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STABILIZATION METHODS FOR UNSTABLE SOLUTIONS OF THE DISCRETE LOGISTIC EQUATION
METODY STABILIZACE NESTABILNÍCH ŘEŠENÍ DISKRÉTNÍ LOGISTICKÉ ROVNICE

MASTER'S THESIS
DIPLOMOVÁ PRÁCE

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Department:	Institute of Mathematics
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Pursuant to Act no. 111/1998 concerning universities and the BUT study and examination rules, you have been assigned the following topic by the institute director Master's Thesis:

Stabilization methods for unstable solutions of the discrete logistic equation

Concise characteristic of the task:

The study of chaotic systems belongs among rapidly developing branches of modern mathematical analysis, especially with respect to many applications in natural sciences and engineering. Besides well–known models based on differential equations, features of chaotic behaviour of solutions appear also in models based on difference equations, having sometimes a very simple form. Classification and stabilization of these chaotic systems belong among basic problems connected with their analysis.

Goals Master's Thesis:

The discrete logistic equation and its interpretations.
Analysis of stabilization methods for unstable steady states.
Analysis of stabilization methods for unstable periodic orbits of higher orders.
Graphical interpretations of a chaotic behaviour and its control.

Recommended bibliography:

ELAYDI, Saber N. Discrete Chaos: With Applications in Science and Engineering. Second edition. Boca Raton: Chapman & Hall/CRC, Taylor & Francis Group, 2008. ISBN 13-978-1-58488-592-4.

STROGATZ, Steven H. Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry and Engineering. Reading: Perseus Books, 1994. ISBN 0-201-54344-3.

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Abstract

The master's thesis deals with a stabilization of a discrete logistic model via several control methods. In particular, the stabilization of equilibria, 2-period orbits and 3-period orbits is performed. For this stabilization purpose, a proportional feedback control, delayed feedback control and prediction based control are utilized. For each of the methods, the stabilization sets for a control gain parameter are derived together with stability ranges of corresponding controlled orbits. Each of the theoretical results is illustrated by a graphical interpretation created in the software MATLAB. The supporting computations are done by the software Maple.

Abstrakt

Diplomová práce pojednává o stabilizaci diskrétního logistického modelu pomocí několika řídících metod. Je zde provedena především stabilizace rovnováh, 2-periodických cyklů a 3-periodických cyklů. Ke stabilizaci systému je využito proporčního zpětně-vazebního řízení, zpětně-vazebního řízení s časovým zpožděním a řízení založeného na predikci. U každé metody je diskutovaná stabilizační množina pro řídící zesilovač spolu s oblastmi stability pro odpovídající kontrolovaná řešení. Všechny teoretické výsledky jsou ilustrovány grafickými interpretacemi v softwaru MATLAB. Podpůrné výpočty jsou provedeny pomocí softwaru Maple.

Keywords

Discrete dynamical model, logistic model, difference equation, delay, equilibrium, period orbit, stability of orbit, chaotic behaviour, bifurcation analysis, stabilization of system, stabilized orbit, proportional feedback control, delayed feedback control, prediction based control.

Klíčová slova

Diskrétní dynamický model, logistický model, diferenční rovnice, zpoždění, rovnováha, periodický cyklus, stabilita řešení, chaotické chování, analýza bifurkace, stabilizace systému, stabilizované řešení, proporční zpětně-vazební řízení, časově zpožděné zpětně-vazební řízení, řízení založené na predikci.

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I declare that I have written my Master's thesis *Stabilization methods for unstable solutions of the discrete logistic equation* independently, under the guidance of my supervisor prof. RNDr. Jan Čermák, CSc., and using the sources listed in references.

May 23, 2019 Lucie Ondrová

I would like to express my sincere gratitude to my supervisor prof. RNDr. Jan Čermák, CSc. for supervising my Master's thesis, for all his support, patience, advice, suggestions and valuable comments.

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Lucie Ondrová

Contents

1 Introduction

The current world can be characterized in a very simple way. Namely, that it offers an incredibly fast development of everything - including the development of technologies, science, industry, medicine. All of this has a positive impact on life standards. On the other hand, it takes its costs. The human population is starting to be overcrowded and so far there is no development that could increase a capacity of the environment, which we are inhabiting. We cannot increase the Earth's area or provide infinity food support.

In fact, the expansion of this problem during the last century was for mathematicians one of the impulses to start studying a discipline called *population biology* in more detail. Among other things, this field studies also a time evolution of size of population. There are specific models describing such an evolution of different species, either of a single-species or the multiple-species. The well known multiple-species model is, e.g, the *predator-prey model* (or the *Lotka–Volterra equations*). The most famous single-species model is the *logistic model* and such a model will be discussed in detail later in this thesis.

One may object, that the logistic model has been already discussed and studied in a lot of works and papers. In spite of this fact, the mathematicians are still amazed at this model. Mainly the version in a discrete time domain is so interesting. Such a model predicts a size of population in n-th time unit. Although its formulation is very simple, it has a rich scale of different types of solutions. According to the input data, the population would tend to its equilibrium, it would keep repeating of some states all over again (i.e., it would behave periodically), or it would even behave totally chaotic.

In [11], we already discussed the types of solutions (mention above) in a very detail, together with their stability ranges. A property of 'to be stable/unstable' is very important in a description of evolution of some species. It determines the ability of model to stay in some state even if the varying input data are (slightly) changed. In the case of logistic model, such a varying parameter is a so-called growth parameter. The results of [11] (i.e., a dependence of the types of solutions on values of this parameter) are summarized in Chapter 3 together with corresponding stability ranges.

In fact, exactly these ranges are the main aims of interest in this thesis. Particularly, we will try to enlarge them as much as possible. In other words, what we will do is the *stabilization* of chosen states (solutions). In general, a stabilization of dynamical models is a relatively new question in a field of modern mathematical analysis (approximately 20 years old). There are already several methods of stabilization, usually based on adding some control to the system. To be honest, the analysis of dynamical systems may be sometimes very difficult, and so do the analysis of corresponding controlled systems.

Later in the text we will use three different control methods described in Chapter 2 (where all the necessary mathematical tools are given), in order to stabilize the logistic model. Naturally, our first step will be the stabilization of equilibrium. For each control method there will be given a discussion about the stabilization set, i.e., the set of all values of control parameter(s), for which we are able to stabilize the system's equilibrium. Moreover, we will determine a new range of stability, formulated by explicit dependence on general growth parameter. It is possible to find it in Chapter 4 or Chapter 7 (there are two different approaches on controls).

Analogously, the same results (i.e., the stabilization set and, if possible, a formulation of new range of stability) will be discussed also for 2-period orbit (see Chapter 5) and 3-period orbit (see Chapter 6). In a lot of papers and works, that studied such a problematic, there is given only the analysis of stabilization of equilibria. In few of them, there is also mentioned a stabilization of period orbits. However, in such a case there are given just concrete results based on numerical experiments, mostly without any analytical justification.

Therefore, the main contribution of this thesis is in its general analytical approach in solving of problem of stability and controllability of the system. General results are then justified by simulations of bifurcation diagrams, that describe each controlled system separately. A stabilization by implementing of different controls into system is, in fact, a practical approach to this problem. In Chapter 8 we will give a theoretical justification of each control method, based mainly on a discussion of the eigenvalues of corresponding system. At the end of this work, we summarize all results and compare them in Chapter 9.

Note that we have chosen a logistic model as a testing model for this thesis just because of several reasons. First, we know this model in detail from [11]. Second, in spite of the simplicity of the model (meaning in comparison with other discrete dynamical models), this model has all types of solutions. Hence, we are able to test a stabilization of any solution. Moreover, this model has also other applications, where these results may be very useful. For example, the application into a traffic flow (so called *dynamical traffic flow model*) is very interesting and it will be described in Chapter 3.

2 Mathematical Background

Herein the chapter, a mathematical background necessary for this thesis is given. The following theory can be found in more details in [5], [6], [14]. As the title suggests, a main part of thesis deals with a logistic model, together with its analysis and stabilization. To assure the reader will understand content of this thesis well and will be able to apply it for more complex problems, the following sections are necessary to be built.

In fact, a logistic model is a discrete chaotic model. For a description of a discrete model we are using the difference equations. In general, a *first-order difference equation* is in form

$$y(n) = f(n, y(n-1)), \qquad f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}, \qquad n \in \mathbb{N},$$

with an initial condition

$$y(0) = y_0.$$

Such an equation is called *nonautonomous* (or time-variant) equation. If the difference equation with an initial condition y_0 is in form

$$y(n) = f(y(n-1)), \qquad f : \mathbb{R} \to \mathbb{R}, \qquad n \in \mathbb{N}, \tag{2.1}$$

it is called *autonomous* (or time-invariant). From now, we will restrict only to the autonomous difference equations, since there will be studied only logistic model, which is explicitly independent on time steps.

In general, a *discrete dynamical model* is a system of m difference equations and it is given by

$$\mathbf{y}(n) = F\left(\mathbf{y}(n-1)\right), \qquad F : \mathbb{R}^m \to \mathbb{R}^m, \qquad m, n \in \mathbb{N}, \tag{2.2}$$

with the system of initial conditions

$$\mathbf{y}(0) = \mathbf{y}_0,$$

where $\mathbf{y}(n) = (y_1(n), y_2(n), \ldots, y_m(n))$. A dynamical model (or system) describes a single-species population if $m = 1$ and such a model is studied deeply in this thesis. On contrary, the system describes a multi-species population if $m > 1$.

The simplest case of (2.2) is a linear system. In general, a *linear nonhomogeneous* discrete system with the initial conditions $\mathbf{y}_0$ is given by

$$\mathbf{y}(n) = A\mathbf{y}(n-1) + B, \qquad n \in \mathbb{N},$$

where A is $m \times m$ real matrix (assumed to be a non-zero matrix) and B is $m \times 1$ real matrix. If the matrix B is a zero matrix, then the system is called *homogeneous.*

As well as the differential equations may be of higher order, the difference equations may be too. We say that the difference equation is of a kth-order if it is in form

$$y(n) = f(y(n-1), y(n-2), \ldots, y(n-k)).$$

The normal form of a kth-order nonhomogeneous linear difference equation is given by

$$y(n) + p_1 y(n-1) + p_2 y(n-2) + \cdots + p_k y(n-k) = g,$$

where $p_i, g \in \mathbb{R}$ for $i = 1, 2, \ldots k$ and $p_k \neq 0$. If $g = 0$, i.e., we have

$$y(n) + p_1 y(n-1) + p_2 y(n-2) + \cdots + p_k y(n-k) = 0, \tag{2.3}$$

we say that it is a homogeneous equation. A specific form of (2.3), written as

$$y(n) = f(y(n-1), y(n-\omega)), \qquad 1 < \omega \leq k, \tag{2.4}$$

meaning that the only non-zero coefficients of (2.3) are p_1 and p_ω, is usually called as a *delayed difference equation.* Here, a symbol ω denotes a *delay* of the difference equation.

Note that in a case of differential equations, we can rewrite a kth-order differential equation into a system of k first-order differential equations. Analogously, the same we can do with a kth-order difference equation (in later, it will be shown in computations). The introduction of delayed equation is important, since it can be very useful for a description of discrete models (as we will see further). In fact, if we describe a model in such a way, it can give us better results in a prediction of the term $y(n)$. However, a delayed term $y(n-\omega)$ represents some information about the system from the past and such an information should be known. Therefore, a number of initial conditions has been increased and thus, it is some kind of payment for getting the better results.

2.1 Equilibrium and Its Stability

There are few important questions in a field of population models (where the logistic model obviously belongs). One of them is whether there exists a steady state in given system. Such a state is called an equilibrium and it is defined consequently:

Definition 2.1. An *equilibrium* $\mathbf{y}^*$ of a given discrete model (2.2) is a point $\mathbf{y}^* \in \mathrm{D}(F)$ satisfying

$$\mathbf{y}^* = F(\mathbf{y}^*).$$

In fact, each equilibrium represents a constant solution of system.

Remark 2.2. In a case of discrete model given by (2.4), the equilibrium $y^* \in \mathrm{D}(f)$ satisfies

$$y^* = f(y^*, y^*).$$

This is an analogy to equilibrium of system, given by the preceding definition.

Once an equilibrium exists in the system, another question arises. Namely, if such a state is stable. A stability of equilibrium describes how much the system is sensitive to disturbances of given data (e.g., in the initial conditions). A precise definition of stability follows.

Definition 2.3. Let $\mathbf{y}^* \in \mathbb{R}^m$ be an equilibrium of system (2.2). We say that the equilibrium $\mathbf{y}^*$ is *stable*, if for each neighbourhood $\mathcal{O}$ of point $\mathbf{y}^* \in \mathbb{R}^m$ there exists a neighbourhood $\mathcal{O}_1 \subseteq \mathcal{O}$ of point $\mathbf{y}^* \in \mathcal{O}$, such that for each solution $\mathbf{y}(n)$ together with an initial condition $\mathbf{y}(0) \in \mathcal{O}_1$, holds $\mathbf{y}(n) \in \mathcal{O}$ for each $n \in \mathbb{N}$.
Moreover, if the following holds

$$\lim_{n\to\infty} \mathbf{y}(n) = \mathbf{y}^*,$$

then the equilibrium $\mathbf{y}^*$ is called *asymptotically stable (attractive).* If the equilibrium $\mathbf{y}^*$ is not stable, then it is called *unstable.*

A determination of stability of y^* can be based on the knowledge of the eigenvalues of Jacobi matrix of corresponding linearized system (more about linearization of system in [5]). A Jacobi matrix of discrete linearized system is given by

$$DF(\mathbf{y}^*) = \begin{pmatrix} \dfrac{\partial f_1}{\partial y_1(n)}(\mathbf{y}^*) & \cdots & \dfrac{\partial f_1}{\partial y_m(n)}(\mathbf{y}^*) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial y_1(n)}(\mathbf{y}^*) & \cdots & \dfrac{\partial f_m}{\partial y_m(n)}(\mathbf{y}^*) \end{pmatrix},$$

where the symbol $\dfrac{\partial f_i}{\partial y_j(n)}$ represents a partial derivative of i-th component of vector function $F = (f_1, ..., f_m)$ by j-th component of vector $\mathbf{y}(n) = (y_1(n), ..., y_m(n))$. A classification of stability of $\mathbf{y}^*$ based on the eigenvalues of $DF(\mathbf{y}^*)$ is given by following theorem.

Theorem 2.4. *Let $\boldsymbol{y}^*$ be an equilibrium and $\lambda_1, ..., \lambda_m$ be the eigenvalues of Jacobi matrix $DF(\boldsymbol{y}^*)$ of system (2.2). If λ_i lies inside a unit circle (i.e., if $|\lambda_i| < 1$) for all $i = 1, ..., m$, then we say the equilibrium $\boldsymbol{y}^*$ is asymptotically stable. If at least one λ_i lies outside a unit circle (i.e., if $|\lambda_i| > 1$) for some $i = 1, ..., m$, then we say the equilibrium $\boldsymbol{y}^*$ is unstable. If at least one λ_i lies on a boundary of unit circle (i.e., if $|\lambda_i| = 1$) and no other λ_j lies outside a unit circle, then we are not able to decide about the stability of equilibrium $\boldsymbol{y}^*$ via this criterion.*

However, in thesis is studied just a logistic model, which is a single-species model. Therefore, in such a case the Jacobi matrix is simplified to a form

$$Df(y^*) = \left(\frac{\partial f}{\partial y(n)}(y^*) \right).$$

Remark 2.5. The stability conditions given by Theorem 2.4 can be simplified due to simpler form of Jacobi matrix in following way:

- the equilibrium y^* of a discrete logistic model is asymptotically stable if $|f'(y^*)| < 1$,
- the equilibrium y^* of a discrete logistic model is unstable if $|f'(y^*)| > 1$.

Later in the text we will deal not only with a classical form of logistic model, but also with its delayed forms (in order to stabilize it). In such a case, i.e., when the system is controlled, we cannot use previous criterion from Remark 2.5 to determine the stability of equilibrium. Therefore, we have to employ another criterion verifying the conditions of Theorem 2.4.

Consider the kth-order difference equation (2.3). Due to its linearity, any solution of (2.3) is stable (asymptotically stable) if and only if the zero solution of (2.3) is stable (asymptotically stable). On this account, when discussing stability of (2.3), we restrict to stability of its zero solution.

The zero solution of (2.3) is asymptotically stable if and only if all characteristic roots λ (sometimes called as zeros) of characteristic polynomial

$$p(\lambda) = \lambda^k + p_1\lambda^{k-1} + ... + p_k \tag{2.5}$$

satisfy $|\lambda| < 1$. For this, the exact expression of λ is needed. In case of higher-order difference equation it may be difficult to find them explicitly. Hence, further we state a useful criterion for determining the stability of zero solution without any knowledge of characteristic roots (see [5]).

Theorem 2.6. ***(Schur-Cohn Criterion).***
The zeros of the characteristic polynomial (2.5) *lie inside the unit circle if and only if the following conditions hold:*

(i) $p(1) > 0$,
(ii) $(-1)^k p(-1) > 0$,
(iii) *the* $(k-1)\times(k-1)$ *real matrices* $B^{\pm}_{k-1}$ *are positive innerwise, where*

$$B^{\pm}_{k-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ p_1 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{k-3} & p_{k-4} & \cdots & 1 & 0 \\ p_{k-2} & p_{k-3} & \cdots & p_1 & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 & \cdots & 0 & p_k \\ 0 & 0 & \cdots & p_k & p_{k-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & p_k & \cdots & p_4 & p_3 \\ p_k & p_{k-1} & \cdots & p_3 & p_2 \end{pmatrix}.$$

The Schur-Cohn criterion gives necessary and sufficient conditions for the coefficients p_i, that assure the asymptotic stability of zero solution of (2.3). Moreover, for a lower-order difference equations there is a possibility to state these conditions in more neat and compact form. These forms are derived in [6]. We will introduce them for the difference equations of order 2, 3 and 4 in following remarks.

Remark 2.7. The zero solution of second-order difference equation

$$y(n) + p_1 y(n-1) + p_2 y(n-2) = 0$$

is asymptotically stable if and only if

$$|p_1| < 1 + p_2 < 2.$$

Remark 2.8. The zero solution of third-order difference equation

$$y(n) + p_1 y(n-1) + p_2 y(n-2) + p_3 y(n-3) = 0$$

is asymptotically stable if and only if

$$|p_1 + p_3| < 1 + p_2 \quad \text{and}$$
$$|p_2 - p_1 p_3| < 1 - (p_3)^2.$$

Remark 2.9. The zero solution of fourth-order difference equation

$$y(n) + p_1 y(n-1) + p_2 y(n-2) + p_3 y(n-3) + p_4 y(n-4) = 0$$

is asymptotically stable if and only if

$$|p_4| < 1,$$
$$|p_1 + p_3| < 1 + p_2 + p_4 \quad \text{and}$$
$$|p_2(1-p_4) + p_4(1-(p_4)^2) + p_1(p_4 p_1 - p_3)| < p_4 p_2 (1-p_4) + (1-(p_4)^2) + p_3(p_4 p_1 - p_3).$$

Nevertheless, the Schur-Cohn criterion is usable only if the order of difference equation is given (i.e., the order/delay is known before the analysis of system is done). As we will see later in the text, there is a possibility to assume a delay in system as a general delay ω, and determine it by the analysis of system in order to be suitable to stabilize the model. For this purpose we state another theorem, usable to determine a stability of zero solution.

Consider a linearized system of delayed difference equations in form

$$\mathbf{y}(n) = A_1\mathbf{y}(n-\rho) + A_2\mathbf{y}(n-\omega), \qquad n \in \mathbb{N}, \tag{2.6}$$

where A_1, A_2 are $m \times m$ real matrices and ρ, ω are positive integer delays, such that $\rho < \omega$. Regarding its stability, the following criterion holds (see [3]).

Theorem 2.10. *Let A_1, A_2 be commutative $m \times m$ real matrices and let ρ, ω be positive integers, such that $\rho < \omega$. Further, let (α_i, β_i), $i = 1, \dots, m$ be simultaneously ordered couples of eigenvalues of A_1, A_2. The zero solution of (2.6) is asymptotically stable if and only if any of the couples (α_i, β_i), $i = 1, \dots, m$ satisfies either*

$$|\alpha_i| + |\beta_i| < 1, \tag{2.7}$$

or

$$|\alpha_i| + |\beta_i| \geq 1, \qquad |\alpha_i| - 1 < |\beta_i| < 1 \qquad \textit{and} \tag{2.8}$$

$$\omega \arccos \frac{1 + |\alpha_i|^2 - |\beta_i|^2}{2|\alpha_i|} + \rho \arccos \frac{1 - |\alpha_i|^2 + |\beta_i|^2}{2|\beta_i|} < j \arccos \left[\cos \frac{\omega \arg(\alpha_i) - \rho \arg(\beta_i)}{j} \right] \tag{2.9}$$

where $j = gcd(\rho, \omega)$.

Because of the logistic model, we state here also version of Theorem 2.10 for a single-species model. In this case, the system (2.6) is simplified into a form

$$y(n) = p_\rho y(n-\rho) + p_\omega y(n-\omega), \qquad \rho < \omega, \ \rho, \omega \in \mathbb{N}, \ p_\rho, p_\omega \in \mathbb{R}. \tag{2.10}$$

Remark 2.11. For the equation (2.10), the conditions given by Theorem 2.10 are simplified in following way:

$$|p_\rho| + |p_\omega| < 1 \qquad \text{or}$$

$$|p_\rho| + |p_\omega| \geq 1, \qquad |p_\rho| - 1 < |p_\omega| < 1, \qquad (p_\rho)^\omega (p_\omega)^\rho < 0 \qquad \text{and}$$

$$\omega \arccos \frac{1 + (p_\rho)^2 - (p_\omega)^2}{2(p_\rho)} + \rho \arccos \frac{1 - (p_\rho)^2 + (p_\omega)^2}{2(p_\omega)} < \pi.$$

2.2 Period Orbits

One may ask whether there exists some other significant states of a model (besides an equilibrium). A natural question arises, namely if there is some repetition in a development of studied species. If it is so, we say that such a model has a periodic solution. Although in a continuous time domain the introduction of periodic solution is intuitive, in a discrete one it might not be so clear. Therefore, further we will show how to deal with this type of solution for discrete systems.

Definition 2.12. We say that a mapping $f : \mathbb{R} \to \mathbb{R}$ has an *orbit of period T* (or *T-period orbit*), if there exist distinct points $\gamma_1, \dots, \gamma_T \in \mathrm{D}(f)$, $T \in \mathbb{N}$, such that

$$f(\gamma_1) = \gamma_2, \qquad f(\gamma_2) = \gamma_3, \qquad \dots \qquad , f(\gamma_{T-1}) = \gamma_T, \qquad f(\gamma_T) = \gamma_1.$$

The numbers $\gamma_1, \dots, \gamma_T$ are called the points of T-period orbit of mapping f.

