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DIFFEOMORPHISMS OF RIEMANNIAN SPACES WITH PRESERVED EINSTEIN TENSOR

Ph.D. Thesis

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Program: P1101 Mathematics:
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I declare that this dissertation is my own work. All the sources that have used have been quoted and acknowledged by means of complete references.

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Abstract

This PhD Thesis is devoted to the study of special (pseudo-) Riemannian spaces and diffeomorphisms (especially conformal, geodesic and holomorphically projective mappings) between them that preserve the Einstein tensor. The subjects of consideration are those properties of pseudo-Riemannian spaces which answer the question on existence or non-existence of Einstein tensor preserving diffeomorphisms under consideration.

We obtained the systems of equations for the theory of conformal, geodesic, and holomorphically projective mappings which preserve the Einstein tensor. We introduce and examine invariants under considered mappings and examine classes of pseudo-Riemanian spaces which are closed under mappings under consideration.

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Abstrakt

Předložená disertační práce je věnována studiu speciálních (pseudo-) Riemannových prostorů a difeomorfismů (konformních, geodetických a holomorfně projektivních zobrazení), při kterých se zachovává Einsteinův tensor. Studují se vlastnosti pseudo-Riemannových prostorů, které odpovídají na otázku existence nebo neexistence invariantnosti Einsteinova tensoru při studovaném difeomorfismu.

Nalezena soustava rovnic pro teorii konformních, geodetických a holomorfně projektivních zobrazení, která zachovávají Einsteinův tensor. Zavádíme a studujeme invarianty uvažovaných zobrazení a zkoumáme třídy pseudo-Riemannových prostorů, které jsou uzavřeny vzhledem k těmto zobrazením.

CONTENTS

Introduction

A few words about the topic

In differential geometry, we investigate geometric spaces generalizing the Euclidean space of dimension n, the so-called differentiable manifolds. For morphisms of such objects, we take diffeomorphisms between our spaces, i.e. invertible and sufficiently differentiable mappings with a differentiable inverse. If the manifold is endowed with a particular geometric structure, we are interested in diffeomorphisms which preserve the structure. Often, the structure is represented by a connection, by metric tensor field, in combination with some additional tensor fields, related to each other by some identities, or with a particular system of curves etc. Such geometry-preserving mappings play a key role in the theory. A lot of works is devoted to investigations of conformal, geodesic and holomorphically projective mappings, see [1]-[34].

Motivations for investigation of such particular diffeomorphisms comes from physics, particularly from mechanics and the theory of relativity, and the reached results have applications in many branches of technical sciences, especially in modelling of dynamical processes.

In modelling by means of diffeomorphism, the so-called invariants, i.e. geometric objects or properties which are preserved, play an important role.

A particular branch of differential geometry is devoted to the study of special Riemannian and Kähler spaces. Especially symmetric spaces and their generalizations play an important role. Attention is paid also to other special classes of pseudo-Riemannian spaces that are distinguished by some algebraic or differential conditions on objects characterizing the inner geometry such as the

Riemannian tensor, Ricci tensor, scalar curvature, or objects constructed from them, such as the Einstein tensor, the tensor of concircular curvature, the tensor of conformal curvature, the Bochner tensor etc.

In this respect, the topic of the dissertation thesis appears to be actual.

Relationship of the thesis with grant projects and projects of international cooperation

The topic is related to grant from Grant Agency of Czech Republic GAČR P201/11/0356 with the title: *Riemannian, pseudo-Riemannian and affine differential geometry*, and from project by the Council of Czech Government MSM 6198959214.

The goal of dissertation is to investigate Einstein tensor preserving diffeomorphisms:

* conformal mappings
* geodesic mappings
* holomorphically-projective mappings.

Objects under consideration are special pseudo-Riemannian spaces and diffeomorphisms between them that preserve the Einstein tensor.

The subject of consideration are those properties of pseudo-Riemannian spaces which answer the question on existence or non-existence of Einstein tensor preserving diffeomorphisms under consideration.

Theoretical and practical applicability of the obtained results.

The investigations are of fundamental theoretical character. The main results of the dissertation may be applied in further development of diffeomorphisms of pseudo-Riemannian spaces, and may be also used in various applications concerning modelling of physical fields, optimalisation of movements of mechanical systems, dynamical processes in electromagnetic fields and gravity in theoretical mechanics and theoretical physics.

Methods

The spaces under consideration are of arbitrary signature of metric. The investigations use local coordinates. We assume that all functions under consideration are sufficiently differentiable, and use tensor methods.

New results obtained in thesis

Among new results obtained in thesis, let us emphasize:

1. We obtained the systems of equations for the theory of conformal, geodesic, and holomorphically projective mappings which preserve the Einstein tensor.

2. We introduce and examine invariants under considered mappings and examine classes of pseudo-Riemanian spaces which are closed under mappings under consideration.

3. We investigate geometric properties of special pseudo-Riemanian spaces in connection with the question whether the spaces admit Einstein tensor preserving conformal, geodesic, holomorphically projective mapping or not.

The Dissertation or its parts were presented at

1. International scientific conference on behalf of *acad. Kravchukh*, Kiev, 2009.
2. International conference *Geometry in Odessa* 2009, 2010, 2011.
3. International conference Geometry in Kislovodsk 2009, 2010.
4. International conference *Aplimat*, Bratislava 2010, 2011, 2012.
5. Regional *Geometrical seminar* in Odessa 2009, 2010, 2011.
6. *Geometrical seminar* in UP Olomouc 2009, 2010, 2011.

Author's contribution to the work

The supervisors formulated the open problems and recomended the necessary apparatus and methods useful to the solution during her studies. The main results of the thesis were obtained by the author herself.

References

The main results of the thesis were published in eight papers and two proceedings from international scientific conferences, see p. 95.

The structure and volume of the thesis

The thesis consist of an introduction, three chapters and a list of references. The chapters are devided into paragraphs. Each chapter is provided with conclusions. The formulas and theorems are enumerated by triples of numbers: the first one means the chapter, the second denotes the paragraph, the third one is the number of the formula in the paragraph. The text consists of **130** pages. The list of references consists of **207** items.

The geometrical problems that we deal are the modern and current. These problems are extended and comprehensive studied. Selection of publications dedicated to it is given in the References.

The content of thesis and recent results

First Chapter is devoted to Einstein tensor preserving conformal mappings of (pseudo-) Riemannian manifolds. We obtained fundamental equations providing tools for decision whether a space admits Einstein tensor preserving conformal mappings or not.

We proved that the mappings under consideration are concircular, that is, preserve geodesic circles. It follows that in a special coordinate system, we are able to give a formula for a metric tensor of spaces admitting Einstein tensor preserving conformal mappings.

In the theory of concircular mappings (introduced by Kentaro Yano [203]), that is conformal mappings preserving geodesic circles, we obtained a new result: *such mappings preserve the Einstein tensor.*

Note that during 1923 – 1925 H.W. Brinkmann [135] studied conformal mappings of Einstein spaces, including conformal mappings between Einstein spaces. In the case that both spaces are Einstein, then the Einstein tensors vanish, i.e. Einstein tensors trivially preserves. Moreover these Einstein spaces are equidistant. We obtained following results:

Theorem I.1.3 *A pseudo-Riemannian space* V_n $(n > 2)$ *admits an Einstein tensor preserving conformal mapping onto some pseudo-Riemannian space* $\bar{V}_n$ *if and only if* V_n *is an equidistant space.*

We constructed objects invariant under conformal mappings and introduced classes of spaces closed under concircular mappings.

Theorem I.2.1 *A Riemannian space admits an Einstein tensor preserving conformal mapping onto a Riemannian space if and only if the mapping under consideration preserves the tensor of concircular curvature.*

We introduced the notion of mobility degree of a pseudo-Riemannian space with respect to concircular mappings. We found lacunas in the distribution of the degrees. We obtained boundaries of the lacuna and found necessary and sufficient conditions for spaces, distinct from spaces of maximal mobility degree, under concircular mappings (see § 3).

Examining these tensor conditions we distinguished the class of weak semisymmetric and Z-semisymmetric spaces, and also introduced three closed disjoint subclasses which are related to the structure of curvature of such spaces.

We proved the following:

Theorem I.5.1 *Let* $Z_{hijk} \equiv R_{hijk} - B\,(g_{hk}g_{ij} - g_{hj}g_{ik})$, $Z_{ij} = Z^{\alpha}_{ij\alpha}$, $Z = Z_{\alpha\beta}g^{\alpha\beta}$. *If the vectors* a_i *and* b_i *from the equations*

$$Z_{hijk} = e\,(a_h b_i - a_i b_h) \cdot (a_j b_k - a_k b_j), \;\; e = \pm 1, \qquad (*)$$

are non-isotropic then the constant B *is non-zero, and moreover, the equations*

$$\frac{1}{2} Z Z_{hijk} = Z_{hk}Z_{ij} - Z_{hj}Z_{ik}$$

are satisfied.

Theorem I.5.2 *If one of the vectors from the equations* $(*)$, *let us say* a_i, *is non-isotropic, and* b_i *is isotropic, then*

$$B = \frac{R}{n(n-1)} \;\; \textit{and} \;\; Z_{ij} = e b_i b_j.$$

Theorem I.5.3 *If both vectors from the equations* $(*)$ *are isotropic, then the space of maximal mobility degree with respect to concircular mappings is an Einstein space.*

The results of Chapter I was published in [C1], [C7], [C9].

Second Chapter is devoted to investigation of geodesic mappings of pseudo-Riemannian spaces. At the beginning, we pay attention to geodesic mappings of spaces that admit concircular mappings. We check that such spaces admit non-trivial geodesic mappings (see Theorem II.6.1 and II.6.2).

Further, we examine spaces admitting *Einstein tensor preserving geodesic mappings.*

In 1980, J. Mikeš [59] proved that the image $\bar{V}_n$ of an Einstein space V_n under a geodesic mapping is the Einstein space again. This result was a starting point for further developments of this topic, for investigations of V.S. Sobchuk [108] who studied harmonic spaces and S. Formella [149] who paid attention to mappings of conformal harmonic spaces.

We have given here basic equations of this theory (Theorem II.6.3), presented methods of determining invariant objects, and distinguished classes of spaces closed under Einstein tensor preserving geodesic mappings (Theorem II.7.1). We studied properties of special spaces admitting the mappings under consideration: spaces of constant curvature, harmonic spaces, spaces of quasiconstant curvature etc. (see § 7 and 8)

In 1953, C. Takeno and M. Ikeda proved that four-dimesional centrally symmetric spaces of non-constant curvature do not admit non-trivial geodesic mappings. A similar result was obtained in 1954 by N.S. Sinyukov for symmetric and recurrent spaces. A problem was posed, what is the cardinality of the class of spaces admitting geodesic mappings. N.S. Sinyukov built the theory of invariants of geodesic preserving mappings. It was proved that if we start with a pair of spaces of constant curvature, it is possible to construct a sequence of pairs of spaces which are related by geodesic mappings, see [71, 170, 178, 96].

We solve an analogous problem in the case when we start from an initial pair of pseudo-Riemannian spaces and consider Einstein tensor preserving geodesic mappings (see § 9).

In § 10 and § 11 we obtained results of special pseudo-Riemannian spaces, including spaces with quasiconstant curvature.

The results obtained in [C2] was cited in [129], the results was used and generalized to geodesic mappings of Weyl spaces.

The results of Chapter II was published in [C2], [C6], [C7], [C8], [C9].

The Third Chapter is devoted to holomorphically projective mappings of Kähler spaces, whose preserve the Einstein tensor.

The theory of Kähler spaces is studied more than 85 years. In 1925, P.A. Shirokov [123] started to investigate special Riemannian spaces, that he himself called *A–spaces*. Independently, the same class of spaces was investigated also by E. Kähler, that is why thay are mostly referred to as Kähler spaces, see [96, 178].

Holomorphically projective mappings of Kähler spaces are natural generalizations of geodesic mappings, these mappings were developed by Otsuki, Ishihara and Tashiro, see [8, 178, 96, 98].

Various aspects of the geometry of Kähler spaces were studied in a great amount of works published by many authors. Some of the results can be found in the monographs and survey papers [8, 72, 96, 170, 178, 204, 206].

From various points of view, attention was paid to special Riemannian and Kähler spaces satisfying additional conditions concerning the Riemannian tensor, the Ricci or Bochner tensor etc., and many papers are devoted to this subject.

We are interested here in those properties of Kähler spaces which are related to holomorphically projective mappings preserving the Einstein tensor. The fundamental equation of this case was found in Theorem III.14.2. Hence we proved

Theorem III.14.3 *The Einstein tensor is preserved under a holomorphically projective mapping if and only if the tensor of holomorphically sectional curvature is preserved.*

In this way, the methods developed in the theory of conformal and geodesic mappings of Riemannian spaces are transferred to the theory of holomorphically projective mappings of Kähler space. We received similar results for hyperbolic Kähler spaces [C4].

The results of Chapter III was published in [C3], [C5], [C10].

CHAPTER I

CONFORMAL MAPPINGS PRESERVING THE EINSTEIN TENSOR

§ 1. Basic equations of the theory of conformal mappings of Riemannian spaces preserving the Einstein tensor

Let V_n and $\bar{V}_n$ $(n > 2)$ be pseudo-Riemannian spaces with the metric tensors $g_{ij}(x)$ and $\bar{g}_{ij}(x)$, respectively.

Definition I.1.1. Under a *conformal mapping* we mean a diffeomorphism of V_n onto $\bar{V}_n$ such that the following is satisfied

$$\bar{g}_{ij}(x) = e^{2\sigma(x)} g_{ij}(x) \tag{I.1.1}$$

where σ is a function in V_n.

If σ is constant then the mapping is called a *homothety*, and, moreower, if $\sigma = 1$ then the mapping is an *isometry*.

In what follows, if not otherwise stated, we restrict ourselves onto non-homothetic mappings.

From (I.1.1) we obtain

$$\bar{g}^{ij} = e^{-2\sigma} g^{ij} \tag{I.1.2}$$

where g^{ij} and $\bar{g}^{ij}$ are components of the matrix inverse to the metric tensor matrix expression of the space V_n and $\bar{V}_n$, respectively. The following is satisfied:

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \delta^h_i \sigma_j + \delta^h_j \sigma_i - \sigma^h g_{ij}; \tag{I.1.3}$$

$$\begin{aligned} \bar{R}^h_{ijk} = \; & R^h_{ijk} + \delta^h_k \sigma_{ij} - \delta^h_j \sigma_{ik} + g^{h\alpha}(\sigma_{\alpha k} g_{ij} - \sigma_{\alpha j} g_{ik}) + \\ & + \Delta_1 \sigma (\delta^h_k g_{ij} - \delta^h_j g_{ik}); \end{aligned} \tag{I.1.4}$$

$$\bar{R}_{ij} = R_{ij} + (n-2)\sigma_{ij} + (\Delta_2 \sigma + (n-2)\Delta_1 \sigma) g_{ij}; \tag{I.1.5}$$

$$\bar{R} = e^{-2\sigma}(R + 2(n-1)\Delta_2 \sigma + (n-1)(n-2)\Delta_1 \sigma). \tag{I.1.6}$$

Here and in what follows Γ^h_{ij} are components of the connection (Christoffel symbols of second type), R^h_{ijk} are components of the Riemannian curvature tensor, R_{ij} are components of the Ricci tensor that are defined by the following formulas

$$R_{ij} \stackrel{\text{def}}{=} R^\alpha_{ij\alpha}; \tag{I.1.7}$$

$R \stackrel{\text{def}}{=} R_{\alpha\beta} g^{\alpha\beta}$ is the scalar curvature, δ^j_i is the Kronecker symbol (a coordinate form of the identity tensor),

$$\sigma_i \equiv \frac{\partial \sigma}{\partial x^i} \equiv \sigma_{,i}, \qquad \sigma^h = \sigma_\alpha g^{\alpha h};$$

$$\sigma_{ij} = \sigma_{,ij} - \sigma_{,i}\sigma_{,\,j}, \tag{I.1.8}$$

$\Delta_1\sigma$ and $\Delta_2\sigma$ are the first and second symbol of Beltrami which are defined by

$$\Delta_1\sigma = g^{\alpha\beta}\sigma_{,\alpha}\sigma_{,\beta}; \qquad \Delta_2\sigma = g^{\alpha\beta}\sigma_{,\alpha\beta}, \tag{I.1.9}$$

and comma denotes the covariant derivative of the metric (Levi-Civita) connection in V_n.

Objects in the space $\bar{V}_n$ that correspond to the objects in V_n under a conformal mapping will be denoted by bar.

The *Einstein tensor* is introduced by

$$E_{ij} = R_{ij} - \frac{R}{n} g_{ij}. \tag{I.1.10}$$

Definition I.1.2. A conformal mapping $f : V_n \to \bar{V}_n$ satisfying

$$\bar{E}_{ij} = E_{ij} \tag{I.1.11}$$

is called *Einstein tensor preserving* conformal mapping.

Let us examine Einstein tensor preserving conformal mappings $f : V_n \to \bar{V}_n$ of psedo-Riemannian manifolds in what follows.

From the equality (I.1.5) we obtain

$$\sigma_{ij} = \frac{1}{n-2}(\bar{R}_{ij} - R_{ij}) + \alpha g_{ij}, \tag{I.1.12}$$

where

$$\alpha \stackrel{\text{def}}{=} -\left(\frac{1}{(n-2)}\Delta_2\sigma + \Delta_1\sigma\right).$$

On the other hand, from (I.1.10) and (I.1.11) we get

$$\bar{R}_{ij} - R_{ij} = \frac{\bar{R}}{n}\bar{g}_{ij} - \frac{R}{n}g_{ij}. \tag{I.1.13}$$

Accounting (I.1.1) and (I.1.13), the equations (I.1.12) read

$$\sigma_{ij} = \beta g_{ij}, \tag{I.1.14}$$

here we put

$$\beta \stackrel{\text{def}}{=} \frac{1}{n}(\Delta_2\sigma - \Delta_1\sigma).$$

Denote $\vartheta \stackrel{\text{def}}{=} -e^{-\sigma}$. Using the formula (I.1.8) of the tensor σ_{ij}, we can write (I.1.14) as

$$\vartheta_{i,\,j} = \rho g_{ij}, \tag{I.1.15}$$

where $\vartheta_i = \vartheta_{,i}$, $\rho \stackrel{\text{def}}{=} e^{-\sigma} \cdot \beta$.

The vector ϑ_i satisfying the equations (I.1.15) is called *concircular*, and the spaces in which such a vector field exists, are called *equidistant*.

So we have in fact proved:

Theorem I.1.1. *If a pseudo-Riemannian space V_n admits a conformal mapping preserving the Einstein tensor then V_n is an equidistant space.*

If $\rho \neq 0$ we say that the equidistant space is of *regular* type while in the case $\rho \equiv 0$ the space will be called *singular*.

If ϑ_i is an isotropic vector, i.e. $\vartheta_\alpha \vartheta_\beta g^{\alpha\beta} = 0$, then the equidistant space is necessarily singular. Equidistant spaces of the basic type are characterized by the property that locally, there exists a coordinate system in which the metric tensor of the equidistant space takes the coordinate form

$$ds_n^2 = dx^{12} + f(x_1) ds_{n-1}^2(x_2, ..., x_n). \tag{I.1.16}$$

Here $f(x^1) \neq 0$ is some function, and ds_{n-1}^2 is a metric of an $(n-1)$-dimensional pseudo-Riemannian space.

According to the Theorem I.1.1, we can formulate the following

Theorem I.1.2. *If a pseudo-Riemannian space V_n $(n > 2)$ admits an Einstein tensor preserving conformal mapping then each point has a neighborhood in which the metric tensor can be presented in the form (I.1.16).*

On the other hand, if in V_n $(n > 2)$, the metric tensor takes locally the form (I.1.16), then in the space, there exists a concircular vector field $\vartheta_i \equiv \vartheta_{,i}$, satisfying (I.1.15). Plugging $\sigma = -\ln(-\vartheta(x))$, from (I.1.5) we obtain

$$\bar{R}_{ij} = R_{ij} + \alpha g_{ij}, \tag{I.1.17}$$

where α is some function.

By (I.1.1) the last formula reads

$$\bar{E}_{ij} \equiv \bar{R}_{ij} - \frac{\bar{R}}{n}\,\bar{g}_{ij} = R_{ij} + \alpha^* g_{ij}, \tag{I.1.18}$$

where α^* is some function.

Contracting (I.1.18) with g^{ij} and using (I.1.1) and (I.1.2), we check that $\bar{E}_{ij} = E_{ij}$, that is, the Einstein tensor is preserved under conformal mapping.

So we have proved

Theorem I.1.3. *A pseudo-Riemannian space V_n $(n > 2)$ admits an Einstein tensor preserving conformal mapping onto some pseudo-Riemannian space $\bar{V}_n$ if and only if V_n is an equidistant space.*

Special conformal mappings for which the tensor σ_i satisfies (I.1.14) are called *concircular mappings*, [203, 96, 170, 178].

Recall that under a *geodesic circle* we mean a curve for which the first curvature is constant and the second curvature vanishes. Concircular mappings are characterized by the property that any geodesic circle in V_n is mapped onto a geodesic circle in $\bar{V}_n$ again.

From the Theorem I.1.3 it follows

Corollary I.1.1. *A conformal mapping of V_n preserves geodesic circles if and only if the mapping preserves the Einstein tensor.*

If the tensor σ_i satisfies

$$\sigma_{i,j} = \alpha\sigma_i\sigma_j + \beta g_{ij}, \tag{I.1.19}$$

where $\alpha = \alpha(\sigma)$ and $\beta = \beta(\sigma)$ are some functions, then the corresponding mapping is called *quasiconcircular*, [50].

Let us examine existence of Einstein tensor preserving quasiconcircular mappings. By the Theorem I.1.3, from (I.1.14) we get the necessary conditions

$$\alpha = 1, \qquad \text{and} \qquad \beta = \frac{1}{n}\,(\Delta_2\sigma - \Delta_1\sigma), \tag{I.1.20}$$

hence the following is proved.

Theorem I.1.4. *Besides concincular mappings, there are no other quasiconcircular mappings of a pseudo-Riemannian space V_n preserving the Einstein tensor.*

The above theorem is a generalization of the resultats by Leyko, [50]. Particularly, as a consequence, we obtain

Corollary I.1.2. *If an Einstein space admits a quasiconcircular map onto an Einstein space then the corresponding map is necessarily concircular.*

Moreover, a quasiconcircular (i.e. torse-forming) vector field in Einstein spaces is concircular, see L. Rachůnek and J. Mikeš [181].

§ 2. Objects invariant under concircular mappings

As well known, [96, 86, 145, 170, 178], the tensor of conformal curvature is invariant under conformal mappings:

$$C^h_{ijk} = \bar{C}^h_{ijk}. \tag{I.2.1}$$

Recall that the tensor of *conformal curvature* C^h_{ijk} is introduced by

$$C^h_{ijk} = R^h_{ijk} - P^h_k g_{ij} + P^h_j g_{ik} - \delta^h_k P_{ij} + \delta^h_j P_{ik}, \tag{I.2.2}$$

where

$$P^h_k = P_{\alpha k} g^{\alpha h},$$

and

$$P_{ij} = \frac{1}{n-2}\left(R_{ij} - \frac{R}{2(n-1)} g_{ij}\right). \tag{I.2.3}$$

The deformation tensor P_{ij} satisfies:

$$\bar{P}_{ij} - P_{ij} = \frac{1}{n-2}\left(\bar{R}_{ij} - R_{ij} - \frac{\bar{R}}{2(n-1)}\bar{g}_{ij} + \frac{R}{2(n-1)} g_{ij}\right). \tag{I.2.4}$$

If the Einstein tensor is preserved under a conformal mapping then using (I.1.11), we can write (I.2.4) as

$$\bar{P}_{ij} - P_{ij} = \frac{\bar{R}}{2n(n-1)}\,\bar{g}_{ij} - \frac{R}{2n(n-1)}\,g_{ij}. \tag{I.2.5}$$

Accounting (I.1.1) and (I.1.2) we obtain

$$
\begin{aligned}
\bar{P}^h_k \bar{g}_{ij} - P^h_k g_{ij} &= \bar{P}_{\alpha k}\bar{g}^{\alpha h}\bar{g}_{ij} - P_{\alpha k} g^{\alpha h} g_{ij} = \\
&= \bar{P}_{\alpha k} g^{\alpha h} g_{ij} e^{-2\sigma} e^{2\sigma} - P_{\alpha k} g^{\alpha h} g_{ij} = \\
&= (\bar{P}_{\alpha k} - P_{\alpha k}) g^{\alpha h} g_{ij} = \\
&= \left(\frac{\bar{R}}{2n(n-1)}\ \bar{g}_{\alpha k} - \frac{R}{2n(n-1)}\ g_{\alpha k} \right) g^{\alpha h} g_{ij} = \qquad \text{(I.2.6)} \\
&= \frac{\bar{R}}{2n(n-1)}\ \bar{g}_{\alpha k} g^{\alpha h} g_{ij} e^{-2\sigma} e^{2\sigma} - \frac{R}{2n(n-1)}\ g_{\alpha k} g^{\alpha h} g_{ij} = \\
&= \frac{\bar{R}}{2n(n-1)}\ \bar{g}_{\alpha k} \bar{g}^{\alpha h} \bar{g}_{ij} - \frac{R}{2n(n-1)}\ g_{\alpha k} g^{\alpha h} g_{ij} = \\
&= \frac{\bar{R}}{2n(n-1)}\ \delta^h_k \bar{g}_{ij} - \frac{R}{2n(n-1)}\ \delta^h_k g_{ij}.
\end{aligned}
$$

By (I.2.5) and (I.2.6) we can write (I.2.1) in the form:

$$\bar{Y}^h_{ijk} = Y^h_{ijk}, \qquad \text{(I.2.7)}$$

where Y^h_{ijk} are components of the concircular curvature tensor

$$Y^h_{ijk} = R^h_{ijk} - \frac{R}{n(n-1)}(\delta^h_k g_{ij} - \delta^h_j g_{ik}). \qquad \text{(I.2.8)}$$

Hence if a Riemannian space V_n admits an Einstein tensor preserving mapping then the tensor of concircular curvature defined by (I.2.8) is necessarily invariant under the mapping under consideration.

