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TENSORS AND THEIR APPLICATIONS IN MECHANICS
TENSORY A JEJICH APLIKACE V MECHANICE

MASTER'S THESIS
DIPLOMOVÁ PRÁCE

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Pursuant to Act no. 111/1998 concerning universities and the BUT study and examination rules, you have been assigned the following topic by the institute director Master's Thesis:

Tensors and their applications in mechanics

Concise characteristic of the task:

The student elaborates a survey of basic concepts and properties of tensors and tensor fields including basic operations over them. The attention will be focused on applications, particularly in mechanics.

Goals Master's Thesis:

The aim is making familiar on deeper level with the theory of tensors and tensor fields from the point of view of differential geometry and their applications.

Recommended bibliography:

KOBAYASHI, N. Foundations of Differential Geometry, Wiley 1996, ISBN 0471157325.

HEINBOCKEL, J. H. Introduction to Tensor Calculus and Continuum Mechanics, Old Dominion Univ. 1996, ISBN 1-55369-133-4

NAIR, S. Introduction to Continuum Mechanics, Cambridge Univ. Press 2009, ISBN 13-97--0521187893

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In Brno,

L. S.

prof. RNDr. Josef Šlapal, CSc. Director of the Institute	doc. Ing. Jaroslav Katolický, Ph.D. FME dean

Abstract

The tensor theory is a branch of Multilinear Algebra that describes the relationship between sets of algebraic objects related to a vector space. Tensor theory together with tensor analysis is usually known to be tensor calculus. This thesis presents a formal category treatment on tensor notation, tensor calculus, and differential manifold. The focus lies mainly on acquiring and understanding the basic concepts of tensors and the operations over them. It looks at how tensor is adapted to differential geometry and continuum mechanics. In particular, it focuses more attention on the application parts of mechanics such as; configuration and deformation, tensor deformation, continuum kinematics, Gauss, and Stokes' theorem with their applications. Finally, it discusses the concept of surface forces and stress vector.

Keywords

Tensors, Manifolds, Differential manifolds, Configuration and deformation, Tensor deformation, Continuum kinematics, Gauss theorem, Stokes' theorem, Surface forces and stress

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I declare that I have worked on this thesis independently under a supervision of doc. RNDr Jiří Tomáš, Ph.D. and using the sources listed in the bibliography.

ADEJUMOBI MUDATHIR ADEWALE

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ADEJUMOBI MUDATHIR ADEWALE

Contents

1. INTRODUCTION TO TENSOR CALCULUS

Tensors were introduced by **Professor Gregorio Ricci** of University of Padua (Italy) in 1887 primarily as an extension of vectors. A quantity having magnitude only is called **scalar** and a quantity that has both magnitude and direction is called **vector** but certain quantities are associated with two or more directions, such a quantity is called **tensor** .

The stress at a point of an elastic solid is an example of a Tensor which depends on two directions one normal to the area and other that of the force on it [19]. Tensors have their applications to Riemannian geometry, mechanics, elasticity, the theory of relativity, electromagnetic theory, and many other disciplines of science and engineering.

An nth-rank tensor in m-dimensional space in a mathematical object that has n indices and m^n component and also obey some certain transformation rules. Generally $m = 3$. Each index of a tensor ranges over the number of dimensions of spaces.

We have tensors of various ranks: Scalar fields are referred to as the tensor field of rank or order zero (i.e has no index), a scalar (density, pressure, temperature, etc.) is a quantity whose specification (in any coordinate system) requires just one number. Vector fields are referred to as tensor fields of rank or order one (i.e has exactly one index), a vector (displacement, acceleration, force, etc.) is a quantity whose specification requires three numbers, namely its components with respect to some basis. A second-order tensor is called a dyad, a third-order tensor is a triad and tensors of order three or higher are called higher-order tensors.

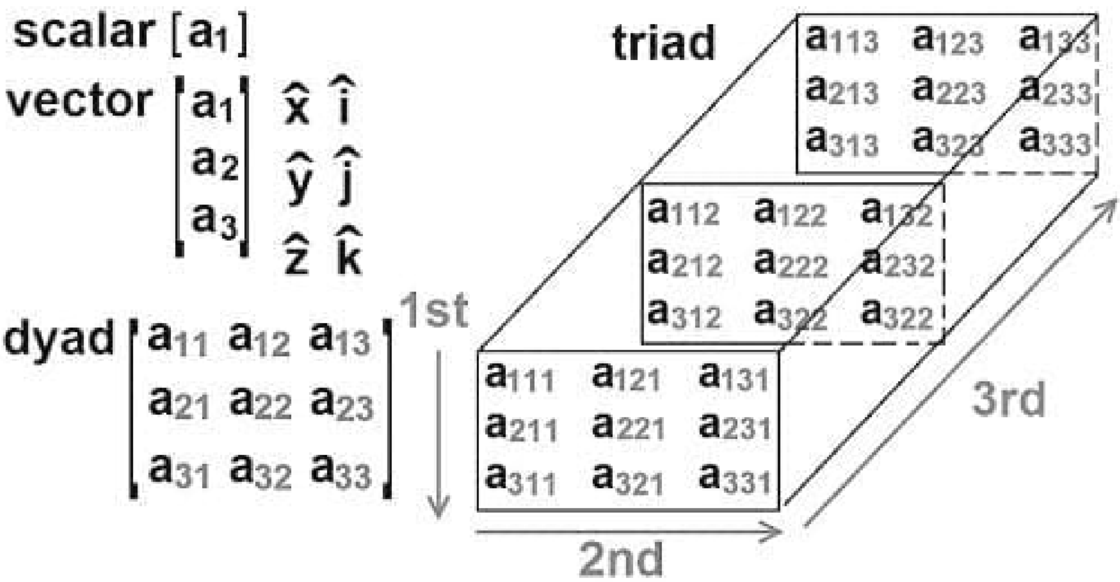

Figure 1.1: Tensor representation

1.1. Index Notation

Quantities which can be represented by a letter with subscripts or superscripts attached are known as a system but when these quantities obey some certain transformation rules as mentioned in the definition above then they are referred to as a tensor system [5].
For example:

$$A_{ij} \quad A_{ij}^{k} \quad A^{ijk} \quad B_j \quad b_{ij} \quad \delta_{ij} \quad e^{ijk}$$

and so on. But we focus our work mainly on the use of A_i^j or $A_{ij}x_j$. We talk about the later index $A_{ij}x_j$ in the next sub-topic. Note that the subscripts and superscripts are known as indices. They must be in lower case and must not be among the listed letters at the end of the English alphabet (u, v, w, x, y, and z) [5].
Index notation uses coordinates x_1, x_2, x_3 to denote x, y and z coordinates respectively. The components of a vector V would be v_1, v_2 and v_3 in $3D$.

1.2. Summation Convention over upper and lower indices

As far as matrix elements are concerned, index notation such as A_{12} is the element in the first row and the second column has been in use for some time. The advantage of index notation in conjunction with the summation convention is that we can write a long mathematical expression in a concise way [9].

Choosing a system of M equation in N unknowns:

$$\begin{gathered}
A_{11}x_1 + A_{12}x_2 + \cdots + A_{1N}x_N = C_1 \\
A_{21}x_1 + A_{22}x_2 + \cdots + A_{2N}x_N = C_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
A_{M1}x_1 + A_{M2}x_2 + \cdots + A_{MN}x_N = C_N
\end{gathered}$$

This system of equations can also be written in the form:

$$\sum_{j=1}^{N} A_{ij}x_j = C_i \quad where \quad i = 1, 2, \ldots..M \quad and \quad j = 1, 2, \ldots, N \tag{1.1}$$

In agreement with the Einstein summation convention we can further simplify the notation by writing:

$$A_{ij}x_j = C_i \qquad where \quad i = 1, 2, \ldots, M; \quad j = 1, 2, \ldots, N \tag{1.2}$$

where summation on the repeated index j is implied. The Einstein summation convention states that whenever there arises an expression where there is an index which occurs twice on the same side of any equation or term within an equation, it is clearly known to represent a summation on these repeated indices [5].

A repeated index is called the summation (**dummy index**) while the unrepeatable index is called the **free index**.

1.3. Symmetry and Anti-symmetry System

A system defined by subscripts and superscripts ranging over a set of values is said to be symmetric in two of its indices if the components are equal upon exchange of the index-values [5]. For example, the second order system T_{ij} is symmetric in the indices i and j if

$$T_{ij} = T_{ji} \text{ for all values of } i \text{ and } j$$
$$i, j = 1, 2, 3$$

A system defined by subscripts and superscripts ranging over a set of values is said to be anti-symmetric (skew-symmetric) in two of its indices if the components are equal but opposite upon exchange of the index-values [5]. For example, the second order system T_{ij} is anti-symmetric in the indices i and j if

$$T_{ij} = -T_{ji} \text{ for all values of } i \text{ and } j$$
$$i, j = 1, 2, 3$$

Note that:
In any skew-symmetric matrix, all the diagonal elements are zero.
Every tensor can be decomposed into sum of symmetric and anti-symmetric tensor.

$$T = T^A + T^S$$
where T^A and T^S are anti-symmetric and symmetric tensor respectively

$$T^S = \frac{T+T^T}{2} \text{ and } T^A = \frac{T-T^T}{2}$$
$$T = \frac{T+T^T}{2} + \frac{T-T^T}{2} = \frac{T+T^T+T-T^T}{2} = T$$

1.4. Order and Type of a System

Order of a system
The number of subscripts and superscripts determine the order of the system
1. A system with one index is a first-order system
2. A system with two indices is called a second-order system
3. A system with N indices s called a Nth order system
4. Lastly, a system with no indices is called a scalar or zeroth-order system

Type of system
The type of system depends on the number of subscripts or superscripts occurring in an expression.
For example, the system:
1. A^k_{ij} and B^n_{lm} are of the same type because they have the same number of subscripts and superscripts.
2. The system A^k_{ij} and C^{qr}_p are not the same type because one system has two superscripts and the other system has only one superscript.

1.5. Kronecker Delta and Permutation Symbol

Kronecker Delta
The Kronecker delta symbol is defined by

$$\delta_{ij} = \begin{cases} 1, & \text{if} \quad i = j \\ 0, & \text{if} \quad i \neq j \end{cases}$$

This definition assumes that i and j are explicit integers, such as $i = 1, 2, 3$ and $j = 1, 2, 3$ and it does not imply $\delta_{ii} = 1$. Hence, elements of the Kronecker delta is the same as the elements of the identity matrix [9].

$$\boldsymbol{I} = [\delta_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Permutation Symbol
The permutation symbol is defined by

$$e_{ijk} = \begin{cases} 1, & \text{if } i, j, k \text{ are even permutations of } 1, 2, 3 \\ -1, & \text{if } i, j, k \text{ are odd permutations of } 1, 2, 3 \\ 0, & \text{otherwise.} \end{cases}$$

From the definition above, we have that:

$$\begin{gathered} e_{ijk} = e_{jki} = e_{kij} = 1 \\ e_{ikj} = e_{jik} = e_{kji} = -1 \\ e_{ijj} = e_{jij} = e_{jij} = 0 \end{gathered}$$

Example

Consider the third-order system a_{prs} , $p, r, s = 1, 2, 3$ which is completely skew-symmetric in all of its indices. Show that the skew-symmetric systems have 27 elements, of which 21 elements are zero [5].

The 6 nonzero elements are all related to the ones given in the above definition

$$\begin{gathered} a_{prs} = a_{rsp} = a_{spr} = 1 \text{ which is the even permutation (clockwise)} \\ a_{psr} = a_{srp} = a_{rps} = -1 \text{ which is the odd permutation (anticlockwise)} \end{gathered}$$

The remaining 21 zero elements are:

$$a_{prr} = a_{rpr} = a_{rrp} = a_{pss} = a_{sps} = a_{ssp} = \ldots = a_{ppp} = a_{rrr} = a_{sss} = 0$$

1.6. Quadratic Forms, Eigenvalue and Eigenvector Problems

From the first part of our introduction chapter, a second-rank tensor (dyad or matrix), a homogeneous quadratic form [9] can be defined:

$$A = A_{ij} x_i x_j = x^T A x \tag{1.3}$$

If given an unsymmetrical (antisymmetric) matrix A_{ij} , we first change the unsymmetrical matrix to a symmetric matrix $A_{(ij)}$ while working with quadratic forms. The function A is called a homogeneous function of the second degree.

We can perform constrained extremization by defining a modified function A^* using a Lagrange multiplier λ in the form

$$A^* = A - \lambda (x_i x_i - 1) \tag{1.4}$$

This system of equations can be put in the form

$$Ax = \lambda x \quad \text{ or } \quad [A - \lambda I]x = 0 \tag{1.5}$$

Recall that, for a nontrivial solution, we require

$$\det[\boldsymbol{A} - \lambda \boldsymbol{I}] = 0 \tag{1.6}$$

Assuming we have a 3×3 matrix, when the preceding determinant is expanded, we get a cubic equation known as the **characteristic equation** of the matrix $\boldsymbol{A}$:

$$-\lambda^3 + I_{A1}\lambda^2 - I_{A2}\lambda + I_{A3} = 0 \tag{1.7}$$

where the coefficients I_{A1}, I_{A2}, and I_{A3} can also be written as

$$I_{A1} = A_{ii}, \quad I_{A2} = \frac{1}{2}(A_{ii}A_{jj} - A_{ij}A_{ij}), \quad I_{A3} = \det A$$

These coefficients are also known as the three invariants of the matrix A. If we denote the three roots of cubic equation by λ_1, λ_2, and λ_3, we have that:

$$\begin{aligned} I_{A1} &= \lambda_1 + \lambda_2 + \lambda_3 \\ I_{A2} &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 \\ I_{A3} &= \det(A) = \lambda_1\lambda_2\lambda_3 \end{aligned}$$

The relation between the extremum values of the quadratic form A and the Lagrange multipliers (eigenvalues) is as follows: Multiply the equation

$$\boldsymbol{A}\boldsymbol{x}^{(1)} = \lambda_1 \boldsymbol{x}^{(1)}$$

by $\boldsymbol{x}^{(1)T}$ to get

$$\boldsymbol{x}^{(1)T}\boldsymbol{A}\boldsymbol{x}^{(1)} = \lambda_1 \boldsymbol{x}^{(1)T}\boldsymbol{x}^{(1)} = \lambda_1$$

Hence, the three eigenvalues are the **extremum values** of the quadratic form along the principal directions. If all the eigenvalues are positive, the quadratic form is called **positive definite** and the matrix is called **a positive-definite matrix.**

1.6.1. Eigenvalue problem

Find the eigenvalues and eigenvectors of the given matrix

$$\boldsymbol{A} = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

The characteristic equation is obtained from

$$\det(A - \lambda I) = 0$$

solving this, we get

$$-(\lambda^3) + 7\lambda^2 - 10\lambda - 6 = 0$$

$$-(\lambda - 3)(\lambda_2 - 4\lambda - 2) = 0$$

$$\lambda_1 = 3, \quad \lambda_2 = -\sqrt{6} + 2 \quad \lambda_3 = \sqrt{6} + 2$$

The invariants of the matrix are

$$\begin{aligned} I_{A1} &= \lambda_1 + \lambda_2 + \lambda_3 = 7 \\ I_{A2} &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 10 \\ I_{A3} &= \det(A) = \lambda_1\lambda_2\lambda_3 = -6 \end{aligned}$$

we see that the eigenvalues are

$$\begin{aligned} \lambda_1 &= 3 \\ \lambda_2 &= -\sqrt{6} + 2 \\ \lambda_3 &= \sqrt{6} + 2 \end{aligned}$$

and the corresponding eigenvectors are

$$\boldsymbol{x}^{(1)} = \begin{Bmatrix} \frac{-1}{2} \\ 0 \\ 1 \end{Bmatrix}, \quad \boldsymbol{x}^{(2)} = \begin{Bmatrix} 2 \\ -\sqrt{6} - 1 \\ 1 \end{Bmatrix}, \quad \boldsymbol{x}^{(3)} = \begin{Bmatrix} 2 \\ \sqrt{6} - 1 \\ 1 \end{Bmatrix}$$

These eigenvectors are called the **principal directions** of matrix A, and the eigenvalues are the **principal values**.

