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University of L'Aquila

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Applied and Interdisciplinary Mathematics

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Master's Thesis

Periodic problem for the Duffing equation

Supervisor
doc. Ing Jiří Šremr

Candidate
Michael Onwona Asante

Student ID (UAQ): 274380
Student ID (BUT): 243839

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BRNO UNIVERSITY OF TECHNOLOGY
VYSOKÉ UČENÍ TECHNICKÉ V BRNĚ

FACULTY OF MECHANICAL ENGINEERING
FAKULTA STROJNÍHO INŽENÝRSTVÍ

INSTITUTE OF MATHEMATICS
ÚSTAV MATEMATIKY

PERIODIC PROBLEM FOR THE DUFFING EQUATION
PERIODICKÁ OKRAJOVÁ ÚLOHA PRO DUFFINGOVU ROVNICI

MASTER'S THESIS
DIPLOMOVÁ PRÁCE

AUTHOR Michael Onwona Asante
AUTOR PRÁCE

SUPERVISOR doc. Ing. Jiří Šremr, Ph.D.
VEDOUCÍ PRÁCE

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AUTOR PRÁCE
AUTHOR

MICHAEL ONWONA ASANTE

VEDOUCÍ PRÁCE
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doc. Ing. JIŘÍ ŠREMR, Ph.D.

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BRNO UNIVERSITY OF TECHNOLOGY

Faculty of Mechanical Engineering

MASTER'S THESIS

Brno, 2022 Michael Onwona Asante

Assignment Master's Thesis

Institut: Institute of Mathematics
Student: **Michael Onwona Asante**
Degree programm: Mathematical Engineering
Branch:
Supervisor: **doc. Ing. Jiří Šremr, Ph.D.**
Academic year: 2021/22

As provided for by the Act No. 111/98 Coll. on higher education institutions and the BUT Study and Examination Regulations, the director of the Institute hereby assigns the following topic of Master's Thesis:

Periodic problem for the Duffing equation

Brief Description:

Ordinary differential equations of various types appear in the mathematical modelling in mechanics. Differential equations obtained are usually rather complicated nonlinear equations. However, using suitable approximations of nonlinearities, one can derive simple equations that are either well known or can be studied analytically. An example of such "approximative" equation is the so–called Duffing equation. Hence, the question on the existence of a periodic solution to the Duffing equation is closely related to the existence of periodic vibrations of the corresponding nonlinear oscillator.

Master's Thesis goals:

Theoretical part:
1) To supplement knowledge in the theory of dynamical systems (in particular, sketch of a phase portrait).
2) To study fundamentals of the qualitative theory of boundary value problems (in particular, method of lower and upper functions for the periodic problem).

Practical part:
1) To analyse the existence of periodic solutions to the considered Duffing equation in the autonomous case.
2) To find conditions guaranteeing the existence of a periodic solution in the nonautonomous case.
3) To make numerical simulations illustrating the obtained results.

Recommended bibliography:

HABETS, P., DE COSTER, C. Two-point boundary value problem: lower and upper solutions. Mathematics in Science and Engineering, 205. Amsterdam: Elsevier B.V., 2006. ISBN 978-0-444-52200-9.

HARTMAN, P. Ordinary differential equations, New York - London - Sydney: John Wiley & Sons, 1964.

KOVACIC, I., (ed.), BRENNAN, M. J. (ed.). The Duffing equation. Nonlinear oscillators and their behaviour. Hoboken, NJ: John Wiley & Sons, Ltd., Publication, 2011. ISBN 978-0-470-71549-9.

PERKO, L. Differential equations and dynamical systems. Text in Applied Mathematics, 7. New York: Springer-Verlag, 2001. ISBN 0-387-95116-4.

Deadline for submission Master's Thesis is given by the Schedule of the Academic year 2021/22

In Brno,

L. S.

prof. RNDr. Josef Šlapal, CSc. Director of the Institute	doc. Ing. Jaroslav Katolický, Ph.D. FME dean

Abstract

In the mathematical modelling of physical systems, ordinary differential equations of various forms are used. Differential equations describing these systems are often complex nonlinear equations, however using suitable approximations of nonlinearity, one can derive simple equations called Duffing equations which can be studied analytically. In mathematical modelling of mechanics, the problem of finding periodic solutions to these Duffing equations is closely related to the existence of periodic vibrations of its corresponding nonlinear oscillator. In this work, the analysis of the solutions and existence of solutions in the autonomous and nonautonomous cases of the considered Duffing equation are carried out supported by simulations in MATLAB.

Keywords

Differential equation, Duffing equation, periodic solution, existence, uniqueness.

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I declare that I have worked on this thesis independently under the supervision of doc. Jiří Šremr, Ph.D. and using the sources listed in the bibliography.

Michael Onwona Asante

I would like to express my gratitude first to my supervisor, doc. Jiří Šremr, Ph.D, without whose guidance this project would have been impossible to complete. I would also like to thank my colleagues and family who supported me and offered deep insight into the study and subsequently the successful completion of my thesis.

Michael Onwona Asante

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1 Introduction

The mathematical modelling of physical systems often lead to the need to use equations that describe how these systems change over time. These equations are known as dynamical systems and they consist of systems of differential equations. The dominant way of modelling how these physical systems change over time is by use of the differential equations. These differential equations often appear in the mathematical modelling of mechanics and are complex and nonlinear. Nevertheless, these nonlinearities can be approximated by simpler equations under some assumptions and these "simpler equations" are the so called *Duffing equations*. In its original form, the Duffing equation has only one extra nonlinear stiffness term compared to the linear second-order differential equation, which is the foundations of the theory of vibrations[2]. The origins of this Duffing equation can be traced back to the original work of the author George Duffing(see [1] for review). Although several real world systems cannot be described accurately by these equations, they can be used to study the behaviour of real world systems qualitatively.

This work is aimed at studying the qualitative behaviour of a physical system particularly a nonlinear oscillator. The goal is to derive the Duffing equation from the chosen nonlinear oscillator and use tools from the theory of dynamical systems to study the qualitative behaviour of the autonomous variant of the system. Then further use the qualitative theory of boundary value problems particularly the method of lower and upper solutions to find conditions guaranteeing the existence of periodic solutions in the nonautonomous case and finally perform simulations to illustrate obtained results.

The organisation of this thesis is as follows;

In the second section, we show how the considered Duffing equation is obtained from a physical system using laws of motion and Taylor approximations.

The third section is the theoretical part where we present definitions and notions in dynamical systems necessary to study the qualitative behaviour of the autonomous variant of the considered Duffing equation. In the same section we introduce the theory of boundary value problems particularly the method of lower and upper functions which will be required in the next section.

The fourth section is dedicated for the qualitative analysis of the autonomous Duffing equation where we obtain the phase portrait from level sets to study the behavior of solutions in the autonomous case. In this same section we find conditions guaranteeing existence of solutions in the nonautonomous case using previously introduced theories.

In the fifth section we perform simulations in MATLAB to illustrate results obtained from the previous sections.

2 Derivation of considered Duffing Equation

In this section, we derive the considered Duffing equation describing a mechanical oscillator. This section is based off the work of [10] and some results from [2].
Duffing equations are usually second order differential equation with a cubic nonlinearity. For simplicity we usually assume no external or damping force and end up with a much simpler general form of the Duffing equation given by

$$y'' \pm \alpha y \pm \gamma y^3 = 0. \tag{2.1}$$

The considered Duffing equation models the oscillator shown in figure 2.0.1 and consists of a unit mass which is restricted in motion to the horizontal x-axis, and two linear springs. We assume in our case that the springs are attached to fixed barriers which may oscillate in the vertical direction.

Let $l(t)$ be the length of the the spring with respect to time. The movement of the mass is restricted to the horizontal axis, so the length $l(t)$ changes with the position of the mass. Let furthermore, k be the spring constant, $x(t)$ be the location of the unit mass at time t, and $d(t)$ the distance of each barrier from the x-axis. Newton's second law of motion

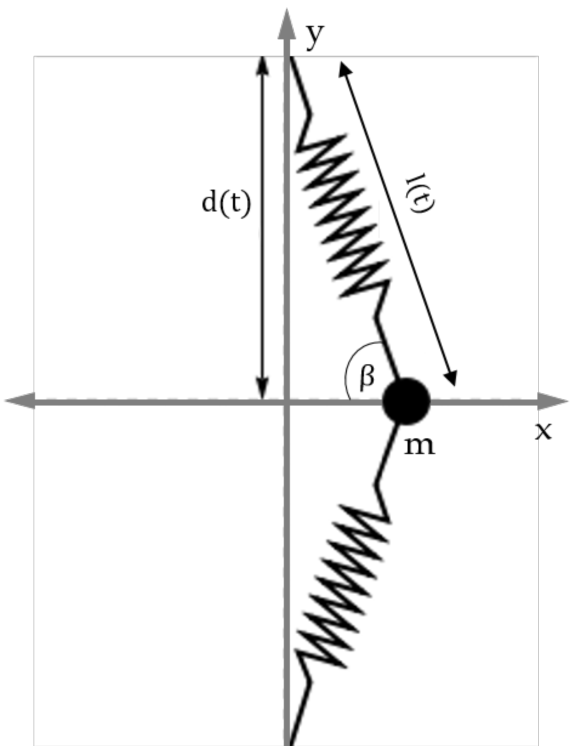


Figure 2.0.1: Nonlinear oscillator

establishes the relation between the force and the product of mass(m) and acceleration($\vec{a}$), where $m = 1$ in our case. Since movement of the mass is in the x direction, all motion is restricted to the horizontal axis hence the force component in the horizontal direction is given by

$$F_x = |\vec{F}| \cos(\beta). \tag{2.2}$$

$|\vec{F}|$ is the magnitude of the resultant acting force and β is the angle the spring makes with the horizontal axis. $|\vec{F}|$ is given by Hooke's law as

$$|\vec{F}| = -kX \tag{2.3}$$

where $X = (l(t) - l_0)$ is the spring stretch, and l_0 is the length of the undeformed spring. By making use of the well known Pythagorean theorem we have

$$\begin{aligned} l^2(t) &= d^2(t) + x^2(t) \\ l(t) &= \sqrt{d^2(t) + x^2(t)}. \end{aligned} \tag{2.4}$$

We consider the positive value only since it is the length of the spring. The cosine of the angle β is given by the equation

$$\begin{aligned} \cos(\beta) &= \frac{x(t)}{l(t)} \\ &= \frac{x(t)}{\sqrt{d^2(t) + x^2(t)}}. \end{aligned} \tag{2.5}$$

From the (2.3) and (2.2), we obtain

$$F_x = 2(-k(l(t) - l_0)\cos\beta). \tag{2.6}$$

From the (2.2)−(2.6), we obtain the second order differential equation

$$\begin{aligned} x''(t) &= 2k\left(l_0 - \sqrt{d^2(t) + x^2(t)}\right)\frac{x(t)}{\sqrt{d^2(t) + x^2(t)}} \\ &= 2kx(t)\left(\frac{l_0}{\sqrt{d^2(t) + x^2(t)}} - 1\right). \end{aligned} \tag{2.7}$$

If $d(t)$ is a constant function, i.e., $d(t) = d$ and we approximate the term $\frac{l_0}{\sqrt{d^2(t)+x^2(t)}}$ by its second-order Taylor expansion we get

$$\frac{l_0}{\sqrt{d^2 + x^2(t)}} \approx \frac{l_0}{d} - \frac{l_0 x^2(t)}{2d^3}.$$

The equation (2.7) becomes

$$x''(t) = 2k\left(\frac{l_0}{d} - 1\right)x(t) - \frac{kl_0}{d^3}x^3(t), \tag{2.8}$$

and if we let $a = 2k\left(\frac{l_0}{d} - 1\right)$ and $b = \frac{kl_0}{d^3}$, we obtain the autonomous Duffing equation

$$x'' = ax - bx^3. \tag{2.9}$$

On the other hand if the barriers oscillate vertically, then $d(t)$ is non constant and we instead replace as functions a, b with $p(t)$ and $h(t)$ respectively, and we obtain the nonautonomous Duffing equation

$$x'' = p(t)x - h(t)x^3. \tag{2.10}$$

3 Theoretical Part

This section is dedicated to present some theoretical concepts for the analysis of the qualitative behavior of solutions to the chosen Duffing equation in the autonomous case and nonautonomous case.

3.1 Concepts from theory of dynamical systems

The idea of dynamical systems is a deterministic process describing a set of conceivable states and a rule of evolution of the state in time[11]. A dynamical system consists of a collection of first order differential equations which is usually derived from some differential equation of higher order.
Consider the system of first-order differential equations

$$\begin{aligned} x_1' &= f_1\left(x_1, x_2, \ldots, x_n\right) \\ x_2' &= f_2\left(x_1, x_2, \ldots, x_n\right) \\ &\vdots \\ x_n' &= f_n\left(x_1, x_2, \ldots, x_n\right). \end{aligned}$$

For some open set $J \subseteq \mathbb{R}^n$, $f_1, f_2, \ldots, f_n : J \to \mathbb{R}$ are continuous functions. This system is called an autonomous system of differential equations since there is no dependence on time. From the theory of differential equations, we know that we can reduce the system to the form

$$x' = f(x), \tag{3.1}$$

where $x = (x_1, x_2, \ldots, x_n)$ and $f = (f_1, f_2, \ldots, f_n)$. For the system of the form $x' = f(x, t)$, we refer to it as a nonautonomous system since it depends on time.

Definition 3.1. A solution to (3.1) on some interval $I \subseteq \mathbb{R}$, is a vector $x = (x_1, x_2, \ldots, x_n)$ of functions with $x_i \in C^1(I), i = 1, \ldots, n$, which satisfies (3.1) on the interval I.

- A solution is a *General solution* if it contains an arbitrary constant. This means we can have several solutions of the form of the general solution depending on the value of the arbitrary constant.
- A *Particular solution* is a solution with no arbitrary constants.

In the case of particular solution, we need to specify some condition(s) on the solution known as *initial condition* or *Cauchy condition.* We represent this condition as

$$x(0) = x_0, \tag{3.2}$$

where $x_0 \in J$. Thus, we assign an initial value of the solution x at a fixed point 0 and we have the Cauchy problem as

$$\begin{cases} x' = f(x) \\ x(0) = x_0. \end{cases} \tag{3.3}$$

Theorem 3.2. *([3, Section 2.4, Theorem 1]). Let J be an open subset of $\mathbb{R}^n$ and assume that $f \in C^1(J)$. Then for each point $x_0 \in J$, there is a maximal interval $I(x_0)$ on which the initial value problem (3.3) has a unique solution, $x(t)$; i.e., if the initial value problem has a solution $y(t)$ on an interval I then $I \subseteq I(x_0)$ and $y(t) = x(t)$ for all $t \in I$. Furthermore, the maximal interval $I(x_0)$ is open; i.e., $I(x_0) = (\alpha, \beta)$.*

Definition 3.3. The interval $I(x_0)$ is the maximal interval of the solution of the Cauchy problem (3.3).

