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STABILITA NEUTRÁLNÍCH DIFERENCIÁLNÍCH ROVNIC SE ZPOŽDĚNÍM A JEJICH DISKRETIZACÍ

STABILITY OF NEUTRAL DELAY DIFFERENTIAL EQUATIONS AND THEIR DISCRETIZATIONS

DIZERTAČNÍ PRÁCE
DOCTORAL THESIS

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Abstrakt

Disertační práce se zabývá asymptotickou stabilitou zpožděných diferenciálních rovnic a jejich diskretizací. V práci jsou uvažovány lineární zpožděné diferenciální rovnice s konstantním i neohraničeným zpožděním. Jsou odvozeny nutné a postačující podmínky popisující oblast asymptotické stability jak pro exaktní, tak i diskretizovanou lineární neutrální diferenciální rovnici s konstantním zpožděním. Pomocí těchto podmínek jsou porovnány oblasti asymptotické stability odpovídajících exaktních a diskretizovaných rovnic a vyvozeny některé vlastnosti diskrétních oblastí stability vzhledem k měnícímu se kroku použité diskretizace. Dále se zabýváme lineární zpožděnou diferenciální rovnicí s neohraničeným zpožděním. Je uveden popis jejích exaktních a diskrétních oblastí asymptotické stability spolu s asymptotickým odhadem jejich řešení. V závěru uvažujeme lineární diferenciální rovnici s více neohraničenými zpožděními.

Summary

The doctoral thesis discusses the asymptotic stability of delay differential equations and their discretizations. The linear delay differential equations with constant as well as infinite lag are considered. The necessary and sufficient conditions describing the asymptotic stability region of both exact and discretized linear neutral delay differential equation with constant lag are derived. We compare asymptotic stability domains of corresponding exact and discretized equations and discuss properties of derived stability regions with respect to a changing stepsize of the utilized discretization. Further, we investigate the linear delay differential equation with the infinite lag. We present the description of its exact and discrete asymptotic stability regions together with asymptotic estimates of its solutions. The linear delay differential equation with several infinite lags is discussed as well.

Klíčová slova

neutrální zpožděná diferenciální rovnice, Θ-metoda, asymptotická stabilita, τ-stabilita, konstantní zpoždění, neohraničené zpoždění

Keywords

neutral delay differential equation, Θ-method, asymptotic stability, τ-stability, constant lag, infinite lag

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I declare that I have written the doctoral thesis Stability of Neutral Delay Differential Equations and Their Discretizations on my own according to the instructions of my diploma thesis supervisor doc. RNDr. Jan Čermák, CSc., and using the sources listed in references.

Ing. Jana Dražková

Hereby, I would like to express my immense thanks to my doctoral thesis supervisor for his helpful advices, patience and many supporting consultations. I could not have imagined having a better advisor.

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Ing. Jana Dražková

Contents

1. Introduction

Delay differential equations are widely used in science and engineering. They arise in modeling of problems where the rate of change of a time–dependent process is determined not only by its present state but also by a certain past state. Then, the use of ordinary differential equations turns out to be insufficient. Problems of this type occur in various fields such as biology, electrodynamics, medicine, economics and many others (various examples are presented in the book by Kolmanovskii and Myshkis [37]).

Unlike for the ordinary differential equations, for which we have several methods to obtain the analytical solution (e.g. separation of variables, variation of constant method etc.), there are no computational methods how to find the analytical solution of delay differential equations, not even in the linear case. Therefore, the qualitative and numerical analysis of these equations is of a great importance.

The basic numerical methods utilized to solve the delay differential equations originates from corresponding procedures for ordinary differential equations (with some additional requirements concerning the delayed terms). Although the methods are based on the same principal, their potential to preserve the qualitative behaviour of the analytic solution may be different.

One of the most important qualitative properties of differential equations is the asymptotic stability. Roughly speaking, this property describes a capability of the equation to eliminate possible errors in input data. The asymptotic stability for the linear delay differential equation can be defined as follows.

Consider the linear delay differential equation

$$x'(t) = a\,x(t) + b\,x(\xi(t))\,, \qquad t \in (t_0, \infty), \tag{1.1}$$

where a, b are real scalars and the function $\xi(t)$ is a continuous function satisfying $\xi(t) < t$ for all $t > t_0$ (some additional assumptions on $\xi(t)$ will be imposed throughout this thesis). Then (1.1) is called asymptotically stable if all its solutions $x(t)$ tend to zero as $t \to \infty$.

This notion can be introduced in the same sense also to other related linear equations such as differential equation with several delays

$$x'(t) = a\,x(t) + \sum_{i=1}^{r} b_i\,x(\xi_i(t))\,, \qquad t \in (t_0, \infty), \tag{1.2}$$

or the delay differential equation of neutral type

$$x'(t) = a\,x(t) + b\,x(\xi(t)) + c\,x'(\xi(t))\,, \qquad t \in (t_0, \infty). \tag{1.3}$$

Since the resulting numerical formulae of numerical methods applied to (1.1)-(1.3) are difference equations, we are interested in their asymptotic stability, too. Analogously to the differential equation, we have the following definition of the asymptotic stability for a linear difference equation of the order k.

Consider the linear difference equation

$$y_n + \alpha_1 y_{n-1} + \alpha_2 y_{n-2} + \cdots + \alpha_k y_{n-k} = 0, \qquad n = 0, 1, 2, \dots\,, \tag{1.4}$$

where α_i are real scalars and k is a positive integer. Then (1.4) is called asymptotically stable if all its solutions y_n tend to zero as $n \to \infty$.

The aim of this thesis is to investigate a potential of some numerical methods to retain the asymptotic stability of the linear delay differential equations. To do so, we have to determine conditions for coefficients a, b under which (1.1) is asymptotically stable and compare them with those for the corresponding discretized equation. It assumes that both the conditions for the exact as well as discretized equation are strong enough in the sense that they are not only sufficient but also necessary.

The thesis is divided into two main parts according to type of the lag $\psi(t) = t - \xi(t)$ occurring in (1.1). In general, delay differential equations can be classified into two categories, namely those with finite time lag, i.e.

$$\limsup_{t\to\infty} \psi(t) < \infty$$

and those with infinite time lag, i.e.

$$\limsup_{t\to\infty} \psi(t) = \infty \, .$$

There are remarkable differences between these two categories of delay differential equations. Let us compare their typical representatives, which are the equations

$$x'(t) = a\,x(t) + b\,x(t-\tau), \qquad t > 0 \tag{1.5}$$

and

$$x'(t) = a\,x(t) + b\,x(qt), \qquad t > 0, \tag{1.6}$$

where a, b, $\tau > 0$ and $q \in (0,1)$ are real numbers. Clearly, (1.5) has a finite lag and (1.6) belongs to the class of equations with the infinite lag. One of the differences between (1.5) and (1.6) consists in the decay rate of their solutions in the asymptotically stable case $a < -|b|$. While the solution of (1.5) decays exponentially, the solution of (1.6) decays algebraically. However, the most significant difference is in storage. In order to calculate all the future values of $x(t)$ beyond some $t^* > 0$, we must remember all the past values in the interval $\langle t^* - \psi(t^*), t^* \rangle$, which is bounded in case of (1.5), but unbounded in the case (1.6) as $t^* \to \infty$ ([44]). This property plays a key role in their numerical discretization. For this reason, we treat them separately.

The thesis is organized as follows. Chapter 2 recalls some results on the asymptotic stability of the difference equations relevant to our further analysis.

Chapter 3 discusses stability of numerical methods for linear delay differential equations with constant lags. We consider (1.5) with $a = 0$ as the simplest case, and (1.3) with $\xi(t) = t - \tau$ as the most general case. The necessary and sufficient conditions for the asymptotic stability of the exact and discretized equations are presented. Based on them, we describe some properties of the discretized equations. The short overview of an equation containing two constant delays is presented, too.

Chapter 4 is devoted to linear delay differential equations with infinite lags. Firstly, we consider the equation (1.6). We introduce a constrained mesh suitable for its discretization and recall results concerning the stability of some numerical formulae for (1.6). The asymptotic estimates of the exact and discretized equations are presented, too. Further, we generalize the results to the equation with a more general lag of the form (1.1). Lastly, the extension to the delay differential equation with several lags (1.2) is investigated. We present the sufficient conditions for the asymptotic stability of both the exact and discretized equations. The asymptotic estimate of the analytical solution is provided, too. Moreover, the necessary and sufficient conditions for the asymptotic stability as well as some asymptotic estimates are derived for the discretization of the differential equation with two iterated lags.

This thesis is based on the papers [7], [8], [27], [28] and [29].

2. Some auxiliary results in difference equations

In this chapter, we recall some results from the theory of difference equations. In the first part, we deal with difference equations with constant coefficients and we present necessary and sufficient criteria for their asymptotic stability. We also derive some consequences for their important particular cases. In the second part, we discuss difference equations with asymptotically constant coefficient. We recall some well-known results providing asymptotic description of the solutions.

2.1. Asymptotic stability of difference equations with constant coefficients

Let us consider the following general linear difference equation of the order k

$$y_n + \alpha_1 y_{n-1} + \alpha_2 y_{n-2} + \cdots + \alpha_k y_{n-k} = 0, \qquad n = 0, 1, 2, \dots ,$$

where α_i, $i = 1, 2, \dots, k$ are real scalars. It is well-known that the problem of its asymptotic stability is equivalent to the problem whether the characteristic polynomial

$$P(\lambda) = \lambda^k + \alpha_1 \lambda^{k-1} + \alpha_2 \lambda^{k-2} + \cdots + \alpha_k \tag{2.1}$$

is of a Schur type, i.e. whether all its zeros are located inside the open unit circle (see Elaydi [17]).

In general, this problem is solved by the Schur-Cohn criterion (see, e.g. Marden [45]), which yields necessary and sufficient conditions for fixed α_1, $\alpha_2, \dots \alpha_k$ and k ensuring the required zero property. However, the form of these conditions does not enable to formulate explicit description of the set of all α_1, $\alpha_2, \dots \alpha_k$ and k such that (2.1) is of a Schur type. The problem of such a description is solved only in some special cases of (2.1), e.g. the case of $\alpha_i < 0$, $i = 1, 2, \dots, k$ for which Stević [54] formulated the following theorem.

Theorem 2.1. *Let $\alpha_i < 0$, $i = 1, 2, \dots, k$. Then (2.1) is of a Schur type if and only if*

$$\sum_{i=1}^{k} |\alpha_i| < 1 \,. \tag{2.2}$$

However, when not all the coefficients are negative then the condition (2.2) is not optimal, but only sufficient (see e.g. Kocić and Ladas [36]).

Theorem 2.2. *The polynomial (2.1) is of a Schur type if*

$$\sum_{i=1}^{k} |\alpha_i| < 1 \,.$$

For the purposes of this work, we are mostly interested in analysis of the following two difference equations

$$y_{n+1} + \alpha y_n + \beta y_{n-k+1} + \gamma y_{n-k} = 0, \qquad n = 0, 1, 2, \dots, \tag{2.3}$$

and

$$y_{n+2} + \mu y_n + \nu y_{n-k} = 0, \quad n = 0, 1, 2, \dots . \tag{2.4}$$

Considering (2.3), the problem of explicit description of asymptotic stability conditions has been discussed in some particular cases. In this connection we can mention the work of Levin and May [41], who derived the explicit necessary and sufficient condition for the case $\alpha = -1$ and $\beta = 0$. Some other results have been obtained by Kuruklis [40] (the case $\beta = 0$) and Dannan and Elaydi [11] (the case $\alpha = 0$), but their formulation requires a solution of an auxiliary non-linear equation. Recently, a system of explicit necessary and sufficient conditions for a general equation (2.3) has been found by Čermák et al. [10]. We note that it can be formulated in a more compact form (see Theorem 3.1 and Theorem 3.2 of [10]), but for its easier treating in the asymptotic stability analysis we prefer the following one.

Theorem 2.3. *Let α, β and γ be real constants and k be a positive integer. Then (2.3) is asymptotically stable if and only if one of the following conditions holds:*

(C1) $1+\alpha+\beta+\gamma>0\,,\quad 1+\alpha-\beta-\gamma>0\,,\quad 1-\alpha+\beta-\gamma>0\,,\quad 1-\alpha-\beta+\gamma>0$ *and k is any positive integer;*

(C2) $1+\alpha+\beta+\gamma>0\,,\quad 1+\alpha-\beta-\gamma=0\,,\quad 1-\alpha+\beta-\gamma>0\,,\quad 1-\alpha-\beta+\gamma>0$ *and k is any positive integer;*

(C3) $1+\alpha+\beta+\gamma>0\,,\quad 1+\alpha-\beta-\gamma>0\,,\quad 1-\alpha+\beta-\gamma=0\,,\quad 1-\alpha-\beta+\gamma>0$ *and k is any positive odd integer;*

(C4) $1+\alpha+\beta+\gamma>0\,,\quad 1+\alpha-\beta-\gamma>0\,,\quad 1-\alpha+\beta-\gamma>0\,,\quad 1-\alpha-\beta+\gamma=0$ *and k is any positive even integer;*

(C5) $1+\alpha+\beta+\gamma>0\,,\quad 1+\alpha-\beta-\gamma<0\,,\quad 1-\alpha+\beta-\gamma>0\,,\quad 1-\alpha-\beta+\gamma>0$ *and k is any positive integer such that*

$$k < \arccos\frac{\alpha^2-\beta^2+\gamma^2-1}{2|\alpha\gamma-\beta|}\bigg/\arccos\frac{\alpha^2-\beta^2-\gamma^2+1}{2|\alpha-\beta\gamma|}\,; \tag{2.5}$$

(C6) $1+\alpha+\beta+\gamma>0\,,\quad 1+\alpha-\beta-\gamma>0\,,\quad 1-\alpha+\beta-\gamma<0\,,\quad 1-\alpha-\beta+\gamma>0$ *and k is any positive odd integer such that (2.5) holds;*

(C7) $1+\alpha+\beta+\gamma>0\,,\quad 1+\alpha-\beta-\gamma>0\,,\quad 1-\alpha+\beta-\gamma>0\,,\quad 1-\alpha-\beta+\gamma<0$ *and k is any positive even integer such that (2.5) holds.*

Further, we provide consequences of Theorem 2.3 for some particular cases of (2.3), which are results of well-known discretizations of some delay differential equations such as the trapezoidal rule or both basic Euler discretizations.