On the other hand, if the Riemannian space V_n admits a conformal mapping onto $\bar{V}_n$ and the conditions (I.2.7) are satisfied then contracting over the indices h and k we check that (I.1.9) holds, another speaking, the Einstein tensor is preserved.

So we establish:

Theorem I.2.1. *A Riemannian space admits an Einstein tensor preserving conformal mapping onto a Riemannian space if and only if the mapping under consideration preserves the tensor of concircular curvature.*

We conclude that invariant objects of Einstein tensor preserving conformal mappings are: the Einstein tensor, the tensor of concircular curvature, and the tensor of conformal curvature. Using these facts we construct the following algebraic invariants (functions) with respect to Einstein tensor preserving conformal mappings:

$$E_{\alpha i} Y^{\alpha}_{jkl} = \underset{1}{K}{}_{ijkl}, \tag{I.2.9}$$

$$E_{\alpha i} C^{\alpha}_{jkl} = \underset{2}{K}{}_{ijkl}, \tag{I.2.10}$$

$$Y^{h}_{\alpha ij} Y^{\alpha}_{klm} = \underset{3}{K}{}^{h}_{ijklm}, \tag{I.2.11}$$

$$Y^{h}_{i\alpha j} Y^{\alpha}_{klm} = \underset{4}{K}{}^{h}_{ijklm}, \tag{I.2.12}$$

$$Y^h_{\alpha ij} C^\alpha_{klm} = \underset{5}{K}{}^{h}{}_{ijklm}, \tag{I.2.13}$$

$$Y^h_{i\alpha j} C^\alpha_{klm} = \underset{6}{K}{}^{h}{}_{ijklm}, \tag{I.2.14}$$

$$Y^\alpha_{ijk} C^h_{\alpha lm} = \underset{7}{K}{}^{h}{}_{ijklm}, \tag{I.2.15}$$

$$Y^\alpha_{ijk} C^h_{l\alpha m} = \underset{8}{K}{}^{h}{}_{ijklm}. \tag{I.2.16}$$

If the function $\underset{i}{K}$ $(i = 1, 2, \ldots, 8)$ for i fixed vanishes in V_n then the space V_n is said to be $\underset{i}{K}$ - *flat* $(i = 1, 2, \ldots, 8)$. The classes of $\underset{i}{K}$ - flat spaces are closed under Einstein tensor preserving conformal mappings. That is, a $\underset{i}{K}$ - flat space admits Einstein tensor preserving conformal mappings only onto $\underset{i}{K}$ - flat spaces.

Let us examine the introduced invariant objects:

$$\begin{aligned} \underset{1}{K} \ : \ E_{\alpha i} Y^\alpha_{jkl} \ = \ & R_{\alpha i} R^\alpha_{jkl} \ - \frac{R}{n} \ R_{ijkl} \ - \\ & -\frac{R}{n(n-1)} \ (R_{il} g_{jk} - R_{ik} g_{jl}) + \frac{R^2}{n^2(n-1)} \ (g_{il} g_{jk} - g_{ik} g_{jl}). \end{aligned} \tag{I.2.17}$$

Then a $\underset{1}{K}$ - flat Riemannian space is characterized by the conditions

$$R_{\alpha i}R^{\alpha}_{jkl} - \frac{R}{n}\ R_{ijkl} - \frac{R}{n(n-1)}\ (R_{il}g_{jk} - R_{ik}g_{jl}) +$$
$$+\frac{R^2}{n^2(n-1)}\ (g_{il}g_{jk} - g_{ik}g_{jl}) = 0. \tag{I.2.18}$$

Using (I.2.11) and (I.2.12) we check that the condition (I.2.18) is satisfied also by the $\underset{2}{K}$ - flat and $\underset{3}{K}$ - flat spaces. The contraction of (I.2.18) yields

$$R_{\alpha i}R^{\alpha}_{l} = \frac{2R}{n}R_{il} - \frac{R^2}{n^2}g_{il}, \tag{I.2.19}$$

$$R^{\alpha}_{\beta}R^{\beta}_{\alpha} = \frac{R^2}{n}. \tag{I.2.20}$$

Plugging (I.1.13) into (I.2.19) and (I.2.20), we verify that also the following is satisfied:

$$\bar{R}_{\alpha i}\bar{R}^{\alpha}_{l} = \frac{2\bar{R}}{n}\bar{R}_{il} - \frac{\bar{R}^2}{n^2}\bar{g}_{il}, \tag{I.2.21}$$

and

$$\bar{R}^{\alpha}_{\beta}\bar{R}^{\beta}_{\alpha} = \frac{\bar{R}^2}{n}. \tag{I.2.22}$$

Consequently, the following holds:

Theorem I.2.2. *The class of pseudo-Riemannian spaces which satisfy (I.2.18), (I.2.19) and (I.2.20), is closed under Einstein tensor preserving conformal mappings.*

§ 3. On the mobility degree of Riemannian spaces with respect to concircular mappings

As we have checked, if a pseudo-Riemannian space admits an Einstein tensor preserving conformal mappings then the function of conformality $\vartheta \overset{\text{def}}{=} -e^{-\sigma}$ satisfies

$$\vartheta_{i,\,j} = \rho g_{ij}. \tag{I.3.1}$$

The integrability conditions of the last set of equations read

$$\vartheta_\alpha R^\alpha_{ijk} = g_{ij}\rho_{,k} - g_{ik}\rho_{,j}. \tag{I.3.2}$$

Using contraction we get:

$$\rho_{,i} = \frac{1}{n-1}\,\vartheta_\alpha R^\alpha_i. \tag{I.3.3}$$

As was shown earlier [33], [36, p. 85-86], these equations are satisfied if $V_n \in C^2$ (i.e. $g_{ij} \in C^2$), $\vartheta_i(x) \in C^2$ and $\varrho(x) \in C^1$.

It is easily see, formula (I.3.1) is true when

$$V_n \in C^2,\ \vartheta_i(x) \in C^1,\ \varrho(x) \in C^0.$$

Then from Lemma 3 (Hinterleitner, Mikeš [154]: *Let $\lambda^h \in C^1$ be a vector field and ρ a function. If $\partial_i\lambda^h - \rho\,\delta^h_i \in C^1$ then $\lambda^h \in C^2$ and $\rho \in C^1$.*) and formula (I.3.1) it follows that: $\vartheta_i(x) \in C^2$, $\varrho(x) \in C^1$. We also have that the equations (I.3.2), (I.3.3) are satisfied.

Moreover, easy to prove that if $V_n \in C^r$ $(r \geq 2)$ and $\vartheta_i(x) \in C^1$, then

$$\vartheta_i(x) \in C^r,\ \varrho(x) \in C^{r-1}.$$

From this viewpoint we specify and generalize the results involving concircular vector fields below.

Definition I.3.1. The upper boundary for the number of substantial parameters in the general solution of the system of equation (I.3.1) is called *the mobility degree* under concircular mappings of the pseudo-Riemannian space.

The system of equations (I.3.1), (I.3.3) is closed. It is a system of linear differential equations with respect to the vector σ_i and function ϱ, of Cauchy type, in first order covariant derivatives with coefficients uniquely determined by the pseudo-Riemannian space V_n .

The system of equations (I.3.1), (I.3.3) in a pseudo-Riemannian space, for any family of initial values of the functions under consideration in the given point, admits at most one solution. Consequently, the number of free parameters in the general solution of the system is at most $n + 1$.

Since the system is linear, it admits at most

$$n + 1$$

linearly independent solutions with constant coefficients. It is obvious that the mobility degree of the space coincides with the cardinality of the system of independent (substantial) equidistant vector fields of the space.

It is known that the only spaces in which a linearly independent vector field exists are just the spaces of constant curvature. Hence under the Einstein tensor preserving conformal mappings, the maximal mobility degree have also just the spaces of constant curvature.

Contracting the condition (I.3.2) with $g^{i\beta}\vartheta_\beta$, we obtain easily

$$\rho_{,k} = B\,\vartheta_k, \tag{I.3.4}$$

where B is some function.

We are going to prove the following:

Theorem I.3.1. *If a pseudo-Riemannian space $V_n \in C^2$ $(n > 2)$ admits at least two linearly independent equidistant vector fields $\vartheta_i(x) \in C^1$ with constant coefficients then B in the equations (I.3.4) is a constant, uniquely determined by the space V_n.*

Remark. In [33] and [36, p. 88] the similar Theorem was published but proof was done only for $V_n \in C^3$, $\vartheta_i(x) \in C^3$ and $\varrho(x) \in C^2$.

Proof. Let in V_n there exist at least two linearly independent equidistant vector fields with constant coefficients ϑ_i and $\tilde{\vartheta}_i$.

Then the following is satisfied:

$$\vartheta_\alpha R^\alpha_{ijk} = B(g_{ij}\vartheta_k - g_{ik}\vartheta_j), \tag{I.3.5}$$

$$\tilde{\vartheta}_\alpha R^\alpha_{ijk} = \tilde{B}(g_{ij}\tilde{\vartheta}_k - g_{ik}\tilde{\vartheta}_j) \tag{I.3.6}$$

where B, $\tilde{B}$ are some functions.

Multiplying (I.3.5) by $\tilde{\vartheta}_k$ and contracting over k we get by (I.3.6)

$$(B - \tilde{B})(g_{ij}\vartheta_\alpha\tilde{\vartheta}^\alpha - \tilde{\vartheta}_i\vartheta_j) = 0.$$

Suppose $B \neq \tilde{B}$. Then $g_{ij}\vartheta_\alpha\tilde{\vartheta}^\alpha - \tilde{\vartheta}_i\vartheta_j = 0$.

From the last formula we get $\vartheta_\alpha\tilde{\vartheta}^\alpha = 0$ and $\tilde{\vartheta}_i\vartheta_j = 0$, a contradiction, since the vector fields are non-zero. Hence $B = \tilde{B}$ holds.

That is, the function B is uniquely defined by the space V_n itself.

Because ϑ_k and $\tilde{\vartheta}_k$ are gradient-like vector fields ($\vartheta_k = \vartheta_{,k}$ and $\tilde{\vartheta}_k = \tilde{\vartheta}_{,k}$) from equation (I.3.4) the fact

$$B = B_1(\vartheta) = B_2(\tilde{\vartheta})$$

follows. Note that ϑ and $\tilde{\vartheta}$ are indenpendent variables, then from this fact it follows: B is constant. This finishes the prove of the Theorem I.3.1.

□

Note that the above theorem is analogous to some results proven earlier under additional assumptions, [200].

Theorem I.3.2. *The are no Riemannian spaces $V_n \in C^2$, distinct from spaces of constant curvature which admit more than $(n-2)$ linearly independent equidistant vector fields $\vartheta_i(x) \in C^1$ with constant coefficients.*

Remark. In [36, p. 86] the similar Theorem was published but proof was done only for $V_n \in C^3$, $\vartheta_i(x) \in C^3$ and $\varrho_i(x) \in C^2$.

Proof. Let us suppose the opposite. Let V_n be a space which is not of constant curvature and yet admits more than $(n-2)$ linearly independent equidistant vector fields with constant coefficients.

The integrability conditions (I.3.1) read

$$\vartheta_\alpha Z^\alpha_{ijk} = 0, \tag{I.3.7}$$

where

$$Z^h_{ijk} \overset{def}{=} R^h_{ijk} - B(\delta^h_k g_{ij} - \delta^h_j g_{ik}). \tag{I.3.8}$$

We can write the tensor Z^h_{ijk} as

$$Z^h_{ijk} = \sum_{s=1}^{m} \underset{s}{b}{}^h \underset{s}{\Omega}{}_{ijk} \tag{I.3.9}$$

where $\underset{s}{b}{}^h$ are some linearly independent vectors, and $\underset{s}{\Omega}{}_{ijk}$ are linearly independent tensors. Since V_n is not of constant curvature, $m \geq 2$ holds.

From the conditions (I.3.7) we obtain, using the representation (I.3.9) of the tensor:

$$\begin{aligned}
\vartheta_\alpha \underset{1}{b}{}^\alpha &= 0,\\
\vartheta_\alpha \underset{2}{b}{}^\alpha &= 0,\\
&\dots\\
\vartheta_\alpha \underset{m}{b}{}^\alpha &= 0.
\end{aligned}$$

Since $m \geq 2$, among the equations of the system (I.3.10) there are at least two substantial equations.

From the previous facts it follows that there exist less or equal $n-2$ linearly independent vector fields ϑ_i, a contradiction.

This proves the Theorem I.3.2. □

The following holds

Theorem I.3.3. *Let* $V_n \in C^2$, $(n > 2)$, *be a pseudo-Riemannian spaces in which there are* $(n-2)$ *linearly independent equidistant vector fields* $\vartheta_i(x) \in C^1$. *Then Riemannian tensor has the following expression*

$$R_{hijk} = B\,(g_{hk}g_{ij} - g_{hj}g_{ik}) + e(a_h b_i - a_i b_h)(a_j b_k - a_k b_j), \tag{I.3.10}$$

where a_i *and* b_i *are non-colinear and pairwise orthogonal vectors,*
$e = \pm 1$, $B = \text{const}$.

Proof. The proof of above theorem follows from the proof of the Theorem I.3.2. The condition is necessary. Indeed, let V_n $(n > 3)$ admit $n-2$ linearly independent equidistant vector fields. It follows from Lemma 3 (Hinterleitner, Mikeš [154]) that $\vartheta_i(x) \in C^2$, $\varrho_i(x) \in C^1$. The equations (I.3.2) and (I.3.3) are satisfied.

Then the Theorem I.3.2 holds, and we can proceed according to its proof. Analysing the system (I.3.10), we see easily that among the vectors $\underset{s}{b}{}^i$ there are at least two independent vectors.

Using $Z^h_{ijk} \neq 0$, (I.3.9), the definition and properties of Z^h_{ijk}, we get the conditions (I.3.10).

Addition these two vector fields, denoted by a_i and b_i, we can choose orthogonal.

□

Finally the following theorem holds.

Theorem I.3.4. *In pseudo-Riemannian spaces* $V_n \in C^3$ $(n > 3)$ *in which there are* $(n-2)$ *linearly independent equidistant vector fields* $\vartheta_i(x) \in C^1$*, and only in such spaces, the following relations are satisfied*

$$R_{hijk} = B(g_{hk}g_{ij} - g_{hj}g_{ik}) + e(a_h b_i - a_i b_h)(a_j b_k - a_k b_j), \qquad \text{(I.3.10)}$$

$$a_{i,j} = \overset{1}{\underset{j}{\xi}} a_i + \overset{2}{\underset{j}{\xi}} b_i + c_i a_j; \qquad \text{(I.3.11)}$$

$$b_{i,\,j} = \overset{3}{\underset{j}{\xi}} a_i + \overset{4}{\underset{j}{\xi}} b_i + c_i b_j; \qquad \text{(I.3.12)}$$

$$c_{i,\,j} = \overset{5}{\underset{j}{\xi}} a_i + \overset{6}{\underset{j}{\xi}} b_i + c_i c_j - B g_{ij}, \qquad \text{(I.3.13)}$$

where a_i and b_i are non-colinear and pairwise orthogonal vectors; $c_i, \overset{s}{\underset{j}{\xi}}$ $(s = 1, \ldots, 6)$ *are some vectors;* $e = \pm 1$, $B = \text{const}$.

Remark. This theorem was proved for

$$V_n \in C^3,\ \vartheta_i(x) \in C^3,\ \varrho(x) \in C^2,$$

in [33]. Detailed proof is contained in [36, p. 88-92].

Proof. It follows from results mentioned above that condition

$$V_n \in C^3 \quad \text{and} \quad \vartheta_i(x) \in C^1$$

imply

$$\vartheta_i(x) \in C^3 \quad \text{and} \quad \varrho(x) \in C^2.$$

Now, the proof follows from [33] and [36, p. 88-92].

□

That is, according to the definition of Z_{hijk}, in pseudo-Riemannian spaces with maximal mobility degree under concircular mappings, the following holds:

$$Z_{hijk} = e(a_h b_i - a_i b_h)(a_j b_k - a_k b_j). \qquad \text{(I.3.14)}$$

Contracting with g^{hk} and accounting orthogonality of a_i and b_i, we find

$$Z_{ij} = -e(a^\alpha a_\alpha b_i b_j - b^\alpha b_\alpha a_i a_j), \tag{I.3.15}$$

$$Z = -2ea^\alpha a_\alpha b^\alpha b_\alpha \tag{I.3.16}$$

where $Z_{ij} \stackrel{def}{=} Z^\alpha_{ij\alpha}$; $\qquad Z \stackrel{def}{=} Z_{\alpha\beta} g^{\alpha\beta}$.

§ 4. On weak symmetric pseudo-Riemannian spaces

Let us examine some special pseudo-Riemannian spaces.

Definition I.4.1. A pseudo-Riemannian space V_n, in which there exists a tensor $A_{i_1 i_2 \dots i_k}$ such that

$$A_{i_1 i_2 \dots i_k, j} = \overset{1}{\tau}_j A_{i_1 i_2 \dots i_k} + \overset{2}{\tau}_{i_1} A_{j i_2 \dots i_k} + \overset{3}{\tau}_{i_2} A_{i_1 j i_3 \dots i_k} + \dots + \overset{k+1}{\tau}_{i_k} A_{i_1 i_2 \dots i_{k-1} j} \tag{I.4.1}$$

holds is called A – *weak symmetric.*

Here $\overset{\alpha}{\tau}_i$ are some vectors.

If the Riemannian tensor satisfies (I.4.1), the space is called *weak symmetric.* If (I.4.1) holds for a tensor Z_{hijk}, defined by (I.3.8):

$$Z^{\alpha}_{ijk} \overset{def}{=} R^h_{ijk} - B(\delta^h_k g_{ij} - \delta^h_j g_{ik}),$$

then (I.4.1) reads

$$Z_{hijk,m} = a_m Z_{hijk} + b_h Z_{mijk} + c_i Z_{hmjk} + d_j Z_{himk} + f_k Z_{hijm}. \tag{I.4.2}$$

Alternating the last equalities over h and i and using algebraic properties of the tensor Z_{hijk}, we get

$$\tau_h Z_{lijk} + \tau_i Z_{lhjk} = 0, \tag{I.4.3}$$

where $\tau_h \overset{def}{=} b_n - c_n$.

Suppose $\tau_h \neq 0$, then we can choose a vector ζ^h satisfying $\tau_\alpha \zeta^\alpha = 1$.

Multiplying (I.4.3) by ζ^h and contracting over h, we get

$$Z_{lijk} + \tau_i \zeta^\alpha Z_{l\alpha jk} = 0. \tag{I.4.4}$$

Once more multiplying by ζ^l and contracting over l we check

$$\zeta^\alpha Z_{\alpha ijk} = 0. \tag{I.4.5}$$

Hence from (I.4.4) we get $Z_{hijk} = 0$, which corresponds to spaces of constant curvature. Consequently, $\tau_i = 0$, that is, $b_n = c_n$.

Similarly, we check $d_n = f_n$.

Hence we proved

Theorem I.4.1. *In Z - weak symmetric spaces, distinct from spaces of constant curvature, the following holds*

$$Z_{hijk\,,\,m} = a_m Z_{hijk} + b_h Z_{mijk} + b_i Z_{hmjk} + d_j Z_{himk} + d_k Z_{hijm}. \tag{I.4.6}$$

Definition I.4.2. *We say that a pseudo-Riemannnian space V_n admits a vector enveloppe relative to a tensor A_{ijkl} if there exists a vector field τ_h in V_n such that*

$$\tau_h A_{ijkl} + \tau_i A_{jhkl} + \tau_j A_{hikl} = 0. \tag{I.4.7}$$

If the tensor A_{ijkl} satisfies the algebraic conditions

$$A_{ijkl} + A_{jikl} = 0, \tag{I.4.8}$$

$$A_{ijkl} - A_{klij} = 0, \tag{I.4.9}$$

$$A_{ijkl} + A_{iklj} + A_{iljk} = 0, \tag{I.4.10}$$

then in V_n there exists at most two linearly independent (non-zero) vector fields satisfying (I.4.7).

For the proof, let us use methods suggested in [27].

Suppose the number of linearly independent vectors $\underset{\alpha}{\tau}$ satisfying (I.4.12) equals three, and choosing a coordinate system in such a way that linearly independent vectors $\underset{\alpha}{\tau}^i$ $(\alpha = 1, 2, 3)$ have components $\underset{\alpha}{\tau}_i = \delta_i^\alpha$ from (I.4.12), accounting (I.4.8), (I.4.9), (I.4.10), we obtain $A_{hijk} = 0$. If $\alpha \leq 2$, then the system of equations (I.4.12) has a non-trivial solution.

Hence we proved

Theorem I.4.2. *The maximal number of linearly independent vectors among vectors belonging to the vector enveloppe of the non-zero tensor A_{ijkl}, is at most two.*

The equation

$$R_{hijk\,,\,l} + R_{hikl\,,\,j} + R_{hilj\,,\,k} = 0 \tag{I.4.11}$$

is called the Bianchi identity.

Let us prove the following:

Theorem I.4.3. *In a pseudo-Riemannian space V_n $(n > 2)$, the tensor Z_{hijk} satisfies the Bianchi-like identity, if and only if $B =$ const.*

Proof. Let $B =$ const, then, differentiating (I.3.8), we obtain

$$Z_{hijk\,,\,l} = R_{hijk\,,\,l} \tag{I.4.12}$$

and, consequently, Z_{hijk} satisfies

$$Z_{hijk\,,\,l} + Z_{hikl\,,\,j} + Z_{hilj\,,\,k} = 0 \tag{I.4.13}$$

Vice versa, let (I.4.13) be satisfied, then by (I.4.11) and (I.3.8), we get

$$\begin{gathered} B_{,\,l}(g_{hk}g_{ij} - g_{hj}g_{ik}) + B_{,\,j}(g_{hl}g_{ik} - g_{hk}g_{il}) + \\ +B_{,\,k}(g_{hj}g_{il} - g_{hl}g_{ij}) = 0. \end{gathered} \tag{I.4.14}$$

Contracting with g^{hk}, we get

$$B_{,\,l}g_{ij} - B_{,\,j}g_{il} = 0. \tag{I.4.15}$$

Finally, contracting with g^{ij}, we check $B_{,\,l} = 0$.

The proof is finished. □

Alternating (I.4.6) over the indices i, j, k we can see that in a pseudo-Riemannian space V_n the following holds:

$$\begin{gathered} (b_i - d_i)Z_{hmjk} + (b_j - d_j)Z_{hmki} + \\ +(b_k - d_k)Z_{hmij} = 0. \end{gathered} \tag{I.4.16}$$

Hence there are three possible types of weak Z - symmetric spaces:

I case: $b_i = d_i$ and

$$Z_{hijk\,,m} = a_m Z_{hijk} + b_h Z_{mijk} + b_i Z_{hmjk} + \\ + b_j Z_{himk} + b_k Z_{hijm}. \tag{I.4.17}$$

II case: $b_i \neq d_i$ and in V_n there exists just one vector envelope relative to the tensor Z_{hijk}, defined by the vector $(b_i - d_i)$

III case: $b_i \neq d_i$ and in V_n there exists another vector envelope relative to the tensor Z_{hijk},
defined by the vector non-collinear with the vector $(b_i - d_i)$.