Remark 2.13. If a mapping f describes a right-hand side of (2.1) and if it has a T-period orbit, then by a suitable choice of initial condition there exists a periodic solution of (2.1) with period T, where $T \in \mathbb{N}$.

Remark 2.14. In a very similar way as in Definition 2.12, the period orbit can be introduced also for a mapping $F : \mathbb{R}^m \to \mathbb{R}^m$. If F describes a right-hand side of (2.2), then by a suitable choice of initial conditions there exists a periodic solution of (2.2) with period T. Analogously, the same can be introduced for (2.4).

Note that a notion of 1-period orbit of mapping f is equivalent to equilibrium of the system described by this mapping. Also, it is possible that the system has more period orbits of different periods. If it is so, there is a rule that gives a specific ordering of their appearance as solutions. Such an ordering was stated by Sharkovsky in 1964 (see [6]).

Theorem 2.15. ***(Sharkovsky Ordering).***
Let f is a continuous map, such that $f : I \to I$ for some interval $I \subseteq \mathbb{R}$ (may be finite or infinite). Let $i, j, T, S \in \mathbb{N}$. The Sharkovsky ordering is the following ordering of natural numbers

$$1 \succ 2 \succ 2^2 \succ ... \succ 2^j \succ ... \succ 7 \cdot 2^i \succ 5 \cdot 2^i \succ 3 \cdot 2^i \succ ... \succ 7 \cdot 2 \succ 5 \cdot 2 \succ 3 \cdot 2 \succ ... \succ 5 \succ 3.$$

Moreover, if the map f has a T-period orbit and $T \prec S$, then the map f has also S-period orbit.

A consequence of Theorem 2.15 was formulated into theorem and independently proved by Li and Yorke in 1975 (see [6]).

Theorem 2.16. ***(Li and Yorke).***
Let $f : I \to I$ be a continuous map on an interval I. If there is a point of 3-period orbit in I, then for every $T = 1, 2, \ldots$ there is a point of T-period orbit in I.

In other words, the preceding theorem says, that if there exists a 3-period orbit of mapping f, then there exist period orbits of all periods. Indeed, even completely *chaotic orbits* exist as well. Therefore, as next we state a definition of chaos (defined according to Devaney, see [6]).

Definition 2.17. A map $f : I \to I$, where I is an interval, is said to be chaotic if:
1. f is transitive,
2. the set of periodic points P is dense in I,
3. f has sensitive dependence on initial conditions.

For more about transitive maps, dense sets and sensitive dependence on initial conditions see [6].

Remark 2.18. To be able to find points of T-period orbit, it is necessary to know a concept of *iteration*: Let $f^2(y)$ be a notation for function $f(f(y))$. Function $f^2(y)$ is called as *second iteration* of function $f(y)$. In a similar way (by induction) we can introduce the notion of *T-th iteration* of function $f(y)$ as $f^T(y)$.

Then, the points creating a T-period orbit of function $f(y)$ can be found in following way: A point $\gamma^* \in \mathrm{D}(f)$ is a point of T-period orbit of f, if holds $f^T(\gamma^*) = \gamma^*$ and $f^S(\gamma^*) \neq \gamma^*$ for all $S = 1, 2, \ldots, T-1$.

Remark 2.19. In whole thesis, by notation $f^T(y)$, where $T \in \mathbb{N}$, we understand T-th iteration of function $f(y)$ and not T-th power of this function. Other notations like a^T we understand in usual way as T-th powers $\big($e.g., $y^T, \gamma^T, (\gamma^*)^T\big)$.

As was argued before with equilibrium, once there exist some periodic solution of system, there arises a question about its stability. Using Remark 2.18, we state here a definition about stability of period orbit.

Definition 2.20. Let f be a continuous function $f : I \to I$ and let $\gamma_1, \dots, \gamma_T$ be the points of T-period orbit. Then, such an orbit is

- asymptotically stable (attractive), if all points $\gamma_1, \dots, \gamma_T$ are asymptotically stable equilibria of its iteration f^T;
- unstable, if all points $\gamma_1, \dots, \gamma_T$ are unstable equilibria of its iteration f^T.

Remark 2.21. It is possible to verify (by direct computation and using Remark 2.5), that if there is one asymptotically stable (unstable) point of T-period orbit of f, then all points of this orbit are asymptotically stable (unstable). A procedure for verification of stability of given period orbit leads from Remark 2.5 and Remark 2.18.

Remark 2.22. If a point γ^* is an equilibrium of (2.1) (i.e., γ^* is a point of 1-period orbit of f), then γ^* is also a point of T-period orbit of f, where $T \in \mathbb{N}$ (if such an orbit exists). Similarly, this property holds also for other cases, e.g., if γ^* is a point of 2-period orbit, then it is also a point of any 2^n-period orbit, where $n \in \mathbb{N}$ (if such an orbit exists).

Remark 2.23. We have introduced a notion of stability for T-period orbit of f via stability of equilibria of T-th iteration of f. A notion of stability of periodic solution of (2.1) may be interpreted just like a stability of corresponding period orbit. Namely, a periodic solution of (2.1) is asymptotically stable, if the corresponding period orbit of f is asymptotically stable, where f describes the right-hand side of (2.1). Conversely, a periodic solution of (2.1) is unstable, if the corresponding period orbit of f is unstable.

2.3 Bifurcation

We have introduced the notions of equilibria and period orbits. They are equivalent to different types of solutions that may occur for some system. In fact, they determine a dynamical behaviour of model. As we have already said, several period orbits with different orders of period can appear as a solution of model.

A phenomenon of exhibiting of new dynamical behaviour from the old one is described by a term *bifurcation*. In fact, bifurcation occurs, when the data characterizing a model are changed. Usually, just a slight modification of some system's parameter causes a bifurcation. If it is so, such a parameter is called as *bifurcation parameter*, and is usually denoted by μ.

Consider, that there is a bifurcation parameter μ in a system described by a map $f(y)$, we denote it by $f_\mu(y)$. Such a one-parameter family may be written as a function $f(\mu, y)$ of two variables, i.e., $f(\mu, y) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. There is a couple of types of bifurcation. We will introduce these types for a function $f(\mu, y)$ (see [6]).

Theorem 2.24. ***(Saddle-Node Bifurcation).***
Suppose that $f_\mu(y) \equiv f(\mu, y)$ is a C^2 one-parameter family of one-dimensional maps (i.e.,

both $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial \mu^2}$ exist and are continuous), and y^* is a fixed point (equilibrium) of f_{μ^*}, with $f'_{\mu^*}(y^*) = 1$. Assume further that

$$A = \frac{\partial f}{\partial \mu}(\mu^*, y^*) \neq 0 \qquad \text{and} \qquad B = \frac{\partial^2 f}{\partial y^2}(\mu^*, y^*) \neq 0.$$

Then there exists an interval I around y^* and a C^2 map $\mu = p(y)$, where $p : I \to \mathbb{R}$, such that $p(y^*) = \mu^*$, and $f_{p(y)}(y) = y$. Moreover, if $AB < 0$, the fixed points exist for $\mu > \mu^*$, and, if $AB > 0$, the fixed points exist for $\mu < \mu^*$. We say that the system has a saddle-node bifurcation at the fixed point y^* for $\mu = \mu^*$.

Remark 2.25. Consider the same assumptions as in preceding theorem with only change in A, namely

$$A = \frac{\partial f}{\partial \mu}(\mu^*, y^*) = 0.$$

Then we say the system has a *transcritical bifurcation* at the fixed point y^* for $\mu = \mu^*$. Furthermore, let also the assumption on B is modified, namely

$$B = \frac{\partial^2 f}{\partial y^2}(\mu^*, y^*) = 0.$$

Then we say the system has a *pitchfork bifurcation* at the fixed point y^* for $\mu = \mu^*$.

Theorem 2.26. ***(Period-Doubling Bifurcation).***
Suppose that

1. *$f_\mu(y^*) = y^*$ for all μ in an interval around μ^*,*
2. *$f'_{\mu^*}(y^*) = -1$,*
3. *$\dfrac{\partial^2 f^2}{\partial \mu \partial y}(\mu^*, y^*) \neq 0$.*

Then, there is an interval I around y^* and a function $p : I \to \mathbb{R}$ such that $f_{p(y)}(y) \neq y$, but $f^2_{p(y)}(y) = y$. We say that the system has a period-doubling bifurcation at the fixed point y^* for $\mu = \mu^*$.

Remark 2.27. In fact, the sign of AB determines the direction of saddle-node bifurcation (see Figure 1a and Figure 1b). The transcritical, pitchfork and period-doubling bifurcations are depicted in Figure 1c, Figure 1d, Figure 1e, respectively.

We introduced these types of bifurcations just for one-dimensional maps. However, it is possible to extend them into multi-dimensional maps (using center manifold theorem, more about this in [5],[6]). Note that although we will study just logistic map that is one-dimensional, we have to assume also two-dimensional maps, because of its further stabilization via delayed terms (it will increase the dimension). For multi-dimensional maps there exists another type of bifurcation, namely the *Neimark-Sacker bifurcation.* Since such a bifurcation appears only when the advanced stabilization is done (see Chapter 7), we will introduce its notion just briefly (especially, just how to determine it).

We will restrict just to two-dimensional maps. Consider the one-parameter map $F(\mu, \mathbf{y})$, $F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is C^r, $r \geq 3$, on some sufficiently large open set in $\mathbb{R} \times \mathbb{R}^2$, with $\mu \in \mathbb{R}$, $\mathbf{y} \in \mathbb{R}^2$. Let $(\mu^*, \mathbf{y}^*)$ be a fixed point, i.e.,

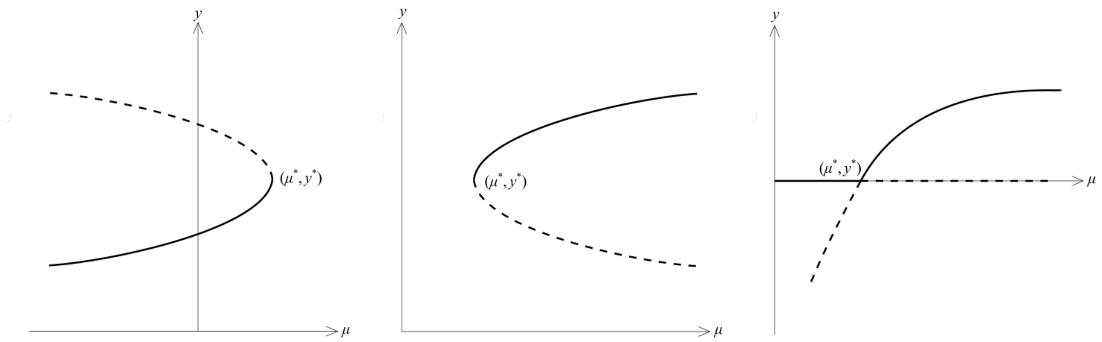
$$F(\mu^*, \mathbf{y}^*) = \mathbf{y}^*.$$

Denote J as linearized map

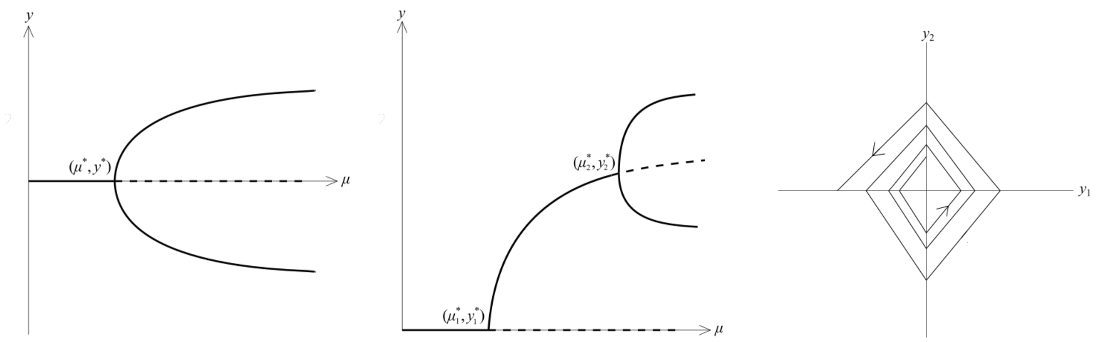
$$J = D_{\mathbf{y}} F(\mu^*, \mathbf{y}^*).$$

Remark 2.28. Following statements were deduced in [6]:

1. Suppose J has an eigenvalue $\lambda = 1$. Then we have either saddle-node, transcritical, or pitchfork bifurcation (depending on fulfilled assumptions stated above).
2. If J has an eigenvalue $\lambda = -1$, then we have a period-doubling bifurcation.
3. Suppose J has a pair of complex conjugate eigenvalues such that $|\lambda| = 1$. Then the Neimark-Sacker bifurcation appears (phase portrait of such a bifurcation is depicted in Figure 1f).



(a) Saddle-node b., $AB > 0$. (b) Saddle-node b., $AB < 0$. (c) Transcritical bif.



(d) Pitchfork bif. (e) Period-doubling bif. (f) Neimark-Sacker bif.

Figure 1: Types of bifurcations.

2.4 Control of System

As we have already mentioned, the main part of thesis is about a stabilization of discrete model. This is usually done by control of the system. A principle of stabilization is based on an implementation of control into the studied system. To make the control useful, it is necessary to set it with convenient parameters. Therefore, later in the text, we derive the conditions assuring the right choice of parameters. A field of mathematics, which studies the problem of control of systems, is called *control theory*. We will introduce here just basics, that are necessary for understanding this problematic. For more about discrete control theory, see for example [5], [12].

A *control* of the system (2.2) is a map U, such that

$$U : \mathbb{R}^m \to \mathbb{R}^k, \qquad m, k \in \mathbb{N},$$

which is used for a stabilization of the system. Usually, a *controlled system* is given by

$$\mathbf{y}(n) = F(\mathbf{y}(n-1)) + U(\mathbf{y}(n-1)). \tag{2.11}$$

Note that using the iteration scheme from the uncontrolled system, the control U may depend also on states $\mathbf{y}(n-2), \dots, \mathbf{y}(n-\omega)$, where $\omega \in \mathbb{N}$ is a delay.

Ideally, a control should be non-invasive, i.e., the equilibria or period orbits remain the same for both the uncontrolled system (2.2) and the controlled system (2.11). Rigorously, let $\mathbf{y}^*$ be the equilibrium of the system, i.e.,

$$\mathbf{y}^* = F(\mathbf{y}^*).$$

A control U is called *non-invasive* (with respect to the equilibrium), if the property

$$\mathbf{y}^* = F(\mathbf{y}^*) + U(\mathbf{y}^*)$$

holds as well. Similarly, we can define non-invasiveness with respect to the orbits of higher periods.

There are several ways how to introduce appropriate controls. Usually, the control depends on two terms:
First, it is a feedback amplitude (the so-called *gain* of the controlled system), denoted as K. In general, a gain $K(\mathbf{y}(n-1)) \in \mathbb{N} \times \mathbb{R}^m \to \mathbb{R}^{m \times m}$ is possibly time-varying and state-dependent. The choice of this matrix gain K is a difficult matter and it is a subject of current investigations. The most frequent choice is a linear dependence of K on $\mathbf{y}(n)$. In the scalar case, this choice is reduced to searching for a suitable gain parameter $K \in \mathbb{R}$. Also in this text, we will discuss an appropriate choice of a constant gain K.

The second control parameter is a delay $\omega \in \mathbb{N}$. There are two ways how the delay can be used. Firstly, we can set $\omega = T$, where T is exactly the period of orbit that we wish to stabilize. On contrary, we can left the delay to be not specified and analyze the controlled system with respect to varying ω.

For these parameters, we will derive the conditions guaranteeing the usefulness of control. We will gather them into the sets of all possible values of parameters of some control, for which the selected orbit can be stabilized. Let us call these sets as *stabilization sets*. There are several control methods (i.e., several definitions of controls), that can be used for a stabilization of discrete system. Let us introduce those, which are used in thesis, together with a notation of corresponding stabilization sets. Note that we will introduce them in a general way, i.e., for the systems. Nevertheless, in this thesis we will apply controls just to difference equations, and in such a case we will denote it a control as $u : \mathbb{R}^\ell \to \mathbb{R}$, where $\ell \in \mathbb{N}$ denotes a number of control state inputs. Also, since we are primarily interested in stabilization of scalar logistic equation, we restrict on the case $y : \mathbb{N} \to \mathbb{R}$ in our next considerations.

Proportional feedback control

A *proportional feedback control* with a gain K is given by

$$u(y(n-\omega)) = K[y(n-\omega) - t], \tag{2.12}$$

where $\omega \in \mathbb{N}$ denotes a delay (the choice $\omega = 1$ is the most frequent one). This control reacts on some targeted value $t \in \mathbb{R}$, that we wish to stabilize. Usually, a target t is taken as an unstable equilibrium or a point of unstable period orbit of the system. Note that for using of such a control, the *a priori* knowledge of t is required. In fact, this may be a problem for some systems (it can be hard to find the point explicitly). In text, this method will be abbreviated as PFC.

A claim about the non-invasiveness of any control in fact means, that when the system reaches the selected orbit, the control is switched off. This is usually done by a specification of type of orbit (meaning by specification of its period). However, in this method, we are supplying the control with the information on the values of t determining the whole orbit. Hence, regardless of the value of a period T of orbit, we will use PFC either given by

$$u(y(n-1)) = K[y(n-1) - t], \tag{2.13}$$

or given by (2.12) with other values of ω (for a comparison if assuming of some information from the past can predict better results).

Delayed feedback control

A *delayed feedback control* with a gain K is usually given by

$$u(y(n-\rho), y(n-\rho-\omega)) = K\left[y(n-\rho-\omega) - y(n-\rho)\right],$$

where $\rho, \omega \in \mathbb{N}$ denote the delays, such that $\rho < \omega$. In text, this method will be abbreviated as DFC. From now we will restrict on $\rho = 1$. Therefore, we get

$$u(y(n-1), y(n-\tilde{\omega})) = K\left[y(n-\tilde{\omega}) - y(n-1)\right], \tag{2.14}$$

where $\tilde{\omega} = \omega + 1$ (for a simpler notation).

Similarly as in a case of PFC, we can supply the control with the information about the orbit, which we wish to stabilize. To do this, we consider a delay ω as a period of corresponding orbit. In this case, let us assign a delay with T (for a distinction between these definitions). We get a control defined by

$$u(y(n-1), y(n-1-T)) = K\left[y(n-1-T) - y(n-1)\right]. \tag{2.15}$$

Since a periodical solution of period T of system satisfies $y(n) = y(n+T)$, the control is switched off after reaching this orbit. Here, *a priori* knowledge of existence of a T-period orbit is required, but *a priori* knowledge of its single points is not required.

Prediction based control

A *prediction based control* is an improvement of DFC given by

$$u(y(n-1), y(n-\omega)) = K\left[y(n-\omega) - f\left(y(n-1)\right)\right], \tag{2.16}$$

where $\omega \in \mathbb{N}$ and $f : \mathbb{R} \to \mathbb{R}$ denotes a right-hand side of system, into which the control is implemented. In text, this method will be abbreviated as PBC. This control compares a delayed term with values along the trajectories of a free (uncontrolled) system.

Similarly, by assigning ω with a period of selected orbit, we get another definition of this control. Namely, we get

$$u(y(n-1), y(n-T)) = K\left[y(n-T) - f\left(y(n-1)\right)\right], \tag{2.17}$$

where T is a corresponding period.

For PBC, the usage of constant or time-varying, state-dependent gain K makes a significant difference (the difference between using constant or non-constant gain in previous control methods is not so significant). By defining the gain K, we can assign to a control the so-called *control law*, that is proposed for a stabilization of specific orbits. Some control laws are proposed in [2]. In fact, such a time-varying, state-dependent gain was just firstly proposed there. In [2], there is introduced also another definition of PBC, namely

$$u(y(n)) = K(y(n-1))(f^T(y(n-1)) - y(n-1)).$$

We will show here just one of the proposed control laws, to see that its formulation (in a vector form) is very complex:

$$K(\mathbf{y}_n) = -\left(D_U F(\mathbf{y}_n, U)|_{U=U(\mathbf{y}_n)}\right)^{-1} D_{\mathbf{y}} F(\mathbf{y}, U(\mathbf{y}_n))|_{\mathbf{y}=\mathbf{y}_n} \left(D_{\mathbf{y}} F^T(\mathbf{y}) - I\right)^{-1}|_{\mathbf{y}=\mathbf{y}_n},$$

where I is $m \times m$ unit matrix and $y_n = y(n)$ (for a neater form). We do not utilize this type of control in our thesis, because it might be useful especially for stabilization of more complex systems.

3 Studied Model

This chapter is for a brief introduction into the problematic of discrete model studied further in this thesis. Therefore, we state here the form of logistic model used in later discussions, and we give the meaning to notation used in this model. We summarize here the results of analysis of a free system (meaning the analysis of system, that is not controlled). In other words, we give here a list of types of possible behaviour of solutions (e.g., equilibria, period orbits) and corresponding ranges of stability.

3.1 Logistic Model

A discrete logistic model is one of the basic models studied in a theory of differential and difference equations, and in a theory of chaos. In [11], there was analyzed in detail the logistic model given by a logistic map in form

$$x(n+1) = x(n) + rx(n)\left(1 - \frac{x(n)}{C}\right). \tag{3.1}$$

For a simplicity and unification with other works and papers, from now we assume the logistic map given as

$$y(n) = \mu y(n-1)\left(1 - y(n-1)\right). \tag{3.2}$$

One can verify, that (3.1) is possible to obtained from (3.2) by using the following substitutions:

$$y(n) = \frac{b}{\mu}x(n), \qquad b = \frac{r}{C}, \qquad \mu = 1 + r.$$

We can interpret the evolution of population as the change of its size between generations. A population size (or population density) in n-th time period, i.e., the n-th generation, is denoted by $y(n)$ (or, in the previous notation, by $x(n)$). A parameter μ (or r) is a bifurcation parameter, denoting the population growth rate from one generation to another. A parameter $C > 0$ denotes the storage capacity of the environment and $b > 0$ denotes the proportional constant of the interaction among members of the species.

It is well-known, that the logistic map (3.2) has different behaviour of solution, with dependence on the bifurcation parameter μ. The solution of model can either tends to its equilibrium, or behaves periodically, or behaves chaotically. The stability dependence of behaviour of solution on the bifurcation parameter μ is shown in the following table:

Discrete Logistic Model	
Type of Solution	**Range of Stability**
Equilibrium $y_1^* = 0$	$0 < \mu < 1$
Equilibrium $y_2^* = 1 - \frac{1}{\mu}$	$1 < \mu \leq 3$
2-period orbit	$3 < \mu < 1 + \sqrt{6}$
4-period orbit	$1 + \sqrt{6} < \mu < 3.544$
8-period orbit	$3.544 < \mu < 3.564$
16-period orbit	$3.564 < \mu < 3.568$
⋮	⋮
3-period orbit	$\mu = 1 + \sqrt{8}$

Table 1: Types of solutions and corresponding ranges of stability.