Definition 3.4. ([3, Section 2.5, Definition 1]). Let J be an open subset of $\mathbb{R}^n$ and let $f \in C^1(J)$. For $x_0 \in J$, let $\phi(t, x_0)$ be the solution of the Cauchy problem (3.3) defined on its maximal interval of existence $I(x_0)$. Then for $t \in I(x_0)$, the set of mappings ϕ_t defined by

$$\phi_t(x_0) = \phi(t, x_0)$$

is called the flow of the differential equation (3.1) or the flow defined by the differential equation (3.1).

Remark 3.5. Under the assumption that the system (3.1) describes a dynamical system $\phi(t, x)$ on J. For a point $x_0 \in J$, the function $\phi(\cdot, x_0) : \mathbb{R} \to J$ defines a *solution curve*, *trajectory*, or *orbit* of the system (3.1) through the point x_0 in J. This trajectory through a point $x_0 \in J$ is the motion along the curve

$$\Gamma_{x_0} = \{x \in J \mid x = \phi(t, x_0), t \in I(x_0)\}.$$

Definition 3.6. The *orbits* of a solution $\phi(\cdot, x_0)$ is a collection of points $\phi(t, x_0)$, where $t \in I(x_0)$.

In multidimensional autonomous systems like (3.1), we sometimes refer to the underlying space $\mathbb{R}^n$ as the *phase space*.

Definition 3.7. The *phase portrait* of a system of differential equations (3.1), is the set of all orbits of (3.1) in the phase plane.

Obviously we cannot draw all the orbits so the phase portrait is just a simplified graph showing several orbits.

Definition 3.8. A *equilibrium point* of the system (3.1) is defined intuitively as a point $\bar{x} = (\bar{x_1}, \bar{x_2}, \ldots, \bar{x_n})$ where there is no change in the system, i.e, a point which satisfies

$$\begin{aligned}
0 &= f_1(\bar{x_1}, \bar{x_2}, \ldots, \bar{x_n}) \\
0 &= f_2(\bar{x_1}, \bar{x_2}, \ldots, \bar{x_n}) \\
&\vdots \\
0 &= f_n(\bar{x_1}, \bar{x_2}, \ldots, \bar{x_n}).
\end{aligned}$$

Points which do not satisfy the above equations are called *regular points.*

3.1.1 Planar Dynamical system

The system (3.1) is defined on the space $\mathbb{R}^n$. In this work, the considered Duffing equation is defined on $\mathbb{R}^2$ as we will see in subsection 4.1. The planar dynamical system is given by

$$\begin{aligned}
x_1' &= f_1(x_1, x_2) \\
x_2' &= f_2(x_1, x_2).
\end{aligned} \tag{3.4}$$

In planar dynamical systems and for the scope of this work, we classify the orbits of (3.4) as *homoclinic orbit*, *heteroclinc orbit* and *periodic orbit.*

Definition 3.9. ([11, Chapter 1, Definition 1.4]). A cycle is a periodic orbit L_0, such that each point $x_0 \in L_0$ satisfies $\phi_{t+T_0}(x_0) = \phi_t(x_0)$ with some $T_0 > 0$, for all $t \in T$.

- A periodic orbit corresponds to closed curves which represents the periodic solutions of the system (3.4).
- Homoclinic orbits are orbits which converge to the same equilibrium point for $t \to \infty$ and $t \to -\infty$.
- Heteroclinic orbit are orbits for which $t \to \infty$ converges to one equilibrium point and $t \to -\infty$ converges to another equilibrium point.

Definition 3.10. Given the system (3.4), the matrix

$$Df(\bar{x}) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1}(\bar{x}) & \dfrac{\partial f_1}{\partial x_2}(\bar{x}) \\ \\ \dfrac{\partial f_2}{\partial x_1}(\bar{x}) & \dfrac{\partial f_2}{\partial x_2}(\bar{x}) \end{bmatrix} \tag{3.5}$$

is called the Jacobian matrix of $f = (f_1, f_2)$ at the point $\bar{x}$.

Definition 3.11. An equilibrium point $\bar{x}$ is called a *hyperbolic equilibrium* point of the planar system (3.4) if none of the eigenvalues of the matrix $Df\,(\bar{x})$ have zero real part.

Remark 3.12. To analyse a nonlinear system, it is useful to determine its equilibrium points and to describe it's behaviour near the equilibrium points. It is shown that the local behaviour of the nonlinear system (3.4) near a *hyperbolic* equilibrium point $\bar{x}$ is qualitatively determined by the behaviour of the linear system

$$x' = Ax, \tag{3.6}$$

where the matrix $A = Df(\bar{x})$, near the origin(see section 2 of [3] for review). The system (3.6) is referred to as the linearization of (3.4) at $\bar{x}$.

Definition 3.13. An equilibrium point $\bar{x}$ of (3.4) is called a sink if all of the eigenvalues of the matrix $Df\,(\bar{x})$ have negative real part; it is called a source if all of the eigenvalues of $Df\,(\bar{x})$ have positive real part; and it is called a saddle if it is a hyperbolic equilibrium point and $Df\,(\bar{x})$ has at least one eigenvalue with a positive real part and at least one with a negative real part.

Definition 3.14. An equilibrium point $\bar{x} \in J$ is *stable* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for each $x_0 \in J$

$$||\bar{x} - x_0|| < \delta$$

implies that for the solution $\phi(\cdot, x_0)$,

$$||\phi(t, x_0) - \bar{x}|| < \epsilon \qquad \forall t > 0.$$

If the equilibrium point does not satisfy these conditions, then it is *unstable*.

Definition 3.15. An equilibrium point $\bar{x}$ is asymptotically stable if it is stable and there exists $\delta > 0$ such that for each $x_0 \in J$ such that

$$\|\bar{x} - x_0\| < \delta$$

implies

$$\lim_{t\to\infty} \|\phi\,(t, x_0) - \bar{x}\| = 0.$$

Remark 3.16. An equilibrium point can be classified as stable or unstable from computations of the Jacobian matrix (3.11) and it is given by the following theorem(see, e.g.,[3]).

Theorem 3.17. *Consider the hyperbolic equilibrium point $\bar{x}$ of (3.4). $\bar{x}$ is stable if all the eigenvalues of the matrix $Df(\bar{x})$ have negative real part and it is unstable if all of the eigenvalues of $Df(\bar{x})$ have positive real part.*

Remark 3.18. We can infer from the above theorem also that a hyperbolic equilibrium point $\bar{x}$ is unstable if the eigenvalues of the matrix $Df(\bar{x})$ are such that, at least one has positive real part and at least one has negative real part.

If the equilibrium point is non hyperbolic, the theorem above does not apply hence we use the next theorem to analyse its stability.

Theorem 3.19. *([3, Section 2.9, Theorem 3]). Let J be open and $J \subseteq \mathbb{R}^2$ such that $x_0 \in J$. Suppose that $f \in C^1(J)$ and that $f(x_0) = 0$. Suppose further that there exists a real valued function $V \in C^1(J)$ satisfying $V(x_0) = 0$ and $V(x) > 0$ if $x \neq x_0$. Then*
a. if $V'(x) \leq 0$ for all $x \in J$, x_0 is stable;
b. if $V'(x) < 0$ for all $x \in J \setminus \{x_0\}$, $\mathbf{x}_0$ is asymptotically stable;
c. if $V'(x) > 0$ for all $x \in J \setminus \{x_0\}$, x_0 is unstable.
Where

$$V'(x) = V'_{x_1}(x_1, x_2) f_1(x_1, x_2) + V'_{x_2}(x_1, x_2) f_2(x_1, x_2) \quad \textit{for} \quad x = (x_1, x_2) \in J.$$

The function $V(x)$ is known as the *Lyapunov* function.

3.1.2 Hamiltonian system in $\mathbb{R}^2$

The Hamiltonian system is a special type of nonlinear dynamical system which is used to describe several physical phenomena. The main advantage of this system is its ability to generate the global phase portrait of a given dynamical system in a more elegant way.

Definition 3.20. Let J be an open subset of $\mathbb{R}^2$ and let $H \in C^2(J)$. A system of the form

$$\begin{aligned} x_1' &= \frac{\partial H(x_1, x_2)}{\partial x_2} \\ x_2' &= -\frac{\partial H(x_1, x_2)}{\partial x_1} \end{aligned} \tag{3.7}$$

is called a Hamiltonian system with 1 degree of freedom on J.

Clearly, (3.7) is a special case of the planar system (3.4) with

$$f_1(x_1, x_2) = \frac{\partial H(x_1, x_2)}{\partial x_2}, \quad f_2(x_1, x_2) = -\frac{\partial H(x_1, x_2)}{\partial x_1}.$$

Second order differential equations of the form

$$y'' + f(y) = 0, \tag{3.8}$$

are known as conservative systems and they are special types of Hamiltonian systems as we will see in section 4.1. Converting (3.8) to a system of differential equations, we can let $x_1 = y$ and $x_2 = y'$. So (3.8) becomes

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -f(x_1). \end{aligned} \tag{3.9}$$

The Hamiltonian of (3.9) is given by

$$H(x_1, x_2) = \frac{x_2^2}{2} + \int_0^{x_1} f(s)\mathrm{d}s. \tag{3.10}$$

In the modelling of physical systems, the Hamiltonian represents the *total energy*.

Definition 3.21. Let $c \in \mathbb{R}$. The *level set* χ_c of the Hamiltonian H is given by

$$\chi_c = \{(x_1, x_2) \in \mathbb{R}^2 : H(x_1, x_2) = c\}.$$

Theorem 3.22. *([3, Section 2.14, Theorem 2]). The total energy $H(x_1, x_2)$ of the Hamiltonian system (3.7) remains constant along orbits of (3.7).*

From Theorem 3.22, if we have a point $x_0 \in J$ where $J \subseteq \mathbb{R}^2$ and $\phi(\cdot, x_0)$ is a solution of the initial value problem (3.7),(3.2) on the interval $I(x_0) \subseteq \mathbb{R}$, then

$$H(\phi(t, x_0)) = H(x_0) \quad \forall t \in I(x_0).$$

Definition 3.23. An equilibrium point $\bar{x}$ of the system

$$x' = f(x)$$

at which $Df(\bar{x})$ has no zero eigenvalues is called a *nondegenerate* equilibrium point of the system, otherwise, it is called a degenerate equilibrium point of the system.

It should be noted however that any nondegenerate equilibrium point of the planar system is either a hyperbolic equilibrium point or a center of the linearized system.

Theorem 3.24. *([3, Section 2.14, Theorem 2]) Any nondegenerate equilibrium point of an analytic Hamiltonian system (3.7) is either a (topological) saddle or a center; furthermore, $\bar{x}$ is a (topological) saddle for (3.7) if and only if it is a saddle of the Hamiltonian function $H(x_1, x_2)$ and a strict local maximum or minimum of the function $H(x_1, x_2)$ is a center for (3.7).*

Remark 3.25. From Theorem 3.24 we can deduce that given the Jacobian matrix of (3.7) at the equilibrium point $\bar{x}$

$$M(\bar{x}) = \begin{bmatrix} \dfrac{\partial^2 H}{\partial x_2 \partial x_1}(\bar{x}) & \dfrac{\partial^2 H}{\partial x_2^2}(\bar{x}) \\ \\ -\dfrac{\partial^2 H}{\partial x_1^2}(\bar{x}) & -\dfrac{\partial^2 H}{\partial x_2 \partial x_1}(\bar{x}) \end{bmatrix}, \tag{3.11}$$

if $det(M(\bar{x})) < 0$ then $\bar{x}$ is saddle of 3.7 and $\bar{x}$ is a center if $det(M(\bar{x})) > 0$.

3.2 Method of Lower and Upper functions

Consider a general periodic nonautonomous second order differential equation of the form

$$u'' = f(t,u) \tag{3.12}$$
$$u(a) = u(b) \quad , \quad u'(a) = u'(b). \tag{3.13}$$

Where $b > a$ and $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be a continuous function of u and t.

Definition 3.26. A solution to (3.12) is a function $u : [a,b] \to \mathbb{R}$ which has continuous derivatives up to the second order and satisfies (3.12) identically.

Definition 3.27. ([4, Chapter 1, Definition 1.1]). A function $\alpha \in \mathcal{C}^2(]a,b[) \cap \mathcal{C}^1([a,b])$ is a lower function of the periodic problem (3.12),(3.13) if

1. For all $t \in]a,b[$, $\alpha''(t) \geq f(t,\alpha(t))$,
2. $\alpha(a) = \alpha(b), \alpha'(a) \geq \alpha'(b)$.

A function $\beta \in \mathcal{C}^2(]a,b[) \cap \mathcal{C}^1([a,b])$ is an upper function of (3.12),(3.13)) if

1. For all $t \in]a,b[$, $\beta''(t) \leq f(t,\beta(t))$,
2. $\beta(a) = \beta(b), \beta'(a) \leq \beta'(b)$.

Theorem 3.28. *Let α and β be lower and upper functions of* (3.12),(3.13) *such that $\alpha \leqslant \beta$, define $E = \{(t,u) \in [a,b] \times \mathbb{R} \mid \alpha(t) \leqslant u \leqslant \beta(t)\}$ and assume $f : E \to \mathbb{R}$ is continuous.*
Then the problem (3.12),(3.13) *has at least one solution $u \in C^2([a,b])$ such that for all $t \in [a,b]$*

$$\alpha(t) \leqslant u(t) \leqslant \beta(t)$$

The proof of the above theorem can be found in [4].

Theorem 3.29. *Let $p,q : [a,b] \to \mathbb{R}$ be continuous functions such that $p(t) > 0, q(t) > 0 \quad \forall t \in [a,b]$ and f satisfies*

$$f(t,z)sgnz \geqslant p(t)|z| - q(t) \qquad \forall t \in [a,b], \ \textit{and} \ \forall z \in \mathbb{R},$$

where

$$\operatorname{sgn} z = \begin{cases} 1 & \textit{for } z > 0 \\ 0 & \textit{for } z = 0 \\ -1 & \textit{for } z < 0. \end{cases}$$

Moreover, let α and β be upper and lower functions of the problem (3.12),(3.13). *Then the problem* (3.12),(3.13) *has at least one solution u such that*

$$\min\{\alpha(t_u), \beta(t_u)\} \leqslant u(t_u) \leqslant \max\{\alpha(t_u), \beta(t_u)\},$$

for some $t_u \in [a,b]$.

This theorem follows from [7] Theorem 1.1, remark 1.2 and also from [8] remark 8.4.

Consider the linear differential equation for free undamped oscillator

$$u'' = p_0(t)u + q_0(t). \tag{3.14}$$

We will need the corresponding homogeneous equation

$$u'' = p_0(t)u. \tag{3.15}$$

The coefficients are continuous on some interval [a,b]. The Fredholm Alternative holds for (3.14),(3.13) and it is presented in the next theorem.

Theorem 3.30. *The problem* (3.14),(3.13) *has a unique solution for every* q_0 *if and only if the problem* (3.15),(3.13) *has only the trivial solution.*

This theorem follows from [5], Chapter XII, Part I, Section I.