If $B = \text{const}$, alternation of (I.4.6) over j, k, m and h, i, m gives

$$(a_m - 2d_m)Z_{hijk} + (a_i - 2d_i)Z_{hikm} + \\ + (a_k - 2d_k)Z_{himj} = 0,$$

$$(a_m - 2b_m)Z_{hijk} + (a_h - 2b_h)Z_{imjk} + \\ + (a_i - 2b_i)Z_{mhjk} = 0.$$

Consequently, if $B = \text{const}$ then weak Z - symmetric spaces can be

I case: *a)* $a_i = 2b_i \quad \Rightarrow$

$$Z_{hijk\,,m} = 2b_m Z_{hijk} + b_h Z_{mijk} + b_i Z_{hmjk} + \\ + b_j Z_{himk} + b_k Z_{hijm} \tag{I.4.18}$$

b) $a_i \neq 2b_i$

and there exists one-vector envelope in V_n relative to the tensor Z_{hijk} generated by $(a_i - 2b_i)$.

II case:

The assumption that both vectors $(a_i - 2d_i)$ and $(a_i - 2b_i)$ are equal zero, which contradicts to $d_i \neq b_i$. Hence among the vectors $(a_i - 2d_i)$ and $(a_i - 2b_i)$, at least one is non-zero.

Suppose $(a_i - 2b_i) = 0$, and $(a_i - 2d_i) \neq 0$, then there are two possible cases:

a) $(a_i = 2b_i)$ and $d_i - b_i = \alpha(a_i - 2d_i)$; (here α is a coefficient of similarity), and, as a consequence,

$$Z_{hijk\,,\,m} = 2b_m Z_{hijk} + b_h Z_{mijk} + b_i Z_{hmjk} + \qquad (I.4.19)$$

$$+d_j Z_{himk} + d_k Z_{hijm},$$

and necessarily, $\alpha = \frac{1}{2}$.

If $(a_i = 2b_i)$ and $d_i - b_i \neq \mu(a_i - 2d_i)$, then two linearly independent vectors are in the envelope with respect to Z_{hijk}, and the space is of the type III.

Let the vectors $(a_i - 2b_i)$ and $(a_i - 2d_i)$ are non-zero, then according to linear dependence or independence of the triple $(d_i - b_i)$, $(a_i - 2b_i)$, $(a_i - 2d_i)$, we get:

b) A triple of linearly dependent vectors:
one vector is in an envelope of the tensor Z_{hijk}.

In the class of spaces of the type III, let us distinguish the following subclasses:

a) $(d_i - b_i)$, $(a_i - 2b_i)$ are non-zero and linearly independent,

$(a_i - 2d_i)$ is either zero, or non-zero and linearly independent of the previous set of vectors *b)* $(d_i - b_i)$, $(a_i - 2d_i)$ are non-zero and linearly independent,

$(a_i - 2b_i)$ is either zero, or non-zero and linearly independent of the previous set of vectors.

Calculating covariant derivative of (I.3.8), accounting the conditions (I.3.10), (I.3.11), (I.3.5) that are satisfied in pseudo-Riemannian spaces with maximal mobility degree under concircular mappings, and using associativity we obtain:

$$Z_{hijk\,,\,m} = 2(\overset{1}{\xi}_m + \overset{4}{\xi}_m) Z_{hijk} + c_h Z_{mijk} + c_i Z_{hmjk} + \tag{I.4.20}$$

$$+ c_j Z_{himk} + c_k Z_{hijm}.$$

That is, such spaces necessarily belong to the type I, and since $B = \text{const}$, they might belong either to the type a), then

$$\overset{1}{\xi}_m + \overset{4}{\xi}_m = c_i, \tag{I.4.21}$$

or to the type b), then $\overset{1}{\xi}_m + \overset{4}{\xi}_m \neq c_i$ and the vector $(\overset{1}{\xi}_m + \overset{4}{\xi}_m - c_i)$ represents a vector envelope relative to the tensor Z_{hijk}.

Obviously, linearly independent vectors a_i, b_i from the equation (I.4.21) form a linear envelope with respect to the tensor which means that

$$\overset{1}{\xi}_m + \overset{4}{\xi}_m = c_i + \alpha a_i + \beta b_i, \tag{I.4.22}$$

for some functions α, β.

So we have given a proof of

Theorem I.4.4. *Pseudo-Riemannian spaces with maximal mobility degree with respect to concircular mappings are Z-weak symmetric, the tensor Z_{hijk} satisfies the conditions (I.4.20), and the vector $(\overset{1}{\xi}_m + \overset{4}{\xi}_m)$ satisfies (I.4.22).*

§ 5. On a tensor indication for pseudo-Riemannian spaces with maximal mobility degree under concircular mappings

Let us prove the following theorems that can be used for finer classification. Due to the following results, we can decompose the class of all spaces with maximal mobility degree under concircular mappings into three disjoint subclasses:

Theorem I.5.1. *If the vectors a_i and b_i satisfying (I.3.14) are non-isotropic, then the constant B is non-zero, and moreover, the following holds*

$$\frac{1}{2}\, Z\, Z_{hijk} = Z_{hk}Z_{ij} - Z_{hj}Z_{ik} \tag{I.5.1}$$

where $Z = Z_{\alpha\beta}g^{\alpha\beta}$.

Proof. Multiplying (**??**) by a_l and alternating, we check

$$a_l Z_{hijk} + a_h Z_{iljk} + a_i Z_{lhjk} = 0. \tag{I.5.2}$$

Multiplying by a^l and contracting over l, we get

$$a^\alpha a_\alpha Z_{hijk} = a_h a_\alpha Z^\alpha_{ijk} - a_i a_\alpha Z^\alpha_{hjk}. \tag{I.5.3}$$

Contracting with g^{hk} we get :

$$a^\alpha a_\alpha Z_{ij} = a^\alpha a^\beta Z_{\alpha ij\beta} + a_i a^\alpha Z_{\alpha j} \tag{I.5.4}$$

and finally,

$$a^\alpha a_\alpha Z = 2a^\alpha a^\beta Z_{\alpha\beta}. \tag{I.5.5}$$

Contracting (I.5.2) with g^{lj} we have

$$a_\alpha Z^\alpha_{khi} + a_h Z_{ik} - a_i Z_{hk} = 0. \tag{I.5.6}$$

Contracting the last tensor formula with g^{kh} we get

$$2a_\alpha Z^\alpha_i = Z a_i. \tag{I.5.7}$$

By (I.5.6), (I.5.3), we can write the above as

$$\begin{gathered} a^\alpha a_\alpha Z_{hijk} = a_h a_k Z_{ij} - a_h a_j Z_{ik} - \\ -a_i a_k Z_{hj} - a_i a_j Z_{hk}. \end{gathered} \tag{I.5.8}$$

Analogously, the vector b_i satisfies

$$\begin{gathered} b^\alpha b_\alpha Z_{hijk} = b_h b_k Z_{ij} - b_h b_j Z_{ik} - \\ -b_i b_k Z_{hj} - b_i b_j Z_{hk}. \end{gathered} \tag{I.5.9}$$

First we multiply (I.5.8) with $b^\alpha b_\alpha$, then (I.5.9) with $a^\alpha a_\alpha$, and create the sum. After some rearranging, with the account of (I.5.5) and (I.5.7), we finally obtain the desired result. □

Theorem I.5.2. *If one of the vectors comming into the equation (I.3.14), let us say a_i, is non-isotropic, and b_i is isotropic, then*

$$B = \frac{R}{n(n-1)} \tag{I.5.10}$$

and

$$Z_{ij} = e b_i b_j. \tag{I.5.11}$$

Proof. By $b_\alpha b^\alpha = 0$, from (I.3.15) we get $Z = 0$, and consequently, due to the definition of Z_{hijk}, the formula (I.5.10) holds.

Similarly, from (I.3.14) it follows (I.5.11), which finishes the proof. □

Pseudo-Riemannian spaces in which the equations (I.5.10), (I.5.11) are satisfied, that is,

$$R_{ij} = \frac{R}{n} g_{ij} + e b_i b_j, \tag{I.5.12}$$

are called *almost-Einstein spaces.*

Note that the vector b_i from (I.5.12) is necessarily isotropic. Hence almost-Einstein spaces form a subclass in the class of spaces with maximal mobility degree under concircular mappings.

Let us pass to the third class, completing the list of possibilities.

Theorem I.5.3. *If both vectors comming to the equation (**??**) are isotropic then the space with maximal mobility degree under concircular mappings is an Einstein space, that is, satisfies*

$$R_{ij} = \frac{R}{n} g_{ij}. \tag{I.5.13}$$

Since the Einstein tensor is invariant under concircular mappings, the spaces satisfying (I.5.13) admit concircular mappings only onto Einstein spaces again.

In a similar way, we can check that also the class of almost-Einstein spaces satisfying (I.5.12) is closed.

Indeed, examine concircular mappings of almost-Einstein spaces. Then

$$\bar{E}_{ij} = E_{ij} = e b_i b_j. \tag{I.5.14}$$

From the above we have

Theorem I.5.4. *The class of almost-Einstein spaces is closed under concircular mappings.*

Since we have distinguished three disjoint subclasses, and two of them are closed, also the remaining third class must be closed.

Definition I.5.1. A tensor $H_{ijklm_1m_2\dots m_{2p-1}m_{2p}}$ is called the *Bochner tensor* of order p of the first type with respect to the tensor A_{ijkl} if

$$H_{ijklm_1m_2\dots m_{2p-1}m_{2p}} = 2^p A_{ijkl\,,\,[m_1m_2][m_3m_4]\dots[m_{2p-1}m_{2p}]}. \tag{I.5.15}$$

If the Riemannian tensor satisfies the condition (I.5.15), we call H simply the *Bochner tensor* of the first type.

Pseudo-Riemannian spaces are classified into classes according to vanishing of the Bochner tensor of order p of the first type.

Definition I.5.2. A pseudo-Riemannian space V_n is called $1/2\,p$*-symmetric* $(p = 1, 2, \dots)$ *with respect to the tensor* A_{ijkl}, if the Bochner tensor of order p of the first type with respect to the tensor A_{ijkl} vanishes on the space.

Particularly, if $p = 1$ then the conditions

$$A_{kl\alpha[j}R^{\alpha}_{i]m_1m_2} + A_{ij\alpha[l}R^{\alpha}_{k]m_1m_2} = 0 \tag{I.5.16}$$

are satisfied, the pseudo-Riemannian space is called A–semisymmetric.

Particularly, when the tensor A_{ijkl} coincides with the Riemannian tensor, the space is called semisymmetric. Geometric properties of semisymmetric spaces were examined by many authors, such spaces play an important role in up-to-date Riemannian geometry.

Plugging the conditions (I.3.14) into (I.5.15) and considering the definition of Z_{ijkl}, we check

Theorem I.5.5. *Pseudo-Riemannian spaces with the maximal mobility degree under concircular mappings are Z–semisymmetric. Particularly, if $B = 0$, they are semisymmetric.*

CHAPTER II

GEODESIC MAPPINGS PRESERVING THE EINSTEIN TENSOR

§ 6. Fundamental equations of the theory of geodesic mappings preserving the Einstein tensor

Definition II.6.1. A diffeomorphism of a pseudo-Riemannian space V_n with the metric tensor g_{ij} onto a pseudo-Riemannian space $\bar{V}_n$ with the metric tensor $\bar{g}_{ij}$ is called a *geodesic mapping* if each geodesic of V_n is mapped onto a geodesic of $\bar{V}_n$.

There exists a geodesic mapping of V_n onto $\bar{V}_n$ if and only if the following equivalently *Levi-Civita equations* are satisfied [161] (see [2, 3, 86, 96, 145, 170, 178, 199, 206]):

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \varphi_i \delta^h_j + \varphi_j \delta^h_i, \tag{II.6.1}$$

$$\bar{g}_{ij},_k = 2\varphi_k \bar{g}_{ij} + \varphi_i \bar{g}_{jk} + \varphi_j \bar{g}_{ik}, \tag{II.6.2}$$

where Γ_{ij}^h ($\bar{\Gamma}_{ij}^h$) are components of the Riemannian connection of V_n or $\bar{V}_n$, respectively (objects in $\bar{V}_n$ corresponding to the given ones under geodesic mappings are denoted by bar), δ_i^h is the Kronecker tensor, " , " denotes covariant derivative in V_n, and φ_i is some vector (necessarily a gradient-like).

If the vector $\varphi_i \neq 0$ we say that the mapping is a *non-trivial* geodesic mapping or *affine.*

Necessary conditions for geodesic mappings to be read:

$$\bar{R}_{ijk}^h = R_{ijk}^h + \varphi_{ij}\delta_k^h - \varphi_{ik}\delta_j^h, \tag{II.6.3}$$

$$\bar{R}_{ij} = R_{ij} + (n-1)\varphi_{ij}. \tag{II.6.4}$$

Here R_{ijk}^h is the Riemannian curvature tensor, R_{ij} is the Ricci tensor, and

$$\varphi_{ij} = \varphi_{i,\,j} - \varphi_i\varphi_j.$$

On the other hand, a pseudo-Riemannian space V_n admits non-trivial geodesic mappings if and only if the following system of Sinyukov equations (see [96, 71, 170, 178]) is solvable in V_n

$$a_{ij},_k = \lambda_i g_{jk} + \lambda_j g_{ik}\ , \tag{II.6.5}$$

$$n\lambda_i,_j = \mu g_{ij} + a_{\alpha i}R_j^\alpha - a_{\alpha\beta}R_{.\,ij\,.}^{\alpha\ \ \ \beta}\ , \tag{II.6.6}$$

$$(n-1)\mu,_k = 2(n+1)\lambda_\alpha R_k^\alpha + a_{\alpha\beta}(2R_{.\,k}^{\alpha},^\beta - R_{..}^{\alpha\beta},_k)\ , \tag{II.6.7}$$

with respect to the tensor regular $a_{ij} = a_{ji} \neq cg_{ij}$, vector $\lambda_i \neq 0$ and a function μ, where

$$R^i_j = R_{\alpha j} g^{\alpha i}; \qquad R^{k\ h}_{.ij.} = R_{\alpha ij\beta} g^{\alpha k} g^{\beta h};$$

$$R^{ij}_{..},_k = R_{\alpha\beta},_k\, g^{\alpha i} g^{\beta j}; \qquad R^i_j,^k = R_{\alpha j},_\beta\, g^{\alpha i} g^{\beta k}.$$

Here g^{ij} are elements of the matrix inverse to g_{ij}.

According to the well-known solution of the above system of differential equations, the metrics of corresponding spaces related by geodesic mappings can be determined from the equations:

$$a_{ij} = e^{2\varphi} \bar{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}; \tag{II.6.8}$$

$$\lambda_i = -e^{2\varphi} \varphi_\alpha \bar{g}^{\alpha\beta} g_{\beta i}. \tag{II.6.9}$$

Let us prove the following:

Theorem II.6.1. *If a pseudo-Riemannian space V_n admits Einstein-preserving conformal mappings then V_n admits non-trivial geodesic mappings.*

Proof. Let a pseudo-Riemannian space V_n admits Einstein-preserving conformal mappings. Then in V_n, the conditions (I.1.15) are satisfied, i.e. V_n is equidistant. As shown by Sinyukov [96], we use this fact for construction of the tensor a_{ij}:

$$a_{ij} = c\, g_{ij} + \vartheta_i \vartheta_j, \tag{II.6.10}$$

where c is some constant.

Covariant derivating of the last equation yields the equations (II.6.5) where

$$\lambda_i = \beta \vartheta_i. \tag{II.6.11}$$

This finishes the proof.

The integrability conditions of (II.7.5) read

$$a_{\alpha(i} R^{\alpha}_{j)kl} = \lambda_{l(i} g_{j)k} - \lambda_{k(i} g_{j)l}, \tag{II.6.12}$$

where $\lambda_{ij} = \lambda_{i,j}$, and from (I.1.15) follows

$$\sigma_\alpha R^{\alpha}_{ijk} = B\,(g_{ij}\sigma_k - g_{ik}\sigma_j).$$

Multiplying (II.6.12) by σ^l and contracting over l, we get:

$$\begin{gathered}(B a_{\alpha i}\sigma^\alpha - \lambda_{\alpha i}\sigma^\alpha) g_{jk} + (B a_{\alpha j}\sigma^\alpha - \lambda_{\alpha j}\sigma^\alpha) g_{ik} + \\ + \sigma_j(\lambda_{ki} - B a_{ki}) + \sigma_i(\lambda_{kj} - B a_{kj}) = 0.\end{gathered} \tag{II.6.13}$$

Alternating the last formula over j and k, switching $i \leftrightarrow k$ in the obtained formula, and then summing with (II.6.13), we get

$$(B\sigma^\alpha a_{\alpha i} - \lambda_{\alpha i}\sigma^\alpha) g_{jk} = (B a_{kj} - \lambda_{kj})\sigma_i. \tag{II.6.14}$$

Contracting with g^{jk} we have

$$n(B\sigma^{\alpha}a_{\alpha i} - \lambda_{\alpha i}\sigma^{\alpha}) = (Ba - \lambda)\sigma_i, \tag{II.6.15}$$

here $a = a_{\alpha\beta}g^{\alpha\beta}$ and $\lambda = \lambda_{\alpha\beta}g^{\alpha\beta}$.

By (II.6.15), we can write (II.6.14) as

$$\sigma_i \left(\frac{\lambda - Ba}{n} g_{jk} + Ba_{jk} - \lambda_{jk}\right) = 0. \tag{II.6.16}$$

As $\sigma_i \neq 0$ it follows

$$\lambda_{i,j} = \mu g_{ij} + Ba_{ij}, \tag{II.6.17}$$

where $\mu = \dfrac{\lambda - Ba}{n}$.

Pseudo-Riemannian spaces admitting non-trivial geodesic mappings such that the vector λ_i satisfies (II.6.17), are denoted by $V_n(B)$.

So we can formulate the result as follows:

Theorem II.6.2. *If a pseudo-Riemannian space V_n admits a conformal mapping preserving the Einstein tensor then the space V_n is of the class $V_n(B)$.*

Definition II.6.2. If a geodesic mapping of the space V_n satisfies

$$\bar{E}_{ij} = E_{ij}, \tag{II.6.18}$$

we speak about an *Einstein tensor preserving geodesic mapping.*

Recall that

$$E_{ij} = R_{ij} - \frac{R}{n}\, g_{ij}. \tag{II.6.19}$$

If this is the case then the deformation tensor of the Ricci tensor satisfies:

$$T_{ij} = \bar{R}_{ij} - R_{ij} = \frac{\bar{R}}{n}\, \bar{g}_{ij} - \frac{R}{n}\, g_{ij}. \tag{II.6.20}$$

On the other hand, accounting (II.6.4), we have

$$T_{ij} = \bar{R}_{ij} - R_{ij} = (n-1)\varphi_{ij}. \tag{II.6.21}$$

Comparing the equations we have:

$$\varphi_{ij} = \frac{\bar{R}}{n(n-1)}\bar{g}_{ij} - \frac{R}{n(n-1)}g_{ij}. \tag{II.6.22}$$

Let us calculate covariant derivative of (II.6.9)

$$\begin{aligned} \lambda_{i,j} &= -e^{2\varphi}\varphi_{\alpha,j}\,\bar{g}^{\alpha\beta}g_{\beta i} + e^{2\varphi}\varphi_\alpha\varphi_\beta\bar{g}^{\alpha\beta}g_{ji} + \\ &+ e^{2\varphi}\varphi_j\varphi_\alpha\bar{g}^{\alpha\beta}g_{\beta i}. \end{aligned} \tag{II.6.23}$$

By (II.6.8) and (II.6.16), we have

$$\lambda_{i,j} = \mu\, g_{ij} + \frac{R}{n(n-1)}\, a_{ij}, \tag{II.6.24}$$

where

$$\mu = e^{2\varphi} \left(\varphi_\alpha \varphi_\beta\, \bar{g}^{\alpha\beta} - \frac{\bar{R}}{n(n-1)}\right). \tag{II.6.25}$$

Obviously, by means of (II.6.18), accounting (II.6.8) and (II.6.9), we obtain from (II.6.17) the formula (II.6.16), and consequently (II.6.12), which proves the theorem:

Theorem II.6.3. *The conditions (II.6.5), (II.6.18) and (II.6.19) are necessary and sufficient conditions in a pseudo-Riemannian space V_n for existence of Einstein tensor preserving geodesic mappings.*

Hence we proved that pseudo-Riemannian spaces V_n, admitting Einstein tensor preserving geodesic mappings, belong to the class $V_n(B)$ where $B = \frac{R}{n(n-1)}$.

If we account the Ricci identity, the integrability conditions for the equations (II.6.5) read

$$a_{i\alpha} R^{\alpha}_{jkl} + a_{j\alpha} R^{\alpha}_{ikl} = \lambda_{i,l}\, g_{jk} + \lambda_{j,l}\, g_{ik} - \lambda_{i,k}\, g_{jl} - \lambda_{j,k}\, g_{il}. \tag{II.6.26}$$

Plugging (II.6.20) into the last equality and accounting (II.6.18), we have:

$$a_{i\alpha} Y^{\alpha}_{jkl} + a_{j\alpha} Y^{\alpha}_{ikl} = 0. \tag{II.6.27}$$

§ 7. Objects invariant under Einstein tensor preserving geodesic mappings

Objects invariant under geodesic mappings are the so-called *Thomas projective parameters*

$$\bar{T}^h_{ij} = T^h_{ij};$$
$$T^h_{ij} = \Gamma^h_{ij} - \frac{1}{n-1}\,(\delta^h_i \Gamma^\alpha_{j\alpha} + \delta^h_j \Gamma^\alpha_{i\alpha}), \tag{II.7.1}$$

and the *Weyl tensor of projective curvature*

$$\bar{W}^h_{ijk} = W^h_{ijk};$$
$$W^h_{ijk} = R^h_{ijk} - \frac{1}{n-1}\,(\delta^h_k R_{ij} - \delta^h_j R_{ik}). \tag{II.7.2}$$

Plugging the last into (II.6.20) and using associativity we get

$$\bar{Y}^h_{ijk} = Y^h_{ijk}. \tag{II.7.3}$$

Here

$$Y^h_{ijk} = R^h_{ijk} - \frac{R}{n(n-1)}\,(\delta^h_k g_{ij} - \delta^h_j g_{ik}) \tag{II.7.4}$$

is the tensor of *concircular curvature.*

So we in fact proved the following:

Theorem II.7.1. *The tensor of concircular curvature is invariant under Einstein tensor preserving geodesic mappings.*

The last formula enables us to verify invariance of the tensors $\underset{1}{K}_{ijkl}$, $\underset{3}{K}_{ijkl}$ and $\underset{4}{K}_{ijkl}$ (introduced in the second paragraph) under Einstein tensor preserving geodesic mappings. If we moreover account invariance of the tensor of projective curvature we have:

$$\underset{1}{G}_{ijkl} = \underset{1}{K}_{ijkl} = E_{\alpha i} Y^{\alpha}_{jkl} \tag{II.7.5}$$

$$\underset{2}{G}_{ijkl} = E_{\alpha i} W^{\alpha}_{jkl} \tag{II.7.6}$$

$$\underset{3}{G}{}^{h}_{ijklm} = \underset{3}{K}{}^{h}_{ijklm} = Y^{h}_{\alpha ij} Y^{\alpha}_{klm} \tag{II.7.7}$$

$$\underset{4}{G}{}^{h}_{ijklm} = \underset{4}{K}{}^{h}_{ijklm} = Y^{h}_{i\alpha j} Y^{\alpha}_{klm} \tag{II.7.8}$$

$$\underset{5}{G}{}^{h}_{ijklm} = Y^{h}_{\alpha ij} W^{\alpha}_{klm} \tag{II.7.9}$$

$$\underset{6}{G}{}^{h}_{ijklm} = Y^{h}_{i\alpha j} W^{\alpha}_{klm} \tag{II.7.10}$$

$$\underset{7}{G}{}^{h}_{ijklm} = Y^{\alpha}_{ijk} W^{h}_{\alpha lm} \tag{II.7.11}$$

$$\underset{8}{G}{}^{h}_{ijklm} = Y^{\alpha}_{ijk} W^{h}_{l\alpha m}. \tag{II.7.12}$$

Let us introduce the object $\underset{2}{G}_{ijkl}$ as follows:

$$\underset{2}{G}: E_{\alpha i}W^{\alpha}_{jkl} = R_{i\alpha}R^{\alpha}_{jkl} - \frac{R}{n}R_{ijkl} - \frac{1}{n-1}(R_{il}R_{jk} - \\ -R_{ik}R_{jl}) + \frac{R}{n(n-1)}(g_{il}R_{jk} - g_{ik}R_{jl}). \tag{II.7.13}$$

Then $\underset{2}{G}$ – flat spaces are characterized by the conditions

$$R_{i\alpha}R^{\alpha}_{jkl} - \frac{R}{n}R_{ijkl} - \frac{1}{n-1}(R_{il}R_{jk} - \\ -R_{ik}R_{jl}) + \frac{R}{n(n-1)}(g_{il}R_{jk} - g_{ik}R_{jl}) = 0. \tag{II.7.14}$$

Contracting the last formula we get

$$R_{\alpha i}R^{\alpha}_{l} = \frac{2R}{n}R_{il} - \frac{R^2}{n^2}g_{il}. \tag{II.7.15}$$

The formula (II.7.15) coincides with he equations (I.2.19), hence the conditions (II.7.15) characterize K_1, K_2, K_3 and $\underset{2}{G}$ flat spaces.