Corollary 2.4. *Let α, β be reals and k be a positive integer. Then*

$$y_{n+1} + \alpha y_n + \beta(y_{n-k+1} + y_{n-k}) = 0, \qquad n = 0, 1, 2, \ldots$$

is asymptotically stable if and only if either

$$2|\beta| < 1 + \alpha, \qquad \alpha < 1 \tag{2.6}$$

or

$$2\beta = 1 + \alpha, \qquad |\alpha| < 1 \tag{2.7}$$

or

$$|1 + \alpha| < 2\beta, \qquad \alpha < 1, \qquad k < \arccos\frac{\alpha + 1}{-2\beta} \Big/ \arccos\frac{\alpha^2 - 2\beta^2 + 1}{2(\beta^2 - \alpha)}\,. \tag{2.8}$$

Proof. Setting $\beta = \gamma$, the conditions (C1), (C2) can be read as (2.6) and (2.7), respectively. Since conditions (C3), (C4) and (C6), (C7) lead to a contradiction, it remains to analyse only condition (C5). After some elementary calculations we can verify that first four inequalities in (C5) occur if and only if $|1 + \alpha| < 2\beta$ and $\alpha < 1$. Finally, the right-hand side of (2.5) can be read for $\beta = \gamma$ as

$$\arccos\frac{\alpha^2 - 1}{2|\alpha - 1||\beta|} \Big/ \arccos\frac{\alpha^2 - 2\beta^2 + 1}{2|\alpha - \beta^2|}\,.$$

Obviously $\alpha - 1 < 0$, $\beta > 0$ and $\alpha < 2\beta - 1 \leq \beta^2$, hence $\alpha - \beta^2 < 0$. This verifies the form of $(2.8)_3$. □

Corollary 2.5. *Let β, γ be reals and k be a positive integer. Then*

$$y_{n+1} - y_n + \beta y_{n-k+1} + \gamma y_{n-k} = 0, \qquad n = 0, 1, 2, \ldots$$

is asymptotically stable if and only if

$$\beta + \gamma > 0, \qquad |\beta - \gamma| < 2, \qquad k < \arccos\frac{\gamma - \beta}{2} \Big/ \arccos\frac{2 - \beta^2 - \gamma^2}{2(1 + \beta\gamma)}\,. \tag{2.9}$$

Proof. Substituting $\alpha = -1$ into (C1)-(C7), the first two inequalities lead to a contradiction in all conditions except for (C5), where they become $\beta + \gamma > 0$. Further, the third and fourth inequalities of (C5) can be read as $|\beta - \gamma| < 2$. Setting $\alpha = -1$ in (2.5) we arrive at

$$k < \arccos\frac{-\beta^2 + \gamma^2}{2|\beta + \gamma|} \Big/ \arccos\frac{2 - \beta^2 - \gamma^2}{2|1 + \beta\gamma|}\,.$$

Since $\beta + \gamma > 0$, the remaining issue is to determine the sign of $1 + \beta\gamma$. Let us consider $\beta > 0$ and $\gamma < 0$. Then $|\beta - \gamma| = \beta - \gamma < 2$, which implies

$$1 + \beta\gamma > 1 + \gamma(2 + \gamma) = (1 + \gamma)^2 \geq 0.$$

The case $\beta < 0$ and $\gamma > 0$ is analogous. If β and γ are of the same sign, then the positivity of $1 + \beta\gamma$ is obvious. □

Corollary 2.6. *Let β be a real and k be a positive integer. Then*

$$y_{n+1} - y_n + \beta(y_{n-k+1} + y_{n-k}) = 0, \qquad n = 0, 1, 2, \dots$$

is asymptotically stable if and only if

$$0 < \beta < \tan \frac{\pi}{4k}\,. \tag{2.10}$$

Proof. If we put $\beta = \gamma$ in stability conditions of Corollary 2.5, then $(2.9)_1$ and $(2.9)_2$ imply $\beta > 0$. Further, $(2.9)_3$ becomes

$$k < \arccos 0 \Big/ \arccos \frac{1-\beta^2}{1+\beta^2} = \frac{\pi}{4 \arctan \beta}\,,$$

which yields (2.10). □

Corollary 2.7. *Let γ be a real and k be a positive integer. Then*

$$y_{n+1} - y_n + \gamma y_{n-k} = 0, \qquad n = 0, 1, 2, \dots$$

is asymptotically stable if and only if

$$0 < \gamma < 2 \cos \frac{k\pi}{2k+1}\,. \tag{2.11}$$

Proof. The stability condition (2.11) follows directly from Corollary 2.5 by use of $\beta = 0$ and $\arccos\left((2-\gamma^2)/2\right) = \pi - 2\arccos(\gamma/2)$. We emphasize that (2.11) was first proved by Levin and May in [41]. □

The asymptotic stability of (2.4) has been discussed in several papers, too. We mention Kipnis and Nigmatullin [35] who described the stability region via some straight lines and certain parametric curves defining its boundary. Another paper on this topic has been presented by Ren [51], who gave a system of necessary and sufficient conditions for asymptotic stability of (2.4), but its formulation needs to solve a non-linear auxiliary equation, similarly to the result of Kuruklis mentioned above. We utilize the assertion by Čermák and Tomášek [9], which formulates the necessary and sufficient conditions in the explicit form with respect to k.

Theorem 2.8. *Let μ, ν be arbitrary reals such that $\mu\nu \neq 0$ and k be a positive integer.*

(a) *Let k be even and $\nu(-\mu)^{k/2+1} < 0$. Then (2.4) is asymptotically stable if and only if*

$$|\mu| + |\nu| < 1\,. \tag{2.12}$$

(b) *Let k be even and $\nu(-\mu)^{k/2+1} > 0$. Then (2.4) is asymptotically stable if and only if*

$$|\mu| + |\nu| \le 1\,, \tag{2.13}$$

or

$$||\mu| - |\nu|| < 1 < |\mu| + |\nu|, \quad k < 2 \arccos \frac{\mu^2 + \nu^2 - 1}{2|\mu\nu|} \Big/ \arccos \frac{\mu^2 - \nu^2 + 1}{2|\mu|} \tag{2.14}$$

holds.

(c) *Let k be odd and $\mu < 0$. Then (2.4) is asymptotically stable if and only if (2.12) holds.*

(d) *Let k be odd and $\mu > 0$. Then (2.4) is asymptotically stable if and only if either (2.13), or*

$$\nu^2 < 1 - \mu < |\nu|, \quad k < 2 \arcsin \frac{1 - \mu^2 - \nu^2}{2|\mu\nu|} \Big/ \arccos \frac{\mu^2 - \nu^2 + 1}{2|\mu|} \tag{2.15}$$

holds.

For our further analysis, it will be useful to mention also the corollary of the previous theorem for the case $|\mu| = 1$ (for the detailed derivation see [9]).

Corollary 2.9. *The equation (2.4) with $|\mu| = 1$ is asymptotically stable if and only if*

$$k \text{ is even}, \qquad \nu(-\mu)^{k/2+1} > 0 \qquad \text{and} \qquad |\nu| < 2 \sin \frac{\pi}{2(k+1)} .$$

2.2. Asymptotic behaviour of difference equations with asymptotically constant coefficients

Let us consider the Poincaré difference equation of the form

$$y_n + (\alpha_1 + \delta_{1,n}) y_{n-1} + \cdots + (\alpha_k + \delta_{k,n}) y_{n-k} = 0, \qquad n = 0, 1, 2, \ldots , \tag{2.16}$$

where $\alpha_k \neq 0$, $\alpha_j \in \mathbb{R}$, $1 \leq j \leq k$ and

$$\lim_{n \to \infty} \delta_{j,n} = 0, \qquad 1 \leq j \leq k . \tag{2.17}$$

The equation (2.16) can be regarded as a perturbation of the limiting constant coefficient difference equation

$$y_n + \alpha_1 y_{n-1} + \cdots + \alpha_k y_{n-k} = 0, \qquad n = 0, 1, 2, \ldots \tag{2.18}$$

having the characteristic polynomial

$$P(\lambda) = \lambda^k + \alpha_1 \lambda^{k-1} + \alpha_2 \lambda^{k-2} + \cdots + \alpha_k . \tag{2.19}$$

It is natural to expect that the solutions of (2.16) retain some properties of the solutions of (2.18). This question has been studied by Poincaré and Perron, whose results can be summarized as follows (see e.g. [16]).

Theorem 2.10. *Suppose (2.17) holds and let the zeros λ_j of characteristic polynomial (2.19) have distinct moduli. Then (2.16) has a fundamental set of solutions $y_n^{(1)}, \ldots, y_n^{(k)}$ such that*

$$\lim_{n \to \infty} \frac{y_{n+1}^{(j)}}{y_n^{(j)}} = \lambda_j, \qquad 1 \leq j \leq k .$$

This result has been improved by Elaydi [16] who derived the asymptotic estimates of the fundamental set of solutions. We state here the results relevant to our further analysis.

Theorem 2.11. *Suppose that the zeros λ_j of characteristic polynomial (2.19) are distinct. If*

$$\sum_{n=0}^{\infty} |\delta_{j,n}| < \infty, \qquad 1 \le j \le k, \tag{2.20}$$

then (2.16) has a fundamental set of solutions $y_n^{(1)}, \ldots, y_n^{(k)}$ such that

$$y_n^{(j)} = (c_j + o(1))\lambda_j^n, \qquad c_j \neq 0, \qquad 1 \le j \le k\,.$$

Theorem 2.12. *Suppose that the characteristic polynomial (2.19) has a double multiple zero $\lambda_1 = \lambda_2$ and the remaining zeros λ_j are distinct. If*

$$\sum_{n=0}^{\infty} n|\delta_{j,n}| < \infty, \qquad 1 \le j \le k,$$

then (2.16) has a fundamental set of solutions $y_n^{(1)}, \ldots, y_n^{(k)}$ such that

$$\begin{aligned} y_n^{(1)} &= (c_1 + o(1))n\,\lambda_1^n, && c_1 \neq 0, \\ y_n^{(j)} &= (c_j + o(1))\lambda_j^n, && c_j \neq 0, \qquad 2 \le j \le k\,. \end{aligned}$$

Another generalization has been provided by Pituk [49] who considered the case when (2.19) has one dominant zero.

Theorem 2.13. *Suppose that the characteristic polynomial (2.19) has a simple dominant root $\tilde{\lambda}$ and let (2.20) hold. Then*

$$\lim_{n\to\infty} ((\tilde{\lambda})^{-n} y_n) < \infty\,.$$

Furthermore, Agarwal and Pituk also presented in [1] the following assertion comparing the growth rates of (2.16) and (2.18).

Theorem 2.14. *Suppose (2.17) holds. If y_n is a solution of (2.16) then the quantity*

$$\rho = \rho(y) = \limsup_{n\to\infty} \sqrt[n]{|y_n|}$$

is equal to the modulus of one of the characteristic zeros of (2.19).

3. Delay differential equation with constant lag

3.1. The equation $x'(t) = b\,x(t-\tau)$

3.1.1. Asymptotic stability of the differential equation

We consider the delay differential equation

$$x'(t) = b\,x(t-\tau), \qquad t > 0, \tag{3.1}$$

where b, $\tau > 0$ are real scalars. The asymptotic stability interval I^*_τ for (3.1) is defined as the set of all reals b for which any solution $x(t)$ of (3.1) tends to zero as $t \to \infty$. It is well-known (see e.g. Kolmanovskii and Myshkis [37]) that I^*_τ for (3.1) is given by

$$I^*_\tau = \left\{ b \in \mathbb{R} : 0 > b > -\frac{\pi}{2\tau} \right\},$$

which yields the necessary and sufficient condition for the asymptotic stability of (3.1). In the sequel, we present the corresponding stability sets for some discretizations of (3.1) and we show their mutual relations as well as their relations with respect to I^*_τ .

3.1.2. Discretization of the differential equation

We consider the uniform mesh of points $t_n = nh$, $n = 0, 1, \ldots$, where $h > 0$ is the stepsize satisfying

$$h = \tau/k, \qquad \text{for a suitable positive integer } k.$$

The involvement of this stepsize constraint is standardly used in discrete approximations of (3.1) with the aim to avoid an interpolation process concerning the appropriate replacement of the delayed term (see, e.g. Bellen and Zennaro [2]). We assume its validity throughout this chapter.

We start with the Θ-method discretization, which is a weighted mean of two basic discrete formulae originating from a replacement of the derivative term in (3.1) by the standard forward and backward difference operator, respectively. Then the application of the Θ-method to (3.1) yields the four-term linear difference equation

$$y_{n+1} - y_n + \beta y_{n-k+1} + \gamma y_{n-k} = 0, \qquad n = 0, 1, 2, \ldots \tag{3.2}$$

with

$$\beta = -\Theta bh, \quad \gamma = -(1-\Theta)bh, \quad k = \tau/h\,. \tag{3.3}$$

We recall that the Θ-method involves, among others, both basic Euler discretizations (the cases $\Theta = 0$ and $\Theta = 1$) as well as the trapezoidal rule discretization (the case $\Theta = 1/2$).

Another possible discretization, apart from the Θ-method, is the midpoint rule (also known as the modified Euler method). It yields the recurrence

$$y_{n+2} - y_n - 2bh\,y_{n-k+1} = 0, \qquad n = 0, 1, 2, \ldots \tag{3.4}$$

Analogously to the continuous case, by the discrete asymptotic stability interval $I_\tau^\Theta(h)$ or $I_\tau^M(h)$ we understand the set of reals b for which any solution y_n of (3.2), (3.3) or (3.4) tends to zero as $n \to \infty$, respectively.

The concept of $\tau(0)$-stability originates from an inclusion relation between I_τ^* and corresponding discrete stability intervals. More precisely, the Θ-method (3.2), (3.3) for delay differential equation (3.1) is called $\tau(0)$-stable if

$$I_\tau^* \subset \bigcap_{k=1}^{\infty} I_\tau^\Theta(h)\,, \qquad h = \tau/k\,.$$

The notion of $\tau(0)$-stability for the midpoint rule is introduced analogously.