Now,

$$\boldsymbol{x}^{(1)T} \boldsymbol{A} \boldsymbol{x}^{(1)} = \lambda_1 \boldsymbol{x}^{(1)T} \boldsymbol{x}^{(1)} = \lambda_1$$

Solving this, we get $3 = \lambda_1$. Hence, the three eigenvalues are the **extremum values** of the quadratic form along the principal directions.

1.6.2. Diagonalization and Polar Decomposition

Using the three eigenvectors as the three columns
We can construct the modal matrix of the given matrix $\boldsymbol{A}$ by denoting the modal matrix by $\boldsymbol{M}$[9]. Then,

$$\boldsymbol{M} = \left[\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \boldsymbol{x}^{(3)}\right] \tag{1.8}$$

It is clear that

$$\boldsymbol{M}^T \boldsymbol{A} \boldsymbol{M} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{1.9}$$

Choosing a principal coordinate system x'_i, using

$$x_i = M_{ij} x'_j \quad \text{or} \quad x' = M^T x \tag{1.10}$$

Multiplying Equation (1.9) by M from the left and by M^T from the right

We can then express our matrix $\boldsymbol{A}$ as

$$\boldsymbol{A} = \sum_{i=1}^{3} \lambda_i \boldsymbol{x}^{(i)} \boldsymbol{x}^{(i)T}$$

This is called the **spectral representation** of $\boldsymbol{A}$

From the previous example above, the modal matrix of matrix A

$$\boldsymbol{M} = \begin{bmatrix} \frac{-1}{2} & 2 & 2 \\ 0 & -\sqrt{6}-1 & \sqrt{6}-1 \\ 1 & 1 & 1 \end{bmatrix}$$

Then

$$\boldsymbol{M}^T\boldsymbol{A}\boldsymbol{M} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -\sqrt{6}+2 & 0 \\ 0 & 0 & \sqrt{6}+2 \end{bmatrix}$$

1.6.3. Polar Decomposition

The polar decomposition applies to any square matrix $\boldsymbol{B}$. Polar decomposition refers to factoring $\boldsymbol{B}$ in the form [9]

$$\boldsymbol{B} = \boldsymbol{R}\boldsymbol{U} \quad \text{or} \quad \boldsymbol{B} = \boldsymbol{V}\boldsymbol{R}$$

where $\boldsymbol{U}$ and $\boldsymbol{V}$ are symmetric matrices and $\boldsymbol{R}$ is an orthogonal (rotation) matrix.

Using $\boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}$, we get

$$\boldsymbol{U}^2 = \boldsymbol{B}^T\boldsymbol{B} \quad \text{and} \quad \boldsymbol{V}^2 = \boldsymbol{B}\boldsymbol{B}^T$$

Find the square root of the matrices on RHS to get $\boldsymbol{U}$ or $\boldsymbol{V}$

Then $\boldsymbol{R}$ can be expressed

$$\boldsymbol{R} = \boldsymbol{B}\boldsymbol{U}^{-1} \quad OR \quad \boldsymbol{R} = \boldsymbol{V}^{-1}\boldsymbol{B}$$

This brings us to finding the square root of a symmetric matrix, say $\boldsymbol{C}$ or, in general, any function $\boldsymbol{F}[\boldsymbol{C}]$ of a matrix.

We begin by assuming the function $\boldsymbol{F}[\boldsymbol{C}]$ has a converging infinite series expansion in C :

$$\boldsymbol{F}[\boldsymbol{C}] = \sum_{0} a_l \boldsymbol{C} \tag{1.11}$$

The functions we have in mind are $\boldsymbol{C}^{1/2}, \sin[\boldsymbol{C}], \exp[\boldsymbol{C}]$, etc. The corresponding functions of a single variable, say x, are $x^{1/2}, \sin x, \exp x$, etc.

The generic matrix function F has the corresponding function of a single variable F. Our symmetric matrix C has three orthogonal eigenvectors $\boldsymbol{x}^{(i)}$ with the corresponding eigenvalues λ_i.

Multiplying Equation (1.11) from the right by $x^{(i)}$, we see that $F(\lambda_i)$ is the eigenvalue of F corresponding to the eigenvector, $\boldsymbol{x}^{(i)}$. The eigenvalues of $[\boldsymbol{C}]^{1/2}, \sin[\boldsymbol{C}]$, and $\exp[\boldsymbol{C}]$ arc $\lambda_i^{1/2}, \sin\lambda_i$, and $\exp\lambda_i$, respectively. Next, we use the Cayley-Hamilton theorem to reduce Equation (1.11) to a quadratic in C (provided we are working with 3×3 matrices):

$$\boldsymbol{F}[\boldsymbol{C}] = c_0\boldsymbol{I} + c_1\boldsymbol{C} + c_2\boldsymbol{C}^2 \tag{1.12}$$

Then the eigenvalues satisfy

$$F(\lambda_i) = c_0 + c_1\lambda_i + c_2\lambda_i^2, \quad i = 1,2,3$$

For example, to find the square root of $\boldsymbol{C}$, we use

$$\lambda_i^{1/2} = c_0 + c_1\lambda_i + c_2\lambda_i^2, \quad i = 1,2,3$$

and solve for c_i.

We substitute these coefficients in Equation (1.12) to get $C^{1/2}$ which is clearly $\boldsymbol{U}$ and then get $\boldsymbol{R}$ and $\boldsymbol{V}$.

Example

[9] Given the matrix B

$$\boldsymbol{B} = \frac{1}{5}\begin{bmatrix} 17 & -11 & 0 \\ 19 & 23 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

We know that

$$\boldsymbol{U}^2 \equiv \boldsymbol{B}^T\boldsymbol{B}$$

The factors $\boldsymbol{U}, \boldsymbol{V}$, and $\boldsymbol{R}$ are obtained from

$$\boldsymbol{U}^2 = \boldsymbol{C} \equiv \boldsymbol{B}^T\boldsymbol{B} = \begin{bmatrix} 26 & 10 & 0 \\ 10 & 26 & 0 \\ 0 & 0 & 9 \end{bmatrix}, \quad \boldsymbol{R} = \boldsymbol{B}\boldsymbol{U}^{-1}, \quad \boldsymbol{V} = \boldsymbol{B}\boldsymbol{R}^T$$

$$-\lambda^3 + 61\lambda^2 - 1044\lambda + 5184 = -(\lambda-9)\left(\lambda^2 - 52\lambda + 576\right) = -(\lambda-9)(\lambda-16)(\lambda-36)$$

The eigenvalues of $\boldsymbol{C}$ are

$$\lambda_1 = 9, \quad \lambda_2 = 16, \quad \lambda_3 = 36$$

Using the expansion

$$\sqrt{\lambda_i} = c_0 + c_1\lambda_i + c_2\lambda_i^2$$

we have the system of equations

$$3 = c_0 + 9c_1 + 9^2c_2, \quad 4 = c_0 + 16c_1 + 16^2c_2, \quad 6 = c_0 + 36c_1 + 36^2c_2$$

Solving this system of equations, we have

$$c_0 = 52/35, \quad c_1 = 23/126, \quad c_2 = -1/630$$

Writing

$$\boldsymbol{C}^{1/2} = c_0\boldsymbol{I} + c_1\boldsymbol{C} + c_2\boldsymbol{C}^2$$

we get

$$\boldsymbol{U} = \boldsymbol{C}^{1/2} = \begin{bmatrix} 5 & 1 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

we can now solve for $\boldsymbol{R}$ and $\boldsymbol{V}$

$$\boldsymbol{R} = \frac{1}{5}\begin{bmatrix} 4 & 3 & 0 \\ -3 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

and

$$\boldsymbol{V} = \frac{1}{25}\begin{bmatrix} 101 & 7 & 0 \\ 7 & 149 & 0 \\ 0 & 0 & 75 \end{bmatrix}$$

2. GENERAL TENSORS

Aside from the number of indices, the position of the indices matters a lot. The coordinates of a vector are enumerated by one upper index, which is called the **contravariant index** . The coordinates of a co-vector are enumerated by one lower index, which is called the **covariant index** . In bilinear form, we use two lower indices hence, bilinear forms are called **twice-covariant tensors**. Linear operators are tensors of mixed type, their components are enumerated by one upper and one lower index. The number of indices and their positions determines the transformation rules. In the general case, any tensor is represented by a multidimensional array with a definite number of upper indices and a definite number of lower indices [12].

Let's denote these numbers by r and s. Therefore, we have a tensor of the type (r, s), (or sometimes the term valency is used). A tensor of type (r, s), (or valency (r, s)) is called an **r-times contravariant**and an **s-times co-variant tensor**.

2.1. Some Definitions

Definition 2.1.1 *If A is any point in space and B is another point then, a directed straight line segment from A to B is called a Line vector.*

Definition 2.1.2 *Vectors are elements of a finite dimensional space V over reals, practically $\mathbb{R}^n$.*

Definition 2.1.3 *Covectors are liner forms. Covectors are function (linear function) $\alpha : V \to \mathbb{R}$ that maps vectors to a real number and also obey the following rules:*
1. $\alpha\left(\vec{v}+\vec{w}\right) = \alpha(\vec{v}) + \alpha(\vec{w})$
2. $\alpha(n\vec{v}) = n\alpha(\vec{v})$

Definition 2.1.4 *Dual Vector is the set of all covectors that act on a vector space V together form the vector space $V^\star$ and these covectors have their own adding and scaling rules:*
1. $(n\alpha)\vec{v} = n\alpha(\vec{v})$
2. $(\beta+\gamma)(\vec{v}) = \beta(\vec{v}) + \gamma(\vec{v})$

Definition 2.1.5 *Dual basis of the dual space V^* is as follows:*

If $\vec{e}_i$ form a basis $\mathcal{E}$ of V, then $\vec{e^i}$ defined by $\vec{e^i}\left(\overrightarrow{e_j} = \delta^i_j\right)$ form the basis of V^*, the so-called dual basis. There are linear isomorphisms $f_{\mathcal{E}} : V \to V^*$ defined by $\vec{e}_i \mapsto \vec{e^*}$ depending on the choice of basis but there are not isomorphisms independent on the choice of basis.

Nevertheless, there is a linear isomorphism $F : V \to (V^*)^*$ defined by $F(\vec{v})(f) = f(\vec{v})$ independent on the choice of a basis (by F we have denoted an element of V^*

2.2. Vector Identities in Cartesian Coordinates

Let $x^1 = x, x^2 = y, x^3 = z$, where superscript variables are employed. Also denote the unit vectors in Cartesian coordinates by $\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2, \widehat{\mathbf{e}}_3$, we try to explain how various vector operations are written by using the index notation [5].

Definition 2.2.1 *Gradient*
In Cartesian coordinates the gradient of a scalar field is

$$\operatorname{grad}\phi = \frac{\partial\phi}{\partial x}\widehat{\mathrm{e}}_1 + \frac{\partial\phi}{\partial y}\widehat{\mathrm{e}}_2 + \frac{\partial\phi}{\partial z}\widehat{\mathrm{e}}_3$$

where

$$\widehat{\mathbf{e}}_j \cdot \operatorname{grad}\phi = \phi_{,j} = \frac{\partial\phi}{\partial x^j}, \quad j = 1, 2, 3$$

The comma subscripts notation is used to denote the derivative
i.e $\quad \phi_{,j} = \frac{\partial\phi}{\partial x^j}, \quad$ *Also* $\quad \phi_{,jk} = \frac{\partial^2\phi}{\partial x^j \partial x^k}$, and so on.

Definition 2.2.2 *Divergence*
In Cartesian coordinates the divergence of a vector field $\vec{A}$ is a scalar field and can be represented by:

$$\nabla \cdot \vec{A} = \operatorname{div}\vec{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

Introducing the Einstein summation convention over upper and lower indices, the divergence in Cartesian coordinates can be represented by

$$\nabla \cdot \vec{A} = \operatorname{div}\vec{A} = A^i_{,i} = \frac{\partial A_i}{\partial x^i} = \frac{\partial A_1}{\partial x^1} + \frac{\partial A_2}{\partial x^2} + \frac{\partial A_3}{\partial x^3}$$

where i is the dummy summation index. In other words with vector component indices up

Definition 2.2.3 *Curl:*
To represent the vector $B = \operatorname{curl}\vec{A} = \nabla \times \vec{A}$ in Cartesian coordinates, we note that the index notation focuses attention only on the components of this vector. The components $B_i, i = 1, 2, 3$ of $\vec{B}$ can be represented by

$$B_i = \widehat{\mathbf{e}}_i \cdot \operatorname{curl}\vec{A} = e_{ijk}A_{k,j}, \quad \textit{for} \quad i, j, k = 1, 2, 3$$

where e_{ijk} is the permutation symbol introduced in the previous chapter and $A^k_j = \frac{\partial A^k}{\partial x^j}$. To verify or check this representation of the curl $\vec{A}$ we need to perform the summations indicated by the repeated indices. We have summing on $j = 1, j = 2$, and $j = 3$ that

$$B_i = e^k_{i1}A^k_1 + e_{i2}A^k_2 + e_{i3}A^k_3$$

Now summing each term on the repeated index k gives:

$$B_i = e^2_{i1}A^2_1 + e^3_{i1}A^3_1 + e^1_{i2}A^1_2 + e^3_{i2}A^3_2 + e^1_{i3}A^1_3 + e^2_{i3}A^2_3$$

We have i as the free index which means that it can takes on any of the values $1, 2$ or 3. we then have

$$\begin{array}{l} \text{For } i=1, \quad B_1 = A_{3,2} - A_{2,3} = \frac{\partial A_3}{\partial x^2} - \frac{\partial A_2}{\partial x^3} \\ \text{For } i=2, \quad B_2 = A_{1,3} - A_{3,1} = \frac{\partial A_1}{\partial x^3} - \frac{\partial A_3}{\partial x^1} \\ \text{For } i=3, \quad B_3 = A_{2,1} - A_{1,2} = \frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2} \end{array}$$

which verifies the index notation representation of curl $\vec{A}$ in the Cartesian coordinates.