Let us examine the question of existence of equidistant vector fields in pseudo-Riemannian spaces satisfying (II.7.15). Accounting (II.7.22) and (II.8.4), let us multiply (II.7.15) by φ^l, and contracting in l, we check $B = \frac{R}{n(n-1)}$.

Therefore we proved

Theorem II.7.2. *If the conditions (II.7.15) are satisfied in a pseudo-Riemannian space that admits an equidistant vector field then*

$$B = \frac{R}{n(n-1)}.$$

Let us consider another way how to construct objects invariant under diffeomorphisms by means of covariant derivative. A covariant derivative in V_n (with respect to the natural Riemannian connection) of a tensor field S of type $\binom{p}{q}$, denoted here by comma, as usual, is in any coordinate system given as follows:

$$\begin{aligned} S^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q\,,\,k}(x) \quad &= \quad \partial_k S^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q}(x) + \\ +\Gamma^{i_1}_{k\alpha}(x) S^{\alpha i_2 \dots i_p}_{j_1 j_2 \dots j_q}(x) \ &+ \dots + \ \Gamma^{i_p}_{k\alpha}(x) S^{i_1 i_2 \dots i_{p-1}\alpha}_{j_1 j_2 \dots j_q}(x) - \\ -\Gamma^{\beta}_{kj_1}(x) S^{i_1 i_2 \dots i_p}_{j_2 j_3 \dots j_q}(x) \ &- \dots - \ \Gamma^{\beta}_{kj_q}(x) S^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_{q-1}\beta}(x). \end{aligned} \tag{II.7.16}$$

In the case of a vector feld Φ, we have

$$\Phi_{i\,\bar{;}\,j} = \partial_j \Phi_i - \Phi_\alpha \bar{\Gamma}^{\alpha}_{ij} \tag{II.7.17}$$

where $\bar{\Gamma}^{i}_{jk}$ are components of the connection of the image of the given pseudo-Riemannian space V_n under the geodesic mapping, and " $\bar{;}$ " denotes the covariant derivative in the image space.

Using (II.6.1) we have

$$\Phi_{i\,\bar{;}\,j} = \Phi_{i\,,\,j} - \Phi_i \varphi_j - \Phi_j \varphi_i. \tag{II.7.18}$$

Alternating the last formula we get

$$K_{ij} = \bar{K}_{ij} \tag{II.7.19}$$

where we denoted

$$K_{ij} \stackrel{def}{=} \Phi_{i\,,\,j} - \Phi_{j\,,\,i}. \tag{II.7.20}$$

Hence we proved

Theorem II.7.3. *The tensor K is invariant under geodesic mappings.*

Examining the tensor field E_{ij}, by (II.7.16) we can write

$$E_{ij\,,\,k} = \partial_k E_{ij} - E_{\alpha j}\Gamma^{\alpha}_{k\,i} - E_{i\alpha}\Gamma^{\alpha}_{kj}. \tag{II.7.21}$$

Assume an Einstein tensor preserving geodesic mapping, then

$$\bar{E}_{ij\,\bar{;}\,k} = E_{ij\,\bar{;}\,k}. \tag{II.7.22}$$

By (II.7.21) and (II.6.1) we get

$$\bar{E}_{ij\,\bar{;}\,k} = E_{ij\,,\,k} - 2\varphi_k E_{ij} - \varphi_i E_{jk} - \varphi_j E_{ik}. \tag{II.7.23}$$

Contraction over h, j in (II.6.1) gives

$$\bar{\Gamma}^{\alpha}_{i\alpha} = \Gamma^{\alpha}_{i\alpha} + (n+1)\varphi_i. \tag{II.7.24}$$

Calculating φ_i from the last formula, plugging into (II.7.23), and then using associativity, we check

$$\bar{E}_{ijk} = E_{ijk} \tag{II.7.25}$$

where

$$E_{ijk} \overset{def}{=} E_{ij\,,\,k} + \frac{1}{n+1}(2\Gamma^{\alpha}_{k\alpha}E_{ij} + \Gamma^{\alpha}_{i\alpha}E_{jk} + \Gamma^{\alpha}_{j\alpha}E_{ik}). \tag{II.7.26}$$

We obtain the following theorem.

Theorem II.7.4. *The object E_{ijk} is invariant under Einstein tensor preserving geodesic mappings of pseudo-Riemannian spaces.*

Note that from the definition of Christoffel symbols and the rules for the derivation of the functional determinant, we obtain the formula

$$\Gamma^{\alpha}_{i\alpha} = \frac{1}{2}\,\partial_i \ln|g| \tag{II.7.27}$$

that holds in any pseudo-Riemannian space V_n. Here $g = \det\|g_{ij}\|$. Hence $\Gamma^{\alpha}_{i\alpha}$ is a gradient vector.

Pseudo-Riemannian space in which the Einstein tensor E_{ij} satisfies

$$E_{ij\,,\,k} = 2u_k E_{ij} + u_i E_{jk} + u_j E_{ik} \tag{II.7.28}$$

are called *weak symmetric with respect to the tensor E_{ij}*.

We can see that the property of weak symmetry is preserved under Einstein tensor preserving geodesic mappings if the following holds:

$$u_i = -\frac{1}{n+1}\,\Gamma^{\alpha}_{i\alpha}. \tag{II.7.29}$$

§ 8. Einstein tensor preserving geodesic mappings of spaces of constant curvature

Let us examine pseudo-Riemannian spaces V_n that admit non-trivial Einstein tensor preserving geodesic mappings, in relation to the scalar curvature R of the space.

Let the scalar curvature R of the space V_n is constant. Then the tensor of concircular curvature satisfies

$$Y^h_{ijk\,,\,l} = R^h_{ijk\,,\,l}. \tag{II.8.1}$$

Under the assumption that the Einstein tensor is preserved under the geodesic mapping, the integrability conditions of the equations (II.6.5) take the form (II.6.27). Differentiation of (II.6.27) can be by (II.8.1) and (II.6.5) written as follows:

$$\begin{gathered}\lambda_i Y_{hjkl} + \lambda_j Y_{hikl} + \lambda_\alpha Y^\alpha_{jkl} g_{ih} + \lambda_\alpha Y^\alpha_{ikl} g_{jh} + \\ + a_{\alpha i} R^\alpha_{jkl\,,\,h} + a_{\alpha j} R^\alpha_{ikl\,,\,h} = 0.\end{gathered} \tag{II.8.2}$$

Alternating over k, l, h and using properties of the tensors Y_{hijk} and $R^h_{ijk\,,\,l}$ we get

$$\lambda_\alpha Y^\alpha_{j(kl} g_{h)i} + \lambda_\alpha Y^\alpha_{i(kl} g_{h)j} = 0. \tag{II.8.3}$$

Here (klh) denotes alternation over indices without division.

Contracting the last equation with g^{hi} we obtain

$$(n-1)\lambda_\alpha Y^\alpha_{jkl} = \lambda_\alpha E^\alpha_l g_{jk} - \lambda_\alpha E^\alpha_k g_{jl}. \tag{II.8.4}$$

Multiplying (II.8.4) by λ^j and contracting in j, we find

$$\lambda_\alpha E^\alpha_l \lambda_k - \lambda_\alpha E^\alpha_k \lambda_l = 0. \tag{II.8.5}$$

Since $\lambda_k \neq 0$, there exists a vector ξ^k such that $\lambda_\alpha \xi^\alpha = 1$. Hence

$$\lambda_\alpha E^\alpha_l = \lambda^\alpha \xi^\beta E_{\alpha\beta} \lambda_l. \tag{II.8.6}$$

Assume $\lambda^\alpha \xi^\beta E_{\alpha\beta} \neq 0$, then, multiplying (II.6.27) by λ^l, we have

$$a_{\alpha j} Y^\alpha_{ik\beta} \lambda^\beta + a_{\alpha i} Y^\alpha_{jk\beta} \lambda^\beta = 0. \tag{II.8.7}$$

By (II.8.4) we get

$$a^\alpha_i \lambda_\alpha g_{kj} + a^\alpha_j \lambda_\alpha g_{ki} - \lambda_j a_{ki} - \lambda_i a_{kj} = 0. \tag{II.8.8}$$

Contraction with g^{kj} yields

$$a^\alpha_i \lambda_\alpha = \frac{a}{n}\, \lambda_i, \tag{II.8.9}$$

where $a = a_{\alpha\beta} g^{\alpha\beta}$.

Plugging (II.8.9) into (II.8.8) and using associativity we get

$$\lambda_i (a_{kj} - \frac{a}{n}\, g_{kj}) + \lambda_j (a_{ki} - \frac{a}{n}\, g_{ki}) = 0. \tag{II.8.10}$$

Alternation over k, i gives:

$$\lambda_i (a_{kj} - \frac{a}{n}\, g_{kj}) - \lambda_k (a_{ij} - \frac{a}{n}\, g_{ij}) = 0. \tag{II.8.11}$$

Interchanging k and j gives

$$\lambda_i (a_{kj} - \frac{a}{n}\, g_{kj}) - \lambda_j (a_{ki} - \frac{a}{n}\, g_{ki}) = 0. \tag{II.8.12}$$

Composing (II.8.12) and (II.8.10) we check

$$\lambda_i(a_{kj} - \frac{a}{n}\, g_{kj}) = 0. \tag{II.8.13}$$

But this is a contradiction under the assumption of non-trivial geodesic mappings, hence the following holds:

$$\lambda^\alpha \xi^\beta E_{\alpha\beta} = 0, \tag{II.8.14}$$

and consequently,

$$\lambda_\alpha E^\alpha_i = 0. \tag{II.8.15}$$

Hence

$$\lambda_\alpha Y^\alpha_{ijk} = 0 \tag{II.8.16}$$

Theorem II.8.1. *If a pseudo-Riemannian space V_n of constant curvature admits non-trivial Einstein tensor preserving geodesic mappings then the conditions (II.8.16) are satisfied in the space.*

Differentiating (II.6.24) and using $R = const$ we have

$$\lambda_{i\,,\,jk} = \mu_k g_{ij} + \frac{R}{n(n-1)}\,(\lambda_i g_{jk} + \lambda_j g_{ik}) \tag{II.8.17}$$

where $\mu_k = \mu_{,\,k} = \partial_k \mu$.

Alternating we get

$$\begin{aligned} \lambda_\alpha R^\alpha_{ijk} \;&=\; \mu_k g_{ij} - \mu_j g_{ik} + \\ &+\; \frac{R}{n(n-1)}\,(\lambda_j g_{ik} - \lambda_k g_{ij}). \end{aligned} \tag{II.8.18}$$

By (II.8.16) we check

$$\frac{2R}{n(n-1)}\,(\lambda_k g_{ij} + \lambda_j g_{ik}) = \mu_k g_{ij} - \mu_j g_{ik}. \tag{II.8.19}$$

Contracting with g^{ij} we obtain

$$\mu_k = 2\cdot\frac{R}{n(n-1)}\,\lambda_k. \tag{II.8.20}$$

Theorem II.8.2. *If a pseudo-Riemannian space V_n of constant curvature admits Einstein tensor preserving geodesic mappings, then in the space, there exists a solution of the equations (II.7.5), (II.7.24), (II.9.20) with respect to a tensor a_{ij}, vector λ_i and a function μ.*

Now let us examine Einstein tensor preserving geodesic mappings under additional assumptions.

Let the vector λ_i from (II.6.24) be of non-zero constant norm, that is,

$$\lambda_\alpha\lambda^\alpha = \text{const} = c \neq 0. \tag{II.8.21}$$

Differentiating and using (II.6.24) we get:

$$\mu\,\lambda_i + \frac{R}{n(n-1)}\,a_{\alpha i}\lambda^\alpha = 0. \tag{II.8.22}$$

Note if $R = 0$ then $\mu = 0$ and consequently, λ_i is covariantly constant.

Assume $R \neq 0$, then (II.8.22) reads

$$a_{\alpha i}\lambda^{\alpha} = \rho\lambda_i, \tag{II.8.23}$$

where

$$\rho \overset{def}{=} -\frac{n(n-1)}{R}\,\mu. \tag{II.8.24}$$

Differentiating (II.8.23) and using (II.8.21), we have

$$c\,g_{ij} + \lambda_i\lambda_j + a_{\alpha i}\lambda^{\alpha}_{\ ,\,j} = \rho_j\lambda_i + \rho\lambda_{i\,,\,j}. \tag{II.8.25}$$

Contracting the last formula wih λ^i:

$$2c\lambda_j = c\rho_j. \tag{II.8.26}$$

If $c \neq 0$ then $\rho_j = 2\lambda_j$, and (II.8.25) reads:

$$\frac{R}{n(n-1)}\,a_{\alpha j}a^{\alpha}_i - \lambda_i\lambda_j = (\rho\mu - c)g_{ij} - 2\mu a_{ij}, \tag{II.8.27}$$

or, denoting

$$A_{ij} = \frac{R}{n(n-1)}\,a_{\alpha j}a^{\alpha}_i - \lambda_i\lambda_j, \tag{II.8.28}$$

$$A_{ij} = (\rho\mu - c)g_{ij} - 2\mu a_{ij}. \tag{II.8.29}$$

Differentiating (II.8.28), accounting the scalar curvature, we get

$$A_{ij\,,\,k} = \frac{R}{n(n-1)}\,(\lambda_\alpha a^{\alpha}_i g_{jk} + \lambda_\alpha a^{\alpha}_k g_{ik}) - \\ -\mu(\lambda_j g_{ik} + \lambda_i g_{jk}). \tag{II.8.30}$$

By (II.8.22) we have

$$A_{ij\,,\,k} = -2\mu(\lambda_j g_{ik} + \lambda_i g_{jk}). \tag{II.8.31}$$

Hence

$$A_{ij\,,\,k} = -2\mu a_{ij\,,\,k}. \tag{II.8.32}$$

Differeniating (II.8.29) we get

$$A_{ij\,,\,k} = (\rho\mu - c)_{,\,k}\, g_{ij} - 2\mu_{,\,k}\, a_{ij} - 2\mu a_{ij\,,\,k}. \tag{II.8.33}$$

Subtracting (II.8.32) from the last formula we have

$$g_{ij}(\rho\mu - c)_{,\,k} = 2\mu_{,\,k}\, a_{ij}. \tag{II.8.34}$$

Contracting with g^{ij} we get

$$n(\rho\mu - c)_{,\,k} = 2a\mu_{,\,k}, \tag{II.8.35}$$

or

$$(\rho\mu - c)_{,\,k} = \frac{2a}{n}\,\mu_{,\,k}. \tag{II.8.36}$$

Plugging (II.8.36) into (II.8.34) we obtain

$$\mu_{,\,k}\,(a_{ij} - \frac{a}{n}\,g_{ij}) = 0. \tag{II.8.37}$$

As the geodesic mapping is non-trivial we have $\mu_{,\,k} = 0$ and $\mu =$ const. But then from (II.8.20) it follows $R = 0$, a contradiction with the assumption $R \neq 0$.

Hence we proved

Theorem II.8.3. *If a pseudo-Riemannian space V_n of constant scalar curvature admits a non-trivial Einstein tensor preserving geodesic mapping and the vector λ_i has a constant norm then the space V_n has a nonvanishing scalar curvature.*

§ 9. Invariants of geodesic mappings

Before the theory of geodesic mappings was developed, a problem was posed on cardinality of the class of spaces admitting non-trivial geodesic mappings. The question was aswered by N.S. Sinyukov who constructed an infinite sequence of disjoint spaces which are related by geodesic mappings. See [96].

Let a pseudo-Riemannian space V_n admits a non-trivial geodesic mapping, corresponding to the vector φ_i, onto a space $\bar{V}_n$. Then in V_n, there exists a solution of the system (II.6.5), satisfying (II.6.8).

Let us assume a_{ij} as a metric tensor of a pseudo-Riemannian space $\overset{1}{V}_n$:

$$a_{ij} \overset{def}{=} \overset{1}{g}_{ij}. \tag{II.9.1}$$

Let us construct the metric tensor of $\overset{1}{\bar{V}}_n$ as follows

$$\overset{1}{\bar{g}}_{ij} = e^{2\varphi} g_{ij}, \tag{II.9.2}$$

that is, we introduce a conformal mapping between V_n and $\overset{1}{\bar{V}}_n$, corresponding to the same vector field φ_i.

Then, as was proved by N.S Sinyukov [96], if a pseudo-Riemannian space V_n with the metric tensor g_{ij} admits a non-trivial geodesic mapping corresponding to the vector φ_i onto a space $\bar{V}_n$ with the metric tensor $\bar{g}_{ij}$, then the pseudo-Riemannian space $\overset{1}{V}_n$ with the metric tensor $\overset{1}{g}_{ij}$, satisfying (II.9.1), admits a geodesic mapping, corresponding to the same vector field φ_i, onto a pseudo-Riemannian space $\overset{1}{\bar{V}}_n$ with the metric tensor $\overset{1}{\bar{g}}_{ij}$, satisfying (II.9.2).

Definition II.9.1. A correspondence, given by the formulas (II.6.8), (II.9.1), (II.9.2), sending a pair of pseudo-Riemannian spaces V_n and $\bar{V}_n$, for which there exists a non-trivial geodesic mapping, onto a pair $\overset{1}{V}_n$ and $\overset{1}{\bar{V}}_n$, related by a non-trivial geodesic mapping again, is called an *invariant geodesics preserving mapping* of pseudo-Riemannian spaces.

The diagram

$$\begin{array}{ccc} V_n & \xrightarrow{gm\ \varphi} & \bar{V}_n \\ \downarrow & \overset{cm\ \varphi}{\searrow} & \\ \overset{1}{V}_n & \xrightarrow{gm\ \varphi} & \overset{1}{\bar{V}}_n \end{array}$$

describes the introduced geodesics preserving mappings, accounting the formulas (II.6.8) and (II.6.9).

Another possibility is to describe the situation as follows

$$\begin{array}{ccc} V_n & \xrightarrow{gm\ \varphi} & \bar{V}_n \\ a^i_j \downarrow & & \downarrow a^i_j \\ \overset{1}{V}_n & \xrightarrow{gm\ \varphi} & \overset{1}{\bar{V}}_n \end{array}$$

where $a^i_j = a_{\alpha j} g^{\alpha i}$, and

$$\overset{1}{g}_{ij} = g_{\alpha i} a^{\alpha}_j, \tag{II.9.3}$$

$$\overset{1}{\bar{g}}_{ij} = \bar{g}_{\alpha i} a^{\alpha}_j. \tag{II.9.4}$$

The last two systems are equivalent. Using the first one, examine the case when the pseudo-Riemannian spaces V_n and $\bar{V}_n$ are related by a non-trivial Einstein tensor preserving geodesic mapping, corresponding to the vector φ_i.

Hence in V_n, the following equations are satisfied

$$\varphi_{ij} = \frac{\bar{R}}{n(n-1)}\bar{g}_{ij} - \frac{R}{n(n-1)}g_{ij}. \tag{II.9.5}$$

Differentiating and accounting (II.6.2), we get

$$\begin{gathered}\varphi_{ij\,,\,k} = \frac{\bar{R}_{,\,k}}{n(n-1)}\bar{g}_{ij} + \frac{\bar{R}}{n(n-1)}(2\varphi_k\bar{g}_{ij} + \\ +\varphi_i\bar{g}_{jk} + \varphi_j\bar{g}_{ik}) - \frac{R_{,\,k}}{n(n-1)}g_{ij}.\end{gathered} \tag{II.9.6}$$

Since V_n and $\overset{1}{\bar{V}}_n$ are related by a conformal mapping (II.9.2), the conditions (I.1.5) must hold:

$$\varphi_{ij} = P_{ij} + \rho g_{ij} \tag{II.9.7}$$

where

$$P_{ij} \overset{def}{=} \frac{1}{n-2}(\bar{R}_{ij} - R_{ij}),$$

$$\rho = \frac{\Delta_2\varphi}{n-2} + \Delta_1\varphi.$$

Differentiating (II.9.8) we have

$$\varphi_{ij\,,\,k} = P_{ij\,,\,k} + \rho_k g_{ij}. \tag{II.9.8}$$

Let us subtract (II.9.8) from (II.9.6):

$$\begin{gathered}P_{ij\,,\,k} = \frac{\bar{R}_{,\,k}}{n(n-1)}\bar{g}_{ij} + \frac{\bar{R}}{n(n-1)}(2\varphi_k\bar{g}_{ij} + \\ +\varphi_i\bar{g}_{jk} + \varphi_j\bar{g}_{ik}) - (\frac{R_{,\,k}}{n(n-1)} - \rho_k)g_{ij}.\end{gathered} \tag{II.9.9}$$

Excluding $\frac{\bar{R}}{n(n-1)}\bar{g}_{ij}$ from the last formula by means of (II.9.5):

$$
\begin{aligned}
P_{ij\,,\,k} \quad &= \frac{\bar{R}_{,\,k}}{n(n-1)}\bar{g}_{ij} + 2\varphi_k\varphi_{ij} + \varphi_i\varphi_{jk} + \\
&+\varphi_j\varphi_{ik} + \frac{R}{n(n-1)}(2\varphi_k g_{ij} + \\
&+ \quad \varphi_i g_{jk} + \varphi_j g_{ik}) - (\frac{R_{,\,k}}{n(n-1)} - \rho_k)g_{ij}.
\end{aligned}
\tag{II.9.10}
$$

We pass to the covariant derivative in $\overset{1}{\bar{V}}_n$, denoted by $\overset{1}{,}$ and, using the formulas (I.1.3), (II.7.16), we get

$$
\begin{aligned}
&P_{ij\overset{1}{,}k} + 2\varphi_k P_{ij} + \varphi_i P_{jk} + \varphi_j P_{ik} - 2P_{\alpha k}\varphi^\alpha g_{ij} = \\
&= \frac{\bar{R}_{,\,k}}{n(n-1)}\bar{g}_{ij} + 2\varphi_k\varphi_{ij} + \varphi_i\varphi_{jk} + \\
&+\varphi_j\varphi_{ik} + \frac{R}{n(n-1)}(2\varphi_k g_{ij} + \\
&+\varphi_i g_{jk} + \varphi_j g_{ik}) - (\frac{R_{,\,k}}{n(n-1)} - \rho_k)g_{ij}.
\end{aligned}
\tag{II.9.11}
$$

Now using (II.9.8) we have

$$
\begin{aligned}
P_{ij\overset{1}{,}k} &= \frac{\bar{R}_{,\,k}}{n(n-1)}\bar{g}_{ij} + (\frac{R}{n(n-1)} - \rho)(2\varphi_k g_{ij} + \\
&+\varphi_i g_{jk} + \varphi_j g_{ik}) - (\frac{R_{,\,k}}{n(n-1)} - \rho_k - 2P_{\alpha k}\varphi^\alpha)g_{ij}.
\end{aligned}
\tag{II.9.12}
$$

Finally, by (II.9.2), we have

$$P_{ij\overset{1}{;}k} = \frac{\bar{R}_{,\,k}}{n(n-1)}\bar{g}_{ij} + (\frac{R}{n(n-1)} - \rho)e^{-2\varphi}(2\varphi_k \overset{1}{\bar{g}}_{ij} +$$

$$+\varphi_i \overset{1}{\bar{g}}_{jk} + \varphi_j \overset{1}{\bar{g}}_{ik}) - e^{-2\varphi}(R_{,\,k} - \rho_k - 2P_{\alpha k}\varphi^{\alpha}) \overset{1}{\bar{g}}_{ij}. \quad \text{(II.9.13)}$$

So we proved:

Theorem II.9.1. *If pseudo-Riemannian spaces V_n and $\bar{V}_n$ are related by a non-trivial Einstein tensor preserving geodesic mapping, then in the space $\overset{1}{\bar{V}}_n$ which is obtained by an invariant geodesics preserving mapping, there exists a tensor P_{ij}, satisfying the conditions (II.9.13).*

If the scalar curvature of the pseudo-Riemannian space $\bar{V}_n$ is constant, then the equations (II.9.13) read

$$P_{ij\overset{1}{;}k} = u_k \overset{1}{\bar{g}}_{ij} + v_i \overset{1}{\bar{g}}_{jk} + v_j \overset{1}{\bar{g}}_{ik}. \quad \text{(II.9.14)}$$

Here u_i and v_i are vectors.

