firstly, we can detect many possible habitable exoplanets for the stellar spectral class G, the stellar spectral class F has by 52% less possible habitable exoplanets than the class G, the stellar spectral class K has by 67% less possible habitable exoplanets than the class G and the stellar spectral class M has by 90% less possible habitable exoplanets than the class G, i. e., the least possible habitable exoplanets. The stellar spectral class G is the best class for possible habitable exoplanets. Secondly, from Table 5.1 it is obvious that the theoretical number of possible exoplanets N_{p} increases with the life-length of star. The number N_{p} increases by 61% between the zero-age of main sequence (ZAMS) and the main-sequence end (MSE).

5.4 Effective temperatures of exoplanets

We can determine the effective temperature of exoplanet T_{eq} by equating the energy received from the star and the energy radiated by the exoplanet. Under the black-body approximation, the flux of the exoplanet F is in the form

$$F = \frac{S(1 - A_{\mathrm{B}})}{4}, \tag{5.17}$$

where S is the flux of the central star and A_{B} is the Bond albedo. We obtain the effective temperature of an exoplanet,

$$T_{\mathrm{eq}}^4 = \frac{R_{\mathrm{s}}^2 T_{\mathrm{eff}}^4}{4r_{\mathrm{p}}^2}, \tag{5.18}$$

where R_{s} is the radius of star, T_{eff} is the effective temperature of stellar surface. The effective temperatures of the Kepler exoplanet candidates have been calculated according to the formula

$$T_{\mathrm{eq}} = T_{\mathrm{eff}} \sqrt{\frac{R_{\mathrm{s}}}{2r_{\mathrm{p}}}} [f(1 - A_{\mathrm{B}})]^{\frac{1}{4}}, \tag{5.19}$$

where f is the factor of full atmospheric thermal circulation [Borucki et al. (2011b)]. The relationship of luminosity and temperature of the extrasolar systems can be found in [Donnison and Williams (2001)]. It has been shown that all extrasolar planets fit on reasonable line in the Hertzsprung-Russell diagram.

We have applied the regression methods for the calculation of effective temperatures of exoplanets to the 2321 Kepler exoplanet candidates [Pintr, Peřinová, Lukš and Pathak (2013)]. We have used equation (5.19), with $f = 1$ and $A_{\mathrm{B}} = 0.3$. We have found the dependence of parameter $r_{\mathrm{p}}^3 T_{\mathrm{eq}}$ on the parameter $v_{\mathrm{p}} r_{\mathrm{p}}$.

For the stellar spectral classes F and G, we can obtain the relation (cf. Fig. 5.1, with the coefficient of determination $R^2 = 0.9987$),

$$y = Cx^5, \tag{5.20}$$

where $C = 5 \times 10^{-43}$ in units [$\mathrm{s}^5\mathrm{Km}^{-7}$] or

$$T_{\mathrm{eq}} = Cv_{\mathrm{p}}^5 r_{\mathrm{p}}^2. \tag{5.21}$$

For the stellar spectral class K, we can obtain the relation (cf. Fig. 5.2, with the coefficient of determination $R^2 = 0.9688$),

$$y = Cx^{4.52}, \tag{5.22}$$

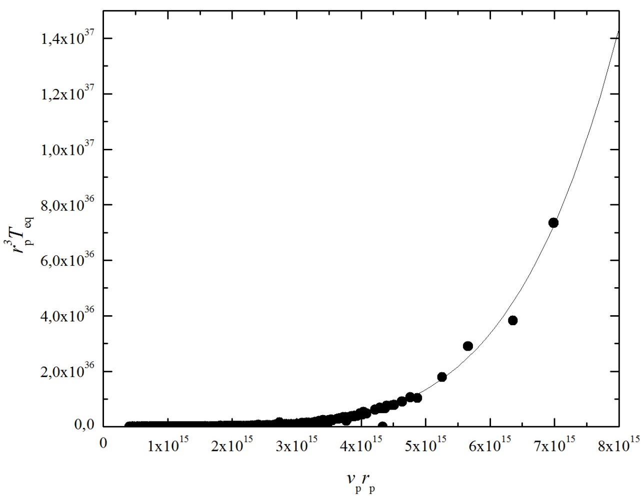

Figure 5.1: Dependence of the parameter $r_{\mathrm{p}}^3 T_{\mathrm{eq}}$ on the parameter $v_{\mathrm{p}} r_{\mathrm{p}}$ for the stellar spectral classes F and G. According to the regression analysis, the best is the power interpolation with the coefficient of determination $R^2 = 0.9987$ for the parameter $r_{\mathrm{p}}^3 T_{\mathrm{eq}}$.