3.1.3. Numerical stability of the Θ-methods and related issues

In this section, we provide the system of necessary and sufficient conditions defining $I_\tau^\Theta(h)$. We discuss their mutual relations as well as a relation between asymptotic stability conditions in the discrete and continuous case. The results presented in this subsection originates from [7] and [27].

Theorem 3.1. *A scalar b belongs to $I_\tau^\Theta(h)$ if and only if*

$$\begin{aligned}
&b < 0, \quad |(1-2\Theta)bh| < 2\,,\\
&\tau \arccos\left(1 - \frac{b^2h^2}{2(1+\Theta(1-\Theta)b^2h^2)}\right) < h \arccos\frac{(2\Theta-1)bh}{2}\,.
\end{aligned} \tag{3.5}$$

Proof. This theorem is a direct consequence of Corollary 2.5 with coefficients $\beta = -\Theta bh$, $\gamma = -(1-\Theta)bh$ and $k = \tau/h$. □

In particular, considering the trapezoidal rule we have

$$I_\tau^{1/2}(h) = \left\{b \in \mathbb{R}:\ -\frac{2}{h}\tan\frac{\pi h}{4\tau} < b < 0\right\}\,,$$

whereas for Euler discretizations it holds

$$I_\tau^{0}(h) = \left\{b \in \mathbb{R}:\ -\frac{2}{h}\cos\frac{\tau\pi}{2\tau+h} < b < 0\right\}$$

and

$$I_\tau^{1}(h) = \left\{b \in \mathbb{R}:\ -\frac{2}{h}\cos\frac{(\tau-h)\pi}{2\tau-h} < b < 0\right\} \tag{3.6}$$

by use of Corollary 2.6 and Corollary 2.7, respectively.

Our next aim is to investigate some basic properties of $I_\tau^\Theta(h)$. Doing so, we first consider the non-delayed (trivial) case $\tau = 0$, i.e. the equation $x'(t) = bx(t)$. Denote by I^* and $I^\Theta(h)$ intervals of exact and discretized asymptotic stability for such an equation. Both these stability intervals can be directly calculated as

$$I^* = \{b \in \mathbb{R}:\ b < 0\}, \qquad I^\Theta(h) = \{b \in \mathbb{R}:\ |1+(1-\Theta)bh| < |1-\Theta bh|\}\,.$$

This implies the following well-known properties of $I^{\Theta}(h)$ with respect to a changing parameter h and I^*: Let $h_1 > h_2 > 0$ be arbitrary. Then

$$\begin{aligned}
&I^{\Theta}(h_1) \subset I^{\Theta}(h_2) \subset I^* && \text{if } 0 \le \Theta < 1/2\,,\\
&I^{\Theta}(h_1) = I^{\Theta}(h_2) = I^* && \text{if } \Theta = 1/2\,,\\
&I^{\Theta}(h_1) \supset I^{\Theta}(h_2) \supset I^* && \text{if } 1/2 < \Theta \le 1\,.
\end{aligned}$$

In particular, the inclusion

$$I^* \subseteq I^{\Theta}(h), \qquad h > 0 \text{ is arbitrary}$$

(defining the notion of A-stability) holds if and only if $1/2 \le \Theta \le 1$.

We are going to discuss a possible validity of such monotony properties also for stability sets $I_\tau^{\Theta}(h)$. First, we investigate the behaviour of $I_\tau^{\Theta}(h)$ as $h \to 0$. Using the L'Hospital rule we can observe that $I_\tau^{\Theta}(h)$ approaches the exact asymptotic stability region I_τ^* as $h \to 0$ for any $0 \le \Theta \le 1$.

Further, considering the backward Euler discretization, we note that a monotony property of $I_\tau^{1}(h)$ has been observed by Kipnis and Levitskaya [34] experimentally, i.e. for several fixed values of parameter h. We show its general validity and, moreover, demonstrate that stability intervals $I_\tau^{1/2}(h)$ and $I_\tau^{0}(h)$ have a certain monotony property, too (see [7], [27]).

Theorem 3.2 (Proposition 4.1 and Proposition 4.3 in [7]). *Let $k_1 < k_2$ be arbitrary positive integers and let $h_1 = \tau/k_1 > \tau/k_2 = h_2$ be corresponding stepsizes. Then*

$$I_\tau^{0}(h_1) \subset I_\tau^{0}(h_2), \qquad I_\tau^{1/2}(h_1) \supset I_\tau^{1/2}(h_2), \qquad I_\tau^{1}(h_1) \supseteq I_\tau^{1}(h_2)\,.$$

Moreover, the equality sign in the last inclusion occurs if and only if $h_1 = \tau$ and $h_2 = \tau/2$.

Proof. We start with $\Theta = 0$. Let $\tilde b \in I_\tau^{0}(h_1)$ be arbitrary, but fixed. We show that $\tilde b \in I_\tau^{0}(h_2)$. On this account we introduce the function

$$\tilde f(h) = -\frac{\tilde b h}{2} - \cos\frac{\tau\pi}{2\tau + h}, \quad 0 \le h \le \tau$$

(we drop the constraint $h = \tau/k$ and we consider $\tilde f(h)$ as a function of a continuous argument h). Obviously $\tilde f(0) = 0$. We consider its derivatives

$$\tilde f'(h) = -\frac{\tilde b}{2} - \frac{\tau\pi}{(2\tau+h)^2}\sin\frac{\tau\pi}{2\tau+h}, \quad \tilde f''(h) = \frac{\tau^2\pi^2}{(2\tau+h)^4}\cos\frac{\tau\pi}{2\tau+h} + \frac{2\tau\pi}{(2\tau+h)^3}\sin\frac{\tau\pi}{2\tau+h}.$$

Since $\tilde f''(h) > 0$ for any $0 \le h \le \tau$, we can claim that $\tilde f(h_1) < 0$ implies $\tilde f(h_2) < 0$ for any $\tau \ge h_1 > h_2 > 0$. Consequently, $\tilde b \in I_\tau^{0}(h_1)$ implies $\tilde b \in I_\tau^{0}(h_2)$.

Further, we show that $I_\tau^{0}(h_1)$ is a sharp subset of $I_\tau^{0}(h_2)$. Let $\bar b < 0$ be such that

$$\frac{\bar b h_1}{2} + \cos\frac{\tau\pi}{2\tau + h_1} = 0\,,$$

i.e. $\bar b \notin I_\tau^{0}(h_1)$. Define

$$\bar f(h) = -\frac{\bar b h}{2} - \cos\frac{\tau\pi}{2\tau + h}, \qquad 0 \le h \le \tau.$$

Since $\bar{f}(0) = \bar{f}(h_1) = 0$ and $\bar{f}''(h) = \tilde{f}''(h) > 0$ for all $0 \le h \le \tau$, we get $\bar{f}(h_2) < 0$ due to $h_1 > h_2$ and this implies $\bar{b} \in I^0_\tau(h_2)$.

A similar approach can be used for $\Theta = 1$. We wish to show here that if $\tilde{b} \in I^1_\tau(h_2)$ then $\tilde{b} \in I^1_\tau(h_1)$. Define

$$\tilde{g}(h) = -\frac{\tilde{b}h}{2} - \cos\frac{(\tau - h)\pi}{2\tau - h}, \qquad 0 \le h \le \tau.$$

We have $\tilde{g}(0) = 0$ and

$$\begin{aligned}
\tilde{g}'(h) &= -\frac{\tilde{b}}{2} - \frac{\tau\pi}{(2\tau - h)^2}\sin\frac{(\tau - h)\pi}{2\tau - h}, \\
\tilde{g}''(h) &= \frac{\tau^2\pi^2}{(2\tau - h)^4}\cos\frac{(\tau - h)\pi}{2\tau - h} - \frac{2\tau\pi}{(2\tau - h)^3}\sin\frac{(\tau - h)\pi}{2\tau - h}.
\end{aligned}$$

Analysis of the sign of $\tilde{g}''(h)$ yields $\tilde{g}''(h) < 0$ for any $0 \le h \le \tau/2$ and $\tilde{g}''(\tau) > 0$. Consequently, $\tilde{g}(h_2) < 0$ implies $\tilde{g}(h_1) < 0$ for $0 < h_2 < h_1 \le \tau/2$, which shows that $\tilde{b} \in I^1_\tau(h_2)$ implies $\tilde{b} \in I^1_\tau(h_1)$. A sharp inclusion between stability domains $I^1_\tau(h_1)$ and $I^1_\tau(h_2)$ can be proved using the same line of arguments as given for $\Theta = 0$. The equality sign between $I^1_\tau(\tau)$ and $I^1_\tau(\tau/2)$ follows immediately from (3.6).

A proof of the monotony property for the trapezoidal rule (the case $\Theta = 1/2$) is analogous to the proof of the same property for a more general, neutral, equation (see Theorem 3.21). Therefore we omit its proof here. $\square$

Further, we show that a generalization of monotony properties of stability intervals $I^0_\tau(h)$, $I^{1/2}_\tau(h)$ and $I^1_\tau(h)$ to other values of Θ is not generally possible. To show that, we first make some observations about the behaviour of $I^\Theta_\tau(h)$ for h close to zero. As it has been remarked above, the discrete stability sets $I^\Theta_\tau(h)$ approach the exact stability set I^*_τ as $h \to 0$. We wish to find a relation between I^*_τ (corresponding to $h = 0$) and $I^\Theta_\tau(h)$ for h close to zero. On this account, we consider the stability condition (3.5) in the form of equality defining an implicit function $b = b(h)$ in a right neighbourhood of zero. This function essentially represents a dependence of the left endpoint of $I^\Theta_\tau(h)$ on h (the right endpoint of $I^\Theta_\tau(h)$ is always zero). Using the implicit differentiation formula and the L'Hospital rule we can observe that

$$\lim_{h\to 0^+} b'(h) = -\frac{(2\Theta - 1)\pi}{2\tau^2}\,.$$

Consequently, starting from $h = 0$ and assuming $0 \le \Theta < 1/2$, the discrete stability intervals $I^\Theta_\tau(h)$ become smaller with h increasing (but sufficiently small), i.e.

$$I^\Theta_\tau(h_1) \subset I^\Theta_\tau(h_2) \subset I^*_\tau \tag{3.7}$$

for any such Θ and any $h_1 = \tau/k_1 > \tau/k_2 = h_2$, where $k_1, k_2 \in \mathbb{Z}^+$ are sufficiently large.

On the other hand, we can easily formulate a condition on Θ guaranteeing that

$$I^*_\tau \subset I^\Theta_\tau(\tau)\,. \tag{3.8}$$

Substituting $h = \tau$ into (3.5) and considering the asymptotic stability condition for (3.1), we can arrive at the relation $\Theta > 1 - 2/\pi$ ensuring the validity of (3.8).

To summarize this, assuming $1-2/\pi < \Theta < 1/2$ and comparing (3.7) with (3.8), we cannot expect a monotony behaviour of $I_\tau^\Theta(h)$ analogous to that described in Theorem 3.2. We illustrate this phenomenon by an example characterizing a typical behaviour of discrete stability intervals $I_\tau^\Theta(h)$ for such values of Θ.

Example 3.3. *We consider the delay differential equation*

$$x'(t) = b\,x(t-1), \quad t > 0$$

and its Θ-method discretization with $\Theta = 0.4 \in (1-2/\pi, 1/2)$ and $h = 1/k$, $k \in \mathbb{Z}^+$, i.e.

$$y_{n+1} - y_n - 0.4bhy_{n-k+1} - 0.6bhy_{n-k} = 0, \qquad n = 0, 1, 2, \dots .$$

Obviously $I_1^* = (-\pi/2, 0)$. The dependence of $I_1^{0.4}(h)$ on changing h is depicted in the following table (we denote here l and r the left and right endpoints of $I_1^{0.4}(h)$, respectively).

h	0	1/1000	1/100	1/10	1/5	1/4	1/3	1/2	1
l	-π/2	-1.5706	-1.5693	-1.5582	-1.5513	-1.5499	-1.5503	-1.5612	-1.6667
r	0	0	0	0	0	0	0	0	0

Table 3.1: The stability intervals for $\Theta = 0.4$

We can observe that for increasing, but small values of h the left endpoints of $I_1^{0.4}(h)$ are decreasing (in modulus). After reaching $h = 1/4$ the situation is reversed, the left endpoints of $I_1^{0.4}(h)$ become to increase (in modulus) up to $h = 1$ when $I_1^{0.4}(1)$ even exceeds I_1^*.

Theorem 3.4. *The Θ-method applied to* (3.1) *is $\tau(0)$-stable if and only if $1/2 \le \Theta \le 1$.*

Proof. The part corresponding to $1/2 \le \Theta \le 1$ follows from conclusions made for a more general equation by Guglielmi in [20]. The part corresponding to $0 \le \Theta < 1/2$ is a direct consequence of (3.7). □

3.1.4. Numerical stability of the midpoint rule and related issues

In this section, we deal with the necessary and sufficient conditions for the asymptotic stability of the midpoint method discretization. Based on these conditions we discuss also some fundamental properties of $I_\tau^M(h)$ concerning its behaviour with respect to changing stepsize h as well as its comparisons with the asymptotic stability interval of the underlying equation I_τ^* and the forward Euler discretization $I_\tau^0(h)$. These results have been presented in [29].

Firstly, we provide the necessary and sufficient conditions describing the asymptotic stability interval $I_\tau^M(h)$.

Theorem 3.5 (Corollary 4 in [29])**.** *A scalar b belongs to $I_\tau^M(h)$ if and only if*

$$k \text{ is odd}, \qquad 0 > b > -\frac{1}{h}\sin\frac{\pi h}{2\tau}.$$

Proof. The proof discusses stability conditions of Corollary 2.9 with parameter $\nu = -2bh$. Let us note, that equation (2.4) is of the order $k+2$, while equation (3.4) is of the order $k+1$. Taking this into account, the first condition of Corollary 2.9 can be read as k is odd and the other two conditions easily implies the inequalities constraining b. □

Theorem 3.6. *The midpoint method applied to* (3.1) *is not* $\tau(0)$*-stable.*

Proof. Since (3.4) is asymptotically stable only for k odd, the condition for $\tau(0)$-stability is not satisfied. □

The behaviour of asymptotic stability regions as $h \to 0$ belongs among the basic properties of numerical methods. Similarly to the Θ-methods, the asymptotic stability interval of the midpoint rule discretization $I_\tau^M(h)$ is approaching I_τ^* as $h \to 0$, which can be verified by the use of the L'Hospital rule.