The theorem II.9.1 is a generalization of the results of N.S. Sinyukov and formulas on invariant geodesics preserving mappings for spaces of constant curvature and Einstein spaces.

§ 10. Einstein tensor preserving geodesic mappings of special pseudo-Riemannian spaces

A pseudo-Riemannian space V_n, in which there exists a tensor $A^h_{i_1 i_2 \dots i_k}$ satisfying

$$A^\alpha_{i_1 i_2 \dots i_k\,,\,\alpha} = 0 \tag{II.10.1}$$

will be called an *A–harmonic* pseudo-Riemannian space.

In the case that the Riemannian tensor of V_n satisfies (II.10.1), the space V_n is called *harmonic*. So according to the type of tensor satisfying (II.10.1), we speak about:

a *harmonic space*, if

$$R^\alpha_{ijk\,,\,\alpha} = 0 \tag{II.10.2}$$

Ricci–harmonic space, if

$$R^\alpha_{i\,,\,\alpha} = 0, \tag{II.10.3}$$

Einstein–harmonic space, if

$$E^\alpha_{i\,,\,\alpha} = 0 \tag{II.10.4}$$

conformally–harmonic space, if

$$C^\alpha_{ijk\,,\,\alpha} = 0 \tag{II.10.5}$$

projectively–harmonic space, if

$$W^\alpha_{ijk\,,\,\alpha} = 0 \tag{II.10.6}$$

concircular–harmonic space, if

$$Y^\alpha_{ijk\,,\,\alpha} = 0 \tag{II.10.7}$$

and, finally, *Brinkmann–harmonic space*, if

$$P^{\alpha}_{i\,,\,\alpha} = 0, \tag{II.10.8}$$

here $P_{ij} = \dfrac{1}{n-2}\left(R_{ij} - \dfrac{R}{2(n-1)} g_{ij}\right)$ is the Brinkmann tensor, $P^i_j = g^{\alpha i} P_{\alpha j}$.

Note the following

Theorem II.10.1. *A pseudo-Riemannian space will be: Ricci–harmonic, Brinkmann–harmonic, Einstein–harmonic, if and only if the scalar curvature of the space is constant.*

From the Bianchi identity, we can easily see that the Ricci tensor of a Ricci–harmonic space V_n satisfies the Codazzi equations:

$$R^{\alpha}_{ijk\,,\,\alpha} = R_{ij\,,\,k} - R_{ik\,,\,j} \;. \tag{II.10.9}$$

Contracting the last formula we check $R_{,\,i} = 0$.

Theorem II.10.2. *A harmonic pseudo-Riemannian space is necessarily a space of constant scalar curvature.*

Considering the definitions of W^h_{ijk} and Y^h_{ijk}, we get

$$R^{\alpha}_{ijk\,,\,\alpha} = \frac{1}{n-1}\,(R_{ij\,,\,k} - R_{ik\,,\,j}) = 0, \tag{II.10.10}$$

$$R^{\alpha}_{ijk\,,\,\alpha} = \frac{1}{n(n-1)}\,(R_{,\,k} g_{ij} - R_{,\,j} g_{ik}) = 0. \tag{II.10.11}$$

Contracting once more we get

$$R^{\alpha}_{k\,,\,\alpha} = \frac{1}{n-1}\,(R_{,\,k} - R^{\alpha}_{k\,,\,\alpha}) = 0, \tag{II.10.12}$$

$$R^{\alpha}_{k\,,\,\alpha} = \frac{1}{n}\,R_{,\,k}. \tag{II.10.13}$$

From (II.10.12), (II.10.13) it follows $R_{,i} = 0$. Consequently we have

Theorem II.10.3. *The classes of harmonic, projective–harmonic and concircular–harmonic pseudo-Riemannian spaces are equivalent.*

Let us examine Einstein tensor preserving geodesic mappings of harmonic spaces.

The integrability conditions of (II.6.5), under the assumption that the geodesic mapping preserves the Einstein tensor, take the form (II.6.27), and for (II.6.24), accounting $R = \text{const}$, read

$$\lambda_\alpha Y^{\alpha}_{ijk} = 0. \tag{II.10.14}$$

Differentiating (II.6.27) and using the above formula, we get

$$\lambda_i Y_{hjkl} + \lambda_j Y_{hikl} + a_{i\alpha} Y^{\alpha}_{jkl\,,\,h} + \alpha_{j\alpha} Y^{\alpha}_{ikl\,,\,h} = 0. \tag{II.10.15}$$

Contracting with g^{hl}, we get

$$\lambda_i E_{jk} + \lambda_j E_{ik} = 0. \tag{II.10.16}$$

Alternating over i and k we find

$$\lambda_i E_{jk} - \lambda_k E_{ji} = 0. \tag{II.10.17}$$

Interchanging $j \leftrightarrow k$, we have

$$\lambda_i E_{kj} - \lambda_j E_{ik} = 0. \tag{II.10.18}$$

Composing the result with (II.10.16), we verify the following

Theorem II.10.4. *If a harmonic space admits a nontrivial Einstein tensor preserving geodesic mapping then the space is an Einstein space.*

As well known, the four-dimensional Einstein space distinct from spaces of constant curvature admits no non-trivial geodesic mappings. Hence we have

Corollary II.10.1. *A four-dimensional harmonic space, distinct from spaces of constant curvature, admits no non-trivial Einstein tensor preserving geodesic mapping.*

Note that in general, a harmonic space may admit a non-trivial geodesic mapping, as proved in the papers by V.S. Sobchuk, J. Mikesh and J. Radulovich [107, 172]: *there were found harmonic spaces admitting non-trivial geodesic mappings onto harmonic spaces.*

§ 11. Geodesic mappings of spaces with quasiconstant curvature

A pseudo-Riemannian space V_n $(n > 2)$ with the metric tensor g_{ij} is called a *space wit quasiconstant curvature* if its Riemannian tensor R^h_{ijk} satisfies:

$$R_{hijk} = \alpha(g_{hj}g_{ik} - g_{hk}g_{ij}) + \beta(\varphi_n\varphi_j g_{ik} - \\ -\varphi_n\varphi_k g_{ij} + \varphi_i\varphi_k g_{hj} - \varphi_i\varphi_j g_{hk}), \tag{II.11.1}$$

where $\alpha,\ \beta$ are vector invariants, and φ_i is the unit vector [25], [163].

Contracting (II.11.1) we check

$$R_{ij} = -(\alpha(n-1) + \beta)g_{ij} - \beta(n-2)\varphi_i\varphi_j. \tag{II.11.2}$$

From (II.11.2), we obtain for the scalar curvature $R = R_{\alpha\beta}g^{\alpha\beta}$:

$$R = -n(\alpha(n-1) + \beta) - \beta(n-2), \tag{II.11.3}$$

and (II.11.2) reads

$$R_{ij} = \frac{R}{n}g_{ij} + \frac{\beta(n-2)}{n}g_{ij} - \beta(n-2)\varphi_i\varphi_j. \tag{II.11.4}$$

From the last formula it follows that that for Einstein spaces, that is, for spaces satisfying $R_{ij} = \dfrac{R}{n}\, g_{ij}$, the function β is necessarily non-vanishing, and hence the space with quasiconstant curvature is necessarily a space of constant curvature.

The integrability conditions of the equations (II.6.5) are:

$$a_{\alpha i} R^{\alpha}_{jkl} + a_{\alpha j} R^{\alpha}_{ikl} = \lambda_{li} g_{jk} + \lambda_{lj} g_{ik} - \lambda_{kj} g_{il} - \lambda_{ki} g_{jl}. \qquad \text{(II.11.5)}$$

Alternating over $(i,\, k,\, l)$, we get

$$a_{\alpha i} R^{\alpha}_{jkl} + a_{\alpha k} R^{\alpha}_{jli} + a_{\alpha l} R^{\alpha}_{jik} = 0. \qquad \text{(II.11.6)}$$

From the last formula, contracting with g^{ij}, we get

$$a_{\alpha l} R^{\alpha}_{h} - a_{\alpha k} R^{\alpha}_{l} = 0. \qquad \text{(II.11.7)}$$

This formula was obtained by Sinyukov [96] with used alternation of formula (II.6.6).

Plugging (II.11.2) into (II.11.7) we verify that under the assumption $\beta \neq 0$ we obtain

$$\varphi^{\alpha} a_{\alpha i} = \rho \varphi_i, \qquad \text{(II.11.8)}$$

where ρ is a function, and $\varphi^h = \varphi_{\alpha} g^{\alpha h}$.

Hence we proved

Theorem II.11.1. *If a pseudo-Riemannian space with a quasiconstant curvature admits a geodesic mapping then the vecor φ_i satisfies the conditions (II.11.8).*

Differentiating (II.11.8), we obtain, using (II.6.5),

$$\varphi^{\alpha} \lambda_{\alpha} g_{ij} + \lambda_i \varphi_j + a_{\alpha i} \varphi^{\alpha}_{,\,j} = \rho_{,\,j} \varphi_i + \rho \varphi_{i,\,j}. \qquad \text{(II.11.9)}$$

Contracting (II.11.9) with φ^i we have

$$\rho_{,\,j} = 2\varphi^\alpha \lambda_\alpha \, \varphi_j, \qquad \text{(II.11.10)}$$

and consequently,

Theorem II.11.2. *The eigenvalue ρ corresponding to the eigenvector φ^α of the matrix a_{ij} is constant if and only if the vectors φ^h and λ_i are orthogonal.*

A solution of the equations (II.6.5), satsfying the conditions

$$a_{ij} = u\, g_{ij} + v\, R_{ij}, \qquad \text{(II.11.11)}$$

is called *canonical* [29].

Let us prove:

Theorem II.11.3. *In a pseudo-Riemannian space of a quasiconstant curvature, the only solutions of the equations (II.6.5) are just canonical solutions.*

Proof. By (II.11.1) and (II.11.8), the equations (II.11.5) read:

$$\begin{aligned} &\beta(\varphi_j\varphi_l a_{ik} - \varphi_j\varphi_k a_{il} + \varphi_i\varphi_l a_{jk} - \varphi_i\varphi_k a_{jl}) = \\ &= \Lambda_{li} g_{jk} + \Lambda_{lj} g_{ik} - \Lambda_{kj} g_{il} - \Lambda_{ki} g_{jl}. \end{aligned} \qquad \text{(II.11.12)}$$

Here we denoted

$$\Lambda_{li} \overset{def}{=} \lambda_{li} + \alpha a_{li} + \rho\beta\varphi_i\varphi_l. \qquad \text{(II.11.13)}$$

Contracting (II.11.13) with g^{jk}, we get

$$a\varphi_i\varphi_l - a_{il} = n\Lambda_{li} - \Lambda g_{il}, \qquad \text{(II.11.14)}$$

where we introduced

$$a \stackrel{def}{=} a_{\alpha\beta} g^{\alpha\beta}; \qquad \Lambda \stackrel{def}{=} \Lambda_{\alpha\beta} g^{\alpha\beta}.$$

We can express Λ_{li} as follows

$$\Lambda_{li} = \frac{\Lambda}{n} g_{li} + \frac{\beta a}{n} \varphi_i \varphi_l - \frac{\beta}{n} a_{il}. \tag{II.11.15}$$

Alternating (II.11.12) over i and k we have

$$\beta(\varphi_j \varphi_l a_{ik} - \varphi_j \varphi_i a_{lk} - \varphi_i \varphi_k a_{jl} + \varphi_l \varphi_k a_{ji}) = \\ = \Lambda_{lj} g_{ik} - \Lambda_{ij} g_{lk} - \Lambda_{ki} g_{jl} + \Lambda_{kl} g_{ji}. \tag{II.11.16}$$

Let us switch j and l in the last formula:

$$\beta(\varphi_l \varphi_j a_{ik} - \varphi_l \varphi_i a_{jk} - \varphi_i \varphi_k a_{jl} + \varphi_j \varphi_k a_{li}) = \\ = \Lambda_{lj} g_{ik} - \Lambda_{il} g_{jk} - \Lambda_{ki} g_{jl} + \Lambda_{kj} g_{li}. \tag{II.11.17}$$

Composing (II.11.17) and (II.11.12) we get:

$$\beta(\varphi_j \varphi_l a_{ik} - \varphi_i \varphi_k a_{jl}) = \Lambda_{lj} g_{ik} - \Lambda_{ik} g_{jl}. \tag{II.11.18}$$

By (II.11.15), the formula (II.11.18) can be written as

$$\varphi_j \varphi_l (n a_{ik} - a g_{ik}) - \varphi_i \varphi_k (n a_{jl} - a g_{jl}) = a_{ik} g_{lj} - a_{lj} g_{ik}. \tag{II.11.19}$$

We contract (II.11.19) with $\varphi^j \varphi^l$ and use (II.11.15). The resulting formula reads

$$(n-1)a_{ik} = (a-\rho)g_{ik} + (n\rho - a)\varphi_i\varphi_j. \tag{II.11.20}$$

If we put

$$u = \frac{\alpha\beta - 2\beta\rho + a\alpha - n\alpha\rho}{\beta(n-2)}, \tag{II.11.21}$$

$$v = \frac{a - n\rho}{\beta(n-1)(n-2)} \tag{II.11.22}$$

we finish the proof by means of (II.11.20) and (II.11.2). □

CHAPTER III

HOLOMORPHICALLY PROJECTIVE MAPPINGS OF KÄHLER SPACES PRESERVING THE EINSTEIN TENSOR

In this part we discuss holomorphically projective mappings of Kähler and pseudo-Kähler spaces which preserve the Einstein tensor. We prove that the tensor of h-concircular curvature is invariant under Einstein tensor-preserving holomorphically projective mappings.

§ 12. Kähler spaces

Recall that a (pseudo-) Riemannian space K_n is called a *(pseudo-) Kähler space* if it is endowed with a metric tensor g (which is positive definite in the Riemannian case and of arbitrary signature in the pseudo-Riemannian case) and at the same time with an affinor structure F (i.e. a tensor field of the type $(1,1)$) satisfying the following relations [72, 178, 96, 98]

$$F^2 = -\mathrm{Id}\,, \qquad g(X, FX) = 0\,, \qquad \nabla F = 0\,.$$

Here X are all tangent vectors of TK_n and ∇ is the metric connection of K_n. The structure F is a complex structure.

In local coordinates, the conditions are

$$F^h_\alpha F^\alpha_i = -\delta^h_i; \quad F_{(ij)} = 0; \quad F^h_{i,j} = 0 \tag{III.12.1}$$

where $F_{ij} = g_{i\alpha}F^\alpha_j$, $(i\ j)$ denotes symmetrization without division over i and j, " , " is a covariant derivative of the natural connection in K_n, and δ^h_i is the Kronecker symbol.

Let us note that first time, Kähler spaces has been studied by P.A. Shirokhov [123] who called them *A-spaces*, and later independently by E. Kähler [155], see [8, 96, 170, 178].

For convenience, we introduce in K_n the operation of conjugation:

$$A^{\cdots}_{\bar{i}\dots} \equiv A^{\cdots}_{\alpha\dots}F^\alpha_i; \qquad B^{\bar{i}\dots}_{\dots} \equiv B^{\alpha\dots}_{\dots}F^i_\alpha. \tag{III.12.2}$$

Here A and B are arbitrary tensors od any type.

According to (III.12.1) and (III.12.2) the following hold:

$$\begin{aligned} &A_{\bar{\bar{i}}} = -A_i; && B^{\bar{\bar{i}}} = -B^i;\\ &A_{\bar\alpha}B^\alpha = A_\alpha B^{\bar\alpha}; && A_{\bar\alpha}B^{\bar\alpha} = -A_\alpha B^\alpha;\\ &(A_{\bar i})_{,j} = A_{\bar i,j}; && (B^{\bar i})_{,j} = B^{\bar i}_{\ ,j}. \end{aligned} \tag{III.12.3}$$

The metric tensor and the Kronecker tensor satisfy

$$g_{\bar i\bar j} = g_{ij}; \quad g_{\bar i j} = -g_{i\bar j}; \quad \delta^h_{\bar i} = \delta^{\bar h}_i = F^h_i; \quad \delta^{\bar h}_{\bar i} = -\delta^h_i. \tag{III.12.4}$$

The Riemannian tensor and the Ricci tensor satisfy the additional identities

$$R_{\bar{h}\bar{i}jk} = R_{hijk}; \quad R^{\alpha}_{\bar{\alpha}jk} = 2R_{j\bar{k}}; \quad R_{\bar{i}\bar{j}} = R_{ij}, \qquad \text{(III.12.5)}$$

where $R_{hijk} = g_{h\alpha}R^{\alpha}_{ijk}$.

In Kähler spaces K_n, the following tensors are defined:

the tensor of holomorphically projective curvature

$$P^h_{ijk} \equiv R^h_{ijk} - \frac{1}{n+2}\,(\delta^h_{[k}R_{j]i} + \delta^h_{[\bar{k}}R_{\bar{j}]i} + 2\delta^h_{\bar{i}}R_{\bar{j}k}); \qquad \text{(III.12.6)}$$

the tensor of holomorphically-sectional curvature of K_n

$$H^h_{ijk} \equiv R^h_{ijk} - \frac{R}{n(n+2)}\,(\delta^h_{[k}g_{j]i} + \delta^h_{[\bar{k}}g_{\bar{j}]i} + 2\delta^h_{\bar{i}}g_{\bar{j}k}); \qquad \text{(III.12.7)}$$

the Bochner tensor of K_n

$$B^h_{ijk} \equiv R^h_{ijk} + (\delta^h_{[j}B_{k]i} + \delta^h_{[\bar{j}}B_{\bar{k}]i} + 2\delta^h_{\bar{i}}B_{\bar{j}k}) + B^h_{[j}g_{k]i} + B^h_{[\bar{j}}g_{\bar{k}]i} + 2B^h_{\bar{i}}g_{\bar{j}k} \qquad \text{(III.12.8)}$$

where

$$\begin{aligned} B^h_i &\equiv g^{h\alpha}B_{\alpha i}, \\ B_{ij} &\equiv \frac{1}{n+4}\left(R_{ij} - \frac{R}{2(n+2)}g_{ij}\right). \end{aligned} \qquad \text{(III.12.9)}$$

In the above formulas as well as in what follows, $[i,\, j]$ denotes alternation without division, $R = R_{\alpha\beta}g^{\alpha\beta}$ is the scalar curvature, and g^{ij} are elements of the matrix inverse to $\|g_{ij}\|$.

For the tensor of holomorphically projective curvature, the following is satisfied:

$$
\begin{aligned}
P_{hijk} = P_{hi\bar{j}\bar{k}} &= P_{\bar{h}\bar{i}jk} = -P_{hikj};\\
P_{h(ijk)} &= 0;\\
P^{\alpha}_{\alpha jk} = P^{\alpha}_{\bar{\alpha}jk} &= P^{\alpha}_{jk\alpha} = P^{\alpha}_{jk\bar{\alpha}} = 0,
\end{aligned}
\tag{III.12.10}
$$

where (i, j, k) denotes cycling in i, j, k. The tensor of holomorphically sectional curvature and the Bochner tensor satisfy [8], [72]:

$$
\begin{aligned}
&H_{hijk} = -H_{ihjk} = H_{jkhi} = H_{\bar{h}\bar{i}jk};\\
&H_{hijk} + H_{hjki} + H_{hkij} = 0;\\
&B_{hijk} = -B_{ihjk} = B_{jkhi} = B_{\bar{h}\bar{i}jk};\\
&B_{hijk} + B_{hjki} + B_{hkij} = 0.
\end{aligned}
\tag{III.12.11}
$$

Tensor of holomorphically projective curvature and Bochner tensor play an important role in the F-decompositions theory, see [159, 160, 169].

§ 13. Fundamental equations of the theory of holomorphically projective mappings

An analytic planar curve L of a Kähler space is a curve given by the equations $x^h = x^h(t)$ such that the following holds

$$\frac{d\xi}{dt} + \Gamma^h_{\alpha\beta}\xi^\alpha\xi^\beta = \varrho_1(t)\xi^h + \varrho_2(t)F^h_\alpha\xi^\alpha \qquad \text{(III.13.1)}$$

where $\xi^h \equiv dx^h(t)/dt$ and $\varrho_1(t)$, $\varrho_2(t)$ are functions.

Definition III.13.1. *A diffeomorphism f between Kähler spaces K_n and $\bar{K}_n$ is called a holomorphically projective mapping if f maps any analytical planar curve of K_n onto an analytical planar curve of the space $\bar{K}_n$.*

We can suppose that $\bar{M} = M$, due to the diffeomorphism f, where M is the "common" manifold on which the metrics g and $\bar{g}$ and the complex structure F of K_n and $\bar{K}_n$ are defined.

Any holomorphically projective mapping f from K_n onto $\bar{K}_n$ preserves the structures and is characterized by the following condition [98], [72]:

$$(\bar{\nabla} - \nabla)(X, Y) = \psi(X)Y + \psi(Y)X - \psi(FX)FY - \psi(FY)FX \qquad \text{(III.13.2)}$$

for any vector fields X, Y, where $\bar{\nabla}$ and ∇ are the metric connections of K_n and $\bar{K}_n$ ψ is a linear form.

Holomorphically projective mappings of K_n onto $\bar{K}_n$ are called *non-trivial* if $\psi \neq 0$. The case when the mapping is affine, i.e. $\psi \not\equiv 0$, is not considered here.

In coordinate system this equation has the following formula:

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \delta^h_i\psi_{j)} - \delta^h_{(\bar{i}}\psi_{\bar{j})}, \tag{III.13.3}$$

where necessarily $\psi_i \equiv \psi_{,i}$ and

$$\bar{F}^h_i = F^h_i.$$

Note that in [180] (see [8, 96, 72, 187, 204]), it was a priori supposed that the structure tensor should be preserved under holomorphically projective mappings. On the other hand, in [98] it was already proved that the structure tensor is necessarily preserved.

The conditions (III.13.3) can be equivalently written as [8, 72, 178, 96, 98]:

$$\begin{aligned}\nabla_Z\bar{g}(X,Y) = 2\psi(Z)\bar{g}(X,Y) + \psi(X)\bar{g}(Y,Z) + \psi(Y)\bar{g}(X,Z) \\ +\bar{\psi}(X)\bar{F}(Y,Z) + \bar{\psi}(Y)\bar{F}(X,Z)\end{aligned} \tag{III.13.4}$$

are satisfed where ∇ is the metric connection of K_n, ψ is a linear form and X, Y, Z are tangent vectors, $\bar{\psi}(X) = \psi(FX)$, $\bar{F}(X,Z) = \bar{g}(X, FZ)$.

Let us rewrite the equations $(III.13.4)$ in local coordinates:

$$\bar{g}_{ij,k} = 2\psi_k\bar{g}_{ij} + \psi_i\bar{g}_{jk} + \psi_j\bar{g}_{ik} + \psi_{\bar{i}}\bar{g}_{\bar{j}\,k} + \psi_{\bar{j}}\bar{g}_{\bar{i}\,k}, \tag{III.13.5}$$

where $\bar{g}_{ij}(x)$ and $\psi_k(x)$ are components of $\bar{g}$ and ψ, respectively, "," is the covariant derivative on K_n, $x = (x^1, x^2, \ldots, x^n)$ is a point of a coordinate neighborhood $U \subset M$. The equations $(III.13.4)$ and $(III.13.5)$ hold when K_n and $\bar{K}_n \in C^1$, i.e. if $g_{ij}(x)$ and $\bar{g}_{ij}(x) \in C^1$ in any coordinate neighbourhood U.