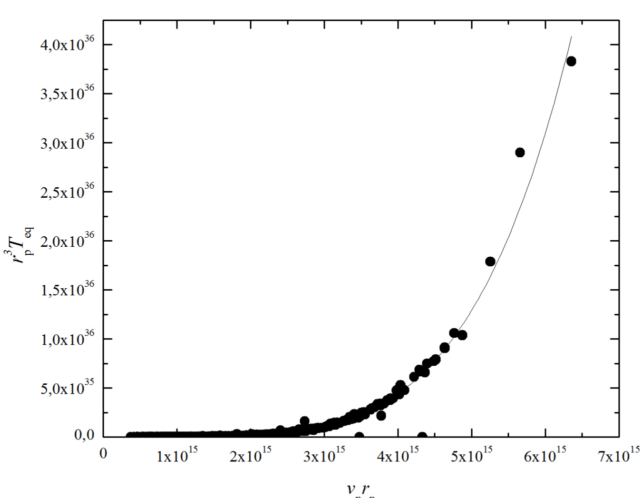

Figure 5.2: Dependence of the parameter $r_{\mathrm{p}}^3 T_{\mathrm{eq}}$ on the parameter $v_{\mathrm{p}} r_{\mathrm{p}}$ for the stellar spectral class K. According to the regression analysis, the best is the power interpolation with the coefficient of determination $R^2 = 0.9688$ for the parameter $r_{\mathrm{p}}^3 T_{\mathrm{eq}}$.

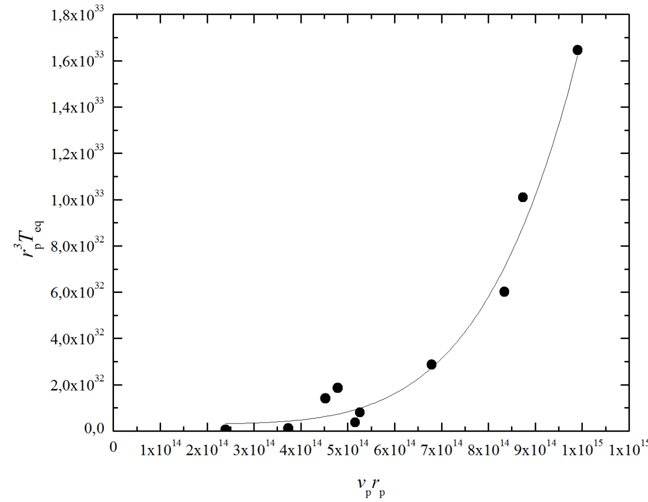

Figure 5.3: Dependence of the parameter $r_{\mathrm{p}}^3 T_{\mathrm{eq}}$ on the parameter $v_{\mathrm{p}} r_{\mathrm{p}}$ for the stellar spectral class M. According to the regression analysis, the best is the power interpolation with the coefficient of determination $R^2 = 0.8366$ for the parameter $r_{\mathrm{p}}^3 T_{\mathrm{eq}}$.

where $C = 1 \times 10^{-35}$ in units [$\mathrm{s}^{4.52}\mathrm{Km}^{-6.04}$] and

$$T_{\mathrm{eq}} = C v_{\mathrm{p}}^{4.52} r_{\mathrm{p}}^{1.52}. \tag{5.23}$$

For the stellar spectral class M, we can obtain the relation (cf. Fig. 5.3, with the coefficient of determination $R^2 = 0.8366$),

$$y = C x^{4.28}, \tag{5.24}$$

where $C = 9 \times 10^{-32}$ in units [$\mathrm{s}^{4.28}\mathrm{Km}^{-5.56}$] and