Being motivated by obtained results in the case of Θ-methods, we investigated also the monotony property of $I_\tau^M(h)$ with respect to changing stepsize h.

Theorem 3.7 (Theorem 5 in [29])**.** *Let* $3 \le k_1 < k_2$ *be arbitrary positive odd integers and let* $h_1 = \tau/k_1 > \tau/k_2 = h_2$ *be corresponding stepsizes. Then*

$$I_\tau^M(h_2) \supset I_\tau^M(h_1).$$

Proof. Likewise in the proof of Theorem 3.2, we investigate the dependence of the left endpoint of $I_\tau^M(h)$ on stepsize h. We define a function

$$f(h) = -\frac{1}{h}\sin\frac{\pi h}{2\tau}, \qquad 0 < h \le \tau/3. \tag{3.9}$$

We assume here that $f(h)$ is a function of a continuous argument h. Then

$$f'(h) = \frac{1}{h^2}\sin\frac{\pi h}{2\tau} - \frac{\pi}{2\tau h}\cos\frac{\pi h}{2\tau}, \qquad 0 < h \le \tau/3.$$

Since $\sin\frac{\pi h}{2\tau} > 0$ for $0 < h \le \tau/3$, then $f'(h) > 0$ if

$$\cot\frac{\pi h}{2\tau} < \frac{2\tau}{\pi h}, \qquad 0 < h \le \tau/3.$$

If we substitute $s = \frac{\pi h}{2\tau}$, the last relation becomes

$$\tan s > s, \qquad 0 < s \le \tau/6. \tag{3.10}$$

Obviously $\tan(0) = 0$ and $(\tan s)' = \cos^{-2} s > 1 = s'$ for $0 < s \le \tau/6$. Therefore (3.10) holds for any $0 < s \le \tau/6$. Thus, we have proved that $f'(h) > 0$ for $0 < h \le \tau/3$ and consequently $I_\tau^M(h_2) \supset I_\tau^M(h_1)$. □

Remark 3.8. We have shown that $-\frac{1}{h}\sin\frac{\pi h}{2\tau}$ is an increasing function for $0 < h \le \tau/3$ (see the proof of Theorem 3.7). Considering also its limit property as $h \to 0$, we conclude that

$$I_\tau^* \supset I_\tau^M(h)$$

for any $h = \tau/k$, where k is odd.

Finally, we discuss a relation between $I_\tau^M(h)$ and the asymptotic stability intervals for the forward Euler discretization of (3.1).

Theorem 3.9 (Theorem 8 in [29]). *Let $k \geq 3$ be an arbitrary positive odd integer and let $h = \tau/k$ be the corresponding stepsize. Then*

$$I_\tau^M(h) \supset I_\tau^0(h).$$

Proof. Since the right endpoints of $I_\tau^M(h)$ and $I_\tau^0(h)$ are zero for any h, we are interested only in the behaviour of the left endpoints with respect to changing stepsize h. We define a function

$$g(h) = -\frac{2}{h}\cos\frac{\tau\pi}{2\tau + h}, \qquad 0 < h \leq \tau,$$

which expresses the dependence of the left endpoint of $I_\tau^0(h)$ on h. We recall that we describe the analogy for $I_\tau^M(h)$ via $f(h)$ defined by (3.9). In the further analysis, we drop the constraint $h = \tau/k$ and consider both functions $f(h)$ and $g(h)$ to be functions of a continuous argument for $0 < h \leq \tau$ (we extend the domain of $f(h)$ to simplify the proof). Our aim is to show that $f(h) - g(h) < 0$ for any $0 < h < \tau$, i.e.

$$-\sin\frac{\pi h}{2\tau} + 2\cos\frac{\pi\tau}{2\tau + h} < 0, \qquad 0 < h < \tau. \tag{3.11}$$

To do this, we introduce next proposition:

Lemma. Let $F \in C^3(\langle a, b\rangle)$ be a function such that $F(a) = F(b) = 0$, $F'(a) \leq 0$, $F'(b) > 0$, $F''(a) < 0$, $F''(b) > 0$ and $F'''(t) > 0$ for all $a \leq t \leq b$. Then $F(t) < 0$ for all $a < t < b$.

Proof. Since $F'''(t) > 0$ for all $a \leq t \leq b$, the function $F''(t)$ is increasing. Since $F''(a) < 0 < F''(b)$, there is a unique point $t_1 \in (a, b)$ such that $F''(t_1) = 0$. Thus, the function $F'(t)$ is decreasing in (a, t_1) and increasing in (t_1, b). Further, since $F'(t_1) < F'(a) \leq 0$ and $F'(b) > 0$, there is a unique point $t_2 \in (a, b)$ such that $F'(t_2) = 0$. Therefore, $F(t)$ is decreasing in (a, t_2) and increasing in (t_2, b). Taking into account $F(a) = F(b) = 0$, we obtain that $F(t) < 0$ for all $a < t < b$. □

Next, we denote $s = 2 + h/\tau$. Then we define

$$\tilde{f}(s) = \sin\frac{\pi s}{2} + 2\cos\frac{\pi}{s}, \qquad 2 < s < 3,$$

which is equivalent to the left-hand side of (3.11). It holds that

$$\tilde{f}(2) = \tilde{f}(3) = 0, \qquad \tilde{f}'(2) = 0, \qquad \tilde{f}'(3) = \frac{\sqrt{3}\pi}{9} > 0,$$

$$\tilde{f}''(2) = -\frac{\pi}{2} < 0, \qquad \tilde{f}''(3) = \frac{\pi^2}{4} - \frac{\pi^2}{81} - 2\sqrt{3}\frac{\pi}{27} > 0.$$

Further,

$$\tilde{f}'''(s) = \frac{12\pi^2}{s^5}\cos\frac{\pi}{s} - \frac{\pi^3}{8}\cos\frac{\pi s}{2} + \frac{2\pi}{s^6}(6s^2 - \pi^2)\sin\frac{\pi}{s} > 0,$$

because each term in the sum is non-negative for all $s \in \langle 2, 3\rangle$. Then, by the previous lemma, we have that $\tilde{f}(s) < 0$ for all $2 < s < 3$ and consequently $f(h) < g(h)$ for $0 < h < \tau$ which concludes the proof. □

3.2. The equation $x'(t) = a\,x(t) + b\,x(t-\tau)$

3.2.1. Asymptotic stability of the differential equation

We consider the delay differential equation

$$x'(t) = a\,x(t) + b\,x(t-\tau), \qquad t > 0, \tag{3.12}$$

where a, b, $\tau > 0$ are real scalars. The asymptotic stability region S_τ^* for (3.12) is then defined as the set of all real couples (a, b) for which any solution $x(t)$ of (3.12) tends to zero as $t \to \infty$. The description of S_τ^* for (3.12) is known either in the form of the line $a + b = 0$ and a parametric curve

$$a = \Phi\cot(\tau\Phi), \qquad b = -\frac{\Phi}{\sin(\tau\Phi)}, \qquad \Phi \in (0, \pi/\tau)$$

defining the stability boundary (see e.g. Kolmanovskii and Myshkis [37]), or directly in the form of explicit conditions on a, b and τ:

$$a \le b < -a \tag{3.13}$$

and

$$|a| + b < 0, \qquad \tau < \frac{\arccos(-a/b)}{(b^2 - a^2)^{1/2}} \tag{3.14}$$

(see e.g. Hayes [25]). Note that (3.13) holds for all positive values of τ. Hence, this condition forms a delay-independent stability region. Contrarily, the condition (3.14) contains a restriction on τ and therefore it defines the so-called delay-dependent stability region.

The asymptotic stability region is depicted in Figure 3.1 to the left from the blue lines. The dashed line divides the delay dependent and independent parts.

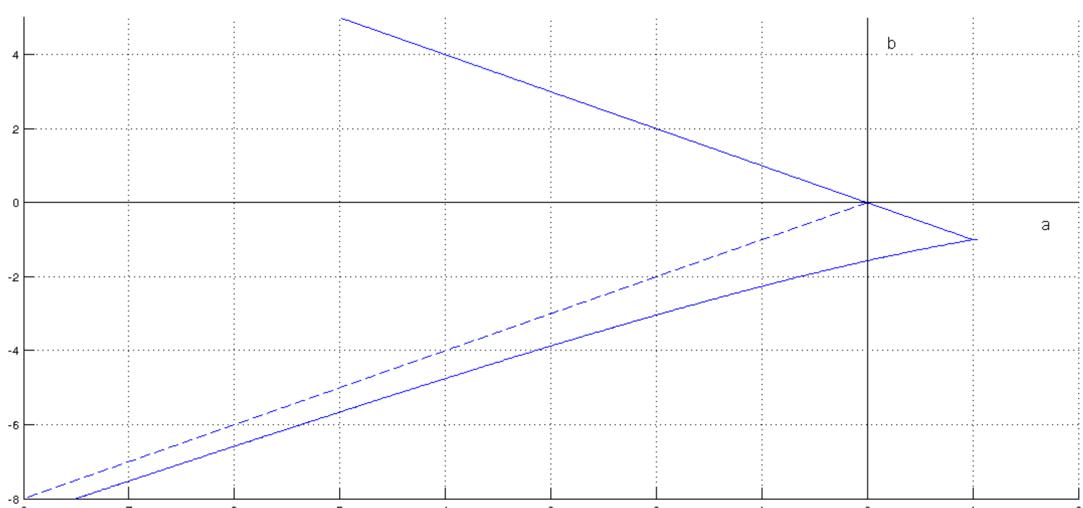

Figure 3.1: The asymptotic stability region S_1^*

3.2.2. Discretization of the differential equation

Similarly to the one-term equation, we consider the Θ-method and the modified midpoint rule to be a tool of discretization.

The Θ-method applied to (3.12) yields the four-term linear difference equation

$$y_{n+1} + \alpha y_n + \beta y_{n-k+1} + \gamma y_{n-k} = 0, \qquad n = 0, 1, 2, \dots , \tag{3.15}$$

with

$$\alpha = -\frac{1 + (1-\Theta)ah}{1 - \Theta ah}, \quad \beta = -\frac{\Theta bh}{1 - \Theta ah}, \quad \gamma = -\frac{(1-\Theta)bh}{1 - \Theta ah}, \quad k = \tau/h \tag{3.16}$$

and the stepsize h satisfying $\Theta ah \neq 1$.

We introduce also the modified midpoint rule, which is another possible discretization of (3.12). The numerical formula is derived by integration over two steps, where the integrals of terms on the right-hand side of (3.12) are approximated via the trapezoidal rule and the midpoint rule, respectively. Moreover, we assume the equidistant mesh with the stepsize $h = \tau/k$, where $k \geq 2$, $k \in \mathbb{Z}^+$. Then, by the application of the modified midpoint rule on (3.12) we obtain the three-term linear difference equation

$$y_{n+2} + \mu y_n + \nu y_{n-k+1} = 0, \qquad n = 0, 1, 2, \dots, \tag{3.17}$$

where

$$\mu = -\frac{1 + ah}{1 - ah}, \qquad \nu = -\frac{2bh}{1 - ah}, \qquad k = \tau/h\,. \tag{3.18}$$

We assume $ah \neq 1$.

By the asymptotic stability region $S_\tau^\Theta(h)$ of the Θ-method discretization of (3.12) we understand the set of real couples (a, b) for which any solution y_n of (3.15), (3.16) tends to zero as $n \to \infty$. Likewise, the asymptotic stability region $S_\tau^M(h)$ of the midpoint rule discretization is formed by all real couples (a, b) for which any solution y_n of (3.17), (3.18) tends to zero as $n \to \infty$.

We say that the Θ-method for (3.12) is $\tau(0)$-stable if it satisfies

$$S_\tau^* \subset \bigcap_{k=1}^{\infty} S_\tau^\Theta(h)\,, \qquad h = \tau/k\,.$$

Analogously, the modified midpoint rule for (3.12) is $\tau(0)$-stable if

$$S_\tau^* \subset \bigcap_{k=2}^{\infty} S_\tau^M(h)\,, \qquad h = \tau/k\,.$$

3.2.3. Numerical stability of the Θ-methods and related issues

First, we state the basic property concerning the stability of Θ-method applied to (3.12).

Theorem 3.10. *The Θ-method applied to* (3.12) *is* $\tau(0)$*-stable if and only if* $1/2 \leq \Theta \leq 1$.

This theorem was proved by Guglielmi in [20] using the boundary locus technique. Such approach leads to a description of $S_\tau^\Theta(h)$ via parametric curves defining its boundary. Utilizing Theorem 2.3 we derive an explicit description of $S_\tau^\Theta(h)$, i.e. a discrete counterpart to (3.13), (3.14). Based on this description, we present also some important properties of Θ-methods extending the $\tau(0)$-stability property. The results in this subsection originate from [7].

First, we introduce the symbols for given a, b and Θ

$$\begin{aligned}\bar{\tau}_1^\Theta(h) =& h\arctan\left(\frac{(b+a)(2+(1-2\Theta)(a+b)h)}{(b-a)(2+(1-2\Theta)(a-b)h)}\right)^{1/2} \Bigg/ \\ & \arctan h\left(\frac{b^2-a^2}{(2+(1-2\Theta)(a+b)h)(2+(1-2\Theta)(a-b)h)}\right)^{1/2}\end{aligned}$$

and

$$\begin{aligned}\bar{\tau}_2^\Theta(h) =& h\arctan\left(\frac{(b+a)(2+(1-2\Theta)(a+b)h)}{(b-a)(2+(1-2\Theta)(a-b)h)}\right)^{(-1)^k/2} \Bigg/ \\ & \operatorname{arccot}\left(\frac{(b^2-a^2)h^2}{(2+(1-2\Theta)(a+b)h)(2+(1-2\Theta)(a-b)h)}\right)^{1/2}.\end{aligned}$$

Then we can formulate the following conditions.

