VYSOKÉ UČENÍ TECHNICKÉ V BRNĚ
BRNO UNIVERSITY OF TECHNOLOGY

FAKULTA STROJNíHO INŽENÝRSTVí
ÚSTAV MATEMATIKY

FACULTY OF MECHANICAL ENGINEERING
INSTITUTE OF MATHEMATICS

THE QUALITATIVE AND NUMERICAL ANALYSIS OF NONLINEAR DELAY DIFFERENTIAL EQUATIONS

KVALITATIVNÍ A NUMERICKÁ ANALÝZA NELINEÁRNÍCH DIFERENCIÁLNÍCH ROVNIC SE ZPOŽDĚNÍM

DISERTAČNí PRÁCE
DOCTORAL THESIS

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BRNO 2011

Abstrakt
Disertační práce formuluje asymptotické odhady řešení tzv. sublineárních a superlineárních diferenciálních rovnic se zpožděním. V těchto odhadech vystupuje řešení pomocných funkcionálních rovnic a nerovností. Dále práce pojednává o kvalitativních vlastnostech diferenčních rovnic se zpožděním, které vznikly diskretizací studovaných diferenciálních rovnic. Pozornost je věnována souvislostem asympotického chování řešení rovnic ve spojitém a diskrétním tvaru, a to v obecném i v konkrétních případech. Studována je rovněž stabilita numerické diskretizace vycházející z θ-metody. Práce obsahuje několik příkladů ilustrujících dosažené výsledky.

Abstract
This thesis formulates the asymptotic estimates of solutions of the so-called sublinear and superlinear differential equations with a delayed argument. These estimates are given in terms of auxiliary functional equations and inequalities. Further this thesis discusses the qualitative properties of some delay difference equations originating from discretizations of studied differential equations. We also deal with the resemblances between asymptotic behaviour of solutions of given equations in the continuous and discrete form, considering general as well as particular cases. We discuss stability properties of the θ-method discretizations, too. Several examples illustrating the obtained results are included in the thesis.

Klíčová slova
Nelineární diferenciální rovnice se zpožděním, funkcionální rovnice a nerovnost, diferenční rovnice, asymptotické chování, stabilita, θ-metoda

Keywords
Nonlinear delay differential equation, functional equation and inequality, difference equation, asymptotic behaviour, stability, the θ-method

DVOŘÁKOVÁ, S. *The qualitative and numerical analysis of nonlinear delay differential equations*. Brno: Vysoké učení technické v Brně, Fakulta strojního inženýrství, 2011. 50 s. Vedoucí disertační práce doc. RNDr. Jan Čermák, CSc.

Prohlašuji, že jsem disertační práci s názvem *The qualitative and numerical analysis of nonlinear delay differential equations* vypracovala samostatně pod vedením doc. RNDr. Jana Čermáka, CSc. s využitím materiálů uvedených v seznamu literatury.

Ing. Stanislava Dvořáková

I would like to sincerely thank my supervisor doc. RNDr. Jan Čermák, CSc. for his helpful advices, leading in the topic, great patience and other discussions. I would like to give special thanks to all my family for supporting me and being there whenever help was needed.

Ing. Stanislava Dvořáková

Contents

1. Introduction

Delay differential equations play an important role in the research field of various applied sciences such as control theory, electrical networks, population dynamics, environment science, biology and life science. Mathematical models employing delay differential equations turn out to be useful especially in the situation, where the description of investigated systems depends not only on the position of a system in the current time, but also in the past. In such cases the use of ordinary differential equations turns out to be insufficient. The presence of a delayed time argument in investigated system may frequently influence properties of solutions.

Delay differential equations in special forms were already investigated by L. Euler. However, the systematic study of these differential equations starts at the beginning of the fifties of the previous century. The survey of the theory related to delay differential equations can be found e.g. in books [3], [6], [18], [28] or [40].

It is known that the exact solution of delay differential equations can be found just in some special cases. There is no unified approach to solve the delayed differential equations, even in the linear case. The theory of ordinary differential equations gives various methods to obtain analytical solution (e.g. the variation of constants method, the separation of variables method and others). But these methods are inapplicable dealing with delay differential equations. Hence qualitative and numerical analysis of these equations gather great importance. It utilizes the classic procedures (modified in some sense) for ordinary differential equations, e.g. variation of constants method, Lyapunov functional method, Taylor and Dirichlet series method, methods of Euler type, trapezoidal rule etc.

Roughly speaking, basic numerical methods for delay differential equations originate from the corresponding procedures for ordinary differential equations, where some additional operations (especially the interpolation of delayed terms) are involved. The resulting formulae are then delay difference equations. Their previous qualitative investigation is rather rare because (on the contrary to delay differential equations) there do not exist many original significant applications for this type of difference equations. Therefore it is just a numerical discretization of delay differential equations which motivates the investigation of delay difference equations.

For successful implementation of numerical methods it is often necessary to have general information about qualitative behaviour of solutions of the corresponding exact equation. In this sense the qualitative and numerical analysis of solutions of delay differential equations influence each other. As an example we can mention a simple initial value problem

$$x'(t) = -x(0.99\,t), \quad t \geq 0, \quad x(0) = 1.$$

Its solution (exact or numerical) takes within a long time interval almost zero values (e.g. $x(t) \approx 10^{-10}$ for $t \in (100; 200)$), consequently the numerical solution gives the identically zero solution after some critical instant due to the rounding errors. But it is contrary to qualitative behaviour of the exact solution, which is not stable (for more details see [35] and [15]). From this point of view, the simultaneous qualitative and numerical investigation of delay differential equations seems to be desirable.

The aim of the thesis is to investigate qualitative (especially asymptotic) properties of some nonlinear delay differential and difference equations.

The first part of this thesis deals with the behaviour of all solutions of the differential equation

$$x'(t) = a(t)x(t) + f(t, x(\tau(t))), \quad t \in I := [t_0, \infty), \tag{1.1}$$

where $x(t)$ represents a given state value, $a(t)$ and $f(t, x)$ are real continuous functions on I and $I \times \mathbb{R}$, respectively, $\tau(t)$ is a real continuous, increasing and unbounded function on I (representing delayed argument), which fulfills conditions $\tau(t) < t$ for all $t > t_0$ and $\tau(t_0) \leq t_0$. In particular, it also involves some special cases, e.g. $\tau(t) = t - \kappa$, $\kappa > 0$ (constant delay) or $\tau(t) = \lambda t$, $0 < \lambda < 1$, $t \geq 0$ (proportional delay).

In this text we focus on the equation (1.1), where the function $f(t, x)$ fulfills the relation

$$|f(t, x)| \leq |b(t)||x|^r + |g(t)|$$

for all $t \in I,\ x \in \mathbb{R}$ and for suitable continuous functions $b(t),\ g(t)$ on I and a suitable real $r > 0$. In this thesis there will be discussed two cases of these equations: the sublinear delay differential equation ($0 < r < 1$) and superlinear delay differential equation ($r > 1$).

The second part of this thesis concerns asymptotic properties of solutions of the delay difference equation

$$\Delta y(n) = p(n)y(n) + \sum_{i=1}^{k} q_i(n)|y(\bar{\tau}_i(n))|^{r_i} \operatorname{sgn} y(\bar{\tau}_i(n)) + d(n), \qquad n \in \mathbb{N}(n_0)\,, \tag{1.2}$$

where $p(n)$, $q_i(n)$, $d(n)$ are sequences of reals and $\bar{\tau}_i(n)$ are nondecreasing unbounded sequences of integers satisfying $\bar{\tau}_i(n) < n$ (and representing lags). This difference equation has been obtained via the numerical discretization of the studied differential equation, where several delays instead of one delay have been considered. For this purpose, we utilized the Euler method as the probably simplest convergent numerical schema. Also other studied discrete equations correspond to selected numerical formulae, which can be used for approximate solutions of analysed equation.

Another task consists in comparisons of the results following from qualitative analysis of studied delay differential equations and corresponding difference equations. Due to these comparisons, we set up conditions on numerical parameters (the stepsize) preserving specific qualitative properties of the underlying equations (stability solutions, asymptotic estimates, etc.).

At the end of the introductory part we mention short comments on the structure of this thesis.

In Chapter 2 we introduce three motivation examples of concrete utilization of linear and nonlinear differential equations, where delay effect is demonstrated. Chapter 3 presents results concerning the asymptotic behaviour of solutions of the delay differential equation (1.1). First, there will be describes asymptotic estimates for a sublinear equation, following by consequences and examples. Secondly, there will be shown asymptotic estimates for a superlinear equation with consequences and examples.

Chapter 4 introduces qualitative analysis of the sublinear delay difference equation (1.2). Chapter 5 discusses selected discretizations of studied differential equations with special emphasis put on θ-methods. This chapter significantly utilizes results of Chapter 4. It involves, among others, the stability analysis of the θ-method. Finally, several examples illustrate the obtained results.

2. Motivational examples

There are introduced three examples of differential equations comprising a delayed argument in the following text. All of these equations were set up as a real problem model (more about modeling via delay differential equations can be found e.g. in [28]). Example 2.1 leads to a linear differential equation with a constant delayed argument. A linear differential equation with a nonconstant (and even unbounded) delay is described in Example 2.2. Finally, Example 2.3 introduces a nonlinear differential equation from the field of electronics.

2.1. Example – Water temperature regulation

We assume that a showering person controls the water temperature by a single lever mixer tap and tries to reach an ideal temperature T_d. The term $T_m(t)$ denotes water temperature at the mixer at time t and h denotes a constant time necessary to deliver the water from the mixer output of the tap towards the head of the showering person.

The water temperature is constant before regulation, i.e. the initial condition is given by

$$T_m(t) - T_d = const \neq 0, \quad t_0 - h \leq t \leq t_0.$$

A showering person adjusts the faucet based on the water temperature at the faucet h seconds ago and so the evolution of the water temperature is described by

$$T_m'(t) = -\kappa(T_m(t-h) - T_d).$$

The constant κ measures reaction rate of showering person to a wrong water temperature. Here it depends on whether the temperament is more phlegmatic or choleric and on how the lever rotates. A phlegmatic person would choose a small value of κ whereas an energetic person would prefer a large value of κ.

The solution properties depend on the product of h and κ. If $h\kappa \leq \frac{1}{\mathrm{e}}$ the temperature T_d will be set very quickly. If $\frac{1}{\mathrm{e}} < h\kappa < \frac{\pi}{2}$ the water temperature oscillates but converges to T_d. If $h\kappa = \frac{\pi}{2}$ the water temperature oscillates around T_d constantly but does not converge to it. And if $h\kappa > \frac{\pi}{2}$ the water temperature oscillations may occur maybe with increasing amplitude leading to burns or frostbite.

A detailed description can be found in [28].

2.2. Example – Pantograph equation

The aim is a mathematical description for the determination of a locomotive collector movement that collects current from the upper trolley wire. It is under constant tension and each wire section of constant length is fixed by a stiff spring. A pantograph model represented by two bodies connected by the spring and the damper is depicted in Figure 2.1. The upper body – the collector is in a permanent contact with the trolley wire. The lower body is affected only by the damper and by a constant upward force that represents the pantograph arm mounted on the locomotive roof.

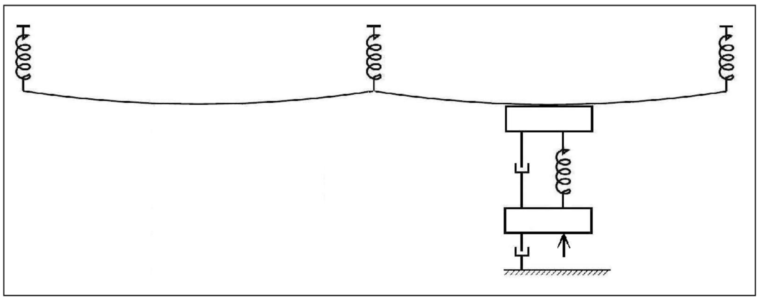

Figure 2.1: Schema of the pantograph

The stiffness and elasticity of the wire in the locations close to the supports are omitted. The contact condition between the collector and the trolley wire is described by system of four differential equations with a proportional delayed argument in the form

$$x'(t) = Ax(t) + Bx(\lambda t), \quad t \geq 0\,,$$

where $0 < \lambda < 1$ is a real scalar and A, B are nonzero matrices of the fourth order. The solution $x(t)$ of this system represents a vertical shift of both pantograph bodies and the contact force of the wire and of the pantograph upper body. This model of pantograph was introduced and discussed in [41]. The qualitative analysis of the pantograph equation and its modifications was presented e.g. in [23], [33], [34] or [42].

2.3. Example – Long line with the Esaki diode

The Esaki diode (or a tunnel diode) behaves in comparison to a common semiconductor diode as a linear resistor with low resistance (this means that it transmits electric current). It has a negative differential resistance in the field of decreasing current. Because of this property, the Esaki diode can be used in fast switches in high-frequency amplifiers and for building oscillators (the diode is able to excite high-frequency oscillations in a resonance circuit).

A long linear conductor with homogeneously distributed parameters is considered in the following text. The energy losses are omitted. An external constant voltage power supply unit E is at one end ($x = 0$) of the wire and the other end ($x = l$) of the wire is grounded by means of the Esaki diode.

The current $i(t, x)$ and the voltage $v(t, x)$ are functions of time t and conductor length x and fulfill the system of telegraph equations

$$L\frac{\partial i}{\partial t} + \frac{\partial v}{\partial x} = 0, \qquad C\frac{\partial v}{\partial t} + \frac{\partial i}{\partial x} = 0. \tag{2.1}$$

Here L is the inductivity and C is the capacity of the conductor per unit length.

The boundary conditions can be formulated as

$$(v + R_0 i)\,|_{x=0} = E, \qquad \left(i - C_l\frac{\partial v}{\partial t} - f(v)\right)\Bigg|_{x=l} = 0, \tag{2.2}$$

where R_0 denotes an input resistance, C_l means the capacity at the output and $f(v)$ is a function of current dependence on the Esaki diode voltage. The behaviour of $f(v)$ is

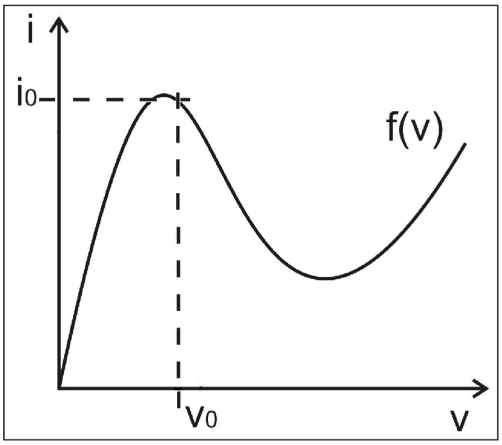

Figure 2.2: Current-voltage characteristic

depicted in Figure 2.2. This empirical characteristic is such that $f(0) = 0$. Then the current increases nearly directly to anode voltage and its local maximum is about 0.1 V.

The system of equations

$$v + R_0 i = E, \quad i - f(v) = 0$$

defines the possible stationary values $v = v_0, i = i_0$, when the diode is "switched on".

It is further assumed that the point $[v_0, i_0]$ lies near the highest point of the graph $i = f(v)$, at the right of it. Then for the working part of the characteristic we can take the relation

$$f(v) = f(v_0) - a(v - v_0) - p(v - v_0)^2, \quad a, p = const > 0.$$

Conditions for the circuit voltage and current can be obtained by application of the d'Alembert rule to the solution of (2.1). These conditions and the boundary ones (2.2) imply the nonlinear differential equation with constant delay

$$x'(t) - kx'(t - \kappa) = mx(t) + knx(t - \kappa) + [x(t) - kx(t - \kappa)]^2, \tag{2.3}$$

where k, m, n are non-dimension constants determined by inductivity, capacity and resistance of the circuit. The problem is described in [28].

It should be noted that according to standard classification of delay differential equations, (2.3) is of neutral type, because the delayed term $t - \kappa$ appears as the argument of the unknown function x and also of its derivative.

3. Asymptotic properties of solutions of some nonlinear delay differential equations

3.1. Some preliminaries

This chapter consists essentially of papers [10] and [16], where we studied the problem of the asymptotic bounds of all solutions for the nonlinear delay differential equation

$$x'(t) = a(t)x(t) + f(t, x(\tau(t))), \quad t \in I := [t_0, \infty), \tag{3.1}$$

where $a : I \to \mathbb{R}$ is a continuous function, $f : I \times \mathbb{R} \to \mathbb{R}$ is given continuous function and $\tau : I \to \mathbb{R}$ is a real continuous, increasing and unbounded function on I, which fulfills conditions $\tau(t) < t$ for all $t > t_0$ and $\tau(t_0) \leq t_0$.

By a solution of (3.1) we understand a real valued function $x(t)$ which is continuous on $[\tau(t_0), \infty)$, continuously differentiable on I and satisfies (3.1) on I.

In this thesis we focus on the equation (3.1), where the function $f(t, x)$ fulfills the relation

$$|f(t, x)| \leq |b(t)||x|^r + |g(t)|, \qquad t \in I \tag{3.2}$$

for a suitable real number $r > 0$, where $x \in \mathbb{R}$ and $b(t)$, $g(t)$ are continuous functions on I. The main results of this chapter are formulated in Subsections 3.2.1 and 3.3.1. In Section 3.2, we derive the asymptotic estimate of all solutions of sublinear delay differential equations, i.e. of the equation (3.1) satisfying (3.2) for some $0 < r \leq 1$ (note that $r = 1$ corresponds to the linear case). The example of such equations is

$$x'(t) = a(t)x(t) + b(t)|x(\tau(t))|^r \operatorname{sgn} x(\tau(t)) + g(t), \quad t \in I, \ 0 < r \leq 1. \tag{3.3}$$

In Section 3.3 we formulate the asymptotic description of all solutions of superlinear delay differential equations, i.e. of the equation (3.1) satisfying (3.2) for some $r > 1$. As an example we can again mention the equation

$$x'(t) = a(t)x(t) + b(t)|x(\tau(t))|^r \operatorname{sgn} x(\tau(t)) + g(t), \quad t \in I, \ r > 1. \tag{3.4}$$

As far as the existence and uniqueness of solutions of (3.1) are concerned, assuming $\tau(t_0) < t_0$ we can apply the method of steps to show that there exists a unique solution of this equation coinciding with a given initial function on the initial interval $[\tau(t_0), t_0]$. But if $\tau(t_0) = t_0$ is valid, then the initial set degenerates to $\{t_0\}$ and instead of the initial function we prescribe the initial condition $x(t_0) = x_0$. To show the existence and uniqueness of the solution of the corresponding initial value problem, we can mention the following result issuing from Theorem 1 and Corollary 6 of [19].

Theorem 3.1. *Consider the equation (3.1) subject to the inequality (3.2). Then (3.1) has a solution on I for any initial value x_0. Furthermore, if $f(t, x)$ is Lipschitz continuous, then this solution is unique.*

Now we proceed to describe the asymptotic properties of solutions of (3.1). It is obvious that the properties depend on the sign of the coefficient $a(t)$. If $a(t)$ is positive, then exponential behaviour of solutions can be expected because (under certain additional assumptions) the equation (3.1) can be treated as a perturbation of the equation

$$x'(t) = a(t)x(t), \quad t \in I.$$

This hypothesis is confirmed by the results obtained for some linear delay differential equations (see, e.g. [2], [13] or [14]).

If $a(t)$ is negative, the solution properties depend on the form of the delay $\tau(t)$ which makes the problem more complicated. Note that if the equation (3.3) in the linear case and with proportional delay is considered, then an algebraic asymptotic behaviour of solution can be observed (see [42]). If we consider the power delay (i.e. $\tau(t) = t^\gamma$, $0 < \gamma < 1$, $t \geq 1$) the asymptotics of solutions is related to logarithmic functions. The precise formulation of the relevant results and their generalization to linear equations with variable coefficients and general variable delay can be found, e.g., in papers [9], [39].