$$T_{\mathrm{eq}} = C v_{\mathrm{p}}^{4.28} r_{\mathrm{p}}^{1.28}. \tag{5.25}$$

5.5 Surface temperature of exoplanets

Stars are a major source of heat in an otherwise cold Universe. When a star radiation strikes a planet and is absorbed and not reflected, it can cause the planet surface to warm up. On the Earth, the water exists in the liquid state at surface temperatures $T_{\rm sur}$ between 273 K and 373 K and the liquid water is essential for the life. Planets that possess it would be probably habitable. But it is very difficult to determine the correct surface temperatures of the exoplanets, because many factors play key role. Star-light and green-house gases are the big ones, but one must also take into account clouds, convection, evaporation of seawater, absorption of sunlight in the atmosphere, etc. A radiative-convective model has been used under the semigrey approximation in [Lorenz and McKay (2003)]. It has been assumed that the atmosphere and gases react in the same way upon the light in the infrared spectral range. A simple grey radiative-convective model has been introduced in the form

$$F_{\rm o} = F(1 + 0.75\tau), \tag{5.26}$$

where $F_{\rm o}$ is the orbital flux of exoplanet and F is the flux of the central star in the approximation of the black body and τ is the infrared optical thickness. The gas optical-thickness can be expressed in the form [Levenson (2012)]

$$\tau_i = k_i \sqrt{P_i}, \tag{5.27}$$

where k_i is the gas proportionality-constant, e. g., $k_{\rm CO_2} = 0.029$, $k_{\rm H_2O} = 0.087$, and P_i is the partial gas-pressure in the form

$$P_i = q_i P, \tag{5.28}$$

where P is the overall atmospheric-pressure, q_i is the volume fraction of gas. The total infrared optical thickness of the exoplanet atmosphere τ is the sum of all gas optical thicknesses τ_i,

$$\tau = \sum_i \tau_i. \tag{5.29}$$

For the determination of correct surface temperature $T_{\rm sur}$, we should also add mechanisms which cool the planetary surface. These mechanisms are the absorption of sunlight by the atmosphere and the sensible heat (conduction and convection). We can define the absorption of sunlight (cooling mechanism) in the form [McKay, Lorenz and Lunine (1999), Lorenz and McKay (2003)]

$$F_{\rm abs} = F[1 - \exp(-\tau_{\rm vis})], \tag{5.30}$$

where $F_{\rm abs}$ is the absorption flux in the atmosphere and $\tau_{\rm vis}$ is the visual optical thickness. We can find a relationship between the visual optical thickness $\tau_{\rm vis}$ and the infrared optical thickness τ in the form [Levenson (2012)]

$$\tau_{\rm vis} = 0.36(\tau - 0.723)^{0.411}. \tag{5.31}$$

From (5.31), it follows that for $\tau < 0.723$, we have to assign $\tau_{\rm vis} = 0$ to avoid imaginary numbers.

For the determination of cooling during the conduction and the convection, we use the formula [Lorenz and McKay (2003)]

$$F_{\mathrm{c}} = \frac{(F - F_{\mathrm{abs}})\tau}{E + D\tau}, \tag{5.32}$$

where F_{c} is the convective flux of exoplanet atmosphere, E and D are constants. The best fit for these constants is $E = 1$ and $D = 1$ [Lorenz and McKay (2003)].

The surface heat flux of the exoplanet F_{sur} is expressed as

$$F_{\mathrm{sur}} = F_{\mathrm{o}} - F_{\mathrm{abs}} - F_{\mathrm{c}} = FQ, \tag{5.33}$$

where

$$Q = 0.75\tau + \frac{\exp(-\tau_{\mathrm{vis}})}{1 + \tau}. \tag{5.34}$$

Using the Stefan-Boltzmann law, we get the surface temperature T_{sur} of the exoplanet in the form

$$T_{\mathrm{sur}} = T_{\mathrm{eq}} Q^{\frac{1}{4}}. \tag{5.35}$$

5.6 Habitable zones for stellar spectral classes

Substituting (5.21) into (5.35) and using the Kepler law

$$v_{\mathrm{p}}^2 = \frac{GM_{\mathrm{s}}^{\mathrm{avg}}}{r_{\mathrm{p}}}, \tag{5.36}$$

where G is the gravitational constant, we obtain the distance of an exoplanet from the central star for the stellar spectral classes F and G,

$$r_{\mathrm{p}} = \frac{C^2 (GM_{\mathrm{s}}^{\mathrm{avg}})^5 Q^{\frac{1}{2}}}{T_{\mathrm{sur}}^2}, \tag{5.37}$$

where $M_{\mathrm{s}}^{\mathrm{avg}} = 2.195 \times 10^{30}$ kg for the stellar spectral class F and $M_{\mathrm{s}}^{\mathrm{avg}} = 1.972 \times 10^{30}$ kg for the stellar spectral class G, $C = 5 \times 10^{-43}$ for the stellar spectral classes F and G.