In contrary to divergence and gradient, curl works only for dimension 3 (or 2 in more trivial cases).

2.3. Transformation Rules

Tensors

In an m-dimensional space, a tensor of rank n is a mathematical object that has n indices and m^n components and also obeys some certain transformation rules.
We now discuss on Transformation rules:
1. A tensor is an object that transforms like a tensor
2. A tensor is an object that is invariant (does not change) under a change of coordinate systems, with components that change according to a special set of mathematical formulas.

2.3.1. Contravariant and covariant Vectors

$$\text{Contravariant component: } A^i$$
$$\text{Covariant component: } B_j$$

Assumptions:
Suppose that V is a vector field defined on a subset of $\mathbb{R}^n$ and suppose that (x^i) and $(\bar{x}^i)$ are two coordinate systems related by the coordinate transformation $\mathcal{T}$: $\bar{x}^i = \bar{x}^i\left(x^1, x^2, x^3, \cdots, x^n\right)$

Contravariant vector

Assume as above
The vector field V is said to be a contravariant tensor of rank 1 (vector) if its components v^i in the (x^i) - coordinate system and $\bar{v}^i$ in the $(\bar{x}^i)$ - coordinate system are related by the following law of Transformation:

$$\bar{v}^i = v^r \frac{\partial \bar{x}^i}{\partial x^r}, \quad where \qquad 1 \leq i \leq n$$

Covariant vector

Also assume as above
The vector field V is said to be a covariant tensor of rank 1 (vector) if its components v_i in the (x^i) - coordinate system and $\bar{v}_i$ in the $(\bar{x}^i)$ - coordinate system are related by the following law of Transformation:

$$\bar{v}_i = v_r \frac{\partial x^r}{\partial \bar{x}^i}, \quad where \qquad 1 \leq i \leq n$$

Invariant

Invariant are mathematical objects that have intrinsic physical entities and laws that obey the transformation rules of tensors.

If v^j represents the components of a contravariant vector and u_j represents the components of a covariant vector, then if the inner product $E \equiv v^j u_j$ is defined in each coordinate system, it is an invariant.

A tensor is an object that is invariant under a change of coordinate systems, with components that change according to a special set of mathematical formula

2.3.2. Contravariant and covariant Tensor

Assumption
Suppose that V is a matrix field of $n \times n$ scalar fields defined over a region of U in $\mathbb{R}^n$. Assume that in the (x^i) coordinate system, the components of V are V^{ij}. Assume also that after a coordinate transformation $\mathcal{T}$: $\bar{x}^i = \bar{x}^i\left(x^1, x^2, x^3, \cdots, x^n\right)$ that takes us to the $(\bar{x}^i)$ - coordinate system, the components of V becomes $\bar{V}^{ij}$.

Contravariant tensor

Assume as above

A matrix field V is said to be a contravariant tensor of rank 2 if its components V^{ij} in the (x^i) - coordinate system and $\bar{V}^{ij}$ in the $(\bar{x}^i)$ - coordinate system obey:

$$\bar{V}^{ij} = V^{mn} \frac{\partial \bar{x}^i}{\partial x^m} \frac{\partial \bar{x}^j}{\partial x^n},$$

where $1 \le i, j \le n$; $\quad \mathbf{m}, \mathbf{n}$ are dummy indices while $\quad \mathbf{i}, \mathbf{j}$ are free indices

Covariant tensor

Assume as above

A matrix field V is said to be a covariant tensor of rank 2 if its components V_{ij} in the (x^i) - coordinate system and $\bar{V}_{ij}$ in the $(\bar{x}^i)$ - coordinate system obey:

$$\bar{V}_{ij} = V_{mn} \frac{\partial x_m}{\partial \bar{x}^i} \frac{\partial x^n}{\partial \bar{x}^j},$$

where $1 \le i, j \le n$; $\quad \mathbf{m}, \mathbf{n}$ are dummy indices while $\quad \mathbf{i}, \mathbf{j}$ are free indices.

Mixed tensor

Lastly, Assume that A is a matrix field of $n \times n$ scalar fields defined over a region of $U \in \mathbb{R}^n$. Assume that (x^i) coordinate system, the components of A are A^i_j. Assume also that after a coordinate transformation $\mathcal{T}$: $\bar{x}^i = \bar{x}^i\left(x^1, x^2, x^3, \cdots, x^n\right)$ that takes us to the $(\bar{x}^i)$ - coordinate system, the components of A becomes $\bar{A}^i_j$.
A matrix field A is said to be a Mixed tensor of rank 2 if its components A^i_j in the (x^i) - coordinate system and $\bar{A}^i_j$ in the $(\bar{x}^i)$ - coordinate system obey:

$$\bar{A}^i_j = A^m_n \frac{\partial \bar{x}_i}{\partial x^m} \frac{\partial x^n}{\partial \bar{x}^j},$$

where $1 \leq i, j \leq n$ $\quad$ $\mathbf{m}, \mathbf{n}$ are dummy indices while $\quad$ $\mathbf{i}, \mathbf{j}$ are free indices.

2.3.3. Ranks of tensor

Suppose V is a tensor with components $V^{i_1,i_2\cdots,i_r}_{j_1,j_2,\cdot,j_s}$. Total rank of V is the sum of the contravariant indices (or rank) and covariant indices (or rank) i.e $T = r + s$. We can call V a $(r, s)-$ tensor where r is the contravariant rank and s is the covariant rank For example: If A is a tensor with components: A^{ij} is a contravariant tensor of rank 2 which is simply written as $(2, 0)$ tensor. A_{ij} is a covariant tensor of rank 2 which is simply written as $(0, 2)$ tensor. Lastly A^i_j is a tensor with contravariant rank 1 and covariant rank 1 which is simply written as $(1, 1)$ tensor and the total rank is 2

On a general case

Assumptions:

Suppose V is an array (1-D, 2-D, 3-D,..., M-D array) field composed of n^m scalar fields (functions) defined over a region U in $\mathbb{R}^n$. Assume that in the (x^i) - coordinate system, the components of V are $V^{i_1,i_2\cdots i_r}_{j_1,j_2,\cdot,j_s}$, where $r + p = m$.

Assume also that after a coordinate Transformation $\mathcal{T}$

$\mathcal{T} : \bar{x}^i = \bar{x}^i\left(x^1, x^2, \ldots, x^n\right)$ that takes us to the $(\bar{x}^i)$ - coordinate system, the components of V become $\bar{V}^{i_1,i_2\cdots i_r}_{j_1,j_2,\cdot,j_s}$.

An array field V is a tensor of rank $m = r + s$ with a contravariant rank of r and covariant rank of s if its components $V^{i_1,i_2\cdots i_r}_{j_1,j_2,\cdot,j_s}$ in the $(\bar{x}^i)$ -coordinate system and $\bar{V}^{i_1,i_2\cdots i_r}_{j_1,j_2,\cdot,j_s}$ in the $(\bar{x}^i)$ - coordinate system obey:

$$\bar{V}^{i_1,i_2\cdots i_r}_{j_1,j_2,\cdot,j_s} = V^{k_1,k_2\cdots k_r}_{l_1,l_2,\cdot,l_s} \frac{\partial \bar{x}^{i_1}}{\partial x^{k_1}} \frac{\partial \bar{x}^{i_2}}{\partial x^{k_2}} \cdots \frac{\partial \bar{x}^{i_r}}{\partial x^{k_r}} \frac{\partial x^{l_1}}{\partial \bar{x}^{j_1}} \frac{\partial x^{l_2}}{\partial \bar{x}^{j_2}} \cdots \frac{\partial x^{l_s}}{\partial \bar{x}^{j_s}}$$

where $1 \leq i, j, k, l \leq n$ $\quad$ $\mathbf{k}, \mathbf{l}$ are dummy indices while $\quad$ $\mathbf{i}, \mathbf{j}$ are free indices.

2.4. Operations on Tensors

Having defined the general concept of tensor over an n -dimensional vector space, let us now introduce the basic arithmetic operations involving tensors [12].

2.4.1. Addition

Two tensors of the same type can be added term by term. The expression

$$C^{i_1\ldots i_r}_{j_1\ldots j_s} = A^{i_1\ldots i_r}_{j_1\ldots j_s} + B^{i_1\ldots i_r}_{j_1\ldots j_s}$$

This means that each coordinate on the L.H.S holds the sum of the corresponding coordinates on the R.H.S. We can simply write tensor addition symbolically as $C = A+B$ [12].

2.4.2. Multiplication by scalar

Each of the coordinates of a tensor can be multiplied by a given scalar to yield a new tensor of the same type. This can be expressed as:

$$C^{i_1 \ldots i_r}_{j_1 \ldots j_s} = cA^{i_1 \ldots j_r}_{j_1 \ldots j_s}$$

We can write tensor multiplication by a scalar symbolically as $C = cA$ [12].

2.4.3. Contraction

Let C be a tensor of type (r, s) at x, with r and s at least 1. Then C has components $C^{j_1, \ldots, j_s}_{i_1, \ldots, i_r}$ as stated earlier. Then there is a tensor of type $(r-1, s-1)$ which has components

$$\sum_{a=1}^{n} C^{j_1 \ldots j_{s-1} a}_{i_1 \ldots i_{r-1} a}$$

This tensor is called a contraction of C [12].

If r and s are large then there will be many such contractions, depending on the indices we choose to sum over.

2.4.4. Inner produt

Here we try to disuss briefly on an important function in the section subsection which is the **inner product** . Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in a real vector space $\mathcal{V}$ and denote by $\mathbf{u} \cdot \mathbf{v}$ a function acting on $\mathbf{u}$ and $\mathbf{v}$ and producing a scalar $a = \mathbf{u} \cdot \mathbf{v}$, such that the following property holds:

i. Bilinearity:

$$(a\mathbf{u}_1 + b\mathbf{u}_2) \cdot \mathbf{v} = a\left(\mathbf{u}_1 \cdot \mathbf{v}\right) + b\left(\mathbf{u}_2 \cdot \mathbf{v}\right)$$
$$\mathbf{u} \cdot (a\mathbf{v}_1 + b\mathbf{u}_2) = a\left(\mathbf{u} \cdot \mathbf{v}_1\right) + b\left(\mathbf{u} \cdot \mathbf{v}_2\right)$$

for all $\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2, \mathbf{v}, \mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$ and $a, b \in \mathbb{R}$ Such a function is called an inner product on the space $\mathcal{V}$.
If the above property holds:

ii. Symmetry:

$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{v}$ for all $\mathbf{u}, \mathbf{v} \in \mathcal{V}$, then the inner product is called symmetric. Also, if the following property holds:

iii. Nondegeneracy:

$$\mathbf{u} \cdot \mathbf{x} = 0 \text{ for all } \mathbf{u} \in \mathcal{V} \Rightarrow \mathbf{x} = 0$$

then the inner product is called nondegenerate.
A vector space equipped with an inner product is called **an inner product space** . Henceforth, we will be considering symmetric nondegenerate inner products as **inner product** without stating it explicitly [12].

2.5. The Metric Tensor

2.5.1. Gram Matrix

Let us express the inner product in some basis $(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$.
Assume $\mathbf{u} = \mathbf{e}_i u^i$ and $\mathbf{v} = \mathbf{e}_i v^i$ be two vectors. Then, using the bilinearity of the inner product, we get:

$$\mathbf{u} \cdot \mathbf{v} = (\mathbf{e}_i \cdot \mathbf{e}_j)\, u^i v^j$$

The entity $(\mathbf{e}_i \cdot \mathbf{e}_j, 1 \leq i, j \leq n)$, consisting of n^2 numbers, is an $n \times n$ matrix and is called the Gram matrix of the basis. We denote this matrix by G.

By symmetry of the inner product mentioned earlier in this section, the matrix G is symmetric[12].

Theorem 2.5.1 (Sylvester theorem) *Every real symmetric matrix G is congruent to a diagonal matrix whose entries have values $+1, -1,$ or 0. The number of $1's$, $-1's$ and zeros, i.e n_+, n_- and n_0 is invariant with respect to the change of the basis in which the discussed $2-form$ is diagonal. The set of numbers n_+, n_- and n_0 is said to be signature. If n_- and n_0 are zeros then we have the case of the classical tensors (euclidean), in other case we have (pseudo)riemannian tensors. In case of $n = 4, n_- = 1$ and n_0 we have Minkowski tensor [12].*

2.5.2. Metric tensor

Define $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$, we have that:

$$\mathbf{u} \cdot \mathbf{v} = g_{ij} u^i v^j$$

The $(0, 2)$ -tensor g_{ij} is called the metric tensor of the inner product space. Like every tensor, it is a geometric object, invariant under change-of-basis transformations.

By Sylvester's theorem, there exists a basis which makes the metric diagonal and reveals the signature of the space [12]. This signature is uniquely defined by the definition of the inner product.
It immediately follows that the inner product is invariant under a change of basis. This is not new to us since the definition of inner product does not depend on a basis. since, by our assumption from the previous theorem, G is nonsingular, it possesses an inverse G^{-1}. The entries of G^{-1} may be viewed as the coordinates of a $(\ 2, 0\ )$-tensor, called the **dual metric tensor**, and it is usually denoted by g^{ij}. It then follows immediately that

$$g_{jk} g^{ki} = \delta^i_j$$

2.5.3. The Minkowski Space

This is an example of non-Euclidean inner product space; Minkowski space. This is a 4-dimensional inner product vector space possessing an orthogonal basis (e_0, e_1, e_2, e_3) and a metric tensor whose coordinates in this orthogonal basis are

$$g_{ij} = \begin{cases} -1, i = j = 0 \\ 1, i = j = 1, 2, 3 \\ 0, i \neq j \end{cases}$$

The metric of this space has signature $n^+ = 3, n^- = 1$.
The Minkowski space is clearly seen to be a non-Euclidean; apparently, this space underlies relativity theory, therefore, it is the space in which our universe exists. The number of the dimensions is starting at 0 instead of 1 [12].
The index 0 is associated with ct, (time multiplied by the speed of light).
The remaining indices are associated with the usual space coordinates x, y, z.
However, relativity theory convention for the coordinates is (x^0, x^1, x^2, x^3) in agreement with tensor notation (we shouldn't confuse contravariant indices with powers)
Now, let x be a vector in the Minkowski space, expressed in the time-space basis of the space. Then we have that:

$$\mathbf{x} \cdot \mathbf{x} = -\left(x^0\right)^2 + \sum_{i=1}^{3} \left(x^i\right)^2$$

We can see that $\mathbf{x} \cdot \mathbf{x}$ is not always non-negative because the inner product of the Minkowski space is not positive.
The following terminology is in use, depending on the sign of $\mathbf{x} \cdot \mathbf{x}$:

$$\mathbf{x} \cdot \mathbf{x} \begin{cases} < 0 : & \text{timelike components} \\ = 0 : & \text{lightlike components (or Null)} \\ > 0 : & \text{spacelike components} \end{cases}$$

2.6. Operation of lowering and raising of indices induced by metric tensors

Let $a^{i_1 \ldots i_r}_{j_1 \ldots j_s}$ be the coordinates of the (r, s) -tensor a in some basis and g_{ij} be the metric tensor in this basis.
Let us form the tensor product $g_{pq} a^{i_1 \ldots i_s}_{j_1 \ldots j_s}$. This tensor has type $(r, s+2)$.