As well known, from (III.13.3) it follows

$$\bar{R}^h_{ijk} = R^h_{ijk} + \delta^h_k \psi_{ji} - \delta^h_j \psi_{ki} + \delta^h_k \psi_{\bar{j}i} - \delta^h_{\bar{j}} \psi_{\bar{k}i} + 2\delta^h_{\bar{i}} \psi_{\bar{j}k}; \tag{III.13.6}$$

$$\bar{R}_{ij} = R_{ij} + (n+2)\psi_{ij}; \tag{III.13.7}$$

where $R^h_{ijk}(\bar{R}^h_{ijk})$ and $R_{ij}(\bar{R}_{ij})$ are correspondingly the tensor of Riemann and Ricci of the space K_n $(\bar{K}_n)$ and

$$\psi_{ij} \equiv \psi_{i,j} - \psi_i \psi_j + \psi_{\bar{i}} \psi_{\bar{j}}. \tag{III.13.8}$$

It follows

$$\psi_{ij} = \psi_{ji} = \psi_{\bar{i}\bar{j}}. \tag{III.13.9}$$

The important invariants under holomorphically projective mappings are the so-called generalized Thomas' parameters

$$\begin{gathered} \bar{T}^h_{ij} = T^h_{ij}; \\ T^h_{ij} = \Gamma^h_{ij} - \tfrac{1}{n+2}(\delta^h_i \Gamma^\alpha_{j\alpha} + \delta^h_j \Gamma^\alpha_{i\alpha} - F^h_i F^\beta_j \Gamma^\alpha_{\beta\alpha} - F^h_j F^\beta_i \Gamma^\alpha_{\beta\alpha}) \end{gathered} \tag{III.13.10}$$

and the tensor of holomorphically projective curvature

$$\begin{gathered} \bar{P}^h_{ijk} = P^h_{ijk}; \\ P^h_{ijk} = R^h_{ijk} - \tfrac{1}{n+2}\,(\delta^h_k R_{ji} - \delta^h_j R_{ki} \\ -F^h_k F^\alpha_j R_{\alpha i} + F^h_j F^\alpha_k R_{\alpha i} + 2F^h_i F^\alpha_j R_{\alpha k}). \end{gathered} \tag{III.13.11}$$

On the other hand (Domashev and Mikeš [18]), if there exists a non-trivial holomorphically projective mapping of K_n onto $\bar{K}_n$ then in K_n the following equation can be solved:

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} + \lambda_{\bar{i}} g_{\bar{j}k} + \lambda_{\bar{j}} g_{\bar{i}k} \tag{III.13.12}$$

with respect to a tensor a_{ij} satisfying the equations

$$a_{ij} = a_{ji}; \quad a_{\bar{i}\bar{j}} = a_{ij}, \quad \det a_{ij} \neq 0 \tag{III.13.13}$$

and a non-zero vector λ_i. This vector must satisfy the equations

$$\lambda_{i,j} = \lambda_{j,i} = \lambda_{\bar{i},\bar{j}}. \tag{III.13.14}$$

Solutions of the equations (III.13.5) and (III.13.12) are related by

$$a_{ij} = e^{2\psi} \bar{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}, \tag{III.13.15}$$

$$\lambda_i = -e^{2\psi} \bar{g}^{\alpha\beta} g_{\alpha i} \psi_\beta, \tag{III.13.16}$$

where $\|\bar{g}^{ij}\| = \|\bar{g}_{ij}\|^{-1}$.

In 1978 J. Mikeš [12, 61] proved that necessary and sufficient condition for existence of non-trivial holomorphically projective mappings of the given pseudo-Kähler space onto pseudo-Kähler spaces is existence of a solution for the system of equations (see [72, 178, 96])

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} + \bar{\lambda}_i F_{jk} + \bar{\lambda}_j F_{ik}, \tag{III.13.17}$$

$$n\lambda_{i,j} = \mu g_{ij} + a_{\alpha i} R^\alpha_j - a_{\alpha\beta} R^\alpha{}_{ij}{}^\beta, \tag{III.13.18}$$

$$\mu_{,k} = 2\lambda_\alpha R^\alpha_k \tag{III.13.19}$$

with respect to a regular symmetric tensor a_{ij}, a co-vector $\lambda_i \neq 0$ and a function μ.

Here $R^i_j = R_{\alpha j} g^{\alpha i}$; $R^k{}_{ij}{}^h = R_{\alpha ij\beta} g^{\alpha k} g^{\beta h}$; $F_{ij} = F^\alpha_i g_{\alpha j}$, and g^{ij} are elements of the matrix inverse to g_{ij}.

§ 14. Basic equations for Einstein tensor – preserving holomorphically projective mappings

We call a holomorphically projective mapping *Einstein tensor-preserving* if it satisfies:

$$\bar{E}_{ij} = E_{ij}, \tag{III.14.1}$$

where

$$E_{ij} = R_{ij} - \frac{R}{n} g_{ij} \tag{III.14.2}$$

is the Einstein tensor and $R = R_{\alpha\beta} g^{\alpha\beta}$ is the scalar curvature.

If this is the case, the deformation tensor for the Ricci tensor takes the form:

$$T_{ij} = \bar{R}_{ij} - R_{ij} = \frac{\bar{R}}{n} \bar{g}_{ij} - \frac{R}{n} g_{ij}. \tag{III.14.3}$$

On the other hand, accounting (III.13.7) we obtain

$$T_{ij} = \bar{R}_{ij} - R_{ij} = (n+2)\psi_{ij}. \tag{III.14.4}$$

Comparing we get:

$$\psi_{ij} = \frac{\bar{R}}{n(n+2)}\, \bar{g}_{ij} - \frac{R}{n(n+2)}\, g_{ij}. \tag{III.14.5}$$

Substituting the last expression into (III.13.6) and using the notation

$$H^h_{ijk} = R^h_{ijk} - \frac{R}{n(n+2)} \left(\delta^h_k g_{ij} - \delta^h_j g_{ik} + F^h_k F^\alpha_i g_{\alpha j} - F^h_j F^\alpha_i g_{\alpha k} + 2\, F^h_i F^\alpha_j g_{\alpha k}\right) \tag{III.14.6}$$

(and similarly with bar) we find

$$\bar{H}^h_{ijk} = H^h_{ijk}. \tag{III.14.7}$$

Here H^h_{ijk} are components of the tensor of *holomorphically-sectional curvature*, where H is an analogue of the tensor of concircular curvature [61, 170, 178, 96, 204].

Hence we have proved:

Theorem III.14.1. *The tensor of holomorphically sectional curvature is invariant under all Einstein tensor-preserving holomorphically projective mappings.*

Let us apply covariant differentiation to the formula (III.12.5):

$$\lambda_{i,j} = -e^{2\psi}\psi_{\alpha,j}\bar{g}^{\alpha\beta}g_{\beta i} + e^{2\psi}\psi_\alpha\psi_\beta\bar{g}^{\alpha\beta}g_{ji} + e^{2\psi}\psi_j\psi_\alpha\bar{g}^{\alpha\beta}g_{\beta i}. \tag{III.14.8}$$

By (III.13.16) and (III.14.5), we get

$$\lambda_{i,j} = \mu g_{ij} + \frac{R}{n(n+2)}a_{ij}, \tag{III.14.9}$$

where

$$\mu = e^{2\psi}\left(\psi_\alpha\psi_\beta\bar{g}^{\alpha\beta} - \frac{\bar{R}}{n(n+2)}\right). \tag{III.14.10}$$

Obviously using (III.13.5), (III.12.5), from (III.14.8) and (III.14.9) we get (III.14.5), and consequently also (III.14.1), hence we have proved:

Theorem III.14.2. *A pseudo-Kähler space admits an Einstein tensor-preserving holomorphically projective mapping if and only if the conditions (III.13.17), (III.14.9) and (III.14.10) are satisfied.*

We say that a Kähler space K_n *belongs to the class* $K_n[B]$ if it admits a geodesic mapping and the corresponding vector satisfies [57, 178]

$$\lambda_{i,j} = \mu g_{ij} + Ba_{ij} \tag{III.14.11}$$

for some function B.

So we have actually proved that a Kähler space K_n admitting Einstein tensor-preserving holomorphically projective mappings belongs to the class $K_n[B]$ where $B = -\frac{R}{n(n+2)}$.

From (III.13.7), for Einstein tensors of the spaces K_n and $\bar{K}_n$ which are determined by

$$E_{ij} = R_{ij} - \frac{R}{n} g_{ij} \tag{III.14.12}$$

it follows

$$\bar{E}_{ij} - E_{ij} + \frac{\bar{R}}{n}\, \bar{g}_{ij} = \frac{R}{n}\, g_{ij} + (n+2)\psi_{ij}. \tag{III.14.13}$$

If the Einstein tensor is preserved under the holomorphically projective mapping, i.e.

$$\bar{E}_{ij} = E_{ij}, \tag{III.14.14}$$

then (III.14.13) reads

$$B\bar{g}_{ij} = Bg_{ij} + \psi_{ij} \tag{III.14.15}$$

where $B = \frac{R}{n(n+2)}$.

Plugging (III.14.15) into (III.13.6), using associativity and (III.12.7), we check

$$\bar{H}^h_{ijk} = H^h_{ijk}. \tag{III.14.16}$$

On the other hand, if the conditions (III.14.16) are satisfied then by contraction, we get (III.14.14). Hence we have proved

Theorem III.14.3. *The Einstein tensor is preserved under a holomorphically projective mapping if and only if the tensor of holomorphically sectional curvature is preserved.*

If we differentiate covariantly (III.13.14) and consider (III.13.13), (III.14.15) we get

$$\lambda_{i,j} = \mu g_{ij} + Ba_{ij}. \tag{III.14.17}$$

Examining the integrability conditions of (III.14.17) in detail we find

$$\mu_{,i} = 2B\lambda_i. \tag{III.14.18}$$

Hence we proved:

Theorem III.14.4. *If the given Kähler space K_n admits an Einstein tensor preserving holomorphically projective mappings then the system of equations* (III.13.12)*,* (III.14.17)*,* (III.14.18) *is solvable in K_n with respect to the tensor a_{ij}, the vector λ_i and the function μ.*

Kähler spaces in which the vector λ_i satisfies (III.14.17) are denoted by $K_n[B]$, and in this case, neessarily $B =$ const holds (see [57, 178]).

Considering the definition of a constant B, we check the following:

Corollary III.14.1. *If a Kähler space K_n admits a non-trivial Einstein tensor preserving holomorphically projective mappings then K_n is a space of constant scalar curvature.*

In this way, the methods developed in the theory of conformal and geodesic mappings of Riemannian spaces are transferred to the theory of holomorphically projective mappings of Kähler space. We received similar results for hyperbolic Kähler spaces [C4].

Publications of Author related to Thesis

C1. Chepurna O., Kiosak V., Mikes J. Conformal mappings of riemannian spaces which preserve the Einstein tensor. Journal of Applied Math., vol. 3, №1, 2010, 253–258.

C2. Chepurna, O.E.; Kiosak, V.A.; Mikeš, J. On geodesic mappings preserving the Einstein tensor. Acta Univ. Palacki. Olomuc., Fac. Rerum Nat., Math. 49, No. 2, 49-52 (2010).

C3. Chepurna, O.E.; Mikeš, J. Holomorphically projective mappings of Kähler spaces preserving the Einstein tensor. Journal of Applied Math., 2011; see: 10th Int. Conf APLIMAT 2011, 661-666.

C4. Chepurna, O.E.; Mikeš, J. Holomorphically projective mappings of hyperbolically Kähler spaces preserving the Einstein tensor. Journal of Applied Math., 2012; see: 11th Int. Conf APLIMAT 2012, 365-370.

C5. Чепурная Е.Е., Киосак В.А. Голоморфно-проективные отображения келеровых пространств с сохранением тензора Эйнштейна. (Holomorphically projective mappings of Kähler spaces preserving the Einstein tensor). Proc. Intern. Geom. Center 2010 3(4) 52-57.

C6. Чепурная Е.Е., Киосак В.А. Инвариантные преобразования с сохранением геодезических. (Invariant transformations with preserved geodesics). Proc. Intern. Geom. Center 2011 4(2) 36-42.

C7. Чепурная Е.Е., Киосак В.А. О слабо конциркулярно симметрических псевдоримановых пространствах. (On weakly concircular symmetric pseudo-Riemannian spaces). Proc. Intern. Geom. Center 2011 4(3) 15-22.

C8. Чепурная Е.Е., Лесечко А.В. Инвариантные объекты и моделирование с использованием специальных диффеоморфизмов псевдоримановых пространств. (Invariant object and modelling with used special diffeomorphisms of pseudo-Riemannian spaces). Informatics and Mathematical Methods in Simulation. 2011. Vol.1, 75-91.

C9. Чепурная Е.Е. О квазиконциркулярных отображениях псевдоримановых пространств с сохранением тензора Эйнштейна. Тез.докл.межд.конф. "Геометрия в Кислововодске-2010". - Кисловодск, 2010, - С.42.// Chepurna, O.E. On quasiconcircular mappings of pseudo-Riemannian spaces preserving the Einstein tensor. Abstr. Thes. of Int. Conf. *Geometry in Kislovods 2010*, Kislovods (Russia), 2010, p. 42.

C10. Чепурная Е.Е. Про голоморфно-проективные отображения келеровых пространств с сохранением тензора Эйнштейна (On holomorphically projective mappings of Kähler spaces preserving the Einstein tensor). Тез. докл. межд. конф. "Геометрия в Одессе-2010". - Одесса, 2010. - С. 28.

REFERENCES

1. Абдуллин В.Н. n-мерные римановы пространства, допускающие поля ковариантно постоянных симметрических тензоров общего типа// Изв. вузов Матем. I: 1970. - 5. - С. 3-13, - 6. С. 3-15.
   Abdullin, V.N. n-dimensional Riemannian spaces that admit covariantly constant symmetric tensor fields of general type. I, II. (Russian) Izv. Vyssh. Uchebn. Zaved., Mat. 1970, No.5(96), 3-13 (1970); ibid. 1970, No.6(97), 3-15 (1970).

2. Аминова А. В. Проективно-групповые свойства некоторых римановых пространств// Тр. геом. семин. - М.: ВИНИТИ, 1974. - 6. - С. 295-316.
   Aminova, A. V. Projective group properties of certain Riemannian spaces. (Russian) Trudy Geometr. Sem. 6 (1974), 295–316

3. Аминова А. В. Группы преобразований римановых многообразий// Итоги науки и техн. Проблемы геометрии. - М.: ВИНИТИ, 1990. - 22. - С. 97-166.
   Aminova, A. V. Transformation groups of Riemannian manifolds. (Russian) Translated in J. Soviet Math. 55 (1991), no. 5, 1996-2041. Itogi Nauki i Tekhniki, Problems in geometry, Vol. 22 (Russian), 97–165, 219, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1990.

4. Аминова А. В., Тогулева Т. П. Проективные и аффинные движения, определяемые конциркулярными векторными полями//Гравитация и теория относительности. - Казань: Изд-во Казанск. ун-та, 1975 (1976). - 10-11. - С. 139-153 .

   Aminova, A. V. Transformation groups of Riemannian manifolds. (Russian) Translated in J. Soviet Math. 55 (1991), no. 5, 1996-2041. Itogi Nauki i Tekhniki, Problems in geometry, Vol. 22 (Russian), 97–165, 219, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1990.

5. Аминова А. В. Инвариантно-групповые методы в теории проективных отображений пространственно-временных многообразий// Дисс....канд. физ.-мат. наук. - М.: 1991. - 390с.

   Aminova, A.V. Invariantly-groupps methods in theory of projective mappings space like times manifolds. DrSc. Thesis. Moscow, 390 p. (1993).

6. Аминова А. В. Псевдоримановы мнообразия с общими геодезическими// Успехи математических наук. 1993. 48, 2. - С. 107-164.

   Aminova, A. V. Pseudo-Riemannian manifolds with general geodesics. (Russian) Uspekhi Mat. Nauk 48 (1993), no. 2(290), 107–164; translation in Russian Math. Surveys 48 (1993), no. 2, 105–160.

7. Бачо Ш. Геодезические отображения финслеровых пространств и их обобщений// Дисс....канд. физ.-мат. наук. Одесса: ОГУ, 1985. - 100с.

   Bácsó, S. Geodesic mappings of Finsler spaces and its generalizations. PhD. Thesis, Odessa Univ., 100 p. (1985). (supervisor N.S. Sinyukov)

8. Беклемишев Д. В. Дифференциальная геометрия пространств с почти комплексной структурой// Итоги науки и техн. Геометрия. 1963, - М.: ВИНИТИ, 1965. - С. 165-212.

   Beklemišev, D.V. Differential geometry of spaces with almost complex structure. (Russian) Akad. Nauk SSSR Inst. Nauıın. Informacii, Moscow, 1965 Geometry, 165-212 (1963).

9. Березовский В.Е. О почти геодезических отображениях пространств аффинной связности// Дисс. ... канд.физ.-мат. наук. М. МПГУ, 1991. Berezovski, V.E. On almost geodesic mappings of spaces with affine connections. PhD. Thesis. Moscow Ped. Univ. (supervisor J. Mikeš) (1991).

10. Вишневский В. В. О параболическом аналоге А-пространств//Изв. вузов. Мат. - 1968. - 1. - С. 29-38.
Vishnevskij, V.V. A parabolic analogue of A-spaces. (Russian) Izv. Vyssh. Uchebn. Zaved., Mat. 1968, No.1(68), 29-38 (1968).

11. Вишневский В. В., Широков А. П., Шурыгин В. В. Пространства над алгебрами. - Казань: Изд-во Казанск. ун-та, 1985. - 262с.
Vishnevskij, V.V.; Shirokov, A.P.; Shurygin, V.V. Spaces over algebras. (Russian). Kazan': Izd. Kazansk. Univ. 263 p. (1985).

12. Гаврилъченко М.Л., Микеш Й. О решениях одной системы дифференциальных уравнений в ковариантных производных// Респ. науч. конф. "Дифференц. и интегр. уравнения и их приложения 1987/ Тез. докл. Ч. 1. - Одесса, ОГУ, 1987,- С. 55-56.
M. L. Gavrilchenko and J. Mikeš, On solutions of a system of differential equations in covariant derivatives, In: Resp. Sci. Conf. “Differ. and Integr. Equations and their Applications,” Abstracts of Reports. Part I, Odessk. Univ., Odessa (1987), pp. 55-56.

13. Голиков В.И. Поля тяготения с общими геодезическими// Дисс.... канд.физ.-мат. наук. Казань, 1963.
Golikov, V. I. Gravitational fields with common geodesics. (Russian) PhD. Thesis. Kazan Univ. 1963. (supervisor A.Z. Petrov).

14. Голиков В.И. Геодезические отображения гравитационных полей общего типа. Труды Семин. вект. и тенз. анализа, 12, 1963, 97-129.
Golikov, V. I. Geodesic mapping of gravitational fields of general type. (Russian) Tr. Semin. Vektor. Tenzor. Anal.

15. Голиков В.И. Поля тяготения с общими геодезическими// Ученые записки Казанского университета. Казань: Изд-во Казанск. ун-та, I: 1963. - 2 С. 72-95, II: 1963. - 12. С. 59-67.
Golikov, V. I. Gravitational fields with common geodesics. (Russian) Uch. spiski Kazanskogo Universit., Kazan, I: 1963, 2, 72-95, II: 1963, 12, 59-67

16. Горбатый Е. 3. О геодезическом отображении эквидистантных римановых пространств и пространств первого класса// Укр. геом. сб. - 1982. - 12. - С. 45-53. Gorbatyi, E.Z. On geodesic mapping of equidistant Riemannian spaces and first class spaces. (Russian) Ukr. geom. sb., 12, 45-53 (1982).

17. Грибков И. В. Одно характеристическое свойство римановых и псевдоримановых метрик с общими геодезическими// VIII Всес. науч. конф. по соврем. пробл. дифференц. геометрии/ Тез. докл. - Одесса, 1984. - С. 39.
Gribkov, I.V. A characteristic property of Riemannian and pseudo-Riemannian metric with general geodesics. (Russian) In abstract: VIII All Union Conf. modern diff. geom., Odessa, p. 39, (1984).

18. Домашев В.В., Микеш Й. К теории голоморфно-проективных отображений келеровых пространств// Мат. заметки. - 1978. - 23, 2. - С. 297-303. Domashev, V.V.; Mikeš, J. Theory of holomorphically projective mappings of Kählerian spaces. (English) Math. Notes 23, 160-163 (1978). Transl. from Mat. Zametki 23, 297-303 (1978).

19. Евтушик Л.Е., Киосак В.А., Микеш Й. О мобильности римановых пространств относительно конформных отображений на пространства Эйнштейна. Изв. вузов. Математка. 2010, No 8, 36-41.
Evtushika L.E., Kiosak V.A,Mikeš J. The mobility of Riemannian spaces with respect to conformal mappings onto Einstein spaces. Russian Math. (Iz. VUZ) 2010, 54, 8, 29-33. Transl. from Izv. vuzov, 2010, No 8, 36-41.

20. Евтушик Л.Е., Лумисте Ю.Г., Остиану Н.М., Широков А.П. Дифференциально-геометрические структуры на многообразиях// Итоги науки и техники. Сер. Проблемы геометрии. М. ВИНИТИ, 1979. - 9. - 246с.
Evtushik, L.E.; Lumiste, Yu.G.; Ostianu, N.M.; Shirokov, A.P. Differential-geometric structures on manifolds. (English. Russian original) J. Sov. Math. 14, 1573-1719 (1980); translation from Itogi Nauki Tekh., Ser. Probl. Geom. 9, 248 p. (1979).

21. Егоров И. П. Движения в пространствах с аффинной связностью. - Казань: Изд-во Казанск. ун-та, 1965. - 179 с.
Egorov, I.P. Motions in affine connected spaces. (Russian) Kazan Univ. Press, 5-179 (1965).

22. Егоров И. П. Движения в обобщенных дифференциально-геометрических пространствах// Итоги науки и техн. Алгебра. Топология. Геометрия. - М.: ВИНИТИ, 1967. - С. 375-428.
Egorov, I.P. Motions in generalized differential-geometric spaces. (English. Russian original) Progress in Mathematics Vol. 6: Topology and Geometry, 171-227 (1970); translation from Itogi Nauki Tekh., Ser. Algebra, Topologiya, Geom. 1965, 375-428 (1967).

23. Егоров И. П. Автоморфизмы в обобщенных пространствах// Итоги науки и техники. Сер. Проблемы геометрии. - М.: ВИНИТИ, 1978. - С. 147-191.
Egorov, I. P. Automorphisms in generalized spaces. (Russian) Problems in geometry, Vol. 10 (Russian), pp. 147–191, 224 (errata insert), VINITI, Moscow, 1978. MR0540266

24. Каган В. Ф. Основы теории поверхностей. I, II. М.-Л.: ОГИЗ, 1948. - 407с.
Kagan, V.F. Foundations of the theory of surfaces. (Russian) Moscow, Leningrad, OGIZ Gostechiz, Part 1, (1947); Part 2, (1948).

25. Каган В. Ф. Субпроективные пространства. - М.: Физматгиз, 1961. - 220с.
Kagan, V.F. Subprojective spaces. (Russian) Biblioteka Russkoi Nauki. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 220p. (1961).

26. Кайгородов В. Р. О римановых пространствах $^*K_n^s$// Тр. геом. семин. - М.: ВИНИТИ, 1974. - 5. - С. 359-373.
Kaigorodov, V. R. Riemannian spaces $^*K_n^s$. (Russian) Trudy Geometr. Sem. 5 (1974), 359-373, 376. (errata insert).

27. Кайгородов В. Р. Структура кривизны пространства-времени//Итоги науки и техн. Сер. Проблемы геометрии. - М.: ВИНИТИ, 1983. - 14. - С. 177-204.
Kaigorodov, V. R. Structure of the curvature of space-time. (Russian) Problems in geometry, Vol. 14, 177–204, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1983.

28. Кармазина А. В., Курбатова И. Н. О некоторых вопросах геодезических отображений почти эрмитовых пространств/ Одесск. пед. ин-т. -

Одесса, 1990. - 14 с. - Деп. в УкрНИИНТИ 12.03.90, 458-Ук90.
Karmasina, A.V.; Kurbatova, I.N. On some problems of geodesic mappings of almost Hermitian spaces. (Russian) Archives at Ukr.NIINTI(Kiev) 12.3.90, No. 458-Uk90, 14 p. (1990).

29. Киосак В. А. О геодезических отображениях специальных римановых пространств "в целом"/ Одесск. ун-т. - Одесса, 1989. - 12 с. - Деп. в УкрНИИНТИ 05.01.89, 176-Ук89.
V. A. Kiosak, On global geodesic mappings of special Riemannian spaces, Odessk. Univ. (1989); Deposited at Ukr. NIINTI, Kiev, 5.1.89, No. 176-Uk89 (1989).

30. Киосак В. А. О римановых пространствах L_n// III Всес. шк. Понтрягинские чтения. Оптим. управл., геометрия и анализ/Тез. докл. - Кемерово, 1990. - С. 33.
V. A. Kiosak, On Riemannian spaces L_n, In: III Vses. Shk. "Pontrjaginskie Chteniya Optim. Upravl., Geometriya i Analiz, Kemerovo (1990).

31. Киосак В. А. Об эквидистантных римановых пространствах// Геометрия обобщенных пространств. - Пенза, 1992. - С. 60-65.
Kiosak, V.A. On equidistant Riemannian spaces. (Russian) Geometria obobschennykh prostranstv, Penza, 60-65 (1992).