Although it is interesting, from a mathematical point of view, to see such a rich scale of behaviour of the system, from a biological point of view it is better to have more stable systems. Therefore it is compulsory to try to stabilize the model (3.2), i.e., to 'enlarge' the range of stability of its steady states as much as possible. To do this, the knowledge of control theory is very useful. Further in the thesis, a stabilization will be done by adding a control u to our free system (3.2). Note that this control should be non-invasive. We will analyze such a new controlled system, to see how it behaves in dependence with changes of bifurcation parameter μ and control parameter(s).

Firstly, we will analyze a stabilization of the equilibria of (3.2) via controls (2.13), (2.15) and (2.17). Next, a stabilization of 2-period orbit and 3-period orbit via the same controls will be discussed. After this, we will stabilize again equilibria, but via controls (2.12), (2.14) and (2.16), in order to compare the different approaches on delay ω.

Doing this, we introduce the following notations:
For PFC given by (2.13), let us denote the stabilization set of all gain parameters K as PFC^T_μ. Here, T denotes a period of a stabilized orbit of (3.2) and $\mu \geq \mu_T$ is arbitrary real parameter, where μ_T is the upper bound of stability interval corresponding to period T (see the right column of Table 1 describing stability of a free system).

Analogously, for DFC given by (2.15), let us denote the stabilization set of all gain parameters K as DFC^T_μ. For PBC given by (2.17), let us denote the stabilization set of all gain parameters K as PBC^T_μ. The argumentation on symbols T and μ is the same as in the previous paragraph.

3.2 Dynamical Traffic Model

Herein the section, we will derive an application of logistic model into a traffic flow to see, where it is possible to utilize the aim and results of this thesis. In general, without any interference and given rules, a traffic flow would be totally unstable and unpredictable, and so completely chaotic. Such a state would be very dangerous, not only for the drivers, but also for everyone in close neighbourhood of a traffic (which is in current world almost everywhere). Hence, the implementation of chaotic control is needed. Further information and description of this model come from [8].

We can consider a traffic flow as a dynamical model, either describing the flow on a single link, or on a more complex network. There are two approaches in a description of a system. First, a microscopic one, in which the system's variables are the position and speed of each vehicle. Second, a macroscopic one, where the system's variables are the total number of trips between two places, the rate of traffic flow, density and speed. We will introduce the model from a macroscopic point of view.

Let q be a traffic flow, d its density and s its speed. Moreover, assume that speed depends on density, i.e., $s = s(d)$. Since the fundamental flow-density-speed diagram is given by $q = ds$, the flow should be therefore also dependent on density. Hence,

$$q(d) = ds(d).$$

Furthermore, assume that the traffic flow satisfies the Greenshield's model. In other words, a relation between speed and density is known, namely

$$s(d) = s_f \left(1 - \frac{d}{d_j}\right),$$

where s_f denotes a speed of free flow, and d_j a density of traffic jam. Next, we assume that the current flow is decided by the traffic conditions in previous time unit. Under these assumptions, we get

$$q(n,d) = d(n-1)s_f\left(1 - \frac{d(n-1)}{d_j}\right). \tag{3.3}$$

Note that the variables are q and d.

Let σ denote an occupancy, which is the ratio of actual occupied time and available time of a certain place. Hence, the value of occupancy is $\sigma \in \langle 0, 1\rangle$. It is defined as

$$\sigma = \frac{q\bar{L}}{\bar{s}} = d\bar{L},$$

where $\bar{s}$ is an average speed and $\bar{L}$ is an average vehicle length. From this, we can rewrite the variables q and d in following way:

$$q = \frac{\sigma\bar{s}}{\bar{L}}, \qquad d = \frac{\sigma}{\bar{L}}.$$

Therefore, the equation (3.3) of two variables q, d can be rewritten into equation

$$\frac{\sigma(n)\bar{s}}{\bar{L}} = s_f\frac{\sigma(n-1)}{\bar{L}}\left(1 - \frac{\sigma(n-1)}{d_j\bar{L}}\right)$$

of a single variable. Denoting $\tilde{\sigma} = \dfrac{\sigma}{d_j\bar{L}}$, we get

$$\tilde{\sigma}(n) = \frac{s_f}{\bar{s}}\tilde{\sigma}(n-1)(1-\tilde{\sigma}(n-1)). \tag{3.4}$$

By comparing the equation (3.2) with (3.4), it is clear that they have the same form. Indeed, a ratio $\frac{s_f}{\bar{s}}$ represents the bifurcation parameter. Therefore, just this ratio may raise a chaotic behaviour.

4 Stabilization of Equilibria

Herein the chapter, the analysis of a stabilization of equilibria of a controlled logistic map is studied. Although there are two equilibria (see Table 1) for (3.2), from a biological point of view it makes sense to study just a stabilization of the equilibrium y_2^*, since we are not interested in the extinction of a species (meaning a stabilization of the equilibrium y_1^*).

As first, we apply controls to a free system; these controls involve the exact period of orbit that should be stabilized. That means, we apply controls (2.13), (2.15) and (2.17) with $T = 1$. For each type of control, the description of the stabilization sets PFC_μ^1, DFC_μ^1 and PBC_μ^1, where $\mu \geq \mu_1 = 3$, will be performed. Moreover, a new (explicit) formulation of ranges of stability will be given.

4.1 Proportional Feedback Control Method

The PFC is a simple control method. Its form (2.13) was proposed, e.g, by Franco and Liz in [7]. In order to stabilize the equilibrium, we set the targeted value on which the control reacts as $t = y_2^*$. The logistic map controlled by a control (2.13) is given by

$$y(n) = \mu y(n-1)(1 - y(n-1)) + K[y(n-1) - y_2^*]. \tag{4.1}$$

The usage of this type of control for a stabilization of the equilibrium is non-invasive with respect to the targeted equilibrium. More precisely, the equilibria of (4.1) are given by

$$\gamma_1^* = \frac{K}{\mu},$$
$$\gamma_2^* = 1 - \frac{1}{\mu}.$$

Thus, the chosen equilibrium remains the same for the controlled and free system, but the other equilibrium is now varying with respect to K and μ.

Notice that the system (4.1) is in form of classical, non-delayed difference equation with a linear control. Thus, in this case, the analysis of stability of y_2^* is analogous to [11]. That means, the stability problem is reduced just to the analysis of first derivative of the right-hand side of (4.1) (see Remark 2.5). Following this, we get that y_2^* is asymptotically stable if

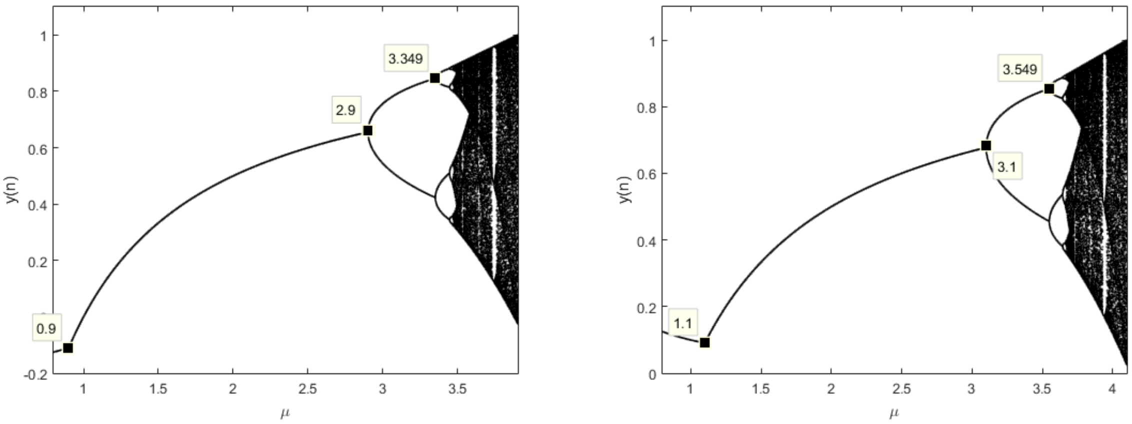
$$|f'(y_2^*)| = |2 - \mu + K| < 1.$$

Solving preceding stability condition, we get that y_2^* of (4.1) is asymptotically stable if

$$1 + K < \mu < 3 + K. \tag{4.2}$$

There is no further specification on value of gain, so there is possible to take it as an arbitrary $K \in \mathbb{R}$. However, one can see that the crucial choice stands on the sign of feedback, i.e, if we set $K > 0$ or $K < 0$. By this we are determining the direction of 'shifting the range of stability' $1 < \mu < 3$, either to the right or left. Let us discuss a bifurcation diagram for (4.1) to see the behaviour of solution in dependence on the sign of K.

In Figure 2a and Figure 2b, there are depicted bifurcation diagrams to (4.1) for chosen values of a gain K. As we said, we will discuss mainly the difference in usage of negative and positive gain K. In both the figures, there are some values that are highlighted. In fact, they are representing the endpoints of intervals of stability of orbits. Particularly, in Figure 2b, the range $1.1 < \mu < 3.1$ represents the asymptotically stable equilibrium y_2^* of (4.1) with $K = 0.1$. On contrary, in Figure 2a, the range $0.9 < \mu < 2.9$ represents the same, but for $K = -0.1$. This verifies the stability condition (4.2), but what can be deduced from it?



(a) $K = -0.1$ (b) $K = 0.1$

Figure 2: Bifurcation diagrams to (4.1).

Let us remind that for the free system (3.2), y_2^* is asymptotically stable for $1 < \mu < 3$ and the overall range of usable growth parameter is $0 < \mu < 4$. However, this overall range has been broadened by using $K > 0$ and, conversely, it has been shortened by using $K < 0$. Particularly, the maximal usable growth parameter for $K = 0.1$ is $\mu = 4.1$ and for $K = -0.1$ it is $\mu = 3.9$. Moreover, the stability of modified equilibrium γ_1^* is lost when $\mu = 1.1$ (the case of $K = 0.1$), or when $\mu = 0.9$ (the case of $K = -0.1$).

All in all, the control (2.13) mainly affects γ_1^*, since its stability range is either reduced (the case $K < 0$) or increased (the case $K > 0$). Paradoxically, the length of the stability range for y_2^* remains the same; it is just shifted by the value K. In fact, the same holds also for the other orbits. For instance, a 2-period orbit of a free system is asymptotically stable for $1 < \mu < \sqrt{6} + 1 \doteq 3.449$, so the labelled values $\mu = 3.549$ (see Figure 2b) and $\mu = 3.349$ (see Figure 2a) confirm previous observation.

A description of stabilizing set PFC_μ^1 highly depends on a formulation of the task. We are not able to answer the question, whether it is possible to enlarge the stability range of y_2^*, since it is not literally enlarged, but just shifted. If we take $K \in PFC_\mu^1$, where

$$PFC_\mu^1 = (0, \infty),$$

the equilibrium y_2^* is asymptotically stable for higher values of growth parameter μ (but the length of the stability range for μ remains the same). As a by-product, the prolonged stability of y_1^* is obtained.

Conversely, if we take $K \in PFC_\mu^1$, where

$$PFC_\mu^1 = (-\infty, 0),$$

the equilibrium y_2^* that is asymptotically stable for lower values of growth parameter μ. We get the shorter stability range of y_1^*. Finally, if we wish to literally enlarge the range

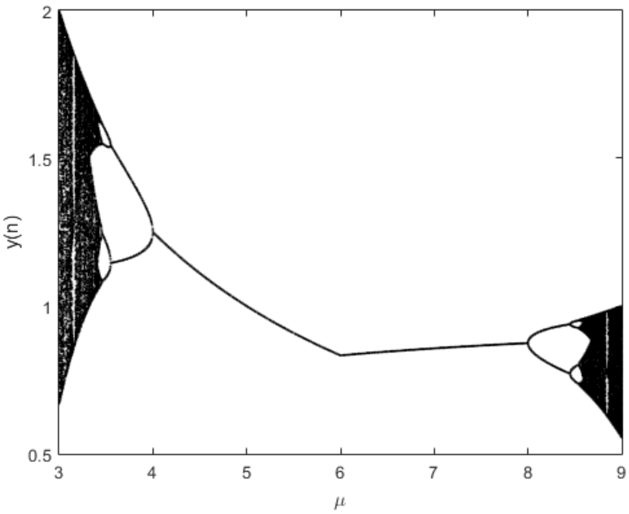


Figure 3: Bifurcation diagram to (4.1) with $K = 5$.

of stability of y_2^*, then

$$PFC_\mu^1 = \emptyset.$$

Moreover, the choice of $K >> 1$ is very specific (see Figure 3). In fact, it causes that the system firstly behaves chaotically, with increasing μ its behaviour is calming down (something like 'reverse' bifurcation), and after a certain value of μ it started bifurcate in a classical way of period-doubling bifurcation. To avoid this, we can restrict a stabilization set, namely to $PFC_\mu^1 \subset (-1, 1)$. Taking this into a consideration, we are able to get a maximal range of stability of controlled y_2^* as

$$2 < \mu < 4 \qquad \text{for } K \to 1, \qquad \text{or} \qquad 0 < \mu < 2 \qquad \text{for } K \to -1,$$

depending on a formulation of the task.

4.2 Delayed Feedback Control Method

The DFC is a control used in a lot of papers and works. It was introduced in the papers of Pyragas [13] and Ushio [16]. We apply a control (2.15) to a free system and in order to stabilize the equilibrium, we set $T = 1$. The controlled logistic map is thus given by

$$y(n) = \mu y(n-1)(1 - y(n-1)) + K[y(n-2) - y(n-1)]. \tag{4.3}$$

The usage of DFC is fully non-invasive, i.e., both equilibria y_1^* and y_2^* remain the same for controlled and uncontrolled system.

Notice that the system (4.3) is now in form of second-order difference equation with the linear control. Therefore, the analysis following a Schur-Cohn criterion (see Theorem 2.6) is needed. In particular, we will use a Remark 2.7. For further studying of condition given in this remark, the coefficients p_1 and p_2 have to be known. We can easily get them from a linearization of (4.3) at y_2^*. We get

$$\frac{\partial f}{\partial y(n-1)}(y_2^*) = \left.\frac{\partial\left[\mu y(n-1)(1-y(n-1)) + K(y(n-2) - y(n-1))\right]}{\partial y(n-1)}\right|_{y(n-1)=y_2^*}$$

$$
\begin{aligned}
&= 2 - \mu - K, \\
\frac{\partial f}{\partial y(n-2)}(y_2^*) &= \left.\frac{\partial \left[\mu y(n-1)(1-y(n-1)) + K(y(n-2)-y(n-1))\right]}{\partial y(n-2)}\right|_{y(n-2)=y_2^*} \\
&= K.
\end{aligned}
$$

Hence, because of the form of equation in Remark 2.7, the coefficients are

$$p_1 = K + \mu - 2 \qquad \text{and} \qquad p_2 = -K.$$

The stability condition gives

$$|K + \mu - 2| < 1 - K < 2.$$

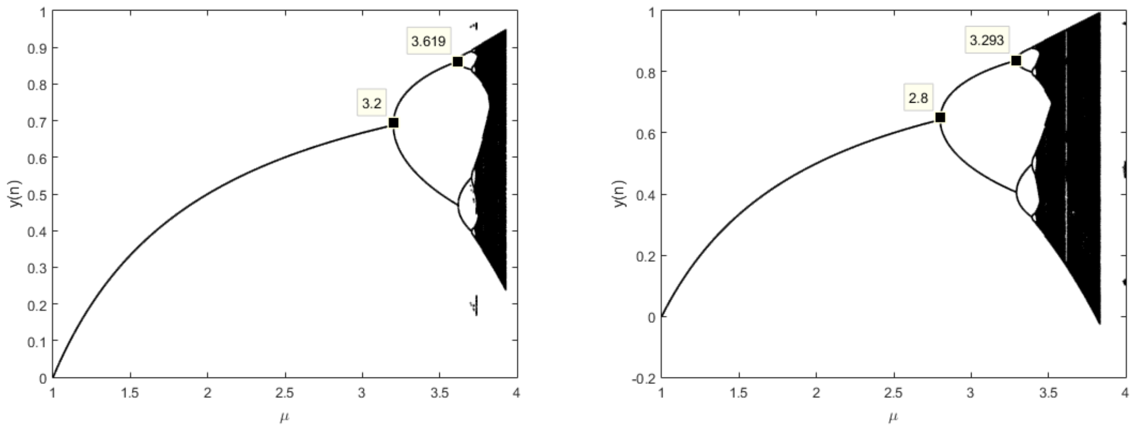
Solving this, we get that y_2^* is asymptotically stable if

$$1 < \mu < 3 - 2K. \tag{4.4}$$

Moreover, a requirement on the values of gain now arises from (4.4). Particularly, it gives $K \in (-1, 1)$. Nevertheless, in order to enlarge the range of stability for the equilibrium y_2^*, the gain should be negative. Therefore, we get the following description of stabilizing set

$$DFC_\mu^1 = (-1, 0).$$

The following bifurcation diagrams to (4.3) confirm this observation. It is obvious that the stability range of y_2^* is enlarged (see Figure 4a) by the control with a negative gain. However, it has an influence on other orbits, namely it causes a contraction of the rest of bifurcation diagram. In fact, the contraction happens no matter on which sign of gain we set (see Figure 4a, 4b). On contrary, the stability range for y_2^* is diminished by the control with a positive gain (see Figure 4b).



(a) $K = -0.1$ (b) $K = 0.1$

Figure 4: Bifurcation diagrams to (4.3).

A natural question arises, namely how much the range of stability can be extended. The control (2.15) with $T = 1$ and $K \in DFC_\mu^1$ stabilizes the equilibrium y_2^* for

$$3 < \mu < 3 - 2K$$

(for $1 < \mu < 3$, this equilibrium is asymptotically stable naturally, i.e., with $K = 0$). Hence, as the most largest values, y_2^* is stabilizes when

$$3 < \mu < 5$$

(as $K \to -1$). It leads to a critical value of μ, we denote it as $\mu_1^* = 5$, for which the set DFC_μ^{cl} becomes an empty set. In other words, for $\mu \geq \mu_1^*$ there exists no value of K such that this control is able to stabilize y_2^*.

4.3 Prediction Based Control Method

The PBC originates from the DFC and it was proposed, e.g., in a paper of Buchner and Zebrowski [1]. Also here, we set $T = 1$ and thus, after application of control (2.17), we get the controlled logistic map in form

$$y(n) = \mu y(n-1)\,(1 - y(n-1)) + K[y(n-1) - \mu y(n-1)\,(1 - y(n-1))]. \tag{4.5}$$

The usage of PBC is again fully non-invasive. The system (4.5) is again in a form of non-delayed equation, but in this case the control is nonlinear.

Since the system is not delayed, it is enough to analyze the stability of y_2^* in the same way as in the case of control of system by PFC in Section 4.1. Therefore, we get the stabilization condition as

$$|f'(y_2^*)| = |K\mu - K - \mu + 2| < 1.$$

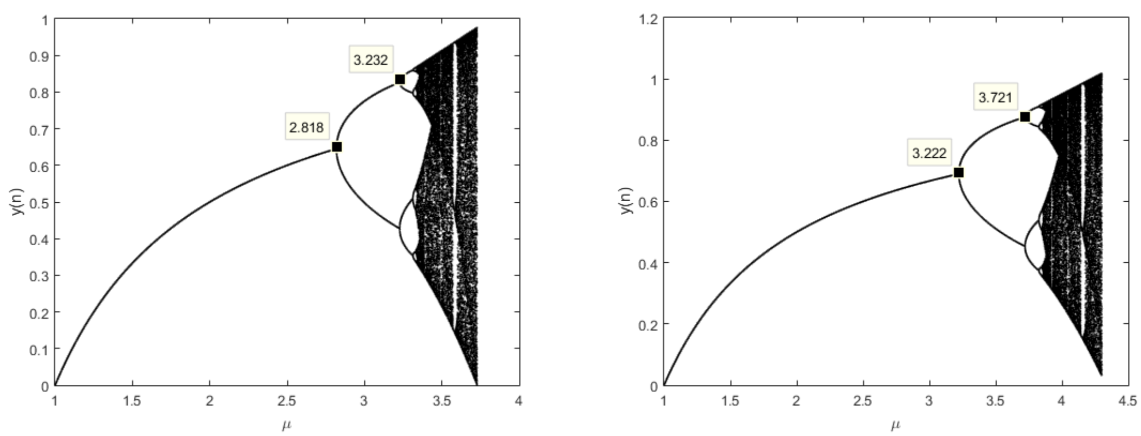
Solving this condition gives that y_2^* is asymptotically stable if

$$1 < \mu < \frac{3-K}{1-K}. \tag{4.6}$$

Moreover, we get that $K < 1$. Nevertheless, in order to extend the range of stability of y_2^*, we put the requirement on gain as $K \in PBC_\mu^{cl}$, where

$$PBC_\mu^{cl} = (0, 1)$$

is the description of stabilizing set of y_2^* for the system (4.5).



(a) $K = -0.1$ (b) $K = 0.1$

Figure 5: Bifurcation diagrams to (4.5).

A claim on a positive gain is verified in Figure 5b. It is clear that the range of asymptotically stable y_2^* is enlarged. Moreover, the overall range of μ is extended, so as a side effect we get that parts of bifurcation diagram corresponding to other asymptotically stable orbits are also expanded a bit. Conversely, the usage of a negative gain causes a contraction of bifurcation diagram (see Figure 5a). This is the reason why such a choice of gain is not suitable for a broadening of stability.

We are able to answer the same question as we put in the previous control method. The control (2.17) with $T = 1$ and $K \in PBC_\mu^1$ stabilizes y_2^* for

$$3 < \mu < \frac{3 - K}{1 - K}.$$

Hence, it stabilizes y_2^* when

$$3 < \mu < \mu_1^*,$$

where $\mu_1^* \to \infty$ as $K \to 1$. So, for any $\mu \geq \mu_1$, there is $K \in PBC_\mu^1$ such that this control is able to stabilize y_2^*. In the limit case (i.e., when $K = 1$) the whole bifurcation part corresponding to period orbits and chaotic part is moved (postponed) to infinity. Thus the discrete system (2.17) behaves like a continuous one (it has just equilibria and has no period orbit).

5 Stabilization of 2-Period Orbits

Herein the section, a stabilization of period orbits of the logistic map starts. Note that for the logistic map there exist orbits of all periods. The ordering of their occurrence as a solution of the system is given by Sharkovsky (see Theorem 2.15). From the summary of results of [11] (see Table 1) it is clear, that when some period orbit looses its stability, the next period orbit (given by the ordering) appears as a asymptotically stable solution. Therefore, one may wonder if this property of a periodic solution of free system is preserved also for the controlled system. In fact, this is true, as we will see in bifurcation diagrams to corresponding controlled systems.