Theorem 3.31. *Let the problem* (3.15),(3.13) *have a non trivial solution. Then the problem* (3.14),(3.13) *is solvable if and only if* q_0 *satisfies*

$$\int_a^b q_0(s)u_0(s)ds = 0$$

for every solution u_0 *to the problem* (3.15),(3.13).

The proof of this theorem can be found in [5].

Theorem 3.32. *A necessary condition for the equation* (3.15) *to have a non trivial solution possessing two zeroes is that*

$$\int_a^b [p_0(s)]_- \, ds > \frac{4}{b-a}.$$

This follows from [5], Corollary 5.1 .
Here, $[p_0(s)]\_$ is called the negative part of the function and it is given by;

$$[p_0(t)]\_ = \frac{|p_0(t)| - p_0(t)}{2}$$

In general, for $x \in \mathbb{R}$ we have

$$[x]_+ = \frac{|x| + x}{2}$$
$$[x]\_ = \frac{|x| - x}{2}.$$

Theorem 3.33. *A necessary condition for the problem* (3.15),(3.13) *to have a non trivial solution possessing two zeroes is that*

$$\int_a^b [p_0(s)]\_ds > \frac{16}{b-a}.$$

This follows from [6], Lemma 3.12.

4 Analysis of solutions to the Duffing Equation

In this section, we consider first the autonomous case of the considered Duffing equation. We will determine the equilibrium points and from that obtain the level sets to draw the phase portrait of the equation. We then further consider the non autonomous case of the Duffing equation where we prove existence and uniqueness of T-periodic solutions to the considered Duffing equation.

4.1 Autonomous Case

Considering the autonomous Duffing equation

$$y'' = ay - by^3 \tag{4.1}$$

where a, b are constant and $a, b > 0$. To describe analytically the orbits, we can represent it as a system of first order differential equations by setting

$$x_1 = y \text{ and } x_2 = y',$$

then

$$x_1' = y' = x_2, \quad x_2' = y'' = ax_1 - bx_1^3.$$

Hence we obtain the system

$$\begin{cases} x_1' = x_2 \\ x_2' = ax_1 - bx_1^3. \end{cases} \tag{4.2}$$

It is clear that the system is of the same form as a conservative system which is a special type of Hamiltonian system(see section 3 for review). The conservative system is of the form

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -f(x_1), \end{aligned}$$

where

$$f(x_1) = -ax_1 + bx_1^3$$

and the Hamiltonian of the conservative system is given by

$$\begin{aligned} H(x_1, x_2) &= \frac{x_2^2}{2} + \int_0^{x_1} f(s)ds \\ &= \frac{x_2^2}{2} + \int_0^{x_1} (-as + bs^3)ds \\ &= \frac{x_2^2}{2} - \frac{ax_1^2}{2} + \frac{bx_1^4}{4}. \end{aligned} \tag{4.3}$$

To determine the equilibrium points, we know that these occur when there is no change in the system, that is, when all derivatives are equated to 0. At equilibrium points

$$\begin{aligned} x_2 &= 0, \\ ax_1 - bx_1^3 &= 0, \end{aligned} \tag{4.4}$$

$\Rightarrow$

$$\begin{aligned} x_2 &= 0, \\ x_1 = 0, \quad x_1 &= \pm\sqrt{\frac{a}{b}}. \end{aligned} \tag{4.5}$$

Hence we obtain the equilibrium points

$$s_1 = (0,0), \qquad s_2 = \left(\sqrt{\frac{a}{b}}, 0\right), \qquad s_3 = \left(-\sqrt{\frac{a}{b}}, 0\right). \tag{4.6}$$

We classify each of the equilibrium points as follows.

CASE $\mathbf{s_1}$: Considering the equilibrium point s_1, the Jacobian matrix (3.11) is given by

$$\begin{aligned} M(s_1) &= \begin{bmatrix} 0 & 1 \\ a - 3bx_1^2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix} \end{aligned}$$

and the eigenvalues are then; $\lambda_1 = \sqrt{a}$ and $\lambda_2 = -\sqrt{a}$.
Hence we can deduce from definition 3.13 and remark 3.25 that s_1 is a hyperbolic equilibrium point and a saddle. It follows from Theorem 3.17 that s_1 is unstable.

CASE $\mathbf{s_2, s_3}$: Considering the equilibrium point s_2 and s_3, the Jacobian matrix (3.11) is given by

$$M(s_2) = M(s_3) = \begin{bmatrix} 0 & 1 \\ -2a & 0 \end{bmatrix}$$

and the eigenvalues are then; $\lambda_1 = i\sqrt{2a}$ and $\lambda_2 = -i\sqrt{2a}$.
We can infer from definition 3.13 and remark 3.25 that s_2, s_3 are non hyperbolic equilibrium points and centres. As for the stability we can use theorem 3.19 to determine the nature.
If we choose the modified Hamiltonian(Total energy) as our Lyapunov function, we obtain

$$\begin{aligned} V(x_1, x_2) &= \frac{x_2^2}{2} - \frac{ax_1^2}{2} + \frac{bx_1^4}{4} + \frac{a^2}{4b} = \frac{x_2^2}{2} + \frac{b}{4}\left(x_1^2 - \frac{a}{b}\right)^2, \\ V'(x_1, x_2) &= (-ax_1 + bx_1^3)x_2 + x_2(ax_1 - bx_1^3), \\ &= 0. \end{aligned}$$

Hence the equilibrium points s_2, s_3 are stable by theorem 3.19.

In order to describe the orbits of the system, we need the level set of H given by

$$\chi_c = \{(x_1, x_2) \in \mathbb{R}^2 : H(x_1, x_2) = c\} \tag{4.7}$$

where c is some admissible constant.

The level curves corresponding to the equilibrium points occurs at $H(s_1) = 0$ and $H(s_2) = H(s_3) = -\frac{a^2}{4b}$ that is at $c = 0$ and $c = -\frac{a^2}{4b}$. We will then analyse the level curves at these values and other regions of admissible c values.

Case 1 : For c $= 0(\chi_0)$
The level curve for this case is given by $H(x_1, x_2) = 0$, hence we obtain

$$\frac{x_2^2}{2} + \frac{b}{4}x_1^4 - \frac{ax_1^2}{2} = 0$$
$$x_2^2 = ax_1^2 - \frac{bx_1^4}{2}$$
$$x_2 = \pm\sqrt{ax_1^2 - \frac{b}{2}x_1^4}$$

with condition that $ax_1^2 - \frac{b}{2}x_1^4 \geqslant 0$,

$$\Rightarrow \quad \frac{b}{2}x_1^4 - ax_1^2 \leqslant 0$$
$$\frac{b}{2}x_1^4 - ax_1^2 + \frac{a^2}{2b} - \frac{a^2}{2b} \leqslant 0$$
$$\frac{b}{2}\left(x_1^2 - \frac{a}{b}\right)^2 \leqslant \frac{a^2}{2b}$$
$$\left(x_1^4 - \frac{2a}{b}x_1^2 + \frac{a^2}{b^2}\right) \leqslant \frac{a^2}{b^2}$$
$$x_1^2 \leqslant \frac{2a}{b}$$
$$|x_1| \leqslant \sqrt{\frac{2a}{b}}$$
$$\Rightarrow -\sqrt{\frac{2a}{b}} \leqslant x_1 \leqslant \sqrt{\frac{2a}{b}}$$

so we have

$$-\sqrt{\frac{2a}{b}} \leqslant x_1 < 0 \quad \text{or} \quad 0 < x_1 \leqslant \sqrt{\frac{2a}{b}} or x_1 = 0.$$

For the interval $-\sqrt{\frac{2a}{b}} \leqslant x_1 < 0$ we have that

$$x_2 = \pm\sqrt{ax_1^2 - \frac{b}{2}x_1^4},$$

for the interval $0 < x_1 \leqslant \sqrt{\frac{2a}{b}}$ we have that

$$x_2 = \pm\sqrt{ax_1^2 - \frac{b}{2}x_1^4},$$

for $x_1 = 0$, we have $x_2 = 0$
The level curve for this case is comprised of two homoclinic orbits and the point $s_1 = 0$ which is shown in the image below

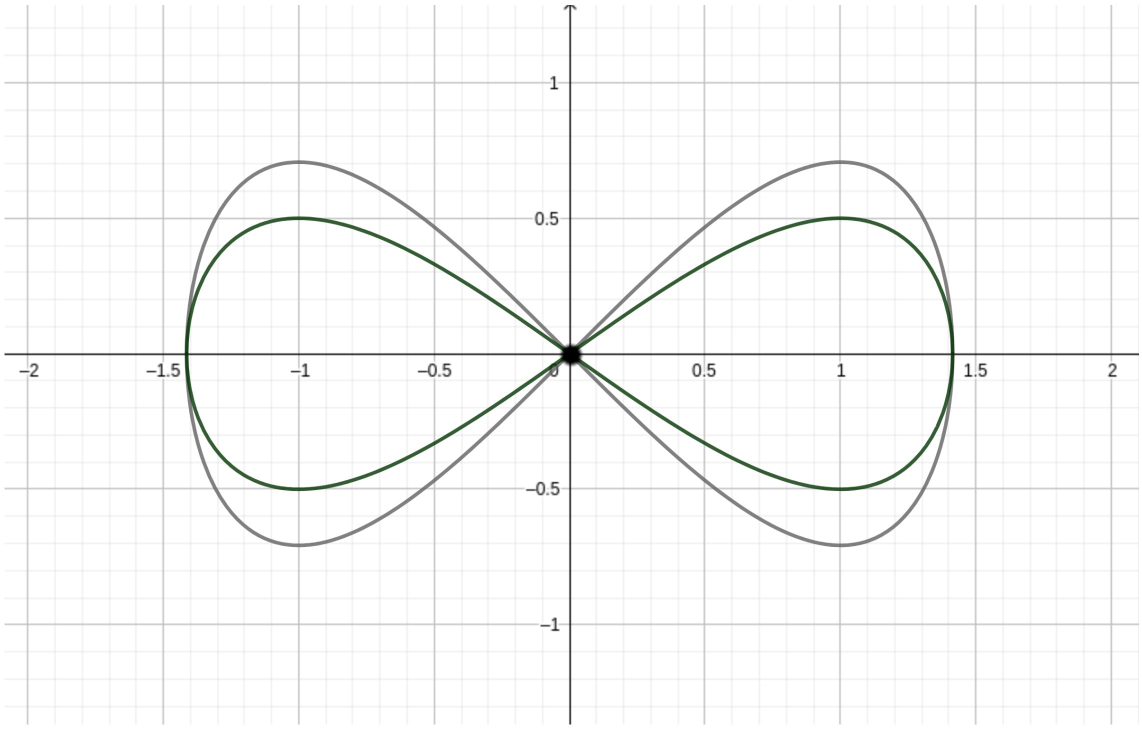


Figure 4.1.2: Level curves of χ_0 for $a = b = 1$ and $a = b = \frac{1}{2}$

Case 2 : For c $= -\frac{a^2}{4b}$, $\chi_{-\frac{a^2}{4b}}$
The level curve for this case is given by $H(x_1, x_2) = -\frac{a^2}{4b}$, hence we obtain

$$\frac{x_2^2}{2} + \frac{b}{4}x_1^4 - \frac{ax_1^2}{2} = -\frac{a^2}{4b}$$
$$x_2^2 + \frac{b}{2}x_1^4 - ax_1^2 = -\frac{a^2}{2b}$$
$$x_2^2 + \frac{b}{2}\left[\left(x_1^2 - \frac{a}{b}\right)^2 - \frac{a^2}{b^2}\right] = -\frac{a^2}{2b}$$
$$x_2^2 + \frac{b}{2}\left(x_1^2 - \frac{a}{b}\right)^2 = 0$$

We have that $a, b > 0$ so $x_2 = 0$ and $\frac{b}{2}\left(x_1^2 - \frac{a}{b}\right)^2 = 0$

$$x_1^2 - \frac{a}{b} = 0$$
$$x_1^2 = \frac{a}{b}$$
$$x_1 = \pm\sqrt{\frac{a}{b}}$$

The level curve for this case consists of the points s_2 and s_3 so we have that $\chi_{-\frac{a^2}{4b}} = \{s_2, s_3\}$ as shown in the image below

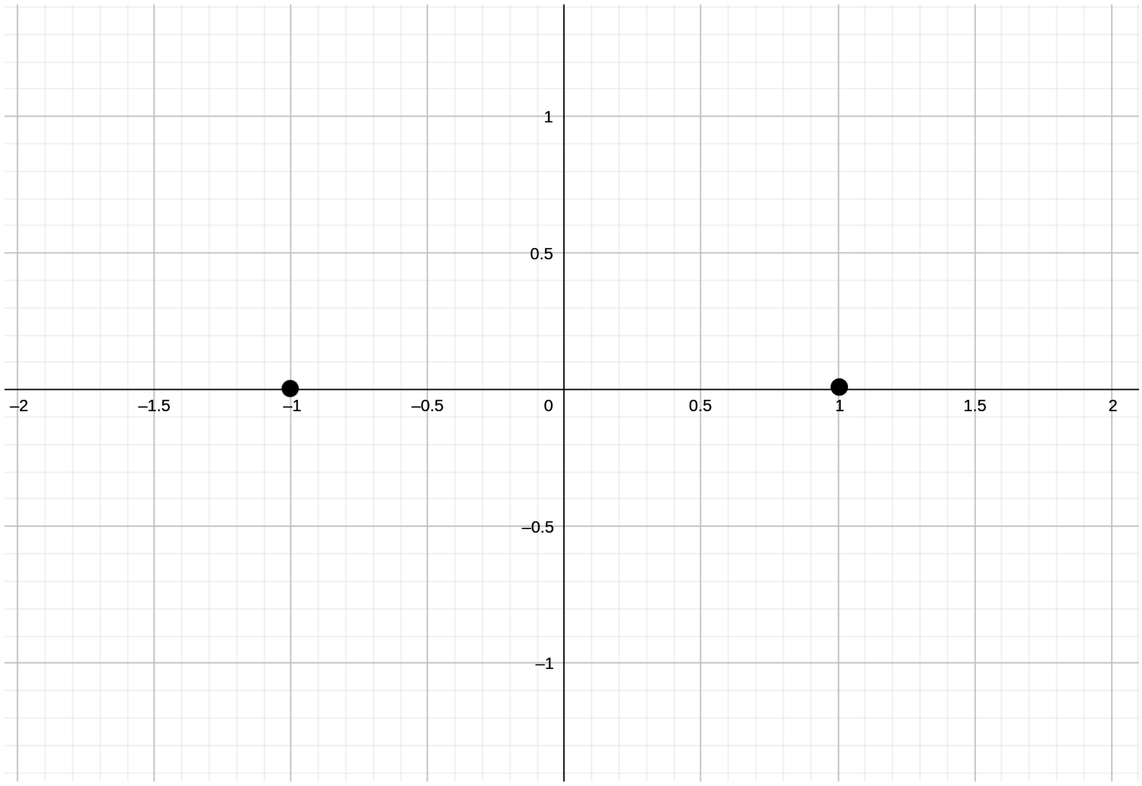


Figure 4.1.3: Level curves of $\chi_{-\frac{a^2}{4b}}$ for $a = b = 1$

Case 3: For $c > 0$
The level curves for this case are given by $H(x_1, x_2) = c$, hence we obtain

$$\begin{aligned} \frac{x_2^2}{2} + \frac{b}{4}x_1^4 - \frac{ax_1^2}{2} &= c, \\ x_2^2 + \frac{b}{2}x_1^4 - ax_1^2 &= 2c. \end{aligned} \tag{4.8}$$