To describe the asymptotic behaviour of solutions of (3.1) with $a(t)$ negative, we introduce the following functional relations, namely the Abel functional equation

$$\psi(\tau(t)) = \psi(t) - 1, \quad t \in I, \tag{3.5}$$

the auxiliary nonlinear functional equation

$$|b(t)|\omega^r(\tau(t)) = |a(t)|\omega(t), \quad t \in I \tag{3.6}$$

and corresponding functional inequality

$$|b(t)|\omega^r(\tau(t)) \leq |a(t)|\omega(t) \quad t \in I. \tag{3.7}$$

The question of the existence and uniqueness of solutions of equations (3.5) and (3.6) can be found, e.g., in the monograph [30]. Here we recall the statement ensuring the existence of solutions of (3.5) which has some differential properties.

Proposition 3.2. *Let $\tau \in C^1(I)$, $\tau(t) < t$ and $\tau'(t) > 0$ for all $t \in I$. Then there exits a solution $\psi \in C^1([\tau(t_0), \infty))$ of (3.5) such that $\psi'(t) > 0$ for all $t \in I$.*

Remark 3.3. *Because of the assumption of Proposition 3.2 throughout this thesis we assume that $\tau(t) < t$ for all $t \in I$ (the case $\tau(t_0) = t_0$ does not enable to solve (3.5) on the whole I). However, we note that all the results presented in this chapter are valid (with some minor modifications) also for lags vanishing at t_0 because we are interested in the asymptotic behaviour of solutions as $t \to \infty$.*

Now we discuss properties of the nonlinear functional equation (3.6) which will be relevant in Section 3.3.

Proposition 3.4. *Consider the functional equation (3.6), where $a, b, \tau \in C^1(I)$, $a(t) < 0$, $b(t) \neq 0$, $\frac{|b(t)|}{|a(t)|}$ is nondecreasing on I, $\tau(t) < t$ for all $t \in I$, $\tau(t)$ is increasing on I and let $M > 0$ be arbitrarily large. Then there exists a positive and nondecreasing solution $\omega \in C^1(I)$ of (3.6) such that $\omega(t) > M$ for all $t \in [\tau(t_0), t_0]$.*

Proof. Let $M > 0$ be such that $|b(t_0)|M^{r-1} \geq |a(t_0)|$, let $\omega_0 \in C^1([\tau(t_0), t_0])$ be a nondecreasing function such that $\omega_0(t) > M$ on $[\tau(t_0), t_0]$ and let

$$|b(t_0)|\omega_0^r(\tau(t_0)) = |a(t_0)|\omega_0(t_0),$$

$$[|b(t)|\omega^r(\tau(t))]'_{t=t_0} = [|a(t)|\omega(t)]'_{t=t_0}.$$

Using the step method we can extend the function $\omega_0(t)$ onto $[\tau(t_0), \infty]$ as the required solution of (3.6). □

3.2. Sublinear delay differential equations

In this section we describe the asymptotic properties of the solution of the sublinear differential equation (3.1) satisfying condition (3.2) for a suitable $0 < r \leq 1$.

First, there will be presented theorems which yield asymptotic estimates of solutions of (3.1), (3.2), where we distinguish the cases $a(t)$ positive and negative. Secondly, we formulate consequences of these estimates in some particular cases.

3.2.1. The general case

An asymptotic description of the solution of (3.1), (3.2) with positive function $a(t)$ will be introduced first. This statement confirms a hypothesis about the expected exponential behaviour of the solution.

Theorem 3.5. *Consider the equation (3.1) subject to the condition (3.2) for a suitable real $0 < r \leq 1$, where $a, b, g, \tau \in C(I)$, $f \in C(I \times \mathbb{R})$, $\tau(t) < t$ for all $t \in I$, $\tau(t)$ is increasing and unbounded on I and let both the integrals*

$$\int_{t_0}^{\infty} |b(t)| \exp\{-(1-r)\int_{t_0}^{\tau(t)} a(u)\,\mathrm{d}u - \int_{\tau(t)}^{t} a(u)\,\mathrm{d}u\}\,\mathrm{d}t,$$

$$\int_{t_0}^{\infty} \exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}\ |g(t)|\,\mathrm{d}t$$

converge. Then for any solution $x(t)$ of (3.1) there exists a constant $L \in \mathbb{R}$ such that

$$\lim_{t\to\infty} x(t)\exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\} = L.$$

Proof. We set

$$z(t) = \exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}x(t)$$

in (3.1) to obtain

$$z'(t) = \tilde{f}(t, z(\tau(t))), \quad t \in I, \tag{3.8}$$

where $\tilde{f}(t,z) = \exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}f(t, z\exp\{\int_{t_0}^{\tau(t)} a(u)\,\mathrm{d}u\})$.

Further, we put $t_j = \sup\{t \geq t_{j-1}, \tau(t) \leq t_{j-1}\}$, $M_j = \max\{|z(t)|, t \in [t_{j-1}, t_j]\}$ and $\bar{M}_j = \max(1, M_j)$ $(j = 1, 2, ...)$. Considering any $t^* \in [t_j, t_{j+1}]$ and by integrating (3.8) over $[t_j, t^*]$ we obtain

$$z(t^*) = z(t_j) + \int_{t_j}^{t^*} \tilde{f}(t, z(\tau(t)))\,\mathrm{d}t. \tag{3.9}$$

To estimate the integral term in modulus we write

$$|\tilde{f}(t, z(\tau(t)))| \leq \exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}\left(|b(t)||z(\tau(t))|^r \exp\{r\int_{t_0}^{\tau(t)} a(u)\,\mathrm{d}u\} + g(t)\right)$$
$$\leq \bar{M}_j|b(t)|\exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u + r\int_{t_0}^{\tau(t)} a(u)\,\mathrm{d}u\} + \exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}|g(t)|$$

by use of (3.2). Then by substituting this relation into (3.9) we have

$$\begin{aligned}|z(t^*)| \leq& \bar{M}_j\left(1 + \int_{t_j}^{t^*} |b(t)|\exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u + r\int_{t_0}^{\tau(t)} a(u)\,\mathrm{d}u\}\,\mathrm{d}t\right)\\ &+ \int_{t_j}^{t^*} \exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}|g(t)|\,\mathrm{d}t\\ \leq& \bar{M}_j\left(1 + \int_{t_j}^{t_{j+1}} |b(t)|\exp\{-(1-r)\int_{t_0}^{\tau(t)} a(u)\,\mathrm{d}u - \int_{\tau(t)}^{t} a(u)\,\mathrm{d}u\}\,\mathrm{d}t\right)\\ &+ \int_{t_j}^{t_{j+1}} \exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}\ |g(t)|\,\mathrm{d}t,\end{aligned}$$

i.e.,

$$\begin{aligned}\bar{M}_{j+1} \leq& \bar{M}_j\left(1 + \int_{t_j}^{t_{j+1}} |b(t)|\exp\{(-1-r)\int_{t_0}^{\tau(t)} a(u)\,\mathrm{d}u - \int_{\tau(t)}^{t} a(u)\,\mathrm{d}u\}\,\mathrm{d}t\right)\\ &+ \int_{t_j}^{t_{j+1}} \exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}\ |g(t)|\,\mathrm{d}t.\end{aligned}$$

By repeating this procedure we arrive at the inequality

$$\begin{aligned}\bar{M}_{j+1} \leq& \bar{M}_1\prod_{k=1}^{j}\left(1 + \int_{t_k}^{t_{k+1}} |b(t)|\exp\{-(1-r)\int_{t_0}^{\tau(t)} a(u)\,\mathrm{d}u - \int_{\tau(t)}^{t} a(u)\,\mathrm{d}u\}\,\mathrm{d}t\right)\\ &+ \sum_{k=1}^{j}\Bigg(\int_{t_k}^{t_{k+1}} \exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}\ |g(t)|\,\mathrm{d}t\\ &\times \prod_{i=k+1}^{j}\left(1 + \int_{t_i}^{t_{i+1}} |b(t)|\exp\{-(1-r)\int_{t_0}^{\tau(t)} a(u)\,\mathrm{d}u - \int_{t_0}^{t} a(u)\,\mathrm{d}u\}\,\mathrm{d}t\right)\Bigg)\\ \leq& \left(\bar{M}_1 + \int_{t_0}^{\infty} \exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}\ |g(t)|\,\mathrm{d}t\right)\\ &\times \prod_{k=1}^{\infty}\left(1 + \int_{t_k}^{t_{k+1}} |b(t)|\exp\{-(1-r)\int_{t_0}^{\tau(t)} a(u)\,\mathrm{d}u - \int_{\tau(t)}^{t} a(u)\,\mathrm{d}u\}\,\mathrm{d}t\right)\end{aligned}$$

which is valid for all $j = 1, 2, ...$ The convergence of the infinite product now implies the boundedness of $\bar{M}_j$ as $j \to \infty$, i.e., $z(t)$ is bounded as $t \to \infty$.

It remains to show that $z(t)$ tends to a finite limit as $t \to \infty$. By integrating (3.8) from $\hat{t}$ to $\bar{t}$ we get

$$\begin{aligned}|z(\bar{t}) - z(\hat{t})| \leq& \int_{\hat{t}}^{\bar{t}} |b(t)|\exp\{-(1-r)\int_{t_0}^{\tau(t)} a(u)\,\mathrm{d}u - \int_{\tau(t)}^{t} a(u)\,\mathrm{d}u\}|z(\tau(t))|^r\,\mathrm{d}t\\ &+ \int_{\hat{t}}^{\bar{t}} \exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}|g(t)|\,\mathrm{d}t.\end{aligned}$$

Now the property $z(t) = O(1)$ as $t \to \infty$ and the convergence of both integral terms yield that considering $\hat{t}, \bar{t}$ sufficiently large we have $|z(\bar{t}) - z(\hat{t})| < \varepsilon$ for any $\varepsilon > 0$. The theorem is proved. □

Remark 3.6. *The positivity of the coefficient $a(t)$ is not strictly required in the previous theorem. However, the convergence requirement put on both integrals would be too strict constraint in the opposite case.*

Now we presented the asymptotic properties of solutions of (3.1), (3.2) provided $a(t)$ is negative.

Theorem 3.7. *Consider the equation (3.1) subject to the condition (3.2) for a suitable real $0 < r \leq 1$, where $a \in C(I)$ is negative and nonincreasing on I, $f \in C(I \times \mathbb{R})$, $b, g \in C(I)$, $\tau \in C^1(I)$, $\tau(t) < t$, $\tau'(t) > 0$ for all $t \in I$ and $\tau(t) \to \infty$ as $t \to \infty$. Let $\psi(t)$ be a solution of (3.5) with the properties guaranteed by Proposition 3.2 such that $\int_{t_0}^{\infty} \frac{\psi'(t)}{-a(t)} \, \mathrm{d}t < \infty$. Further assume that there exists a positive function $\omega \in C^2(I)$ fulfilling the inequality (3.7) such that $\omega' - \omega a > 0$ on I, $\omega'_*/(\omega' - \omega a)$ is nonincreasing on I and*

$$\int_{t_0}^{\infty} \frac{\omega'_*(t)}{\omega'(t) - \omega(t)a(t)} \, \psi'(t) \, \mathrm{d}t < \infty,$$

where $\omega'_*(t) = (|\omega'(t)| - \omega'(t))/2$, $t \in I$. If $g(t) = O(\omega(t))$ as $t \to \infty$, then

$$x(t) = O(\omega(t)) \qquad \text{as } t \to \infty$$

for any solution $x(t)$ of (3.1).

Proof. Let $x(t)$ be a solution of (3.1). If we put

$$z(t) = \frac{x(t)}{\omega(t)},$$

then the equation (3.1) becomes

$$z'(t) = \frac{f(t, \omega(\tau(t))z(\tau(t)))}{\omega(t)} - \frac{\omega'(t) - \omega(t)a(t)}{\omega(t)} z(t). \tag{3.10}$$

The equation (3.10) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}[\omega(t) \exp\{-\int_{t_0}^{t} a(u) \, \mathrm{d}u\} z(t)] = \exp\{-\int_{t_0}^{t} a(u) \, \mathrm{d}u\} f(t, \omega(\tau(t))z(\tau(t))). \tag{3.11}$$

Similarly as in the proof of Theorem 3.5 we put $t_j = \sup\{t \geq t_{j-1}, \tau(t) \leq t_{j-1}\}$, $M_j = \max\{|z(t)|, t \in [t_{j-1}, t_j]\}$ and $\bar{M}_j = \max(1, M_j)$ $(j = 1, 2, ...)$.

Let $t^* \in [t_j, t_{j+1}]$. By integrating (3.11) over $[t_j, t^*]$ we obtain

$$\omega(t) \exp\{-\int_{t_0}^{t} a(u) \, \mathrm{d}u\} z(t)|_{t_j}^{t^*} = \int_{t_j}^{t^*} \exp\{-\int_{t_0}^{t} a(u) \, \mathrm{d}u\} f(t, \omega(\tau(t))z(\tau(t))) \, \mathrm{d}t.$$

Hence,

$$\begin{aligned} z(t^*) = &\exp\{\int_{t_j}^{t^*} a(u) \, \mathrm{d}u\} \frac{\omega(t_j)}{\omega(t^*)} z(t_j) + \frac{\exp\{\int_{t_0}^{t^*} a(u) \, \mathrm{d}u\}}{\omega(t^*)} \\ &\times \int_{t_j}^{t^*} \exp\{-\int_{t_0}^{t} a(u) \, \mathrm{d}u\} f(t, \omega(\tau(t))z(\tau(t))) \, \mathrm{d}t. \end{aligned}$$

By using (3.2), (3.7) and the asymptotic property $|g(t)| \le L\omega(t)$ for all $t \in I$ and a suitable real $L > 0$ we get

$$
\begin{aligned}
|z(t^*)| \le & M_j \exp\{\int_{t_j}^{t^*} a(u)\,\mathrm{d}u\}\frac{\omega(t_j)}{\omega(t^*)} + \frac{\exp\{\int_{t_0}^{t^*} a(u)\,\mathrm{d}u\}}{\omega(t^*)} \\
& \times \int_{t_j}^{t^*} \exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}(|b(t)|\omega^r(\tau(t))|z(\tau(t))|^r + |g(t)|)\,\mathrm{d}t \\
\le & M_j \exp\{\int_{t_j}^{t^*} a(u)\,\mathrm{d}u\}\frac{\omega(t_j)}{\omega(t^*)} + \frac{\exp\{\int_{t_0}^{t^*} a(u)\,\mathrm{d}u\}}{\omega(t^*)} \\
& \times \left[-M_j^r \int_{t_j}^{t^*} a(t)\omega(t)\exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}\,\mathrm{d}t + L\int_{t_j}^{t^*} \omega(t)\exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}\,\mathrm{d}t\right],
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
|z(t^*)| \le & \bar{M}_j \exp\{\int_{t_j}^{t^*} a(u)\,\mathrm{d}u\}\frac{\omega(t_j)}{\omega(t^*)} \\
& + \bar{M}_j \frac{\exp\{\int_{t_0}^{t^*} a(u)\,\mathrm{d}u\}}{\omega(t^*)} \int_{t_j}^{t^*} \omega(t)\frac{\mathrm{d}}{\mathrm{d}t}[\exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}]\,\mathrm{d}t \\
& - L\frac{\exp\{\int_{t_0}^{t^*} a(u)\,\mathrm{d}u\}}{\omega(t^*)} \int_{t_j}^{t^*} \frac{1}{a(t)}\omega(t)\frac{\mathrm{d}}{\mathrm{d}t}[\exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}]\,\mathrm{d}t.
\end{aligned}
$$

Since $a(t)$ is nonincreasing, we arrive at the estimate

$$
\begin{aligned}
|z(t^*)| \le & \bar{M}_j \exp\{\int_{t_j}^{t^*} a(u)\,\mathrm{d}u\}\frac{\omega(t_j)}{\omega(t^*)} \\
& + \left(\bar{M}_j - \frac{L}{a(t_j)}\right)\frac{\exp\{\int_{t_0}^{t^*} a(u)\,\mathrm{d}u\}}{\omega(t^*)} \int_{t_j}^{t^*} \omega(t)\frac{\mathrm{d}}{\mathrm{d}t}[\exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}]\,\mathrm{d}t.
\end{aligned}
\tag{3.12}
$$

To estimate the last integral we write

$$
\begin{aligned}
& \int_{t_j}^{t^*} \omega(t)\frac{\mathrm{d}}{\mathrm{d}t}[\exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}]\,\mathrm{d}t \\
& \quad \le \omega(t)\exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}|_{t_j}^{t^*} + \int_{t_j}^{t^*} \omega_*'(t)\exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}\,\mathrm{d}t \\
& \quad \le \omega(t)\exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}|_{t_j}^{t^*} + \frac{\omega_*'(t)}{\omega'(t)-\omega(t)a(t)}\omega(t)\exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}|_{t_j}^{t^*} \\
& \quad\quad + \frac{-\omega_*'(t)}{\omega'(t)-\omega(t)a(t)}|_{t_j}^{t^*}\omega(t^*)\exp\{-\int_{t_0}^{t^*} a(u)\,\mathrm{d}u\} \\
& \quad \le \omega(t)\exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}|_{t_j}^{t^*}\left(1 + \frac{\omega_*'(t_j)}{\omega'(t_j)-\omega(t_j)a(t_j)}\right).
\end{aligned}
$$

By substituting this estimate into (3.12) we get

$$
\begin{aligned}
|z(t^*)| \leq & \bar{M}_j \exp\{\int_{t_j}^{t^*} a(u)\,\mathrm{d}u\}\frac{\omega(t_j)}{\omega(t^*)} + \left(\bar{M}_j - \frac{L}{a(t_j)}\right)\frac{\exp\{\int_{t_0}^{t^*} a(u)\,\mathrm{d}u\}}{\omega(t^*)} \\
& \times \left(1 + \frac{\omega_*'(t_j)}{\omega'(t_j) - \omega(t_j)a(t_j)}\right)\left(\omega(t)\exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}|_{t_j}^{t^*}\right) \\
\leq & \bar{M}_j \left(1 + \frac{\omega_*'(t_j)}{\omega'(t_j) - \omega(t_j)a(t_j)}\right) - \frac{L}{a(t_j)}\left(1 - \exp\{\int_{t_j}^{t^*} a(u)\,\mathrm{d}u\}\frac{\omega(t_j)}{\omega(t^*)}\right) \\
& \times \left(1 + \frac{\omega_*'(t_j)}{\omega'(t_j) - \omega(t_j)a(t_j)}\right) \\
\leq & \left(\bar{M}_j - \frac{L}{a(t_j)}\right)\left(1 + \frac{\omega_*'(t_j)}{\omega'(t_j) - \omega(t_j)a(t_j)}\right).
\end{aligned}
$$

Since $t^* \in [t_j, t_{j+1}]$ was arbitrary, we have the estimate

$$
\begin{aligned}
\bar{M}_{j+1} & \leq \left(\bar{M}_j - \frac{L}{a(t_j)}\right)\left(1 + \frac{\omega_*'(t_j)}{\omega'(t_j) - \omega(t_j)a(t_j)}\right) \\
& \leq \left(\bar{M}_1 - \sum_{k=1}^{j} \frac{L}{a(t_k)}\right)\prod_{k=1}^{j}\left(1 + \frac{\omega_*'(t_k)}{\omega'(t_k) - \omega(t_k)a(t_k)}\right)
\end{aligned}
$$

which is valid for all $j = 1, 2, \dots$.

To prove the boundedness of $\bar{M}_j$ as $j \to \infty$ it remains to show that both the series

$$
\sum_{k=1}^{j} \frac{-1}{a(t_k)}, \quad \sum_{k=1}^{j} \frac{\omega_*'(t_k)}{\omega'(t_k) - \omega(t_k)a(t_k)}
$$

converge as $j \to \infty$. We set $s = \psi(t)$ and denote $s_0 = \psi(t_0)$. Then

$$
\psi(t_k) = \psi(\tau^{-k}(t_0)) = \psi(t_0) + k = s_0 + k, \qquad k = 1, 2, \dots,
$$

where $\tau^{-k}(t)$ is the k-th iterate of the inverse function $\tau^{-1}(t)$. Hence,

$$
\sum_{k=1}^{j} \frac{-1}{a(t_k)} = \sum_{k=1}^{j} \frac{-1}{a(\psi^{-1}(s_0 + k))}
$$

and the convergence of this series (as $j \to \infty$) follows from the convergence of the corresponding improper integral $\int_{t_0}^{\infty} \frac{\psi'(t)}{-a(t)}\,\mathrm{d}t$ by use of the integral criterion and the substitution rule. The convergence of the latter series can be proved quite similarly.