Theorem 3.11 (Theorem 3.1 and Theorem 3.2 in [7])**.** (a) *Let $0 \le \Theta < \frac{1}{2}$. Then a real couple (a,b) belongs to $S_\tau^\Theta(h)$ if and only if one of the following conditions holds:*

$$\begin{array}{lll}
|b|+a<0, & 2+(1-2\Theta)(a-|b|)h>0\,; & \\
a=b<0, & 1+(1-2\Theta)ah>0\,; & \\
a<(-1)^{k+1}b<0, & 2+(1-2\Theta)ah=(-1)^k(1-2\Theta)bh\,; & \\
|a|+b<0, & 2+(1-2\Theta)(a+b)h>0, & \tau<\bar{\tau}_1^\Theta(h)\,; \\
a+(-1)^k b<0, & |2+(1-2\Theta)ah|<(-1)^k(1-2\Theta)bh, & \tau<\bar{\tau}_2^\Theta(h)\,.
\end{array}$$

(b) *Let $\Theta = \frac{1}{2}$. Then a real couple (a,b) belongs to $S_\tau^\Theta(h)$ if and only if either*

$$a \le b < -a \tag{3.19}$$

or

$$|a|+b<0, \qquad \tau\arctan\left(\frac{h}{2}(b^2-a^2)^{1/2}\right) < \frac{h}{2}\arccos\frac{a}{|b|}\,. \tag{3.20}$$

(c) *Let $\frac{1}{2} < \Theta \le 1$. Then a real couple (a,b) belongs to $S_\tau^\Theta(h)$ if and only if one of the following conditions holds:*

$$\begin{array}{lll}
a\le b<-a\,; & & \\
|(2\Theta-1)bh|<(2\Theta-1)ah-2\,; & & \\
(-1)^k(2\Theta-1)bh=(2\Theta-1)ah-2\,; & & \\
|a|+b<0, & 2+(2\Theta-1)(b-a)h>0, & \tau<\bar{\tau}_1^\Theta(h)\,; \\
a+(-1)^{k+1}b>0, & |2-(2\Theta-1)ah|<(-1)^k(2\Theta-1)bh, & \tau<\bar{\tau}_2^\Theta(h)\,.
\end{array}$$

Proof. A technique of the proof is analogous to the technique used in the proof of Theorem 3.17, hence we leave it out here. We provide here only the proof of the case $\Theta = 1/2$, because the coefficient β and γ coincides for the trapezoidal rule discretization of (3.12) and therefore it is enough to reformulate conditions (2.6)-(2.8) of Corollary 2.4 with respect to (3.16).

If $2 - ah > 0$ then some simple calculations imply that the condition (2.6) can be converted into $|b| < -a$. If $2 - ah < 0$, then the latter relation of (2.6) cannot occur, hence $|b| < -a$ is an equivalent expression of (2.6). Analogously, the substitution of (3.16) into (2.7) leads to $a = b < 0$, which along with $|b| < -a$ defines the delay independent stability region of $S_\tau^{1/2}(h)$ via the condition (3.19).

Now we consider (2.8). Substituting (3.16) into (2.8) we get that the first two inequalities of (2.8) can be read as $|a| + b < 0$, $2 - ah > 0$. Furthermore, the last relation of (2.8) is convertible into the form

$$\tau \arccos \frac{4 - (b^2 - a^2)h^2}{4 + (b^2 - a^2)h^2} < h \arccos \frac{a}{|b|} .$$

Using the formula

$$2 \arctan s = \arccos \frac{1 - s^2}{1 + s^2} = 2 \operatorname{arccot} \frac{1}{s}, \qquad s > 0,$$

we can simplify this delay restriction to the form $(3.20)_2$ which, along with $|a| + b < 0$ implies $2 - ah > 0$. Thus, we have fully described the stability region for (3.15), (3.16) and $\Theta = 1/2$ in terms of the conditions (3.19), (3.20). □

Similarly to the case $a = 0$ discussed in the previous section, the asymptotic stability region of the discretization $S_\tau^\Theta(h)$ approaches the exact asymptotic stability region S_τ^* as $h \to 0$. This can be verified by investigation of the limit form of Theorem 3.11.

Further, we focus on the trapezoidal rule discretization of (3.12) which is the only Θ-method of the order 2. Note that its description is also the only one invariant to the parity of k. We confirm that the trapezoidal rule keeps the inclusion property of stability regions, which was observed also for purely delayed equation (see Theorem 3.2).

Theorem 3.12 (Proposition 4.1 in [7]). *Let $k_1 < k_2$ be arbitrary positive integers and let $h_1 = \tau/k_1 > \tau/k_2 = h_2$ be corresponding stepsizes. Then*

$$S_\tau^{1/2}(h_1) \supset S_\tau^{1/2}(h_2) \supset S_\tau^*.$$

Proof. We refer to the proof of Theorem 3.21, where this property is derived for a more general equation and can be easily simplified to obtain this result. □

The monotony property for a delay $\tau = 1$ is depicted in Figure 3.2 . To the left from the blue curves there is the asymptotic stability region for the differential equation (the dashed line divides the delay-dependent and independent parts). The red curve (together with the line $a + b = 0$) is defining the stability area $S_1^{1/2}(1)$, while the green one is a boundary of $S_1^{1/2}(1/2)$.

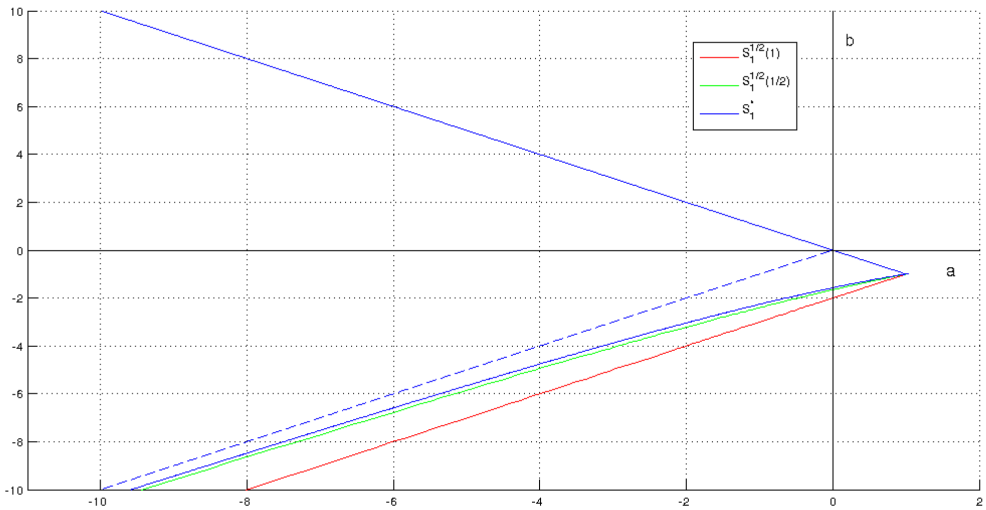

Figure 3.2: The asymptotic stability regions $S_1^{1/2}(h)$

3.2.4. Numerical stability of the modified midpoint rule and related issues

In this section, we provide a set of necessary and sufficient conditions guaranteeing the asymptotic stability of (3.17), (3.18) as they were derived in [29]. The analysis of (3.17), (3.18) falls naturally into two parts according to the parity of k. For an effective and clear formulation of the main result, we introduce the symbols

$$\bar{\tau}_1^M(h) = h + 2h \arcsin \frac{a + b^2 h}{(1 + ah)|b|} \Big/ \arccos \frac{1 + a^2h^2 - 2b^2h^2}{a^2h^2 - 1},$$
$$\bar{\tau}_2^M(h) = h + 2h \arccos \frac{a + b^2 h}{|(1 + ah)b|} \Big/ \arccos \frac{1 + a^2h^2 - 2b^2h^2}{|a^2h^2 - 1|},$$

which are utilized in these two parts, respectively.

Theorem 3.13 (Theorem 3 in [29])**.** (a) *Let $k \geq 2$ be even. Then a real couple (a, b) belongs to $S_\tau^M(h)$ if and only if one of the following conditions holds:*

$$|bh| \leq 1, \qquad |b| + a < 0; \tag{3.21}$$
$$2 < 2b^2h^2 < 1 - ah, \qquad \tau < \bar{\tau}_1^M(h). \tag{3.22}$$

(b) *Let $k \geq 3$ be odd and $\ell = (k-1)/2$. Then a real couple (a, b) belongs to $S_\tau^M(h)$ if and only if one of the following conditions holds:*

$$a \leq b < -a, \qquad |bh| < 1; \tag{3.23}$$
$$|b| + a < 0, \qquad (-1)^\ell bh = 1; \tag{3.24}$$
$$b + |a| < 0, \qquad bh > -1, \qquad \tau < \bar{\tau}_2^M(h); \tag{3.25}$$
$$(-1)^\ell b + a < 0, \qquad (-1)^\ell bh > 1, \qquad \tau < \bar{\tau}_2^M(h); \tag{3.26}$$
$$(-1)^\ell b + a > 0, \qquad (-1)^{\ell+1} bh > 1, \qquad \tau < \bar{\tau}_2^M(h). \tag{3.27}$$

Proof. The proof is based on application of Theorem 2.8 to (3.17), (3.18) and the ensuing analysis of the obtained conditions. We point out that the equation (2.4) is of the order $k+2$, while equation (3.17) is of the order $k+1$.

Case (a): Investigating the case k even, we utilize parts (c) and (d) of Theorem 2.8. Firstly, we focus on the condition (c): Considering the coefficients (3.18), the assumption $\mu < 0$ implies $|ah| < 1$. Thus, (2.12) is equivalent to $|b| + a < 0$. Therefore, condition (c) coincides with $|bh| < -ah < 1$.

Now, we analyse the condition (d): The assumption $\mu > 0$ implies $|ah| > 1$. Hence, (2.13) gives $|bh| \le 1$ providing $ah < -1$, while relation (2.13) cannot occur for the case $ah > 1$. We now turn to (2.15). Relation $(2.15)_1$ can be read as $2 < 2b^2h^2 < 1 - ah$. Furthermore, the restriction $(2.15)_2$ becomes

$$k - 1 < 2\arcsin\frac{-a-b^2h}{|(1+ah)b|}\Big/\arccos\frac{1+a^2h^2-2b^2h^2}{|a^2h^2-1|}.$$

Since $|ah| > 1$ and $k = \tau/h$, it can be written as $\tau < \bar{\tau}_1^M(h)$. Therefore, condition (d) is satisfied if and only if either $|bh| \le 1$, $ah < -1$ or (3.22).

Finally, Theorem 2.8 does not cover the case of $\mu\nu = 0$ (i.e. $ah = -1$ or $b = 0$). In our case, we do not consider the eventuality $b = 0$ because we deal with discretization of (3.12). Accordingly, for $ah = -1$ equation (3.17), (3.18) turns to

$$y_{n+1} - bhy_{n-k+1} = 0, \qquad n = 0, 1, \ldots$$

and the necessary and sufficient condition for its asymptotic stability is $|bh| < 1$. Summarizing above discussion we conclude that if k is even, then (3.17), (3.18) is asymptotically stable if either (3.21) or (3.22) holds.

Case (b): For k odd we consider the conditions (a) and (b) of Theorem 2.8. Condition (a) can be rewritten as

$$\frac{-2bh}{1-ah}\left(\frac{1+ah}{1-ah}\right)^{1+\ell} < 0, \qquad \left|\frac{1+ah}{1-ah}\right| + \left|\frac{2bh}{1-ah}\right| < 1 \tag{3.28}$$

by use of (3.18). With respect to the parity of power in the first relation we obtain a set of conditions equivalent to (3.28) as

$$\begin{aligned} &|ah| < 1, \qquad b > 0, \qquad a < -b; \\ &ah < -1, \qquad b > 0, \qquad bh < 1 \end{aligned} \tag{3.29}$$

for ℓ odd and (3.29),

$$ah < -1, \qquad b < 0, \qquad -bh < 1$$

for ℓ even.

In the case (b) of Theorem 2.8 condition (2.13) can be reformulated as

$$\frac{-2bh}{1-ah}\left(\frac{1+ah}{1-ah}\right)^{1+\ell} > 0, \qquad \left|\frac{1+ah}{1-ah}\right| + \left|\frac{2bh}{1-ah}\right| \le 1. \tag{3.30}$$

Analogous analysis as above shows that for ℓ odd (3.30) is equivalent to

$$\begin{aligned} &|ah| < 1, \qquad b < 0, \qquad a \le b; \\ &ah < -1, \qquad b < 0, \qquad -bh \le 1. \end{aligned} \tag{3.31}$$

In the case ℓ even condition (3.30) is satisfied if and only if (3.31) or

$$ah < -1, \qquad b > 0, \qquad bh \le 1$$

holds. The above discussion including the case $\alpha = 0$ (i.e. $ah = -1$, $|bh| < 0$, see *Case (a)*) can be jointly written as (3.23)–(3.24).