Since $c > 0$

$$\begin{aligned} \Rightarrow x_2^2 &= -\frac{b}{2}x_1^4 + ax_1^2 + 2c, \\ x_2 &= \pm\sqrt{ax_1^2 - \frac{b}{2}x_1^4 + 2c}, \end{aligned} \tag{4.9}$$

with condition that

$$ax_1^2 - \frac{b}{2}x_1^4 + 2c \geq 0$$

$$\begin{aligned} \Rightarrow \frac{b}{2}\left(x_1^2 - \frac{a}{b}\right)^2 &\leqslant 2c + \frac{a^2}{2b} \\ \left(x_1^2 - \frac{a}{b}\right)^2 &\leqslant \frac{4c}{b} + \frac{a^2}{b^2} \\ \left(x_1^2 - \frac{a}{b}\right)^2 &\leqslant \frac{4bc + a^2}{b^2} \end{aligned}$$

for $c > 0$

$$\left|x^2 - \frac{a}{b}\right| \leqslant \sqrt{\frac{4bc + a^2}{b^2}}$$

$$\frac{a}{b} - \sqrt{\frac{4bc + a^2}{b^2}} \leqslant x_1^2 \leqslant \frac{a}{b} + \sqrt{\frac{4bc + a^2}{b^2}}$$

$$|x_1| \leqslant \sqrt{\frac{a}{b} + \sqrt{\frac{4bc + a^2}{b^2}}}$$

Hence, we have the orbit

$$x_2 = \pm\sqrt{ax_1^2 - \frac{b}{2}x_1 + 2c} \quad , \qquad -\sqrt{\frac{a}{b} + \sqrt{\frac{4bc + a^2}{b^2}}} \leqslant x_1 \leqslant \sqrt{\frac{a}{b} + \sqrt{\frac{4bc + a^2}{b^2}}}$$

The curve for this case is shown in the image below

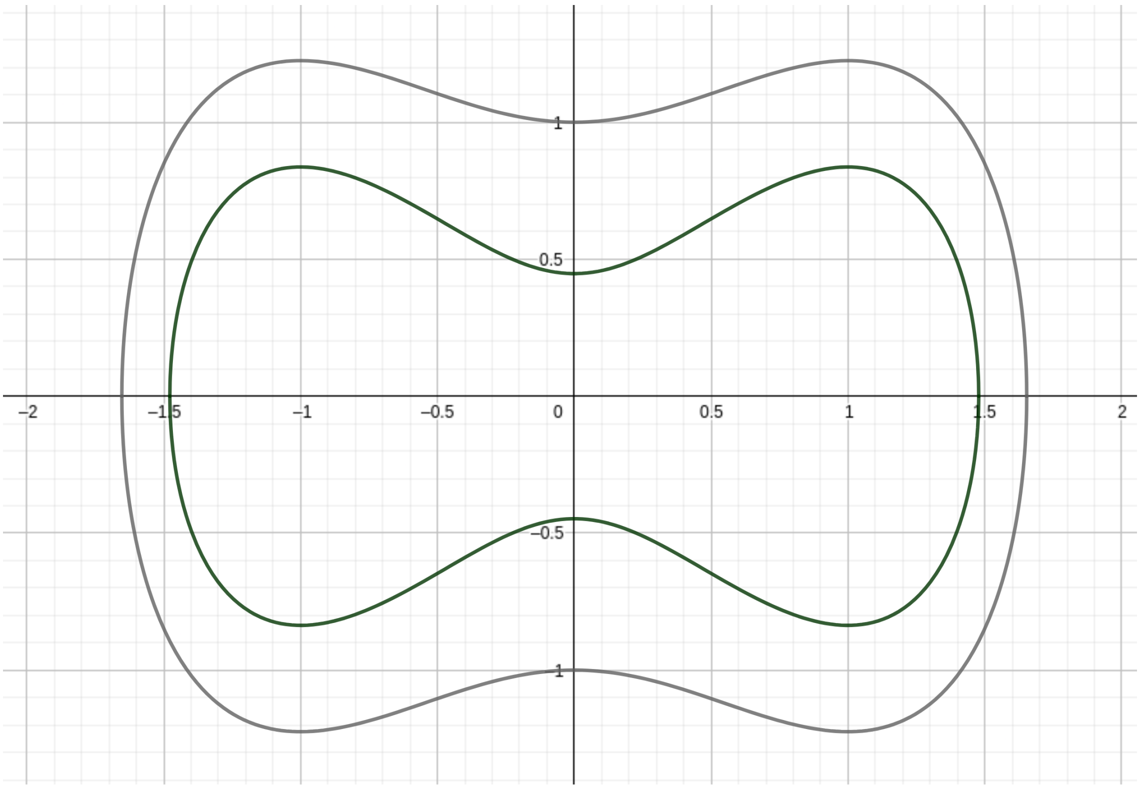


Figure 4.1.4: Level curves with $c > 0$ for $a = b = 1, c = \frac{1}{2}$ and $c = \frac{1}{10}$

Case 4: For $-\frac{a^2}{4b} < c < 0$
From equations (4.8) and (4.9)

$$x_2^2 = ax_1^2 - \frac{b}{2}x_1^4 + 2c$$
$$x_2 = \pm\sqrt{ax_1^2 - \frac{b}{2}x_1^4 + 2c}$$

with the condition that

$$ax_1^2 - \frac{b}{2}x_1^4 + 2c \geqslant 0,$$
$$\left|x_1^2 - \frac{a}{b}\right| \leqslant \sqrt{\frac{4bc + a^2}{b^2}},$$

so we have

$$\begin{aligned} -\sqrt{\frac{4bc + a^2}{b^2}} \leqslant \left(x_1^2 - \frac{a}{b}\right) \leqslant \sqrt{\frac{4bc + a^2}{b^2}}, \\ \Rightarrow \frac{a}{b} - \sqrt{\frac{4bc + a^2}{b^2}} \leqslant x_1^2 \leqslant \frac{a}{b} + \sqrt{\frac{4bc + a^2}{b^2}}. \end{aligned} \tag{4.10}$$

Since $c < 0$, then $\frac{a}{b} - \sqrt{\frac{4bc+a^2}{b^2}} > 0$, thus equation (4.10) yields

$$\sqrt{\frac{a}{b} - \sqrt{\frac{4bc + a^2}{b^2}}} \leqslant |x_1| \leqslant \sqrt{\frac{a}{b} + \sqrt{\frac{4bc + a^2}{b^2}}}.$$

We have two orbits for $-\frac{a^2}{4b} \leqslant c < 0$. For the interval $\sqrt{\frac{a}{b} - \sqrt{\frac{4bc+a^2}{b^2}}} \leqslant x_1 \leqslant \sqrt{\frac{a}{b} + \sqrt{\frac{4bc+a^2}{b^2}}}$ we have

$$x_2 = \pm\sqrt{ax_1^2 - \frac{b}{2}x_1^4 + 2c}.$$

For the interval $-\sqrt{\frac{a}{b} + \sqrt{\frac{4bc+a^2}{b^2}}} \leqslant x_1 \leqslant -\sqrt{\frac{a}{b} - \sqrt{\frac{4bc+a^2}{b^2}}}$ we have

$$x_2 = \pm\sqrt{ax_1^2 - \frac{b}{2}x_1^4 + 2c}.$$

The image for this case is shown on the next page

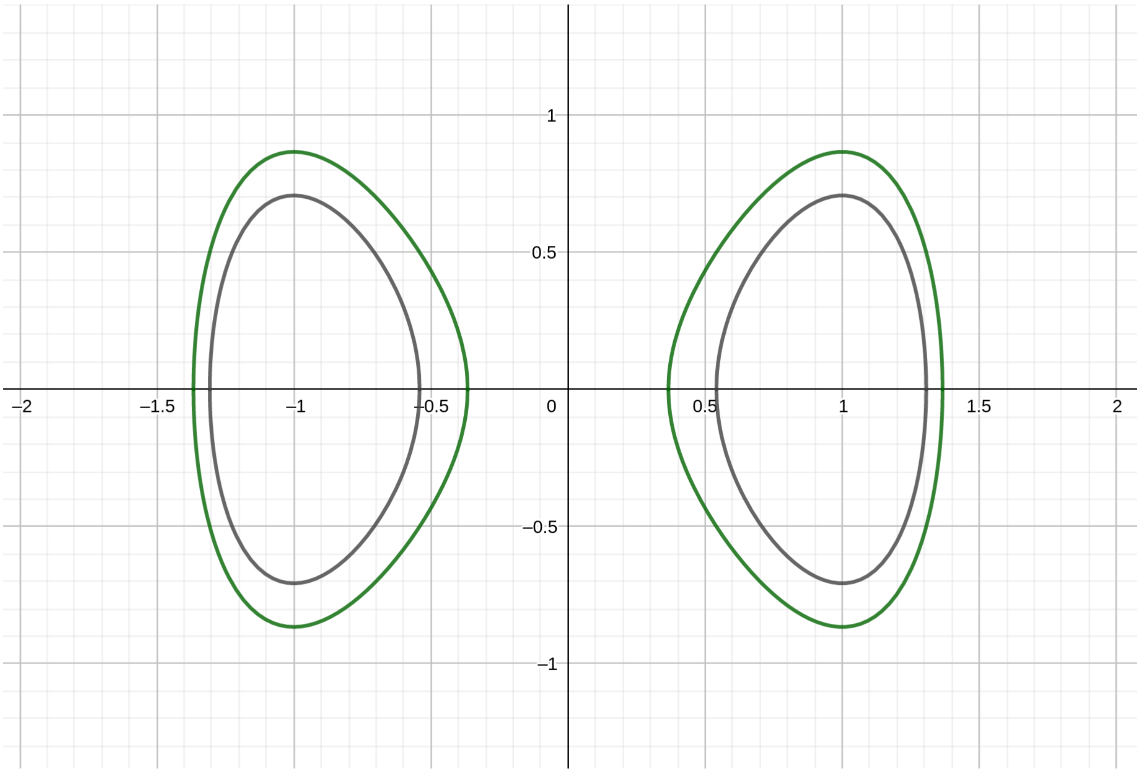


Figure 4.1.5: Level curves with $-\frac{a^2}{4b} < c < 0$ for $a = b = 2, c = -\frac{1}{8}$ and $c = -\frac{1}{4}$

Case 5: For $c < -\frac{a^2}{4b}$
For this case, we assume that $\chi_c \neq \phi$. It follows from (4.9) that

$$\begin{aligned} ax_1^2 - \frac{b}{2}x_1^4 + 2c &\geqslant 0, \\ \frac{b}{2}\left(x_1^2 - \frac{a}{b}\right)^2 &\leqslant 2c + \frac{a^2}{2b}, \\ \Rightarrow c + \frac{a^2}{4b} &\geqslant 0. \end{aligned}$$

This is a contradiction, hence for $c < -\frac{a^2}{4b}$ we have $\chi_c = \phi$.

From the cases established, we obtain the phase portrait of the autonomous differential equation (4.1) as follows

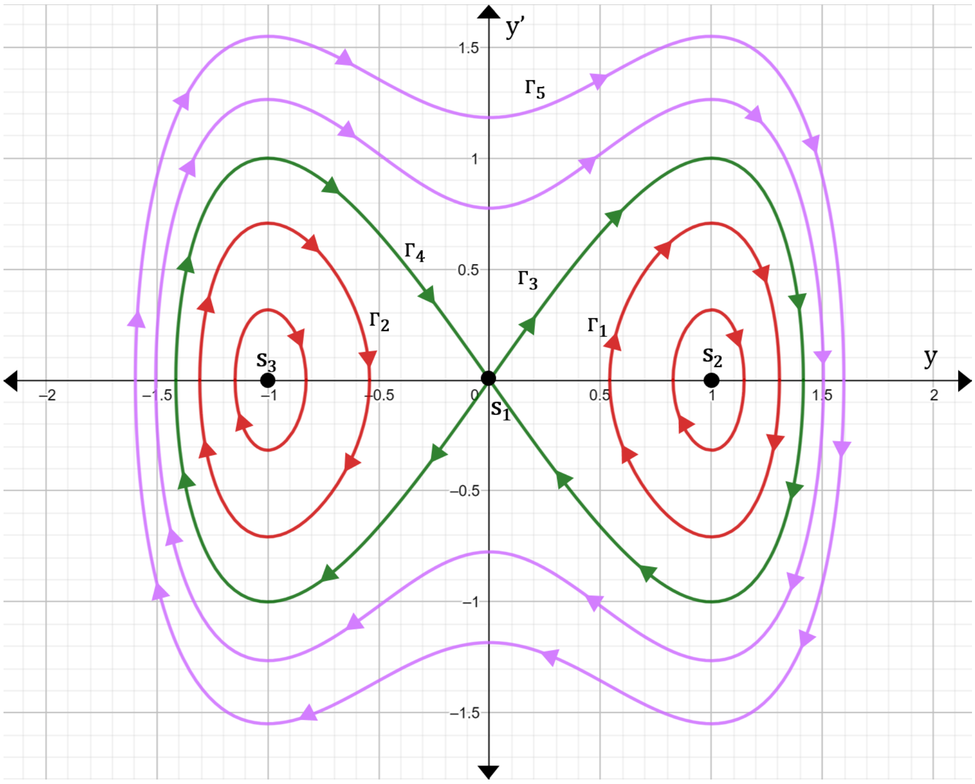


Figure 4.1.6: Phase portrait of (4.1) with $a = b = 2$

The phase portrait consists of the three equilibria(s_1, s_2 and s_3), two homoclinic orbits and and closed periodic orbits.

1. Γ_1 and Γ_2 consists of periodic orbits corresponding to positive and negative periodic solutions of (4.1). The level of Hamiltonian for this case is $H(x_1, x_2) = c$ with $-\frac{a^2}{4b} < c < 0$.
2. Γ_3 and Γ_4 consists of two homoclinic orbits and the equilibrium point s_1 and forms the separatrix cycle of the phase portrait. The separatrix cycle divides the phase portrait into closed periodic orbits and sign changing orbits. The level of Hamiltonian for this case $H(x_1, x_2) = c$ with $c = 0$ and correspond to non constant solutions such that $\lim\limits_{t\to\infty} y(t) = s_1$ and $\lim\limits_{t\to-\infty} y(t) = s_1$.
3. Γ_5 consists of sign changing orbits which correspond to periodic sign changing solutions. The level of Hamiltonian for this orbit is $H(x_1, x_2) = c$ with $c > 0$.
4. The points s_1 and s_2 which are the equilibrium points of (4.1) are the constant solutions of the system.