Substituting (5.23) into (5.35) and using the Kepler law (5.36), we obtain the distance of an exoplanet from the central star for the stellar spectral class K in the form

$$r_{\mathrm{p}} = \left[\frac{C (GM_{\mathrm{s}}^{\mathrm{avg}})^{2.26} Q^{\frac{1}{4}}}{T_{\mathrm{sur}}} \right]^{1.351}, \tag{5.38}$$

where $M_{\mathrm{s}}^{\mathrm{avg}} = 1.214 \times 10^{30}$ kg and $C = 1 \times 10^{-35}$.

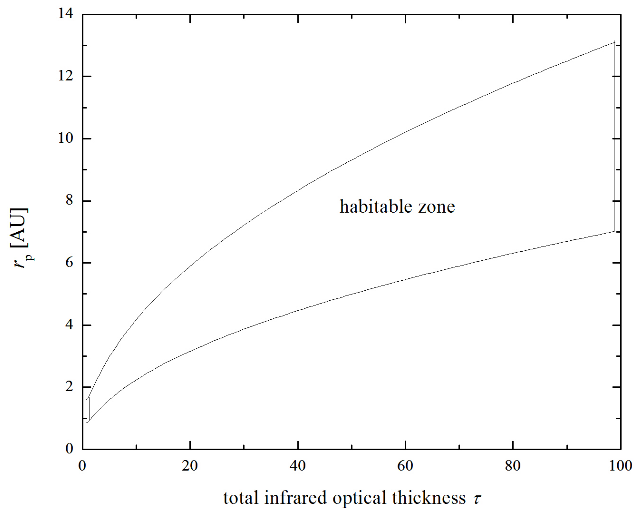

Figure 5.4: Dependence of the distance of the habitable exoplanet from the central star $r_{\rm p}$ on the total infrared optical thickness τ for the stellar spectral class F.

Substituting (5.25) into (5.35) and using the Kepler law (5.36), we obtain the distance of an exoplanet from the central star for the stellar spectral class M in the form

$$r_{\rm p} = \left[\frac{C(GM_{\rm s}^{\rm avg})^{2.14}Q^{\frac{1}{4}}}{T_{\rm sur}}\right]^{1.163}, \tag{5.39}$$

where $M_{\rm s}^{\rm avg} = 8.101 \times 10^{29}$ kg and $C = 9 \times 10^{-32}$.

We can determine the boundaries of habitable zones under the condition for the liquid water that the surface temperature of exoplanets $T_{\rm sur}$ should be between 273 K and 373 K (cf., Figs. 5.4, 5.5, 5.6, 5.7).

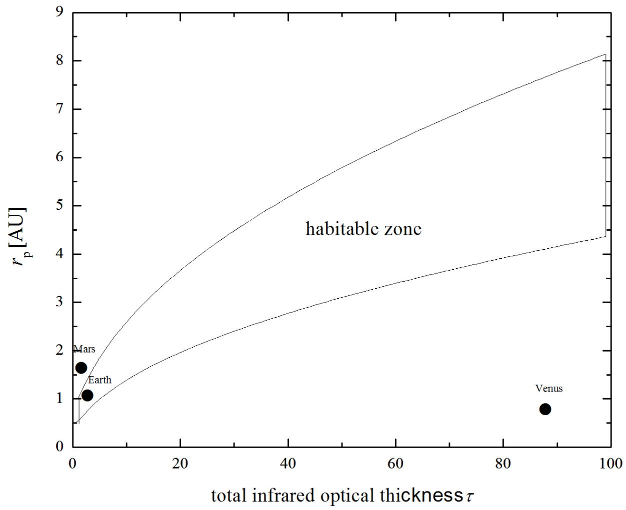

Figure 5.5: Dependence of the distance of the habitable exoplanet from the central star r_{p} on the total infrared optical thickness τ for the stellar spectral class G. For Venus $\tau = 88.010$, for the Earth $\tau = 1.888$, for Mars $\tau = 0.745$, for the minimum boundary $\tau = 0.723$.