Summarizing this, the sequence $\bar{M}_j$ is bounded as $j \to \infty$. Hence, $z(t)$ is bounded as $t \to \infty$ and using the backward substitution $x(t) = z(t)\omega(t)$ we obtain the required asymptotic estimate. □

Remark 3.8. *This theorem essentially says that, under certain constraints, the solution $x(t)$ of the delay differential equation (3.1) can be estimated by a solution $\omega(t)$ of a functional nondifferential equation (3.7). However, finding such a solution is not a simple matter, in general.*

Remark 3.9. *It follows from the proof of Theorem 3.7 that considering (3.2) with $g(t)$ identically zero on I we can omit the assumptions that $a(t)$ is nonincreasing and $\int_{t_0}^{\infty} \frac{\psi'(t)}{-a(t)}\,\mathrm{d}t < \infty$. Similarly, if $g(t)$ is not identically zero on I and both the mentioned assumptions are replaced by $a(t) \le a < 0$ for all $t \in I$, then using the same line of arguments as given in the proof of Theorem 3.7 we can modify the result of Theorem 3.7 as*

$$x(t) = O(\omega(t)\psi(t)) \text{ as } t \to \infty$$

for any solution $x(t)$ of (3.1).

3.2.2. Applications to particular cases

In the sequel we give some applications of Theorem 3.5 and Theorem 3.7. Particularly, we consider the equation (3.3) under various assumptions on $a(t)$, $b(t)$ and $\tau(t)$ and show that the previous assertions can yield effective asymptotic results. These results are illustrated by several examples, where the choices of the delayed argument $\tau(t)$ enable to solve the Abel equation (3.5) explicitly.

Corollary 3.10. *Consider the sublinear delay differential equation*

$$x'(t) = a(t)x(t) + b(t)|x(\tau(t))|^r \operatorname{sgn} x(\tau(t)) + g(t), \quad t \in I,\ 0 < r < 1, \tag{3.13}$$

where $a, b, g \in C(I)$, $\tau \in C^1(I)$, $0 < |b(t)| \le K|a(t)|$, $\tau(t) < t$, $\tau'(t) > 0$ for all $t \in I$ and a suitable real $K > 0$ and $\tau(t) \to \infty$ as $t \to \infty$.

- *(i) If $a(t)$ is positive on I and $\int_{t_0}^{\infty} \exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}|g(t)|\,\mathrm{d}t < \infty$, then for any solution $x(t)$ of (3.13) there exists a constant $L \in \mathbb{R}$ such that*

$$\lim_{t\to\infty} x(t) \exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\} = L.$$

- *(ii) If $a(t)$ is negative and $g(t)$ is identically zero on I, then any solution $x(t)$ of (3.13) is bounded on I.*
- *(iii) If $a(t)$ is negative and nonincreasing on I, $g(t)$ is bounded on I and $\int_{t_0}^{\infty} \frac{\psi'(t)}{-a(t)}\,\mathrm{d}t$ converges, where $\psi(t)$ is a solution of the Abel equation (3.5) with the properties guaranteed by Proposition 3.2, then any solution $x(t)$ of (3.13) is bounded on I.*

Proof. To prove the case (i) it is enough to verify the assumptions of Theorem 3.5, particularly the first integral condition. Indeed,

$$\begin{aligned}
&\int_{t_0}^{\infty} |b(t)| \exp\{-(1-r)\int_{t_0}^{\tau(t)} a(u)\,\mathrm{d}u - \int_{\tau(t)}^{t} a(u)\,\mathrm{d}u\}\,\mathrm{d}t \\
&\quad \le K\int_{t_0}^{\infty} a(t) \exp\{-(1-r)\int_{t_0}^{\tau(t)} a(u)\,\mathrm{d}u - \int_{\tau(t)}^{t} a(u)\,\mathrm{d}u\}\,\mathrm{d}t \\
&\quad \le K\int_{t_0}^{\infty} a(t) \exp\{-(1-r)\int_{t_0}^{t} a(u)\,\mathrm{d}u\}dt \\
&\quad = -\frac{K}{1-r}\int_{t_0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}[\exp\{-(1-r)\int_{t_0}^{t} a(u)\,\mathrm{d}u\}]dt < \infty.
\end{aligned}$$

Now let $a(t) < 0$ on I. Under the assumptions of Theorem 3.7, we can relate the asymptotic bounds of any solution of (3.13) to a solution of (3.7). It is easy to verify that under the assumption $|b(t)| \le K|a(t)|$ the inequality (3.7) admits the positive constant solution, namely $\omega(t) \equiv K^{\frac{1}{1-r}}$. Then $\omega'_* \equiv \omega' \equiv 0$ on I and the corresponding assumptions of Theorem 3.7 become trivial. Now both cases (ii) and (iii) follow immediately from Theorem 3.7 with the respect to Remark 3.9. $\square$

Example 3.11. We investigate the asymptotic behaviour of the solutions of

$$x'(t) = ax(t) + b|x(\tau(t))|^r \operatorname{sgn} x(\tau(t)) + g(t), \quad t \in I,\ 0 < r < 1, \tag{3.14}$$

where $a, b \neq 0$ are real constants, $g \in C(I)$, $\tau \in C^1(I)$, $\tau(t) < t$, $\tau'(t) > 0$ for all $t \in I$ and $\tau(t) \to \infty$ as $t \to \infty$.

(i) If $a > 0$ and $\int_{t_0}^{\infty} \exp\{-at\}|g(t)|\,\mathrm{d}t < \infty$, then for any solution $x(t)$ of (3.14) there exists a constant $L \in \mathbb{R}$ such that

$$\lim_{t\to\infty} x(t)\exp\{-at\} = L.$$

(ii) If $a < 0$ and $g(t) \equiv 0$ on I, then any solution $x(t)$ of (3.14) is bounded on I.

(iii) If $a < 0$ and $g(t)$ is bounded on I, then, by Remark 3.9,

$$x(t) = O(\psi(t)) \quad \text{as } t \to \infty,$$

where $\psi(t)$ is a solution of the Abel equation (3.5) with the properties guaranteed by Proposition 3.2.

In the sequel we consider only the cases when $a(t)$ is negative and $g(t)$ is identically zero on I in (3.13). The extension of the next results also to $a(t)$ positive and $g(t)$ nonzero can be easily done by use of Theorem 3.5 and Theorem 3.7.

Corollary 3.12. *Consider the equation without forcing term*

$$x'(t) = a(t)x(t) + b(t)|x(\tau(t))|^r \operatorname{sgn} x(\tau(t)), \quad t \in I,\ 0 < r < 1, \tag{3.15}$$

where $I = [t_0, \infty)$ with $t_0 > 0$, $a, b \in C(I)$, $\tau \in C^1(I)$, $a(t) < 0$, $b(t) \neq 0$, $\tau(t) < t$, $\tau(t_0) > 0$, $\tau'(t) > 0$ for all $t \in I$, $\tau(t) \to \infty$ as $t \to \infty$ and assume that $0 < |b(t)| \le$ $\le K|a(t)|t^{\alpha}(\tau(t))^{-r\alpha}$ for suitable $K, \alpha \in \mathbb{R}$, $K > 0$ and all $t \in I$.

(i) If $\alpha \ge 0$, then

$$x(t) = O(t^{\alpha}) \quad \text{as} \quad t \to \infty \tag{3.16}$$

for any solution $x(t)$ of (3.15).

(ii) If $\alpha < 0$, $a(t) < \frac{\alpha}{t}$ for all $t \in I$, $a(t)t$ is nonincreasing on I and

$$\int_{t_0}^{\infty} \frac{\psi'(t)}{\alpha - a(t)t}\,\mathrm{d}t < \infty,$$

where $\psi(t)$ is a solution of (3.5) with the properties guaranteed by Proposition 3.2, then (3.16) holds for any solution $x(t)$ of (3.15).

Proof. We can easily check that the function

$$\omega(t) = K^{\frac{1}{1-r}} t^{\alpha}$$

defines a positive solution of the auxiliary functional inequality (3.7). The assumptions of Corollary 3.12 are now the reformulation of those presented in Theorem 3.7 with the respect to this solution $\omega(t)$. □

Example 3.13. We consider the equation

$$x'(t) = (\frac{a}{\sqrt{t}} - 1)x(t) + \frac{b}{t}\sqrt{|x(\lambda t)|}\,\operatorname{sgn} x(\lambda t), \quad t \in [2, \infty), \tag{3.17}$$

where $a < 0$, $b \neq 0$, $0 < \lambda < 1$ are real constants. Then the Abel equation (3.5) can be read as

$$\psi(\lambda t) = \psi(t) - 1$$

and admits the solution $\psi(t) = \frac{\log t}{\log \lambda^{-1}}$ having the required properties. Now it follows from Corollary 3.12 (with $r = \frac{1}{2}$, $\alpha = -2$, $K = |b|/\lambda$) that

$$x(t) = O\left(\frac{1}{t^2}\right) \quad \text{as } t \to \infty$$

for any solution $x(t)$ of (3.17).

Corollary 3.14. *Consider the equation (3.15), where $a, b \in C(I)$, $\tau \in C^1(I)$ such that $a(t) < 0$, $b(t) \neq 0$, $\tau(t) < t$, $\tau'(t) > 0$ for all $t \in I$, $\tau(t) \to \infty$ as $t \to \infty$. Let $\psi(t)$ be a solution of (3.5) with the properties guaranteed by Proposition 3.2. Further assume that $0 < |b(t)| \leq LK^{\psi(t)}|a(t)|$ for suitable real constants $L, K > 0$ and all $t \in I$.*

(i) If $K \geq 1$, then

$$x(t) = O\left(K^{\frac{\psi(t)}{1-r}}\right) \quad \text{as} \quad t \to \infty \tag{3.18}$$

for any solution $x(t)$ of (3.15).

(ii) If $0 < K < 1$, $a(t) < \frac{\log K}{1-r}\psi'(t)$ for all $t \in I$, $a(t)/\psi'(t)$ is nonincreasing on the interval I and, moreover,

$$\int_{t_0}^{\infty} \frac{(\psi'(t))^2}{(\log K)\psi'(t) - (1-r)a(t)}\,\mathrm{d}t < \infty,$$

then (3.18) holds for any solution $x(t)$ of (3.15).

Proof. By substituting into (3.7) we can verify that the function

$$\omega(t) = MK^{\frac{\psi(t)}{1-r}}, \quad M = (LK^{\frac{-r}{1-r}})^{\frac{1}{1-r}}$$

defines a solution of the auxiliary inequality (3.7). Now it is easy to verify that this function $\omega(t)$ fulfills all the assumptions introduced in Theorem 3.7. □

Example 3.15. We consider the equation

$$x'(t) = -(\log t)x(t) + b(t)\sqrt{|x(\sqrt{t})|}\ \operatorname{sgn} x(\sqrt{t}), \quad t \in [2, \infty), \tag{3.19}$$

where $b(t)$ is bounded on $[2, \infty)$. In this case we investigate the equation with a power lag. Therefore the corresponding Abel equation (3.5) has the form

$$\psi(\sqrt{t}) = \psi(t) - 1$$

and admits the solution $\psi(t) = \frac{\log \log t}{\log 2}$ having the required properties. Now it follows from Corollary 3.14 (with $r = K = \frac{1}{2}$ and $|b(t)| \le L$ for all $t \ge 2$) that

$$x(t) = O\left(\frac{1}{\log^2 t}\right) \quad \text{as } t \to \infty$$

for any solution $x(t)$ of (3.19).

3.3. Superlinear delay differential equations

In this section we derive the asymptotic properties of the solution of the superlinear differential equation (3.1) satisfying condition (3.2) for a suitable $r > 1$.

Comparing with the sublinear case we impose two restrictions. We assume that the function $g(t)$, appearing in (3.2), is identically zero, and, furthermore, $a(t)$ is negative.

3.3.1. The general case

Theorem 3.16. *Let $x(t)$ be a solution of (3.1) and (3.2) holds for a suitable real $r > 1$, where $g(t) \equiv 0$, $a(t)$, $b(t)$ are continuously differentiable functions on I such that $a(t)$ is negative, $b(t)$ is nonzero and $\frac{|b(t)|}{|a(t)|}$ is nondecreasing on I. Further, let $\tau \in C^1(I)$, $\tau(t) < t$ and $\tau'(t) > 0$ for all $t \in I$ and let $\psi(t)$ be a solution of (3.5) with the properties guaranteed by Proposition 3.2. Finally assume that $\omega \in C^1(I)$ is a positive and nondecreasing function satisfying (3.6) and let $M_0 = \sup\left\{\frac{|x(t)|}{\omega(t)}, t \in [\tau(t_0), t_0]\right\}$.*

(i) If $M_0 \le 1$, then $|x(t)| \le \omega(t)$ for all $t \ge t_0$.

(ii) If $M_0 > 1$, then $|x(t)| \le \omega(t)\ M_0^{r^{\psi(t)+1-\psi(t_0)}}$ for all $t \ge t_0$.

Proof. Let $x(t)$ be a solution of (3.1). If we set

$$z(t) = \frac{x(t)}{\omega(t)},$$

then the equation (3.1) becomes

$$z'(t) = \frac{f(t, \omega(\tau(t))z(\tau(t)))}{\omega(t)} - \frac{\omega'(t) - \omega(t)a(t)}{\omega(t)} z(t). \tag{3.20}$$

The equation (3.20) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}[\omega(t) \exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\} z(t)] = \exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\} f(t, \omega(\tau(t))z(\tau(t))).$$

Now similarly as in the proof of Theorem 3.7, we put $t_j = \sup\{t \geq t_{j-1}, \tau(t) \leq t_{j-1}\}$ and $M_j = \max\{|z(t)|, t \in [t_{j-1}, t_j]\}$ $(j = 1, 2, \dots)$. Considering arbitrary $t^* \in [t_j, t_{j+1}]$ and integrating the previous equation over $[t_j, t^*]$ we have

$$\omega(t)\exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}z(t)|_{t_j}^{t^*} = \int_{t_j}^{t^*} \exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}f(t, \omega(\tau(t))z(\tau(t)))\,\mathrm{d}t.$$

Hence

$$\begin{aligned} z(t^*) =& \exp\{\int_{t_j}^{t^*} a(u)\,\mathrm{d}u\}\frac{\omega(t_j)}{\omega(t^*)}z(t_j) + \frac{\exp\{\int_{t_0}^{t^*} a(u)\,\mathrm{d}u\}}{\omega(t^*)} \\ &\times \int_{t_j}^{t^*} \exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}f(t, \omega(\tau(t))z(\tau(t)))\,\mathrm{d}t. \end{aligned}$$

Using the condition (3.2) and the functional equation (3.6) we arrive at the estimate

$$\begin{aligned} |z(t^*)| \leq& M_j \exp\{\int_{t_j}^{t^*} a(u)\,\mathrm{d}u\}\frac{\omega(t_j)}{\omega(t^*)} \\ &- M_j^r \frac{\exp\{\int_{t_0}^{t^*} a(u)\,\mathrm{d}u\}}{\omega(t^*)} \int_{t_j}^{t^*} a(t)\omega(t)\exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}\,\mathrm{d}t. \end{aligned}$$

We can rewrite the last integral and integrate by parts. We obtain

$$\begin{aligned} \int_{t_j}^{t^*} a(t)\omega(t)\exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}\,\mathrm{d}t =& -\int_{t_j}^{t^*} \omega(t)\frac{\mathrm{d}}{\mathrm{d}t}[\exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}]\,\mathrm{d}t \\ =& -\omega(t)\exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}|_{t_j}^{t^*} - \int_{t_j}^{t^*} \omega'(t)\exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}\,\mathrm{d}t. \end{aligned}$$

Since the function $\omega(t)$ is nondecreasing on I, we obtain the estimation

$$-\int_{t_j}^{t^*} a(t)\omega(t)\exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}\,\mathrm{d}t \leq \omega(t)\exp\{-\int_{t_0}^{t} a(u)\,\mathrm{d}u\}|_{t_j}^{t^*}.$$

Hence, we can estimate $|z(t^*)|$ as

$$|z(t^*)| \leq M_j \exp\{\int_{t_j}^{t^*} a(u)\,\mathrm{d}u\}\frac{\omega(t_j)}{\omega(t^*)} + M_j^r\left(1 - \exp\{\int_{t_j}^{t^*} a(u)\,\mathrm{d}u\}\frac{\omega(t_j)}{\omega(t^*)}\right)$$

and then

$$|z(t^*)| \leq M_j^r + \left(M_j - M_j^r\right)\exp\{\int_{t_j}^{t^*} a(u)\,\mathrm{d}u\}\frac{\omega(t_j)}{\omega(t^*)}. \tag{3.21}$$

First let $M_0 \leq 1$. We put $j = 0$ in (3.21). Since $a(t)$ is a negative function and $\omega(t)$ is nondecreasing, we get $|z(t^*)| \leq M_0$, i.e., $M_1 \leq M_0$. Repeating this we obtain the boundedness of (M_j) as $j \to \infty$, namely $M_j \leq 1$ $(j = 0, 1, \dots)$. Hence, we have $|z(t)| \leq 1,\ t \in I$ and after using the backward substitution

$$|x(t)| \leq \omega(t) \quad \text{for all } t \geq t_0.$$

Now let $M_0 > 1$. Similarly to the previous case we put $j = 0$ in (3.21) and we get $|z(t^*)| \leq M_0^r$, i.e., $M_1 \leq M_0^r$. This implies $M_j \leq M_0^{r^j}$ $(j = 1, 2, \dots)$. By the definition of M_j, $|z(t)| \leq M_0^{r^{\psi(t)+1-\psi(t_0)}}$, $t \in I$, i.e.,

$$|x(t)| \leq \omega(t)M_0^{r^{\psi(t)+1-\psi(t_0)}} \quad \text{for all } t \geq t_0.$$

□

Remark 3.17. *It follows from the proof of Theorem 3.16 that the functional equation (3.6) can be replaced by the functional inequality (3.7) and the assertion of Theorem 3.16 remains valid. Moreover, in this case it is not necessary to require the differentiability of* $a(t)$, $b(t)$ *and we can omit the monotonic assumption on* $\frac{|b(t)|}{|a(t)|}$.

Remark 3.18. *The conclusions of Theorem 3.16 can be modified in the following way. By Proposition 3.4, we are able to make the function* ω *arbitrarily large on* $[\tau(t_0), t_0]$. *This implies, among others, that we can choose a solution* $\omega(t)$ *of (3.7) such that* $M_0 \le 1$. *However, it does not mean that this procedure automatically yields the "better" estimate than the original result of Theorem 3.16 (corresponding to the case* $M_0 > 1$*) yields. More details concerning this question will be discussed in the next section.*

3.3.2. Applications to particular cases

Corollary 3.19. *Consider the superlinear delay differential equation*

$$x'(t) = a(t)x(t) + b(t)|x(\tau(t))|^r \operatorname{sgn} x(\tau(t)), \quad t \in I, \ r > 1, \tag{3.22}$$

where $a, b \in C(I)$, $\tau \in C^1(I)$, $a(t) < 0$, $0 < |b(t)| \le K|a(t)|$, $\tau(t) < t$ *and* $\tau'(t) > 0$ *for all* $t \in I$ *and a suitable real* $K > 0$. *Let* $x(t)$ *be a solution of the equation (3.22) and let* $X_0 = \sup\{|x(t)|, t \in [\tau(t_0), t_0]\}$.

(i) If $X_0 \le K^{\frac{1}{1-r}}$, *then* $|x(t)| \le K^{\frac{1}{1-r}}$ *for all* $t \ge t_0$.