Now it remains to analyse the condition (2.14) adapted for equation (3.17), (3.18), i.e.

$$\left| \left| \frac{1+ah}{1-ah} \right| - \left| \frac{2bh}{1-ah} \right| \right| < 1 < \left| \frac{1+ah}{1-ah} \right| + \left| \frac{2bh}{1-ah} \right|, \qquad \tau < \bar{\tau}_2^M(h)$$

under the assumption $\frac{-2bh}{1-ah} \left(\frac{1+ah}{1-ah} \right)^{1+\ell} > 0$. In the same manner as above we get the equivalence to the following set of conditions

$$\begin{array}{llllll}
|ah| < 1, & b < 0, & 1+ah \le -2bh, & b < a, & bh > -1, & \tau < \bar{\tau}_2^M(h); \qquad (3.32)\\
|ah| < 1, & b < 0, & 1+ah > -2bh, & b < -|a|, & \tau < \bar{\tau}_2^M(h; & \qquad (3.33)\\
ah > 1, & b > 0, & 1+ah \le 2bh, & b < a, & \tau < \bar{\tau}_2^M(h); & \qquad (3.34)\\
ah > 1, & b > 0, & 1+ah > 2bh, & bh > 1, & \tau < \bar{\tau}_2^M(h); & \qquad (3.35)\\
ah < -1, & b < 0, & 1+ah < 2bh, & bh < -1, & \tau < \bar{\tau}_2^M(h); & \qquad (3.36)\\
ah < -1, & b < 0, & 1+ah \ge 2bh, & bh < -1, & a < b, & \tau < \bar{\tau}_2^M(h) \qquad (3.37)
\end{array}$$

for ℓ odd and (3.32), (3.33),

$$\begin{array}{llllll}
ah > 1, & b < 0, & 1+ah \le -2bh, & b > -a, & \tau < \bar{\tau}_2^M(h); & \qquad (3.38)\\
ah > 1, & b < 0, & 1+ah > -2bh, & bh < -1, & \tau < \bar{\tau}_2^M(h); & \qquad (3.39)\\
ah < -1, & b > 0, & 1+ah < -2bh, & bh > 1, & \tau < \bar{\tau}_2^M(h); & \qquad (3.40)\\
ah < -1, & b > 0, & 1+ah \ge -2bh, & bh > 1, & b < -a, & \tau < \bar{\tau}_2^M(h) \qquad (3.41)
\end{array}$$

for ℓ even. These conditions are jointly expressed by (3.25)–(3.27). In fact, (3.25) coincides with (3.32), (3.33). Condition (3.26) is equivalent to (3.36), (3.37) and (3.40), (3.41) for ℓ odd and ℓ even, respectively. Finally, (3.27) is the same as (3.34), (3.35) for ℓ odd and (3.38), (3.39) for ℓ even. The proof is complete. $\square$

Concerning the relation between the exact and discretized stability region, we investigate a limit form of Theorem 3.13 as $h \to 0$. For k even, the asymptotic stability region for (3.17), (3.18) becomes $|b| + a < 0$. Let us note that, with the exception of the boundary, this region corresponds to (3.13), i.e. the delay-independent stability region of S_τ^*. Considering $k \ge 3$ odd, it may be shown (by the L'Hospital rule) that the asymptotic stability conditions turn into

$$\begin{gathered}
a \le b < -a, \\
|a| + b < 0, \qquad \tau < \arccos(-a/b)/(b^2 - a^2)^{1/2}\,.
\end{gathered}$$

These relations are equivalent to the conditions defining S_τ^*, i.e. (3.13), (3.14).

Theorem 3.14. *The midpoint method applied to* (3.12) *is not* $\tau(0)$*-stable.*

Proof. It follows from the above mentioned limit property for k even, that there exists sufficiently small h such that $S_\tau^* \supset S_\tau^M(h)$. Hence, the method is not $\tau(0)$-stable. $\square$

3.3. The equation $x'(t) = a\,x(t) + b\,x(t-\tau) + c\,x'(t-\tau)$

Finally, we deal with the neutral delay differential equation. Our main interest is the $N\tau(0)$-stability of its Θ-method discretization (a precise specification of this notion will be introduced later). It is known, that the Θ-method discretization is not $N\tau(0)$-stable for $0 \le \Theta < 1/2$ and $1/2 < \Theta \le 1$. The $N\tau(0)$-stability of the trapezoidal rule discretization (the case $\Theta = 1/2$) has not been sufficiently clarified in the existing literature (see e.g. Guglielmi [21]), especially with respect to the so-called asymptotically critical case $|c| = 1$. Therefore, in this section we provide the necessary and sufficient conditions for the asymptotic stability of the differential equation as well as for its discretization. By their comparison, we resolve the question of the $N\tau(0)$-stability for the trapezoidal rule and we mention some other consequences following from these conditions. These results have been presented in [8].

3.3.1. Asymptotic stability of the differential equation

We consider the neutral delay differential equation

$$x'(t) = a\,x(t) + b\,x(t-\tau) + c\,x'(t-\tau), \qquad t > 0\,, \tag{3.42}$$

where a, b, c and $\tau > 0$ are real scalars. The asymptotic stability region Σ^*_τ for (3.42) is then defined as the set of all real triplets (a, b, c) for which any solution $x(t)$ of (3.42) tends to zero as $t \to \infty$.

The standard way how to describe the asymptotic stability region for linear autonomous functional differential equations consists in analysis of zeros of the corresponding characteristic quasi-polynomial. In the case of (3.42) this quasi-polynomial becomes

$$P(\lambda) \equiv \lambda - a - b\mathrm{e}^{-\lambda\tau} - c\lambda\mathrm{e}^{-\lambda\tau}\,. \tag{3.43}$$

This analysis was used in several papers to obtain asymptotic stability conditions for (3.42) in the pure delayed case ($c = 0$ - see the results of Section 3.1.1 and 3.2.1) as well as in the neutral case ($c \neq 0$). While the case $|c| > 1$ easily implies instability of (3.42), the case $|c| < 1$ is closely related to stability investigations of (3.12). It is well-known that the asymptotic stability property for (3.12) can be equivalently expressed as

$$\Re(\lambda) \le \delta < 0 \qquad \text{for a real scalar } \delta \text{ and any zero } \lambda \text{ of (3.43)}\,. \tag{3.44}$$

We recall, that using this fact, descriptions of the exact stability region for (3.12) are known either in the form of parametric curves defining the stability boundary or directly in the form of explicit conditions on a, b and τ (for their precise formulation see Section 3.2.1). For other types of stability conditions we refer to the recent paper by Huang [30]. The description of the stability boundary via parametric curves is convenient especially for two-parameter equations, because it enables to depict the stability picture in the plane of these parameters. However, considering a multi-parameter equation, an explicit system of conditions seems to be more useful. In this section, we utilize such a system.

The case $|c| = 1$ turns out to be the most problematic in stability analysis of (3.42) (sometimes it is called the asymptotically critical case). More precisely, if $|c| = 1$ and $a + |b| < 0$ then all the zeros of (3.43) have negative real parts (see e.g. Freedman and Kuang [18]), but as observed by Gromova in [19], there always exists a sequence of zeros of (3.43) tending to the imaginary axis. Consequently, the condition (3.44), required in asymptotic stability analysis of the pure delayed case, is not satisfied. On this account, some authors involve into the exact stability set Σ_τ^* only cases corresponding to the condition $|c| < 1$, regarding it as the necessary condition for the asymptotic stability of (3.42) (see, e.g. Theorem 7.7.1 of [39] and its proof). However, such a description of Σ_τ^* is not precise. Based on the paper by Freedman and Kuang [18], the stability set Σ_τ^* can be described via the following necessary and sufficient conditions for the asymptotic stability of (3.42).

Theorem 3.15 (Theorem 2.1 in [8]). *A triplet (a, b, c) belongs to Σ_τ^* if and only if either*

$$a \le b < -a, \qquad |c| < 1\,, \tag{3.45}$$

or

$$a + |b| < 0, \qquad |c| = 1\,, \tag{3.46}$$

or

$$|a| + b < 0, \qquad |c| < 1, \qquad \tau < \tau^*\,, \tag{3.47}$$

where

$$\tau^* = \left(\arccos \frac{a - bc}{ac - b}\right) \Big/ \left(\frac{b^2 - a^2}{1 - c^2}\right)^{\frac{1}{2}}\,. \tag{3.48}$$

Proof. The prevailing part of this assertion is covered by Theorem 3.1 of [18] and consecutive discussions. It remains to dispose with some subcases corresponding to the asymptotically critical case $|c| = 1$, which are in [18] either omitted ($a > -b \operatorname{sgn} c$ and $a = b \operatorname{sgn} c < 0$), or their proof seems to be incomplete ($a + |b| < 0$).

First we analyse the subcases $a > -b \operatorname{sgn} c$ and $a = b \operatorname{sgn} c < 0$. Since $P(0) = -a - b$ and $P(\lambda) \to \infty$ as $\lambda \to \infty$, the polynomial (3.43) has a positive real zero if $a + b > 0$, which immediately yields instability of (3.42) in such a case. Consequently, regarding the case $c = 1$, it is enough to prove that if $a = b < 0$ then (3.42) is not asymptotically stable. Doing this, we show that (3.43) has a purely imaginary zero $\lambda = \mathrm{i}v$ whenever $a = b < 0$. Substituting this into (3.43) we get

$$\mathrm{i}v - a - a\mathrm{e}^{-\mathrm{i}v\tau} - \mathrm{i}v\mathrm{e}^{-\mathrm{i}v\tau} = 0\,.$$

Equating real and imaginary parts we obtain two equations for a and v. Both these equations are equivalent to

$$a = -v \tan(v\tau/2)\,. \tag{3.49}$$

Since the right-hand side of (3.49) is a continuous bijection of $(0, \pi/\tau)$ onto $(-\infty, 0)$, we have that for any $a < 0$ there exists $v \in (0, \pi/\tau)$ such that (3.49) holds. In other words, for any $a = b < 0$ the polynomial (3.43) has a purely imaginary zero, hence (3.42) is not asymptotically stable.

Further let $c = -1$. Using the same line of arguments as given above one can show that (3.42) is not asymptotically stable in the cases $a + b > 0$ and $a + b = 0$. It remains to discuss the case $|a| + b < 0$. It is easy to check that, under the conditions $c = -1$ and $|a| + b < 0$, (3.43) has a (unique) negative zero. We show that all imaginary zeros of (3.43) have non-negative real parts. Put $c = -1$ in (3.43) and rewrite the corresponding characteristic equation as

$$1 - \frac{\lambda(1 + \mathrm{e}^{-\lambda\tau})}{a + b\mathrm{e}^{-\lambda\tau}} = 0. \tag{3.50}$$

Substituting $\lambda = u + \mathrm{i}v$, $v \neq 0$ into (3.50) and separating the imaginary part we arrive at

$$v(b\mathrm{e}^{-\tau u} + a\mathrm{e}^{\tau u}) + (b-a)u\sin(\tau v) + (b+a)v\cos(\tau v) = 0\,. \tag{3.51}$$

For arbitrary fixed $v \neq 0$, we consider the left-hand side of (3.51) as a function of variable u and denote it as $f_v(u)$. Let $v > 0$. Then

$$f_v(0) = v(b+a)(1+\cos(\tau v)) \leq 0$$

and

$$\frac{\mathrm{d}}{\mathrm{d}u} f_v(u) = \tau v(-b\mathrm{e}^{-\tau u} + a\mathrm{e}^{\tau u}) + (b-a)\sin(\tau v) > \tau v(-b)\left(\mathrm{e}^{-\tau u} + \mathrm{e}^{\tau u} - 2\right) > 0$$

for any $u < 0$. Consequently, the condition (3.51) is not satisfied for any $u < 0$. Analogously we can dispose with the sign variant $v < 0$. Summarizing this, the stability quasi-polynomial (3.43) has imaginary zeros with non-negative real parts, hence (3.42) is not asymptotically stable.

Finally, we are going to discuss the asymptotically critical case $a+|b| < 0$, $|c| = 1$. The proof procedure performed in [18] revealed that zeros of (3.43) have negative real parts for all $\tau > 0$ provided (3.46) holds. However, it is shown by Snow [53] that even if all the zeros of characteristic polynomials of linear autonomous neutral differential equations may have negative real parts, it is still possible to observe instability of such equations. In particular, this phenomenon may occur when (3.44) is not true. We have already noted in the introductory part of this section that it is just the case (3.46), when all the zeros of (3.43) lie in the left half plane, but are asymptotic to the imaginary axis. From this point of view, the proof procedure based only on the fact that (3.43) has zeros with negative real parts seems to be insufficient. Therefore, we give an argumentation confirming the asymptotic stability of (3.42) in the critical case (3.46), but not analysing zeros of (3.43).

Let (3.46) hold. The following procedure is due to Junca and Lombard [32], where the energy method was applied to show the asymptotic stability property for a special non-linear delay differential equation of neutral type. If $x(t)$ is the solution of (3.42), then

$$(x'(t) - a\,x(t))^2 = (x'(t-\tau) \pm b\,x(t-\tau))^2\,, \tag{3.52}$$

where the sign $\pm$ corresponds to $c = \pm 1$, respectively. One can easily check that

$$(x'(t) \pm b\,x(t))^2 = (x'(t) - a\,x(t))^2 - (a^2 - b^2)(x(t))^2 + 2(a \pm b)x'(t)x(t)\,.$$

Substituting this into (3.52) we have

$$\begin{aligned} &(x'(t) - a\,x(t))^2 - (x'(t-\tau) - a\,x(t-\tau))^2 \\ &+(a^2 - b^2)(x(t-\tau))^2 - 2(a \pm b)x'(t-\tau)y(t-\tau) = 0\,. \end{aligned}$$

If we denote by $g(t)$ an initial function for (3.42), defined and differentiable in $[-\tau, 0]$, then integration of the last relation over $[0, t]$ yields

$$\begin{aligned} &\int_{t-\tau}^{t} (x'(s) - a\,x(s))^2\,\mathrm{d}s + (a^2 - b^2)\int_0^{t-\tau} (x(s))^2\,\mathrm{d}s - (a \pm b)(x(t-\tau))^2 \\ &= \int_{-\tau}^{0} (g'(s) - a\,g(s))^2\,\mathrm{d}s - (a^2 - b^2)\int_{-\tau}^{0} (g(s))^2\,\mathrm{d}s - (a \pm b)(g(-\tau))^2\,. \end{aligned}$$

Since $a+|b|<0$ and t was arbitrary, we get

$$\int_0^\infty (x(s))^2\,\mathrm{d}s < \frac{1}{a^2-b^2}\int_{-\tau}^0 (g'(s)-a\,g(s))^2\,\mathrm{d}s - \int_{-\tau}^0 (g(s))^2\,\mathrm{d}s - \frac{1}{a\mp b}(g(-\tau))^2\,. \tag{3.53}$$

This proves the asymptotic stability property of $x(t)$ under the condition (3.46). □

Remark 3.16. (i) The conditions (3.45)–(3.46) describe the delay-independent stability region for (3.42), i.e. the set of all real triplets (a,b,c) such that the solution $x(t)$ of (3.42) tends to zero as $t\to\infty$ for all lags $\tau>0$. We emphasize that this delay-independent stability region involves also the asymptotically critical case (3.46). One can observe its certain specific property in the frame of this region, namely the fact that the solution $x(t)$ of (3.42) is no longer decaying exponentially due to the lack of (3.44). A certain information on the decay rate of the solution $x(t)$ in this critical case is provided by the inequality (3.53).