Let us remind how the analysis of period orbits is performed. Usually, it is done by the analysis of fixed points of a corresponding iteration of map. That means, for a T-period orbit we use a T-th iteration of studied map. Note that in [11] we were able to do a discussion about stability of period orbits in detail just for the orbits of period 2, 4 and 3, as the other periods lead to the analysis of polynomials of high order (for a T-period orbit we have to analyze a 2^T-iteration).

Due to the Sharkovsky ordering, we start with the analysis of 2-period orbit. In general, a 2-period orbit is given by two generations, that are alternating from one to the other. For the studied uncontrolled system (3.2), the 2-period orbit is given by

$$\gamma_1 = \frac{\mu + 1 - \sqrt{\mu^2 - 2\mu - 3}}{2\mu}, \qquad \gamma_2 = \frac{\mu + 1 + \sqrt{\mu^2 - 2\mu - 3}}{2\mu}.$$

Also here, we will firstly restrict on the stabilization via controls given by (2.13), (2.15) and (2.17). As the results of this section we will describe the stabilization sets PFC^2_μ, DFC^2_μ and PBC^2_μ, where $\mu \geq \mu_2 = 1 + \sqrt{6}$. Moreover, we will determine a new range of stability for each of controlled systems.

5.1 Proportional Feedback Control

For the purpose of stabilization of 2-period orbit via PFC, we choose as the targeted value t one of the points that generate this orbit, i.e., either γ_1 or γ_2. The application of control (2.13) leads to the controlled system in form

$$y(n) = \mu y(n-1)(1 - y(n-1)) + K[y(n-1) - \gamma_1]. \tag{5.1}$$

Note that it does not matter if we choose γ_1 or γ_2 as a target t, the results will remain same for both cases (it follows from Remark 2.21).

However, for this purpose, the usage of such a control is invasive. It modifies not even the forms of equilibria, but also forms of γ_1 and γ_2. To see this, let us find the fixed points of a second iteration of (5.1). Denote a whole right-hand side of (5.1) as $f(y)$. The second iteration $f^2(y)$ is given by

$$f^2(y) = \mu f(y)(1 - f(y)) + K[f(y) - \gamma_1] = \sum_{i=0}^{4} a_i y^i,$$

where the coefficients a_i are

$$a_4 = -\mu^3,$$

$$
\begin{aligned}
a_3 &= 2\mu^2(\mu + K),\\
a_2 &= -\mu[2K\mu\gamma_1 + (K+\mu)(K+\mu+1)],\\
a_1 &= (K+\mu)(2K\mu\gamma_1 + K + \mu),\\
a_0 &= -K\gamma_1(K\mu\gamma_1 + K + \mu + 1).
\end{aligned}
$$

The fixed points of $f^2(y)$ are

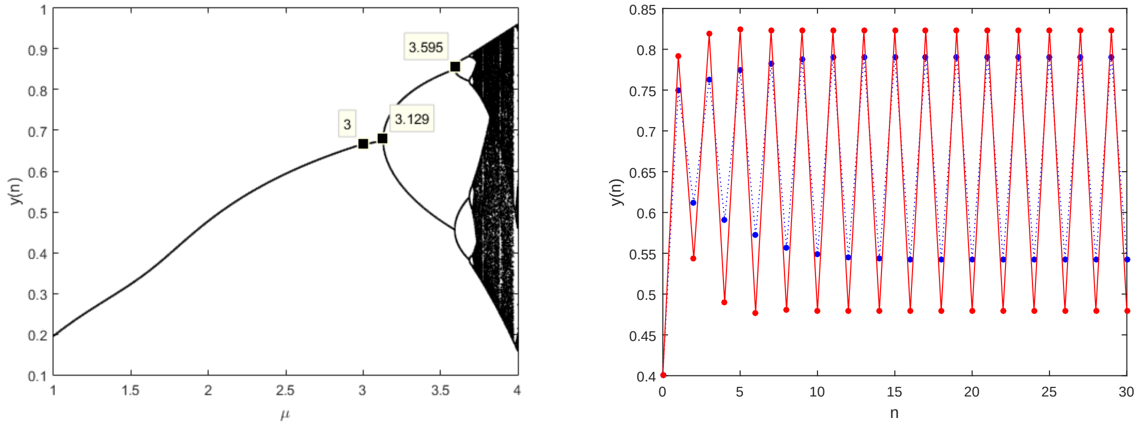
$$
\begin{aligned}
\gamma_1^* &= \frac{K+\mu-1-\sqrt{-4K\mu\gamma_1 + K^2 + 2K\mu - 2K + \mu^2 - 2\mu + 1}}{2\mu},\\
&= \frac{K+\mu-1-\sqrt{K^2 - 2K\sqrt{\mu^2-2\mu-3} - 4K + \mu^2 - 2\mu + 1}}{2\mu},\\
\gamma_2^* &= \frac{K+\mu-1+\sqrt{-4K\mu\gamma_1 + K^2 + 2K\mu - 2K + \mu^2 - 2\mu + 1}}{2\mu},\\
&= \frac{K+\mu-1+\sqrt{K^2 - 2K\sqrt{\mu^2-2\mu-3} - 4K + \mu^2 - 2\mu + 1}}{2\mu},\\
\gamma_3^* &= \frac{K+\mu+1-\sqrt{-4K\mu\gamma_1 + K^2 + 2K\mu - 2K + \mu^2 - 2\mu - 3}}{2\mu},\\
&= \frac{K+\mu+1-\sqrt{K^2 - 2K\sqrt{\mu^2-2\mu-3} - 4K + \mu^2 - 2\mu - 3}}{2\mu},\\
\gamma_4^* &= \frac{K+\mu+1+\sqrt{-4K\mu\gamma_1 + K^2 + 2K\mu - 2K + \mu^2 - 2\mu - 3}}{2\mu},\\
&= \frac{K+\mu+1+\sqrt{K^2 - 2K\sqrt{\mu^2-2\mu-3} - 4K + \mu^2 - 2\mu - 3}}{2\mu}.
\end{aligned}
$$

For $K = 0$ (meaning that there is no control of system), we get $\gamma_1^* = y_1^*$, $\gamma_2^* = y_2^*$, $\gamma_3^* = \gamma_1$ and $\gamma_4^* = \gamma_2$. It is obvious, that their expressions are more complicated under the presents of a control.

Nevertheless, a breaking of the assumption on non-invasiveness of control does not explicitly mean that such a control is wrong and unusable. From Figure 6a we see that the range of stability of 2-period orbit is 'enlarged', in the same sense as in the stabilization of equilibrium via PFC, i.e., the range is shifted to the right. However, the length of range of stability of 2-period orbit remains almost the same as the length of uncontrolled 2-period orbit. A labelled part between $3 < \mu < 3.129$ corresponds to asymptotically stable γ_2^*, i.e., to modified y_2^*. Up to $\mu = 3$, the orbit γ_1^* is asymptotically stable, i.e., modified y_1^*. Hence, as main results, we obtain significantly prolonged stability range of γ_1^* and significantly shortened stability range of γ_2^*.

A graphical interpretation of invaded points is depicted in Figure 6b. A 2-period orbit of uncontrolled system (3.2) is depicted as red points. The invasiveness of control in system (5.1) is clear from this figure. The blue points, corresponding to points γ_3^* and γ_4^* creating controlled 2-period orbit, should coincide with the red ones (in a case of non-invasive control), but this is obviously not the case. The parameters used in Figure 6b are $K = 0.1$ and $\mu = 3.3$ with initial condition $y(0) = 0.4$, the same for both the systems.

Although the mathematical correctness is not satisfied due to invasive control, it is possible to deduce some contribution of it. Think about the system (either (3.2) or (5.1))



(a) Bifurcation diagram to (5.1), $K = 0.1$.

(b) 2-period orbit of (3.2) and (5.1).

Figure 6

as it is the prediction of n-th generation of some population. Then, by predicting the population has behaviour as 2-period orbit given by generations ζ, ξ we mean that after a while, there will be just a repetition of these two generations. The difference $|\zeta - \xi|$ means how many individuals are predicted to be born/to die during a single time unit. Notice that

$$|\gamma_3^* - \gamma_4^*| < |\gamma_1 - \gamma_2|,$$

i.e., the generations γ_3^* and γ_4^* are closer to each other than generations γ_1 and γ_2. Therefore, the controlled system (5.1) predicts not so shocking behaviour as the system (3.2) predicts (shocking in sense that many of individuals will die/be born during a time unit).

Since we have just given a logical justification of the usage of invasive control, it makes sense to analyze a stability of controlled 2-period orbit precisely. The Jacobi matrix of f^2 at point γ_3^* is given by

$$\begin{aligned} Df^2(\gamma_3^*) &= \frac{\partial f^2}{\partial y}(\gamma_3^*) \\ &= -K^2 + 2K\sqrt{\mu^2 - 2\mu - 3} + 4K - \mu^2 + 2\mu + 4. \end{aligned}$$

Note that the analysis has to be done for a point γ_3^* and not for a point γ_1, because it is modified. The stability condition is given by

$$|-K^2 + 2K\sqrt{\mu^2 - 2\mu - 3} + 4K - \mu^2 + 2\mu + 4| < 1. \tag{5.2}$$

This condition is depicted in Figure 7. We highlighted just area of our interest, i.e., the area, where $\mu > 0$. It is clear that (5.2) is satisfied for $\mu \geq \mu_2$ if $K > 0$. The boundaries of hatched area are

$$\begin{aligned} K &= \sqrt{\mu^2 - 2\mu - 3} + 2 - \sqrt{6 + 4\sqrt{\mu^2 - 2\mu - 3}} = \bar{K}_\mu && \text{(purple colour)}, \\ K &= \sqrt{\mu^2 - 2\mu - 3} + 2 - 2\sqrt{1 + \sqrt{\mu^2 - 2\mu - 3}} = \tilde{K}_\mu && \text{(red colour)}, \\ K &= \sqrt{\mu^2 - 2\mu - 3} + 2 + 2\sqrt{1 + \sqrt{\mu^2 - 2\mu - 3}} && \text{(blue colour)}, \\ K &= \sqrt{\mu^2 - 2\mu - 3} + 2 + \sqrt{6 + 4\sqrt{\mu^2 - 2\mu - 3}} && \text{(green colour)}. \end{aligned}$$

Also, as we discussed in Section 4.1, it is enough to assume $|K| < 1$. Hence, the stabilization set $PFC^2_\mu \subset (0, 1)$. Particularly, for $\mu = \mu_2$, we get $PFC^2_{\mu_2} = (0, \tilde{K}_\mu)$. With increasing μ, the stabilization set is given as

$$PFC^2_\mu = (\bar{K}_\mu, \tilde{K}_\mu),$$

hence, in other words, an unstable 2-period orbit of (3.2) can be stabilized via (2.13) only if

$$\bar{K}_\mu < K < \tilde{K}_\mu < 1.$$

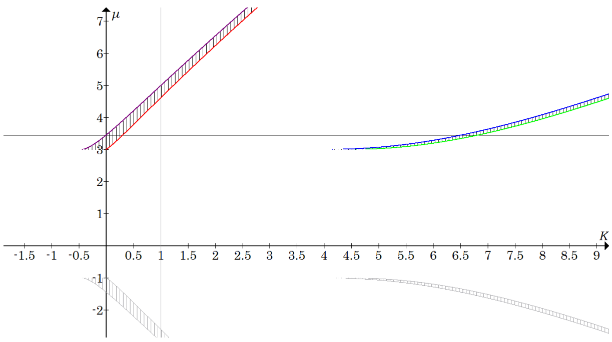


Figure 7: Graphical interpretation of condition (5.2).

Conversely, from the condition (5.2), it is possible to derive a range of stability of μ for $K \in PFC^2_\mu$. It yields

$$3 < \bar{\mu}_K < \mu < \tilde{\mu}_K,$$

where

$$\bar{\mu}_K = 1 + \sqrt{4 + K^2 + 4K + 4K^{\frac{3}{2}}},$$
$$\tilde{\mu}_K = 1 + \sqrt{6 + K^2 + 2K\sqrt{4K + 2} + 4K}.$$

Therefore, by applying a control (2.13) with $K \in PFC^2_\mu$, we are able to stabilize an unstable 2-period orbit at most for

$$3 < \bar{\mu}_K < \mu < \tilde{\mu}^*_K,$$

where $\tilde{\mu}^*_K$ is a critical value that is computed in $K = 1$, i.e.,

$$\tilde{\mu}^*_K = 4.987352441.$$

5.2 Delayed Feedback Control

Let us apply a control (2.15), for which we set $T = 2$ in order to stabilize a 2-period orbit. Therefore, we get a controlled system in form

$$y(n) = \mu y(n-1)(1 - y(n-1)) + K[y(n-3) - y(n-1)]. \tag{5.3}$$

For a further analysis of stability of controlled 2-period orbit we need the second iteration of (5.3). However, in the case of higher-order difference equation, one may get confused in a construction of the iteration. Note that the problem of discussing the stability of equilibrium of higher-order difference equation is equivalent to the same problem for corresponding system of first-order difference equations. Hence, we derive the required second iteration precisely and rigorously from the system of difference equations.

By rewriting (5.3) into the system of three first-order difference equations, we get

$$\begin{aligned} y_1(n) &= y_2(n-1) \\ y_2(n) &= y_3(n-1) \\ y_3(n) &= \mu y_3(n-1)(1-y_3(n-1)) + K[y_1(n-1) - y_3(n-1)] \\ &= f(y_3(n-1)) + K[y_1(n-1) - y_3(n-1)]. \end{aligned}$$

Thus, for some vector $\mathbf{y} = (y_1, y_2, y_3)^{\mathrm{T}} \in \mathbb{R}^3$, a mapping $F : \mathbb{R}^3 \to \mathbb{R}^3$ is defined as

$$F(\mathbf{y}) = (y_2, y_3, f(y_3) + K[y_1 - y_3])^{\mathrm{T}}.$$

The second iteration of F is then given by

$$\begin{aligned} F^2(\mathbf{y}) &= F(F(\mathbf{y})) = F(y_2, y_3, f(y_3) + K[y_1 - y_3]) \\ &= (y_3, f(y_3) + K[y_1 - y_3], f(f(y_3) + K[y_1 - y_3]) + K[y_2 - f(y_3) - K[y_1 - y_3]]). \end{aligned}$$

For a neat form of F^2, let us make the substitutions

$$\begin{aligned} Y_1 &= y_3, \\ Y_2 &= f(Y_1) + K(y_1 - Y_1), \\ Y_3 &= f(Y_2) + K(y_2 - Y_2). \end{aligned}$$

Therefore, we have $F^2(\mathbf{y}) = (Y_1, Y_2, Y_3)^T$. One may easily check from F^2 that the usage of (2.15) for stabilization of 2-period orbit is fully non-invasive.

To be able to study a stability of γ_1 or γ_2 (thus to study a stability of 2-period orbit), it is necessary to express the coefficients p_i of the characteristic polynomial. These coefficients are required as entries to Remark 2.8, which cover the following discussion. The Jacobi matrix of F^2 at $\gamma = (\gamma_1, \gamma_2)^{\mathrm{T}}$ is given as

$$DF^2(\gamma) = \begin{pmatrix} \dfrac{\partial Y_1}{\partial y}(\gamma) \\ \dfrac{\partial Y_2}{\partial y}(\gamma) \\ \dfrac{\partial Y_3}{\partial y}(\gamma) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ K & 0 & a_1 - K \\ K(a_2 - K) & K & (a_1 - K)(a_2 - K) \end{pmatrix},$$

where $a_1 = -1 - \sqrt{\mu^2 - 2\mu - 3}$, $a_2 = -1 + \sqrt{\mu^2 - 2\mu - 3}$. In fact, a_1 and a_2 can be explicitly derived as

$$a_1 = Df(\gamma_1) \qquad \text{and} \qquad a_2 = Df(\gamma_2).$$

The characteristic polynomial of $DF^2(\gamma_1)$ has the form

$$p(\lambda) = \lambda^3 + \lambda^2(K - a_1)(a_2 - K) + \lambda[K(K - a_2) + K(K - a_1)] - K^2$$

and corresponds exactly to the characteristic polynomial of the second iteration of (5.3). The required coefficients are

$$p_1 = (K - a_1)(a_2 - K), \quad p_2 = K(K - a_2) + K(K - a_1), \quad p_3 = -K^2,$$

so we are ready to use a Remark 2.8 for a discussion of stability of γ_1.

The stability conditions given by this remark after assigning p_1, p_2, p_3 are in form

$$0 < \mu^2 - 2\mu - 3, \tag{5.4}$$
$$0 < 4K^2 + 4K - \mu^2 + 2\mu + 5, \tag{5.5}$$
$$0 < 2K^3 - (\mu^2 - 2\mu - 2)K^2 - 2K + 1, \tag{5.6}$$
$$0 < -2K^4 - 2K^3 + (\mu^2 - 2\mu - 2)K^2 + 2K + 1. \tag{5.7}$$

First, let us discuss some preliminary facts on the form of DFC^2_μ. Obviously, (5.4) is equivalent to $\mu > 3$ (it allows also $\mu < -1$, but since $y(n)$ should be non-negative, we omit this case). Further, we denote by $F_\mu(K)$, $G_\mu(K)$ and $H_\mu(K)$ the right-hand sides of (5.5), (5.6) and (5.7), respectively. It is clear that

$$0 < H_\mu(K) = -2K^4 + 2 - G_\mu(K),$$

hence if $G_\mu(K) > 0$, then for the validity of $0 < H_\mu(K)$ it is necessary to have

$$0 < -2K^4 + 2.$$

In other words, if $K \in DFC^2_\mu$ for appropriate K and $\mu \geq \mu_2$ (in order to have satisfied condition (5.6)), then $K \in (-1, 1)$. Furthermore, if (5.5) occurs for some K and $\mu \geq \mu_2$, then either $K > 0$, or $K < -1$. Summarizing these facts,

$$DFC^2_\mu \subset (0; 1).$$

Now we show that the validity of (5.6) implies the validity of (5.7). Equivalently, we show that if $G_\mu(K) > 0$ for $K \in (0, 1)$ and $\mu \geq \mu_2$, then $G_\mu(K) < -2K^4 + 2$. Since

$$\begin{aligned} G_\mu(1) &= -\mu^2 + 2\mu + 3 < 0, \\ G_\mu(0) &= 1 < 2, \\ G'_\mu(0) &= -2 < 0, \\ G''_\mu(0) &= -2(\mu^2 - 2\mu - 2) < 0, \end{aligned}$$

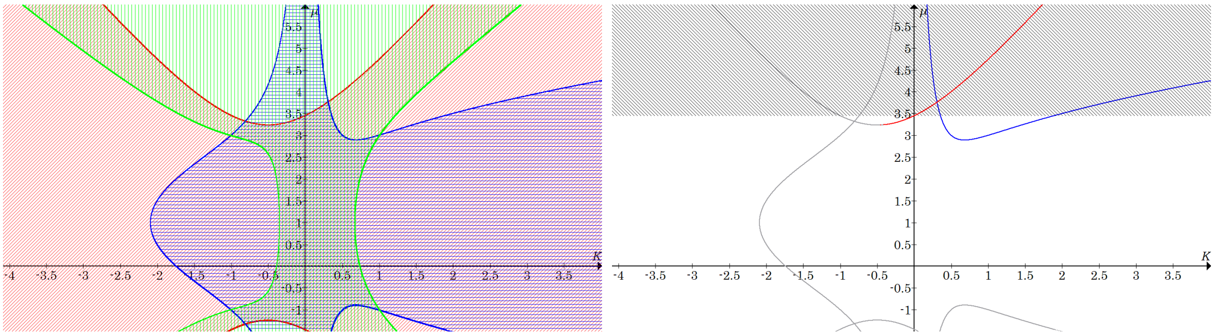
this property is evident. Consequently, when analyzing (5.4)–(5.7), it is enough to restrict to (5.5), (5.6). This conclusion we can also observe from Figure 8a, where the red colour corresponds to condition (5.5), the blue one to (5.6) and the green one to (5.7). Since we are interested in area of $\mu \geq \mu_2$, such an observation is obvious (see Figure 8b).

Let us describe the set DFC^2_μ more precisely. Firstly, we denote as $\tilde{K}_\mu$ such values $K > 0$ that satisfy $G_\mu(K) = 0$ for $\mu > 0$ (depicted by blue colour in Figure 8b). The quadratic polynomial $F_\mu(K)$ has two real roots, we denote the right of them by $\bar{K}_\mu$, i.e.,

$$\bar{K}_\mu = \frac{-1 + \sqrt{\mu^2 - 2\mu - 4}}{2}$$

(depicted by red colour in Figure 8b). It is clear that starting from μ_2, we get $DFC^2_{\mu_2} = (0, \tilde{K}_{\mu_2})$. With increasing μ we get that $DFC^2_\mu = (\bar{K}_\mu, \tilde{K}_\mu)$. In other words, an unstable 2-period orbit of (3.2) can be stabilized via (2.15) only if

$$\bar{K}_\mu < K < \tilde{K}_\mu.$$



(a) Conditions (5.4)–(5.7). (b) Conditions (5.5) and (5.6).

Figure 8: Graphical interpretations.

Obviously, in the area of interest (i.e., when $\mu \geq \mu_2$) the values $\bar{K}_\mu$ and $\tilde{K}_\mu$ are increasing and decreasing, respectively, with respect to increasing μ. Thus the stabilization interval DFC_μ^2 becomes smaller and smaller.

A natural question arises, namely what is the smallest value of μ when DFC_μ^2 becomes empty (we denote such a critical value by μ_2^*). Obviously, $\mu = \mu_2^*$ just when $\bar{K}_\mu = \tilde{K}_\mu$. This characterization leads to a nonlinear planar system

$$\begin{aligned} 4K^2 - 4K - \mu^2 + 2\mu + 5 = 0, \\ 2K^3 + (\mu^2 - 2\mu - 2)K^2 - 2K - 1 = 0 \end{aligned} \tag{5.8}$$

for unknowns μ and K. Solving (5.8), one can set up an algebraic equation of the fourth-order for the unknown K (it is enough to express the term $\mu^2 - 2\mu$ from the first equation of (5.8) and substitute it into the second equation of (5.8)). Thus, as its root, we obtain the value K_2^* as a critical value of gain for which γ_1 has the widest range of stability with respect to μ. This root can be evaluated as

$$K_2^* = 0.3090169944,$$

and an appropriate critical value of μ becomes

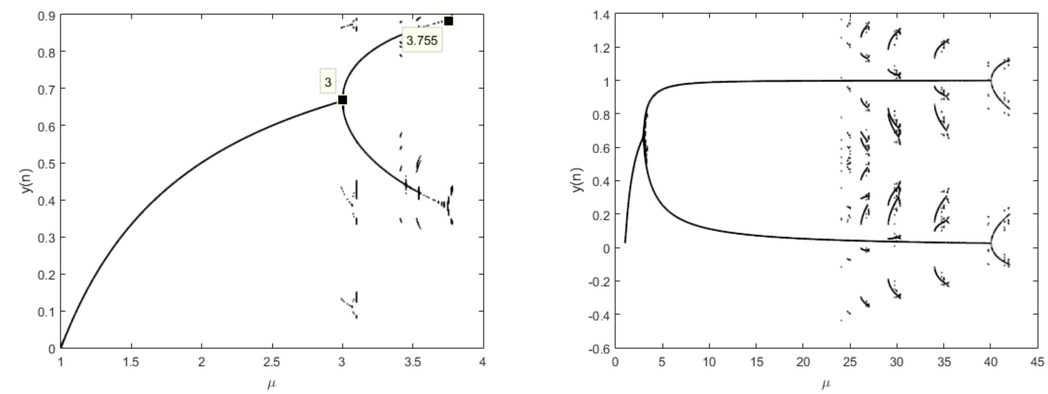
$$\mu_2^* = 3.76007862.$$

An almost critical state is depicted in Figure 9a. The choice of $K \to K_2^*$ causes that it is hard to make the bifurcation diagram 'clean' (meaning without ambient noise).