(ii) If $X_0 > K^{\frac{1}{1-r}}$, *then* $|x(t)| \le K^{\frac{1}{1-r}} \left(X_0 K^{\frac{-1}{1-r}}\right)^{r^{\psi(t)+1-\psi(t_0)}}$ *for all* $t \ge t_0$, *where* $\psi(t)$ *is a solution of (3.5) with the properties guaranteed by Proposition 3.2.*

Proof. It is easy to verify that under the assumption $0 < |b(t)| \le K|a(t)|$ the inequality (3.7) admits the positive constant solution, namely $\omega(t) \equiv K^{\frac{1}{1-r}}$. Then the cases (i) and (ii) follow immediately from Theorem 3.16 with the respect to Remark 3.17. ☐

Example 3.20. We investigate asymptotic behaviour of solutions of the superlinear delay differential equation with constant coefficients and a constant delay

$$x'(t) = ax(t) + b|x(t-\kappa)|^r \operatorname{sgn} x(t-\kappa), \quad t \in I, \ r > 1, \kappa > 0, \tag{3.23}$$

where $a < 0$, $b \ne 0$ are real constants.

The corresponding Abel equation (3.5) becomes $\psi(t-\kappa) = \psi(t) - 1$ and admits $\psi(t) = \frac{t}{\kappa}$ as a solution, which satisfies assumptions mentioned in Proposition 3.2. If we put $K = |\frac{b}{a}|$, then the estimate

$$|x(t)| \le \left|\frac{b}{a}\right|^{\frac{1}{1-r}} \left[X_0 \left|\frac{b}{a}\right|^{\frac{-1}{1-r}}\right]^{r^{\frac{1}{\kappa}(t-t_0)+1}}, \quad t \in I$$

holds for any solution $x(t)$ of (3.23) (see Corollary 3.19). Moreover, if $X_0 \le \left|\frac{b}{a}\right|^{\frac{1}{1-r}}$, then $|x(t)| \le \left|\frac{b}{a}\right|^{\frac{1}{1-r}}$ for all $t \in I$.

Example 3.21. Consider the superlinear delay differential equation with constant coefficients and with a proportional delay, i.e. the equation

$$x'(t) = ax(t) + b|x(\lambda t)|^r \operatorname{sgn} x(\lambda t), \quad t \geq t_0 > 0, \quad r > 1, \ 0 < \lambda < 1, \tag{3.24}$$

where $a < 0$, $b \neq 0$ are real constants.

In the case of the proportional delay, the function $\psi(t) = \frac{\log t}{\log \lambda^{-1}}$ is a solution of the Abel equation $\psi(\lambda t) = \psi(t) - 1$ and fulfills assumptions described in Proposition 3.2. If $K = |\frac{b}{a}|$, then, by Corollary 3.19,

$$|x(t)| \leq \left|\frac{b}{a}\right|^{\frac{1}{1-r}} \left[X_0 \left|\frac{b}{a}\right|^{\frac{-1}{1-r}}\right]^{r^{\frac{\log t - \log t_0}{\log \lambda^{-1}}+1}}, \quad t \in I$$

for any solution $x(t)$ of (3.24). Moreover, if $X_0 \leq \left|\frac{b}{a}\right|^{\frac{1}{1-r}}$, then $|x(t)| \leq \left|\frac{b}{a}\right|^{\frac{1}{1-r}}$ for all $t \in I$.

Remark 3.22. *Consider the equation (3.22) under assumptions of Corollary 3.19. Substituting into (3.7), we can easily verify that the system*

$$\omega(t) = K^{\frac{1}{1-r}} \exp\{\alpha r^{\psi(t)}\}, \quad t \in I, \ r > 1, \tag{3.25}$$

where α is a real parameter and $\psi(t)$ is a solution of the Abel equation (3.5), forms the one-parameters family of solutions of (3.7). Hence (3.25) satisfies the auxiliary functional inequality (3.7). Moreover, choosing α large enough we can fulfill the required inequality $\omega(t) > M$ for all $t \in [\tau(t_0), t_0]$ and M being arbitrarily large (see also Proposition 3.4). In particular, if

$$\alpha = r^{1-\psi(t_0)} \log\left(X_0 K^{-\frac{1}{1-r}}\right), \qquad X_0 = \sup\{|x(t)|, t \in [\tau(t_0), t_0]\},$$

then $M_0 = \sup\left\{\frac{|x(t)|}{\omega(t)}, t \in [\tau(t_0), t_0]\right\} \leq 1$ and, by Theorem 3.16, the estimate

$$|x(t)| \leq \omega(t) = K^{\frac{1}{1-r}} \left(X_0 K^{\frac{-1}{1-r}}\right)^{r^{\psi(t)+1-\psi(t_0)}} \qquad t \in I$$

holds for any solution $x(t)$ of (3.22). It may be interesting to note, that this estimate coincides with the result obtained in Corollary 3.19 (ii).

Corollary 3.23. *Consider the equation (3.22) subject to the condition*

$$0 < |b(t)| \leq K|a(t)| \exp\{c(t - r\tau(t))\}, \quad t \in I, \ r > 1, \tag{3.26}$$

where $a, b \in C(I)$, $\tau \in C^1(I)$, $a(t) < 0$, $\tau(t) < t$ and $\tau'(t) > 0$ for all $t \in I$ and $K > 0$, $c \geq 0$ are suitable real constants. Further, we assume that $X_0 = \sup\{|x(t)|, t \in [\tau(t_0), t_0]\}$.

(i) If $X_0 \leq K^{\frac{1}{1-r}} \exp\{c\tau(t_0)\}$, then for a solution $x(t)$ of (3.22) holds

$$|x(t)| \leq K^{\frac{1}{1-r}} \exp\{ct\} \quad \text{for all } t \geq t_0.$$

(ii) If $X_0 > K^{\frac{1}{1-r}}\exp\{c\tau(t_0)\}$, then for a solution $x(t)$ of (3.22) holds

$$|x(t)| \le K^{\frac{1}{1-r}}\exp\{ct\}\left(Y_0 K^{\frac{-1}{1-r}}\exp\{-c\tau(t_0)\}\right)^{r^{\psi(t)+1-\psi(t_0)}}$$

for all $t \ge t_0$, where $\psi(t)$ is a solution of (3.5) with the properties guaranteed by Proposition 3.2.

Proof. We can check that the function

$$\omega(t) = K^{\frac{1}{1-r}}\exp\{ct\}, \quad t \in I,\ r > 1,\ c \ge 0,\ K > 0$$

is a solution of inequality (3.7). By substituting into Theorem 3.16 we obtain the required estimate presented in Corollary 3.23. □

Example 3.24. We assume the equation

$$x'(t) = a(t)x(t) + b(t)\left(x\left(t/4\right)\right)^2, \quad t \ge 1, \tag{3.27}$$

where $a, b \in C(I)$, $a(t) < 0$, $0 < |b(t)| \le K|a(t)|\exp\{t/2\}$, $K > 0$ for all $t \ge 1$. Based on Corollary 3.23 (if we put $\tau(t) = t/4$, $r = 2$ and $c = 1$), the asymptotic behaviour of a solution $x(t)$ of (3.27) is going to be estimated.

The corresponding Abel equation (3.5) is $\psi(t/4) = \psi(t) - 1$ and admits the solution $\psi(t) = \frac{\log t}{\log 4}$. We assume that $x(t)$ is a solution of (3.27) and $X_0 = \sup\{|x(t)|, t \in [\frac{1}{4}, 1]\}$.

(i) If $X_0 \le K^{-1}\exp\{\frac{1}{4}\}$ then

$$|x(t)| \le K^{-1}\exp\{t\}, \quad t \ge 1.$$

(ii) If $X_0 > K^{-1}\exp\{\frac{1}{4}\}$ then

$$|x(t)| \le K^{-1}\exp\{t\}\left(X_0 K\exp\{-1/4\}\right)^{2\sqrt{t}}, \quad t \ge 1.$$

Remark 3.25. *Consider the equation (3.22) subject to the condition (3.26). It can be verified easily that the one-parameter system*

$$\omega(t) = K^{\frac{1}{1-r}}\exp\{ct + \alpha r^{\psi(t)}\}, \quad t \in I,\ r > 1,\ c \ge 0,\ K > 0,$$

where α is a real parameter and $\psi(t)$ is a solution of the Abel equation (3.5), solves the auxiliary functional inequality (3.7). We will proceed similarly as in Remark 3.22. Choosing α large enough we can ensure the validity of $\omega(t) > M$ for all $t \in [\tau(t_0, t_0]$ (see also Proposition 3.4). In particular, if we put

$$\alpha = r^{1-\psi(t_0)}\log\left(X_0 K^{\frac{-1}{1-r}}\exp\{-c\tau(t_0)\}\right),$$

where $X_0 = \sup\{|x(t)|, t \in [\tau(t_0), t_0]\}$, then $M_0 = \sup\left\{\frac{|x(t)|}{\omega(t)}, t \in [\tau(t_0), t_0]\right\} \le 1$ and, by Theorem 3.16, we can estimate any solution $x(t)$ of the equation (3.22) as

$$|x(t)| \le K^{\frac{1}{1-r}}\exp\{ct\}\left(X_0 K^{\frac{-1}{1-r}}\exp\{-c\tau(t_0)\}\right)^{r^{\psi(t)+1-\psi(t_0)}} \quad t \in I.$$

This estimate and the estimate presented in Corollary 3.23 (ii) are the same ones.

Remark 3.26. *The term $\psi(t) - \psi(t_0)$ appearing in the previous asymptotic estimates will be further studied. It holds that if $\psi(t)$ is a solution of the Abel equation (3.5), then $\psi(t) + \alpha, \alpha \in \mathbb{R}$ is also a solution of (3.5). This implies that without the loss of validity we can choose the solution $\psi(t)$ of (3.5) with the property $\psi(t_0) = 0$ and then omit $\psi(t_0)$ in all formulae involving the term $\psi(t) - \psi(t_0)$.*

4. Asymptotic estimates of solutions of linear or sublinear difference equations

4.1. Some preliminaries

The content of this chapter is based on the paper [11], which discusses some asymptotic properties of the delay difference equation

$$\Delta y(n) = p(n)y(n) + \sum_{i=1}^{k} q_i(n)|y(\bar{\tau}_i(n))|^{r_i} \operatorname{sgn} y(\bar{\tau}_i(n)) + d(n), \qquad n \in \mathbb{N}(n_0)\,, \tag{4.1}$$

where $n_0 \in \mathbb{Z}$ is nonnegative, $\mathbb{N}(n_0) = \{n_0, n_0+1, n_0+2, \dots\}$, $0 < r_i \le 1$ are real scalars, $p(n)$, $q_i(n)$, $d(n)$ are sequences of reals and $\bar{\tau}_i(n)$ are nondecreasing unbounded sequences of integers such that $\bar{\tau}_i(n) < n$ for all $n \in \mathbb{N}(n_0)$ $(i = 1, \dots, k)$. The forward difference operator Δ is defined as usually, i.e. $\Delta y(n) = y(n+1) - y(n)$. The equation (4.1) is a discrete analogue of the delay differential equation

$$x'(t) = a(t)x(t) + \sum_{i=1}^{k} b_i(t)|x(\tau(t))|^{r_i} \operatorname{sgn} x(\tau(t)) + g(t), \qquad t \ge t_0.$$

This differential equation with $k = 1$ has been discussed in the previous chapter. Considering the discrete case, we consider $k \in \mathbb{N}$ to generalize some known results of the qualitative theory of difference equations. Note that the extension of the results of Chapter 3 for the case of several delays is only a technical matter.

First, we briefly mention some known qualitative properties of the equation (4.1). The form (4.1) involves both the linear case ($r_i = 1$ for all $i = 1, \dots, k$) and the sublinear case ($0 < r_i < 1$ for some i). Probably the simplest (nontrivial) particular case of (4.1) is provided by the choice $r_i = 1$, $p(n) \equiv p$, $q_0 = 1 + p$, $q_i(n) \equiv q_i$ and $\bar{\tau}_i(n) = n - i$ $(i = 1, \dots, k)$ and $f(n) \equiv 0$, when (4.1) becomes

$$y(n+1) = \sum_{i=0}^{k} q_i y(n-i), \qquad n \in \mathbb{N}(n_0)\,. \tag{4.2}$$

The problem of necessary and sufficient conditions for the asymptotic stability of (4.2) has attracted the attention of many mathematicians. From a theoretical viewpoint, this problem is solved by the Schur-Cohn criterion (see [17]), but explicit conditions of asymptotic stability of (4.2) are known only in special cases (see [31]). We recall that the basic sufficient (in some particular cases also necessary) condition guaranteeing asymptotic stability of (4.2) for any $k \in \mathbb{N}$ is

$$\sum_{i=0}^{k} |q_i| < 1\,. \tag{4.3}$$

This condition can be extended to more general linear difference equations (see, e.g., [38]). We note that a stability condition for the nonautonomous equation (4.2) analogous to the condition (4.3) appears as a special consequence of our more general result proved by use of a different technique.

The asymptotic investigation of sublinear difference equations is less developed. Some related results can be found in [22] and [29], where properties of the equation

$$\Delta y(n) = f(y(n-\kappa)), \qquad n \in \mathbb{N}(n_0), \quad \kappa \in \mathbb{N}$$

have been reported. The description of asymptotics of (4.1) was stated in [45], where the author established a condition under which the behaviour at infinity of solutions of (4.1) with $d(n) \equiv 0$ can be related to the behaviour of a solution of the difference equation

$$\Delta y(n) = p(n)y(n), \qquad n \in \mathbb{N}(n_0)\,. \tag{4.4}$$

We recall this result here because of its relevance to our investigations.

Theorem 4.1 ([45, Theorem 2]). *Assume that $1 + p(n) \neq 0$, $d(n) \equiv 0$, $n \in \mathbb{N}$ and*

$$\sum_{n=n_0}^{\infty} \frac{1}{\prod_{j=0}^{n} |1+p(j)|} \sum_{i=1}^{k} |q_i(n)| \prod_{j=0}^{\bar{\tau}_i(n)-1} |1+p(j)|^{r_i} < \infty\,. \tag{4.5}$$

Then for any solution $y(n)$ of (4.1) there exists a solution $\bar{y}(n)$ of (4.4) such that $y(n)$ is either asymptotically equivalent to $\bar{y}(n)$, or $y(n)$ is of asymptotic order less than $\bar{y}(n)$. Conversely, for any solution $\bar{y}(n)$ of (4.4) there exists a solution $y(n)$ of (4.1) asymptotically equivalent to $\bar{y}(n)$.

Note that the condition (4.5) is "natural" especially for equations (4.1) with $|1+p(n)| > 1$, $n \in \mathbb{N}(n_0)$. If $|1+p(n)| < 1$, then (4.5) can result in a considerable restriction on coefficients $q_i(n)$ which must be very small (in modulus). For other related results we refer also to [1], [4] or [8].

The main goal is to formulate a general asymptotic bound for all solutions of (4.1) provided $|1+p(n)| < 1$ for all $n \in \mathbb{N}(n_0)$. Using this estimate we present some effective asymptotic criterions for solutions $y(n)$ of (4.1) in the linear and sublinear case. Throughout the whole chapter let us assume that $\bar{\tau}_i(n)$ $(i = 1, \dots, k)$ satisfy conditions introduced at the beginning of this chapter, i.e. $\bar{\tau}_i(n)$ are nondecreasing unbounded sequences of integers such that $\bar{\tau}_i(n) < n$ for all $n \in \mathbb{N}(n_0)$ $(i = 1, \dots, k)$.

4.2. The main result

Let $n_{-1} = \min\{\bar{\tau}_i(n_0) : i = 1, \dots, k\}$. By a solution of (4.1) we mean a sequence $y(n)$ of real numbers which is defined for $n \geq n_{-1}$ and satisfies (4.1) for $n \geq n_0$. It is easy to see that for any given $n_0 \in \mathbb{N}(0)$ and initial conditions $y(n) = y_0(n)$, $n_{-1} \leq n \leq n_0$, the equation (4.1) has a unique solution satisfying these initial conditions.

In the sequel, we formulate an upper bound for solutions $y(n)$ of (4.1). Before doing this, we introduce some necessary notations and auxiliary relations. Put

$$\begin{aligned} \sigma_{-1} &= n_{-1}, \\ \sigma_0 &= n_0, \\ \sigma_{m+1} &= \max\{n \in \mathbb{N}(n_0) :\ \bar{\tau}_i(n) \leq \sigma_m \text{ for all } i = 1, \dots, k\}, \quad m = 0, 1, 2, \dots \end{aligned}$$

and consider two difference inequalities

$$\bar{\psi}(\sigma_{m+1}) \geq \bar{\psi}(\sigma_m) + 1, \qquad m = 0, 1, 2, \dots \tag{4.6}$$

and

$$\sum_{i=1}^{k} |q_i(n)|(\bar{\omega}(\bar{\tau}_i(n)))^{r_i} \leq (1 - |1 + p(n)|)\bar{\omega}(n), \qquad n \in \mathbb{N}(n_0)\,. \tag{4.7}$$

Note that previous inequalities correspond to the auxiliary relations used in Chapter 3. More precisely, the relation (4.6) is an analogue of the Abel equation (3.5) and the inequality (4.7) is consistent with the auxiliary functional relation (3.7). In addition, sequences $\bar{\psi}(n)$, $\bar{\omega}(n)$ are discrete analogues of functions $\psi(t)$, $\omega(t)$, respectively.

Further, for $m = 0, 1, 2, \dots$ we denote

$$u(m) = \min\{\frac{\Delta\bar{\omega}(\nu)}{1 - |1 + p(\nu)|} : \sigma_m \leq \nu \leq \sigma_{m+1}\} \tag{4.8}$$

$$v(m) = \max\{\frac{|d(\nu)|}{(1 - |1 + p(\nu)|)\bar{\omega}(\nu)} : \sigma_m \leq \nu \leq \sigma_{m+1}\}\,. \tag{4.9}$$

Theorem 4.2. *Consider the equation (4.1), where $|1 + p(n)| < 1$ for all $n \in \mathbb{N}(n_0)$. Further, let $\bar{\omega}(n)$ be a positive monotonous sequence satisfying (4.7), let $\bar{\psi}(n)$ be a positive increasing sequence satisfying (4.6) and let $u(m)$, $v(m)$ be given by (4.8) and (4.9), respectively.*

(i) If $\bar{\omega}(n)$ is nondecreasing, then there exists a constant $L > 0$ such that

$$|y(n)| \leq \Big(L + \sum_{i=0}^{\lfloor\bar{\psi}(n)\rfloor} v(i)\Big)\bar{\omega}(n) \tag{4.10}$$

for any solution $y(n)$ of (4.1) and all $n \in \mathbb{N}(n_0)$. (The symbol $\lfloor\ \rfloor$ means an integer part.)

(ii) If $\bar{\omega}(n)$ is decreasing, then there exists a constant $L > 0$ such that

$$|y(n)| \leq \Big(L + \sum_{i=0}^{\lfloor\bar{\psi}(n)\rfloor} v(i)\Big)\bar{\omega}(n) \prod_{s=0}^{\lfloor\bar{\psi}(n)\rfloor} \Big(1 - \frac{u(s)}{\bar{\omega}(\sigma_{s+1})}\Big) \tag{4.11}$$

for any solution $y(n)$ of (4.1) and all $n \in \mathbb{N}(n_0)$.

Proof. Using the transformation $z(n) = y(n)/\bar{\omega}(n)$ we convert (4.1) into

$$\begin{aligned}\bar{\omega}(n+1)z(n+1) =& (1 + p(n))\bar{\omega}(n)z(n) + \sum_{i=1}^{k} q_i(n)(\bar{\omega}(\bar{\tau}_i(n)))^{r_i}|z(\bar{\tau}_i(n))|^{r_i} \\ &\times \operatorname{sgn} z(\bar{\tau}_i(n)) + d(n)\,.\end{aligned} \tag{4.12}$$

Now we denote $M(s) = \max\{|z(\nu)|,\ \nu \in \mathbb{Z}, \sigma_{-1} \leq \nu \leq \sigma_s\}$ and $\bar{M}(s) = \max(M(s), 1)$, $s = 0, 1, 2, \dots$.