(ii) The value τ^* given by (3.48) defines the stability switch for (3.42), i.e. the critical value of a lag such that, assuming $|a|+b<0$ and $|c|<1$, the solution $x(t)$ of (3.42) tends to zero as $t\to\infty$ if and only if $\tau<\tau^*$. The explicit expression of such a value is important for theoretical as well as practical reasons and it is a subject of current investigations also for other types of delay differential equations (see e.g. Matsunaga [46] and Matsunaga and Hashimoto [47]).

(iii) The problem of necessary and sufficient conditions for the asymptotic stability of (3.42) was discussed also by Ren [50]. The conclusions presented in this paper seem to be consistent with ours. Since its content is not generally accessible for language reasons (and also because of the above mentioned vagueness concerning the asymptotically critical case), we have preferred to discuss this matter in details.

3.3.2. Discretization of the differential equation

For the neutral delay differential equation (3.42), we consider merely the Θ-method discretization. It yields the following recurrence

$$y_{n+1}+\alpha y_n+\beta y_{n-k+1}+\gamma y_{n-k}=0, \qquad n=0,1,2,\dots \tag{3.54}$$

with

$$\alpha=-\frac{1+(1-\Theta)ah}{1-\Theta ah}, \quad \beta=-\frac{\Theta bh+c}{1-\Theta ah}, \quad \gamma=-\frac{(1-\Theta)bh-c}{1-\Theta ah}, \quad k=\tau/h\,. \tag{3.55}$$

We assume $\Theta ah\neq 1$. By the asymptotic stability region $\Sigma_\tau^\Theta(h)$ of the Θ-method discretization of (3.42) we understand the set of all real triplets (a,b,c) for which any solution y_n of (3.54), (3.55) tends to zero as $n\to\infty$.

Further, we say that the Θ-method for (3.42) is $N\tau(0)$-stable if

$$\Sigma_\tau^*\subset\bigcap_{k=1}^\infty \Sigma_\tau^\Theta(h)\,, \qquad h=\tau/k\,. \tag{3.56}$$

3.3.3. Numerical stability of the Θ-method discretization and related issues

We begin this section by discussion on asymptotic stability of the trapezoidal rule discretization, which is the only Θ-method of the order 2 with interesting stability properties within the considered class (see the previous sections). Our first aim is to describe the stability region $\Sigma_\tau^{1/2}(h)$ in the form of necessary and sufficient conditions imposed on a, b, c, τ and h. Having such a description, we can discuss not only some other significant properties of the trapezoidal rule, but also come back to the issue of its $N\tau(0)$-stability, especially with respect to the asymptotically critical case.

Further, we provide the necessary and sufficient conditions for the asymptotic stability of (3.54), (3.55) for $\Theta \neq 1/2$, too. Analogous to the trapezoidal rule, we mention also some consequences following from such a description. These consequences concern only the case $\frac{1}{2} < \Theta \leq 1$ because the case $0 \leq \Theta < \frac{1}{2}$ is not interesting from the stability viewpoint.

At the end of this section, we deal with the forward Euler discretization for $x'(t) = b\,x(t-\tau)+c\,x'(t-\tau)$. Besides the description of its asymptotic stability region, we discuss also a monotony property with respect to changing h. The results for the particular case $a = 0$ have been published in [28].

For given a, b, c, we introduce the symbol

$$\tilde{\tau}^{1/2}(h) = \left(h\arccos\frac{a-bc}{|ac-b|}\right) \Big/ \left(2\arctan\left(\frac{h}{2}\left(\frac{b^2-a^2}{1-c^2}\right)^{\frac{1}{2}}\right)^{\omega}\right), \qquad \omega = \operatorname{sgn}(1-|c|)\,.$$

Using this notation we have

Theorem 3.17 (Theorem 3.2 in [8]). *A triplet (a,b,c) belongs to $\Sigma_\tau^{1/2}(h)$ if and only if one of the following conditions holds:*

$$a \leq b < -a, \qquad |c| < 1\,; \tag{3.57}$$

$$a + |b| < 0, \qquad (-1)^{k+1}c = 1\,; \tag{3.58}$$

$$|a| + b < 0, \qquad |c| < 1, \qquad \tau < \tilde{\tau}^{1/2}(h)\,; \tag{3.59}$$

$$-|a| + |b| < 0, \qquad \operatorname{sgn}(a)(-1)^k c > 1, \qquad \tau < \tilde{\tau}^{1/2}(h)\,. \tag{3.60}$$

Proof. The trapezoidal rule discretization of (3.42) yields the recurrence (3.54) with

$$\alpha = -\frac{2+ah}{2-ah}, \quad \beta = -\frac{bh+2c}{2-ah}, \quad \gamma = -\frac{bh-2c}{2-ah}\,. \tag{3.61}$$

By Theorem 2.3, we have to analyse conditions (C1)-(C7) with α, β and γ given by (3.61).

We start with (C1) and (C2). Substituting (3.61), we distinguish two cases with respect to the sign of $2 - ah$. While the case $2 - ah < 0$ leads to a contradiction, for $2 - ah > 0$ the conditions (C1), (C2) become

$$\begin{aligned} a + |b| &< 0, \qquad |c| < 1, \\ a = b &< 0, \qquad |c| < 1\,, \end{aligned}$$

which can be written jointly as (3.57) (we note that the inequality $2 - ah > 0$ is involved here implicitly due to $a < 0$). Similarly, conditions (C3) and (C4) yield

$$\begin{aligned} a + |b| &< 0, \qquad c = 1 \qquad && \text{for } k \text{ odd}\,, \\ a + |b| &< 0, \qquad c = -1 \qquad && \text{for } k \text{ even}\,, \end{aligned}$$

which is equivalent to (3.58). The delay-independent part of $\Sigma_\tau^{1/2}(h)$ is complete.

Conditions (C5)–(C7) already contain the restriction on k, hence they define a delay-dependent part of $\Sigma_\tau^{1/2}(h)$. First we omit (2.5), substitute (3.61) into the four sign conditions of (C5)–(C7) and similarly to the delay-independent part we obtain

$$|a| + b < 0, \qquad |c| < 1, \qquad 2 - ah > 0, \qquad k \text{ is arbitrary}\,, \tag{3.62}$$

$$-|a| + |b| < 0, \qquad \operatorname{sgn}(-a)\, c > 1, \qquad \operatorname{sgn}(-a)\,(2 - ah) > 0, \qquad k \text{ is odd}$$

and

$$-|a| + |b| < 0, \qquad \operatorname{sgn}(a)\, c > 1, \qquad \operatorname{sgn}(-a)\,(2 - ah) > 0, \qquad k \text{ is even}\,,$$

respectively. Of course, the last two relations can be captured jointly as

$$-|a| + |b| < 0, \quad \operatorname{sgn}(a)(-1)^k c > 1, \quad \operatorname{sgn}(-a)\,(2 - ah) > 0, \quad k \text{ is arbitrary}. \tag{3.63}$$

Now we discuss the form of (2.5). Using (3.61) we can rewrite it as

$$\tau \ \arccos \frac{4(1 - c^2) + (a^2 - b^2)h^2}{|4(1 - c^2) - (a^2 - b^2)h^2|} < h \ \arccos \frac{a - bc}{|b - ac|}. \tag{3.64}$$

The left-hand side of (3.64) can be treated by use of the relation

$$\arccos s = 2 \arctan \frac{(1 - s^2)^{1/2}}{1 + s}, \qquad -1 < s \le 1\,, \tag{3.65}$$

which results either in

$$2\tau \arctan \left(\frac{h}{2} \left(\frac{b^2 - a^2}{1 - c^2} \right)^{\frac{1}{2}} \right) \qquad \text{if} \quad 4(1 - c^2) - (a^2 - b^2)h^2 > 0\,,$$

or in

$$2\tau \operatorname{arccot} \left(\frac{h}{2} \left(\frac{b^2 - a^2}{1 - c^2} \right)^{\frac{1}{2}} \right) \qquad \text{if} \quad 4(1 - c^2) - (a^2 - b^2)h^2 < 0\,.$$

Obviously, $4(1 - c^2) - (a^2 - b^2)h^2 > 0$ if (3.62) holds, and $4(1 - c^2) - (a^2 - b^2)h^2 < 0$ if (3.63) holds. Then, after some simple calculations, we get that (3.64) is equivalent to $\tau < \tilde{\tau}^{1/2}(h)$.

Finally, we show that the sign conditions involving the term $2 - ah$ in (3.62) and (3.63) are superfluous if we consider (3.62) and (3.63) along with $\tau < \tilde{\tau}^{1/2}(h)$. Doing this, we use again (3.65) to transform the right-hand side of (3.64) into

$$2h\arctan\left(\frac{(b+a)(1-c)}{(b-a)(1+c)}\right)^{\frac{1}{2}} \qquad \text{if} \quad b-ac<0\,,$$

or into

$$2h\,\mathrm{arccot}\left(\frac{(b+a)(1-c)}{(b-a)(1+c)}\right)^{\frac{1}{2}} \qquad \text{if} \quad b-ac>0\,.$$

Note that $b-ac<0$ if $|a|+b<0$, $|c|<1$ or $-|a|+|b|<0$, $\operatorname{sgn}(a)c>1$, whereas $b-ac>0$ if $-|a|+|b|<0$, $\operatorname{sgn}(-a)c>1$. Now let (3.62) hold. Since the condition $2-ah>0$ is trivial if $a\le 0$, in the sequel we consider only the case $a>0$. Using the previous calculations we can rewrite (3.64) as

$$2\tau\arctan\left(\frac{h}{2}\left(\frac{b^2-a^2}{1-c^2}\right)^{\frac{1}{2}}\right) < 2h\arctan\left(\frac{(b+a)(1-c)}{(b-a)(1+c)}\right)^{\frac{1}{2}}.$$

A necessary condition for the validity of this relation for a given h is

$$\frac{h}{2}|b-a|<|1-c|\,.$$

Since $|a|+b<0$ and $|c|<1$, it is equivalent to

$$2-ah>2c-bh\,.$$

On the other hand, for $a>0$ it holds

$$2c-bh>ah-2\,.$$

Comparing the last two relations we get that $2-ah>0$. In other words, if $2-ah<0$ then $\tau<\tilde{\tau}^{1/2}(h)$ does not hold for a given h.

Analogously, it can be proved that the condition $\operatorname{sgn}(-a)\,(2-ah)>0$ is superfluous in (3.63) provided $\tau<\tilde{\tau}^{1/2}(h)$. Thus, we have fully described the delay-dependent part of $\Sigma_\tau^{1/2}(h)$ and the proof is complete. $\square$

In Figure 3.3 and Figure 3.4 we illustrate $\Sigma_\tau^{1/2}(h)$ for a fixed parameter c. Figure 3.3 depicts this region in the case $|c|>1$, which is described by (3.60). In the left part, we can see $\Sigma_1^{1/2}(1/2)$ for $c=-1.1$, while the right part corresponds to $\Sigma_1^{1/2}(1/2)$ for $c=1.1$. Since the underlying differential equation is not asymptotically stable for any $|c|>1$, these regions do not have their continuous counterparts. In Figure 3.4 we illustrate the case $|c|<1$. The line $a-b=0$ divides the delay-dependent and independent parts. While the delay-independent part (above this line) is common for both Σ_τ^* and $\Sigma_\tau^{1/2}(h)$, the delay-dependent part (below this line) is larger for $\Sigma_\tau^{1/2}(h)$ (Σ_τ^* is restricted by the dashed curve). Note the resemblance of the stability region $\Sigma_\tau^{1/2}(h)$ for $|c|<1$ to the stability region $S_\tau^{1/2}(h)$ (see Figure 3.2). The case $|c|=1$ is discussed below.

Remark 3.18. Theorem 3.17 can be taken for a direct discrete counterpart to Theorem 3.15. In particular, the value $\tilde{\tau}^{1/2}(h)$ defines the stability switch for (3.54), (3.61) as a discrete analogue of the value $\tilde{\tau}^*$ given by (3.48). One can easily check that

$$\tilde{\tau}^{1/2}(h)\to\tilde{\tau}^* \quad \text{as} \quad h\to 0\,.$$

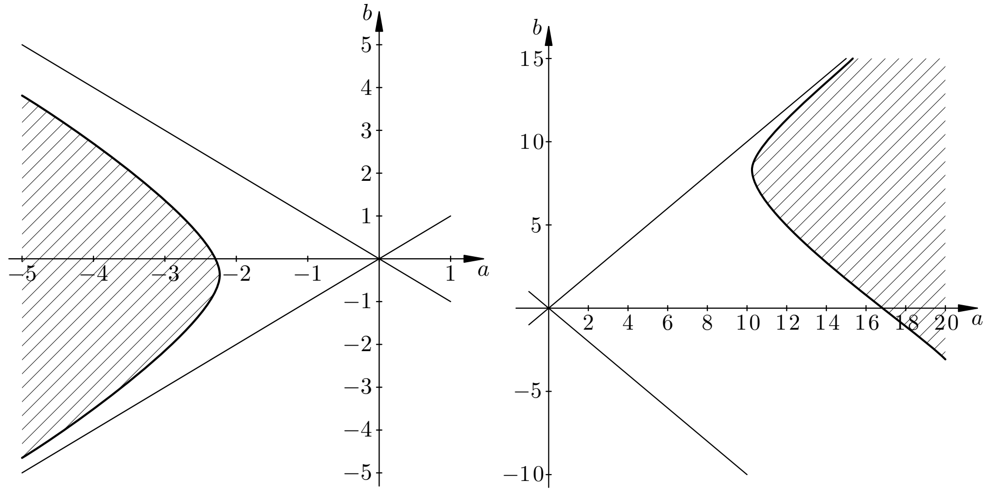

Figure 3.3: Stability regions $\Sigma_1^{1/2}(1/2)$ for $c = -1.1$ (the left part) and $c = 1.1$ (the right part)