4.2 Nonautonomous Case

Consider

$$u'' = p(t)u - h(t)u^3 \tag{4.11}$$

such that $p, h : \mathbb{R} \to \mathbb{R}$ are continuous and T-periodic functions.

Theorem 4.1. *Let $p(t) > 0, h(t) > 0 \quad \forall t \in \mathbb{R}$, then the equation (4.11) has at least one positive T-periodic solution.*

To prove this theorem, we need to first establish the following lemmas.

Lemma 4.2. *If $p(t) > 0, h(t) > 0 \quad \forall t \in \mathbb{R}$ then there exists upper and lower function α and β of the problem*

$$u'' = p(t)u - h(t)u^3,$$
$$u(0) = u(T) \quad , \quad u'(0) = u'(T)$$

such that

$$0 < \beta(t) \leqslant \alpha(t) \qquad \forall t \in [0, T].$$

Proof. Since $p(t) > 0, h(t) > 0 \quad \forall t \in \mathbb{R}$, there exists a positive constant c such that

$$c \geqslant \sqrt{\frac{p(t)}{h(t)}} \qquad \forall t \in \mathbb{R}. \tag{4.12}$$

If we put $\alpha(t) = c \quad \forall t \in \mathbb{R}$, then since $\alpha(t)$ is constant then it satisfies the boundary conditions

$$\alpha(0) = \alpha(T) = c,$$
$$\alpha'(0) = \alpha'(T) = 0$$

Moreover, (4.12) yields

$$h(t)c^2 \geqslant p(t)$$

rearranging and multiplying both sides by c

$$0 \geqslant p(t)c - h(t)c^3$$

but $\alpha(t) = c$ and $\alpha''(t) = 0$ hence

$$\alpha''(t) \geqslant p(t)\alpha(t) - h(t)\alpha^3(t) \quad \forall t \in [0, T]$$

$\Rightarrow \alpha(t)$ is a lower function by definition.

Similarly, there exists another constant $d > 0$ such that

$$d \leqslant \sqrt{\frac{p(t)}{h(t)}} \quad \forall t \in \mathbb{R}. \tag{4.13}$$

If we put $\beta(t) = d \quad \forall t \in \mathbb{R}$, since $\beta(t)$ is constant, then it satisfies the boundary conditions and

$$\beta'(t) = 0, \Rightarrow \beta''(t) = 0 \qquad \forall t \in \mathbb{R}.$$

Moreover, from (4.13) we get

$$h(t)d^2p(t),$$

and multiplying both sides by d and rearranging

$$0 \leqslant p(t)\beta(t) - h(t)\beta^3(t) \qquad \in [0,T]$$

since $\beta''(t) = 0$

$$\beta''(t) \leqslant p(t)d - h(t)d^3(t).$$

So by definition, β(t) is an upper function and we can deduce from from (4.12) and (4.13) that

$$0 < \beta(t) \leqslant \sqrt{\frac{p(t)}{h(t)}} \leqslant c = \alpha(t)$$
$$\Rightarrow 0 < \beta(t) \leqslant \alpha(t) \quad \forall t \in \mathbb{R}$$

□

Lemma 4.3. *Let $p, u : [0,T] \to \mathbb{R}$ be both continuous functions such that u has continuous derivatives up to the second order and $u(t) > 0 \quad \forall t \in [0,T], u(0) = u(T), u'(0) = u'(T)$, u satisfies*

$$u''(t) \leqslant p(t)u(t) \qquad \forall t \in [0,T].$$

Then

$$M \leqslant me^{\sqrt{\frac{T}{4}\int_0^T p(s)ds}},$$

where

$$M = \max\{u(t) : t \in [0,T]\},$$
$$m = \min\{u(t) : t \in [0,T]\}.$$

The proof of this can be found in [6].

Now we present the proof of theorem 4.1.

Proof. It follows from lemma 4.2 that $\exists\ \alpha, \beta$ such that

$$0 < \beta(t) \leqslant \alpha(t) \qquad \forall t \in [0,T] \tag{4.14}$$

and

$$\begin{aligned} \alpha''(t) \geqslant p(t)\alpha(t) - h(t)\alpha^3(t)\forall t \in [0,T] \\ \alpha(0) = \alpha(T), \alpha'(0) \geqslant \alpha'(T). \end{aligned} \tag{4.15}$$

$$\begin{aligned} \beta''(t) \geqslant p(t)\beta(t) - h(t)\beta^3(t)\forall t \in [0,T] \\ \beta(0) = \beta(T), \beta'(0) \leqslant \beta'(T) \end{aligned} \tag{4.16}$$

We introduce a function to bound the right hand side of (4.11). To do this first we define

$$\delta := \max\{\alpha(t) : t \in [0,T]\} \cdot e^{\sqrt{\frac{T}{4}\int_0^T p(s)ds}}.$$

We introduce the cutting function $\varphi(z) := [z]_+ - [z-\delta]_+$. Clearly,

$$0 \leqslant \varphi(z) \leqslant \delta \qquad \forall z \in \mathbb{R}. \tag{4.17}$$

Consider the auxiliary periodic problem

$$\begin{cases} u'' = p(t)u - h(t)[\varphi(u)]^3 \\ u(0) = u(T) \\ u'(0) = u'(T). \end{cases} \tag{4.18}$$

Since

$$\operatorname{sgn} z = \begin{cases} 1 & \text{for } z > 0 \\ 0 & \text{for } z = 0 \\ -1 & \text{for } z < 0, \end{cases}$$

we have

$$\left(p(t)z - h(t)[\varphi(z)]^3\right) \operatorname{sgn} z = p(t)z \operatorname{sgn} z - h(t)[\varphi(z)]^3 \operatorname{sgn} z.$$

From equation (4.17)

$$[\varphi(z)]^3 \operatorname{sgn} z \leqslant [\varphi(z)]^3 \leqslant \delta^3,$$

hence

$$\left(p(t)z - h(t)[\varphi(z)]^3\right) \operatorname{sgn} z \geqslant p(t) \mid z \mid -h(t)\delta^3 \qquad \forall z \in \mathbb{R}, \forall t \in [0,T].$$

We know that $0 < \beta(t) \leqslant \alpha(t) \leqslant \delta \quad \forall t \in [0,T]$

It follows from (4.14)-(4.16) and the definition of φ that $\varphi(\alpha(t)) = \alpha(t), \varphi(\beta(t)) = \beta(t) \quad \forall t \in [0,T]$
$\alpha(t)$ and $\beta(t)$ are lower and upper functions of the auxiliary problem (4.18)

Therefore all hypotheses of Theorem 3.29 are satisfied with $a := 0$ and $b := T$, $f(t,z) := p(t)z - h(t)[\varphi(z)]^3$ and $q(t) := \delta^3 h(t)$

Hence the auxiliary problem (4.18) has a solution u such that

$$0 < \beta(t_u) \leqslant u(t_u) \leqslant \alpha(t_u) \quad \text{for some } t_u \in [0,T] \tag{4.19}$$

We will extend the function u T-periodically on the whole real axis

1. We first show that

   $$u(t) \geqslant 0 \quad \forall t \in \mathbb{R}$$

   Suppose on the contrary that there exists $t_0 \in \mathbb{R}$ such that $u(t_0) < 0$, (4.19) yields that

   $$\exists\, a \in \mathbb{R}, b \in]a, a+T[$$

   such that

   $$u(t) < 0 \quad \forall t \in]a,b[, \quad u(a) = 0, u(b) = 0.$$

   Therefore

   $$\varphi(u(t)) = 0 \quad \forall t \in [a,b],$$

from equation (4.18)

$$\begin{aligned} u''(t) &= p(t)u(t) - h(t)[\varphi(u(t))]^3 \\ &= p(t)u(t) \quad \forall t \in [a,b] \end{aligned}$$

u is a solution to the equation

$$u'' = p(t)u$$

with two zeroes on the interval $[a,b]$.
It follows from Theorem 3.32 with $p_0(t) = p(t)$, that

$$\int_a^b [p(s)]_- ds > \frac{4}{b-a}, \tag{4.20}$$

but we assume that $p(t) > 0 \quad \forall t \in \mathbb{R}$, hence $[p(t)]_- = 0 \quad \forall t \in \mathbb{R}$ which is a contradiction to equation (4.20).

2. We show that $u(t) > 0 \quad \forall t \in \mathbb{R}$.
   Suppose on the contrary that there exists $t_0 \in \mathbb{R}$ such that $u(t_0) = 0$,
   in view of the above proved item (1), there exists $a, b \in \mathbb{R}$ s.t $a < t_0 < b$ and

$$\begin{aligned} 0 \leqslant u(t) \leqslant \delta \quad \forall t \in [a,b] \quad u(a) > 0, \\ \Rightarrow \varphi(u(t)) = u(t) \quad \forall t \in [a,b]. \end{aligned}$$

   By equation (4.18)

$$\begin{aligned} u''(t) &= p(t)u(t) - h(t)[\varphi(u(t))]^3 \\ &= p(t)u(t) - h(t)u^3(t) \qquad \forall t \in [a,b] \\ u''(t) &= \left(p(t) - h(t)u^2(t)\right) u(t) \qquad \forall t \in [a,b]. \end{aligned}$$

   u is a solution to the linear equation

$$w'' = (p(t) - h(t)u^2(t))u \tag{4.21}$$

   $u(t) \geq 0 \quad \forall t \in [a,b], u(t_0) = 0$.
   u is continuously differentiable function, hence $u'(t_0) = 0$ and u is a solution to (4.21) satisfying initial conditions $u(t_0) = 0, u'(t_0) = 0$

   The initial value problem for the equation (4.21) is uniquely solvable and its solution is $w(t) = 0 \quad$ for $t \in [a,b]$ which implies that $u(t) = 0 \quad \forall t \in [a,b]$ but $u(a) > 0$ which is a contradiction.

3. Finally we prove that

$$u(t) \leqslant \delta \quad t \in \mathbb{R}. \tag{4.22}$$

   From (4.18)

$$u''(t) = p(t)u(t) - h(t)[\varphi(u(t))]^3 \quad \forall t \in \mathbb{R}$$

   Moreover φ is everywhere non-negative and thus,

$$\varphi(u(t)) \geqslant 0 \qquad \forall t \in \mathbb{R}.$$

On the other hand,

$$h(t) > 0 \quad \forall t \in \mathbb{R},$$

hence $u''(t) \leqslant p(t)u(t) \qquad \forall t \in \mathbb{R}$
From lemma 4.3

$$\max\{u(t) : t \in [0,T]\} \leqslant \min\{u(t) : t \in [0,T]\} \cdot e^{\sqrt{\frac{T}{4}\int_0^T p(s)ds}}. \tag{4.23}$$

Since $u(t_u) \leq \alpha(t_u)$, it follows from equation(4.19) that

$$\min\{u(t) : t \in [0,T]\} \leqslant \max\{\alpha(t) : t \in [0,T]\}$$

Hence by equation (4.23)

$$\max\{u(t) : t \in [0,T]\} \quad \leqslant \quad \max\{\alpha(t) : t \in [0,T]e^{\sqrt{\frac{T}{4}\int_0^T p(s)ds}}.$$

Therefore, (4.22) holds and

$$u(t) > 0 \Rightarrow \varphi(u(t)) = u(t) \qquad \forall t \in \mathbb{R}$$

Hence u is a solution to (4.11) and $u > 0$ and is T-periodic.

□

Remark 4.4. In the autonomous case we have shown that equation (4.1) has three equilibia which are zero, positive and negative. This corresponds to constant solutions(periodic solutions with any period T). Moreover we have two homoclinic orbits, periodic solutions with positive and negative orbits and sign changing solutions. In the non autonomous case, we have proved the existence of at least one T-periodic positive solution which coincides with the facts known in the autonomous case.

Next, we will show that given two distinct positive solutions T-periodic in non-autonomous case, they are not ordered but oscillates around each other.

Theorem 4.5. *Let $h(t) > 0 \quad \forall t \in \mathbb{R}$. Then for any distinct positive T-periodic solutions u_1 and u_2 to equation* (4.11)*, the conditions*

$$\min\{u_2(t) - u_1(t) : t \in [0,T]\} < 0 \tag{4.24}$$

and

$$\max\{u_2(t) - u_1(t) : t \in [0,T]\} > 0 \tag{4.25}$$

holds.

Proof. Let u_1 and u_2 be distinct positive T-periodic solutions to (4.11).
Suppose on the contrary that either (4.24) or (4.25) is not satisfied so without loss of generality, we can assume

$$0 < u_2(t) \leqslant u_1(t) \qquad \forall t \in [0,T], u_2(t) \not\equiv u_1(t) \tag{4.26}$$

$$\Rightarrow u_1''(t) = p(t)u_1(t) - h(t)u_1^3(t)$$
$$= \left(p(t) - h(t)u_1^2(t)\right) u_1(t) \qquad \forall t \in [0,T]$$

put $p_0(t) = p(t) - h(t)u_1^2(t)$ for $t \in [0,T]$

$$u_1''(t) = p_0(t)u_1(t) \qquad \forall t \in [0,T]$$

u_1 is a solution to the equation

$$z'' = p_0(t)z$$

and satisfies

$$u_1(0) = u_1(T), u_1'(0) = u_1'(T).$$

On the other hand

$$u_2''(t) = p(t)u_2(t) - h(t)u_2^3(t)$$
$$= \left[p(t) - h(t)u_1^2(t)\right] u_2(t) + h(t)\left[u_1^2(t) - u_2^2(t)\right] u_2(t)$$

if we put

$$q_0(t) = h(t)\left[u_1^2(t) - u_2^2(t)\right] u_2(t) \qquad \forall t \in [0,T]$$

Then

$$u_2''(t) = p_0(t)u_2(t) + q_0(t) \qquad \forall t \in [0,T]$$

u_2 is a solution to the equation

$$z'' = p_0(t)z + q_0(t) \qquad \forall t \in [0,T],$$

and satisfies

$$u_2(0) = u_2(T), u_2'(0) = u_2'(T).$$

Theorem 3.31 guarantees that u_1 is orthogonal to q_0, hence

$$\int_0^T h(s)\left[u_1^2(s) - u_2^2(s)\right] u_2(s)u_1(s)ds = 0$$

□

Now we prove uniqueness, under the additional assumption that $p(t) = h(t)$. So the equation (4.1) is of the form

$$\begin{aligned} u'' &= p(t)u - p(t)u^3 \\ u'' &= p(t)u(1-u^2) \end{aligned} \tag{4.27}$$

A solution to (4.27) is $u(t) = 1$ which is T-periodic for every $T > 0$

Theorem 4.6. *Let* $p(t) > 0 \quad \forall t \in \mathbb{R}$ *and*

$$\int_0^T p(s)ds \leqslant \frac{8}{T} \tag{4.28}$$

Then the constant solution $u(t) = 1$ *is the unique* T *- periodic positive solution to equation* (4.27)

To prove this we first state the Hölder's inequality as a lemma.