Let $n^* \in \mathbb{N}(n_0)$ and let $m \in \mathbb{N}$ be such that $\sigma_m < n^* \le \sigma_{m+1}$. We wish to express and estimate $z(n^*)$ in term of $z(s)$, where $\sigma_{-1} \le s \le \sigma_m$. Hereafter it is necessary to distinguish the following three cases in the proof:

(a) Let $1+p(n) \neq 0$ for any $\sigma_m \le n \le n^*-1$. Multiply relation (4.12) by $1/\prod_{\ell=\sigma_m}^{n}(1+p(\ell))$ to get

$$\Delta\Big(\frac{\bar\omega(n)z(n)}{\prod_{\ell=\sigma_m}^{n-1}(1+p(\ell))}\Big) = \frac{\sum_{i=1}^k q_i(n)(\bar\omega(\bar\tau_i(n)))^{r_i}|z(\bar\tau_i(n))|^{r_i}\operatorname{sgn} z(\bar\tau_i(n)) + d(n)}{\prod_{\ell=\sigma_m}^{n}(1+p(\ell))}.$$

By applying the discrete Newton-Leibniz formula we have

$$\frac{\bar\omega(n^*)z(n^*)}{\prod_{\ell=\sigma_m}^{n^*-1}(1+p(\ell))} - \bar\omega(\sigma_m)z(\sigma_m) = \sum_{j=\sigma_m}^{n^*-1}\frac{\sum_{i=1}^k q_i(j)(\bar\omega(\bar\tau_i(j)))^{r_i}|z(\bar\tau_i(j))|^{r_i}\operatorname{sgn} z(\bar\tau_i(j)) + d(j)}{\prod_{\ell=\sigma_m}^{j}(1+p(\ell))},$$

i.e.

$$\begin{aligned} z(n^*) =& \frac{\bar\omega(\sigma_m)}{\bar\omega(n^*)}z(\sigma_m)\prod_{\ell=\sigma_m}^{n^*-1}(1+p(\ell)) + \frac{1}{\bar\omega(n^*)}\sum_{j=\sigma_m}^{n^*-1}\Big(\sum_{i=1}^{k} q_i(j)(\bar\omega(\bar\tau_i(j)))^{r_i}|z(\bar\tau_i(j))|^{r_i}\operatorname{sgn} z(\bar\tau_i(j)) \\ &+ d(j)\Big)\prod_{\ell=j+1}^{n^*-1}(1+p(\ell)). \end{aligned}$$

The relation (4.7) implies

$$\begin{aligned} |z(n^*)| \le& \bar M(m)\frac{\bar\omega(\sigma_m)}{\bar\omega(n^*)}\prod_{\ell=\sigma_m}^{n^*-1}|1+p(\ell)| + \frac{1}{\bar\omega(n^*)}\sum_{j=\sigma_m}^{n^*-1}\Big(\bar M(m)(1-|1+p(j)|)\bar\omega(j) \\ &+ |d(j)|\Big)\prod_{\ell=j+1}^{n^*-1}|1+p(\ell)| \\ =& \bar M(m)\frac{\bar\omega(\sigma_m)}{\bar\omega(n^*)}\prod_{\ell=\sigma_m}^{n^*-1}|1+p(\ell)| + \frac{1}{\bar\omega(n^*)}\sum_{j=\sigma_m}^{n^*-1}\Big(\bar M(m) + \frac{|d(j)|}{(1-|1+p(j)|)\bar\omega(j)}\Big) \\ &\times \bar\omega(j)\Delta\prod_{\ell=j}^{n^*-1}|1+p(\ell)|, \end{aligned}$$

where the difference operator Δ is considered with respect to the variable j. Then using (4.9) we get

$$|z(n^*)| \le \bar M(m)\frac{\bar\omega(\sigma_m)}{\bar\omega(n^*)}\prod_{\ell=\sigma_m}^{n^*-1}|1+p(\ell)| + \frac{\bar M(m)+v(m)}{\bar\omega(n^*)}\sum_{j=\sigma_m}^{n^*-1}\bar\omega(j)\Delta\prod_{\ell=j}^{n^*-1}|1+p(\ell)|.$$

By applying the summation by parts we arrives at

$$|z(n^*)| \le \Big(\bar M(m)+v(m)\Big)\Big(1-\frac{1}{\bar\omega(n^*)}\sum_{j=\sigma_m}^{n^*-1}\Delta\bar\omega(j)\prod_{\ell=j+1}^{n^*-1}|1+p(\ell)|\Big). \tag{4.13}$$

If $\bar\omega(n)$ is nondecreasing, then (4.13) can be reduced to

$$|z(n^*)| \le \bar M(m) + v(m). \tag{4.14}$$

Obviously

$$\bar{M}(m) \le \bar{M}(0) + \sum_{i=0}^{m-1} v(i), \qquad \text{hence} \qquad |z(n^*)| \le \bar{M}(0) + \sum_{i=0}^{m} v(i)\,. \tag{4.15}$$

To estimate m in terms of n^* we recall that $\sigma_m < n^* \le \sigma_{m+1}$. Since $\bar{\psi}(\sigma_m) \ge \bar{\psi}(\sigma_0) + m$, we get

$$m \le \bar{\psi}(\sigma_m) - \bar{\psi}(\sigma_0) \le \lfloor \bar{\psi}(n^*) \rfloor\,. \tag{4.16}$$

Now substituting back $y(n) = \bar{\omega}(n) z(n)$ into $(4.15)_2$ we can deduce the validity of (4.10).

If $\bar{\omega}(n)$ is decreasing, then (4.13) becomes

$$\begin{aligned} |z(n^*)| &\le \big(\bar{M}(m) + v(m)\big)\Big(1 - \frac{1}{\bar{\omega}(n^*)} \sum_{j=\sigma_m}^{n^*-1} \frac{\Delta\bar{\omega}(j)}{1 - |1+p(j)|}\ \Delta \prod_{\ell=j}^{n^*-1} |1 + p(\ell)|\Big) \\ &\le \big(\bar{M}(m) + v(m)\big)\Big(1 - \frac{u(m)}{\bar{\omega}(n^*)}\Big) \end{aligned} \tag{4.17}$$

by use of (4.8). Repeated application of this estimate yields

$$\begin{aligned} \bar{M}(m) &\le \bar{M}(0) \prod_{s=0}^{m-1} \Big(1 - \frac{u(s)}{\bar{\omega}(\sigma_{s+1})}\Big) + \sum_{i=0}^{m-1} v(i) \prod_{s=i}^{m-1} \Big(1 - \frac{u(s)}{\bar{\omega}(\sigma_{s+1})}\Big) \\ &\le \Big(\bar{M}(0) + \sum_{i=0}^{m-1} v(i)\Big) \prod_{s=0}^{m-1} \Big(1 - \frac{u(s)}{\bar{\omega}(\sigma_{s+1})}\Big)\,. \end{aligned}$$

Now the backward substitution $y(n) = \bar{\omega}(n) z(n)$ along with (4.16) implies (4.11).

(b) Let $1 + p(n^* - 1) = 0$. Then

$$z(n^*) = \frac{1}{\bar{\omega}(n^*)}\Big(\sum_{i=1}^{k} q_i(n^*-1)\bar{\omega}(\bar{\tau}_i(n^*-1))^r |z(\bar{\tau}_i(n^*-1))|^r \operatorname{sgn} z(\bar{\tau}_i(n^*-1)) + d(n^*-1)\Big).$$

Taking absolute values and using (4.7) and (4.9) we get

$$|z(n^*)| \le \bar{M}(m)\frac{\bar{\omega}(n^*-1)}{\bar{\omega}(n^*)} + \frac{|d(n^*-1)|}{\bar{\omega}(n^*)} \le \big(\bar{M}(m) + v(m)\big)\frac{\bar{\omega}(n^*-1)}{\bar{\omega}(n^*)}\,. \tag{4.18}$$

If $\bar{\omega}(n)$ is nondecreasing then (4.18) can be reduced to $|z(n^*)| \le \bar{M}(m) + v(m)$. This relation corresponds to the relation (4.14) as it was proceeded in the part (a).

If $\bar{\omega}(n)$ is decreasing then by use of (4.8) we get

$$|z(n^*)| \le \big(\bar{M}(m) + v(m)\big)\Big(1 - \frac{\Delta\bar{\omega}(n^*-1)}{\bar{\omega}(n^*)}\Big) \le \big(\bar{M}(m) + v(m)\big)\Big(1 - \frac{u(m)}{\bar{\omega}(n^*)}\Big).$$

This relation corresponds exactly to the equation (4.17). The proof continues as in the previous part (a).

(c) Let $1 + p(n^*-1) \ne 0$ and $1 + p(\nu) = 0$ for some $\nu \in \mathbb{N}(n_0)$ such than $\sigma_m \le \nu < n^* - 1$. The proof technique applied in this case is a combination of procedures utilized in cases (a) and (b) and therefore we present only the main idea. Denote

$$\sigma := \max\{\nu \in \mathbb{N}(n_0), \sigma_m \le \nu < n^* - 1 \text{ and } 1 + p(\nu) = 0\}.$$

Then multiply the equation (4.12) by $1/\prod_{\ell=\sigma+1}^{n}(1+p(\ell))$ and sum from $\sigma+1$ to n^*-1 to obtain

$$z(n^*) = \frac{\bar{\omega}(\sigma+1)}{\bar{\omega}(n^*)} z(\sigma+1) \prod_{\ell=\sigma+1}^{n^*-1}(1+p(\ell)) + \frac{1}{\bar{\omega}(n^*)} \sum_{j=\sigma+1}^{n^*-1} \Big(\sum_{i=1}^{k} q_i(j)(\bar{\omega}(\bar{\tau}_i(j)))^{r_i} |z(\bar{\tau}_i(j))|^{r_i}$$
$$\times \operatorname{sgn} z(\bar{\tau}_i(j)) + d(j) \Big) \prod_{\ell=j+1}^{n^*-1}(1+p(\ell)).$$

The definition of σ implies $1+p(\sigma)=0$, hence, by the case (b), we can use the estimate

$$|z(\sigma+1)| \le \big(\bar{M}(m)+v(m)\big)\frac{\bar{\omega}(\sigma)}{\bar{\omega}(\sigma+1)}.$$

Then the application of relation (4.7) yields

$$|z(n^*)| \le \big(\bar{M}(m)+v(m)\big)\frac{\bar{\omega}(\sigma)}{\bar{\omega}(n^*)} \prod_{\ell=\sigma+1}^{n^*-1}|1+p(\ell)| + \frac{1}{\bar{\omega}(n^*)} \sum_{j=\sigma+1}^{n^*-1} \Big(\bar{M}(m)(1-|1+p(j)|)\bar{\omega}(j)$$
$$+ |d(j)|\Big) \prod_{\ell=j+1}^{n^*-1}|1+p(\ell)|.$$

Using (4.9) we obtain

$$|z(n^*)| \le \big(\bar{M}(m)+v(m)\big)\frac{\bar{\omega}(\sigma)}{\bar{\omega}(n^*)} \prod_{\ell=\sigma+1}^{n^*-1}|1+p(\ell)| + \frac{\bar{M}(m)+v(m)}{\bar{\omega}(n^*)} \sum_{j=\sigma+1}^{n^*-1} \bar{\omega}(j) \prod_{\ell=j+1}^{n^*-1}|1+p(\ell)|$$
$$\le \big(\bar{M}(m)+v(m)\big)\Big(1-\frac{1}{\bar{\omega}(n^*)}\sum_{j=\sigma}^{n^*-1}\Delta\bar{\omega}(j)\prod_{\ell=j+1}^{n^*-1}|1+p(\ell)|\Big).$$

The right-hand side of this inequality is a modification of the corresponding term involved in (4.13) with σ_m replaced by σ. Using the same line of arguments as given in the case (a) we arrive at (4.10) for $\bar{\omega}(n)$ nondecreasing and (4.11) for $\bar{\omega}(n)$ decreasing. □

Remark 4.3. *If the product in (4.11) converges as $n \to \infty$, it is useless to solve the auxiliary relation (4.6) and the estimate (4.11) becomes (4.10) (see also Corollary 4.10).*

Remark 4.4. *By Theorem 4.2, any solution $y(n)$ of the delay difference equation (4.1) with a forcing term $d(n)$, can be estimated in terms of solutions of difference inequalities (4.6) and (4.7). Moreover, if $d(n)$ is identically zero, then $v(i)$ is also identically zero and both the estimates (4.10) and (4.11) are significantly simplified.*

Remark 4.5. *The asymptotics of solutions of (4.1), described by estimates (4.10) and (4.11) under the assumption $|1+p(n)|<1$, is quite different from that presented in Theorem 4.1. In particular, contrary to Theorem 4.1, we are able to formulate conditions for boundedness of solutions $y(n)$ of (4.1), or discuss their convergency to zero including the rate of this convergency.*

4.3. Applications to particular cases

In this part we apply our general asymptotic result to some important particular cases to demonstrate, how it can be turned into effective asymptotic criterions.

Corollary 4.6. *Consider the equation (4.1), where $d(n) \equiv 0$ and let $r = \max\{r_1, \dots, r_k\}$. Then any solution $y(n)$ of (4.1) is bounded if either*

$$r = 1 \qquad \text{and} \qquad |1 + p(n)| + \sum_{i=1}^{k} |q_i(n)| \le 1, \quad n \in \mathbb{N}(n_0) \tag{4.19}$$

or

$$0 < r < 1 \qquad \text{and} \qquad 0 < \frac{\sum_{i=1}^{k} |q_i(n)|}{1 - |1 + p(n)|} < K, \quad n \in \mathbb{N}(n_0), \tag{4.20}$$

where K is a suitable scalar.

Proof. If (4.19) holds, then for any constant $\bar{\omega} \ge 1$ the sequence $\bar{\omega}(n) \equiv \bar{\omega}$ is a positive constant solution of (4.7). Let (4.20) holds and let $K \ge 1$. Then

$$\sum_{i=1}^{k} |q_i(n)| K^{\frac{r_i}{1-r}} \le (1 - |1 + p(n)|) K^{\frac{1}{1-r}}, \qquad n \in \mathbb{N}(n_0)\,,$$

hence $\bar{\omega}(n) \equiv K^{1/(1-r)}$ is also a positive constant solution of (4.7). The statement now follows immediately from (4.10) with respect to $v(i) \equiv 0$. ☐

Remark 4.7. *The condition (4.19) corresponds to the known stability results for linear difference equations with a constant delay (in particular, $(4.19)_2$ is consistent with (4.3)). However, considering the sublinear case, a region of coefficients guaranteeing boundedness of all solutions of (4.1) is much larger. The following example illustrates it.*

Example 4.8. The linear difference equation

$$y(n+1) = q_0 y(n) + \sum_{i=1}^{k} q_i y(\bar{\tau}_i(n)), \qquad n \in \mathbb{N}(n_0)$$

has all its solutions bounded if $\sum_{i=0}^{k} |q_i| \le 1$. The sublinear difference equation

$$y(n+1) = q_0 y(n) + \sum_{i=1}^{k} q_i |y(\bar{\tau}_i(n))|^{r_i} \operatorname{sgn} y(\bar{\tau}_i(n)), \quad 0 < r_i < 1, \qquad n \in \mathbb{N}(n_0)$$

has all its solutions bounded if $|q_0| < 1$ (the values of $q_1, \dots, q_k$ may be arbitrary). It can be verified that the constant $K = \max\left(1, \frac{\sum_{i=1}^{k} |q_i|}{1-|q_0|}\right)$ satisfies relation $(4.20)_2$ and the sequence $\bar{\omega}(n) \equiv K^{1/(1-r)}$ is the solution of inequality (4.7). Then boundedness of the solution $y(n)$ follows from the previous Corollary 4.6.

Example 4.9. Now we extend our illustrations by involving a nonzero term $d(n)$ and show its influence on boundedness and asymptotics of solutions. Consider the sublinear difference equation

$$\Delta y(n) = p y(n) + \sum_{i=1}^{k} q_i |y(\bar{\tau}_i(n))|^{r_i} \operatorname{sgn} y(\bar{\tau}_i(n)) + d(n), \qquad n \in \mathbb{N}(n_0), \tag{4.21}$$

where $0 < r_i < 1$ $(i = 1, \dots, k)$, p, $q_1, \dots, q_k$ are real scalars. Assume that $|1+p| < 1$ and $d(n)$ is bounded (an analogous discussion can be performed for the corresponding linear equation satisfying $|1+p| + \sum_{i=1}^{k} |q_i| \le 1$). By (4.8), the sequence $u(i)$ is zero and, by (4.9), the sequence $v(i)$ is bounded. Then Theorem 4.2 implies that

$$y(n) = O(\bar{\psi}(n)) \qquad \text{as } n \to \infty \tag{4.22}$$

for any solution $y(n)$ of (4.21), where the sequence $\bar{\psi}(n)$ satisfying (4.6).

To make this estimate quite explicit, we have to specify $\bar{\tau}_i(n)$. Let $\bar{\tau}_i(n) = n - \kappa_i$, $\kappa_i \in \mathbb{N}$ $(i = 1, \dots, k)$, then $\sigma_m = \sigma_0 + m\kappa$, where $\kappa = \min\{\kappa_1, \dots, \kappa_k\}$ and it can be easily checked that $\bar{\psi}(n) = n/\kappa$ is a positive and nondecreasing sequence satisfying (4.6). Hence, (4.22) becomes

$$y(n) = O(n) \qquad \text{as } n \to \infty\,.$$

Further let $\bar{\tau}_i(n) = \lfloor \lambda_i n \rfloor$, $0 < \lambda_i < 1$ $(i = 1, \dots, k)$ and let $\lambda := \max\{\lambda_1, \dots, \lambda_k\}$. Then $\sigma_m = \lfloor \frac{\sigma_{m-1}}{\lambda} \rfloor$ and we can choose $\bar{\psi}(n) = \frac{\log(n - \lambda/(1-\lambda))}{\log \lambda^{-1}}$. In such a case, (4.22) becomes

$$y(n) = O(\log n) \qquad \text{as } n \to \infty\,.$$

To obtain a boundedness condition, assume that $d(n) = O(1/n)$ as $n \to \infty$. Analogously as in the previous example, if $\bar{\tau}_i(n) = \lfloor \lambda_i n \rfloor$, $0 < \lambda_i < 1$ $(i = 1, \dots, k)$, then any solution $y(n)$ of (4.21) is already bounded.

As another consequence, we discuss the sublinear difference equation without a forcing term

$$\Delta y(n) = p(n)y(n) + q(n)|y(\lfloor \lambda n \rfloor)|^r \operatorname{sgn} y(\lfloor \lambda n \rfloor), \qquad n \in \mathbb{N}(n_0), \quad 0 < \lambda, r < 1\,, \tag{4.23}$$

originating from the numerical discretization of the sublinear pantograph equation. We present conditions under which all its solutions tend to zero and derive also the rate of this convergency.

Corollary 4.10. *Consider the equation (4.23), where $|1+p(n)| \le \bar{p} < 1$ for all $n \in \mathbb{N}(n_0)$ and $|q(n)| = O(n^{\alpha(1-r)})$ as $n \to \infty$ for a real scalar α. Then*

$$y(n) = O(n^\alpha) \qquad \text{as } n \to \infty \tag{4.24}$$

for any solution $y(n)$ of (4.23).