The stability of 2-period orbit of (5.3) was already discussed in paper [10] in a similar way as here. Nevertheless, in [10] the results were based mostly on the numerical experiments. Moreover, there is given just an estimate of μ_2^* as

$$3.76 < \mu_2^* < 3.77.$$

On contrary, we gave here a precise analytical discussion of this problem, supported by graphical interpretations. We were able to derive the exact value μ_2^*, and even the value K_2^*. We were following this paper in such a way mainly to show our contribution to it (thus, to make it for a reader easily comparable).



(a) Bif. diagram to (5.3) with $K = 0.3$. (b) Bif. diagram to (5.9) with $K = 0.95$.

Figure 9: Bifurcation diagrams.

5.3 Prediction Based Control

Applying a control (2.17) with $T = 2$, we get a controlled system given as

$$y(n) = \mu y(n-1)(1-y(n-1)) + K[y(n-2) - \mu y(n-1)(1-y(n-1))]. \tag{5.9}$$

Also here it is necessary to analyze the second iteration of (5.9) instead of the equation itself. Following the same argumentation as in the previous method, we will derive now F^2.

A system of two first-order difference equations corresponding to (5.9) is

$$\begin{aligned} y_1(n) &= y_2(n-1) \\ y_2(n) &= \mu y_2(n-1)(1-y_2(n-1)) + K[y_1(n-1) - \mu y_2(n-1)(1-y_2(n-1))] \\ &= f(y_2(n-1)) + K[y_1(n-1) - f(y_2(n-1))]. \end{aligned}$$

Hence, for some vector $\mathbf{y} = (y_1, y_2)^{\mathrm{T}} \in \mathbb{R}^2$, a mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as

$$F(\mathbf{y}) = (y_2, f(y_2) + K[y_1 - f(y_2)])^{\mathrm{T}}.$$

The second iteration F^2 is given by

$$F^2(\mathbf{y}) = (f(y_2)+K[y_1-f(y_2)], f(f(y_2)+K[y_1-f(y_2)])+K[y_2-f(f(y_2)+K[y_1-f(y_2)])]).$$

The following substitutions

$$\begin{aligned} Y_1 &= f(y_2) + K[y_1 - f(y_2)], \\ Y_2 &= f(Y_1) + K[y_2 - f(Y_1)], \end{aligned}$$

give us a neat form of $F^2(\mathbf{y}) = (Y_1, Y_2)^{\mathrm{T}}$. It is easy to check the non-invasiveness of (2.17).

Since (5.9) is a second-order difference equation, to discuss a stability of its 2-period orbit we use Remark 2.7. The Jacobi matrix of F^2 at $\gamma = (\gamma_1, \gamma_2)$ is given by

$$DF^2(\gamma) = \begin{pmatrix} \dfrac{\partial Y_1}{\partial y}(\gamma) \\ \dfrac{\partial Y_2}{\partial y}(\gamma) \end{pmatrix} = \begin{pmatrix} K & (1-K)a_2 \\ (1-K)Ka_1 & (1-K)^2 a_1 a_2 + K \end{pmatrix},$$

where $a_1 = Df(\gamma_1) = -1 - \sqrt{\mu^2 - 2\mu - 3}$, $a_2 = Df(\gamma_2) = -1 + \sqrt{\mu^2 - 2\mu - 3}$. The characteristic polynomial is given by

$$p(\lambda) = \lambda^2 - \lambda\left(2K + (1-K)^2 a_1 a_2\right) + K^2.$$

Hence, the needed coefficients for Remark 2.7 are

$$p_1 = -2K - (1-K)^2 a_1 a_2, \qquad p_2 = K^2.$$

Assigning p_1 and p_2 to this remark gives us the stability conditions

$$0 < 1 - K^2, \tag{5.10}$$
$$0 < (\mu^2 - 2\mu - 3)(1-K)^2, \tag{5.11}$$
$$0 < 2(1+K^2) - (\mu^2 - 2\mu - 3)(1-K)^2. \tag{5.12}$$

Firstly, some preliminary facts on the form of PBC^2_μ will be discussed. Clearly, (5.10) is equivalent to $-1 < K < 1$ and (5.11) results to $\mu > 3$ ($\mu < -1$ is omitted with the same argumentation as in previous method). We denote by $I_\mu(K)$ the right-hand side of (5.12). Further, rewriting (5.12) into

$$0 < 2(1-K)^2 - (\mu^2 - 2\mu - 3)(1-K)^2 + 4K$$

by completion the first term $I_\mu(K)$ on its square, we get

$$0 < (1-K)^2(-\mu^2 + 2\mu + 5) + 4K. \tag{5.13}$$

Therefore, if (5.13) occurs for some K and $\mu \geq \mu_2$, then $K > 0$. Summarizing these facts,

$$PBC^2_\mu \subset (0,1).$$

Let us describe the set PBC^2_μ in detail. The quadratic polynomial $I_\mu(K)$ has two real roots, we denote the left of them by $\bar{K}_\mu$, i.e.,

$$\bar{K}_\mu = \frac{\mu^2 - 2\mu - 3 - 2\sqrt{\mu^2 - 2\mu - 4}}{\mu^2 - 2\mu - 5}.$$

Starting from μ_2 we get $PBC^2_{\mu_2} = (0,1)$. By increasing μ we get $PBC^2_\mu = (\bar{K}_\mu, 1)$. An unstable 2-period orbit of (3.2) can be stabilized via (2.17) only if

$$\bar{K}_\mu < K < 1.$$

Obviously, the stabilization interval becomes smaller and smaller, but (in theoretical point of view) there exists no $\mu \geq \mu_2$ such that this interval becomes empty.

Conversely, let us assume $I_K(\mu)$ instead of $I_\mu(K)$ and denote its left root as $\bar{\mu}_K$, i.e.,

$$\bar{\mu}_K = \frac{K - 1 - \sqrt{6K^2 - 8K + 6}}{K - 1}.$$

The control (2.17) with $T = 2$ and $K \in PBC^2_\mu$ stabilizes an unstable 2-period orbit of (3.2) for

$$\mu_2 = 1 + \sqrt{6} < \mu < \bar{\mu}_K.$$

Hence, such a period orbit can be at most stabilized for

$$\mu_2 < \mu < \mu_2^*,$$

where $\mu_2^* \to \infty$ as $K \to 1$. An almost critical state is depicted in Figure 9b. In this case the bifurcation diagram is even more surrounded by a noise, that is hard to clear up.

6 Stabilization of 3-Period Orbits

Because of the Sharkovsky ordering (see Theorem 2.15), it should be appropriate now to be interested in analysis of stabilization of 4-period orbit. However, the analysis of such an orbit leads to computations with a polynomial of order 16, which is a very tedious matter. Moreover, by pure mathematical analysis, we are not able to get the final result on stability, since for this we need the exact expression of points creating the orbit. In [11], this problem was discussed by graphical approach, i.e., by studying the values of slope of corresponding iteration in points crossing a first-quadrant axis. Hence, we omit the analysis of stabilization of 4-period orbit and we will show just numerical experiments simulating it.

Although even the points of 3-period orbit are not explicitly expressible, the computations with a third iteration, i.e., with a polynomial of order 8, are more pleasant. Note that till now we were applying all introduced methods (PFC, DFC, PBC). Unfortunately, in this case, we are not able to use a PFC method, since we do not know the exact form of period points (so we do not know the target of stabilization). We will state at least stability conditions of corresponding controlled systems in general form. We will try to specify the stabilization sets DFC^3_μ, PBC^3_μ as much as possible for $\mu > \mu_3 = 1 + \sqrt{8}$.

One may argue that a PFC is usable, because of the fact that a 3-period orbit of uncontrolled system is asymptotically stable only if $\mu = 1 + \sqrt{(8)}$ (see Table 1). Hence, it is possible to directly compute its points, i.e.,

$$\gamma_1 = 0.1599288184, \qquad \gamma_2 = 0.5143552771, \qquad \gamma_3 = 0.9563178420.$$

Therefore, one may take as the targeted value t, needed for a stabilization via PFC, one of the computed points γ_1, γ_2 or γ_3. Nevertheless, for orbits of period higher than 1, this method is invasive (as we have observed in stabilization of 2-period orbit). So in further analysis on stability of controlled system via PFC we have no point, around which we could make a linearization. Thus, we are not able to get any closer specification on μ or K, for which an unstable 3-period orbit can be stabilized, and so a PFC method is unusable.

6.1 Delayed Feedback Control

Using a second method, i.e., applying a control (2.15) with setted $T = 3$, we get a controlled system in the form

$$y(n) = \mu y(n-1)(1 - y(n-1)) + K[y(n-4) - y(n-1)]. \tag{6.1}$$

Analogously, as in the case of stabilization of 2-period orbit, we derive a third iteration of (6.1) in order to be able to discuss a stability of controlled 3-period orbit.

A rewriting of fourth-order difference equation into a system of four first-order difference equations is done in a similar way as before. Therefore, for some vector $\mathbf{y} = (y_1, y_2, y_3, y_4)^{\mathrm{T}} \in \mathbb{R}^4$, there is a mapping $F : \mathbb{R}^4 \to \mathbb{R}^4$ defined as

$$F(\mathbf{y}) = (y_2, y_3, y_4, f(y_3) + K[y_1 - y_4])^{\mathrm{T}},$$

where the mapping f is again the logistic map. A third iteration of F is given by

$$F^3(\mathbf{y}) = F^2\left(y_2, y_3, y_4, f(y_3) + K[y_1 - y_4]\right)$$

$$
\begin{aligned}
&= F\left(y_3, y_4, f(y_4) + K[y_1 - y_4], f\left(f(y_4) + K[y_1 - y_4]\right) + K\left[y_2 - f(y_4) - K[y_1 - y_4]\right]\right) \\
&= (Y_1, Y_2, Y_3, Y_4),
\end{aligned}
$$

where the substitutions are

$$
\begin{aligned}
Y_1 &= y_4, \\
Y_2 &= f(Y_1) + K[y_1 - Y_1], \\
Y_3 &= f(Y_2) + K[y_2 - Y_2], \\
Y_4 &= f(Y_3) + K[y_3 - Y_3].
\end{aligned}
$$

We can check a non-invasiveness of this control. Let a vector $\gamma^* = (\gamma_1^*, \gamma_2^*, \gamma_3^*, \gamma_4^*)^{\mathrm{T}}$ be an equilibrium of $F^3(\mathbf{y})$. We will show that components γ_1^*, γ_2^*, γ_3^* are equilibria of third iteration of free system (3.2), in other words that they are points of uncontrolled 3-period orbit. Assumption on a vector γ^* to be an equilibrium means that $F(\gamma^*) = \gamma^*$, i.e.,

$$
\begin{aligned}
&Y_1 = \gamma_4^* = \gamma_1^*, && && \\
&Y_2 = f(Y_1) + K[\gamma_1^* - Y_1] = \gamma_2^* && \longrightarrow && f(\gamma_1^*) = \gamma_2^*, \\
&Y_3 = f(Y_2) + K[\gamma_2^* - Y_2] = \gamma_3^* && \longrightarrow && f(\gamma_2^*) = \gamma_3^*, \\
&Y_4 = f(Y_3) + K[\gamma_3^* - Y_3] = \gamma_4^* && \longrightarrow && f(\gamma_3^*) = \gamma_4^* = \gamma_1^*.
\end{aligned}
$$

Therefore, it is clear (from the right column) that γ_1^*, γ_2^*, γ_3^* are actually the points of uncontrolled 3-period orbit and that the control is non-invasive. We collect them into a vector $\gamma = (\gamma_1, \gamma_2, \gamma_3)^{\mathrm{T}}$.

For determining the stability conditions given by Remark 2.9, we have to find required coefficients p_i. Let

$$
a_1 = Df(\gamma_1) = -2\mu\gamma_1 + \mu, \qquad a_2 = Df(\gamma_2) = -2\mu\gamma_2 + \mu, \qquad a_3 = Df(\gamma_3) = -2\mu\gamma_3 + \mu.
$$

Then, by straightforward computation, one may get the Jacobi matrix of F^3 at γ given by

$$
DF^3(\gamma) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ K & 0 & 0 & a_1 - K \\ K(a_2 - K) & K & 0 & (a_1 - K)(a_2 - K) \\ K(a_2 - K)(a_3 - K) & K(a_3 - K) & K & (a_1 - K)(a_2 - K)(a_3 - K) \end{pmatrix}.
$$

The characteristic polynomial of $DF^3(\gamma)$ is

$$
p(\lambda) = \lambda^4 + p_1\lambda^3 + p_2\lambda^2 + p_3\lambda + p_4,
$$

where the coefficients are

$$
\begin{aligned}
p_1 &= -(a_1 - K)(a_2 - K)(a_3 - K), \\
p_2 &= -K\left[(a_1 - K)(a_2 - K) + (a_1 - K)(a_3 - K) + (a_2 - K)(a_3 - K)\right], \\
p_3 &= -K^2[(a_1 - K) + (a_2 - K) + (a_3 - K)], \\
p_4 &= -K^3.
\end{aligned}
$$

The stability conditions given by Remark 2.9 after assigning p_1, p_2, p_3, p_4 are

$$
-1 < K < 1, \tag{6.2}
$$

$$0 < 1 - a_1a_2a_3, \tag{6.3}$$
$$0 < 1 + a_1a_2a_3 - 2K^3 - 2K\left[(a_1-K)(a_2-K) + (a_1-K)(a_3-K) + (a_2-K)(a_3-K)\right], \tag{6.4}$$
$$0 < -(K^3+1)^2p_2 - (K^3p_1+p_3)(p_3-p_1) + (K^3+1)(1-K^6), \tag{6.5}$$
$$0 < -(K^6-1)p_2 - (K^3p_1+p_3)(p_3+p_1) + (K^3-1)(K^6-1). \tag{6.6}$$

Let us discuss some preliminaries, which will be useful later. Clearly, from (6.2) we have $DFC^3_\mu \subset (-1,1)$. Furthermore, for $\mu = \mu_3$ we are able to compute exact values of a_i, i.e.,

$$a_1^{\mu_3} = 2.603875472, \qquad a_2^{\mu_3} = -0.1099162645, \qquad a_3^{\mu_3} = -3.493959201,$$

so $a_1^{\mu_3}a_2^{\mu_3}a_3^{\mu_3} = 1$. It is natural to assume that with a small increase μ the values of γ_1, γ_2, γ_3 are not changed rapidly, so neither the values of a_1, a_2, a_3 are. This assumption will be later confirmed experimentally. However, all these small changes together imply that a_2 becomes a very small, but positive number, and that the condition (6.3) holds. Moreover, for $\mu > \mu_3$, we get that $-1 < a_1a_2a_3 < 0$, so $0 < 1 + a_1a_2a_3 < 1$.

Under preceding assumptions we are able to specify closer the stabilization set DFC^3_μ, namely the sign of K. It leads from preliminaries that the condition (6.4) holds if

$$0 < -2K^3 - 2K\left[(a_1-K)(a_2-K) + (a_1-K)(a_3-K) + (a_2-K)(a_3-K)\right] \tag{6.7}$$

holds. Using the assumption, one may find that (6.7) holds only for $K > 0$. Hence,

$$DFC^3_\mu \subset (0,1).$$

Note that $a_1 - K > 0$ and $a_3 - K < 0$ for every $K \in DFC^3_\mu$, because $a_1 > 1$ and $a_3 < 0$. However, since $0 < a_2 < 1$, we can have either $a_2 - K > 0$ (if $a_2 > K$) or $a_2 - K < 0$ (otherwise). From these facts we may deduce following observations:

- $0 < p_1 << 1$ if $a_2 > K$, otherwise $p_1 << -1$;
- $p_2 > 0$ for every $K \in DFC^3_\mu$;
- $p_3 > 0$ for every $K \in DFC^3_\mu$.

Using preceding observations, we are able to discuss conditions (6.5), (6.6).

It is obvious from (6.5), that its first term is negative for all $K \in DFC^3_\mu$ and the last one is positive for all $K \in DFC^3_\mu$. The middle term requires a more attention. If p_1 is positive, then K is a very small (in the absolute value) number. Therefore, p_3 gives also a small number, and even smaller then p_1 gives (because of the K^2 term involved in p_3). Hence, the middle term is positive for $K < a_2$. On contrary, let p_1 be negative. We have observed that in this case $p_1 << -1$, which in fact causes $K^3p_1 + p_3 < 0$. All in all, the middle term is positive for all $K \in DFC^3_\mu$. The only problem in validity of (6.5) may cause the first term. To avoid this, we have to choose very small K (the first and the last term will have almost same value, just with opposite signs). This fact is satisfied just by the case of positive p_1, hence $a_2 > K$ and so

$$DFC^3_\mu = (0, a_2).$$

The first and the last term of (6.6) are obviously positive for all $K \in DFC^3_\mu$. For the middle term, there is an argumentation in a very similar way. The both cases of sign p_1

leads to the negative sign of term, and so the problematic term in validity of (6.6) may be only this term. However, it results to the same consequence as in the previous condition. A choice of K being sufficiently small (i.e., $K < a_2$) ensures that even though the middle term is negative, it is very small and thus it does not cause any problem in validity of this condition.

We are not able to discuss the range of stability of controlled 3-period orbit, because without explicit expression of its points we are not able to proceed further in analysis. Therefore, we will show it just in some numerical experiments. Unfortunately, in bifurcation diagrams the range of stability of 3-period orbit (either of controlled or uncontrolled) is not clear so much. In Figure 10 we have enlarged the part corresponding to 3-period orbit. It clarifies the above assumption about values of points of this orbit. A graph of evolution of $y(n)$ verifies that usage of negative gain causes the 3-period orbit of (6.1) (depicted by red colour in Figure 11a) became unstable for $\mu = 1 + \sqrt{8}$. On contrary, a Figure 11b verifies that usage of positive gain stabilizes this orbit for $\mu > 1 + \sqrt{8}$. Specially, for $K = 0.05$ we get its range of stability as $1 + \sqrt{8} < \mu < 3.859$ (see Figure 10).

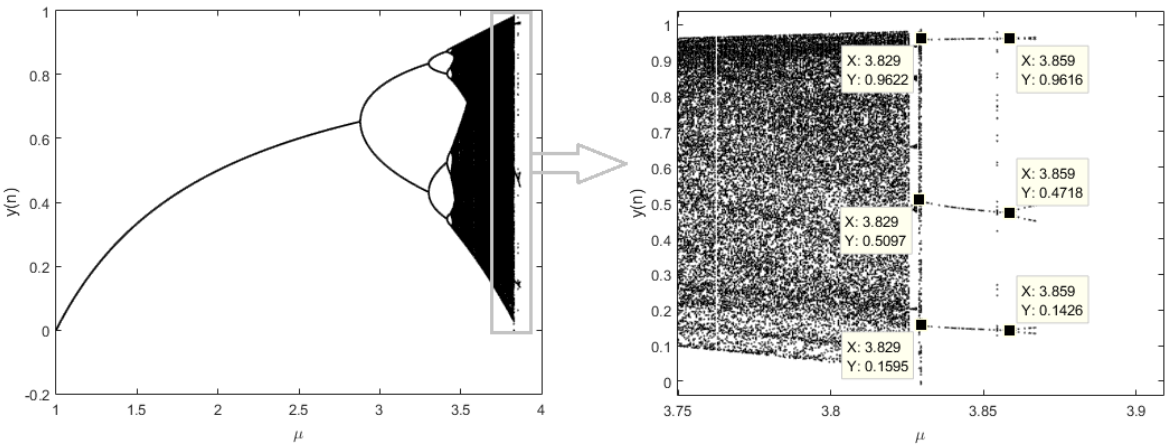
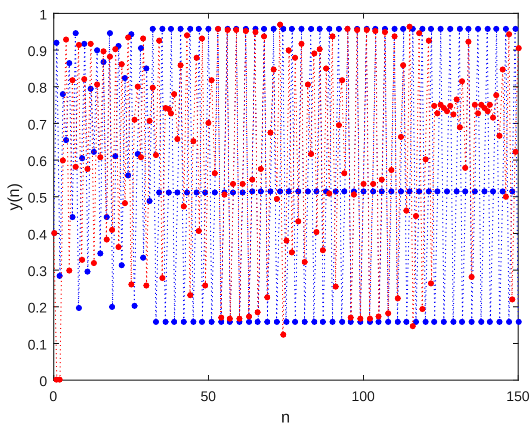
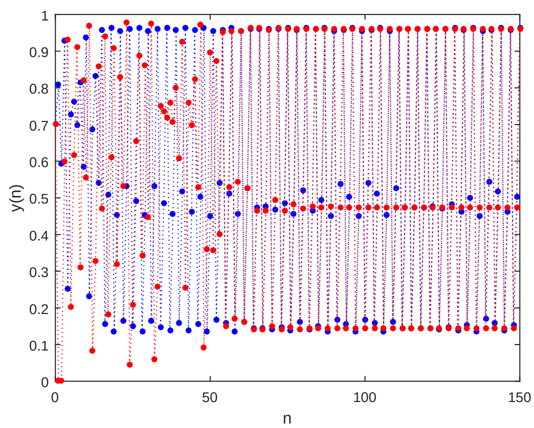


Figure 10: Bifurcation diagram to (6.1), $K = 0.05$.



(a) $K = -0.05$, $\mu = 1 + \sqrt{8}$



(b) $K = 0.05$, $\mu = 3.855$

Figure 11: Evolution of $y(n)$ for system (3.2) (blue) and (6.1) (red).

6.2 Prediction Based Control

Application of the third method for stabilization of 3-period orbit leads to analysis of controlled system given by

$$y(n) = \mu y(n-1)(1-y(n-1)) + K[y(n-3) - \mu y(n-1)(1-y(n-1))]. \tag{6.8}$$

Analogously, as in Chapter 5.3 we derive a third iteration of (6.8), and then discuss a stability of 3-period orbit controlled via (2.17).