2.6.1. Lowering of tensors

We now choose one of the contravariant coordinates of a, say i_k and replace i_k by q and then perform contraction with respect to q. So q will disappear and we will be left with a tensor of type $(r-1, s+1)$ which is written simply in the form $T^r_s \to T^{r-1}_{s+1}$

$$A^{i_1 \ldots i_{k-1} i_{k+1} \ldots i_r}_{p j_1 \ldots j_s} \to g_{pq} A^{i_1 \ldots i_{k-1} q i_{k+1} \ldots i_r}_{j_1 \ldots j_s}$$

This operation is called **lowering**. Lowering acts do decrease the contravariance valency by 1 and increase the covariant valency by 1. There are r possible lowerings, depending on the choice of k. Note that the new covariant index of the result, p in the equation above is placed in the first position [12].

2.6.2. Raising of tensor

Raising is the opposite of lowering. We begin with the dual metric tensor g^{pq} and form the tensor product $g^{pq}a^{i_1\ldots i_r}_{j_1\ldots j_s}$. Now we choose an index j_k and replace j_k by q and perform contraction with respect to q, we then obtain

$$A^{pi_1\ldots i_r}_{j_1\ldots j_{k-1}j_{k+1}\cdots j_s} \to g^{pq}A^{i_1\ldots i_r}_{j_1\ldots j_{k-1}qj_{k+1}\cdots j_s}$$

This operation is called **raising**.
Raising acts to increase the contravariance valency by 1 and decrease the covariant valency by 1, so the resulting tensor has type $(r+1, s-1)$, which is simply written in the form $T^r_s \to T^{r+1}_{s-1}$. There are s possible raising, depending on the choice of k [12].

Note that the new contravariant index of the result, p in the above equation is placed in the first position.
A common use of lowering and raising is in moving between vectors and covectors [12]. If v^i is a vector in some basis, we define its corresponding covector v_i through the relationships

$$v_i = g_{ik}v^k, \quad and \quad v^i = g^{ik}v_k$$

These relationships establish a natural isomorphism between the given vector space $\mathcal{V}$ and its dual space of covectors $\mathcal{V}^*$.

2.7. The Levi-Civita Symbol

The Levi-Civita Symbol $\epsilon_{i_1i_2,\ldots,i_n}$ is a function of n indices, each taking values from 1 to n. It is therefore fully defined by n^n values, one for each choice of indices. The definition of the Levi-Civita symbol is as follows [12].

$$\epsilon_{i_1i_2\ldots i_n} = \begin{cases} 1, i_1i_2\ldots i_n \text{ is an even permutation of } 12\ldots n \\ -1, i_1i_2\ldots i_n \text{ is an odd permutation of } 12\ldots n \\ 0, i_1i_2\ldots i_n \text{ is not a permutation of } 12\ldots n \end{cases}$$

We can see that:
$\epsilon_{i_1i_2\ldots i_n}$ is 1 in $n!/2$ cases out of n^n
e.g $n = 3 \implies 3!/2 = 3$
$\epsilon_{i_1i_2\ldots i_n}$ is -1 in $n!/2$ cases, and
$\epsilon_{i_1i_2\ldots i_n}$ is 0 in all other cases. Let A be an $n \times n$ matrix. Using the Levi-Civita symbol, we can express the determinant of A as

$$\det A = \epsilon_{i_1i_2\ldots i_n}A^{i_1}_1A^{i_2}_2\ldots A^{i_n}_n$$

with implied summation over all indices.

2.8. Symmetry and Anti-symmetry

From our previous knowledge, we know that; $A_{j_1j_2\ldots j_s}$ is a $(0, s)$ -tensor, but can also be a symbol such as the Levi-Civita symbol or a pseudotensor such as the volume tensor. We say that $A_{j_1j_2\ldots j_s}$ is symmetric with respect to a pair of indices p and q if

$$A_{j_1j_2\ldots p\ldots q\ldots j_s} = A_{j_1j_2\ldots q\ldots p\ldots j_s} \tag{2.1}$$

We say that $A_{j_1j_2\ldots j_s}$ is anti-symmetric with respect to a pair of indices p and q if

$$A_{j_1j_2\ldots p\ldots q\ldots j_s} = -A_{j_1j_2\ldots q\ldots p\ldots j_s} \tag{2.2}$$

We note that each of (2.1) and (2.2) above involves transposition of p and q; hence, symmetry and anti-symmetry are defined by the behavior of the coordinates under transpositions [12].
A tensor is called completely symmetric if it exhibits symmetry under all possible transpositions
A tensor is called completely anti-symmetric if it exhibits anti-symmetry under all possible transpositions.

Theorem 2.8.1 *A tensor $a_{j_1j_2\ldots j_s}$ is completely symmetric if and only if $A_{k_1k_2\ldots k_s} = A_{j_1j_2\ldots j_s}$ for any permutation $k_1, k_2, \ldots, k_n$*

Proof: Trivial

3. DIFFERENTIABLE MANIFOLD

As much as a material body is the fundamental object of Continuum Mechanics, a differentiable manifold is the fundamental object of Differential Geometry. We will be presenting the general definition of this differential manifolds. There are many different ways of defining a differentiable manifold; an object whose main feature looks locally like the Euclidean space $\mathbb{R}^n$ [4].

There are only two possible 1-dimensional manifolds. Continuous plane curves that do not self-intersect (lines) and circles are topological 1 - dimensional manifolds.

Also, the 2 - dimensional manifolds are known as surfaces. Examples include spheres, tori, and hyperboloids. These are topological 2- dimensional manifolds.

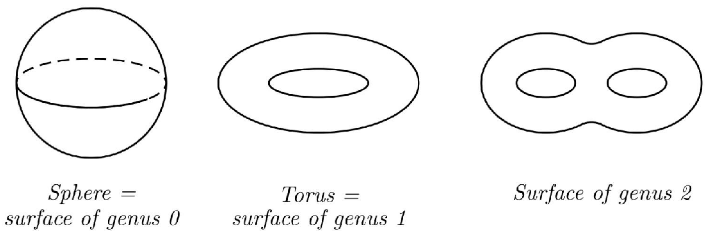

Figure 3.1: 2-dimensional manifolds

3.1. Manifold

Definition 3.1.1 *A Hausdorff topological space with a countable basis is said to be a manifold of order r or ∞ if the following claims are satisfied:*

(i) There is a system $\mathcal{A} = (U_\alpha, \varphi_\alpha)_{\alpha \in I}$, I at most countable such that the system U_α consists of open subsets and cover M and the so-called local maps $\varphi_\alpha : U_\alpha \to \varphi(U_\alpha) \subseteq \mathbb{R}^n$ are homeomorphisms.

(ii) the transition maps $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ are smooth (differentiable up to order r or ∞).

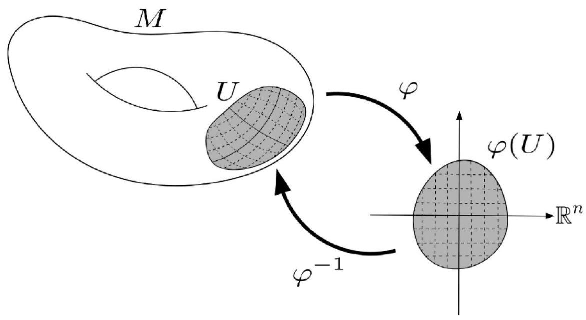

Figure 3.2: From definition 9

The system $\mathcal{A}$ is said to be an atlas on M

3.1.1. Differentiable Manifold

Definition 3.1.2 *A manifold of class C^r and dimension n is a topological Hausdorff space M with a fixed complete atlas compatible and where r is a positive integer or infinity [4].*

A manifold will be called smooth when it is of class C^∞. In concise, smooth manifold (is infinitely differentiable manifold). Henceforth, we will only be dealing with smooth manifolds.

In simple definition: smooth manifolds are geometrical objects that locally look like some Euclidean space and on which we can do calculus. [18]

With smooth we mean infinitely differentiable (smooth manifold), i.e, for a map $f : U \to \mathbb{R}$ open, all the partial derivatives of f need to exist and need to be continuous on U. When f is smooth we use the notation $f \in \mathcal{C}^\infty(U)$ [18].
In general, a function $F : U \to \mathbb{R}^k$ with $U \subset \mathbb{R}^l$ open is said to be smooth if each component function F_i of $F = (F_1, \ldots, F_k)$ is smooth.

Definition 3.1.3 *Suppose $U \subset \mathbb{R}^l$ and $V \subset \mathbb{R}^k$ are open subsets. A map $F : U \to V$ is called a diffeomorphism if it is a bijective smooth map with a smooth inverse [18].*

Note that when F is a diffeomorphism, it is definitely homeomorphism.

Definition 3.1.4 *Two charts (U, φ) and (V, ψ) on M are said to be compatible if either the intersection $U \cap V$ is disjoint or the transition map*

$$\psi \circ \varphi^{-1}\big|_{\varphi(U\cap V)} : \varphi(U \cap V) \to \psi(U \cap V)$$

is a diffeomorphism [18].

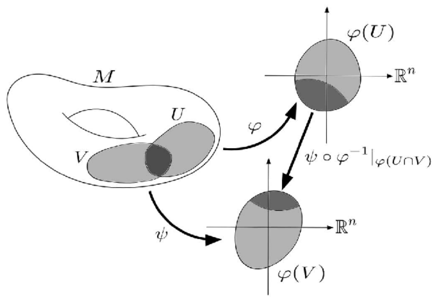

Figure 3.3: Compatible

Definition 3.1.5 *Consider two differentiable manifolds (M, N) and a mapping $f : M \to N$. A mapping $f : M \to N$ is said to be differentiable if for every chart (U_i, φ_i) of M and every chart (V_k, ψ_k) of N such that $f(U_i) \subset V_k$ the mapping*

$$\psi_k \circ f \circ \varphi_i^{-1} : \varphi_i (U_i) \to \psi_k (V_k)$$

is differentiable (i.e smooth differentiable) [4].

Furthermore, in order to develop differential calculus on a manifold and to be able to calculate derivatives in a specific direction, we try to introduce the concept of a tangent vector to a differentiable manifold.

3.2. Tangent vector space

Definition 3.2.1 *Tangent vector*
A linear map $X : \mathcal{C}^\infty(M) \to \mathbb{R}$ is said to be a derivation of M at p when it satisfies the Leibniz condition

$$X(fg) = f(p)Xg + g(p)Xf$$

for all $f, g \in \mathcal{C}^\infty(M)$.

The tangent space of M at p, denoted by T_pM, is the set of all **derivations of M at p** . An element of T_pM is also called **a tangent vector of M at** p [4].

Clearly a tangent space is an R-vector space. Furthermore note that the Leibniz condition is some kind of product rule, hence an essential example of a derivation is the directional derivative of a function along a smooth path.

Particularly, let $\gamma : \mathbb{R} \to M$ be some smooth path with the property $\gamma(0) = p$. Then the map X acts as

$$X(f) = \left.\frac{d}{dt}\right|_{t=0} (f \circ \gamma)$$

for all $f \in \mathcal{C}^\infty(M)$ defines a derivation at p. This follows quite directly by noting the useful equality $fg \circ \gamma = (f \circ \gamma)(g \circ \gamma)$. The converse is actually true as well [15].

3.3. Vector Fields

Let M be an n -dimensional smooth manifold. Before we discuss on vector fields, we will first have to define the notion of a tangent bundle.

3.3.1. Tangent bundle

In the above subsection, we defined the tangent space T_pM at each point p on M. Now, let us consider a collection of tangent bundles over every point on M

$$TM = \cup_{p \in M} T_pM$$

Which is clearly a manifold.

For a given coordinate chart (U_i, φ_i) , we choose to define coordinates on $\cup_{p \in U_i} T_pM$ as (x^α, v^α) , where (x^α) are coordinates on U_i and we parametrize a tangent vector as

$$v = v^\alpha \frac{\partial}{\partial x^\alpha}$$

This defines differential structure on TM (TM is a differential manifold). Hence, TM is called a tangent bundle [4] [15].

Definition 3.3.1 *We define the space T_xM as the set of all $j_0^1\gamma$ for curves $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = x$. Then $TM = \cup_x T_xM$ is said to be the tangent bundle on M.*

For a smooth map $f : M \to N$ define the map Tf by $Tf\,(j_0^1\gamma) = j_0^1(f \circ \gamma)$ [6] [17].

Proposition 3.3.1 *There is a structure of a smooth manifold on TM, locally diffeomorphic to $\mathbb{R}^n \times \mathbb{R}^n$.*

Furhter, there is a projection $p_M : TM \to M$ mapping every element from T_xM to x. Clearly, Tf is a smooth map.

Definition 3.3.2 *A local section $X : M \to TM$ is said to be a vector field on M. Local section: $p \circ X = \mathrm{id}_M$*

3.3.2. Vector field on manifold

A vector field on manifold is a result of selecting at each point of a manifold a tangent vector.

Definition 3.3.3 *A vector field on the manifold M is a mapping*

$$X : M \to \bigcup_{p \in M} T_pM$$

such that $X(p) \in T_pM$ for every $p \in M$.

X is called a differentiable vector field if, for every differentiable function f, $X(f)$, viewed as a real-valued function on M, is differentiable in a neighbourhood of every point.
i.e, a vector field X is differentiable if, for every point $p \in M$, there exists an open neighbourhood U_p, such that $X_q(f)$ is differentiable at every $q \in U_p$, and for every $f \in \mathcal{F}(q)$.
Denoting X_p in the given coordinate induced basis $\frac{\partial}{\partial u^{\tau}}\Big|_{p'} \cdots , \frac{\partial}{\partial u^{w}}\Big|_{p}$ as

$$X_p = \sum_{j=1}^{n} \xi_j(p) \frac{\partial}{\partial w}\Bigg|_p$$

where ξ_j is the components of the vector field X in the coordinate system $u^1, \ldots, u^n$ and these components are real-valued functions on the manifold M [4]

Proposition 3.3.2 *A vector field X on a manifold M is differentiable if and only if its components in one and therefore in every, coordinate system are differentiable functions on M.*

The set $\mathcal{X}(M)$ of all differentiable vector fields on M is a real vector space with pointwise addition and multiplication by scalars. To be precise, it is an algebra with bracket operation defined by

$$[X, Y](f) \equiv X(Y(f)) - Y(X(f))$$

for any differentiable function $f : M \to \mathbb{R}$, and any pair of vector fields $X, Y \in \mathcal{X}(M)$. X, Y are vector fields [15].

The above proposition leads us to consider vector fields as the element of a Lie Algebra.