Proof. The proof is divided in two cases, according to whether α is nonnegative or negative.
(a) Let $\alpha \ge 0$ and let K_1 be the upper bound of the sequence $|q(n)|n^{-\alpha(1-r)}$. We verify that $\bar{\omega}(n) = K_2 n^\alpha$, where $K_2 = (K_1 \lambda^{\alpha r}/(1-\bar{p}))^{\frac{1}{1-r}}$ is a solution of (4.7). Indeed, by substituting this form into (4.7) we get

$$|q(n)| K_2^r \lfloor \lambda n \rfloor^{\alpha r} \le K_1 K_2^r \lambda^{\alpha r} n^{\alpha(1-r)+\alpha r} = (1-\bar{p}) K_2 n^\alpha \le (1 - |1+p(n)|) K_2 n^\alpha\,.$$

Then (4.24) follows from (4.10).
(b) If $\alpha < 0$, we have to discuss the estimate (4.11). Here, similarly, $\bar{\omega}(n) = K_3 n^\alpha$ is the solution of (4.7) for a suitable $K_3 > 0$. Furthermore, by using the mean value theorem and the relation

$$\lfloor \lambda^{-1} \sigma_s \rfloor \le \sigma_{s+1} \le \lfloor \lambda^{-1}(\sigma_s + 1) \rfloor + 1$$

we obtain

$$|u(s)| = \max\{\frac{-\Delta\bar{\omega}(\nu)}{1-|1+p(\nu)|} : \sigma_s \le \nu \le \sigma_{s+1}\} \le \frac{K_3|\alpha|(\sigma_s)^{\alpha-1}}{1-\bar{p}}$$

and $\bar{\omega}(\sigma_{s+1}) = K_3(\sigma_{s+1})^\alpha \ge K_3\lambda^{-\alpha}(\sigma_s)^\alpha$. From here we get

$$\frac{|u(s)|}{\bar{\omega}(\sigma_{s+1})} = \frac{|\alpha|\lambda^\alpha}{(1-\bar{p})\sigma_s}.$$

Further it holds that

$$\sigma_s \ge \left\lfloor \frac{\sigma_{s-1}}{\lambda} \right\rfloor \ge \frac{\sigma_{s-1}}{\lambda} - 1 \ge \frac{\sigma_0}{\lambda^{s-1}} - \sum_{i=0}^{s-1} \frac{1}{\lambda^i} \ge \frac{\sigma_0}{\lambda^{s-1}}$$

and

$$\frac{|u(s)|}{\bar{\omega}(\sigma_{s+1})} = O(\lambda^s) \qquad \text{as } s \to \infty\,.$$

Consequently, the product in (4.11) converges as $n \to \infty$ and the estimate (4.11) becomes (4.24). □

Example 4.11. Assume the following difference equation

$$\Delta y(n) = -p\Big(1+\frac{1}{\sqrt{n}}\Big)y(n) + \frac{q}{n}|y(\lfloor \lambda n \rfloor)|^r \operatorname{sgn} y(\lfloor \lambda n \rfloor), \qquad n \in \mathbb{N}(n_0), \tag{4.25}$$

where $n_0 \ge 1$, $0 < \lambda, r < 1$, p, q are real constants such that $q \ne 0$, $p \in \left(0, \frac{2\sqrt{n_0}}{1+\sqrt{n_0}}\right)$. It is easy to verify that there exists $\bar{p} \in \mathbb{R}$ such that $|1 - p(1+\frac{1}{\sqrt{n}})| \le \bar{p} < 1$ holds for all $n \in \mathbb{N}(n_0)$. Further, $|q|/n = O(n^{\alpha(1-r)})$ as $n \to \infty$ for $\alpha = \frac{-1}{1-r}$. Then (4.24) implies

$$y(n) = O\left(n^{\frac{-1}{1-r}}\right) \qquad \text{as } n \to \infty$$

for any solution $y(n)$ of (4.25).

5. Some discretizations of sublinear delay differential equations

The importance of numerical solutions of delay differential equations has still been stated in the first chapter. Problems of numerical methods for (linear as well as nonlinear) delay differential equations have been investigated in many papers (see e.g. [7], [24], [25], [35], [37] or [43]). General reference is [6], where the overview of basic results from numerical analysis for differential equations with delayed argument can be found. In particular, numerical investigations of the θ-method for some linear delay differential equations are the subject of papers [5], [12], [20], [21], [32] and [44]. It can be stated that the analysis of the θ-method for nonlinear delay differential equations is just at its beginning.

In the first section of this chapter, the Euler formula for a sublinear delay differential equation is derived. Further, this formula is extended to the θ-method. Applications of qualitative results derived in Chapter 4 to these numerical discretizations are presented in the following two sections. The stability of these discretizations is discussed as well. Numerical experiments displayed in the last section of this chapter illustrate theoretical results.

5.1. The derivation of numerical formulae

For the illustration of derivations of corresponding difference relations we provide numerical discretization for the equation (3.1) in its special cases (3.3) and (3.4), i.e.

$$x'(t) = a(t)x(t) + b(t)|x(\tau(t))|^r \operatorname{sgn} x(\tau(t)) + g(t), \quad t \in I,\ r > 0, \tag{5.1}$$

where $a(t)$, $b(t)$, $g(t)$ are real continuous functions on I, r is a suitable constant and $\tau(t)$ is a function of delay satisfying assumptions introduced in the first chapter. Note that the derivation of appropriate discretizations for more general nonlinear equations is only a technical problem.

We set the discretization equidistant grid $t_n := t_0 + nh$, where $n \in \mathbb{N}$ and $h > 0$ is the stepsize, and let $t \in [t_n, t_{n+1}]$ be an arbitrary point. Then from the integration of (5.1) over $[t_n, t_{n+1}]$ we obtain

$$\begin{aligned} x(t_{n+1}) - x(t_n) = &\int_{t_n}^{t_{n+1}} a(u)x(u)\,\mathrm{d}u \ + \int_{t_n}^{t_{n+1}} b(u)|x(\tau(u))|^r \operatorname{sgn} x(\tau(u))\,\mathrm{d}u \ + \\ &+ \int_{t_n}^{t_{n+1}} g(u)\,\mathrm{d}u. \end{aligned} \tag{5.2}$$

To approximate integrals on the right-hand side we use some standard numerical integrations methods. First we approximate integrals on the right-hand side of (5.2) using the rectangular formula with the left grid point, i.e.

$$\int_{t_n}^{t_{n+1}} a(u)x(u)\,\mathrm{d}u \approx ha(t_n)x(t_n),$$

$$\int_{t_n}^{t_{n+1}} b(u)|x(\tau(u))|^r \operatorname{sgn} x(\tau(u))\,\mathrm{d}u \approx hb(t_n)|x(\tau(t_n))|^r \operatorname{sgn} x(\tau(t_n)),$$

$$\int_{t_n}^{t_{n+1}} g(u)\,\mathrm{d}u \approx hg(t_n).$$

Now let $y(n) \approx x(t_n)$. The replacement of $x(\tau(t_n))$ (necessary in the second relation) is not evident to provide, because the value of $\tau(t_n)$ is not usually a grid point. This replacement can be done by several ways. At first, we show the simplest of them. We perform the piecewise constant interpolation, i.e., we replace this value by the nearest left grid point:

$$x(\tau(t_n)) \approx y(\bar{\tau}(n)), \qquad \bar{\tau}(n) := \left\lfloor \frac{\tau(t_n) - t_0}{h} \right\rfloor,$$

where symbol $\lfloor\ \rfloor$ means an integer part. Substituting this into (5.2) we get the forward Euler method in the form

$$\Delta y(n) = p(n)y(n) + q(n)|y(\bar{\tau}(n))|^r \operatorname{sgn} y(\bar{\tau}(n)) + d(n), \tag{5.3}$$

where

$$p(n) := ha(t_n), \qquad q(n) := hb(t_n), \qquad d(n) := hg(t_n). \tag{5.4}$$

Note that (5.3) has the form of the delay difference equation (4.1) with one lag (i.e. $k = 1$) investigated in Chapter 4.

Another standard way of discretization of (5.1) utilizes the fact that integrals on the right-hand side of (5.2) can be approximated using the rectangular formula with the right grid point. Thus we arrive at the backward Euler formula.

The linear combinations of both Euler methods implies the θ-method formula ($0 \leq \leq \theta \leq 1$). In this case, the replacement of all integrations on the right-hand side of (5.2) is

$$\int_{t_n}^{t_{n+1}} a(u)x(u)\,\mathrm{d}u \approx h\Big((1-\theta)a(t_n)x(t_n) + \theta a(t_{n+1})x(t_{n+1})\Big)$$
$$\approx h\Big((1-\theta)a(t_n)y(n) + \theta a(t_{n+1})y(n+1)\Big),$$

$$\int_{t_n}^{t_{n+1}} b(u)|x(\tau(u))|^r \operatorname{sgn} x(\tau(u))\,\mathrm{d}u \approx h\Big((1-\theta)b(t_n)|x(\tau(t_n))|^r \operatorname{sgn} x(\tau(t_n)) +$$
$$+ \theta b(t_{n+1})|x(\tau(t_{n+1}))|^r \operatorname{sgn} x(\tau(t_{n+1}))\Big),$$

$$\int_{t_n}^{t_{n+1}} g(u)\,\mathrm{d}u \approx h\Big((1-\theta)g(t_n) + \theta g(t_{n+1})\Big).$$

The replacement of the values of $x(t)$ at the points $\tau(t_n)$, $\tau(t_{n+1})$ can be done similarly as in the previous part. We use the piecewise linear interpolation utilizing the left and right neighbours of $\tau(t_n)$, namely

$$x(\tau(t_n)) \approx y^h(\bar{\tau}(n)) := (1-s_n)y(\bar{\tau}(n)) + s_n y(\bar{\tau}(n)+1), \tag{5.5}$$

where

$$s_n := \frac{\tau(t_n) - t_0}{h} - \left\lfloor \frac{\tau(t_n) - t_0}{h} \right\rfloor .$$

The interpolation value of point $\tau(t_{n+1})$ is performed analogously. Then (5.2) becomes

$$\begin{aligned}\Delta y(n) = h\Big(&(1-\theta)a(t_n)y(n) + \theta a(t_{n+1})y(n+1) + (1-\theta)b(t_n)|y^h(\bar\tau(n))|^r \\ &\times \operatorname{sgn} y^h(\bar\tau(n)) + \theta b(t_{n+1})|y^h(\bar\tau(n+1))|^r \operatorname{sgn} y^h(\bar\tau(n+1)) \\ &+ (1-\theta)g(t_n) + \theta g(t_{n+1})\Big),\end{aligned}$$

where we substitute the term from (5.5) instead of $y^h(\bar\tau(n)$, $y^h(\bar\tau(n+1)$.

Let $1 - \theta h a(t_{n+1}) \neq 0$. Then the previous equation can be also rewritten as difference equation

$$\begin{aligned}\Delta y(n) = p(n)y(n) &+ q(n)\Big|\mu(n)y(\bar\tau(n)) + \eta(n)y(\bar\tau(n)+1)\Big|^r \operatorname{sgn}\Big(\mu(n)y(\bar\tau(n)) \\ &+ \eta(n)y(\bar\tau(n)+1)\Big) + \hat q(n)\Big|\hat\mu(n)y(\bar\tau(n+1)) + \hat\eta(n)y(\bar\tau(n+1)+1)\Big|^r \\ &\times \operatorname{sgn}\Big(\hat\mu(n)y(\bar\tau(n+1)) + \hat\eta(n)y(\bar\tau(n+1)+1)\Big) + d(n),\end{aligned} \tag{5.6}$$

where

$$\begin{aligned} p(n) &:= \frac{(1-\theta)ha(t_n) + \theta h a(t_{n+1})}{1-\theta h a(t_{n+1})}, & q(n) &:= \frac{hb(t_n)}{1-\theta h a(t_{n+1})}, \\ d(n) &:= \frac{(1-\theta)hg(t_n) + \theta h g(t_{n+1})}{1-\theta h a(t_{n+1})}, & \hat q(n) &:= \frac{hb(t_{n+1})}{1-\theta h a(t_{n+1})} \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} \eta(n) &:= (1-\theta)^{\frac{1}{r}}\left(\frac{\tau(t_n)-t_0}{h} - \left\lfloor\frac{\tau(t_n)-t_0}{h}\right\rfloor\right), & \mu(n) &:= (1-\theta)^{\frac{1}{r}} - \eta(n), \\ \hat\eta(n) &:= \theta^{\frac{1}{r}}\left(\frac{\tau(t_{n+1})-t_0}{h} - \left\lfloor\frac{\tau(t_{n+1})-t_0}{h}\right\rfloor\right), & \hat\mu(n) &:= \theta^{\frac{1}{r}} - \hat\eta(n) . \end{aligned} \tag{5.8}$$

Note that if $\theta = 0$ we get the forward Euler formula, if $\theta = 1$ we get the backward Euler formula and if $\theta = \frac{1}{2}$ we get the trapezoidal rule. Of course if we use the piecewise constant interpolation, the formula (5.6) is simplified (in particular for $\theta = 0$ it becomes (5.3)).

5.2. Asymptotic estimates for the Euler discretization of (3.3)

As we have shown, the simplest discretization of (3.3) has the form

$$\Delta y(n) = p(n)y(n) + q(n)|y(\bar\tau(n))|^r \operatorname{sgn} y(\bar\tau(n)) + d(n), \qquad n \in \mathbb{N}(0),\ 0 < r \le 1\,, \tag{5.9}$$

where $y(n)$ approximates the value of $x(t_n)$ at $t_n = t_0 + nh$, $h > 0$ is the stepsize, $p(n), q(n), d(n)$ are given by (5.4), $\bar\tau(n) = \left\lfloor \frac{\tau(t_n)-t_0}{h} \right\rfloor$ and $0 < r \le 1$ is a real scalar. We can easily check that the assumptions imposed on $\tau(t)$ in Chapter 3 ensure that properties assumed on $\bar{\tau}(n)$ in Chapter 4 are valid. Then (5.9) is a particular case of the difference equation (4.1) considered in the previous chapter.

Our aim is to show that asymptotic bounds of solutions valid in the continuous case hold (under some restrictions) also in the corresponding discrete case. Doing this, we formulate an upper bound for solutions $y(n)$ of (5.9), which corresponds to the results mentioned in Theorem 3.7.

First we state some notes on the initial conditions and necessary auxiliary relations. Similarly to Chapter 4, we set $n_{-1} := \bar{\tau}(0)$. The equation (5.9) has a unique solution satisfying initial conditions

$$y(n) = y_0(n), \qquad n \in \mathbb{Z},\ n_{-1} \le n \le 0.$$

These initial conditions originate from the prescribed initial functions defined on the initial interval $[\tau(t_0), t_0]$. It is obvious that if $\tau(t_0) = t_0$, then $n_{-1} = 0$ and the initial condition is $y(0) := x(t_0)$ $(= x_0)$.

Furthermore we put $\sigma_{-1} = n_{-1}$, $\sigma_0 = 0$, $\sigma_{m+1} = \max\{n \in \mathbb{N}(0) : \bar{\tau}(n) \le \sigma_m\}$, $m = 0, 1, 2, \ldots$. We consider the following auxiliary difference inequality

$$|q(n)|\bar{\omega}(\bar{\tau}(n))^r \le (1 - |1 + p(n)|)\bar{\omega}(n), \qquad n \in \mathbb{N}(0)\,. \tag{5.10}$$

This relation is a simplification of difference inequality (4.7). In addition, note that here we have to use relation (4.6) mentioned in the previous chapter, too. The sequences $u(m)$, $v(m)$ given by (4.8) and (4.9), respectively, for $m = 0, 1, 2, \ldots$ are also used.

Theorem 5.1. *Let $p(n), q(n), d(n)$ be given by (5.4) and let $|1+p(n)| < 1$ for all $n \in \mathbb{N}(0)$. Further, let $u(m)$, $v(m)$ be given by (4.8) and (4.9), respectively. Let $\bar{\omega}(n)$ be a positive monotonous sequence satisfying (5.10) and $\bar{\psi}(n)$ a positive and increasing sequence satisfying (4.6). Finally let $y(n)$ be a solution of (5.9).*

(i) If $\bar{\omega}(n)$ is nondecreasing, then there exists a constant $L > 0$ such that

$$|y(n)| \le \Big(L + \sum_{i=0}^{\lfloor \bar{\psi}(n) \rfloor} v(i)\Big)\bar{\omega}(n) \qquad \text{for all } n \in \mathbb{N}(0).$$

(ii) If $\bar{\omega}(n)$ is decreasing, then there exists a constant $L > 0$ such that

$$|y(n)| \le \Big(L + \sum_{i=0}^{\lfloor \bar{\psi}(n) \rfloor} v(i)\Big)\bar{\omega}(n) \prod_{s=0}^{\lfloor \bar{\psi}(n) \rfloor} \Big(1 - \frac{u(s)}{\bar{\omega}(\sigma_{s+1})}\Big) \qquad \text{for all } n \in \mathbb{N}(0).$$

Proof. The proof follows immediately from Theorem 4.2. □

Corollary 5.2. *Consider the equation (5.9) under the assumptions of Theorem 5.1, where $d(n) \equiv 0$.*

(i) Let $r = 1$ and

$$|1 + p(n)| + |q(n)| \le 1, \qquad n \in \mathbb{N}(0).$$

Then any solution $y(n)$ of (5.9) is bounded.

(ii) Let $0 < r < 1$. Assume that there exists an arbitrary $K \geq 0$ such that

$$0 < \frac{|q(n)|}{1 - |1 + p(n)|} \leq K, \quad n \in \mathbb{N}(0). \tag{5.11}$$

Then any solution $y(n)$ of (5.9) is bounded.

Proof. The proof follows from Corollary 4.6. □

Remark 5.3. *Note that condition (5.11) is a discrete analogue of the condition $0 < |b(t)| \leq K|a(t)|$ from Corollary 3.10 formulated for the exact delay differential equation. Hence, the following two examples supplement Example 3.11.*

Example 5.4. We consider the sublinear differential equation

$$x'(t) = ax(t) + b|x(\tau(t))|^r \operatorname{sgn} x(\tau(t)), \qquad t \geq t_0,\ 0 < r < 1, \tag{5.12}$$

where $a < 0$, $b \neq 0$ are real constants. As mentioned in Example 3.11 (ii), any solution $x(t)$ of (5.12) is bounded. Now we describe the asymptotic estimate of solutions of the corresponding Euler discretization

$$\Delta y(n) = hay(n) + hb|y(\bar{\tau}(n))|^r \operatorname{sgn} y(\bar{\tau}(n)), \qquad n \in \mathbb{N}(0), \quad 0 < r < 1 \tag{5.13}$$

with the stepsize $h > 0$ and $\bar{\tau}(n) = \left\lfloor \frac{\tau(t_n) - t_0}{h} \right\rfloor$. If $|1 + ha| < 1$ and b is arbitrary, then any solution $y(n)$ of (5.13) is bounded due to Corollary 5.2.

Example 5.5. Now we consider the sublinear delay differential equation with constant coefficients and with forcing term

$$x'(t) = ax(t) + b|x(\tau(t))|^r \operatorname{sgn} x(\tau(t)) + g(t), \quad t \geq t_0, \quad 0 < r < 1\,. \tag{5.14}$$

Assume that $a < 0$ and $g(t)$ is bounded. By Example 3.11 (iii),

$$x(t) = O(\psi(t)) \qquad \text{as } t \to \infty \tag{5.15}$$

for any solution $x(t)$ of (5.14), where $\psi(t)$ is a solution of the Abel equation (3.5).

In particular, if $\tau(t) = t - \kappa$, $\kappa \in \mathbb{R}$ (constant delay), then $\psi(t) = t/\kappa$ is the solution of the Abel equation (3.5) and (5.15) becomes

$$x(t) = O(t) \qquad \text{as } t \to \infty. \tag{5.16}$$

Alternatively, if $\tau(t) = \lambda t$, $0 < \lambda < 1$ (the proportional delay), then the Abel equation (3.5) admits the solution $\psi(t) = \frac{\log t}{\log \lambda^{-1}}$ and from (5.15) we get

$$x(t) = O(\log t) \qquad \text{as } t \to \infty. \tag{5.17}$$

The Euler discretization of (5.14) is

$$\Delta y(n) = hay(n) + hb|y(\bar{\tau}(n))|^r \operatorname{sgn} y(\bar{\tau}(n)) + d(n), \qquad n \in \mathbb{N}(0),\ 0 < r < 1, \tag{5.18}$$

where $d(n) = hg(t_n)$, $\bar{\tau}(n) = \left\lfloor \frac{\tau(t_n) - t_0}{h} \right\rfloor$ and $y(n)$ is an approximation of $x(t)$ at $t = t_n$.