The system of three first-order difference equations equivalent to (6.8) is given by mapping $F : \mathbb{R}^3 \to \mathbb{R}^3$ such that for some vector $\mathbf{y} = (y_1, y_2, y_3)^{\mathrm{T}}$ this map is defined by $F(\mathbf{y}) = (y_2, y_3, f(y_3) + K[y_1 - f(y_3)])^{\mathrm{T}}$. The mapping f represents again a logistic map. Therefore, the third iteration of F is given by

$$\begin{aligned}
F^3(\mathbf{y}) &= F^2(y_2, y_3, f(y_3) + K[y_1 - f(y_3)]) \\
&= F(y_3, f(y_3) + K[y_1 - f(y_3)], f(f(y_3) + K[y_1 - f(y_3)]) \\
&\quad + K[y_2 - f(f(y_3) + K[y_1 - f(y_3)])]) \\
&= (Y_1, Y_2, Y_3),
\end{aligned}$$

where the substitutions are

$$\begin{aligned}
Y_1 &= f(y_3) + K[y_1 - f(y_3)], \\
Y_2 &= f(Y_1) + K[y_2 - f(Y_1)], \\
Y_3 &= f(Y_2) + K[y_3 - f(Y_2)].
\end{aligned}$$

We can quickly check a non-invasiveness of this control. Let $\gamma^* = (\gamma_1^*, \gamma_2^*, \gamma_3^*)^{\mathrm{T}}$ be an equilibrium of $F^3(y)$. Therefore, $F(\gamma^*) = \gamma^*$, i.e.,

$$\begin{aligned}
Y_1 &= f(\gamma_3^*) + K[\gamma_1^* - f(\gamma_3^*)] = \gamma_1^*, \\
Y_2 &= f(\gamma_1^*) + K[\gamma_2^* - f(\gamma_1^*)] = \gamma_2^*, \\
Y_3 &= f(\gamma_2^*) + K[\gamma_3^* - f(\gamma_2^*)] = \gamma_3^*.
\end{aligned}$$

If the control is truly non-invasive, then γ_1^*, γ_2^*, γ_3^* should be exactly equal to γ_1, γ_2, γ_3, respectively (it means that they are the points creating a 3-period orbit). Hence, because of Definition 2.12, there should hold

$$f(\gamma_1^*) = \gamma_2^*, \qquad f(\gamma_2^*) = \gamma_3^*, \qquad f(\gamma_3^*) = \gamma_1^*.$$

Clearly, the control is non-invasive. We collect these points in a vector $\gamma = (\gamma_1, \gamma_2, \gamma_3)^{\mathrm{T}}$.

Let a_1, a_2, a_3 be the same as in Chapter 6.1. The Jacobi matrix of F^3 at γ is given by

$$DF^3(\gamma) = \begin{pmatrix} K & 0 & (1-K)a_3 \\ (1-K)Ka_1 & K & (1-K)^2 a_1 a_3 \\ (1-K)^2 K a_1 a_2 & (1-K)Ka_2 & (1-K)^3 a_1 a_2 a_3 + K \end{pmatrix}.$$

One may check it by straightforward computation using the Definition 2.12 mentioned above on points of period orbit. The characteristic polynomial of $DF^3(\gamma)$ is

$$p(\lambda) = \lambda^3 + (K^3 a_1 a_2 a_3 - 3K^2 a_1 a_2 a_3 + 3K a_1 a_2 a_3 - a_1 a_2 a_3 - 3K)\lambda^2 + 3K^2\lambda - K^3,$$

so the coefficients p_i needed to discussion of stability conditions given by Remark 2.8 are

$$\begin{aligned} p_1 &= a_1a_2a_3(K^3 - 3K^2 + 3K - 1) - 3K, \\ p_2 &= 3K^2, \\ p_3 &= -K^3. \end{aligned}$$

The stability conditions after assigning the values p_1, p_2, p_3 are given by

$$a_1a_2a_3(1-K)^3 < (1-K)^3, \tag{6.9}$$

$$a_1a_2a_3(K-1)^3 < (K+1)^3, \tag{6.10}$$

$$a_1a_2a_3K^3(K-1)^3 < (1-K^2)^3, \tag{6.11}$$

$$a_1a_2a_3K^3(1-K)^3 < (1-K)(K+1)(K^4+4K^2+1). \tag{6.12}$$

Recall from Section 6.1, that the condition (6.3) holds for $\mu > \mu_3$, and that a_1, a_2, a_3 remain unchanged. Therefore, the condition (6.9) should hold as well. It gives that $(1-K)^3 > 0$, so $K < 1$ and $PBC_\mu^3 \subset (-\infty, 1)$. Also, we already know that $-1 < a_1a_2a_3 < 0$ for $\mu > \mu_3$, and $(K-1)^3 < 0$ for $K \in PBC_\mu^3$. Hence, the left-hand side of condition (6.10) is positive, so this condition is satisfied when $0 < (K+1)^3$. Consequently, we get $-1 < K < 1$, i.e., $PBC_\mu^3 \subset (-1, 1)$. Furthermore, the right-hand sides of remaining conditions (6.11), (6.12) are positive for $K \in PBC_\mu^3$. Nevertheless, the only thing that we are able to deduce is that for $K > 0$, a validity of these conditions is ensured in both cases.

Unfortunately, for $K > 0$ the numerical experiments showed that the chaotic part (including also a part of 3-period orbit) has been suppressed. Therefore, the set of points creating a 3-period orbit is a void set. Since the void set may have any property, then it is also possible to stabilize it trivially. Hence, the deduction on a specification of stabilizing set PBC_μ^3 is correct from a theoretical point of view. Practically, the stabilization of 3-period orbit via (2.17) does not make sense, unless its main aim was not to suppress the chaotic behaviour. If yes, then such a control works perfectly.

7 Advanced Stabilization of Equilibria

Herein the chapter, a stabilization of equilibria via controls PFC, DFC, PBC will be done, but with an emphasise put on delay ω. In other words, we will state here three different controlled systems with two varying parameters K, ω. For this purpose, let us introduce following notations:

For PFC given by (2.12), let us denote the stabilization set of all admissible couples (K, ω), where K is a gain parameter and ω is a delay, as PFC_μ. Here, $\mu \geq \mu_1$ is an arbitrary real parameter, where μ_1 is the upper bound of stability interval corresponding to equilibrium y_2^* (see the right column of Table 1 describing stability of a free system).

Similarly, for DFC given by (2.14), let us denote the stabilization set of all admissible couples (K, ω), where K is a gain parameter and ω is a delay, as DFC_μ. For PBC given by (2.16), we denote the stabilization set as PBC_μ. Also here, $\mu \geq \mu_1$ is considered as an arbitrary real parameter.

Therefore, our main aim in this chapter will be the description of stabilization sets PFC_μ, DFC_μ, PBC_μ. Note that the limitations on values of ω will arise from Theorem 2.10. After this, we will be able to discuss whether the implementation of information about the past to system may be useful for getting better results in stabilization.

7.1 Proportional Feedback Control

The application of control is done in a similar way as in Section 4.1 and is also non-invasive. However, now we implement a general ω to the system, where $\omega > 1$. Thus, the controlled system has the form

$$y(n) = \mu y(n-1)(1 - y(n-1)) + K[y(n-\omega) - y_2^*]. \tag{7.1}$$

As we have already indicated in the mathematical background part, for varying ω we are not able to use Schur-Cohn criterion when analyzing the stability. Therefore, we use Theorem 2.10 for this purpose (see also [4]).

Since the controlled system (7.1) is nonlinear, we make a linearization around y_2^*:

$$\begin{aligned}
\frac{\partial f}{\partial y(n-1)}(y_2^*) &= \left.\frac{\partial \left[\mu y(n-1)(1-y(n-1)) + K[y(n-\omega) - y_2^*]\right]}{\partial y(n-1)}\right|_{y(n-1)=y_2^*} \\
&= 2 - \mu, \\
\frac{\partial f}{\partial y(n-\omega)}(y_2^*) &= \left.\frac{\partial \left[\mu y(n-1)(1-y(n-1)) + K[y(n-\omega) - y_2^*]\right]}{\partial y(n-\omega)}\right|_{y(n-\omega)=y_2^*} \\
&= K.
\end{aligned}$$

This leads to a linearized controlled system

$$y(n) = (2-\mu)y(n-1) + Ky(n-\omega). \tag{7.2}$$

Comparing the equation (7.2) with the equation (2.6), we get $A_1 = 2 - \mu$, $A_2 = K$ with a dimension $m = 1$ and delays $\rho = 1$ and a general ω. Hence, the eigenvalues are

$$\alpha_1 = 2 - \mu, \qquad \beta_1 = K.$$

Note that (7.2) satisfies all assumptions of Theorem 2.10 (see also Remark 2.11). Thus, application of this theorem to (7.2) enables the analysis of stability of y_2^*. We already know that y_2^* is asymptotically stable for $1 < \mu \le 3$. However, in the previous argumentation of case $\omega = 1$ we have shown that the lower bound of stability can be slightly modified. Our aim will be now to discuss whether there exists some pair of control parameters (K, ω) such that y_2^* is asymptotically stable when $\mu > \mu_1 = 3$.

Analysis of stability condition (2.7)

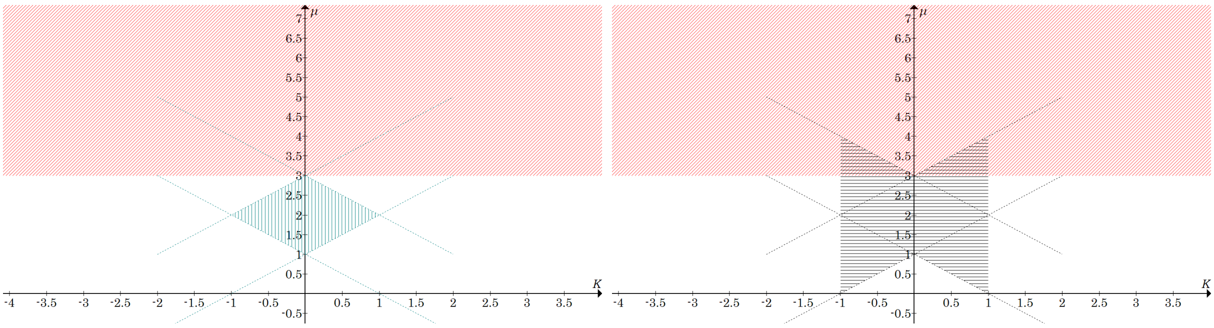
Assigning the eigenvalues α_1, β_1 into the condition (2.7) we get

$$|2 - \mu| + |K| < 1.$$

This inequality is depicted in Figure 12a as the blue area bounded by functions

$$\mu = 3 - K, \qquad \mu = 3 + K \qquad \text{and} \qquad \mu = 1 - K, \qquad \mu = 1 + K.$$

From Figure 12a it is clear that this inequality is never satisfied for $\mu > 3$ (depicted as the red area).



(a) Condition (2.7) for system (7.1). (b) Condition (2.8) for system (7.1).

Figure 12

Analysis of stability conditions (2.8), (2.9)

Clearly, for $\mu > 3$, the first part of the condition (2.8)

$$|2 - \mu| + |K| \ge 1$$

is satisfied for any $K \in \mathbb{R}$ since its area may be depicted as the complement of the blue region (see Figure 12a). The next part of this condition with assigned eigenvalues α_1, β_1 has the form

$$|2 - \mu| - 1 < |K| < 1.$$

It is depicted in Figure 12b as the black area, so we can easily get the range of stability of y_2^*. There are two relations between K and μ

$$-1 < K < 3 - \mu < 0 \qquad \text{or} \qquad 0 < \mu - 3 < K < 1,$$

so either the control parameter K is positive, or it is negative. Since we are interested in the range of stability of y_2^*, we get

$$\mu < 3 - K \qquad \text{for } K < 0 \qquad \text{or} \qquad \mu < 3 + K \qquad \text{for } K > 0.$$

However, for the stability of y_2^* also the condition (2.9) needs to hold. It is clear that $\arg(\alpha_1) = \pi$ and $\arg(\beta_1) = 0$ if $0 < K < 1$ or $\arg(\beta_1) = \pi$ if $-1 < K < 0$. Note that $\gcd(l_1, l_2) = \gcd(1, \omega) = 1$. Thus, we get

$$\omega \arccos \frac{1 + (2-\mu)^2 - K^2}{2(\mu - 2)} + \arccos \frac{1 - (2-\mu)^2 + K^2}{2|K|} < \arccos\left[\cos(\omega\pi - \arg(K))\right].$$

Note that the left-hand side of this inequality is always positive. For $K > 0$ the above inequality has no solution for an even ω, because in this case the right-hand side is zero. On the other hand, for $K < 0$ there is no solution for an odd ω.

Therefore, for a stabilization of y_2^*, we need to have the controlled system (7.1) either with the control parameters

$$-1 < K < 0 \qquad \text{and} \qquad \omega < \frac{\arccos \dfrac{1 - (2-\mu)^2 + K^2}{2K}}{\arccos \dfrac{1 + (2-\mu)^2 - K^2}{2(\mu-2)}} = \bar{\omega}$$

such that ω is an even positive integer, or with the control parameters

$$0 < K < 1 \qquad \text{and} \qquad \omega < \frac{\arccos \dfrac{(2-\mu)^2 - 1 - K^2}{2K}}{\arccos \dfrac{1 + (2-\mu)^2 - K^2}{2(\mu-2)}} = \bar{\bar{\omega}}$$

such that ω is an odd positive integer. Note that when expressing ω we have used the relation $\arccos(-x) = \pi - \arccos(x)$. Hence, we get two stabilization sets, namely

$$\begin{aligned} PFC_\mu &= \{(K, \omega); -1 < K < 0 \text{ and even } \omega < \bar{\omega}\} \qquad \text{or} \\ PFC_\mu &= \{(K, \omega); 0 < K < 1 \text{ and odd } \omega < \bar{\bar{\omega}}\}. \end{aligned}$$

For the controlled system (7.1), we get that the equilibrium y_2^* is asymptotically stable on the interval

$$\begin{aligned} 1 + K < \mu < 3 - K \qquad &\text{for } K < 0 \qquad \text{or} \\ 1 - K < \mu < 3 + K \qquad &\text{for } K > 0, \end{aligned}$$

where $(K, \omega) \in PFC_\mu$. Note that the lower bounds of stability ranges can be confirmed by Figure 12b. In fact, we are able to stabilize this equilibrium at most for

$$0 < \mu < 4$$

using $(K, \omega) \in PFC_\mu$, where either $K \to 1$ or $K \to -1$.

7.2 Delayed Feedback Control

Analogously as in the previous section, after application of non-invasive control (2.14), the controlled logistic map has the form

$$y(n) = \mu y(n-1)(1 - y(n-1)) + K[y(n - \tilde{\omega}) - y(n-1)]. \tag{7.3}$$

Let $\tilde{\omega} > 2$ (otherwise we are applying control (2.15) with $T = 1$). Due to the same argumentation as in the case of previous control method, we make a linearization of system (7.3) around y_2^*:

$$\begin{aligned}
\frac{\partial f}{\partial y(n-1)}(y_2^*) &= \left.\frac{\partial\left[\mu y(n-1)\left(1-y(n-1)\right) - Ky(n-1) + Ky(n-\tilde{\omega})\right]}{\partial y(n-1)}\right|_{y(n-1)=y_2^*} \\
&= 2-\mu-K, \\
\frac{\partial f}{\partial y(n-\tilde{\omega})}(y_2^*) &= \left.\frac{\partial\left[\mu y(n-1)\left(1-y(n-1)\right) - Ky(n-1) + Ky(n-\tilde{\omega})\right]}{\partial y(n-\tilde{\omega})}\right|_{y(n-\tilde{\omega})=y_2^*} \\
&= K,
\end{aligned}$$

leading to the linearized controlled system

$$y(n) = (2-\mu-K)y(n-1) + Ky(n-\tilde{\omega}). \tag{7.4}$$

Comparing (7.4) with (2.6) we get $A_1 = 2-\mu-K$, $A_2 = K$ with a dimension $m = 1$ and delays $\rho = 1$ and a general $\tilde{\omega}$. Hence, the eigenvalues needed for the application of Theorem 2.10 are

$$\begin{aligned}
\alpha_1 &= 2-\mu-K \\
\beta_1 &= K.
\end{aligned}$$

Again, with the same argumentation as in the previous control method, we apply this theorem on (7.4) to analyze the stability of y_2^* for $\mu \geq \mu_1 = 3$.

Analysis of stability condition (2.7)

Assigning the eigenvalues α_1, β_1 into the condition (2.7) we get

$$|2-\mu-K| + |K| < 1.$$

This inequality is depicted in Figure 13a as the blue area, which is bounded by functions

$$\mu = 1-2K \qquad \text{and} \qquad \mu = 3-2K.$$

From Figure 13a it is clear that this inequality is again never satisfied for $\mu > 3$ (depicted as the red area).

Analysis of stability conditions (2.8), (2.9)

Clearly, for $\mu > 3$, the first part of the condition (2.8)

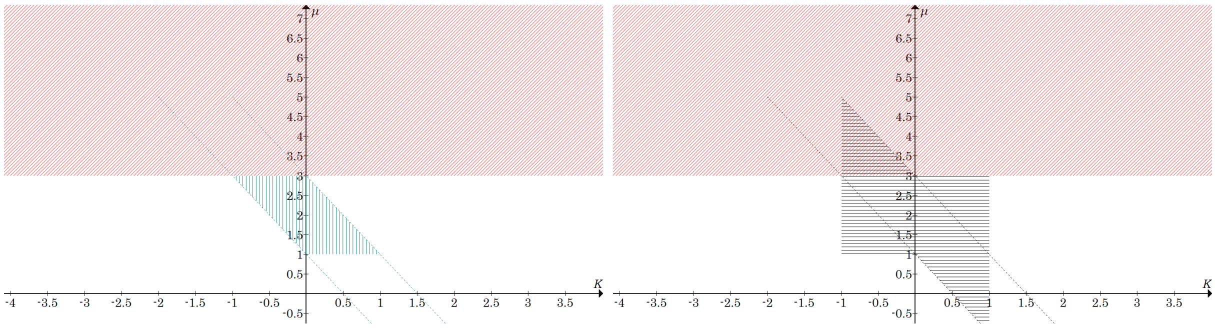
$$|2-\mu-K| + |K| \geq 1$$

is satisfied for any $K \in \mathbb{R}$. The next part of this condition with assigned eigenvalues α_1, β_1 has the form

$$|2-\mu-K| - 1 < |K| < 1.$$

It is depicted in Figure 13b as the black area, so we can easily find the range of stability of y_2^*. The relation between K and μ is (see also Figure 13b)

$$-1 < K < \frac{3-\mu}{2} < 0,$$



(a) Condition (2.7) for model (7.4).

(b) Condition (2.8) for system (7.4).

Figure 13

and since we are interested in the range of the stability of the equilibrium y_2^*, we get

$$\mu < 3 - 2K.$$

Also, here it is necessary to analyze the condition (2.9), since the stability depends on ω. Now $\arg(\alpha_1) = \pi$ and $\arg(\beta_1) = \pi$ as $-1 < K < 0$. Thus we get

$$\tilde{\omega} \arccos \frac{1 + (2-\mu-K)^2 - K^2}{2(K+\mu-2)} + \arccos \frac{1 - (2-\mu-K)^2 + K^2}{-2K} < \arccos\left[\cos(\tilde{\omega}\pi - \pi)\right].$$

Therefore, since $\tilde{\omega} = \omega + 1$, we get

$$\tilde{\omega} \arccos \frac{1 + (2-\mu-K)^2 - K^2}{2(K+\mu-2)} + \arccos \frac{1 - (2-\mu-K)^2 + K^2}{-2K} < \arccos\left[\cos(\omega\pi)\right].$$

For an even ω (thus for an odd $\tilde{\omega}$), we get no solution of this inequality. Therefore, for a stabilization of y_2^* we need to have the controlled system (7.3) with the control parameters

$$-1 < K < 0 \qquad \text{and} \qquad \tilde{\omega} < \frac{\arccos \dfrac{1 - (2-\mu-K)^2 + K^2}{2K}}{\arccos \dfrac{1 + (2-\mu-K)^2 - K^2}{2(K+\mu-2)}} = \bar{\omega}$$

such that $\tilde{\omega}$ is an even positive integer. Hence, the stabilization set is given by

$$DFC_\mu = \{(K, \omega); -1 < K < 0 \text{ and even } \omega < \bar{\omega}\}.$$

For the controlled system (7.3), we get that the unstable equilibrium y_2^* can be stabilized when

$$\mu_1 = 3 < \mu < 3 - 2K$$

for $(K, \omega) \in DFC_\mu$. Consequently, it can be stabilized at most for

$$\mu_1 < \mu < 5,$$

using control parameters $(K, \omega) \in DFC_\mu$, where $K \to -1$.

7.3 Prediction Based Control

The logistic map controlled by a control (2.17) has the form

$$y(n) = (1-K)\,\mu y(n-1)\,(1-y(n-1)) + Ky(n-\omega). \tag{7.5}$$

Note that the non-invasiveness is satisfied also here. Let $\omega > 1$. After the linearization of system (7.5) around y_2^*, where

$$\begin{aligned}
\frac{\partial f}{\partial y(n-1)}(y_2^*) &= \left.\frac{\partial\left[(1-K)\,\mu y(n-1)\,(1-y(n-1)) + Ky(n-\omega)\right]}{\partial y(n-1)}\right|_{y(n-1)=y_2^*} \\
&= (1-K)(2-\mu), \\
\frac{\partial f}{\partial y(n-\omega)}(y_2^*) &= \left.\frac{\partial\left[(1-K)\,\mu y(n-1)\,(1-y(n-1)) + Ky(n-\omega)\right]}{\partial y(n-\omega)}\right|_{y(n-\omega)=y_2^*} \\
&= K,
\end{aligned}$$

we get the linearized controlled system

$$y(n) = (1-K)(2-\mu)y(n-1) + Ky(n-\omega). \tag{7.6}$$

Comparing (7.6) with (2.6) we get $A_1 = (1-K)(2-\mu)$, $A_2 = K$, a dimension $m = 1$ and delays $\rho = 1$ and a general ω. Hence, the eigenvalues needed for application of Theorem 2.10 are

$$\begin{aligned}
\alpha_1 &= (1-K)(2-\mu) \\
\beta_1 &= K.
\end{aligned}$$

In the sequel, we apply to Theorem 2.10 on (7.6) when analyzing the stability of y_2^* for $\mu \geq \mu_1 = 3$.