Lemma 4.7. *Let $f, g : [a, b] \to \mathbb{R}$ continuous functions and $\lambda, \mu > 0,$ such that $\dfrac{1}{\lambda} + \dfrac{1}{\mu} = 1.$ Then*

$$\int_a^b \mid f(s)g(s) \mid ds \leq \left(\int_a^b \mid f(s) \mid^\lambda ds\right)^{1/\lambda} \left(\int_a^b \mid g(s) \mid^\mu ds\right)^{1/\mu}$$

Now we can present the proof of Theorem 4.6 which is motivated by the results of [6].

Proof. Suppose on the contrary that u is a positive T-periodic solution to equation (4.27) such that $u(t) \not\equiv 1$.
It follows from Theorem 4.5 that

$$\begin{aligned} &\min\{u(t) - 1 : t \in [0, T]\} < 0 \\ &\max\{u(t) - 1 : t \in [0, T]\} > 0 \end{aligned} \tag{4.29}$$

Equation (4.27) implies that

$$\int_0^T u''(s)ds = \int_0^T p(s)u(s)ds - \int_0^T p(s)u^3(s)ds \tag{4.30}$$

but the left hand side yields $u'(T) - u'(0)$ and from boundary conditions we know that $u'(T) = u'(0)$ hence the left hand side reduces to 0.

$$\Rightarrow \int_0^T p(s)u(s)ds = \int_0^T p(s)u^3(s)ds \tag{4.31}$$

Now making use of lemma 4.7 with $\lambda = 3/2,\ \mu = 3,\ f(t) = p^{2/3}(t),\ g(t) = p^{1/3}(t)u(t),$ recall that

$$\begin{aligned} \int_0^T p(s)u(s)ds &= \int_0^T p^{2/3}(s)\left(p^{1/3}(s)u(s)\right)ds \\ &\leqslant \left(\int_0^T p(s)ds\right)^{2/3} \left(\int_0^T p(s)u^3(s)ds\right)^{1/3} \end{aligned}$$

from equation 4.31

$$\int_0^T p(s)u(s)ds \leqslant \left(\int_0^T p(s)ds\right)^{2/3} \left(\int_0^T p(s)u(s)ds\right)^{1/3}$$

dividing both sides by the term $\left(\int_0^T p(s)u(s)ds\right)^{1/3}$

$$\Rightarrow \left(\int_0^T p(s)u(s)ds\right)^{2/3} \leq \int_0^T (p_0 p(s)ds)^{2/3}$$

$$\Rightarrow \int_0^T p(s)u(s)ds \leq \int_0^T p(s)ds \tag{4.32}$$

Again from lemma 4.7 with $\lambda = 3,\ \mu = 3/2,\ f(t) = p^{1/3}(t),\ g(t) = p^{2/3}(t)u^2(t)$, recall that

$$\begin{aligned}\int_0^T p(s)u^2(s)ds &= \int_0^T p^{1/3}(s)\left(p^{2/3}(s)u^2(s)\right)ds \\ &\leqslant \left(\int_0^T p(s)ds\right)^{1/3}\left(\int_0^T p(s)u^3(s)ds\right)^{2/3}.\end{aligned}$$

Since (4.31) holds,

$$\Rightarrow \int_0^T p(s)u^3(s)ds = \int_0^T p(s)u(s)ds$$

The inequality becomes

$$\int_0^T p(s)u^2(s)ds \leqslant \left(\int_0^T p(s)ds\right)^{1/3}\left(\int_0^T p(s)u(s)ds\right)^{2/3}$$

from equation (4.32)

$$\int_0^T p(s)u^2(s)ds \leqslant \left(\int_0^T p(s)ds\right)^{1/3}\left(\int_0^T p(s)ds\right)^{2/3}$$

$$\int_0^T p(s)u^2(s)ds \leqslant \int_0^T p(s)ds \tag{4.33}$$

If we consider the difference $u(t) - 1$, it satisfies

$$\begin{aligned}(u(t)-1)'' &= p(t)(u(t))\left(1-(u(t))^2\right) \\ &= [-p(t)u(t)(1+u(t))](u(t)-1) \qquad \forall t \in \mathbb{R}\end{aligned}$$

If we set $-p(t)u(t)(1+u(t)) = p_0(t)$ and let $w(t) = u(t) - 1$, then we have

$$w''(t) = p_0(t)w(t) \qquad \forall t \in \mathbb{R}$$

w is a solution to the linear homogeneous equation

$$z'' = p_0(t)z$$

w is T-periodic because it is the difference between two T-periodic functions.
Equation (4.29) implies

$$\begin{aligned}&\min\{w(t) : t \in [0,T]\} < 0 \\ &\max\{w(t) : t \in [0,T]\} > 0\end{aligned}$$

From Theorem 3.33

$$\int_0^T [p_0(s)]_-\,ds > \frac{16}{b-a}.$$

Since $p(t) > 0$, then $p_0(t)$ is everywhere negative

$$[p_0(t)]_- = p(t)u(t)(1+u(t)) \quad \forall t \in R$$

$$\Rightarrow \quad \int_0^T p(s)(1+u(s))u(s)ds > \frac{16}{T}. \tag{4.34}$$

If we consider equations (4.32) and (4.33) we obtain

$$\begin{aligned}\int_0^T p(s)u(s)(1+u(s))ds &= \int_0^T p(s)u(s)ds + \int_0^T p(s)u^2(s)ds \\ &\leq 2\int_0^T p(s)ds\end{aligned} \tag{4.35}$$

From (4.34) and (4.35)

$$\int_0^T p(s)ds > \frac{8}{T}.$$

This is a contradiction to (4.28) hence $u(t) = 1$ is a unique T-periodic solution to equation (4.27). □

Corollary 4.8. *Let y be a non constant positive periodic solution to the equation*

$$y'' = ay - by^3 \tag{4.36}$$

with $a, b > 0$
Then the minimum period T of y satisfies

$$T > \frac{2\sqrt{2}}{\sqrt{a}} \tag{4.37}$$

Proof. Put

$$Y(t) = \sqrt{\frac{b}{a}}y(t)$$

Then

$$\begin{aligned} Y''(t) &= aY(t) - aY^3(t) \\ \Rightarrow Y''(t) &= aY(t)\left[1 - Y^2(t)\right]\end{aligned} \tag{4.38}$$

since Y is positive non constant T-periodic function, using Theorem 4.6 with $p(t) = a$, then

$$\begin{aligned}\int_0^T ads &> \frac{8}{T} \\ aT &> \frac{8}{T} \\ T^2 &> \frac{8}{a}\end{aligned}$$

If we take the square root of both sides, since both sides are positive we get

$$T > \frac{2\sqrt{2}}{\sqrt{a}}.$$

□

5 Simulations

To support our results, we perform simulations with MATLAB™ on the considered Duffing equation which gives a better illustration of the facts known in the qualitative analysis of both the autonomous and non autonomous case. The MATLAB™ function *ode*45 is used to solve the Duffing equation numerically to obtain the desired results which corresponds to the results already obtained. This function is MATLAB's standard solver for ordinary differential equations and uses the Runge-Kutta method with a variable time step for efficient computation(see [9] for review).

5.1 Autonomous case

There is a continuous dependence of initial conditions on the solution of a differential equation and as a result of this, the trajectory corresponding to a solution of the differential equation will depend on the chosen initial condition. The constants a and b are chosen to be $a = 1$ and $b = 2$, so the equation (4.1) becomes

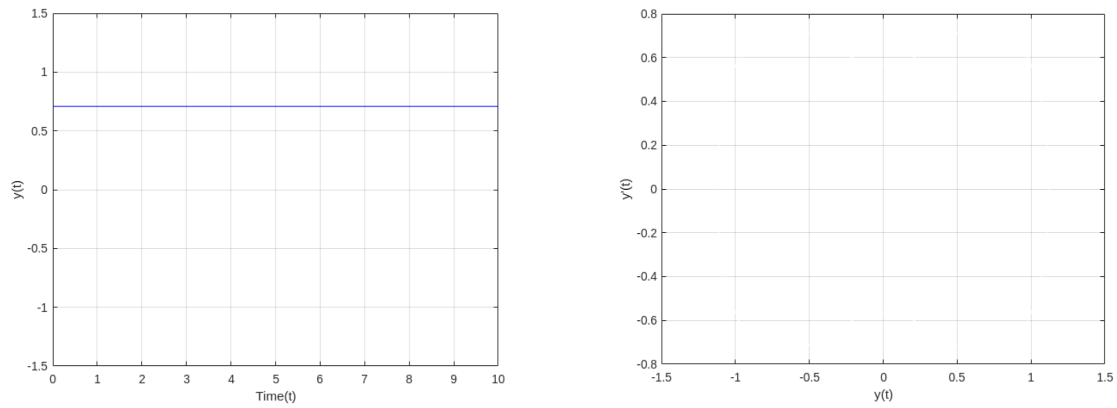
$$y'' = y - 2y^3, \tag{5.1}$$

which can be represented as

$$\begin{cases} x_1' = x_2 \\ x_2' = x_1 - 2x_1^3. \end{cases} \tag{5.2}$$

To obtain better accuracy, the relative tolerance and absolute tolerance are both set to $1e-8$. In subsection 4.1, we presented four cases representing different orbits of the solution to the Duffing equation. In this subsection, we present again four cases corresponding solutions of (5.1) based on different initial conditions.

Case I: For this case we choose the initial condition $y(0) = \sqrt{\frac{1}{2}},\ y'(0) = 0$ and we can see that we obtain constant solution which corresponds to the equilibrium point $\mathbf{s}_2$(see (4.6)). The image is shown below.

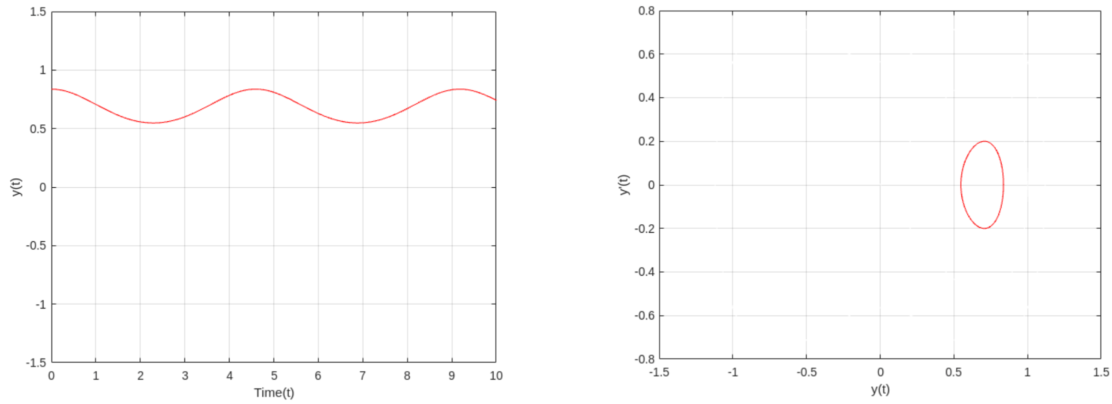


(a) y(t) against time

(b) Phase plane plot

Figure 5.1.7: Solutions of (5.1) corresponding to $y(0) = \sqrt{\frac{1}{2}},\ y'(0) = 0$

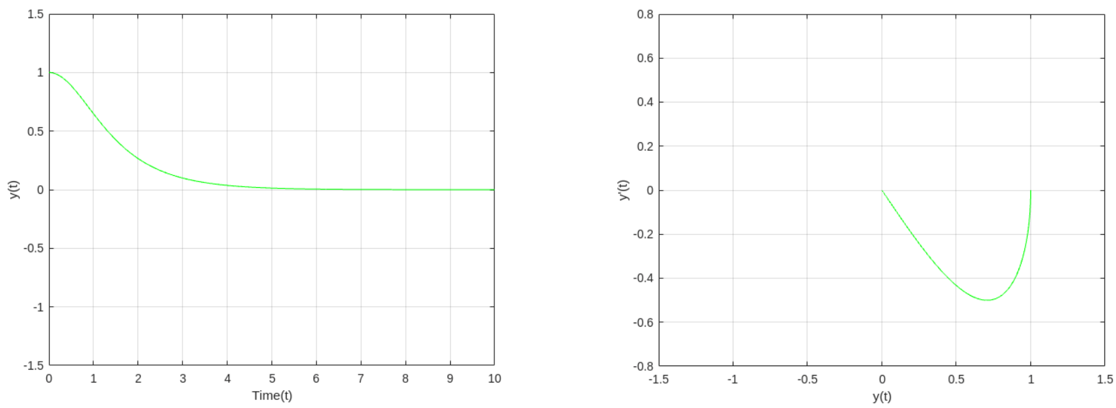
Case II: Here, we choose the initial condition $y(0) = \sqrt{\frac{7}{10}},\ y'(0) = 0$ and we obtain periodic solution which correspond to the periodic orbits obtained in **Case4** of subsection 4.1. The results are shown below



(a) y(t) against time (b) Phase plane plot

Figure 5.1.8: Solutions of (5.1) corresponding to $y(0) = \sqrt{\frac{7}{10}},\ y'(0) = 0$

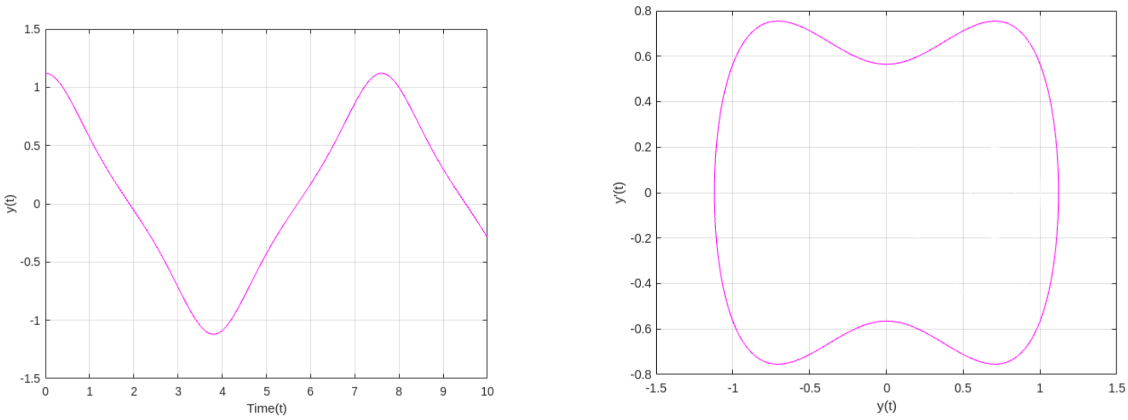
Case III: We choose the initial condition $y(0) = 1,\ y'(0) = 0$ and we obtain again periodic solutions which converges to 0 as $t \to +\infty$. This is due to the fact that the case under consideration analogous to the autonomous case consisted of homoclinic orbits and the equillibrium point at the origin. The image is shown below