We set the stepsize h such that $|1 + ha| < 1$. It is clear that the sequence $d(n)$ is bounded. Then from (4.9) the sequence $v(m)$ is bounded too. From (4.8) the sequence $u(m)$ is zero. Then Theorem 5.1 implies that

$$y(n) = O(\bar{\psi}(n)) \qquad \text{as } n \to \infty \tag{5.19}$$

for any solution $y(n)$ of (5.18), where the sequence $\bar{\psi}(n)$ satisfies (4.6).

If we consider the constant delay, then $\bar{\tau}(n) = n - \left\lfloor \frac{\kappa}{h} \right\rfloor$, $\kappa \in \mathbb{R}$ and $\sigma_m = \left\lfloor \frac{\kappa}{h} \right\rfloor m$. It can be easily checked that $\bar{\psi}(n) = n / \left\lfloor \frac{\kappa}{h} \right\rfloor$ is a positive and nondecreasing sequence satisfying (4.6). Hence, the estimate (5.19) becomes

$$y(n) = O(n) \qquad \text{as } n \to \infty\,. \tag{5.20}$$

If we consider the proportional delay, then $\bar{\tau}(n) = \lfloor \lambda n \rfloor$, $0 < \lambda < 1$ and $\sigma_m = \lfloor \frac{\sigma_{m-1}}{\lambda} \rfloor$. We can choose $\bar{\psi}(n) = \frac{\log(n - \lambda/(1-\lambda))}{\log \lambda^{-1}}$. This sequence satisfies (4.6) and the estimate (5.19) becomes

$$y(n) = O(\log n) \qquad \text{as } n \to \infty\,. \tag{5.21}$$

Comparing (5.16), (5.20) and (5.17), (5.21), we can observe the resemblance of the asymptotics of solutions of (5.14) and (5.18) provided that $|1 + ha| < 1$.

5.3. Asymptotic estimates for the θ-method discretization of (3.3)

As we have already mentioned, the standard discretization of (3.3) is the θ-method (5.6) involving Euler methods and the trapezoidal rule as its particular cases. In this section, we derive conditions which imply that the solution sequence of the θ-method discretization of (3.3) has asymptotic behaviour analogous to the behaviour of the exact solution. These conditions depend on coefficients $a(t)$, $b(t)$, the stepsize h and the parameter θ.

We consider a sublinear differential equation with general delay (3.3) and its discretization obtained from the θ-method (5.6) with $p(n)$, $q(n)$, $\hat{q}(n)$, $d(n)$ given by (5.7) and $\eta(n)$, $\mu(n)$, $\hat{\eta}(n)$, $\hat{\mu}(n)$ given by (5.8).

As in the previous section, we introduce a sequence σ_m and an auxiliary relation. Put $\sigma_{-1} = n_{-1} = \bar{\tau}(0)$, $\sigma_0 = 0$, $\sigma_{m+1} = \max\{n \in \mathbb{N}(0) : \bar{\tau}(n+1) + 1 \le \sigma_m\}$, $m = 0, 1, 2, \ldots$ and consider a difference inequality

$$\begin{aligned} &|q(n)| \cdot \left|\mu(n)\bar{\omega}(\bar{\tau}(n)) + \eta(n)\bar{\omega}(\bar{\tau}(n)+1)\right|^r \\ &+ |\hat{q}(n)| \cdot \left|\hat{\mu}(n)\bar{\omega}(\bar{\tau}(n+1)) + \hat{\eta}(n)\bar{\omega}(\bar{\tau}(n+1)+1)\right|^r \le (1 - |1 + p(n)|)\bar{\omega}(n) \end{aligned} \tag{5.22}$$

for all $n \in \mathbb{N}(0)$. This relation corresponds to auxiliary inequality (4.7). The following theorem is a direct consequence of Theorem 4.2.

Theorem 5.6. *Let $p(n)$, $q(n)$, $\hat{q}(n)$, $d(n)$ given by (5.7) and $\eta(n)$, $\mu(n)$, $\hat{\eta}(n)$, $\hat{\mu}(n)$ given by (5.8) and let $|1+p(n)| < 1$ for all $n \in \mathbb{N}(0)$. Further, let $u(m)$, $v(m)$ be given by (4.8) and (4.9), respectively. Let $\bar{\omega}(n)$ be a positive monotonous sequence satisfying (5.22) and $\bar{\psi}(n)$ a positive and increasing sequence satisfying (4.6). Finally let $y(n)$ be a solution of (5.6).*

(i) If $\bar{\omega}(n)$ is nondecreasing, then there exists a constant $L > 0$ such that

$$|y(n)| \le \Big(L + \sum_{i=0}^{\lfloor \bar{\psi}(n) \rfloor} v(i)\Big)\bar{\omega}(n) \qquad \text{for all } n \in \mathbb{N}(0).$$

(ii) If $\bar{\omega}(n)$ is decreasing, then there exists a constant $L > 0$ such that

$$|y(n)| \le \Big(L + \sum_{i=0}^{\lfloor \bar{\psi}(n) \rfloor} v(i)\Big)\bar{\omega}(n) \prod_{q=0}^{\lfloor \bar{\psi}(n) \rfloor} \Big(1 - \frac{u(q)}{\bar{\omega}(\sigma_{q+1})}\Big) \qquad \text{for all } n \in \mathbb{N}(0).$$

Using the notation

$$\bar{q}(n) := \max(|q(n)|, |\hat{q}(n)|)$$

and the property

$$|\mu(n) + \eta(n)|^r + |\hat{\mu}(n) + \hat{\eta}(n)|^r = 1$$

valid for all $n \in \mathbb{N}(0)$ we get the following corollary.

Corollary 5.7. *Consider the equation (5.6) under the assumptions of Theorem 5.6, where $d(n) \equiv 0$.*

(i) Let $r = 1$ and

$$0 < |1 + p(n)| + \bar{q}(n) \le 1, \quad n \in \mathbb{N}(0).$$

Then any solution $y(n)$ of (5.6) is bounded.

(ii) Let $0 < r < 1$. Assume that there exists an arbitrary $K \ge 0$ such that

$$0 < \frac{\bar{q}(n)}{1 - |1 + p(n)|} \le K, \quad n \in \mathbb{N}(0).$$

Then any solution $y(n)$ of (5.6) is bounded.

Example 5.8. We discuss the same problem as in Example 5.4. Instead of the Euler discretization we consider the θ-method discretization with the piecewise linear interpolation. By Example 3.11 (ii), all solutions $x(t)$ of (5.12) are bounded provided $a < 0$. The θ-method discretization of (5.12) yields

$$\begin{aligned}\Delta y(n) = py(n) + q\Big(&\big|\mu(n)y(\bar{\tau}(n)) + \eta(n)y(\bar{\tau}(n)+1)\big|^r \operatorname{sgn}(\mu(n)y(\bar{\tau}(n)) \\ &+\eta(n)y(\bar{\tau}(n)+1)) + \big|\hat{\mu}(n)y(\bar{\tau}(n+1)) + \hat{\eta}(n)y(\bar{\tau}(n+1)+1)\big|^r \\ &\times \operatorname{sgn}(\hat{\mu}(n)y(\bar{\tau}(n+1)) + \hat{\eta}(n)y(\bar{\tau}(n+1)+1))\Big),\end{aligned} \tag{5.23}$$

where $n \in \mathbb{N}(0)$, $r \in \mathbb{R}$, $0 < r < 1$,

$$p = \frac{ha}{1 - \theta ha}, \qquad q = \frac{hb}{1 - \theta ha}$$

and $\eta(n)$, $\mu(n)$, $\hat{\eta}(n)$, $\hat{\mu}(n)$ are given by (5.8). By Corollary 5.7 (with $p(n) \equiv p$, $\bar{q}(n) \equiv |q|$ and $K = \frac{|q|}{1-|1+p|}$), the solution $y(n)$ of (5.23) is bounded if

$$|1 + p| = \Big|1 + \frac{ha}{1 - \theta ha}\Big| < 1$$

and q is arbitrary. In the next section, this particular example helps us to illustrate conditions on the parameters (especially on the parameter θ and the stepsize h) under which the θ-method (5.23) is stable.

5.4. Stability analysis of the θ-method discretization of (3.3)

The aim of this section is to analyse the stability of the numerical method originating from the θ-method discretization of (3.3). This analysis substantially utilizes qualitative properties of the studied differential equations and their discretizations (from related papers we refer to [26], [27], [36] and [38]).

We consider the test equation

$$x'(t) = ax(t) + b|x(\tau(t))|^r \operatorname{sgn} x(\tau(t)), \quad t \geq t_0, \quad a, b, r \in \mathbb{R},\ a, b \neq 0,\ 0 < r < 1 \tag{5.24}$$

and its θ-method discretization

$$\begin{aligned} \Delta y(n) = py(n) + q\Big(&\big|\mu(n)y(\bar{\tau}(n)) + \eta(n)y(\bar{\tau}(n)+1)\big|^r \operatorname{sgn}\big(\mu(n)y(\bar{\tau}(n)) \\ &+ \eta(n)y(\bar{\tau}(n)+1)\big) + \big|\hat{\mu}(n)y(\bar{\tau}(n+1)) + \hat{\eta}(n)y(\bar{\tau}(n+1)+1)\big|^r \\ &\times \operatorname{sgn}\big(\hat{\mu}(n)y(\bar{\tau}(n+1)) + \hat{\eta}(n)y(\bar{\tau}(n+1)+1)\big)\Big), \end{aligned} \tag{5.25}$$

$n \in \mathbb{N}(0)$, where

$$p = \frac{ha}{1-\theta ha}, \quad q = \frac{hb}{1-\theta ha} \tag{5.26}$$

and $\eta(n)$, $\mu(n)$, $\hat{\eta}(n)$, $\hat{\mu}(n)$ are given by (5.8) and $h > 0$ is the stepsize. We do not consider the pure delayed case ($a = 0$) in this section.

An important theoretical question on these numerical approximations is a problem whether the numerical and exact solutions have the related asymptotic behaviour on the unbounded domain. More precisely, if all solutions of a given differential equation have certain asymptotic properties, then we investigate if the solutions of corresponding discretization have the same properties (regardless of the stepsize h and the delayed argument $\tau(t)$). We pay a specially attention to boundedness property.

Example 3.11 () (ii) implies

Theorem 5.9. *Let $x(t)$ be a solution of (5.24), where $a < 0$ and $b \neq 0$. Then $x(t)$ is bounded as $t \to \infty$.*

The following property is taken from the standard notions of stability of numerical methods for linear equations.

Definition 5.10. *The numerical method (5.25) is called* stable *if any application of the method to the equation (5.24), where $a < 0$, generates a numerical solution $y(n)$ that is bounded for any $h > 0$.*

Procedures performed in Example 5.8 can be summarized as follows.

Theorem 5.11. *Let $y(n)$ be a solution of the θ-method discretization (5.25) with p, q given by (5.26), $\eta(n)$, $\mu(n)$, $\hat{\eta}(n)$, $\hat{\mu}(n)$ are given by (5.8), where $a, b \neq 0$ and*

$$0 < |1 - \theta ha| - |1 + (1-\theta)ha|. \tag{5.27}$$

Then $y(n)$ is bounded.

Remark 5.12. *The case when $r = 1$ (the linear equation) will not be the subject of our investigation here. For particular cases of $\tau(t)$, the corresponding results on linear equations can be found in [12], [24] or [26], where the asymptotic stability property is studied as well.*

We look in detail at the condition (5.27). This condition essentially gives a restriction on the stepsize h (which does not depend on the coefficient b).

If $a < 0$, then we distinguish two cases: if

$$(1-\theta)h|a| \leq 1$$

holds, then (5.27) is satisfied trivially. On the contrary, if

$$(1-\theta)h|a| > 1$$

is valid, then (5.27) is reduced to

$$(1-2\theta)h|a| < 2. \tag{5.28}$$

This relation holds for $a < 0$ and any $h > 0$ if and only if $\frac{1}{2} \leq \theta \leq 1$. If we consider $0 \leq \theta < \frac{1}{2}$, then (5.28) yields the restriction on stepsize h in the form

$$h < \frac{2}{(2\theta-1)a}. \tag{5.29}$$

We summarize these considerations into the following theorem.

Theorem 5.13. *Let $a < 0$, $b \neq 0$. The θ-method discretization (5.25) is stable if and only if*

$$\frac{1}{2} \leq \theta \leq 1.$$

5.5. Numerical experiments

In this section, several comparisons and numerical consequences concerning the asymptotic estimates of solutions of exact equations and their discretizations are presented. Throughout this section $y(n)$ means the approximation of the exact solution $x(t)$ at $t_n = t_0 + nh$.

This subsection includes five examples. The first example deals with a difference equation with no relation to a differential equation. The second example investigates the sublinear delay differential equation and its Euler discretization with respect to the boundedness of the solutions. The third example demonstrates the case when the solutions of differential and difference equation are tending to zero. The fourth example shows an equation with bounded forcing term. The fifth example illustrates the stability analysis.

Example 5.14. We illustrate Example 4.8 as a particular initial value problem here. We consider an autonomous difference equation with one constant and one proportional delayed arguments

$$y(n+1) = 0.7y(n) + 1.3\sqrt{|y(\lfloor n/2\rfloor)|}\,\operatorname{sgn} y(\lfloor n/2\rfloor) + 0.2\sqrt[3]{y(n-3)}, \quad n = 3, 4, \dots \tag{5.30}$$

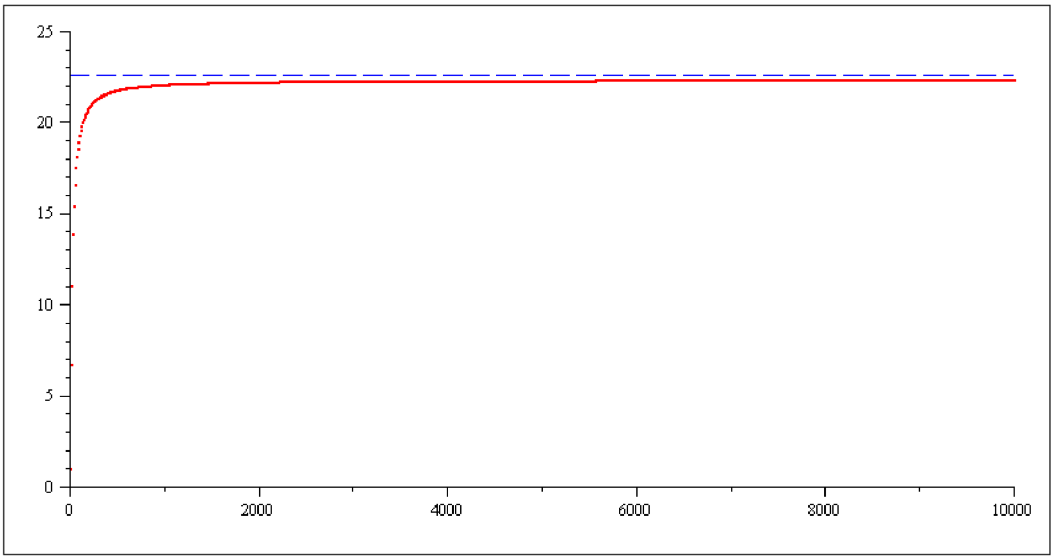

Figure 5.1: The solution $y(n)$ and its constant upper bound

with initial conditions

$$y(0) = y(1) = y(2) = y(3) = 1\,.$$

It is easy to verify that conditions (4.20) are valid, for $r = 1/2$, $p(n) = -0.3$, $q_1(n) = 1.3$, $q_2(n) = 0.2$ and $K \geq 6$ (see also Example 4.8). Figure 5.1 shows that the solution $y(n)$ of (5.30) is actually bounded, as it follows from Corollary 4.6.

Example 5.15. We consider the initial value problem for the equation with nonconstant coefficients

$$x'(t) = (\sin t - 2)x(t) + (\sin 2t + 2)\sqrt[3]{x(t/4)}, \qquad t \geq 0, \quad x(0) = 1. \tag{5.31}$$

It is easy to verify that coefficients in (5.31) satisfy the condition

$$0 < |b(t)| \leq K|a(t)|, \qquad t \geq 0$$

from Corollary 3.10 with $K \geq 3$. It follows from Corollary 3.10 that any solution $x(t)$ of (5.31) is bounded. The discretization of (5.31) via the Euler formula (5.9) with the stepsize $h = 0.5$ becomes

$$\begin{aligned} y(0) &= 1, \\ y(n+1) &= \frac{\sin(n/2)}{2}\, y(n) + \left(1 + \frac{\sin n}{2}\right)\sqrt[3]{y(\lfloor n/4 \rfloor)}, \qquad n = 0, 1, 2, \ldots\,. \end{aligned} \tag{5.32}$$

It is obvious that the condition (5.11) holds for coefficients

$$p(n) = \frac{\sin(n/2)}{2} - 1, \qquad q(n) = 1 + \frac{\sin n}{2}$$

and a suitable constant $K \geq 3$. Then, by Corollary 5.2, the solution $y(n)$ of (5.32) is also bounded (see Figure 5.2). Since the forward Euler method is not stable, the choice of h can not be arbitrary if we wish to preserve the boundedness property.

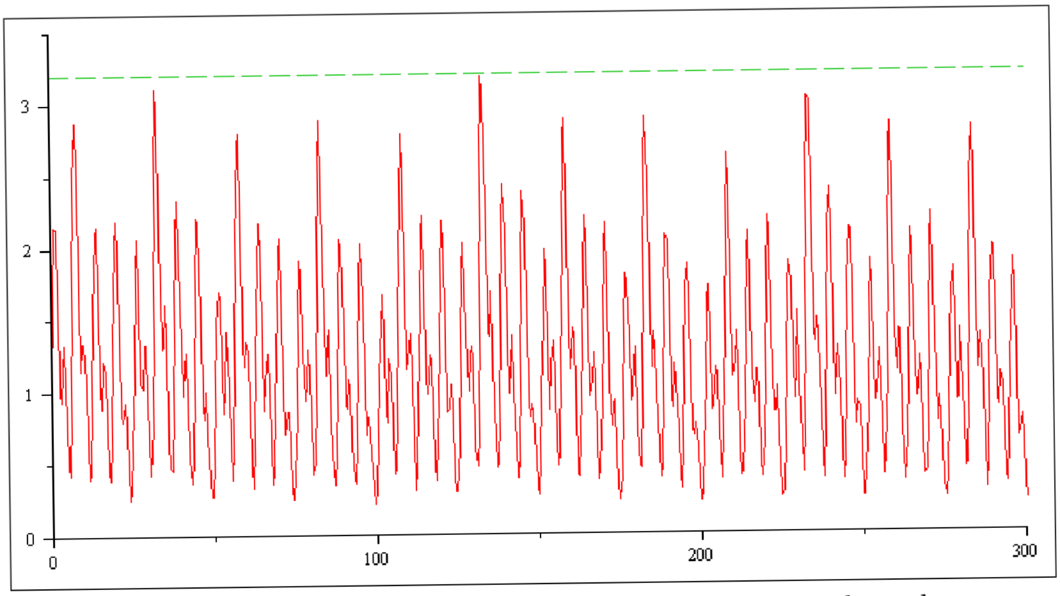

Figure 5.2: The solution $x(t)$ and its constant upper bound

Example 5.16. In this example, we show a delay differential equation whose solution is tending to zero. We consider the equation

$$x'(t) = -(1+\frac{1}{\sqrt{t}})x(t) + \frac{1}{t}\sqrt{|x(t/3)|}\ \mathrm{sgn}\, x(t/3), \qquad t \geq 2 \tag{5.33}$$

with the initial condition

$$x(t) = 1 \qquad 2/3 \leq t \leq 2.$$

By Example 3.13 (where $a = -1$, $b = 1$ and $\lambda = 1/3$), it holds

$$x(t) = O(1/t^2) \text{ as } t \to \infty$$

for the solution $x(t)$ of (5.33).