Analysis of stability condition (2.7)

Assigning the eigenvalues α_1, β_1 into the condition (2.7), we get the inequality

$$|(1-K)(2-\mu)| + |K| < 1.$$

This inequality is depicted as the blue area (see Figure 14a), where its left borders are made of two functions

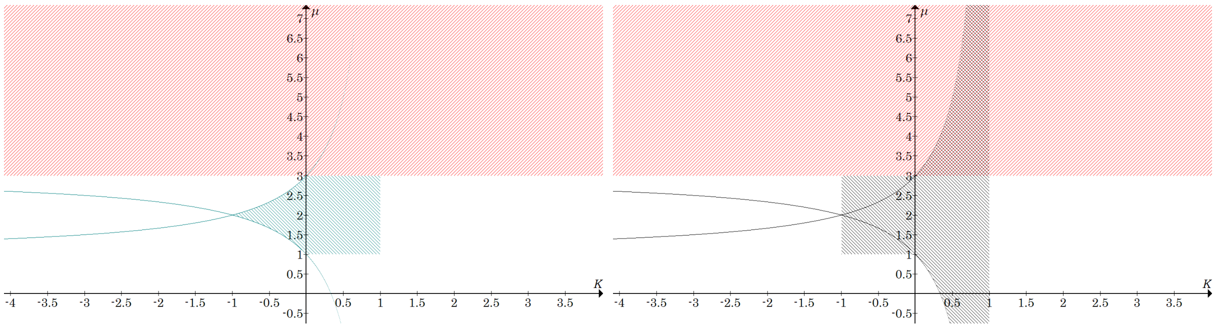
$$\mu = \frac{3-K}{1-K} \qquad \text{and} \qquad \mu = \frac{1-3K}{1-K}.$$

From Figure 14a it is clear that this condition is never satisfied for $\mu > 3$ (depicted as the red area).

Analysis of stability conditions (2.8), (2.9)

Clearly, for $\mu > 3$, the first part of the condition (2.8)

$$|(1-K)(2-\mu)| + |K| \geq 1$$



(a) Condition (2.7) for system (7.6). (b) Condition (2.8) for system (7.6).

Figure 14

is satisfied for any $K \in \mathbb{R}$ since its area can be depicted as the complement of the blue region in Figure 14a. By assigning the eigenvalues α_1, β_1 into the next part of the condition (2.8), we get the inequality

$$|(1-K)(2-\mu)| - 1 < |K| < 1.$$

Figure 14b depicts this inequality as the black region, so we can easily get the relation between μ and K for which the equilibrium y_2^* is asymptotically stable. The relation is

$$0 < \frac{3-\mu}{1-\mu} < K < 1,$$

and since we are looking for a range of the stability of the equilibrium y_2^*, we get

$$\mu < \frac{3-K}{1-K}.$$

It is clear that $\arg(\alpha_1) = \pi$, $\arg(\beta_1) = 0$ as $0 < K < 1$. Thus, we get the condition (2.9) in the form

$$\omega \arccos \frac{1+K+(1-K)(2-\mu)^2}{2(\mu-2)} + \arccos \frac{1-(1-K)^2(2-\mu)^2+K^2}{2K} < \arccos\left[\cos(\omega\pi)\right].$$

For an even delay, this inequality has no solution. For an odd ω, the right-hand side of the inequality is equal to π. Therefore, for a stabilization of y_2^* we need to have the controlled system (7.5) with the control parameters

$$0 < K < 1 \qquad \text{and} \qquad \omega < \frac{\arccos \dfrac{(1-K)^2(2-\mu)^2-1-K^2}{2K}}{\arccos \dfrac{1+K+(1-K)(2-\mu)^2}{2(\mu-2)}} = \bar{\omega}$$

such that ω is an odd positive integer. The stabilization set is given as

$$PBC_\mu = \{(K,\omega); 0 < K < 1 \text{ and } \omega < \bar{\omega} \text{ being odd}\}.$$

For the controlled system (7.5), we get that the unstable equilibrium y_2^* can be stabilized when

$$\mu_1 = 3 < \mu < \frac{3-K}{1-K}$$

for $(K,\omega) \in PBC_\mu$. Hence, it can be stabilized at most for

$$\mu_1 < \mu_1^*,$$

where $\mu_1^* \to \infty$ as $K \to 1$.

8 Parity of Delay and Its Connection to Eigenvalues

Up to now, we were able to get some stabilizing results based mostly on straightforward (but tedious) computations. We have been interested in a reaction of the system on a given control and we have decided if the reaction was desired (the system was stabilized) or not. It was in fact rather a practical approach to the problem of stabilization.

In this chapter, we show a more general background. It was inspired by [1], but it is enhanced by some detailed comments and graphical interpretations. The following considerations will be performed with a general delay ω (the second approach may be obtained from this by setting $\omega = T$). Also, because of the complexity of justifications for orbits of higher periods, we will justify only results on stability of equilibrium.

8.1 Justification of PBC

We have shown that y_2^* may be stabilized via the control (2.17) with $\omega > 1$ only if it is taken with an odd delay. For the even delays, there is no possibility how to stabilize it via (2.17). This fact can be verified by the analysis of eigenvalues of the linearized model. By this theoretical approach we will show also another interesting fact, which remains hidden in the previous approach, namely a so-called quasi-periodic behaviour of solution. We understand this notion in an intuitive sense; we have been motivated especially by numerical outputs displaying a periodic-like behaviour.

Firstly, we take the system (7.5) and we rewrite it into the system of ω difference equations

$$\begin{aligned}
y_1(n) &= (1-K)\mu y_1(n-1)(1-y_1(n-1)) + Ky_2(n-1)\\
y_2(n) &= y_3(n-1)\\
&\ \vdots\\
y_\omega(n) &= y_1(n-1).
\end{aligned}$$

Note that a rewriting to a system done in previous sections is equivalent to this one. It is clear that this system has the Jacobi matrix of linearization around $\mathbf{y}_2^* = (y_2^*, \dots, y_2^*)$ in the form

$$DF(\mathbf{y}_2^*) = \begin{pmatrix} (1-K)(2-\mu) & K & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 0 & 1\\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Obviously, its determinant is $\det(D) = K$. Since the system should be dissipative, we get the condition $|K| < 1$. The characteristic equation of the matrix D has the form

$$[\lambda - (1-K)(2-\mu)]\,\lambda^{\omega-1} - K = 0.$$

There are eigenvalues λ_1, λ_2, ..., λ_ω that can be hard to find explicitly for higher values of the delay ω, but still it is possible to get them numerically. Using the general Vieta's formula for the product of all roots we get

$$(-1)^{\omega+1}K = \prod_{j=1}^{\omega} \lambda_j.$$

The eigenvalues can be either real or complex-conjugate pairs and K is a real number such that $|K| < 1$, thus we can rewrite the above equation in the sense of complex numbers as

$$|(-1)^{\omega+1}K|e^{\mathrm{i}\arg((-1)^{\omega+1}\mathrm{K})} = \prod_{j=1}^{\omega} |\lambda_j| e^{\mathrm{i}\arg(\lambda_\mathrm{j})}.$$

Since Lyapunov exponents are defined as $\ln|\lambda_j|$, by applying of logarithm on the above equation, we get

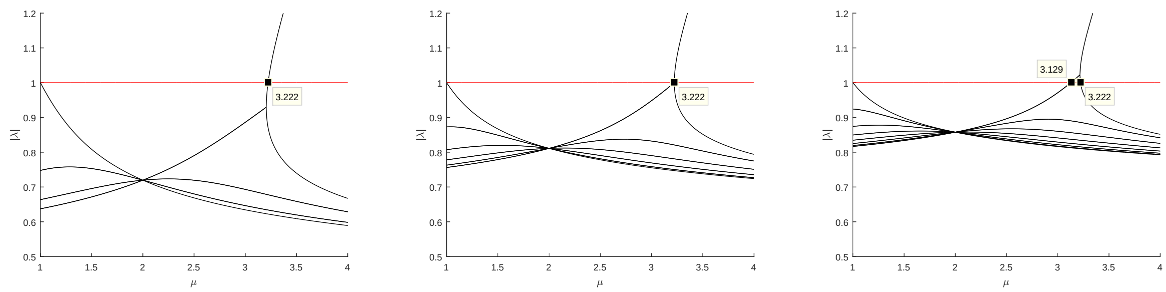
$$\ln|(-1)^{\omega+1}K| + \mathrm{i}\arg((-1)^{\omega+1}\mathrm{K}) = \sum_{\mathrm{j}=1}^{\omega} \ln|\lambda_\mathrm{j}| + \mathrm{i}\sum_{\mathrm{j}=1}^{\omega} \arg(\lambda_\mathrm{j}).$$

Comparing of the imaginary parts, we can conclude that

$$\sum_{j=1}^{\omega} \arg(\lambda_j) = \begin{cases} 0 & \text{if} \quad (-1)^{\omega+1}K \geq 0, \\ \pi & \text{if} \quad (-1)^{\omega+1}K < 0. \end{cases}$$

Thus, there exist two types of solution of how the eigenvalues can look like. For the first type, we have either a control with the odd delay ω and the feedback amplitude $K > 0$ or, on the contrary, a control with the even delay ω and the feedback amplitude $K < 0$. For the second type, we have either a control with the even delay ω and the feedback amplitude $K > 0$ or, on the contrary, a control with the odd delay ω and the feedback amplitude $K < 0$. Since we have shown that for the control (2.16) we can have only $K > 0$, in the next discussion we omit the cases with $K < 0$.

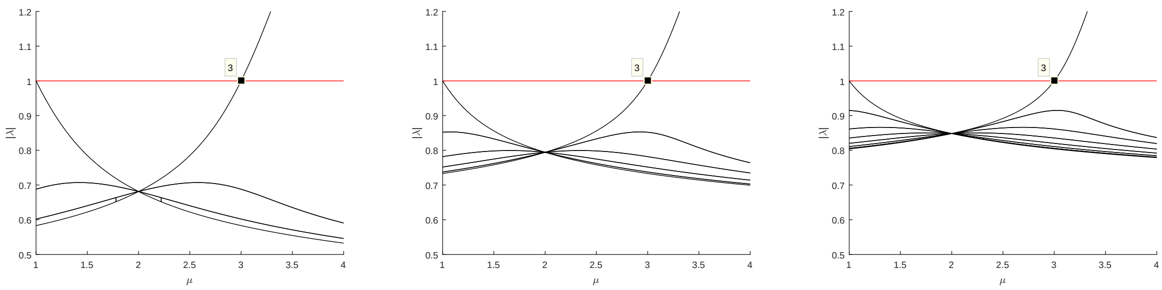
We will analyze now how the eigenvalues behave in dependence on the parameters μ and ω. Let us analyze instead of the eigenvalues their magnitudes, since for an asymptotically stable equilibrium we need to have all eigenvalues with $|\lambda_i| < 1$. The magnitudes of eigenvalues are depicted in the following pictures with respect to dependence on the growth parameter μ together with unit circle, meaning $|\lambda| = 1$ (depicted as the red line).



(a) $\omega = 7$ (b) $\omega = 11$ (c) $\omega = 15$

Figure 15: Modules of eigenvalues of (7.5) for given odd ω and $K = 0.1$.

For the PBC with an odd delay ω (see Figure 15) we get that there is one real eigenvalue and $\frac{\omega-1}{2}$ complex conjugate pairs of eigenvalues. Later, one pair of them is converted into two real eigenvalues. Here, the tangent bifurcation takes place. From Figure 15 we can conclude that for $\omega \in \{3, 5, 7, 9, 11\}$ just the modulus of one of these two real eigenvalues is greater than 1 after $\mu = 3.\overline{22}$. Until this value, the equilibrium $\mathbf{y}_2^*$ is asymptotically stable and after exceeding this value, it becomes unstable, where the first period-doubling

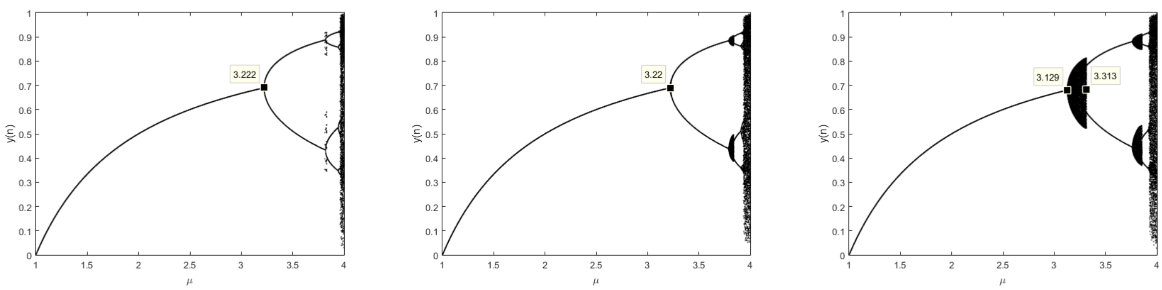


(a) $\omega = 6$ (b) $\omega = 10$ (c) $\omega = 14$

Figure 16: Modules of eigenvalues of (7.5) for given even ω and $K = 0.1$.

bifurcation occurs. It means that after such a value we get a solution behaving as a 2-period orbit. We can see that for $\omega > 11$ both modules of these real eigenvalues exceed 1 for a while. The phenomenon of exceeding of branch with the two modules of real eigenvalues before the tangent bifurcation occurs (see Figure 15c) is called as a Neimark-Sacker bifurcation. While there is the Neimark-Sacker bifurcation, a solution behaves quasi-periodically. When the quasi-periodical solution looses its stability, an asymptotically stable 2-period orbit is found. We can see that in all cases of chosen odd delays ω we enlarge a bit the range of stability of the equilibrium $\mathbf{y}_2^*$.

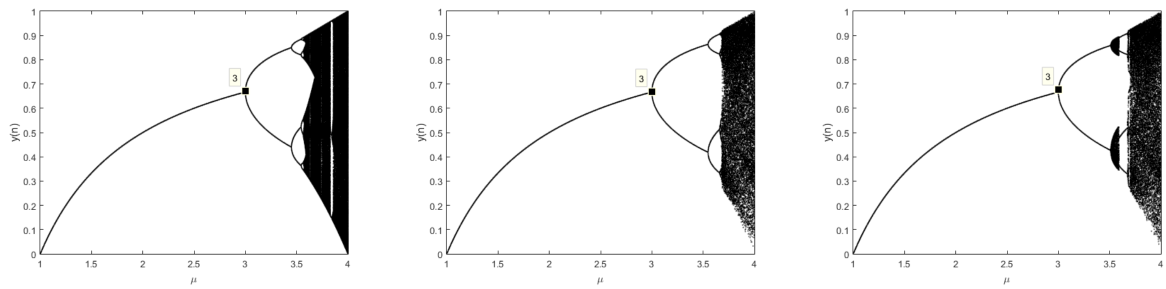
For the PBC with an even delay ω (see Figure 16), we get that there are two real eigenvalues and $\frac{\omega-2}{2}$ complex conjugate pairs. The modulus of one of the real eigenvalues exceeds 1 in $\mu = 3$ regardless of the value of ω. Thus, this is why we are not able to enlarge the range of stability of $\mathbf{y}_2^*$ with any even delay ω. For $\mu > 3$ the equilibrium $\mathbf{y}_2^*$ looses its stability and a solution behaves as an asymptotically stable 2-period orbit.



(a) $\omega = 7$ (b) $\omega = 11$ (c) $\omega = 15$

Figure 17: Bifurcation diagrams to (7.5) for given odd ω and $K = 0.1$.

Let us now take a look on the bifurcation diagrams for chosen delays. We have shown the existence of a new type of behaviour of solution for odd $\omega > 15$. This fact is possible to see in Figure 17c. There is a dark area between values $\mu = 3.129$ and $\mu = 3.313$, which corresponds exactly to the quasi-periodical solution. However, we can see that such a solution appears also in other bifurcation diagrams, namely before the 4-period orbit occurs and even in a case of the even ω (see Figures 17b, 17c, 18c). By comparing Figure 17 and Figure 18 we can clearly see how much the system is stabilized by using the odd delay. The whole ‘messy’ part of a bifurcation diagram to the uncontrolled system (see Figure 18a) is significantly suppressed in controlled system with an odd delay (Figure 17). On contrary, there is no significant difference between the ‘messy’ parts of



(a) Uncontrolled system (b) $\omega = 10$ (c) $\omega = 14$

Figure 18: Bifurcation diagrams to (7.5) for given even ω and $K = 0.1$.

the uncontrolled system and controlled system with an even delay (Figure 18).

One can object that we skipped the case with $K < 0$ only because of the results from previous approach, even though the approach in eigenvalues tells us just that $|K| < 1$ and no other specifications on K. Let us discuss now what happens with the system if we use $K < 0$ as a parameter of PBC. In this case, the modulus of the largest eigenvalue exceeds 1 around $\mu = 2.82$ for odd ω and for even ω it exceeds 1 around $\mu = 2.98$. Thus, the range of stability of $\mathbf{y}_2^*$ is diminished. So we get even worst results on stability.

8.2 Justification of DFC

Analogously, we can verify the practical approach for control (2.14) by the theoretical one. The model (7.3), rewritten into the system of $\tilde{\omega}$ difference equations, is in form

$$\begin{aligned} y_1(n) &= \mu y_1(n-1)(1 - y_1(n-1)) + K[y_2(n-1) - y_1(n-1)] \\ y_2(n) &= y_3(n-1) \\ &\vdots \\ y_{\tilde{\omega}}(n) &= y_1(n-1). \end{aligned}$$

The system has the Jacobi matrix of linearization around $\mathbf{y}_2^* = (y_2^*, \dots, y_2^*)$ in the form

$$DF(\mathbf{y}_2^*) = \begin{pmatrix} (2-\mu-K) & K & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

with a characteristic equation

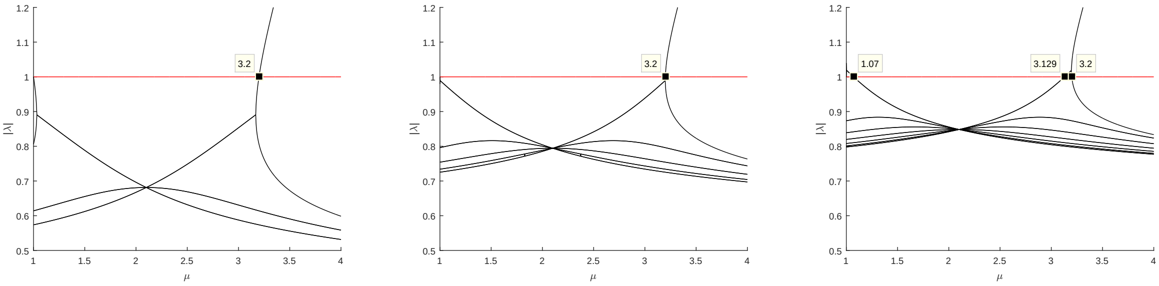
$$[\lambda - (2 - \mu - K)]\, \lambda^{\tilde{\omega}-1} - K = 0.$$

Following the same steps as in the first method, we get

$$\sum_{j=1}^{\tilde{\omega}} \arg(\lambda_j) = \begin{cases} 0 & \text{if } \ (-1)^{\tilde{\omega}+1} K \geq 0, \\ \pi & \text{if } \ (-1)^{\tilde{\omega}+1} K < 0. \end{cases}$$

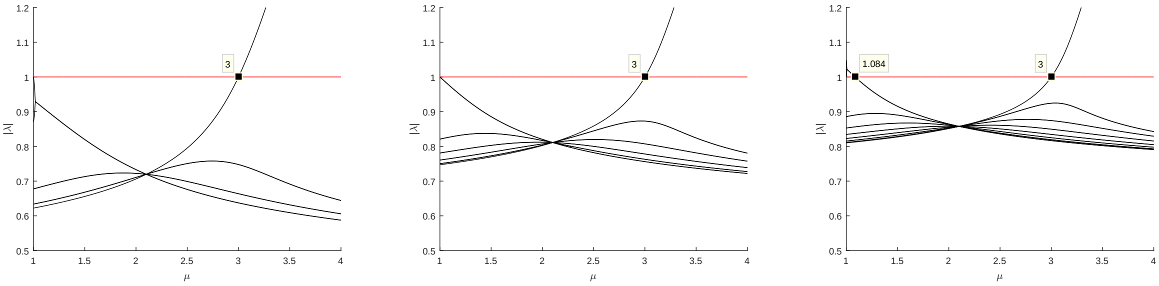
Notice, that in a practical approach we have shown that for stabilization of $\mathbf{y}_2^*$ we need a control with $K < 0$, so we restrict only to this case. Therefore, the first type of solution of the eigenvalues leads from a control with an even delay $\tilde{\omega}$ and the second leads from a control with an odd delay $\tilde{\omega}$. We will analyze now the dependence of the modules of eigenvalues on parameters μ and $\tilde{\omega}$.

For the DFC with an even delay $\tilde{\omega}$ (see Figure 19) we get that all eigenvalues are complex conjugate pairs. Later, one pair of them is again converted into two real eigenvalues. For $\tilde{\omega} \in \{2, 4, 6, 8, 10\}$ we get that also in this case the modulus of one of the two real eigenvalues exceeds 1, but when $\mu \geq 3.2$. For $\tilde{\omega} > 10$ we get that both modules of real eigenvalues exceeds 1 for a while, so the Neimark-Sacker bifurcation (and thus the quasi-periodical solution) appears also here. We can see that also with the DFC, taking an even $\tilde{\omega}$, we have enlarged the range of stability of $\mathbf{y}_2^*$.



(a) $\tilde{\omega} = 6$ (b) $\tilde{\omega} = 10$ (c) $\tilde{\omega} = 14$

Figure 19: Modules of eigenvalues of (7.3) for given even $\tilde{\omega}$ and $K = -0.1$.



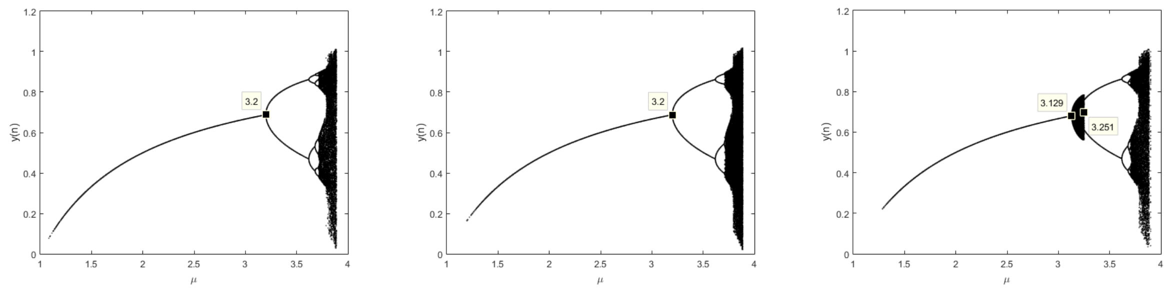
(a) $\tilde{\omega} = 7$ (b) $\tilde{\omega} = 11$ (c) $\tilde{\omega} = 15$

Figure 20: Modules of eigenvalues of (7.3) for given odd $\tilde{\omega}$ and $K = -0.1$.