3.3.3. Lie Algebra

Let $[x, y]$ be a vector field on a manifold M provided that X and Y are vector fields on M. Then $\mathcal{X}(M)$ is said to be a Lie algebra if there exists a bilinear map $[-,-]: \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{X}(M)$, called the Lie bracket [1], such that
1. The Lie bracket $[-,-]$ is skew-symmetric:

$$[X, Y]=-[Y, X], \quad \forall X, Y \in \mathcal{X}(M)$$

2. The Jacobi identity is satisfied:

$$[X,[Y, Z]]+[Z,[X, Y]]+[Y,[Z, X]]=0, \quad \forall X, Y, Z \in \mathcal{X}(M)$$

The space $\mathcal{X}(M)$ is indeed a Lie algebra over the set of real numbers.

We may also regard $\mathcal{X}(M))$ as a module over the algebra $\mathcal{F}(M)$ of differentiable functions on M as follows [15] :
If f is a function and X is a vector field on M, then $f X$ is a vector field on M defined by $(f X)_{p}=f(p) X$ for $p \in M$. Then

$$\begin{gathered}
[f X, g Y]=f g[X, Y]+f(X g) Y-g(Y f) X \\
f, g \in \mathcal{F}(M), \quad X, Y \in \mathcal{X}(M)
\end{gathered}$$

For a point p of M, the dual vector space $T_{p}^{*}(M)$ of the tangent space $T_{p}(M)$ is called the space of covectors at p.
An assignment of a covector at each point p is called a 1 -form (diferential form of degree 1).
For each function f on M, the total differential (df) of f at p is defined by:

$$\left\langle(d f)_{p}, X\right\rangle=X f \quad \text { for } X \in T_{p}(M)$$

where (,) denotes the value of the first entry on the second entry as a linear functional on T$_{p}$(M). If $u^{1}, \ldots, u^{\mathrm{a}}$ is a local [15].

As we mentioned earlier, the tangent space of the manifold M at the point $p \in M$ is a real vector space. Its dual space $T_{p}^{*} M$ is called the space of covectors at p. A smooth field of co-vectors $\omega: M \rightarrow \bigcup_{p \in M} T_{p}^{*} M$ such that $\omega_{p} \equiv \omega(p) \in T_{p}^{*} M$ is called a 1 -form on M [15]. In other words, a 1 -form ω on M is a linear mapping from the space $\mathcal{X}(M)$ of all vector fields on M into the algebra of all differentiable functions $\mathcal{F}(M)$ on M such that

$$\omega(X)(p)=\omega_{p}\left(X_{p}\right), \quad X_{p} \in \mathcal{X}(M), \quad \omega_{p} \in T_{p}^{*} M, \quad p \in M$$

In particular, given a differentiable function $f: M \rightarrow \mathbb{R}$, its total differential is the 1 -form $d f$ defined at each $p \in M$ by

$$d f_{p}\left(X_{p}\right) \equiv X_{p}(f)$$

for every $X_{p} \in T_{p} M$. If $u^{1}, \ldots, u^{n}$ is a local coordinate system in a neighbourhood of p, the total differentials $d u_{p}^{1}, \ldots, d u_{p}^{n}$ form a basis of $T_{p}^{*} M$. Moreover, according to the definition of a differential

$$d u_{p}^{j}\left(X_{p}\right)=d u_{p}^{j}\left(\xi^{k}(p) \left.\frac{\partial}{\partial u^{k}}\right|_{p}\right)=\xi^{j}(p)$$

for any

$$X_p = \sum_{k=1}^{n} \xi^k(p) \frac{\partial}{\partial u^k}\bigg|_p$$

and any $j = 1, \ldots, n$. Thus, given a coordinate system $u^1, \ldots, u^n$ in an neighbourhood of a point p, any 1 -form ω can be represented locally as

$$\omega = \sum_{k=1}^{n} f_k du^k$$

where the functions f_k, called components of ω in the coordinates $u^1, \ldots, u^n$ are differentiable (in the neighbourhood of p) real-valued functions [4].

Generalizing the concept of a 1 -form we say that a (differentiable) r form on a n -dimensional manifold M is a skew-symmetric r-linear mapping of the Cartesinn product $\times^r \mathcal{X}(M) \equiv \mathcal{X}(M) \times \cdots \times \mathcal{X}(M)$ (r-times) into $\mathcal{F}(M)$. We denote by $D^r(M)$ the set of all differentiable r-forms on M, where $r = 0, 1, \ldots, n$, and where by convention $D^0(M) \equiv \mathcal{F}(M)$[4]. Each set $D^r(M)$ is a real vector space as well as an $\mathcal{F}(M)$ -module. Namely, if $f \in \mathcal{F}(M)$ and $\omega \in D^r(M)$ then $f\omega \in D^r(M)$ is viewed as an r -form such that $(f\omega)_p = f(p)\omega_p$ for any $p \in M$. An alternative way of defining an r -form is to consider the differentiable manifolds exterior algebra $\wedge T_p^* M$ with an alternating product $\wedge$ defined as follows".

If $\omega_1, \ldots, \omega_r$ are 1 -forms on M and if $X_p^1, \ldots, X_p^r$ are vectors at $p \in M$, then

$$(\omega_1 \wedge, \cdots, \wedge \omega_r)_p \left(X_p^1, \ldots, X_p^r\right) \equiv \det\left\{\omega_j\left(X_p^k\right)\right\}, \quad j, k = 1, \ldots, r$$

An r -form ω evaluated at $p \in M$ is an element of degree r in $\Lambda T_p^* M$. In a local coordinate system $u^1, \ldots, u^n$ the form ω can therefore be expressed uniquely as

$$\omega = \sum_{i_1 < i_2 < \cdots < i_r} f_{i_1 \ldots i_r} du^{i_1} \wedge \cdots \wedge du^{i_r}$$

Let $D(M)$ denote the totality of differential forms on M. The exterior differential d : $D(M) \rightarrow D(M)$ is a linear mapping such that:
1. $d\left(D^r(M)\right) \subset D^{r+1}(M)$
2. If $f \in \mathcal{F}(M)$ then df is the total differential of f
3. If $\omega \in D^r(M)$ and $\lambda \in D^*(M)$ then

$$d(\omega \wedge \lambda) = d\omega \wedge \lambda + (-1)^r \omega \wedge d\lambda$$

4. $d^2 \equiv d \circ d = 0$ The concept of a differential form can be generalized further to include differential forms with values in a vector space. That is, let V be an $m-$ dimensional real vector space. A V -valued r -form at $p \in M$ is a skewsymmetric r -linear mapping ω of the product $\times^r T_p M$ into V. Given a basis $v^1, \ldots, v^m$ in V one can write

$$\omega_p = \sum_{j=1}^{m} \omega_j(p) v^j$$

where ω_j are usual r -forms on M. Indeed,

$$\omega_p\left(X_p^1, \cdots, X_p^m\right)=\sum_{j=1}^m \omega_j(p)\left(X_p^1, \cdots, X_p^m\right) v^j$$

for any $X_p^1, \ldots, X_p^m \in T_p M$. The exterior derivative of ω is simply

$$d\omega \equiv \sum_{j=1}^m d\omega_j(p) v^j$$

By definition, the form ω is differentiable if each form ω_j is differentiable [4]. In what follows we will only consider differentiable forms, both real and vector valued.

3.3.4. Tensor products

Let A and B be two tensors at x of types (r, s) tensor and (p, q) tensor respectively. Then the tensor product $A \otimes B$ is the tensor at x of type $(r+p, s+q)$ defined by

$$A \otimes B\left(v_1, \ldots, v_{r+p}, \omega_1, \ldots, \omega_{s+q}\right)=A\left(v_1, \ldots, v_r, \omega_1, \ldots, \omega_s\right) \quad \cdot \quad B\left(v_{r+1}, \ldots, v_{r+p}, \omega_{s+1}, \ldots, \omega_{s+q}\right)$$

for all vectors $v_1, \ldots, v_{r+p} \in T_x M$ and all covectors $\omega_1, \ldots, \omega_{s+q} \in T_x^* M$ [12].

Definition 3.3.4 *A covector ω at $x \in M$ is a linear map from $T_x M \rightarrow \mathbb{R}$. The set of covectors at x forms an n -dimensional vector space, which we denote $T_x^* M$. A tensor of type (k, l) at x is a multilinear map which takes k vectors and l covectors and gives a real number [12]*

$$T_x: \underbrace{T_x M \times \ldots \times T_x M}_{k \text{ times}} \times \underbrace{T_x^* M \times \ldots \times T_x^* M}_{i \text{ time}} \rightarrow \mathbb{R}$$

Note that a covector is just a tensor of type $(1,0)$, and a vector is a tensor of type $(0,1)$, since a vector v acts linearly on a covector ω by $v(\omega):=\omega(v)$ Multilinearity means that

$$\begin{aligned} & T\left(\sum_{i_1} c^{i_1} v_{i_1}, \ldots, \sum_{i_k} c^{i_k} v_{i_k}, \sum_{j_1} a_{j_1 \omega} \omega^{j_1} \ldots, \sum_{j_i} a_{j_l} \omega^{j_l}\right) \\ = & \sum_{i_1, \ldots, i_k, j_1, \ldots, j_l} c^{i_1} \ldots c^{i_k} a_{j_1} \ldots a_{j_l} T\left(v_{i_1}, \ldots, v_{i_k}, \omega^{j_1}, \ldots, \omega^{j i}\right)\end{aligned}$$

3.4. Submanifolds

Consider the space $\mathbb{R}^m$ supported by the standard inner product, which will be denoted by $\mathbb{E}_m$.

Definition 3.4.1 *A subset $M \subseteq \mathbb{R}^n$ is said to be an n-dimensional submanifold of $\mathbb{R}^m$ if for any $x \in M$ there is a neighbourhood W of $x \in \mathbb{R}^m$ and a diffeomorphism $f: W \rightarrow V \subseteq \mathbb{R}^m$ such that $f(W \cap M)=V$ and $f^{n+1}=\cdots=f^m=0$ [17].*

Consider its local parametrization

$$f(u^1, \dots, u^n) \text{ for } (u^1, \dots, u^n) \in U \subseteq \mathbb{R}^n$$

Since $f : U \to M$ is a local diffeomorphism the so-called coordinate tangent vectors

$$f_i = \tfrac{\partial f}{\partial u^i} = (\tfrac{\partial f^p}{\partial u^i}), \quad i = 1, \dots, n, \; p = 1, \dots, m$$

are linearly independent and form the basis of the tangent space T_aM where $a = f(u)$.

Let $A = \Sigma_{i=1}^n a^i f_i$ and $B = \Sigma_{i=1}^n b^i f_i$. We want to emphasize the extension of indices up to n, otherwise we could apply the Einstein summation convention. Then the value of their inner product is

$$(A, B) = \Sigma_{i=1,j=1}^n a^i b^j (f_i, f_j).$$

Setting $g_{ij}(u) = (f_i(u), f_j(u)) = g_{ij}(u) = g_{ji}(u)$ we obtain

$$(A, B) = \Sigma_{i,j=1}^n g_{ij}(u) a^i b^j = g_{ij}(u) a^i b^j,$$

applying the Einstein summation convention in the last expression. We have obviously obtained the symmetric bilinear form

$$g_{ij}(u) dx^i dx^j, \tag{1}$$

which is a $(0, 2)$-tensor field (briefly tensor), i.e. a symmetric 2-form.

In the domain of parameters, let us define a path $u^i = u^i(t), \quad i = 1, \dots, n$. In $\mathbb{E}_m$, we have the path $f(u^i(t))$. Its tangent vectors satisfy $\frac{df(u(t))}{dt} = f_i \frac{du^i}{dt}$. For its square we have

$$(ds)^2 = g_{ij}(u(t)) \frac{du^i}{dt} \frac{du^j}{dt}, \tag{2}$$

which is a quadratic form with the associated symmetric bilinear form (1).

Remark 3.4.1 *In what follows we apply the concept of a curve, which is only the trajectory of path determined by a map $f : I \to M$. Unlike curve, paths contain the complete kinematic history. In other words, a curve can be identified with all possible reparametrizations of a given path determining the same trajectory.*

Consider a curve $C \subseteq \mathbb{E}_m$ detrmined by $x^i(t)$ for $t \in \langle a, b \rangle$. Its length is given by the expression as follows

$$\int_a^b || \frac{dx^i}{dt} || dt = \int_a^b \sqrt{(\frac{dx^1}{dt})^2 + \dots + (\frac{dx^n}{dt})^2} dt = \int_a^b ds, \tag{3}$$

see (2).

Definition 3.4.2 *The quadratic form (2) is said to be the first fundamental form.*

Remark 3.4.2 *(a) The quadratic form* (2) *is the generalization of the fundamental form from the classical differential geometry. Two coordinate tangent vectors correspond to the parametrization of a surface and the formula (3) yields the length of a curve with the parametrization $x^i(t)$ lying on a given surface.*

(b) All properties depending on the first fundamental form only are said to be the properties of the inner geometry of a submanifold while the others like normal curvature are called the external geometry.

3.5. Paralel transport

Definition 3.5.1 *Let M be a submanifold in $\mathbb{E}_m$. By the normal space N_xM of M we call the orthogonal complement to T_xM in $T_x\mathbb{E}_m$. The union $\bigcup_{x\in N} N_xM$ is said to be the normal bundle of $M \subseteq \mathbb{E}_m$ [7].*

The following, obvious definition defines the parallel transport of a system of vectors along a curve in $\mathbb{E}_m$.

Definition 3.5.2 *Let $p(t)$ be a path in $\mathbb{E}_m$ and $v(t)$ be a system of vectors considered as bounded vectors coming out from the points of $p(t)$. Then $v(t)$ is said to be paralelly transported along $p(t)$ if and only if $\frac{dv}{dt} = 0$ [7].*

It is easy to verify that any reparametrization of $p(t)$ giving the same trajectory does not affect the parallel transport of $v(t)$. We generalize the recent definition corresponding to the case of $n = m$ to a general submanifold N as follows.

Definition 3.5.3 *A system of vectors $v(t) \in \mathbb{R}^m$ is said to be parallely transported along a path $p(t) \subseteq M$ if and only if $\frac{dv}{dt} \in N_{p(t)}M$.*

Let us describe the parallel transport in formulas. Clearly, $T\mathbb{E}_m = \mathbb{E}_m \times \mathbb{R}^m$. The vectors f_i can then be considered as a map $U \to \mathbb{R}^m$ (U being the domain of parameters). Then $f_{ij} = \frac{\partial f_i}{\partial u^j}$. The definition of the coordinate tangent vectors f_i yields, $f_{ij} = f_{ji}$. Further, let $n_s(u)$ be a basis of a normal space at u, $s = 1, \dots, m-n = m - \dim M$. Then

$$f_{ij} = \Sigma_{k=1}^{n}\Gamma_{ij}^{k}(u)f_k(u) + \Sigma_{s=1}^{m-n}b_{ij}^{s}(u)n_s(u). \tag{4}$$

In coordinates, let $p(t)$ be expressed by $u^i = p^i(t)$ and $v(t) = v^i(t)f_ip(t)$, i.e. $v(t) = v^i(t)f_i(p(t))$, applying the Einstein summation convention. Taking the derivative of the recent equality we have

$$\frac{dv}{dt} = \frac{dv^i}{dt}f_i(p(t)) + v^i(t)f_{ij}(p(t))\frac{dp^j}{dt}. \tag{5}$$

The parellel transport corresponds to the zero projection of $\frac{dv}{dt}$ to T_xM, which is equivalent to the zero values of $(f_l, \frac{dv}{dt})$. Substituting (4) to (5) we obtain

$$(f_l, \tfrac{dv^i}{dt} + \Gamma_{jk}^{i}f_i\tfrac{dp^k}{dt}) = 0.$$

Then the regularity and consequently the invertibility of the inner product matrix yields the following formula and Proposition [7].