We perform the discretization of this equation using the forward Euler method, i.e. we use relations (5.3), (5.4) and we put $t_n = 2 + nh$. We choose the stepsize h so that the condition $|1 + p(n)| < 1$ formulated in Theorem 5.1 holds, i.e. we choose e.g. $h = 0.1$. The equation (5.9) thus becomes

$$\begin{aligned} y(n) &= 1, \qquad n = -14, -13, \ldots, 0, \\ y(n+1) &= \Big(\frac{9}{10} - \frac{\sqrt{10}}{10\sqrt{20+n}}\Big)y(n) \\ &\quad + \frac{1}{20+n}\sqrt{|y(\lfloor n/3 - 40/3 \rfloor)|}\ \mathrm{sgn}\, y(\lfloor n/3 - 40/3 \rfloor), \qquad n = 0, 1, \ldots . \end{aligned} \tag{5.34}$$

It is easy to check that all assumptions of Corollary 4.10 are fulfilled (with $p(n) = -\frac{1}{10} - \frac{\sqrt{10}}{10\sqrt{20+n}}$, $q(n) = \frac{1}{20+n}$, $r = \frac{1}{2}$, $\alpha = -2$ and $\bar{p} = 0.9$). Moreover, then from (4.24) we get that

$$y(n) = O\left(\frac{1}{n^2}\right) \qquad \text{as } n \to \infty$$

for any solution $y(n)$ of (5.34). This estimate again corresponds to the continuous case.

By Figure 5.3, the solution $y(n)$ is actually tending to zero as $n \to \infty$. For better comparison of the solution and its estimation we present Figure 5.4. This figure plots the value $(nh, \log(|y(nh)| + \varepsilon))$, where $\varepsilon = 2.23 \times 10^{-308}$.

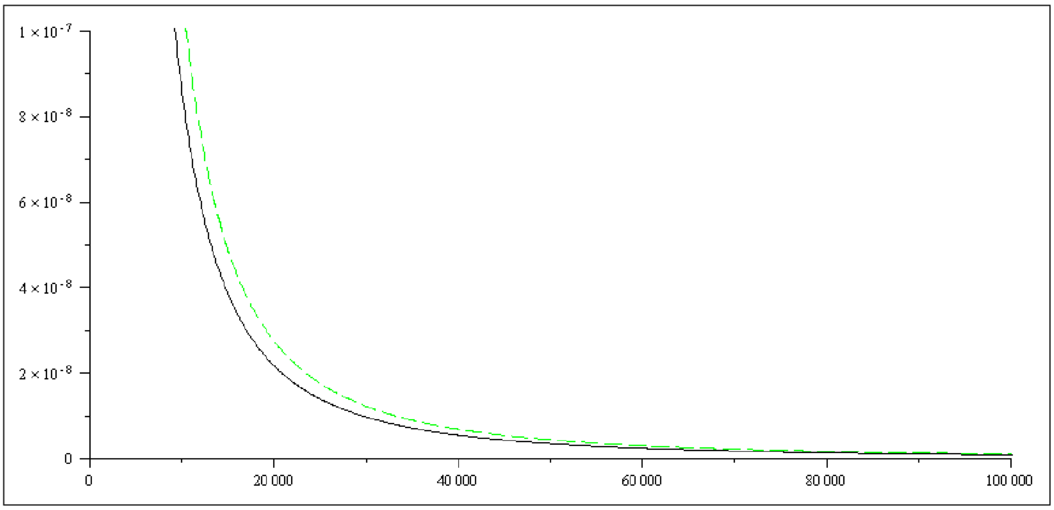

Figure 5.3: The solution $y(n)$ and its upper bound

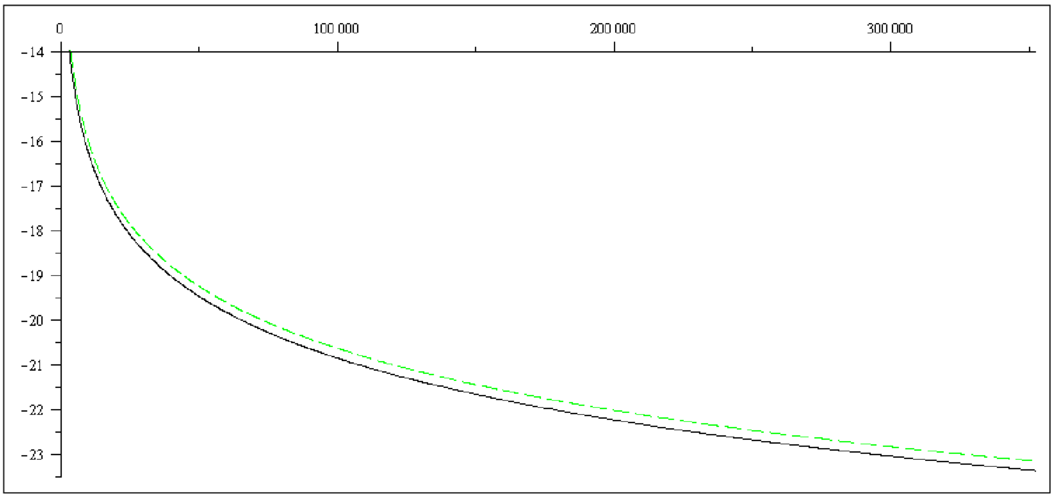

Figure 5.4: The solution $y(n)$ and its upper bounds on the logarithmic scale

Example 5.17. This example shows the equation with nonzero forcing term. We consider a sublinear differential equation with constant delayed argument

$$x'(t) = -x(t) + 10\Big(x\Big(t - \frac{3}{2}\Big)\Big)^{\frac{2}{3}} \operatorname{sgn} x\Big(t - \frac{3}{2}\Big) + g(t), \qquad t \geq 0 \tag{5.35}$$

with the initial condition

$$x(t) = 1 \quad \text{for} - 3/2 \leq t \leq 0.$$

In addition we assume that $g(t)$ is bounded.

In this case of constant delay $\tau(t) = t - 3/2$, the function $\psi(t) = 2t/3$ is a solution of the Abel equation (3.5). By Example 3.11 (iii) (where $a = -1$, $b = 10$, $r = 2/3$), the asymptotic estimate of any solution $x(t)$ of (5.35) is

$$x(t) = O(t) \qquad \text{as } t \to \infty.$$

After the Euler discretization of (5.35) we obtain

$$\begin{aligned} y(n) &= 1, \qquad n = -3, -2, -1, 0, \\ y(n+1) &= \frac{1}{2}y(n) + 5|y(n-3)|^{\frac{2}{3}} \operatorname{sgn} y(n-3) + \frac{1}{2}g\Big(\frac{n}{2}\Big), \qquad n = 0, 1, 2, \dots, \end{aligned} \tag{5.36}$$

where we choose the stepsize $h = 0.5$ so that the condition $|1 + p(n)| < 1$ formulated in Theorem 5.1 holds. By Example 5.5 (with $a = -1$, $b = 10$, $r = 2/3$ and $\kappa = 3/2$), the estimate (5.20) holds, i.e.

$$y(n) = O(n) \qquad \text{as} \quad n \to \infty$$

for any solution $y(n)$ of the corresponding difference equation (5.36).

Figure 5.5 shows solutions $x(t)$ of (5.35) under choices $g(t) = \arctan(t)$, $g(t) = \sin(t)$ and $g(t) = -1$.

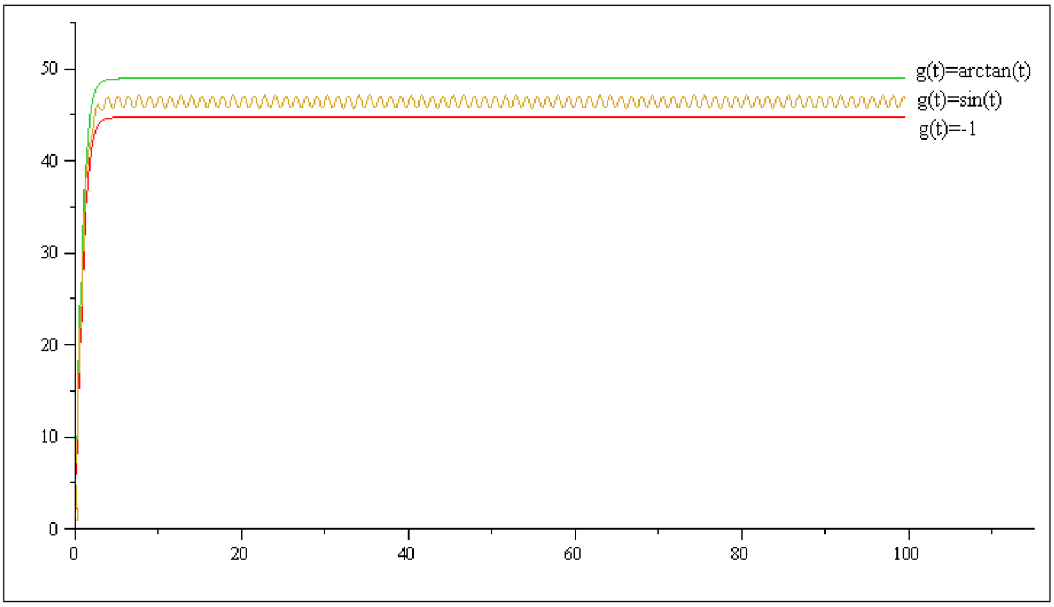

Figure 5.5: The solution $x(t)$ for $g(t) = \arctan(t)$; $\sin(t)$; -1

Example 5.18. Consider the initial value problem for the delay differential equation with constant coefficients

$$x'(t) = ax(t) + b|x(\lambda t)|^r \operatorname{sgn} x(\lambda t), \qquad t \geq 0, \quad x(0) = 1, \tag{5.37}$$

where a, b, λ, r are real scalars such that $a < 0$, $b \neq 0$ and $0 < \lambda, r < 1$. By Theorem 5.9, the solution $x(t)$ of (5.37) is bounded.

We illustrate stability analysis stated in the previous subsection by this equation. We recall that the asymptotic properties (in particular boundedness) depend on the coefficient a, on the parameter θ and on the stepsize h. We put $a = -4$, $b = 1$, $r = \lambda = 1/2$ and discuss the boundedness of the corresponding discretization with respect to changing $h > 0$ and $0 \leq \theta \leq 1$. We perform the θ-method discretization (5.6) to obtain the initial value problem

$$\begin{aligned} y(0) =& 1 \\ y(n+1) =& (1+p)y(n) + q\sqrt{|\mu(n)y(\lfloor n/2 \rfloor) + \eta(n)y(\lfloor n/2 \rfloor + 1)|} \\ & \times \operatorname{sgn}\Big(\mu(n)y(\lfloor n/2 \rfloor) + \eta(n)y(\lfloor n/2 \rfloor + 1)\Big) \\ & + q\sqrt{|\hat{\mu}(n)y(\lfloor (n+1)/2 \rfloor) + \hat{\eta}(n)y(\lfloor (n+1)/2 \rfloor + 1)|} \\ & \times \operatorname{sgn}\Big(\hat{\mu}(n)y(\lfloor (n+1)/2 \rfloor) + \hat{\eta}(n)y(\lfloor (n+1)/2 \rfloor + 1)\Big), \end{aligned} \tag{5.38}$$

$n = 0, 1, \ldots$, where

$$\begin{aligned} p &= \frac{-4h}{1+4\theta h}, & q &= \frac{h}{1+4\theta h}, \\ \eta(n) &= (1-\theta)^2(n/2 - \lfloor n/2 \rfloor), & \mu(n) &= (1-\theta)^2(1 - n/2 + \lfloor n/2 \rfloor), \\ \hat{\eta}(n) &= \theta^2((n+1)/2 - \lfloor (n+1)/2 \rfloor), & \hat{\mu}(n) &= \theta^2(1 - (n+1)/2 + \lfloor (n+1)/2 \rfloor). \end{aligned}$$

By Theorem 5.11, the solution $y(n)$ of (5.38) is bounded if the condition

$$0 < 1 + 4\theta h - |1 - 4h(1-\theta)|$$

holds. First we consider $1/2 \le \theta \le 1$, e.g. $\theta = 0.8$. In this case, by Theorem 5.13, the solution $y(n)$ is bounded for all $h > 0$ and the method (5.38) is stable. The situation is illustrated by Table 5.1 and Figure 5.6.

Further we assume $0 \le \theta < 1/2$, e.g. $\theta = 0.3$. It follows from (5.29) that the solution of (5.38) is bounded provided

$$h < \frac{1}{2(1-2\theta)} = 1.25.$$

Table 5.2 and Figure 5.7 demonstrate the strictness of this stepsize condition.

$h\backslash nh$	50	150	500	1000
0.01	0.06419	0.06306	0.06267	0.06258
0.1	0.06482	0.06327	0.06273	0.06261
0.5	0.06787	0.06414	0.06301	0.06275
1	0.06992	0.06513	0.06321	0.06285
5	0.10704	0.07189	0.06484	0.06366
10	0.21264	0.08504	0.06626	0.06435
50	1	0.06832	0.11672	0.08539

Table 5.1: The solution $x(nh)$ for $\theta = 0.8$

$h\backslash nh$	50	150	500	1000
0.01	-0.05831	0.06234	0.06244	-0.06247
0.5	-0.06773	-0.06388	-0.06203	0.06226
1	0.09807	-0.06138	-0.05605	-0.05931
1.24	0.9753	0.4786	0.09373	-0.02712
1.2499	1.3398	1.3434	1.3666	1.1991
1.255	-0.8951	-1.1907	2.7191	11.9956
2	-779.004	-2.857 E9	2.689 E32	1.776 E65

Table 5.2: The solution $x(nh)$ for $\theta = 0.3$

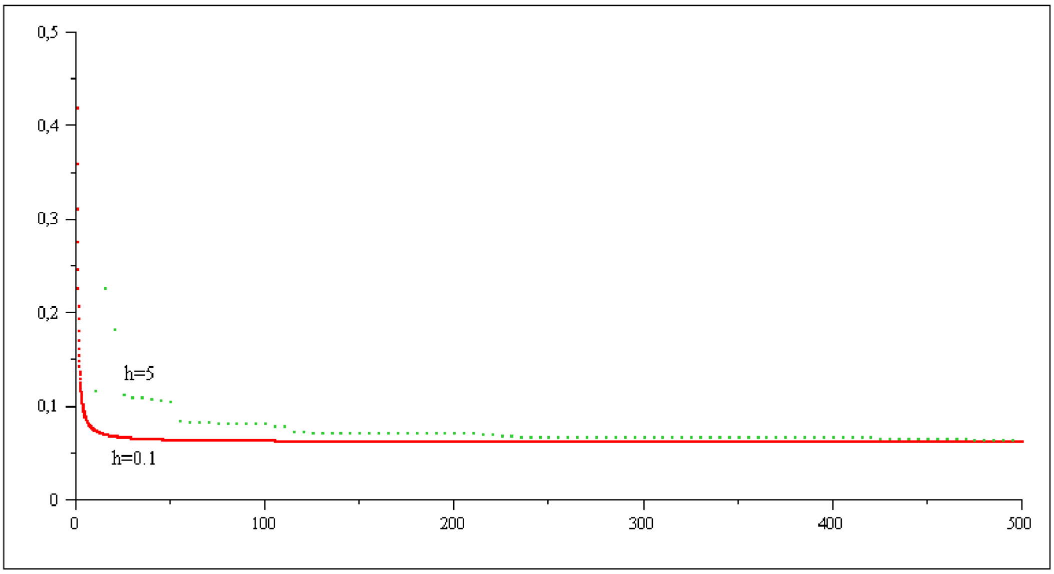

Figure 5.6: The solution $y(n)$ for $\theta = 0.8$, $h = 0.1$ and $h = 5$

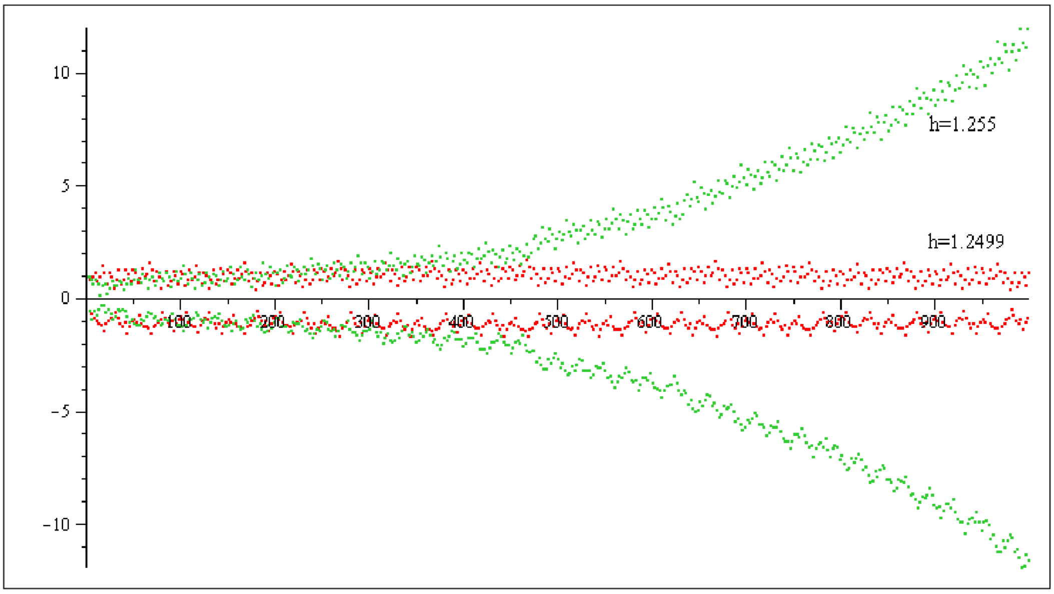

Figure 5.7: The solution $y(n)$ for $\theta = 0.3$

6. Conclusion

In this thesis, there are presented the results concerning with the asymptotic behaviour of the nonlinear delay differential equation

$$x'(t) = a(t)x(t) + f(t, x(\tau(t))), \qquad t \in [t_0, \infty),$$

where the right-hand side fulfills the relation

$$|f(t, x)| \leq |b(t)||x|^r + |g(t)|, \qquad t \in [t_0, \infty), \quad r > 0.$$

We derived two different types of asymptotics, which depends on the sign of the function $a(t)$ provided that the studied equation is of sublinear type ($0 < r < 1$). Asymptotic estimates of the superlinear equation ($r > 1$) were provided only for negative values of $a(t)$. Obtained results were demonstrated by several corollaries and illustrating examples.

Since the searching for an analytical solution of the studied nonlinear equations turned out to be impossible, we need to discuss the numerical solution. The appropriate numerical formulae are constructed as difference equations. Consequently, the second part of this thesis is already concerned with problems of sublinear difference equations (with one or more delays). Considering these equations, main qualitative properties (especially asymptotic) were derived. Using these results we discussed the stability property of the θ-method discretization. It was shown, that for $\frac{1}{2} \leq \theta \leq 1$ this method is stable. Further, in several examples we compared asymptotic estimates of both exact and numerical solutions.

There are several directions, where the results obtained in this work can be further developed. It can be useful to focus on improvement of some asymptotic estimates. Some numerical experiments indicate that some of these estimates can be improved. Another development may consist in considering the corresponding differential equations of neutral type. Finally, obtained results for differential and difference equations can be unified and generalized in the frame of the time scale theory.

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7. List of abbreviations and symbols

$\mathbb{R}$	set of real numbers
$\mathbb{Z}$	set of integers
$\mathbb{N}$	set of positive integers
$\mathbb{N}(0) := \{0, 1, 2, \dots\}$	set of nonnegative integers
$\mathbb{N}(n_0) := \{n_0, n_0 + 1, n_0 + 2, \dots\}$	set of integers $\geq n_0$
$C(I)$	set of continuous functions on I
$C^1(I)$	set of continuous and continuously differentiable functions on I
$[a, b]$	closed interval of real numbers
$I := [t_0, \infty)$	interval of real numbers
$I \times R$	Cartesian product
$x'(t)$, $\frac{\mathrm{d}x(t)}{\mathrm{d}t}$	first derivative of $x(t)$ with respect to t
$\Delta y(n) := y(n+1) - y(n)$	forward difference operator
$\operatorname{sgn} a$	signum function
$\exp\{a\}$	natural exponential function e^a
$\lvert a \rvert$	absolute value of a
$\lfloor a \rfloor$	integer part of a
$O(g(t))$	Omicron notation, upper asymptotic estimate