For the control DFC with an odd delay $\tilde{\omega}$ (see Figure 20) we get that there is one real eigenvalue and $\frac{\tilde{\omega}-1}{2}$ complex conjugates. We can see that regardless of $\tilde{\omega}$, a modulus of the real eigenvalue always exceeds 1 exactly at $\mu = 3$.

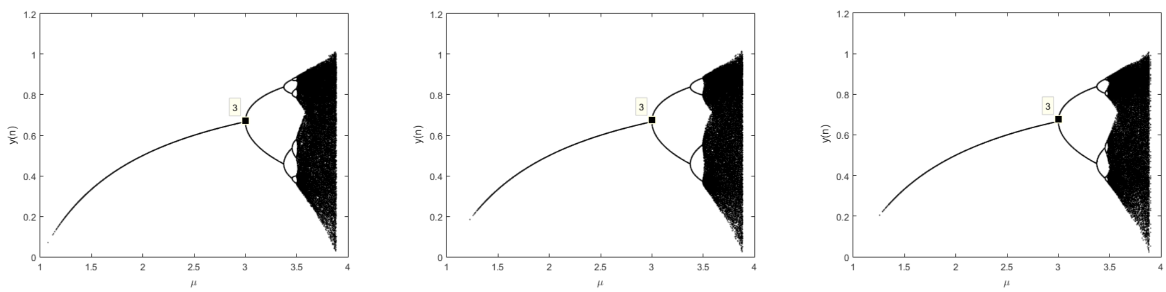
Let us discuss the bifurcation diagrams for chosen delays. Firstly, notice that control (2.14) slightly shortened the overall range of μ. On the other hand, quasi-periodic behaviour (see Figure 21c) appears in solution for more delayed control (2.14) (compared with PBC). The 'messy' part is again suppressed a lot (see Figure 21) for an even $\tilde{\omega}$, but it is broader than in Figure 17. There is again no significant difference between the 'messy' parts of the uncontrolled system and controlled system by DFC with an odd delay $\tilde{\omega}$ (see Figure 22).

Using $K > 0$ as a parameter of this control leads again to diminishing of the range of stability of $\mathbf{y}_2^*$. The modulus of the real eigenvalue exceeds 1 around $\mu = 2.8$ for even $\tilde{\omega}$ and for odd $\tilde{\omega}$ around $\mu = 2.95$.



(a) $\tilde{\omega} = 6$ (b) $\tilde{\omega} = 10$ (c) $\tilde{\omega} = 14$

Figure 21: Bifurcation diagrams to (7.3) for given even $\tilde{\omega}$ and $K = -0.1$.



(a) $\tilde{\omega} = 7$ (b) $\tilde{\omega} = 11$ (c) $\tilde{\omega} = 15$

Figure 22: Bifurcation diagrams to (7.3) given odd $\tilde{\omega}$ and $K = -0.1$.

8.3 Justification of PFC

We can briefly illustrate the previous approach for control (2.12). The model (7.1), rewritten into the system of ω difference equations, has the form

$$\begin{aligned} y_1(n) &= \mu y_1(n-1)(1-y_1(n-1)) + K\left[y_2(n-1) - \left(1 - \frac{1}{\mu}\right)\right] \\ y_2(n) &= y_3(n-1) \\ &\vdots \\ y_\omega(n) &= y_1(n-1). \end{aligned}$$

The system has the Jacobi matrix of linearization around $\mathbf{y}_2^* = (y_2^*, \ldots, y_2^*)$ in the form

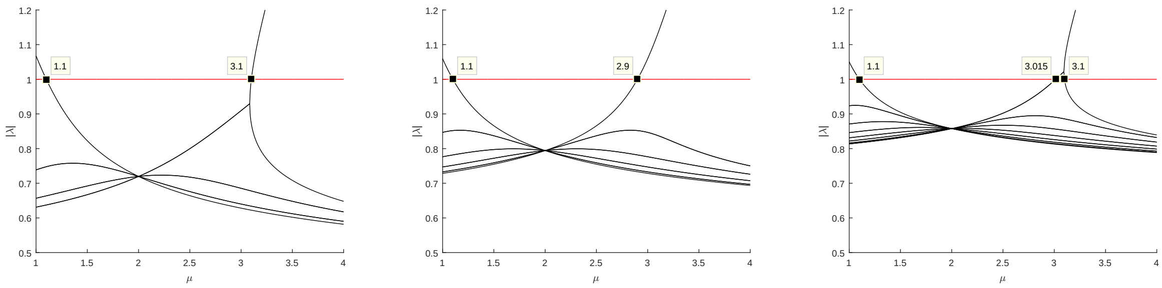
$$DF(\mathbf{y}_2^*) = \begin{pmatrix} (2-\mu) & K & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

with a characteristic equation

$$\left[\lambda - (2-\mu)\right]\lambda^{\omega-1} - K = 0.$$

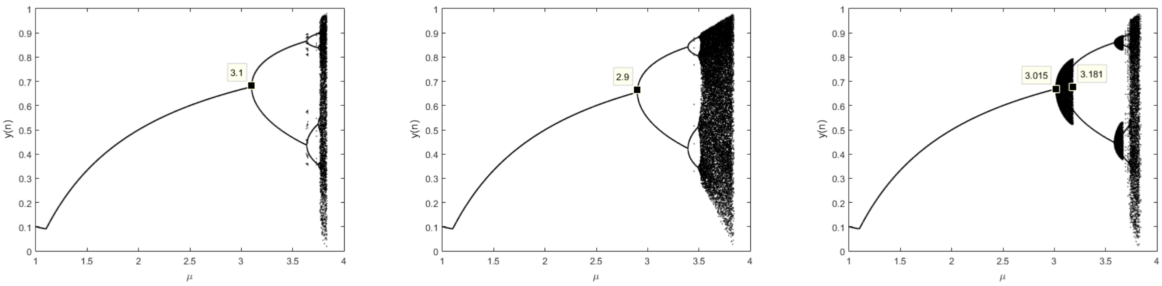
Following the same steps and discussions as for the PBC and DFC, we get that it is necessary to analyze both cases $K > 0$ and $K < 0$.

Analysis of the case $K > 0$:
For the PFC with an odd ω there is one real eigenvalue and $\frac{\omega-1}{2}$ complex conjugates. Later, one pair is again converted into two real eigenvalues. For $\omega \in \{3, 5, 7, 9, 11\}$, a modulus of one of two real eigenvalues exceeds 1 after $\mu = 3.1$ (see Figure 23a). For $\omega > 11$ both modules of eigenvalues exceeds 1 for a while. Thus, the Neimark-Sacker bifurcation occurs (see Figure 23c). It is clear that PFC with $K > 0$ and with an odd ω has enlarged the range of stability of $\mathbf{y}_2^*$. On contrary, PFC with an even ω diminished the range of stability (see Figure 23b). There are two real eigenvalues and $\frac{\omega-2}{2}$ complex conjugates. For arbitrary even ω, a modulus of one real eigenvalue exceeds 1 after $\mu = 2.9$.



(a) $\omega = 7$ (b) $\omega = 10$ (c) $\omega = 15$

Figure 23: Modules of eigenvalues of (7.1) for given ω and $K = 0.1$.

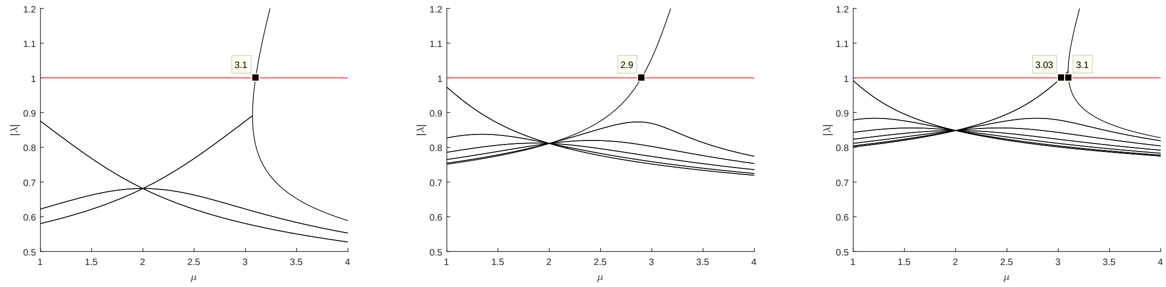


(a) $\omega = 7$ (b) $\omega = 10$ (c) $\omega = 15$

Figure 24: Bifurcation diagrams to (7.1) for given ω and $K = 0.1$.

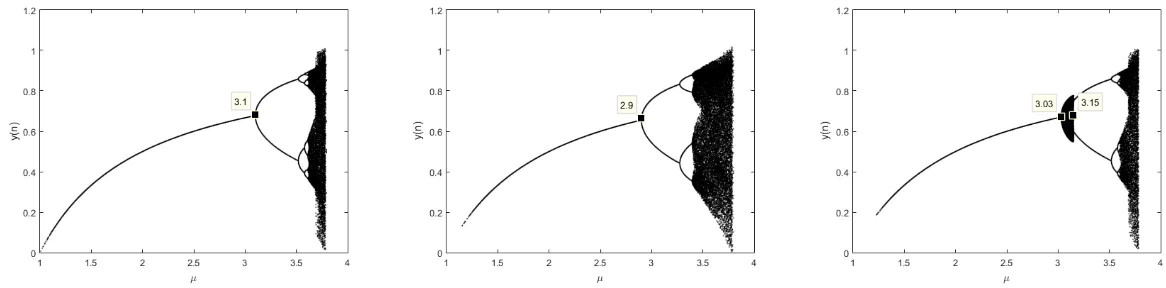
From the bifurcation diagrams (see Figure 24) we can conclude that also PFC slightly modifies a behaviour of $\mathbf{y}(n)$ with respect to μ. We can see that the overall range of μ is even more shortened.

Analysis of the case $K < 0$:
A discussion about how the eigenvalues behave (see Figure 25) is the same as in the case $K > 0$, just here the roles of even ω and odd ω are switched (an even ω enlarged the range of stability, an odd ω diminished it). The difference is just that for an even ω all eigenvalues are complex conjugates, where one pair later converts into two real eigenvalues (see Figure 25c). For an odd ω, there is one real eigenvalue and the rest are complex conjugates.



(a) $\omega = 6$ (b) $\omega = 11$ (c) $\omega = 14$

Figure 25: Modules of eigenvalues of (7.1) for given ω and $K = -0.1$.



(a) $\omega = 6$ (b) $\omega = 11$ (c) $\omega = 14$

Figure 26: Bifurcation diagrams to (7.1) for given ω and $K = -0.1$.

9 Comparison of Control Methods

Throughout this thesis, various stabilization results have been derived. Therefore, we create here a summarizing table (see Table 2) displaying the most important results. Moreover, once they are together in a table, we can compare them each other. Firstly, we can compare methods between themselves, and next, especially in the case of stabilization of equilibrium, we can compare the same controls with respect to the value of its delay. Particularly, we decide whether or not the implementation of general delay has some contributions in stabilization process compared to a stabilization with implemented exact period of orbit.

		PFC		**DFC**		**PBC**	
Orbit		**Stab.** K	**Crit.** μ	**Stab.** K	**Crit.** μ	**Stab.** K	**Crit.** μ
EQ	N-I	✓		✓		✓	
	$\omega = T$	PFC_μ^1	4	DFC_μ^1	5	PBC_μ^1	∞
2-PO	N-I	✗		✓		✓	
	$\omega = T$	PFC_μ^2	$\tilde{\mu}_K^*$	DFC_μ^2	μ_2^*	PBC_μ^2	∞
3-PO	N-I	✗		✓		✓	
	$\omega = T$	- - -	- - -	DFC_μ^3	?	PBC_μ^3	$\emptyset$

Table 2: Significant results summary.

We add some explanations to Table 2:
In the preceding table, the equilibrium, 2-period orbit and 3-period orbit are abbreviated as EQ, 2-PO and 3-PO, respectively. Further, there is emphasised whether the given control method is non-invasive (✓) or invasive (✗) for corresponding stabilized orbit. The actual stability sets for K and critical values of μ can be specified as follows:

$$PFC_\mu^1 = (0, \infty),$$
$$DFC_\mu^1 = (-1, 0),$$
$$PBC_\mu^1 = (0, 1),$$
$$PFC_\mu^2 = (\bar{K}_\mu, \tilde{K}_\mu), \qquad \bar{K}_\mu = \sqrt{\mu^2 - 2\mu - 3} + 2 - \sqrt{6 + 4\sqrt{\mu^2 - 2\mu - 3}},$$
$$\tilde{K}_\mu = \sqrt{\mu^2 - 2\mu - 3} + 2 - 2\sqrt{1 + \sqrt{\mu^2 - 2\mu - 3}},$$
$$\tilde{\mu}_K^* = 4.987352441,$$
$$DFC_\mu^2 = (\bar{K}_\mu, \tilde{K}_\mu), \qquad \bar{K}_\mu = \frac{-1 + \sqrt{\mu^2 - 2\mu - 4}}{2},$$

for the description of value $\tilde{K}_\mu$, see page 41,

$$\mu_2^* = 3.76007862,$$
$$PBC_\mu^2 = (\bar{K}_\mu, 1), \qquad \bar{K}_\mu = \frac{\mu^2 - 2\mu - 3 - 2\sqrt{\mu^2 - 2\mu - 4}}{\mu^2 - 2\mu - 5},$$
$$DFC_\mu^3 = (0, a_2), \qquad a_2 \approx 0.08,$$
$$PBC_\mu^3 \subset (0, 1),$$

Since PFC is not usable for stabilization of 3-PO, we put there - - - . We put a question mark into a critical value μ of 3-PO corresponding to DFC, since this matter remains an open problem (because of its computational difficulty).

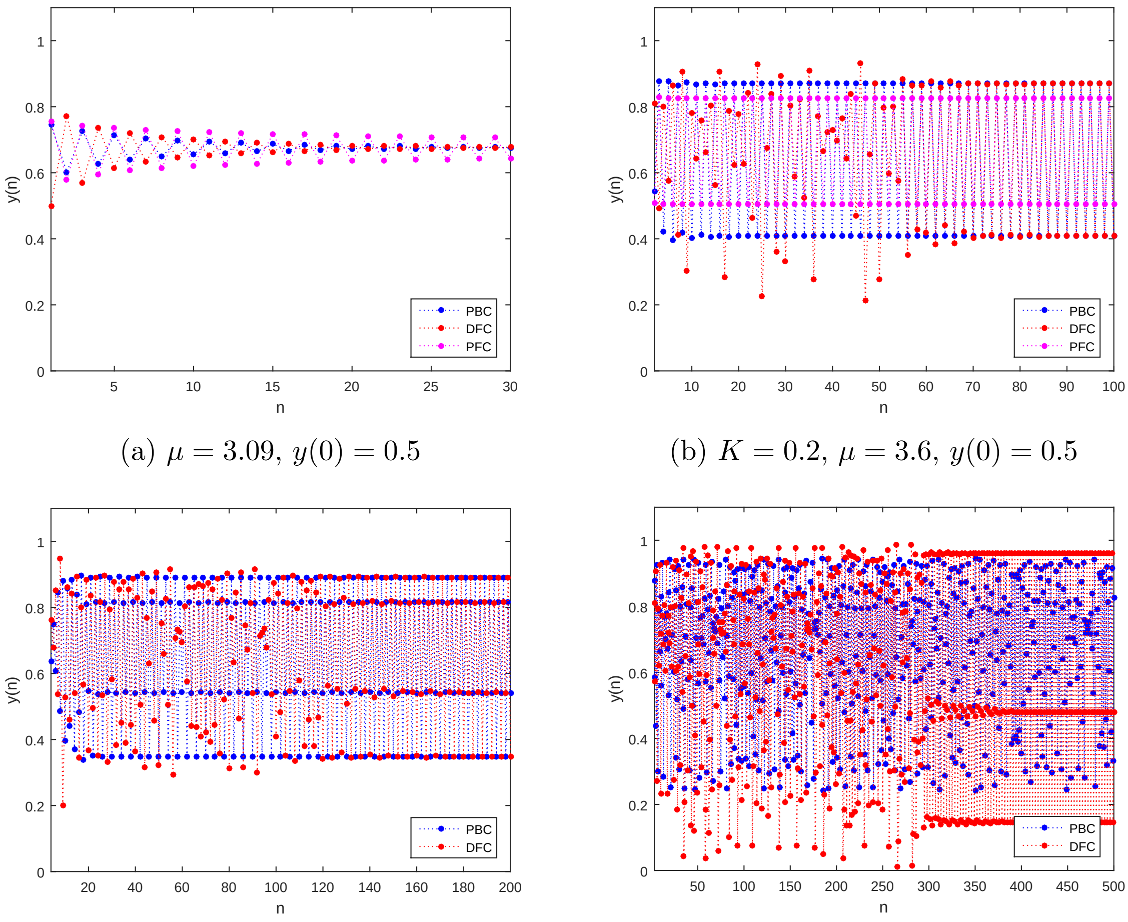
The control methods used for a stabilization of equilibrium are compared in Figure 27a. For this simulation, we use $K = 0.1$ for PFC (depicted by magenta colour) and PBC method (depicted as blue colour), and $K = -0.1$ for DFC method (depicted by red colour). Other parameters like bifurcation parameter μ and initial condition $y(0)$ are given under the graph. It is clear that control PFC is the slowest control. In fact, it needs at least a hundreds of iterations to get completely into the equilibrium. On the other hand, controls DFC and PBC have approximately the same speed of stabilization (for such a chosen parameters). However, with increasing K, the speed of stabilization is faster for both the methods and remain still similar. In fact, for $|K| = 0.3$, both the methods are able to stabilize the equilibrium just in less then 7 iterations. With such a parameter K, PFC needs around 60 iterations.

A speed of stabilization of 2-period orbit is compared in Figure 27b. Setted parameters for this simulation are given under this figure. Firstly, notice the invasiveness of PFC. Although it is clear that this method is now the fastest one, this observation is not relevant and is not comparable with a speed of rest of methods, since it was settled down in different 2-period orbit. Further, notice that DFC is now significantly slower than PBC. Its speed is getting even slower for higher periods. In Figure 27c there is shown an experiment on stabilization of 4-period orbit. For this, $K = 0.1$ was used for PBC and $K = -0.1$ for DFC. Note that we are not able to simulate a stabilization of 4-period orbit via PFC, since we do not know its points.

As the period of orbit is increasing, a time required for its stabilization is increasing as well. In fact, for a stabilization of 3-period orbit we need at least 400 iterations. It is simulated in Figure 27d. Note that we determined in Section 6.1 that in order to stabilize a 3-period orbit via DFC, K should be very small (in the absolute value). Indeed, in this experiment we used $K = 0.06$. However, we were not able to get range of stability larger than $1 + \sqrt{8} < \mu < 1.03 + \sqrt{8}$. In fact, this experiment shows also a stabilization of chaos that depicted in first 200 iterations by red. On the other hand, the blue points depict an orbit of very high period. It is not a chaos literally, since we already know that PBC is suppressing a chaotic behaviour.

In Table 2, we are not considering the stabilization of orbits via controls with a general delay ω, which was done in Chapter 7. It is especially because of the fact, that the implementation of these controls does not provide a significant difference in stabilization results compared to ones obtained for controls with a fixed delay (equal to the order of a period). Based on numerical experiments and some results of Chapter 8, controls involving a suitable delay are suppressing a chaotic behaviour more then controls stated in Table 2. On the other hand, the arise of quasi-periodical behaviour may occur.

One may already conjecture that the worst stabilization method is PFC. Although its simpleness, the requirements on prior knowledge of data cause this method unusable for stabilization of orbits of higher periods. On contrary, PBC is very useful method. It is also a very fast method as we seen, and it suppresses a chaos. Hence, some orbits may disappear (e.g., a 3-period orbit). From this point of view, DFC is very interesting method. It may stabilize orbits of any period, even a 3-period orbit, as we seen.



(a) $\mu = 3.09$, $y(0) = 0.5$

(b) $K = 0.2$, $\mu = 3.6$, $y(0) = 0.5$

(c) $\mu = 3.59$, $y(0) = 0.5$

(d) $K = 0.06$, $\mu = 1.02 + \sqrt{8}$, $y(0) = 0.5$

Figure 27: Simulations of stabilizations.

10 Conclusion

As the main result of this work, we can consider Table 2 summarizing significant results obtained throughout the thesis. These results are determined analytically, using just knowledges of modern mathematical analysis. Moreover, all of them are immediately justified by graphical interpretations and simulations. All of the figures were specially made for this thesis. Most of them are done in MATLAB, few of them in Graph or Inkscape. Because of the computational difficulty of particular problems, we mostly compute them in Maple.

Our aim was to present results general enough in the sense that, for any data, we are able to get immediately particular results on stability or controllability of the system. The discussion on comparisons of the used controls was presented as well. All our theoretical conclusions have been supported by numerous experiments.

A specific contribution of this thesis consists a further development of the topic discussed in [10]. In this paper, there is given an estimate of critical value of bifurcation parameter μ, for which an unstable 2-period orbit becomes unstabilizable via DFC. In addition to this result, we have derived here a precise value of this critical parameter as a solution of the algebraic equation of the fourth-order. In fact, this paper inspired us to a current form of Chapter 5.

This thesis provides a base for a further investigation in this area. In particular, we can analyze more advanced discrete chaotic models, including multiple-species models, epidemiology models or Saturn's rings model. Also, a potential of delay parameter on stabilization property of a given control is far from being fully explored.

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List of Symbols

$\mathbb{N}$	Set of positive integers (natural numbers)
$\mathbb{Z}$	Set of integers
$\mathbb{R}$	Set of real numbers
$\mathbb{R}^m$	Cartesian product $\mathbb{R} \times \cdots \times \mathbb{R}$ (m-times)
f	Right-hand side of difference equation
F	Right-hand side of system of difference equations
K	Gain of control
t	Targeted value of PFC
y^*	Equilibrium of system
A, B	Matrix
f^T, F^T	T-th iteration of map f (or F)
T, S	Period of orbit
U, u	Control of system
μ	Bifurcation parameter
γ, γ_i	Point of period orbit
γ^*, γ_i^*	Fixed point of orbit
ω, ρ	Delay of system
α_i, β_i, λ_i	Eigenvalue of system
$\mathrm{D}(f)$, $\mathrm{D}(F)$	Domain of dependence of map f (or map F)
$Df(y^*)$, $DF(\gamma^*)$	Jacobi matrix of f at y^* (or F at γ^*)

Abbreviations

PFC	Proportional feedback control
DFC	Delayed feedback control
PBC	Prediction based control
PFC_μ^T	Stabilization set of PFC according to period T
PFC_μ	Stabilization set of PFC according to general delay ω
DFC_μ^T	Stabilization set of DFC according to period T
DFC_μ	Stabilization set of DFC according to general delay ω
PBC_μ^T	Stabilization set of PBC according to period T
PBC_μ	Stabilization set of PBC according to general delay ω