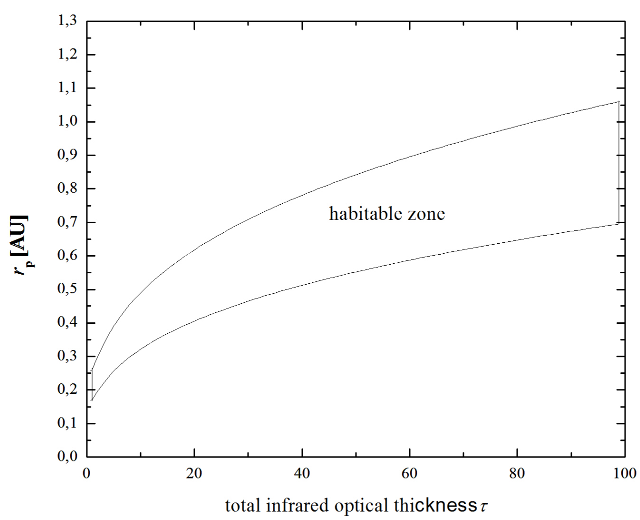

Figure 5.6: Dependence of the distance of the habitable exoplanet from the central star r_{p} on the total infrared optical thickness τ for the stellar spectral class K.

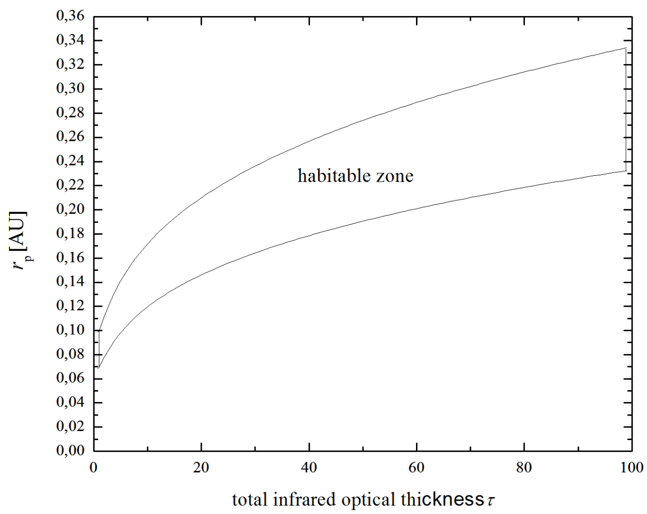

Figure 5.7: Dependence of the distance of the habitable exoplanet from the central star $r_{\rm p}$ on the total infrared optical thickness τ for the stellar spectral class M.

Chapter 6

Conclusions

This doctoral thesis provides a summary of results concerning the detailed investigation of regularities in planetary systems and also statistical analysis of extrasolar systems.

We have concerned with a specific astronomical theme in chapter 2, the regularities in the distances of planets from the respective central bodies. Historically, this topic is restricted to appropriate mentions of the Titius–Bode law. We have taken into account that this narrow theme has been elaborated on in recent years due to the advances in the discoveries of the extrasolar planetary systems. The doubt that the regularity has emerged accidentally is an alternative to the hypothesis that some law is in force in conjunction with causes of some deviations from it. A relative certainty can be achieved by mathematical methods of the statistical decision making. But it entails to try to define a complicated random sample.

Stochasticity elements can be expected in the dynamics of the rotating nebula. We have not dealt with such scenarios, but with some acceptable estimates of the mass distribution in the primordial nebula. The present dynamical theories of the origin of the planetary system assume a mixture of gas and condensed matter. We have admitted that a number of authors are searching for a shortcut of the pathway leading from the assumptions of the model to the regularities of the planetary orbits. We have considered a formal division of the nebula into rings defined by optimum, "convenient", assignment of the angular momentum to circular loops to be easy to describe.