(a) y(t) against time (b) Phase plane plot

Figure 5.1.9: Solutions of (5.1) corresponding to $y(0) = 1,\ y'(0) = 0$

Case IV: Finally, we chose the initial condition $y(0) = 1.12,\ y'(0) = 0$ and we obtain again sign changing periodic solutions. This corresponds to the result obtained in **Case 3** of Section 4.1. The results are shown below in the figure 5.1.10
The plot of solutions for all considered cases is shown in figure 5.1.11



(a) y(t) against time
(b) Phase plane plot

Figure 5.1.10: Solutions of (5.1) corresponding to $y(0) = 1.12,\ y'(0) = 0$

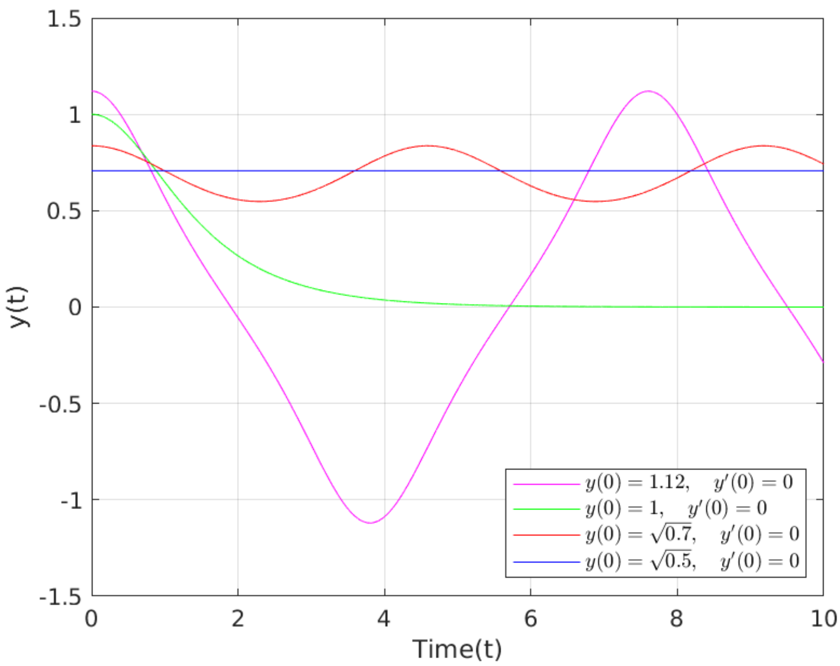


Figure 5.1.11: Plot of all solutions(y(t) against time).

5.2 Nonautonomous case

We now consider the non-autonomous equation

$$y'' = p(t)y - h(t)y^3. \tag{5.3}$$

We choose functions $p(t)$ and $h(t)$ which are T-periodic functions. The functions chosen as

$$\begin{aligned} p(t) &= 1 + A\sin\left(\frac{2\pi}{T}t\right) \\ h(t) &= 2 + B\sin\left(\frac{2\pi}{T}t\right) \end{aligned} \tag{5.4}$$

with different amplitudes(A and B) and T has the same period as **Case 4** of the autonomous case in subsection 4.1. In the **Case II** of the autonomous simulation, we obtained periodic solutions and we simulated with initial condition $y(0) = \sqrt{\frac{7}{10}},\ y'(0) = 0$. To obtain the period of the solution in this case, we use the formula

$$T = \oint_\Gamma \left(\frac{1}{x_2}, 0\right) d\vec{s},$$

where Γ is the closed curve in the figure 5.1.8(b). We have found in **Case 4** of section 4.1 that

$$x_2 = \pm\sqrt{ax_1^2 - \frac{b}{2}x_1^4 + 2c},$$

and since we are considering the positive initial condition, the interval for this solution is on the positive x_1 axis and is given by $\sqrt{\frac{a}{b} - \sqrt{\frac{4bc+a^2}{b^2}}} \leqslant x_1 \leqslant \sqrt{\frac{a}{b} + \sqrt{\frac{4bc+a^2}{b^2}}}$.
We know the values of a and b so to determine the value of c, we substitute the initial condition into $x_2 = \sqrt{ax_1^2 - \frac{b}{2}x_1^4 + 2c}$ and obtain

$$\begin{aligned} 0 &= \sqrt{\left(\sqrt{\frac{7}{10}}\right)^2 - \left(\sqrt{\frac{7}{10}}\right)^4 + 2c} \\ 0 &= \sqrt{\frac{7}{10} - \left(\frac{7}{10}\right)^2 + 2c} \\ 0 &= \frac{7}{10} - \left(\frac{7}{10}\right)^2 + 2c \\ 0 &= 0.21 + 2c \\ c &= -0.105. \end{aligned}$$

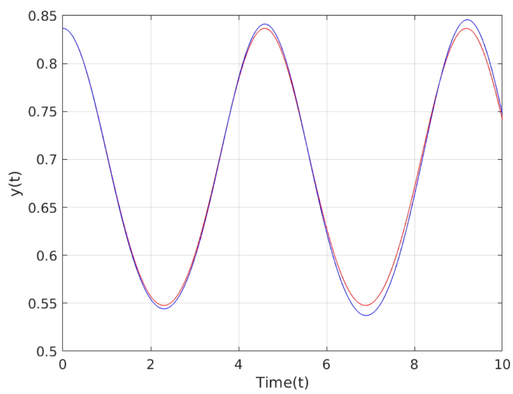
The period T is then

$$\begin{aligned} T &= 2 \cdot \int_{\sqrt{0.3}}^{\sqrt{0.7}} \frac{1}{\sqrt{x_1^2 - x_1^4 + 2(-0.105)}}\, dx_1 \\ &= 2 \cdot \int_{\sqrt{0.3}}^{\sqrt{0.7}} \frac{1}{\sqrt{x_1^2 - x_1^4 - 0.21}}\, dx_1. \end{aligned}$$

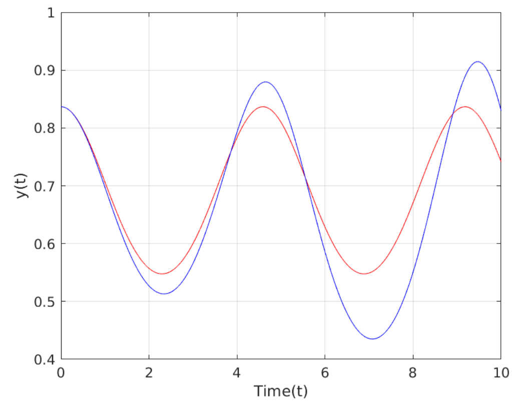
Solving the integral numerically, we obtain

$$T \approx 4.5893.$$

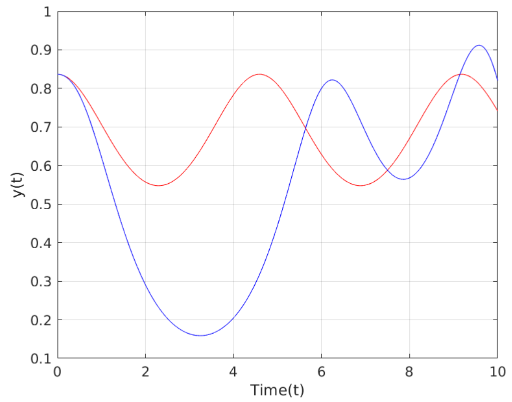
We now substitute (5.4) into (5.3) with different A and B values and compare the solutions(denoted by blue line) to the solution of **Case II**(denoted by red line) of the autonomous case and the results are as follows



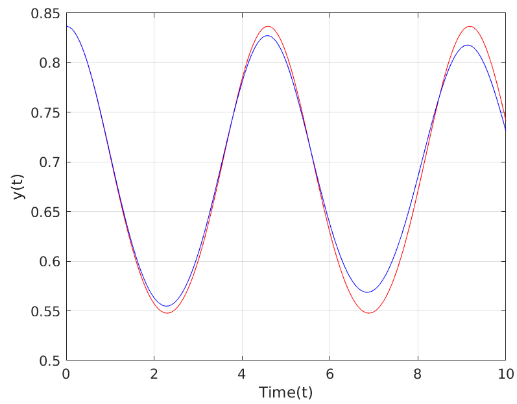
(a) Solution with A = 0, B = 0.01



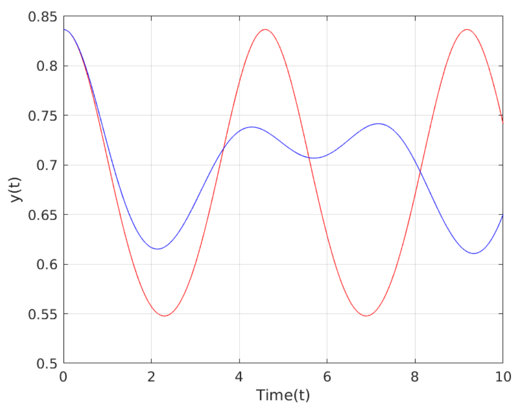
(b) Solution with A = 0, B = 0.1



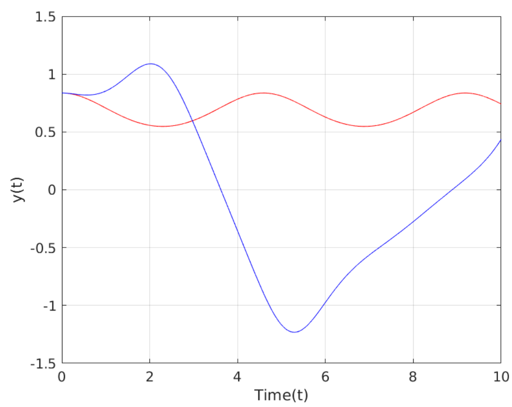
(c) Solution with A = 0, B = 1



(d) Solution with A = 0.01, B = 0

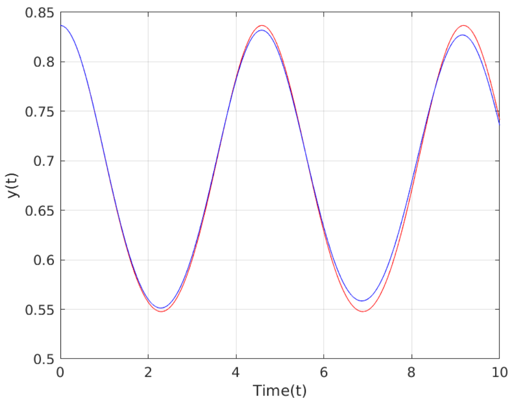


(e) Solution with A = 0.1, B = 0

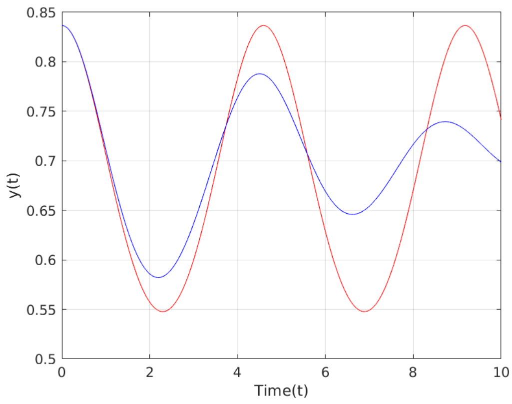


(f) Solution with A = 1, B = 0

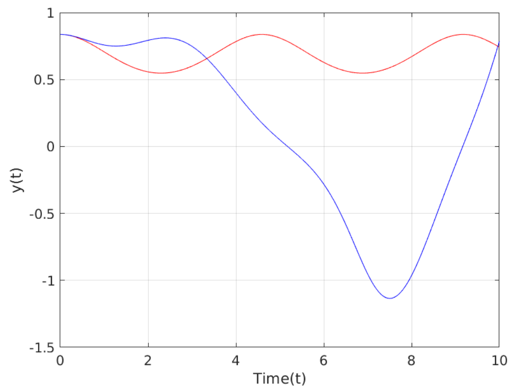
Figure 5.2.12: Plots of nonautonomous case with $y(0) = \sqrt{\frac{7}{10}}, \quad y'(0) = 0$



(a) Solution with A = 0.01, B = 0.01



(b) Solution with A = 0.1, B = 0.1



(c) Solution with A = 1, B = 1

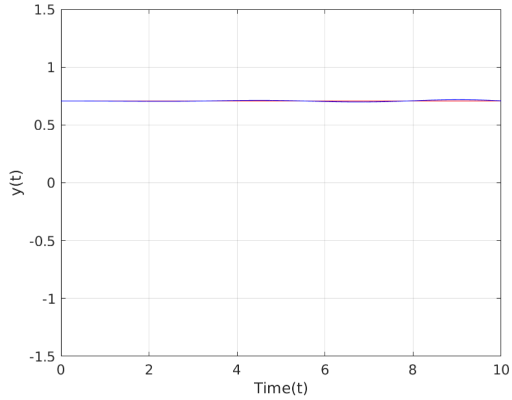
Figure 5.2.13: Plots of nonautonomous case with $y(0) = \sqrt{\frac{7}{10}}, \quad y'(0) = 0$

On the other hand, if $A = 0$ and $B = 0$, equation (5.3) has the constant $y(t) = \sqrt{\frac{1}{2}}$ which is clearly T-periodic will with period $T = 4.5893$. We compare the solutions of the initial value problem

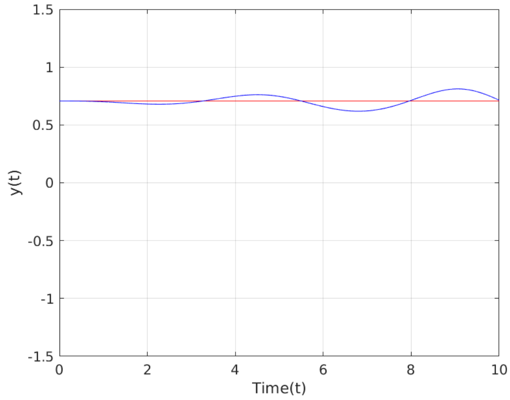
$$y'' = p(t)y - h(t)y^3$$

$$y(0) = \sqrt{\frac{1}{2}} \quad , \quad y'(0) = 0$$

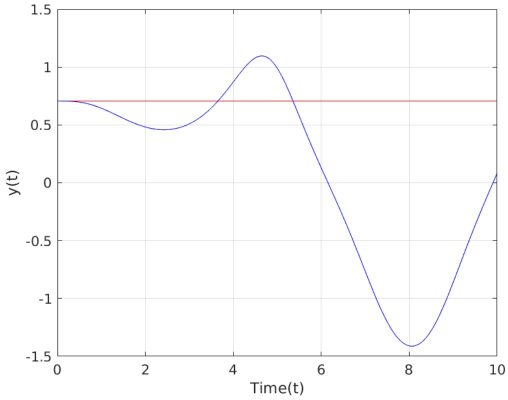
with the solution $y(t) = \sqrt{\frac{1}{2}}$ of equation (5.1) and the results are as follows