Proposition 3.5.1 *The system of vector $v(t)$ is parallely transported along the path $p(t)$ if and only if the following formula is satisfied*

$$\tfrac{dv^i}{dt} + \Gamma_{jk}^{i}(p(t))v^j\tfrac{dp^k}{dt} = 0.$$

where Christoffel symbols Γ_{jk}^{i} are defined by the formula

$$\Gamma_{jk}^{i} = \tfrac{1}{2}\Sigma_{e=1}^{n}g^{ie}\left(\tfrac{\partial g_{je}}{\partial u^k} + \tfrac{\partial g_{ek}}{\partial u^j} - \tfrac{\partial g_{jk}}{\partial u^e}\right)$$

By g^{ie} we denote the inverse matrix to g. We have unified the indices in formula [7].

It is easy to verify that the recent formula remains valid if we reparametrize the path $p(t)$. Thus Proposition (3.5.1) can be reformulated for a curves instead paths (paths being representatives of curves).

Definition 3.5.4 *Let $x = p(t)$ be an integral curve of a vector field X. Then we can defined the parallel transport of a vector field Y on M along the curve field X by the following condition:*

$$\frac{dY^i}{dX^k}\frac{\partial X^k}{\partial t} + \Gamma^i_{jk}(p(t))Y^jX^k = 0, \text{ i.e, } \left(\frac{\partial Y^i}{\partial X^j} + \Gamma^i_{jk}(X)Y^j(x)\right)X^k = 0$$

Definition 3.5.5 *Let X be a vector field on a submanifold $M \subset \mathbb{R}^n$. Then the map $\nabla_x : \chi(M) \to \chi(M)$ defined by $Y \to \left(\frac{\partial Y^i}{\partial X^k} + \Gamma^i_{jk}(x)Y^j(x)\right)X^k(x)$ is said to be a covariant derivative of a vector field Y along the vector field X.*

Corollary 3.5.0.1 *Parallel transport is the inner property of a submanifold. In other words, it depends only on the first fundamental form of the given submanifold $M \subset \mathbb{E}_m$.*

Remark 3.5.1 *Given a curve along which we parallely transport, the vector field is parallely transported in a unique way. The system of differential equation from Proposition* (3.5.1) *is uniquely defined.*

Furthermore, parallel transport preserves linear combinations of vectors [17].

Examples:
Consider the unit sphere, more exactly $\frac{1}{8}$ of the unit sphere

$$\text{(a) } \gamma_1 : f(t) = (r\cos t, r\sin t, 0), \gamma_2 = (-r\sin t, r\cos t, 0), \gamma_3 = (0, 0, 1)$$

Its tangent vector is $N(t) = (-r\sin t, r\cos t, 0) \in N_{f(t)}M$
(b) Let us further transport paralelly the vector $v(0)$ along γ_2, Clearly $\frac{dv}{dt} = 0$
(c) Finally, let us transport $v(t)$ paralelly along γ_3.
The situation is quite analogous to (a); the main (principal) circle on the sphere.

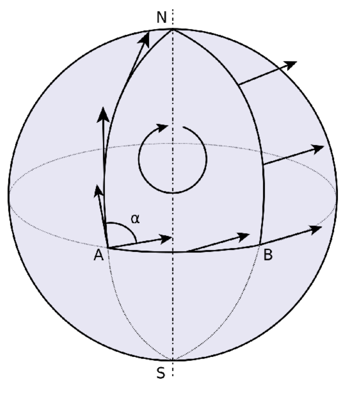

Figure 3.4: Parallel transport of a vector around a closed loop (from A to N to B and back to A) on the sphere [20]

If we transport $v(0)$ from A to be along γ_1, then the result vector in B is different from that obtained by parallel transport from A along γ_2 and along γ_3. Thus parallel tansport of a vector from A to B depends on the curve.

Theorem 3.5.1 *For arbitrary vector fields x, y on M and any function $f : M \to \mathbb{R}$ the following holds:*
(i) $\nabla_X(Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$
(ii) $\nabla_X (fY) = (Xf)Y + f\nabla_X Y$
(iii) $\nabla_{X_1+X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y$
(iv) $\nabla_{fX} Y = f\nabla_X Y$

The above are the so-called **Koszul Axioms.**

Definition 3.5.6 *A curve $\gamma(t)$ is said to be a geodesic curve if there is a parametrization (i.e. a path $p(t)$) such that the system of its tangent vectors is paralelly transported along $p(t)$.*

4. Application of Tensor in Continuum Mechanics

We start by briefing us through Gauss theorem and Stokes' theorem

4.1. Gauss Theorem

In vector calculus, Gauss theorem which is also known as divergence theorem is a result that is related to the flux of a vector field in an enclosed volume [11].

Theorem 4.1.1 *Let F be a continuously differentiable vector field defined in a volume V where V is a subset of $\mathbb{R}^n$ i.e $n = 3$. Let S be the closed surface forming the boundary of V and let n be the unit outward normal to S. Then, the Gauss theorem states that*

$$\int\int\int_V (\nabla \cdot \mathbf{F})dV = \oint\oint_S (\mathbf{F} \cdot \mathbf{n})dS$$

Mathematically speaking, the Gauss theorem states that the total amount of expansion of F within the volume V is equal to the flux of F out of the surface S.

The left side is a volume integral over the volume V, the right side is the surface integral over the boundary of the volume V [21].

Now, let us consider a convex region V bounded by a smooth surface S in $3-D$. Let $A\,(x_1, x_2, x_3)$ be a differentiable function defined in V. We start by defining the integral

$$\int\int\int_V (\nabla \cdot A)dV$$

$$\text{where } \nabla = \tfrac{\partial}{\partial x}$$

$$I = \int\int\int_V \frac{\partial A}{\partial x_1} dx_1 dx_2 dx_3$$

Integrating with respect to x_1, we have that

$$I = \int\int A dx_2 dx_3 |_{S^{**}}^{S^*}$$

$$I = \int\int_{S^*} A dx_2 dx_3 - \iint_{S^{**}} A dx_2 dx_3$$

let $dx_2 dx_3 = n_1^* dS$ on S^*

and

let $dx_2 dx_3 = -n_1^{**} dS$ on S^{**} then we get

$$I = \int_S A n_1 dS$$

On a general case

$$\int_V \partial_i A dV = \int_S n_i A dS$$

4.1.1. Applications of Gauss Theorem

Conservation of mass for a fluid

As an example of the application of the divergence theorem, this section presents the derivation of the law of conservation of mass for a fluid of variable density [11].

Consider a fluid with density $\rho(r,t)$ flowing with velocity $u(r,t)$. Let V be an arbitrary volume fixed in space, with surface S and outward normal n. Then the total mass of the fluid contained in V is the volume integral of ρ :

$$\text{Mass of fluid in } V = \iiint_V \rho dv \tag{4.1}$$

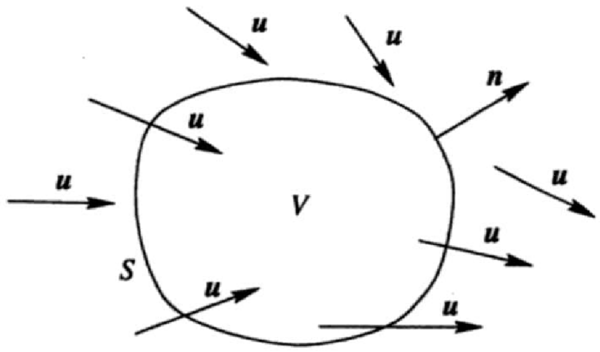

Figure 4.1: Fluid flows with velocity u through region V

The rate at which mass enters V is equal to the surface integral of the flux ρu

$$\text{Rate of mass flow into } V = -\oint_s \rho u \cdot ndS$$

where the minus sign is signifying n points outward, so mass enters V if

$$u \cdot n < 0$$

We can now apply the physical law that mass is conserved: the rate of change of the mass in V must equal the rate at which mass enters V.

Mathematically, we have

$$\frac{d}{dt}\int\int\int_V \rho dV = -\oint\oint_S \rho u \cdot ndS \tag{4.2}$$

The surface integral on the RHS can now be written as a volume integral using the divergence theorem.

Also, the order of the derivative and the integral on the LHS can be interchanged:

$$\int\int\int_V \frac{\partial \rho}{\partial t} dV = -\int\int\int_V \nabla \cdot (\rho u) dV \tag{4.3}$$

where the time derivative has become a partial derivative since ρ is a function of space and time.

Now we combine the two integrals into one, we then have

$$\int\int\int_V \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) dV = 0 \tag{4.4}$$

We have obtained the result without any restrictions on the volume V. Thus it is true for any arbitrary volume V. The only way that this can be true is if the integrand (the quantity inside the integral) is zero everywhere. If there were some point where the integrand were non-zero, a small volume could be drawn around that point, which would contradict (4.4)

Therefore the law for conservation of mass of a fluid is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \tag{4.5}$$

This conservation law takes the following form: the rate of change of the density plus the divergence of the flux is zero.

Many other conservation laws can also be written in this form: conservation of energy or conservation of electric charge [11].

By expanding the divergence of ρu, in equation (4.5). It can be written in the form

$$\frac{\partial \rho}{\partial t} + u \cdot \nabla \rho + \rho \nabla \cdot u = 0 \tag{4.6}$$

If the density of the fluid is constant and uniform, i.e. independent of time and space, then this equation simplifies to

$$\nabla \cdot \mathbf{u} = 0 \tag{4.7}$$

A fluid satisfying equation (4.7) is said to be **incompressible**.

4.2. Stokes Theorem

In vector calculus and differential geometry, Stokes' theorem is a statement about the integration of differential forms on manifolds, which both simplifies and generalizes several theorems from vector calculus [21].

Stokes's theorem gives an alternative expression for the surface integral of the curl of a vector field. This is analogous to the divergence theorem, so Stokes's theorem could be referred to as the **'curl theorem'**. The proof of the theorem is very similar to that for the divergence theorem, being based on the definition of curl in terms of a line integral [11].

Theorem 4.2.1 *Let C be a closed curve which forms the boundary of a surface S. Then for a continuously differentiable vector field u,* **Stokes's theorem** *states that*

$$\int\int_S \nabla \times u \cdot n dS = \oint_C u \cdot dr$$

where the direction of the line integral around C and the normal n are oriented in a right-handed sense see (Figure 4.2).

As we did in the Gauss theorem above, we express the same here.
Now, consider a $2-D$ convex region S bounded by the curve C in the x_1, x_2 plane. Let A be differentiable inside S. Then the area integral is defined by:

$$I = \int\int_S \frac{\partial A}{\partial x_1} dx_1 dx_2$$

Integrating with respect to x_1 we have that

$$I = \int A dx_2 |_{C^{**}}^{C^*}$$

$$I = \int_{C^*} A dx_2 - \int_{C^{**}} A dx_2$$

$$I == \oint_C A dx_2$$

We have shown the two points P_1 and P_2, which are located at the minimum and maximum values of x_2, dividing the curve C into C^* and C^{**}.

Similarly, we do the same for function B

$$J = \int\int_S \frac{\partial B}{\partial x_2}$$

Integrating with respect to x_2, we then have

$$J = \int B dx_1 |_{C''}^{C'}$$

$$J = \int_{C'} B dx_1 - \int_{C''} B dx_1$$

$$J = -\oint_C B dx_1$$

where C' and C'' are gotten by dividing the curve C Using the minimum and maximum values of x_1 and summing I and J together, we have that

$$\int\int_S \left[\frac{\partial A}{\partial x_1} - \frac{\partial B}{\partial x_2}\right] dx_1 dx_2 = \oint_C [A dx_2 + B dx_1]$$

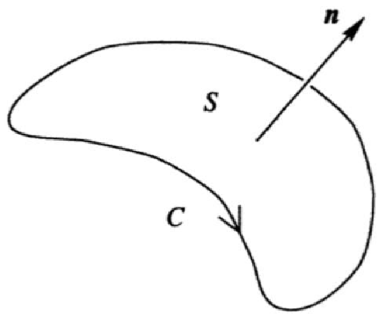

Figure 4.2: Orientation of curve C and surface S

Equating A and B to vector components,

$$i.e \quad A = A_2, \quad and \quad B = A_1$$

we have that

$$\int\int_S \left[\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right] dx_1 dx_2 = \oint_C [A_1 dx_1 + A_2 dx_2]$$

This above result is known as the Stokes theorem.

The left-hand side of this equation has the e_3 component of the curl of the vector A. We may write this as

$$\int\int_S \boldsymbol{e}_3 \cdot (\nabla \times \boldsymbol{A})\, dx_1 dx_2 = \oint_C \boldsymbol{A} \cdot d\boldsymbol{x}$$

This shows that the integral of the normal component of the *curl* of a vector field on the surface S is equal to the integral of the tangential component of the same field around the closed curve C [9].

4.2.1. Applications of Stokes' Theorem

Stokes's theorem can be useful for evaluating integrals, by converting line integrals to surface integrals or vice versa [11].

Example: Show that any irrotational vector field is conservative.

Proof: Suppose that u is irrotational, so $\nabla \times u = 0$. Then for any closed curve C

$$\oint_C \boldsymbol{u} \cdot \boldsymbol{dr} = \iint_S \boldsymbol{\nabla} \times \boldsymbol{u} \cdot \boldsymbol{n} dS = 0$$

where S is any surface spanning C.

Thus u is a conservative vector field.

4.2.2. Ampère's Law

Ampere's law states that the total flux of electric current flowing through a loop is proportional to the line integral of the magnetic field around the loop [11].

Using Stokes's theorem to obtain an alternative form of this law that does not involve any integrals.

Let $\boldsymbol{B}$ be the magnetic field strength and $\boldsymbol{j}$ be the current density. The constant of proportionality is μ_0 in SI units. Then, Ampère's law states that

$$\oint_C \boldsymbol{B} \cdot d\boldsymbol{r} = \mu_0 \iint_S \boldsymbol{j} \cdot \boldsymbol{n} dS$$

for any surface S that spans the loop C.

Rewriting the LHS using Stokes' theorem, we have

$$\int\int_S \nabla \times B \cdot ndS = \mu_0 \int\int_S j \cdot ndS$$

Now if this is true for any loop C, and so any surface S, it follows that

$$\nabla \times B = \mu_0 j$$

$$(\nabla \times B) - \mu_0 j = 0$$

which is therefore the differential form of Ampère's law and is one of Maxwell's equations.

4.3. Configuration and deformation

By a material body we mean an open submanifold $\mathcal{B} \subseteq \mathbb{R}^3$. Let $\mathbb{E}^3$ be the euclidean space, i.e, the affine space supported by the standard inner product over real vectors, which is also called a physical space. Such space can be also considered as the space $\mathbb{R}^3$ if we choose a frame (formed by the origin and some orthonormal basis).