Even though we have not aimed just at detailed calculations, we have taken into account that they are feasible at present, also due to the competition with the research demonstrating the chaotic behaviour and migration of the planets. The possibility of the chaotic behaviour has already been admitted in the past and it seems to be a reason why other researchers moved away the quantization of a theory with an unusual kinetic energy somewhere to the frontier of the observable universe. We have included a formal use of the quantization to discretization of orbits and various efforts for finding a physical underpinning of such an advantage. In extremity, the vibration modes of a circular membrane with orbits of planets or moons can be compared. The mechanical vibrations model a specific magnetohydrodynamical behaviour, which is expected in the region of the birth of stars and their planetary systems.

Basing on the numerical agreement between calculation and real data in chapter 3, we have found suitable numbers of the planets and a proportionality constant of their areal velocities to this numbers. We have derived that the distance of the planet to the central star is directly proportional to the square of this number. In this way we have obtained the first possible orbit of a planet at the distance 0.052 AU in the solar system and 0.000484 AU in the system HD10180. Analogously, we have got the first possible orbit of a moon at the distance 7890.79 km in the Jovian system of moons and 1723.28 km in the Uranian moon system.

We conclude that the distances of the planets and moons in gravitational systems can be obtained as follows:

Areal velocities of planets relate to integral numbers, viz., suitable numbers of the planets. These velocities are directly proportional to the appropriate numbers of the planets with a proportionality constant $K^{(\mathrm{approx})}$.

Distances of the planets in the gravitational system are directly proportional to the squares of the numbers of the planets and the proportionality constant, the radius of a possible first orbit $\bar{a}_1$ depends on the parameter $K^{(\mathrm{approx})}$.

After a normalization of all extrasolar systems in chapter 4, we have found the areas, where the density of exoplanets is higher and have also conjectured that these exoplanets form in two groups with the Gaussian distributions. For five multi-planet extrasolar systems, we have found that the best collective physical parameter is the dependence of the parameter $r_{\mathrm{p}}T_{\mathrm{eff}}$ on $v_{\mathrm{p}}r_{\mathrm{p}}$. For 759 exoplanets detected by the method of radial velocities, we have found that the distances of exoplanets from the central star obey, in general, the Schmidt law and these distances r_{p} depend on the stellar surface temperature T_{eff}. Each stellar spectral class has a little different regression of $r_{\mathrm{p}}T_{\mathrm{eff}}$ on $v_{\mathrm{p}}r_{\mathrm{p}}$. The star luminosity L and the star irradiance J have no effect on the regression of distances of exoplanets. We have applied these characteristics also to 2321 exoplanet candidates from the Kepler mission. We have confirmed all regressions of exoplanets for $r_{\mathrm{p}}T_{\mathrm{eff}}$, $r_{\mathrm{p}}L$ and $r_{\mathrm{p}}J$. We have confirmed the results based on the power n_{i}^2 distribution, with the constant initial distribution of planetesimals for $p = 0$ in [Laskar (2000)]. All semi-major distances of exoplanets follow an n_{i}^2 - power distribution for spectral classes F, G, K and M. We get the best power fit of exoplanet distribution for the spectral class F. We get the worse power fit of exoplanet distribution for the spectral class K.

Taking into account the planets of the solar system in chapter 5, the only planets that could be in the habitable zone are the Earth and Mars (in the future). As to the extrasolar planets, there are many candidates for the habitable ones. Using the regression analysis, we can find many possible habitable exoplanets for the stellar spectral class G. The stellar spectral class F could have by 52% less possible habitable exoplanets than the class G, the stellar spectral class K could have by 67% less possible habitable exoplanets than the class G. The stellar spectral class M could have by 90% less possible habitable exoplanets than the class G, i. e., the least possible habitable exoplanets. The stellar spectral class G is the most promising class for possible habitable exoplanets. This finding is expected to be useful for the missions dedicated to the detection of habitable planets. To be precise, as the possibility of detecting a habitable planet is much higher for the star from the stellar spectral class G, the exoplanet detection missions can be focused on the region around the stars of this spectral class. Using the model of planetary atmospheres, we have also found habitable zones for the stellar spectral classes F, G, K and M. We show that the main parameter is the total infrared optical thickness.

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