32. Киосак В. А., Курбатова И. Н., Микеш Й., Яблонская Н. В. Геодезические отображения и некоторые их обобщения// Тез. докл. межобл. науч.-практ. конф. мол. ученых, посвящ. 60-й годовщине образования СССР/ Тез. докл. Ч. П. – Одесса: ОГУ, 1983. - С. 6-7.
V. A. Kiosak, I. N. Kurbatova, J. Mikeš, and N. V. Yablonskaya, "Geodesic mappings and some of their generalizations,"In: Interregional Scientific

Working Conference of Young Scientists Dedicated to the 60th Anniversary of USSR, Abstracts of Reports, Part II (1983), pp. 6-7.

33. Киосак В. А., Микеш Й. О степени подвижности римановых пространств относительно геодезических отображений// Геометрия погруженных многообразий. - М., 1986. - С. 35-39.
V. A. Kiosak and J. Mikeš, On the degree of mobility of Riemannian spaces with respect to geodesic mappings, In: Geometriya Pogruzh. Mnogoobraz., Moscow (1986), pp. 35-39.

34. Киосак В. А., Микеш Й. Геодезические отображения и проективные преобразования римановых пространств// Межвуз. сб. науч. тр. "Движение в обобщенных пространствах". - Рязань: Рязан. гос. пед. ин-т, 1988. - С. 29-31.
V. A. Kiosak and J. Mikeš, Geodesic mappings and projective transformations of Riemannian spaces, In: Int. Sci. Proc.: Motion in General Spaces, Ryazansk. Ped. Inst., Ryazan' (1988), pp. 29-31.

35. Kiosak, V.A. Geodesic mappings of Riemannian spaces. (Russian) PhD. Thesis, Moscow Ped. Inst., (1994). (supervisor J. Mikeš)

36. Kiosak, V.A. Geodesic mappings of Riemannian spaces. (Russian and Czech) PhD. Thesis, Olomouc, (2002). (supervisor J. Mikeš)

37. Kiosak, V.A.; Haddad M. On A-harmonic spaces. (Russian) In: Geom. Obobshch. Prostr., Penza, 41-45 (1992).

38. Кручкович Г. И. Римановы и псевдоримановы пространства//Итоги науки и техн. Алгебра. Топология. Геометрия. 1966. - М., ВИНИТИ, 1968. - С. 191-220.
Kruchkovich, G.I. Riemannian and pseudo-Riemannian spaces. (Russian)

1968 Algebra. Topology. Geometry. 1966 (Russian) Akad. Nauk SSSR Inst. Naucn. Informacii, Moscow, 191–220 (1966).

39. Кручкович Г. И. О пространствах $V(K)$ и их геодезических отображениях// Тр. Всес. заочн. энергетич. ин-та. Мат. - 1967. - 33. - С. 3-18. Kruchkovich, G.I. On spaces $V(K)$ and their geodesic mappings. Trudy Vsesoyuzn. Zaochn. Energ. Inst., Moscow, 33. 3-18 (1967).

40. Leiko S.G. Mechanical interpretation of isoperimetric extremals of rotations on surfaces. (Russian. English summary) Visn. Odes. Derzh. Univ., Ser. Fiz.-Mat. Nauky 4, No.4, 79-82 (1999).

41. Leiko S.G. Isoperimetric rotation extremals on surfaces in Euclidean space E^3. (English. Russian original) Russ. Math. 40, No.6, 22-29 (1996); translation from Izv. Vyssh. Uchebn. Zaved., Mat. 1996, No.6(409), 25-32 (1996).

42. Leiko S.G. Infinitesimal rotational transformations and deformations of surfaces in Euclidean space. (English. Russian original) Dokl. Math. 52, No.2, 190-192 (1995); translation from Dokl. Akad. Nauk, Ross. Akad. Nauk 344, No.2, 162-164 (1995).

43. Leiko S.G. P-geodesic sections of tangent bundle. (English. Russian original) Russ. Math. 38, No.1, 29-39 (1994); translation from Izv. Vyssh. Uchebn. Zaved., Mat. 1994, No.1(380), 32-42 (1994).

44. Leiko S.G. Extremals of rotation functionals of curves in a pseudo-Riemannian space, and trajectories of spinning particles in gravitational fields. (English. Russian original) Russ. Acad. Sci., Dokl., Math. 46, No.1, 84-87 (1993); translation from Dokl. Akad. Nauk, Ross. Akad. Nauk 325, No.4, 659-663 (1992).

45. Leiko S.G. Rotational diffeomorphisms on Euclidean spaces. (English. Russian original) Math. Notes 47, No. 3, 261-264 (1990); translation from Mat. Zametki 47, No. 3, 52-57 (1990).

46. Leiko S.G. Planar connections on manifolds with commutative algebra of two almost product structures. (Russian. English summary) Uch. Zap. Kazan. Gos. Univ., Ser. Fiz.-Mat. Nauki 147, No. 1, 121-131 (2005).

47. Leiko S.G. Isoperimetric problems for rotation functionals of the first and second orders in (pseudo) Riemannian manifolds. (English, Russian) Russ. Math. 49, No. 5, 45-51 (2005); translation from Izv. Vyssh. Uchebn. Zaved., Mat. 2005, No. 5, 49-55 (2005).

48. Leiko S.G., Fedchenko, Yu.S. Infinitesimal rotary deformations of surfaces and their application to the theory of elastic shells. (Ukrainian, English) Ukr. Mat. Zh. 55, No. 12, 1697-1703 (2003); translation in Ukr. Math. J. 55, No. 12, 2031-2040 (2003).

49. Leiko S.G., Fedchenko Yu.S. Displacement vectors of rotary-conformal deformation on surfaces of rotation. (Ukrainian. English summary) Visn. Odes. Derzh. Univ., Ser. Fiz.-Mat. Nauky 8, No. 2, 50-54 (2003).

50. Leiko S.G., Sami Al-Hussin. Rotary quasiconcircular diffeomorphisms on the (pseudo) Riemannian spaces. (Ukrainian) Mat. Metody Fiz.-Mekh. Polya 44, No. 1, 22-27 (2001).

51. Leiko S.G. On the geodesic flow on a spherical tangent bundle of a two-dimensional manifold with the Sasaki metric. (Russian) Izv. Vyssh. Uchebn. Zaved., Mat. 2001, No.3, 33-38 (2001).

52. Leiko S.G., Vinnik A.V. Rotational conformal transformations of the Lobachevskij plane. (English. Russian original) Russ. Math. 44, No.9, 77-79

(2000); translation from Izv. Vyssh. Uchebn. Zaved., Mat. 2000, No.9, 79-81 (2000).

53. Vinnik, A.V.; Leiko S.G. Isoperimetric extremals of rotation functionals on two-dimensional connected Lie groups with invariant Riemannian metrics. (English. Russian original) Russ. Math. 44, No.7, 1-3 (2000); translation from Izv. Vyssh. Uchebn. Zaved., Mat. 2000, No.7, 3-5 (2000).

54. Лумисте Ю: Г. Полусимметрические многообразия// Итоги науки и техн. Проблемы геометрии. - М.: ВИНИТИ, 1991. - 23. - С. 3-28.
Lumiste, Yu. G. Semisymmetric submanifolds. (Russian) Translated in J. Math. Sci. 70 (1994), no. 2, 1609-1623. Itogi Nauki i Tekhniki, Problems in geometry, Vol. 23 (Russian), 3-28, 187, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1991. MR1152584

55. Микеш Й. Геодезические отображения полусимметрических римановых пространств/ Одесск. ун-т. - Одесса, 1976. - 19 с. - Деп. в ВИНИТИ 11.11.76, 3924-75Деп.
Mikeš, J. Geodesic mappings of semisymmetric Riemannian spaces. (Russian) Odessk. Univ. Moscow: Archives at VINITI, 11.11.76, No. 3924-76, 19 p. (1976).

56. Микеш Й. О некоторых классах римановых пространств замкнутых относительно геодезических отображений// VII Всес. конф. по соврем. пробл. геометрии/ Тез. докл. - Минск, 1979. - С. 126.
J. Mikeš, "On some classes of Riemannian spaces closed under to geodesic mappings,"In: VII All-Union Conf. Modern Diff. Geom., Minsk (1979).

57. Mikeš, J. Geodesic and holomorphically projective mappings of special Riemannian space. (Russian) PhD. Thesis, Odessa Univ., 107 p. (1979). (supervisor N.S. Sinyukov)

58. Микеш Й. О геодезических отображениях Риччи 2-симметрических римановых пространств// Мат. заметки. - 1980. - 28, 2. - С. 313-317.
Mikeš, J. On geodesic mappings of 2-Ricci symmetric Riemannian spaces. Math. Notes 28, 622-624 (1981). Transl. from Mat. Zametki 28, 313-317 (1980).

59. Микеш И. О геодезических отображениях пространств Эйнштейна// Мат. заметки. - 1980. - 28, 6. - С. 935-939.
Mikeš, J. On geodesic mappings of Einstein spaces. Math. Notes 28, 922-923 (1981). Transl. from Mat. Zametki 28, 935-938 (1980).

60. Микеш Й. Проективно симметрические и проективно рекуррентные пространства аффинной связности// Тр. геом. семин. - Казань: Казанск. ун-т, 1981.- 13. - С.61-62.
Mikeš, J. Projective-symmetric and projective-recurrent affine connection spaces. (Russian) Tr. Geom. Semin. 13, 61-62 (1981).

61. Микеш Й. О голоморфно-проективных отображениях келеровых пространств// Украинский геом. сборник. - 1980. - 23. - С. 90-98.
Mikesh, J. On holomorphically projective mappings of Kählerian spaces. (Russian) Ukr. Geom. Sb. 23, 90-98 (1980).

62. Микеш Й. Об эквидистантных келеровых пространствах// Мат.заметки. - 1985. - 38, 4. - С. 627-633.
Mikeš, J. Equidistant Kähler spaces. Math. Notes 38, No.4, 855-858 (1985). Transl. from Mat. Zametki 38, No.4, 627-633 (1985).

63. Микеш Й. О сасакиевых и эквидистантных келеровых пространствах// Докл. АН СССР. - 1986. - 291, 1. - С. 33-36.
Mikeš, J. On Sasaki spaces and equidistant Kähler spaces. Sov. Math., Dokl. 34, 428-431 (1987). Transl. from Dokl. Akad. Nauk SSSR 291, 33-36 (1986).

64. Микеш Й. О конциркулярных векторных полях "в целом"на компактных римановых поверхностях// Одесск. ун-т. - Одесса, 1988. -10 с. - Деп. в УкрНИИНТИ 02.03.88, 615-Ук88.
Mikeš, J. On global concircular vector fields on compact Riemannian spaces. Archives at Ukr. NIINTI, N. 615-Uk88, (1988).

65. Микеш Й. Об оценках порядков групп проективных преобразований римановых пространств// Мат. заметки. - 1988. - 43, 2. - С. 256-262.
Mikeš, J. Estimates of orders of groups of projective transformations of Riemannian spaces. Math. Notes 43, No.2, 145-148 (1988). Transl. from Mat. Zametki 43, No.2, 256-262 (1988).

66. Микеш Й. О существовании n-мерных компактных римановых пространств, допускающих нетривиальные проективные преобразования "в целом"// Докл. АН СССР. - 1989. - 305, 3. - С. 534-536.
Mikeš, J. On the existence of n-dimensional compact Riemannian spaces admitting nontrivial global projective transformations. Sov. Math., Dokl. **39**, 315-317 (1989). Transl. from Dokl. Akad. Nauk SSSR **305**, 534-536 (1989).

67. Микеш Й. О геодезических и квазигеодезических отображениях "в целом"римановых пространств с краем// Всес совещ. мол. ученых по дифференц. геометрии, посвящ. 80-летию Н. В. Ефимова/Тез. докл. - Абрау Дюрсо, Ростов-на-Дону, 1990. - С. 69.
J. Mikeš, On geodesic and quasigeodesic global mappings of Riemannian spaces with boundary, In: All-Union Colloquium of Young Scientists Dedicated to the 80th Anniversary of Birth of N.I. Efimov, Abstracts of Reports, Abrau Dyurso, Rostov-on-Don (1990).

68. Микеш Й. О геодезических и голоморфно-проективных отображениях обобщенно m-рекуррентных римановых пространств/Новосибирск, 1991. - 14 с. - Деп. в ВИНИТИ 12.03.91, 1059-В91.
J. Mikeš, On geodesic and holomorphically projective mappings of generalized m-recurrent Riemannian spaces, Sib. Mat. Zh. Novosibirsk (1991); Deposited at VINITI, 12.03.91, No. 1059-B91 (1991).

69. Микеш Й. О геодезических отображениях специальных римановых пространств// Респ. научно-метод. конф., посв. 200-летию со дня рождения Н. И. Лобачевского/ Тез. докл. Часть I. - Одесса, 1992. - С. 80.
J. Mikeš, On geodesic mappings of special Riemannian spaces, In: Resp. Scientific Working Conference Dedicated to the 200th Anniversary of Birth of N.I. Lobachevski, Abstracts of Reports, Part I, Odessa (1992), p. 80.

70. Микеш Й. О геодезических и голоморфно-проективных отображениях компактных полусимметрических и риччи-полусимметрических пространств// Междунар. науч. конф./Тез. докл. Часть I. - Казань: Казанск. ун-т, 1992. - С. 63-64.
J. Mikeš, On geodesic and holomorphically projective mappings of compact semisymmetric and Ricci semi-symmetric spaces, In: International Scientific Conference, Abstracts of Reports, Part. I, Kazan' (1992), pp. 63-64.

71. Микеш Й. Геодезические отображения аффинносвязных и римановых пространств// Итоги науки и техн. Сер. Соврем. мат. и ее прил. Темат. обз., 2002. - 11. С. 121-162. Перевод: Mikeš J. Geodesic mappings of affine-connected and Riemannian spaces. (English) J. Math. Sci., New York 78, No.3, 311-333 (1996).

72. Микеш Й. О геодезических и голоморфно-проективных отображениях компактных полусимметрических и риччи-полусимметрических про-

странств// Итоги науки и техн. Сер. Соврем. мат. и ее прил. Темат. обз., 2002. - 30. С. 258-289.

73. Микеш Й., Киосак В. А. О геодезических отображениях четырехмерных пространств Эйнштейна/ Одесск. ун-т. - Одесса, 1982. - 19 с. - Деп. в ВИНИТИ 09.04.82, 1678-82Деп.
Mikeš and V. A. Kiosak, On geodesic mappings of four-dimensional Einsteinian spaces, Odessk. Univ. (1982); Deposited at VINITI, 9.04.82, No. 1678-82 (1982).

74. Микеш Й., Киосак В. А. О геодезических отображениях специальных римановых пространств/ Одесск. ун-т. - Одесса, 1985. - 24 с. - Деп. в УкрНИИНТИ 05.05.85, 904-Ук85.
J. Mikeš and V. A. Kiosak, On geodesic mappings of special Riemannian spaces, Odessk. Univ. (1985), Deposited at Ukr. NIINTI, Kiev, 5.5.85, No. 904-85 (1985).

75. Микеш Й., Киосак В. А., Султанов А. Я. О специальных проективных векторах// Межвуз. науч.-практ. конф. мол. ученых/Тез. докл. Часть III. - Одесса: Одесск. ун-т, 1987. - С. 22.
J. Mikeš, V. A. Kiosak, and A. Ya. Sultanov, On special projective vectors, In: International Scientific Working Conference of Young Scientists, Abstracts of Reports, Part III, Odessa (1987).

76. Микеш Й., Молдобаев Д. О некоторых алгебраических свойствах тензоров// Исследования по неевклидовой геометрии и топологии/Сб. науч. тр. - Фрунзе: Изд-во Киргиз, ун-та, 1982. - С. 60-64.
J. Mikeš and Dj. Moldobayev, "On some algebraic properties of tensors,"In: Issled. po Neevklid. Geom., Kirghyz. Univ., Frunze (1982), pp. 60-64.

77. Микеш Й., Молдобаев Д. О распределении порядков групп конформных преобразований римановых пространств// Изв. вузов. Мат. - 1991. - 12. - С. 24-29.

Mikesh, I.; Moldobaev, D. Distribution of the orders of groups of conformal transformations of Riemannian spaces. (Russian) Izv. Vyssh. Uchebn. Zaved. Mat. 1991, 12, 24-29; translation in Soviet Math. (Iz. VUZ) 35 (1991), 12, 24-29. MR1205019

78. Микеш Й., Молдобаев Д., Сабыканов А. О рекуррентных эквиаффинных проективно-евклидовых пространствах// Исследования по топологии и геометрии/Сб. науч. тр. - Бишкек, 1991. - С. 46-52.

J. Mikeš, Dj. Moldobayev, and A. Sabykanov, On recurrent equiaffine projectively Euclidean spaces, In: Issled. po Topol. i Geometrii, Kirghyz. Univ., Bishkek (1991), pp. 46-52.

79. Микеш Й., Покасъ С. М. Группы Ли преобразований второго порядка в ассоциированных римановых пространствах/ Одесск. ун-т. - Одесса, 1981. - С. 21. - Деп. в ВИНИТИ 30.10.81, 4988-81Деп.

J. Mikeš and S. M. Pokas, "Lie-groups of second order transformations in associated Riemannian spaces,"Odessk. Univ. (1981); Deposited at VINITI, 30.10.81, No. 4988-81 (1981).

80. Микеш Й., Радулович Ж., Хаддад М. Геодезические и голоморфно-проективные отображения m-псевдо и m-квази симметрических римановых пространств// Изв. вузов. Мат. - 1996. - 3. - С. 30-35.

Mikesh, J.; Radulović, Ž.; Haddad, M. Geodesic and holomorphically projective mappings of m-pseudo- and m-quasisymmetric Riemannian spaces. (English. Russian original) Russ. Math. 40, No.10, 28-32 (1996); translation from Izv. Vyssh. Uchebn., Mat 1996, No.10(413), 30-35 (1996).

81. Микеш Й., Собчук В. С. О геодезических отображениях 3-симметрических римановых пространств// Укр. геом. сб. - 1991. - 34. - С. 80-83.
Mikesh, I.; Sobchuk, V. S. Geodesic mappings of 3-symmetric Riemannian spaces. (Russian) Ukrain. Geom. Sb. No. 34 (1991), 80-83, iii; translation in J. Math. Sci. 69 (1994), no. 1, 885-887. MR1224668

82. Микеш Й., Старко Г. А. О гиперболически сасакиевых и эквидистантных гиперболически келеровых пространствах// Укр. геом. сб. - 1989. - 32. - С. 92-98.
Mikesh, I.; Starko, G. A. On hyperbolically Sasakian and equidistant hyperbolically Kählerian spaces. (Russian) Ukrain. Geom. Sb. No. 32 (1989), 92-98; translation in J. Soviet Math. 59 (1992), no. 2, 756-760.

83. Микеш Й., Толобаев О. С. Симметрические и проективно-симметрические пространства аффинной связности// Исследования по топологии и обобщенным пространствам/ Сб. науч. тр. - Фрунзе: Изд-во Киргиз, ун-та, 1988. - С. 58-63.
J. Mikeš and O. S. Tolobayev, Symmetric and projectively symmetric spaces of an affine connection, In: Issled. po Topol. i Obobshch. Prostranstvam, Kirghyz. Univ., Frunze (1988), pp. 58-63.

84. Мирзоян В. А. Ric-полусимметрические подмногообразия// Итоги науки и техн. Проблемы геометрии. - М.: ВИНИТИ, 1991. - 23. - С. 29-66.
Mirzoyan, V.A. Ric-semisymmetric submanifolds. J. Math. Sci., New York 70, No.2, 1624-1646 (1994). Transl. from Itogi Nauki Tekh., Ser. Probl. Geom. 23, 29-66 (1991).

85. Норден А. П. Пространства аффинной связности. - М.; Наука, 1976. - 432 с.

Norden, A.P. Spaces of Affine Connection. Nauka, Moscow (1976).

86. Петров А. 3. Новые методы в общей теории относительности. - М.: Наука, 1966. - 495 с. Petrov, A.Z. New methods in the general theory of relativity. Moscow, Nauka, (1966).

87. Пилипосян В. А. О геодезическом отображении касательных расслоений римановых многообразий с метрикой полного лифта $(TM,^c g)$// Тр. геом. семин. - Казань: Казанск. ун-т, 1988. - 18. - С. 69-89.
V. A. Piliposyan, "On geodesic mappings of tangential stratifications of Sem., Vol. 18, Kazan' (1988), pp. 69-89.

88. Погорелое А. В. Об одной теореме Бельтрами// Докл. АН СССР. - 1991. - 316, 2. - С. 297-299.
Pogorelov, A.V. On a theorem of Beltrami. Dokl. Akad. Nauk SSSR, 316, No.2, 297-299 (1991).

89. Покасъ С. М., Яблонская К. В. О специальных почти геодезических отображениях афинносвязных и римановых пространств//Abstr. Colloq. Differ. Geom. Eger (Hung). - 1987-1988. - С. 45-50.

90. Радулович Ж., Микеш Й. О геодезических и F-планарных отображениях конформно-келеровых пространств// Междунар. науч. конф. "Лобачевский и современная геометрия"/ Тез. докл. Часть I. - Казань: Казанск. ун-т, 1992. - С. 81.

91. Рашевский П. К. Скалярное поле в расслоенном пространстве//Тр. семин. по вект. и тензорн. анализу. - 1948. - 6.
Rashevsky, P.A. Scalar fields in fibered space. Tr. Sem. Vekt. Tens. Anal., No. 6, Moscow State Univ. (1948).

92. Розенфельд Д. И. Геодезическое соответствие конформно-плоских римановых пространств// Укр. геометр, сб. - 1968 - 5-6. - С. 139-146.
Rosenfeld, D.I. Geodesic correspondence of conformally-flat Riemannian spaces. Ukr. geom. sb., 6, 5-6, 139-146 (1968).

93. Розенфельд Д. И., Горбатый Е. 3. О геодезических отображениях римановых пространств на конформно-плоские римановы пространства// Укр. геометр, сб. - 1972. - 12. - С. 115-124.
Rosenfeld, D.I.; Gorbaty, E.Z. On geodesic mappings of Riemannian spaces onto conformally flat Riemannian spaces. Ukr. geom. sb., 12, 115-124 (1972).

94. Сабыканов А., Микеш Й., Молдобаев Д. Рекуррентные эквиаффинные проективно-евклидовы пространства// Респ. науч.-метод. конф., посв. 200-летию со дня рождения Н. И. Лобачевского/Тез. докл., Часть I. - Одесса, 1992. - С. 80.
Sabykanov, A., MikeJ̌., Moldobayev Dj. Recurrent equiaffine projective-Eucleidian spaces. Resp. Conf. Odessa, 1992, p. 80

95. Sabykanov, A. Projective Euclidian and holomorphically projective flat recurrent spaces with affine connection. PhD. Thesis. Bishkek/Kyrgystan, (1995). (supervisors J. Mikeš and D. Moldobayev)

96. Синюков Н. С. Геодезические отображения римановых пространств. - М.: Наука, 1979. - 255 с.
Sinyukov, N.S. Geodesic mappings of Riemannian spaces. Moscow, Nauka, 256 p. (1979).

97. Синюков Н. С. Группы Ли проективных преобразований эквидистантных пространств// IX Всес. геометр, конф./ Тез. докл. - Кишинев, 1988. - С. 285-286.

Sinyukov, N.S. Lie-groups of projective transformations of equidistant spaces. In abstract: IX All Union Conf. of Geom., Kishineu, 285-286 (1988).

98. Синюков Н. С., Курбатова И. Н., Микеш Й. Голоморфно-проективные отображения келеровых пространств. - Одесса: Одесск. ун-т, 1985. - 69 с.
Sinyukov N.S.; Kurbatova, I.N.; Mikeš, J. Holomorphically projective mappings of Kähler spaces. Odessk. Univ., 69 p. (1985).

99. Синюков Н. С. Микеш Й. О геодезических и голоморфно-проективных отображениях некоторых римановых пространств//V Прибалт. геометр. конф./ Тез. докл. - Друскининкай, 1978. - С. 78.
N. S. Sinyukov and J. Mikeš, On geodesic and holomorphically projective mappings of some Riemannian spaces, In: Fifth Pribalt. Geom. Conf., Abstracts of Reports, Druskininkay (1978).

100. Синюков Н. С., Покась С. М. Группы движений второй степени в ассоциативном римановом пространстве// Движения в обобщенных пространствах/ Межвуз. сб. науч. тр. - Рязань, 1985. - С. 30-36.
Sinyukov, N.S.; Pokas', S.M. Groups of second degree motions in an associated Riemannian space. (Russian) Dvizheniya v obobshchenn. prostranstvakh, Ryazan', (1985).

101. Синюков Н. С., Синюкова Е. Н. О геодезических отображениях в целом обобщенно дважды рекуррентных римановых пространств//VII Прибалт. конф. по соврем. пробл. дифференц. геометрии/ Тез. докл. - Таллин, 1984. - С. 106-107.
Sinyukov, N.S.; Sinyukova, E.N. Holomorphically projective mappings of special Kähler spaces. Math. Notes 36, 706-709 (1984); . Transl. from Mat. Zametki 36, No.3, 417-423 (1984).