By a configuration, we mean a local map $\kappa : \mathbb{B} \to \mathbb{E}^3$. A selected configuration κ_0 is said to be the reference configuration. In terms of the refernece configuration we transmit the metric structure from the physical space to the body $\mathbb{B}$. By a deformation we mean any map $\kappa \circ \kappa_0^{-1} : \mathbb{R}^3 \to \mathbb{R}^3$. Let us denote the coordinates with respect to the reference configuration by X^α (the so-called Lagrange coordinates) while the coordinates with respect to a generic configuration κ by x^i. They are said to be **Euler or space coordinates**[17].

Convention : For the Lagrange coordinates (reffering to the reference configuration) we use capital letters while for the Euler (spatial) coordinates we use ordinary letters.

As for indices, in [9] there are applied greek letters for indices of Lagrange coordinates and latin letters for indices of the Euler coordinates in order to differ between these kinds of coordinates. Nevertheless, this part of the convention will not be obligatory, particularly in some deductions below where keeping only one notation of indices is more convenient.

If we define the map $\chi_\kappa = \chi = \kappa \circ \kappa_0^{-1}$ then its Jacobi matrix (non-singular) $\partial x^i X^\alpha$ represents the deformation gradient $\mathbf{F}$. This is a $(1,1)$ tensor since it obeys the tensor transformation rules if we change the coordinate system on $\mathcal{B}$.

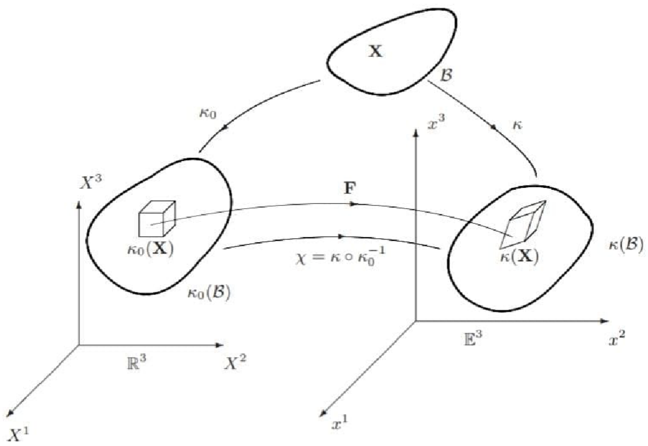

Figure 4.3: The deformation and its gradient

4.4. Continuum kinematics

The motion of a material body can be investigated point-wise. This means that we can describe the motion of the individual particles of the body. There are two basis descriptions; the Lagrangian one and the Eulerian one.

Lagrange description:

$$x^i = x^i\left(X^\alpha, t\right).$$

The last equation corresponds to the motion of a particle which in the initial time $t = 0$ occupies the place with the coordinates X^α.

Euler description: is determined by the equation as follows:

$$v^i = v^i(x^j, t).$$

This yields the velocity of a particle depending on time t and the location x^i. As a matter of fact, a field of velocities is given. Under reasonable assumption on velocity field like smoothness, non-degeneracy one can aggregate the system of curves on the tangent vectors of which take the same direction as an element of a velocity field at a given point. The curves of this kind are said to be the stream lines, [9].

The general form for of a stream line is given by the equation as follows

$$\frac{dx^i}{ds} = k(s)v_i$$

The following concept of integral curve on a manifold M [15] corresponds to the concept of a path line in [9].

Definition 4.4.1 *Given a vector field X on a manifold M, a curve $\gamma(t)$ on M is said to be the integral curve of X if $\gamma'(t) = X(\gamma(t)$ where $\gamma'(t)$ denotes $T_t(1)\gamma$ for $\gamma : I \to M$ and the unit tangent vector $1 \in T_tM$ to γ.*

The following theorem guarantees for any $x \in X$ at least locally the existence of integral curve of the vector field X intersecting X.

Theorem 4.4.1 *Let X be a vector field on a manifold M and $x \in M$. Then there is an integral curve $\gamma_x : I_x \to M$ of X satisfying $\gamma_x(0) = x$ in some neighbourhood $I_x \subseteq \mathbb{R}$ containing 0. If I_x is a maximal interval of this property then γ_x is uniquely determined.*

4.4.1. Helmholtz theorem

Starting from the Euler description (equations for v^i depending on a location x^j) and assuming the velocity $v^i\left(x^j, t\right)$ at a point x^j and time t we obtain at the location $x^j + dx^j$ the velocity $v_i(x^j + dx^j,\, t)$ as follows:

$$v^i(x^j + dx^j, t) = v^i(x^j, t) + \frac{\partial v^i}{\partial x^j}dx^j.$$

The expression $\frac{\partial v^i}{\partial x^j}$ can be written in the form:

$$\frac{\partial v^i}{\partial x^j} = \frac{1}{2}(\frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i}) + \frac{1}{2}(\frac{\partial v^i}{\partial x^j} - \frac{\partial v^j}{\partial x^i})$$

The first bracket corresponds to the symmetric part and the second bracket the skew-symmetric part. Substituting this to the recent equation we obtain:

$$v^i(x^j + \mathrm{d}x^j,\, t) = v^i(x^j,\, t) + \tfrac{1}{2}(\tfrac{\partial v^i}{\partial x^j} - \tfrac{\partial v^j}{\partial x^i})dx^j + \tfrac{1}{2}(\tfrac{\partial v^i}{\partial x^j} + \tfrac{\partial v^j}{\partial x^i})$$

The first and second term express the motion of the continuum itself. The first one represent the translation and the second one the vorticity. The third term represents the proper deformation characterizing the change of distance of the individual particles during the motion. In case of the zero deformation term we are speaking about the *rigid motion* or a *a rigid body* .The first term corresponds to the translation velocity while the second one the vorticity velocity.

The recent deductions can be summarized to the the classical Helmholtz theorem as follows:

A motion of a a continuum body can be uniquely decomposed to the translation, rotation and proper deformation.

4.4.2. Tensors of deformation

In what follows let us focus our attention to searching for changes of distances during a fixed time interval. Let us consider the initial locations of particles tied with the reference configuration while the present location tied with the spatial configuration are of the form:

$$x^j = x^j\,(X^\alpha, t = 0) = X^j \quad and \quad x^j = x^j\,(X^\alpha, t)$$

The time dependency of deformations will be not considered and thus we will only write

$$x^j = x^j(X^\alpha) \tag{4.8}$$

In the following deduction we drop the convention from the end of Subsection **1.1.** regarding greek and latin indices. The recent equation gives an assignment of a particle from the location X^α in the reference configuration to the location x^i in common (spatial) coordinates. Of course, we assume the uniqueness of such assignment. Consider a *displacement vector* $\mathbf{u} = \mathbf{x} - \mathbf{X}$, in coordinates $u^i = x^i - X^i$,
In other words

$$x^j = x^j(X^i) = X^j + u^j(X^i)$$

It is easy to see that we can write

$$dx^j = dx^j + du^j = dx^j + \frac{\partial u^j}{\partial x^i}dx^i \tag{4.9}$$

We are to evaluate the following expression characterizing the distance between the initial and the common location

$$dx^j dx^j - dX^j dX^j \tag{4.10}$$

Clearly, we have

$$dx^j dx^j = (dX^j + du^j)(dX^j + du^j) = (dX^j + \frac{\partial u^j}{\partial X^l} dX^l)(dx^j + \frac{\partial u_j}{\partial X^k} dX^k) \tag{4.11}$$

By means of the Kronecker symbol δ^i_j, the expression $\mathrm{d}X^j + (\frac{\partial u^j}{\partial X^l}\mathrm{d}X^l)$ can be rewritten to the form $(\delta^j_l + \frac{\partial u^j}{\partial X^l})\mathrm{d}X^l$. In the analogous way we can write $\mathrm{d}X^j + \frac{\partial u^j}{\partial X^k}\mathrm{d}X^k = (\delta^j_k + \frac{\partial u^j}{\partial X^k})\mathrm{d}X^k$ and consequently

$$dx^j dx^j = (\delta^j_l + \frac{\partial u^j}{\partial X^l})dX^l(\delta^j_k + \frac{\partial u^j}{\partial X^k})dX^k = (\delta^j_l + \frac{\partial u^j}{\partial X^l})(\delta^j_k + \frac{\partial u^j}{\partial X^k})dX^l dX^k = \tag{4.12}$$

$$= (\delta^j_l \delta^j_k + \frac{\partial u^j}{\partial X^l}\delta^j_k + \delta^j_l \frac{\partial u^j}{\partial X^k} + \frac{\partial u^j}{\partial X^l}\frac{\partial u^j}{\partial X^k})dX^l dX^k \tag{4.13}$$

Notice that in the recent term of the bracket we have applied the convention summing repeating indices. Nevertheless, the next equation below regarding ϵ_{lk} will be not the case.

Taking into account that $\delta^j_l \delta^j_k dX^l dX^k = dX^j dX^j$ a $\frac{\partial u^j}{\partial x^l}\ \delta^j_k = \frac{\partial u^k}{\partial X^l}$, $\delta^j_l\ \frac{\partial u^j}{\partial X^k} = \frac{\partial u^l}{\partial X^k}$ we can write $\mathrm{d}x_j dx_j = dX^j dX^j + (\frac{\partial u^k}{\partial X^l} + \frac{\partial u^l}{\partial X^k} + \frac{\partial u^j}{\partial X^l}\ \frac{\partial u^j}{\partial X^k})\mathrm{d}X^l\mathrm{d}X^k$ and so we have

$$dx^j dx^j - dX^j dX^j = 2\epsilon_{lk} dX^l dX^k, \tag{4.14}$$

where

$$2\epsilon_{lk} = \frac{\partial u^k}{\partial X^l} + \frac{\partial u^l}{\partial X^k} + \frac{\partial u^j}{\partial X^l}\frac{\partial u^j}{\partial X^k} \tag{4.15}$$

Thus we have the second order covariant tensor ((0,2)-tensor, which is obviously symmetric. Clearly, ϵ_{kl} and consequently the tensor under discussion depends on X^i. The recently described tesor of deformation, which has been considered with respect to the reference configuration is said to be *Green tensor.*

Now we reverse our deductions in the sense of $dx^i dx^i - dX^i dX^i$ depending on the common (spatial) coordinates x^i and defined the so-called *Almansi tensor.* Taking the difference between the initial and common location in the form of

$$X^i = x^i - u^i(x^j), \tag{4.16}$$

where u^j denotes the displacement vector we write $\mathrm{d}X^i$ as

$$dX^i = dx^i - \frac{\partial u^i}{\partial x^j}dx^j = (\delta^i_j - \frac{\partial u^i}{\partial x^j})dx^j, \tag{4.17}$$

which implies

$$dX^i dX^i = (\delta^i_k - \frac{\partial u^i}{\partial x^k})(\delta^i_l - \frac{\partial u^i}{\partial x^l})dx^k dx^l = \tag{4.18}$$

$$= \mathrm{d}x^i\mathrm{d}x^i - \frac{\partial u_l}{\partial x^k}\mathrm{d}y_k\mathrm{d}y_l - \frac{\partial u^k}{\partial x^l}\mathrm{d}x^k\mathrm{d}x^l + \frac{\partial u^i}{\partial x^k}\ \frac{\partial u^i}{\partial x^l}\mathrm{d}x^k\mathrm{d}x^l$$

Then we have

$$dx^i dx^i - dX^i dX^i = (\frac{\partial u^l}{\partial x^k} + \frac{\partial u^k}{\partial x^l} - \frac{\partial u^i}{\partial x_k}\frac{\partial u^i}{\partial y^l})dx^k dx^l \tag{4.19}$$

Introducing the notation

$$2\bar{\epsilon}_{lk} = \frac{\partial u^l}{\partial x^k} + \frac{\partial u^k}{\partial x^l} - \frac{\partial u^i}{\partial x^k}\frac{\partial u^i}{\partial x^l}. \tag{4.20}$$

we have

$$dx^i dx^i - dX^i dX^i = 2\bar{\epsilon}_{kl} d^k dx^l. \tag{4.21}$$

Since u^i is considered as a function x^j, it is obvious that $\bar{\epsilon}_{kl}$ depends on the spatial coordinates x^j as well:

$$\bar{\epsilon}_{kl} = \bar{\epsilon}_{kl}(x^j). \tag{4.22}$$

The first tensor has been defined with respect to ther reference configuration and Lagrange coordinates while the second one with respect to the spatial (common) configuration and Euler (spatial) coordinates.

Both of the discusses tensors are modeling all kinds of deformations including the big ones and so they are sometimes said to be *tensors of big deformations* instead the more brief name of *tensor of deformation* or *deformation tensor.*

Tensors of small deformations

In what follows, suppose that the deformations are small. More exactly, suppose that the components of the displacement vector $\vec{u}$ as well as the partial derivatives $\frac{\partial u^j}{\partial X^i}$ are small. Neglecting the last term in the coefficients ϵ_{ij} reduces the deformation tensor to the so-called tensor of small deformations with the coefficients defined as follows

$$e_{ij} = \frac{1}{2}(\frac{\partial u^i}{\partial X^j} + \frac{\partial u^j}{\partial X^i}), \tag{4.23}$$

For the assumptions imposed on displacement vectors and its partial derivatives above we obviously obtain

$$dx^i dx^i - dX^I dX^I = 2e_{lk} dX^l dX^k. \tag{4.24}$$

Conversely, the tensor of small deformations can be expressed by

$$\bar{e}_{ij} = \frac{1}{2}(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i}) \tag{4.25}$$

provided we start from the deformed configuration and express the tensor with respect to the spatial coordinates. In formulas we have

$$dx^i dx^i - dX^i dX^i = 2\bar{e}_{lk} dx_l dx_k \tag{4.26}$$

If we consider only small deformations then the displacements dX^I and dx^i can be identified, which follows that it is not necessary to distinct between them and consequently we can identify $\epsilon_{lk}(x_j)$ and $\bar{\epsilon}_{lk}(y_j)$. This finally implies the unique notation e_{ij} for the coefficients of both tensors.

4.5. Mechanical meaning of the coefficients ϵ_{ij} of the deformation tensor

In the first step, consider the relative prolongation of a segment $d\mathbf{X}$ by a deformation defined by $\frac{|d\mathbf{x}|-|d\mathbf{X}|}{|d\mathbf{X}|}$. Let $d\mathbf{X}^1 = (\mathrm{d}\mathbf{X}^1, 0, 0)$ and $d\mathbf{x}^1$ be the corresponding deformed segment.

Then $dx^1 = \sqrt{1+2\epsilon_{11}}dX^1$ implies $\frac{dx^1 - dX^1}{dX^1} = \sqrt{1+2\epsilon_{11}} - 1$ and ϵ_{11} corresponds to the relative prolongation in direction of the first axis. Analogously we can do for the relative prolongations of the segments in the directions of the remaining axes.