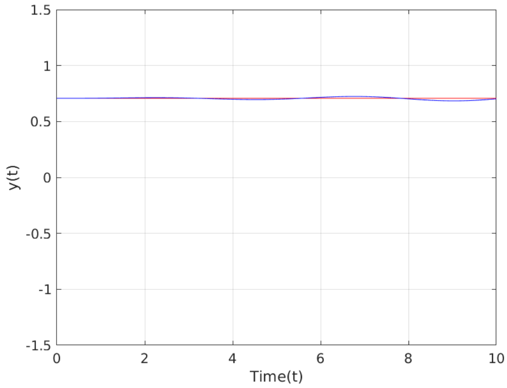
(a) Solution with A = 0, B = 0.01



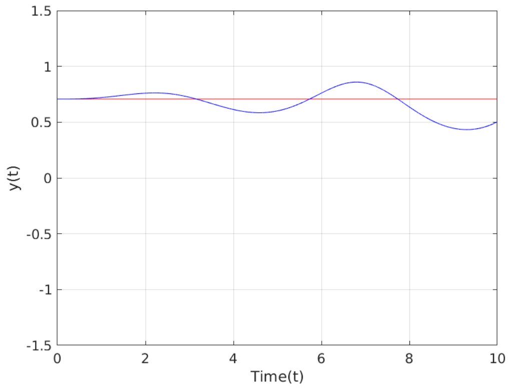
(b) Solution with A = 0, B = 0.1



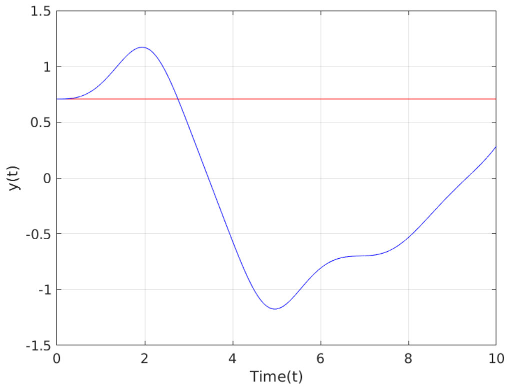
(c) Solution with A = 0, B = 1



(d) Solution with A = 0.01, B = 0

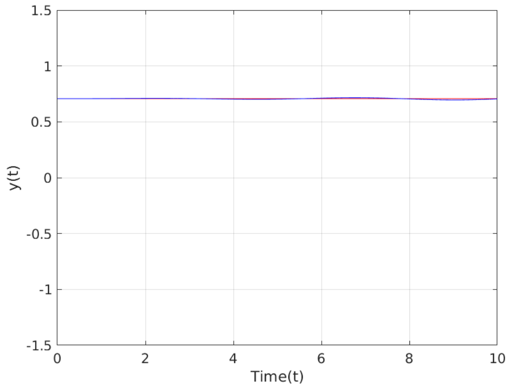


(e) Solution with A = 0.1, B = 0

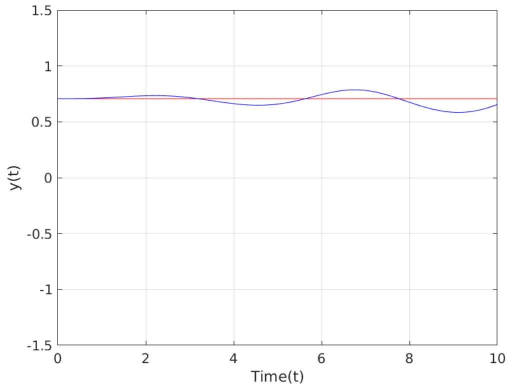


(f) Solution with A = 1, B = 0

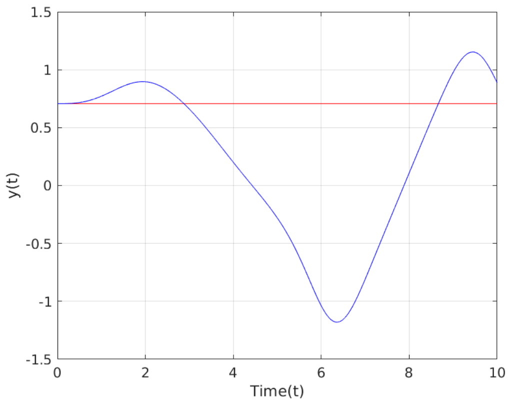
Figure 5.2.14: Plots of nonautonomous case with $y(0) = \sqrt{\frac{1}{2}}, \quad y'(0) = 0$



(a) Solution with A = 0.01, B = 0.01



(b) Solution with A = 0.1, B = 0.1



(c) Solution with A = 1, B = 1

Figure 5.2.15: Plots of nonautonomous case with $y(0) = \sqrt{\frac{1}{2}}, \quad y'(0) = 0$

5.3 Summary of results

The results obtained from all the simulations performed with the various initial conditions and

5.3.1 Simulation for autonomous case

In the autonomous case we chose values for a, b and performed simulations with the different initial values which corresponded to different solution curves or trajectories. It can be observed that the results obtained analytically, corresponds to the simulations under the condition that the simulation is performed under suitable tolerance levels and time intervals. It was however discovered that there are some numerical inaccuracies in solving numerically the autonomous differential equation with the initial condition which yields homoclinic orbits. The *ode45* solver in MATLAB, on longer time intervals gives a solution which appears to be periodic for the case of homoclinic orbit which is not valid if solved analytically.

5.3.2 Simulation for nonautonomous case

In the nonautonomous case, we chose periodic functions for $p(t)$ and $h(t)$ in (5.4). The simulation for this case was done with varied amplitudes of the periodic functions and the period is set to the same value as the known period of the **Case II** of the simulation for the autonomous case. It can be observed that the period of the solutions in this case approaches the period of the **Case II** of the autonomous simulation as the amplitude approaches 0. However, for higher amplitudes, the solutions begin to exhibit chaotic behavior and are no more periodic.

Conclusion

This work was aimed at analyzing the solutions to the Duffing equation considered and further perform simulations in MATLAB to see how the analysis compares with numerical solutions.

In the second section we derived the Duffing equation from a nonlinear oscillator using the Newton's second law of motion, Hooke's law and Taylor approximation.

In the third section, we presented some theory from dynamical systems necessary for the analysis of solutions to the autonomous variant of the considered Duffing equation. Further, we presented some theory on method of lower and upper functions for a periodic problem which was necessary for the finding conditions guaranteeing existence of solutions in the nonautonomous case.

In the fourth section we analyzed qualitatively the solutions to the considered Duffing equation. In the autonomous case, we found the equilibrium points of (4.2), and with that derived all levels of hamiltonian corresponding to the solution of (4.1). In the last part of subsection 4.1, we proved that for $c < -\frac{a^2}{2b}$ of the hamiltonian level, (4.7) does not correspond to any solution of (4.1) and then generated the phase portrait from all the levels of hamiltonian considered. In the nonautonomous case of this section we proved existence of periodic solutions and uniqueness by making using of the method of upper and lower functions introduced in the third section.

In the last section, we performed simulations of results from the previous sections. Due to dependence of solutions of differential equations on initial conditions, we were able to simulate the solutions corresponding to the different levels of hamiltonian derived in the previous section. In the nonautonomous case we derived the period for the **Case 4** of subsection 4.1 and compared solutions of this case to the solution of (5.3) with $p(t)$ and $h(t)$ defined in (5.4). The solutions corresponding to the equilibria in the autonomous case was also compared with solutions of (5.3) having conditions (5.4). The results were then summarised for all the simulations performed.

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Appendices

A Autonomous case

```
clear all
close all

%We define relative tolerance and absolute tolerance for the
    simulation
options = odeset('RelTol',1e-8,'AbsTol',1e-8);

%Definition of initial values for considered Duffing equation
xsqrt = [sqrt(0.5),0];
xsqrt_1 = [sqrt(0.7),0];
xone = [1,0];
xg1 = [1.12, 0];
dx= @(t,x)[x(2);(x(1)-2*x(1)^3)];

%Numerical Solution of the Duffing equation on timespan(tspan)
tspan = linspace(0,10,10000);
[t1,x1]=ode45(dx,tspan,xsqrt,options);
[t2,x2]=ode45(dx,tspan,xsqrt_1,options);
[t3,x3]=ode45(dx,tspan,xone,options);
[t4,x4]=ode45(dx,tspan,xg1,options);

%Auxiliary to plot solutions and phase potraits at different
    initial values
colors = ['b','r','g','m'];
time_interv = [t1,t2,t3,t4];
sol = [x1,x2,x3,x4];

%Plots of solutions and corresponding phase potrait for each
    initial value
for i=1:4
    figure
    xt = sol(1:end,(i*2)-1:i*2);
    tm = time_interv(1:end,i);
    plot(tm,xt(:,1),colors(i))
    ylim([-1.5 1.5])
    xlabel('Time(t)')
    ylabel('y(t)')
    grid on
    figure
    plot(xt(:,1),xt(:,2),colors(i))
    xlim([-1.5 1.5])
    ylim([-0.8 0.8])
    xlabel('y(t)')
```

```
    ylabel("y'(t)")
    grid on
end

%Plots of all solutions against time
figure
plot(t4,x4(:,1),'m')
hold on
plot(t3,x3(:,1),'g')
hold on
plot(t2,x2(:,1),'r')
hold on
plot(t1,x1(:,1),'b')
xlabel('Time(t)')
ylabel('y(t)')
legnd = legend("$y(0)=1.12,\quad y'(0)=0$","$y(0)=1,\quad y'(0)
    =0$", "$y(0)=\sqrt{0.7},\quad y'(0)=0$","$y(0)=\sqrt{0.5},\
    quad y'(0)=0$");
set(legnd,'Interpreter','latex');
legend('Location','southeast')
grid on
hold off



%End of Code
```

B Non Autonomous case I

```
clear all
close all

%Set the relative tolerance and absolute tolerance
options = odeset('RelTol',1e-8,'AbsTol',1e-8);

%Define the initial values of the duffing euation
xsqrt_1 = [sqrt(0.7) ,0];


%Define time lenght and period T
tspan = linspace(0,10,100000);
T = 4.5893;

%Vector of Amplitude values of A and B to be tested
itervar = [0.01,0.1,1];

%Solution ofthe initial value probelm of the autonomous cas
dx= @(t,x)[x(2);(x(1)-2*x(1)^3)];
[t2,x2]=ode45(dx,tspan,xsqrt_1,options);


%Comparison of solutions in autonomous case with the
    nonautonmous case
for i=1:length(itervar)
    A = 0;
    B = itervar(i);
    dx2= @(t,x)[x(2);(1+A*sin(2*(pi/T)*t))*x(1)-(2+B*sin(2*(pi/T
        )*t))*x(1)^3];
    [t21,x21]=ode45(dx2,tspan,xsqrt_1,options);

    %Plot of the solution of the autonomous case with the
        different nonautonomous
    %cases
    figure
    plot(t2,x2(:,1),'r')
    hold on
    plot(t21,x21(:,1),'b')
    xlabel('Time(t)')
    ylabel('y(t)')
    grid on
    hold off
end


```

```
for i=1:length(itervar)
      A = itervar(i);
      B = 0;
      dx2= @(t,x)[x(2);(1+A*sin(2*(pi/T)*t))*x(1)-(2+B*sin(2*(pi/T
          )*t))*x(1)^3];
      [t21,x21]=ode45(dx2,tspan,xsqrt_1,options);

      %Plot of the solution of the autonomous case with the
          different nonautonomous
      %cases
      figure
      plot(t2,x2(:,1),'r')
      hold on
      plot(t21,x21(:,1),'b')
      xlabel('Time(t)')
      ylabel('y(t)')
      grid on
      hold off
end


for i=1:length(itervar)
      A = itervar(i);
      B = itervar(i);
      dx2= @(t,x)[x(2);(1+A*sin(2*(pi/T)*t))*x(1)-(2+B*sin(2*(pi/T
          )*t))*x(1)^3];
      [t21,x21]=ode45(dx2,tspan,xsqrt_1,options);

      %Plot of the solution of the autonomous case with the
          different nonautonomous
      %cases
      figure
      plot(t2,x2(:,1),'r')
      hold on
      plot(t21,x21(:,1),'b')
      xlabel('Time(t)')
      ylabel('y(t)')
      grid on
      hold off
end

%End of code.
```

C Non Autonomous case II

```
clear all
close all

%Set the relative tolerance and absolute tolerance
options = odeset('RelTol',1e-8,'AbsTol',1e-8);

%Define the initial values of the duffing euation
xsqrt = [sqrt(0.5),0];


%Define time length and period T
tspan = linspace(0,10,100000);
T = 4.5893;


%Vector of Amplitude values of A and B to be tested
itervar = [0.01,0.1,1];

%Solution ofthe initial value probelm of the autonomous cas
dx= @(t,x)[x(2);(x(1)-2*x(1)^3)];
[t1,x1]=ode45(dx,tspan,xsqrt,options);


%Comparison of solutions in autonomous case with the
    nonautonmous case
for i=1:length(itervar)
     A = 0;
     B = itervar(i);
     dx2= @(t,x)[x(2);(1+A*sin(2*(pi/T)*t))*x(1)-(2+B*sin(2*(pi/T
         )*t))*x(1)^3];
     [t21,x21]=ode45(dx2,tspan,xsqrt,options);

     %Plot of the solution of the autonomous case with the
         different nonautonomous
     %cases
     figure
     plot(t1,x1(:,1),'r')
     hold on
     plot(t21,x21(:,1),'b')
     ylim([-1.5 1.5])
     xlabel('Time(t)')
     ylabel('y(t)')
     grid on
     hold off
end
```

```
for i=1:length(itervar)
    A = itervar(i);
    B = 0;
    dx2= @(t,x)[x(2);(1+A*sin(2*(pi/T)*t))*x(1)-(2+B*sin(2*(pi/T)*t))*x(1)^3];
    [t21,x21]=ode45(dx2,tspan,xsqrt,options);

    %Plot of the solution of the autonomous case with the different nonautonomous
    %cases
    figure
    plot(t1,x1(:,1),'r')
    hold on
    plot(t21,x21(:,1),'b')
    ylim([-1.5 1.5])
    xlabel('Time(t)')
    ylabel('y(t)')
    grid on
    hold off
end


for i=1:length(itervar)
    A = itervar(i);
    B = itervar(i);
    dx2= @(t,x)[x(2);(1+A*sin(2*(pi/T)*t))*x(1)-(2+B*sin(2*(pi/T)*t))*x(1)^3];
    [t21,x21]=ode45(dx2,tspan,xsqrt,options);

    %Plot of the solution of the autonomous case with the different nonautonomous
    %cases
    figure
    plot(t1,x1(:,1),'r')
    hold on
    plot(t21,x21(:,1),'b')
    ylim([-1.5 1.5])
    xlabel('Time(t)')
    ylabel('y(t)')
    grid on
    hold off
end

%End of code.
```