102. Синюкова Е. Н. О геодезических отображениях некоторых специальных римановых пространств// Мат. заметки. - 1981. - 30, 6. - С. 889-894. Sinyukova, E.N. Geodesic mappings of certain special Riemannian spaces. Math. Notes 30, 946-949 (1982). Transl. from Mat. Zametki 30, 889-894 (1981).

103. Синюкова Е. Н. О геодезических отображениях "в целом"некоторых специальных римановых пространств/ Одесск. ун-т. - Одесса, 1982. - 15 с. - Деп. в ВИНИТИ 20.07.82, 3892-82Деп.
Sinyukova E.N. On global geodesic mappings of some special Riemannian spaces. (Russian) Archives at VINITI 20.07.82, 3892–82, 15p. (1982).

104. Синюкова Е. Н. Геодезические отображения пространств L_n//Изв. вузов. Мат. - 1982. - 3. - С. 57-61.
Sinyukova, E.N. Geodesic mappings of L_n. Sov. Math. 26, No.3, 71-77 (1982). Transl. from Izv. Vuzov. Mat. 3, 57-61 (1982).

105. Синюкова Е.Н. О геодезических отображениях "в целом"римановых пространств, удовлетворяющих некоторым условиям типа неравенств// Межобл. науч.-практ. конф. мол. ученых, посвящ. 60-й годовщине образования СССР/ Тез. докл. Часть П. - Одесса: Одесск. ун-т, 1983. - С. 16-18.
Sinyukova, E.N.; Sinyukov, N.S. Infinitesimally small F-planar deformations of the metrics of Riemann spaces. (Russian) USSR Conf. on Geom. and Analysis, Novosibirsk, p. 75 (1989).

106. Синюкова Е. Н. Некоторые вопросы теории геодезических отображений римановых пространств "в целом"// IX Всес. геометр. конф./ Тез. докл. - Кишинев, 1988. - С. 286-287.
E. N. Sinyukova, Some problems of the theory of global geodesic mappings

of Riemannian spaces, In: Ninth All- Union Conference on Geometry, Abstracts of Reports, Kishinev (1988), pp. 286-287.

107. Собчук В. С. Риччи-обобщенно-симметрические римановы пространства допускают нетривиальные геодезические отображения//Докл. АН СССР. - 1982. - 267, 4. - С. 793-795.
Sobchuk, V.S. Ricci generalized symmetric Riemannian spaces admit nontrivial geodesic mappings. Sov. Math., Dokl. 26, 699-701 (1982). Transl. from Dokl. Akad. Nauk SSSR 267, 793-795 (1982).

108. Собчук В. С. Исследования по теории геодезических отображений и их обобщений// Дифференц.-геометр. шк./ Черновиц. ун-т, - Черновцы, 1991. - С. 196-208 Библиогр. 16 назв. Деп. в М.: ВИНИТИ 05.02.91, 562-В91.
V. S. Sobchuk, Investigations on the theory of geodesic mappings and their generalizations, In: Differents. - Geometr. Struct. na Mnogoobr., Chernovits. Univ., Chernovtzy (1991); Deposited at VINITI, 05.02.91, No. 562-91B (1991). pp. 196-208. 562-B91.

109. Собчук В. С. О геодезическом отображении Риччи 4-симметрических римановых пространств// Изв. вузов. Мат. - 1991. - 4. - С. 69-70.
Sobchuk, V.S. On the Ricci geodesic mapping of 4-symmetric Riemannian spaces. Sov. Math. 35, No.4, 68-69 (1991). Transl. from Izv. Vyssh. Uchebn. Zaved., Mat. 1991, No.4(347), 69-70 (1991).

110. Собчук В. С., Лайтарчук Д. Н., Стангрет О. Г. Геодезические отображения m-симметрических и обобщенно m-рекуррентных римановых пространств// Всес. конф. по геометрии "в целом"/Тез. докл. - Новосибирск, 1988. – С. 114.
V. S. Sobchuk, D. R. Laitarchuk, and O. G. Stangret, Geodesic mappings of

m-symmetric and generally m-recurrent Riemannian spaces, In: All-Union Conference on Geometry in the Large, Abstracts of Reports, Novosibirsk (1988), p. 114.

111. Солодовников А. С. Геодезические классы пространств V(K)//Докл. АН СССР. - 1956. - 3, 1. - С. 33-36.
A. S. Solodovnikov, Geodesic classes of spaces $V(K)$, Dokl. Akad. Nauk SSSR, 3, No. 1, 33-36 (1956).

112. Солодовников А. С. Геометрическое описание всевозможных представлений римановой метрики в форме Леви-Чивита// Тр. семин. по вект. и тензорн. анализу. - 1963. - 12. - С. 131-173.
A. S. Solodovnikov, Geodesic description of all possible forms of Riemannian metrics in the form of Levi-Civita, In: Tr. Ser. Vect. Tenz. Anal., Vol. 12 (1963), 131-173.

113. Схоутен И. А., Стройк Д. Дж. Введение в новые методы дифференциальной геометрии. - М.-Л.: Гостехиздат, 1939.
I. A. Schouten and D. J. Struik, Introduction to New Methods of Differential Geometry, Ch. 1 [In Russian], Gostekhizdat, Moscow-Leningrad (1939).

114. Ферапонтов Е. В. Автопреобразования по решению и гидродинамические симметрии// Диффер. уравн. - 1991. - 27, 7. - С. 1250-1263.
E. V. Ferapontov, "Autotransformations by solution and hydrodynamical symmetries,"Differents. Uravn., 27, No. 7, 1250-1263 (1991).

115. Фомин В. Е. О геодезическом отображении бесконечномерных римановых пространств на симметрические пространства аффинной связности// Тр. геом. семин. - Казань: Казанск. ун-т, 1979. - 11. - С. 93-99.
V. E. Fomin, On geodesic mappings of infinite-dimensional Riemannian

spaces onto symmetric spaces of an affine connection, In: Tr. Geometr. Sam., Vol. 11, Kazan' (1979), pp. 93-99. 84.

116. Фомин В. Е. Пара бесконечных пространств Леви-Чивита может не иметь общих геодезических// Тр. геом. семин. - Казань: Казанск. ун-т, 1986. - 17. - С. 79-83.
V. E. Fomin, A pair of infinite-dimensional Levi-Civita spaces can have no general geodesic, In: Tr. Geom. Semin., Vol. 17, Kazan' (1986), pp. 79-83.

117. Цыганок И. И. Торсообразующее векторное поле и группа аффинных гомотетий// Ткани и квазигруппы. - Калинин, 1988. - С 114-119.
I. I. Tsiganok, Torse-Forming Vector Fields and Groups of A ne Homotheties, Webs, and Quasigroup8 [In Russian], Kalinin (1988).

118. Шадыев Х. Проективные преобразования синектической связности в касательном расслоении// Изв. вузов. Мат. - 1987. - 9. - С. 75-77.
Shadyev, Kh. Projective transformations of a synectic connection in a tangent bundle. (English. Russian original) Sov. Math. 31, No.9, 91-95 (1987); translation from Izv. Vyssh. Uchebn. Zaved., Mat. 1987, No.9(304), 75-77 (1987).

119. Шандра И. Г. К вопросу о геодезических отображениях римановых пространств// Респ. научно-метод. конф., посв. 200-летию со дня рождения Н. И. Лобачевского/ Тез. докл. Часть I - Одесса, 1992. - С. 99-100.
I. G. Shandra, On the problem of geodesic mappings of Riemannian spaces, In: Rasp. Scientific Working Conference Dedicated to the 200th Anniversary of Birth of N.I. Lobachevski, Abstracts of Reports. Part I, Odessa (1992), pp. 99-100. 88.

120. Шандра И. Г. Пространства $V_n(K)$ и йордановы алгебры// Тр. геом. семин. Юбилейный сб. "Памяти Лобачевского посвящается". - Казань:

Казанск. ун-т, 1992. - С. 99-104.
Shandra, I.G. Spaces $V_n(K)$ and Jordan algebra. Dedicated to the memory of Lobachevskij, Kazan', N. 1, 99-104 (1992).

121. Шапиро Я. Л. О геодезических полях многомерных направлений//Докл. АН СССР. - 1941. - 32, 4. - С. 237-239.
Shapiro, Ya.L. On geodesic fields of many-dimensional directions. Dokl. Akad. Nauk SSSR, 32, No. 4, 237-239 (1941).

122. Широков А. П. Структуры на дифференцируемых многообразиях//Итоги науки и техн. Алгебра. Топология. Геометрия. 1967. - М.: ВИНИТИ, 1969. - С. 127-188.
Shirokov, A.P. Structures on differentiable manifolds. Itogi nauki: Algebra, topologiya, geometriya, 1967, VINITI, Moscow, pp. 121-188 (1969). Transl. from Progress in Math., vol. 9, Plenum Press, New York, (1971).

123. Широков П. А. Избранные работы по геометрии. - Казань: Изд-во Казанск. ун-та, 1966. - 432 с.
Shirokov, P.A. Selected investigations on geometry. Kazan' University press, 1966, 432 p. (1966).

124. Шиха М. Геодезические и голоиорфно-проективные отображения параболически келеровых пространств// Дисс. на соиск. уч. степени к.физмат. наук. Москва, 1992.

125. Shiha, M. Geodesic and holomorphically projective mappings of parabolically Kählerian spaces. PhD. Thesis, Moscow Ped. Inst. (1992). (supervisor J. Mikeš)

126. Шиха М., Микеш Й. Об эквидистантных параболически келеровых пространствах//Тр. геом. семин. 22, 97-107 (1994).

Shiha, M.; Mikeš, J. On equidistant, parabolically Kählerian spaces. (Russian) Tr. Geom. Semin. 22, 97-107 (1994).

English References

127. Afwat M., Svec A, Global differential geometry of hypersurfaces//Rozpr. CSAV. - 1978. - 88, 7. - C. 1-75.

128. Akbar-Zadeh H., Couty R. Espaces a tenseur de Ricci parailele admetant des transformations projectives// Rend. Mat. - 1978. - 11, 1. - C. 85-96.

129. Arsan G.G., Civi G. Geodesic mappings preserving the Einstein tensor of Weyl spaces. Journal of Applied Math., vol. 3, №1, 2012; see: 11th Int. Conf APLIMAT, 2012, 333-338.

130. Beltrami E. Risoluzione del probleraa: riportari i punti di una superficie sopra un piano in modo che le linee geodetiche vengano rappresentante da linee rette. Opere 1// Ann. Mat. - 1865. - 1, 7

131. Beltrami E. Teoria fondamente degli spazii di curvatura constante//Ann. Mat. - 1868. - 2, 2. - C. 232-255

132. Bochner S. Curvature in hermitian metric// Bull. Am. Math. Soc. - 1947. - 53. - C. 179-195

133. Brinkmann, H. W. Einstein spaces mapped conformally on each other. Ebenda [American M. S. Bull. 29], 215 (1923).

134. Brinkmann H. W. Riemann spaces conformal to Einstein spaces. Math. Ann. 91, 269-278 (1924).

135. Brinkmann H. W. Einstein spaces which are mapped conformally on each other. Math. Ann. 94, 119-145 (1925).

136. Cartan E. Locons sur la theorie des spaces a connexion projective. - Paris, 1931

137. Cartan E. Sur une classe remarquable d'espaces de Riemann//Bull. Soc. Math, France. - 1926. - 54. - C 214-264; 1927. - 55. - C. 114-134 (Пер. на рус. яз.: Картан Э. Геометрия групп Ли и симметрических пространств// Сб. работ. - М.: ИЛ, 1949)

138. Civi G.,Arsan G.G. On holomorphically projective mappings of Kähler-Weyl spaces. Journal of Applied Math., vol. 3, №1, 2012; see: 11th Int. Conf APLIMAT, 2012, 359-364

139. Coburn N. Unitary spaces with corresponding geodesic// Bull. Am. Math. Soc. - 1941. - 47. - C. 901-910

140. Couty R. Transformations projectives sur un espace d'Emstem complect// C. R. Acad. Sci. - 1961. - 252, 8. - C. 1096-1097

141. Couty R. Transformations projectives des varietes presque kahleriennes// C. R. Acad. Sci. - 1962. - 254, 24. - C. 4132-4134

142. Defever F., Deszcz R. A note on geodesic mappings of pseudosymmetric Riemannian manifolds// Colloq. Math. - 1991. - 62. - C. 313-319

143. Deszcz R., Hotlos M. On geodesic mappings in pseudo-symmetric manifolds// Bull. Inst. Math. Sinica. - 1988. - 16, 3. - C. 251-262

144. Dini U. Sobre un problema che is presenta nella theoria generale delle rappresentazioni geografiche di una superficie su di un'altra//Ann. Mat.- 1869. - 3

145. Eisenhart L. P. Riemannian geometry. - Princeton Univ. Press, 1926 (Пер. на рус. яз.: Эйзенхарт Л. П. Риманова геометрия. - М.: ИИЛ, 1948)

146. Fedorova A., Kiosak V., Matveev V., Rosemann S. The only Kähler manifold with degree of mobility at least 3 is $(CP(n), g_{Fubini-Study})$. Proc. London Math. Soc. Page 1 of 36, 2012. - C. 443-473

147. Fialkow A. Conformal geodesic// Trans. Am. Math. Soc. - 1939. - 45. - C. 443-473

148. Formella S. On gedesic mappings in some riemannian and pseudoriemannian manifolds// Tensor. - 1987. - 46. - C. 311-315

149. Formella S. Generalized Einstein manifolds// Rend. Circ. Mat. Palermo. - 1990. - 22. - C. 49-58

150. Fubini G. Sui gruppi transformazioni geodetiche// Mem. Acc. Torino. - 1903. - 2. - C. 261-313

151. Gavrilcenko M. L. Geodesic deformations of Riemannian spaces//Diff. Geom. and Its Appl. Int. Conf. Brno, 1989. - Singapore, 1990. - C. 47-53

152. Glodek W. A note on riemannian spaces with recurrent projective curvature// Pr. nauk. Inst, matem. i fiz. teor. Ser. stud, mater. - 1970. - 1. - C. 9-12

153. Gray A., Hervella L. M. The sixteen classes of almost Hermitian manifolds and their linear invariants// Ann. Mat. Pura Appl. - 1980. - 123, 4. - C. 35-58

154. Hinterleitner I., Mikeš J. Geodesic Mappings and Einstein Spaces. arXiv:1201.2827v1 [math.DG] 2012.

155. Kahler E. Uber eine bemerkenswerte Hermitische Metrik// Abh. Math. Semin. Hamburg. Univ. 1933. 9. 173-186.

156. Kiosak V. Matveev V. Complete Einstein metrics are geodesically rigid. Commun. Math. Phys. 289, No. 1, 383-400 (2009).

157. Kobayashi S. Transformations groups in differential geometry. - Berlin: Springer-Verlag, 1972. - 182 с. (Пер. на рус. яз.: Кобаяси Ш. Группы преобразований в дифференциальной геометрии. - М.: Наука, 1986. - 224 с.)

158. Kobayashi S., Nomizu K. Foundation of differential geometry. Vol 1. - N.Y.-L.: Interscience, 1963; Vol 2. - N.Y.–L.: Interscience, 1969. (Пер на рус. яз.: Кобаяси Ш., Номидзу К. Основы дифференциальной геометрии. - М.: Наука, 1981. - 1. – 344 с; 2. -. 416 с.)

159. Lakomá L., Jukl M. The decomposition of tensor spaces with almost complex structure. The proceedings of the 23th winter school "Geometry and physics Srní, Czech Republic, January 18–25, 2003. Palermo: Circolo Matema'tico di Palermo. Suppl. Rend. Circ. Mat. Palermo, II. Ser. 72, 145-150 (2004).

160. Lakomá L.; Mikeš J., Mikušová, L. The decomposition of tensor spaces. Differential geometry and applications. Proceedings of the 7th international conference, DGA 98, and satellite conference of ICM in Berlin, Brno, Czech Republic, August 10-14, 1998. Brno: Masaryk University. 371-378 (1999).

161. Levi-Civita T. Sulle transformationi delle equazioni dinamiche//Ann. Mat. Milano, Ser. 2. - 1896. - 24. - C. 255–300

162. Lichnerowicz A. Courbure, nombres de Betti et espaces symmetriques// Am. Math. Soc. - 1952. - 2. - C. 216–223

163. Lie Jian-cheng, Du Li. A note on the 2-harmonic submanifolds of quasi constant curvature spaces.// J. Northw. Norm. Univ. Natur. Sci. 2008. T. 44. №2. C. 18–21.

164. Mikeš J. O geodetickych transformacich polosymmetrickych nemannovych variet// Zb. anotacii, Kosice (Czech.). - 1977

165. Mikeš J. Geodesic mappings of special Riemannian spaces// Colloq. Math. Soc. J. Bolyai/ Top. in diff. geom. Debrecen, 1984 - Amsterdam, 1988. - 46, 2. - C. 793–813

166. Mikeš J. On an order of special transformation of Riemannian spaces// Proc. Conf. Diff. Geom. Appl. Dubrovnik, 1988. - C. 199-208

167. Mikeš J. On existence of nontrivial global geodesic mappings of n-dimensional compact surfaces of revolution// Diff. Geom. and Its Appl./ Proc. conf. Brno, 1989. - Singapore: World Scientific, 1990. - C. 129-137

168. Mikeš J., Berezovski V. Geodesic mappings of affine-connected spaces onto Riemannian spaces// Colloq. Math. Soc. J. Bolyai. Diff. geom. Eger (Hung.), 1989. - C. 491-494.

169. Mikeš J., Jukl M., Juklová L. Some results on traceless decomposition of tensors. J. of Math. Sci. 174, 5, 627-640. 2011.

170. Mikeš J., Kiosak V., Vanžurová A. Geodesic Mappings of Manifolds with Affine Connection. Olomouc: UP, 2008. 220p.

171. Mikeš, J.; Radulović, Ž. Geodesic mappings of conformally Kählerian spaces. Russ. Math. 38, No.3, 48-50 (1994). Transl. from Izv. Vyssh. Uchebn. Zaved., Mat. 1994, No.3 (382), 50-52 (1994).

172. Mikeš, J.; Radulović, Ž. Concircular and torse-forming vector fields “on the whole". (Russian) Math. Montisnigri 4, 43-54 (1995).

173. Mikeš, J.; Radulović, Ž. On geodesic and holomorphically projective mappings of generalized recurrent spaces. (Russian) Publ. Inst. Math., Nouv. Sér. 59(73), 153-160 (1996).

174. Mikeš, J.; Radulović, Ž. On projective transformations of Riemannian spaces with harmonic curvature. New developments in differential geometry, Budapest 1996. Dordrecht: Kluwer Acad. Publ. 279-283 (1999).

175. Mikeš, J.; Radulović, Ž.; Haddad, M. Geodesic and holomorphically projective mappings of m-pseudo- and m-quasisymmetric Riemannian spaces. Russ. Math. 40, No.10, 28-32 (1996). Transl. from Izv. Vyssh. Uchebn., Mat 1996, No.10(413), 30-35 (1996).

176. Mikeš, J.; Rachůnek, L. T-semisymmetric spaces and concircular vector fields. Palermo: Circolo Matematico di Palermo, Suppl. Rend. Circ. Mat. Palermo, II. Ser. 69, 187-193 (2002).

177. Mikeš, J.; Rachůnek, L. Torse-forming vector fields in T-semisymmetric Riemannian spaces. Proc. of the colloquium on diff. geom., Debrecen, Hungary, July 25-30, 2000. Debrecen: Univ. Debrecen, Institute of Mathematics and Informatics, 219-229 (2001).

178. Mikeš J., Vanžurová A., Hinterleitner I.. Geodesic mappings and some generalizations. Palacky University Press, Olomouc, 2009.

179. Mitsuru K. On projective diffeomorphismus not necessarily preserving complex structure// Math. J. Okayama Univ. - 1977. - 19, 2. - C. 183-191

180. Otsuki, T., Tashiro, Y. On curves in Kaehlerian spaces. Math. J. Okayama Univ. 4, 57-78 (1954).

181. Rachůnek, L.; Mikeš, J. On tensor fields semiconjugated with torse-forming vector fields. Acta Univ. Palacki. Olomuc., Fac. Rerum Nat., Math. 44, 151-160 (2005).

182. Reynolds R. F., Thompson A. H. Projective-symmetric spaces// J. Austral. Math. Soc. - 1967. - 7, 1. - C. 48-54

183. Roter W. A note infinitesimal projective transformations in recurrent spaces of second order// Zesz. Nauk. Politechn.Wroclawsk. - 1968. - 197. - C. 87-94

184. Simonescu C. Varietati Riemann in corepondenta geodezica definite pe un suport compact// Lucr. sti. Inst, politech. Brasov. Fac. mec. - 1961. - 5. - C. 15-19

185. Tachibana, Shun-Ichi; Ishihara, S. On infinitesimal holomorphically projective transformations in Кдhlerian manifolds. Tohoku Math. J., II. Ser. 12, 77-101 (1960).

186. Takeno H., Ikeda M. Theory of the spherically symmetric spacetimes. VII. Space-times with corresponding geodesies// J. Sci. Hiroshima Univ. - 1953. - A17, 1. - C. 75-81

187. Tashiro, Y. On a holomorphically projective correspondence in an almost complex space. Math. J. Okayama Univ. 6, 147-152 (1957).

188. Thomas T. Y. On projective and equiprojective geometries of paths// PWC. Nat. Acad. Sci. USA. - 1925. - 11 - C. 198-203

189. Venzi P. On geodesic mapping on Riemannian and pseudoriemannian manifolds// Tensor. - 1978. - 32, 2.- C. 192-198

190. Venzi P. Geodatische Abbildungen in Riemanscher Mannigfaltigkeiten// Tensor. - 1979. - 33. - C. 313-321

191. Venzi P. On concircular mapping in Riemannian and pseudo-Riemannian manifolds with symmetry conditions// Tensor. - 1979. - 33. - C. 109-113

192. Venzi P. On geodesic mappings in Riemannian and pseudo-Riemannian manifolds// Tensor. - 1979. - 33. - C. 23-28

193. Venzi P. Geodatische Abbildungen mit $\lambda_{ij} = \Delta g_{ij}$// Tensor. - 1979. - 34, 2. - C. 230-234

194. Venzi P. On q-projectively recurrent spaces// Rend. Circ. Math. Palermo. - 1981. - 30, 3. - C. 421-434

195. Venzi P. Uber konforme und geodatische Abbildungen// Result. Math. - 1982. - 5, 2. - C. 184-198

196. Venzi P. The metric $ds^2 = F(u)du^2 + G(u)d\sigma^2$ and an application to concircular mappings// Util. Math. - 1982. - 22. - C. 221-233

197. Venzi P. Klassifikation der geodatischen Abbildungen mit $\bar{Ric} - Ric = \Delta g$ // Tensor. - 1982. - 37. - C. 137-147

198. Venzi P. The geodesic mappings in Riemannian and pseudo-Riemannian manifolds// Stochastic processes in classical and quantum system/ Proc. Inst. Int. Ascona, Switz. 1985. Lect. Notes Phys. - 1986. - 262. - C. 512-516

199. Vranceanu G. Proprietati globale ale spatiilor bui Riemann cu conexiune abina constanta// Stud. si cerc. mat. Acad. RPR. - 1963. - 14, 1. - C. 7-22

200. Vries H. L. Uber Riemannsche Raume die infinitesimale konforme Transformationen gestatten// Math. Z. - 1954. - 60, 3. - C. 328-347

201. Westlake W. J. Hermitian spaces in geodesic correspondence//Proc. Am. Math. Soc. - 1954. - 5, 2. - C. 301-303

202. Weyl H. Zur Infinitesimalgeometrie Einordnung der projectiven und der konformen Auffassung// Gottinger Nachtr. - 1921. - C. 99-112

203. Yano K. Concircular geometry, I-IV// info Proc. Imp. Acad. Tokyo. - 1940. - 16. - C. 195-200; 354-360; 442-448; 505-511

204. Yano K. Differential geometry on complex and almost complex spaces. - Oxford: Pergamon Press, 1965. - 326 c.

205. Yano K. Sur la correspondence projective entre deux espaces pseudohermitens// C. R. Acad. Sci. - 1956. - 239. - C. 1346-1348

206. Yano K., Bochner S. Curvature and Betti numbers. - Princeton, New Jersey, 1953. - 190 c. (Пер. на рус. яз.: Яно К., Бохнер С. Кривизна и числа Бетти. М.: ИЛ, 1957. - 159 c.)

207. Yano K., Nagano T. Some theorems on projective and conformal transformations// Koninkl NederL Akad. Wet. - 1957. - A 60, 4. - C. 451-458