As for the mixed coefficient ϵ_{12}, take $d\mathbf{X}^1$, $d\mathbf{x}^1$ and $d\mathbf{X}^2 = (0, \mathrm{d}\mathbf{X}^2, 0)$ together with the corresponding deformed segment $d\mathbf{x}^2$. letting $d\mathbf{x}^1 = (\mathrm{d}\mathbf{x}^1, 0, 0)$, $d\mathbf{x}^2 = (0, \mathrm{d}\mathbf{x}^2, 0)$, $d\mathbf{X}^1 = (\mathrm{d}\mathbf{X}^1, 0, 0)$ and $d\mathbf{X}^2 = (0, \mathrm{d}\mathbf{X}^2, 0)$ In the subsection devoted to deriving the deformation tensor (in the Green form) we have deduced that:s

$$dx^i = (\delta^i_k + \frac{\partial u^i}{\partial x^k})dX^k$$

Taking the inner product $d\mathbf{x}^1 \cdot \mathrm{d}\mathbf{x}^2$ of the deformed segments to the originally perpendicular segments $d\mathbf{X}^1$ and $d\mathbf{X}^2$ we obtain

$$d\mathbf{x}^1 \cdot \mathrm{d}\mathbf{x}^2 = \frac{\partial \mathrm{u}^2}{\partial \mathrm{x}^1} + \frac{\partial \mathrm{u}^1}{\partial \mathrm{x}^2} + \frac{\partial \mathrm{u}^\mathrm{i}}{\partial \mathrm{x}^1}\frac{\partial \mathrm{u}^\mathrm{i}}{\partial \mathrm{x}^2} = 2\epsilon_{12}\mathrm{d}\mathrm{X}^1\mathrm{d}\mathrm{X}^2.$$

Since $d\mathbf{x}^1 \cdot \mathrm{d}\mathbf{x}^2 = |\mathrm{d}\mathbf{x}^1||\mathrm{d}\mathbf{x}^2|\cos\varphi$ we obtain the formula:

$$\sin\alpha_{12} = \cos\varphi = \frac{2\epsilon_{12}}{\sqrt{1+2\epsilon_{11}}\sqrt{1+2\epsilon_{22}}},$$

where $\alpha_{12} = \frac{\pi}{2} - \varphi$ denotes the change of the originally right angle between the segments $d\mathbf{X}^1$ and $d\mathbf{X}^2$.

5. Surface forces and a stress vector

Consider a body $\mathcal{B}$ in a deformed configuration and take a volume element together with the surface element surrounding it (Figure 5.1). Let $\vec{\nu}$ be the unit normal vector determining the orientation of the surface, which directs out from the volume.

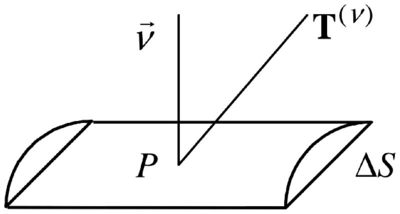

Figure 5.1: surface forces

Clearly, $\vec{\nu}$ enables to define the positive and negative side of the oriented surface element [10].

We make some assumptions analogous to the rigid body working for sufficiently small volume and surface elements. An effect of surface forces over such element can be replaced by a force vector $\mathbf{H}$ at any point $P \in \Delta S$ and the same holds for the force momentum $\mathbf{G}$. Then we define the so-called stress vector $\mathbf{T}^{\nu}$ as the limit as follows:

$$\mathbf{T}^{\nu} = \lim_{\mathrm{S}\to 0} \frac{\mathbf{F}}{\mathrm{S}}$$

while

$$\lim_{\Delta S\to 0} \frac{\Delta \mathbf{G}}{\Delta S} = 0.$$

By the principle of the action and reaction, the force $-T^{\nu}$ acts to the reverse (negative) side of the surface [10].

In order to emphasize the orientation of the surface by the unit normal $\vec{\nu}$ we involve it to the notation.

Analogously we can do with a volume element and volume forces. An effect of volume forces over the element ΔV can be reduced to the choice of a point $P \in \Delta V$ and one force vector $\mathbf{K}$ such that $\lim_{\Delta V\to 0} \frac{\Delta \mathbf{K}}{\Delta V} = \mathbf{F}$. Analogously to the case of surface forces we have $\lim_{\Delta V\to 0} \frac{\Delta \mathbf{L}}{\Delta V} = 0$ for the force momentum.

As for the stress vector at a point P it would be completely described in case of knowledge of its values over all infinitesimal surface containing P, which would be rather complicated. Fortunately, it suffices to know the values of the stress vector under discussion only on three elementary surface elements.

Consider the i-th elementary surface ΔS perpendicular to the i-th axis with the normal determining its orientation coincides i-th unit vector $\vec{e}_i$. The components of the stress acting on this surface let us denote by T_1^i, T_2^I, T_3^i. Let us denote by τ_{ij} the j-th component of the stress vector acting to the i-th elementary surface. Clearly, to the reverse (negative) side of the i-th elementary surface, the stress vector is formed by the components $-\tau_{i1}$, $-\tau_{i2}$, $-\tau_{i3}$ [10].

(see figure 5.2)

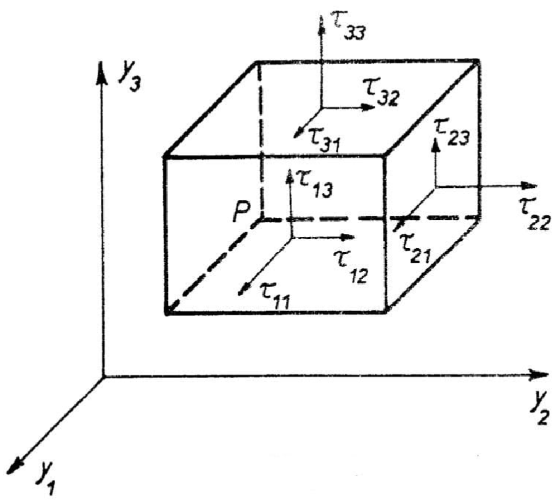

Figure 5.2: the surface elementary to the i-th axis

It is easy to see that by means of nine components τ_{ij} one can determine the stress vector acting on an arbitrary infinitesimal surface with the normal $\vec{\nu}$. To deduce the equilibrium equation for surface and volume forces, consider the elementary tetrahedron, three surfaces of which are parallel with coordinate planes and the fourth one is in the distance h from the point P, (see Figure 5.3).

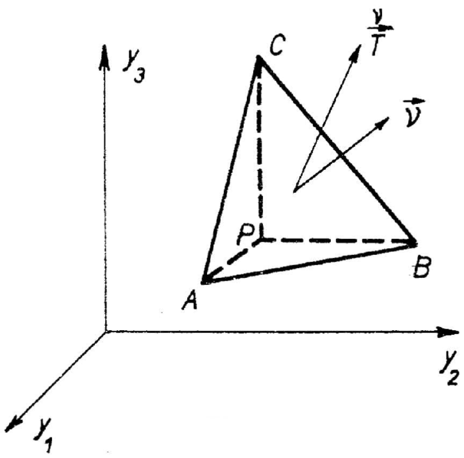

Figure 5.3: Tetrahedron

It is easy to see that provided σ is the area of the surface ABC in this picture then the area of the i-th surface is $\sigma_i = \sigma\nu_i$, since ν_i are exatly the direction cosines of the normal ν related to the surface ABC [10].

The equilibrium equation for the surface and volume forces (the first impulse theorem) is of the form:

$$\int\int_S \mathbf{T}^{\nu}\mathrm{dS} + \int\int\int_{\mathrm{V}} \mathbf{F}\mathrm{dV} = \frac{\mathrm{d}}{\mathrm{dt}}\int\int\int \rho\mathbf{v}\mathrm{dV} \tag{1}$$

while the equlibrium for forces momentum (the second impulse theorem) is of the form

$$\int\int_S \mathbf{y}\times\mathbf{T}^{\nu} + \int\int\int_{\mathrm{V}} \mathbf{y}\times\mathbf{F}\mathrm{dV} = 0. \tag{2}$$

If there are not inertial forces in the first equilibrium equation (i.e. the right hand side equals to zero) then we obtain

$$T_i^{\nu} - \tau_{ji}\sigma_j + \frac{1}{3}\sigma h F_i = 0$$

The orientation of the tetrahedron surface is determined by a normal directing outside and therefore the negative signs by the i-th surface). Taking the limit with respect to $h \to 0$ yields only

$$T_i^{\nu} = \tau_i^j\sigma_j$$

since the limit of the volume of the tetrahedron draws near to zero faster than the surfaces [10].

We outline that in the recent two equations, we have used the convention for summing over the repeating indices. As a matter of fact, $\mathbf{T}^{\nu}$ is a vector (the index ν being not a tensor index but indicating the unit normal determining the orientation of the surface). Since the last equation transforms normal vectors to vectors, it is a linear map and so a $(1,1)$-tensor. Coming back to the Einstein convention we rewrite $T_i^{\nu} = \tau^{ji}\sigma_j$ to

$$T^i = \tau_j^i\sigma^j,$$

omitting the symbol ν. We remark that the recent form of the tensor has the discussed mechanical sense over unit vectors only but taking other vectors ν, T is still a $(1,1)$ tensor.

Now we present the equilibrium equation in the differential form. Consider the first "negative" elementary surface ($i = 1$) of the cube from figure (5.2) above (i.e. the surface elementary to the 1-st axis) where the stress vector is of the form:

$$(-\tau_1^i(x^1,x^2,x^3), -\tau_2^i(x^1,x^2,x^3), -\tau_3^i(x^1,x^2,x^3))$$

while on the positive the value of the stress is:

$$(\tau_1^i(x^1+dx^1,x^2,x^3), \tau_2^i(x^1+dx^1,x^2,x^3), \tau_3^i(x^1+dx^1,x^2,x^3).$$

The corresponding forces to the individual surfaces of the cube are

$$(-\tau_1^i(x^1,x^2,x^3), -\tau_2^i(x^1,x^2,x^3), -\tau_3^i(x^1,x^2,x^3))dx^2dx^3$$

and

$$(\tau_1^i(x^1+dx^1,x^2,x^3), \tau_2^i(x^1+dx^1,x^2,x^3), \tau_3^i(x^1+dx^1,x^2,x^3)dx^2dx^3.$$

Replacing the force vectors on the "positive" surface by the first order Taylor polynomial and subtracting the "negative" we obtain the equilibrium condition for forces acting on the sides perpendicular to the first canonical vector as follows

$$\frac{\partial \tau_1^1}{\partial x^1} + \frac{\partial \tau_1^2}{\partial x^2} + \frac{\partial \tau_1^3}{\partial x^3} + F^1 = 0.$$

Further, the volume force is of the form $F^1 dx^1 dx^2 dx^3$. The same can be done for the remaining couples of mutually parallel surfaces of the cube and we finally obtain:

$$F^i + \frac{\partial \tau_i^j}{\partial x^j} = 0 \quad or \quad F_i + div\tau_i = 0$$

or, finally

$$\mathbf{F} + \mathrm{Div}\tau = 0,$$

by the capital symbol *Div* indicating matrices instead vectors corresponding to entries of *div*.

5.1. Principal Stresses

On an arbitrary plane with normal n, the traction $\sigma^{(n)}$ can be obtained [9] as

$$\boldsymbol{\sigma}^{(n)} = \sigma_{ij} n_i \boldsymbol{e}_j \tag{5.1}$$

We may resolve this with one component perpendicular to the plane and the remainder tangential to the plane. Denoting the perpendicular component by N, we have

$$N = \boldsymbol{n} \cdot \boldsymbol{\sigma}^{(n)} = \sigma_{ij} n_i n_j \tag{5.2}$$

For a given stress tensor σ_{ij}, as the direction of the plane changes, the value of N changes according to the preceding relation. We call N the normal stress on the plane. It is important to know the maximum value of the normal stress and the corresponding plane when we design and analyze structures [9].

We should find the extremum value of N with respect to n_i with the constraint $n_i n_i = 1$. The procedure is identical to the one we used for extremum stretch, and it results in an eigenvalue problem. However, as we will see, the stress tensor is not symmetric when the body moments ℓ are present. This fact appears to make this eigenvalue problem different from the case of the maximum stretch. But an inspection of the quadratic form in n_i shows that, because of the symmetry of $n_i n_j$ its coefficient is $\sigma_{ij} + \sigma_{ji}$. Thus we are free to use the symmetric form of the matrix σ_{ij} in computing the eigenvalues. Assuming the new matrix is the symmetric version $\sigma_{(ij)}$, the eigenvalue problem results in the system of equations

$$\sigma_{(ij)} n_j = \sigma n_i \tag{5.3}$$

where σ is the eigenvalue and n is the eigenvector. For nontrivial solutions of this homogeneous system, we have to have

$$\left|\sigma_{(ij)} - \sigma \delta_{ij}\right| = 0 \tag{5.4}$$

Suppose $\sigma^{(k)}$ and $n^{(k)}$ be the eigenvalues and eigenvectors of this problem.

Multiplying Equation (5.3) by n_i, we see that the eigenvalues are the stationary values of N, i.e..

$$N = \sigma_{(ij)} n_j^{(k)} = \sigma^{(k)} n_i^{(k)} n_i^{(k)} = \sigma^{(k)} \tag{5.5}$$

Hence, the three eigenvectors are mutually orthogonal, and these directions are called the **principal directions** and the corresponding normal tractions $\sigma^{(k)}$ are the **principal stresses**. The three invariants of the matrix $\sigma_{(ij)}$ are

$$\begin{aligned}
I_{\sigma 1} &= \sigma^{(1)} + \sigma^{(2)} + \sigma^{(3)} = \sigma_{(ii)} \\
I_{\sigma 2} &= \sigma^{(1)}\sigma^{(2)} + \sigma^{(2)}\sigma^{(3)} + \sigma^{(3)}\sigma^{(1)} = \tfrac{1}{2}\left[\sigma_{(ii)}\sigma_{(ji)} - \sigma_{(ij)}\sigma_{(ij)}\right] \\
I_{\sigma 3} &= \sigma^{(1)}\sigma^{(2)}\sigma^{(3)} = \left|\sigma_{(ij)}\right|
\end{aligned}$$

6. Conclusion

In this thesis, we discuss the basic concepts and properties of tensors. We studied the basic operations over them. The attention focused on the part of the application. Two main ideas of this thesis are structured on tensors and tensor fields and the last idea is explored on the application part of the thesis.

The first idea is to introduce us to the formulation of the tensor (which is an extension of the vector). We studied how the index notation and in conjunction with the Einstein summation convention is implied. I also studied different kinds of tensors we have and how the transformation rules are implemented. We concluded the first main part of this work by explaining the operation on tensor (i.e addition, multiplication by scalar, contraction, and inner product) and the operation of raising and lowering of indices by metric tensors.

The second major idea in this thesis is focused on the differentiable manifold and vector fields. I also introduce the concept of submanifold, and parallel transport.

In the applications part of the thesis, we have implemented the knowledge gained in the previous parts for the formulation of Gauss and Stokes theorem and a series of mechanical concepts. For example deformation, configuration, tensors of deformation, and mechanical meaning on it. We also studied the stress tensor and used the idea gained in the first chapter of this work in getting the principal directions and the principal stresses.

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