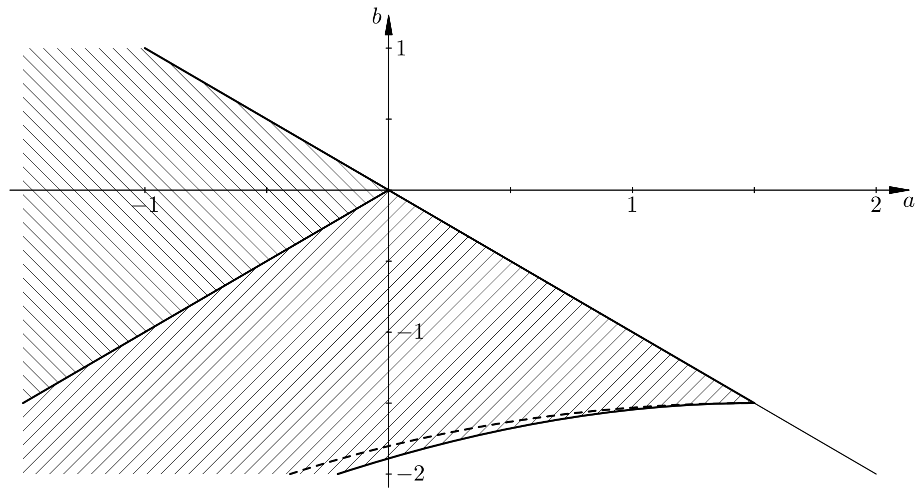

Figure 3.4: Stability regions $\Sigma_1^{1/2}(1/3)$ and Σ_1^* for $c = -0.5$

Later we show that this convergence is monotonous. Therefore, it might be natural to expect that $\Sigma_\tau^{1/2}(h)$ becomes Σ_τ^* as $h \to 0$. However, a more detailed insight shows that this is not true. The problem appears at a part of the stability boundary corresponding to the asymptotically critical case $|c| = 1$. More precisely, triplets (a, b, c) satisfying (3.46) belong to the exact stability region Σ_τ^* due to Theorem 3.15, but their involvement to $\Sigma_\tau^{1/2}(h)$ is restricted by the additional requirement $(-1)^{k+1}c = 1$ (see Theorem 3.17). Its fulfilment depends on parity of k, hence the limit as $h \to 0$ cannot be considered. Moreover, this fact has another consequence concerning the property (3.56) defining $N\tau(0)$-stability of the numerical method. In view of the previous discussion, this property cannot be obviously satisfied. It implies

Corollary 3.19. *The trapezoidal rule is not $N\tau(0)$-stable.*

This conclusion does not agree with the existing results on this topic (see e.g. Theorem 5.2 of [21]). The explanation of this discrepancy is clear. Assertions confirming $N\tau(0)$-stability of the trapezoidal rule utilize the description of the exact stability region Σ_τ^* involving only triplets (a, b, c) with $|c| < 1$. However, as pointed out in Theorem 3.15, the triplets (a, b, c) satisfying (3.46) belong to Σ_τ^* as well. Although they are lying on the boundary of Σ_τ^*, a rigorous approach to the definition of $N\tau(0)$-stability yields the above mentioned conclusion. In this sense, the Θ-method for the neutral equation (3.42) is not $N\tau(0)$-stable for any $0 \le \Theta \le 1$.

In this connection, one can observe the following interesting fact. The exact equation (3.42) is under the condition (3.46) asymptotically stable, but the decay rate of its solutions is only algebraic, not exponential. If we consider its trapezoidal rule discretization (3.54), (3.61) under the condition (3.46), then considering only the fact that (3.54) is a linear homogeneous autonomous difference equation, we can expect just two possibilities: this discretization is either asymptotically stable with an exponential decay rate of its solutions (i.e. it is exponentially stable), or it is not asymptotically stable; nothing "between" like in the continuous case. Our analysis summarized in Theorem 3.17 shows that the discretization (3.54), (3.61) in the asymptotically critical case respects, in a certain sense, this dilemma: it is asymptotically stable for (a, b, c) satisfying (3.46) and $k \in \mathbb{Z}^+$ if and only if $(-1)^{k+1}c = 1$. In such a case this stability is even exponential.

The following example illustrates this dependence of the asymptotic stability property on parity of k in the critical case $c = 1$.

Example 3.20. *We consider the equation*

$$\begin{aligned} x'(t) &= a\,x(t) + x'(t-\tau), && t > 0\,, \\ x(t) &= g(t), && -\tau \le t \le 0, \end{aligned} \tag{3.66}$$

$a < 0 < \tau$, whose thorough stability analysis was performed by Snow in [53]. This analysis revealed, among others, a rate of approach of the characteristic zeros to the imaginary axis and described an algebraic decay of the solutions $x(t)$ to zero via the function $t^{-\varkappa}$ (as $t \to \infty$), where $\varkappa > 0$ depends upon the smoothness of $g(t)$.

The following table presents the numerical solution y_n of (3.66) with $a = -1$ and $\tau = 1$ when applied the trapezoidal rule with $k = 2, 3, \ldots, 7$. Since the characteristic equation of this discretization becomes

$$\lambda^{k+1} + \frac{1-2k}{1+2k}\lambda^k - \frac{2k}{1+2k}\lambda + \frac{2k}{1+2k} = 0\,, \tag{3.67}$$

one can check by use of (C3) in Theorem 2.3 that all the zeros of (3.67) are located inside the unit circle for k odd, but considering k even, a simple zero $\lambda = -1$ appears. In our case, this zero is dominating (in the absolute value), hence the numerical solution y_n is eventually oscillatory and $|y_n|$ tends to a non-zero finite limit. These observations correspond to the theoretical conclusions of Theorem 3.17 and are supported by the data in Table 3.2 (we set here $y_0 = \cdots = y_{-k} = 1$).

Notice also that an exponential rate of convergence of y_n to zero becomes smaller with respect to increasing (odd) k which corresponds to the fact that the decay of the exact solution $x(t)$ is not exponential.

Theorem 3.17 implies other important properties of $\Sigma_\tau^{1/2}(h)$. One of them describes an inclusion property of these stability regions with respect to changing h. This property was shown in the delayed case $c = 0$ (see Theorem 3.12) and we confirm its validity (up to switches of parity of k) also in the neutral delay case ($c \neq 0$).

t_n	$\lvert y_n\rvert$					
	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
10	4.4893E-2	6.2780E-2	6.7592E-2	6.9577E-2	7.0594E-2	7.1186E-2
100	5.8824E-2	4.6100E-2	4.2130E-2	5.9877E-3	9.2572E-3	4.6205E-3
1000	5.8824E-2	1.4998E-5	1.5390E-2	9.6526E-3	1.8766E-3	4.7342E-3
5000	5.8824E-2	2.9242E-11	1.5385E-2	1.5357E-3	6.9534E-3	4.6110E-3
10000	5.8824E-2	4.3482E-20	1.5385E-2	8.7120E-5	6.8967E-3	1.0461E-3
15000	5.8824E-2	7.3859E-29	1.5385E-2	3.1290E-6	6.8966E-3	3.5255E-4

Table 3.2: The values of $|y_n|$ for $a=-1$, $\tau=1$

Theorem 3.21. *Let $k_1 < k_2$ be arbitrary positive integers of the same parity and let $h_1 = \tau/k_1 > \tau/k_2 = h_2$ be corresponding stepsizes. Then*

$$\Sigma_\tau^{1/2}(h_1) \supset \Sigma_\tau^{1/2}(h_2)\,.$$

Proof. First we show that if $(\tilde{a},\, \tilde{b},\, \tilde{c}) \in \Sigma_\tau^{1/2}(h_2)$ then $(\tilde{a},\, \tilde{b},\, \tilde{c}) \in \Sigma_\tau^{1/2}(h_1)$. Since the conditions (3.57) and (3.58) are independent of h, it is enough to consider the delay-dependent part of $\Sigma_\tau^{1/2}(h)$ represented by (3.59) and (3.60). More precisely, we are going to analyse the inequality $\tau < \tilde{\tau}^{1/2}(h)$.

Assume that $(\tilde{a}, \tilde{b}, \tilde{c})$ belongs to the delay-dependent part of $\Sigma_\tau^{1/2}(h_2)$ defined by (3.59). It particularly means that

$$\frac{1}{h}\arctan\left(\frac{h}{2}\left(\frac{\tilde{b}^2-\tilde{a}^2}{1-\tilde{c}^2}\right)^{\frac{1}{2}}\right) < \frac{1}{2\tau}\arccos\frac{\tilde{a}-\tilde{b}\tilde{c}}{|\tilde{a}\tilde{c}-\tilde{b}|}\,. \tag{3.68}$$

Obviously, $(\tilde{a}, \tilde{b}, \tilde{c})$ belongs to the delay-dependent part of $\Sigma_\tau^{1/2}(h_1)$ defined by (3.59), if the last inequality is true also for h_1. To prove this, it is enough to show that the function

$$\tilde{f}(s) = \frac{1}{s}\arctan\left(\frac{sr}{2}\right), \qquad 0 < s \le \tau$$

is decreasing in s for any $r \ge 0$. Then

$$\tilde{f}'(s) = \frac{1}{s^2}\left(\frac{2sr}{4+s^2r^2} - \arctan\left(\frac{sr}{2}\right)\right).$$

Obviously, $\tilde{f}'(s) < 0$ for all $0 < s \le \tau$ if and only if $\tilde{g}(s) < 0$ for all $0 < s \le \tau$, where

$$\tilde{g}(s) = \frac{2sr}{4+s^2r^2} - \arctan\left(\frac{sr}{2}\right) \qquad 0 < s \le \tau\,.$$

It holds $\tilde{g}(0) = 0$ and

$$\tilde{g}'(s) = -\frac{4s^2r^3}{(4+s^2r^2)^2} < 0, \qquad 0 < s \le \tau\,.$$

This implies $\tilde{g}(s) < 0$ for all $0 < s \le \tau$, hence $\tilde{f}(s)$ is decreasing in $(0, \tau\rangle$. In particular, if (3.68) holds for a given triplet $(\tilde{a},\, \tilde{b},\, \tilde{c})$ and a given stepsize h_2, then (3.68) holds for

$(\tilde{a},\,\tilde{b},\,\tilde{c})$ and any stepsize h_1 such that $h_2 < h_1 \le \tau$. Consequently, if $(\tilde{a},\,\tilde{b},\,\tilde{c}) \in \Sigma_\tau^{1/2}(h_2)$ then $(\tilde{a},\,\tilde{b},\,\tilde{c}) \in \Sigma_\tau^{1/2}(h_1)$.

It remains to show that the set equality $\Sigma_\tau^{1/2}(h_1) = \Sigma_\tau^{1/2}(h_2)$ cannot occur for any $0 < h_2 < h_1 \le \tau$. Consider a real triplet $(\bar{a},\,\bar{b},\,\bar{c})$ lying on the delay-dependent stability boundary of $\Sigma_\tau^{1/2}(h_2)$ and satisfying

$$|\bar{a}| + \bar{b} < 0, \qquad |\bar{c}| < 1, \qquad \tau = \tilde{\tau}^{1/2}(h_2)\,.$$

It follows immediately from the above observed monotony property of $\tilde{f}(s)$ that

$$|\bar{a}| + \bar{b} < 0, \qquad |\bar{c}| < 1, \qquad \tau < \tilde{\tau}^{1/2}(h_1)$$

for any $h_1 > h_2$. Thus $(\bar{a},\,\bar{b},\,\bar{c}) \in \Sigma_\tau^{1/2}(h_1)$, whereas $(\bar{a},\,\bar{b},\,\bar{c}) \notin \Sigma_\tau^{1/2}(h_2)$.

For the delay-dependent part of $\Sigma_\tau^{1/2}(h_2)$ defined by (3.60) is the proof procedure analogous. $\square$

Now, we consider $\Theta \neq 1/2$. We state the necessary and sufficient conditions describing the asymptotic stability regions. Doing that, we introduce the symbols for given a, b, c and Θ

$$\begin{aligned}\tilde{\tau}_1^\Theta(h) =& h \arctan\left(\frac{(b+a)(2(1-c)+(1-2\Theta)(a+b)h)}{(b-a)(2(1+c)+(1-2\Theta)(a-b)h)}\right)^{1/2} \Bigg/ \\ & \arctan\left(\frac{(b^2-a^2)h^2}{(2(1-c)+(1-2\Theta)(a+b)h)(2(1+c)+(1-2\Theta)(a-b)h)}\right)^{1/2}\end{aligned}$$

and

$$\begin{aligned}\tilde{\tau}_2^\Theta(h) =& h \arctan\left(\frac{(b+a)(2(1-c)+(1-2\Theta)(a+b)h)}{(b-a)(2(1+c)+(1-2\Theta)(a-b)h)}\right)^{(-1)^k/2} \Bigg/ \\ & \operatorname{arccot}\left(\frac{(b^2-a^2)h^2}{(2(1-c)+(1-2\Theta)(a+b)h)(2(1+c)+(1-2\Theta)(a-b)h)}\right)^{1/2}.\end{aligned}$$

Then we can formulate the following conditions with respect to $0 \le \Theta < \frac{1}{2}$ and $\frac{1}{2} < \Theta \le 1$.

Theorem 3.22. (a) *Let $0 \le \Theta < \frac{1}{2}$. A triplet (a,b,c) belongs to $\Sigma_\tau^\Theta(h)$ if and only if one of the following conditions holds:*

$$\begin{aligned}
&a \le b < -a, && 2+(1-2\Theta)ah > |2c-(1-2\Theta)bh|;\\
&a+|b| < 0, && 2+(1-2\Theta)ah = (-1)^{k+1}\,(2c-(1-2\Theta)bh) > 0;\\
&|a|+b < 0, && 2+(1-2\Theta)ah > |2c-(1-2\Theta)bh|, \quad \tau < \tilde{\tau}_1^\Theta(h);\\
&|a|-|b| > 0, && |2+(1-2\Theta)ah| < \operatorname{sgn}(a)\,(-1)^k\,(2c-(1-2\Theta)bh)\,, \quad \tau < \tilde{\tau}_2^\Theta(h)\,.
\end{aligned}$$

(b) *Let $\frac{1}{2} < \Theta \le 1$. A triplet (a,b,c) belongs to $\Sigma_\tau^\Theta(h)$ if and only if one of the following conditions holds:*

$$\begin{aligned}
&a \le b < -a, && 2-(2\Theta-1)ah > |2c+(2\Theta-1)bh|;\\
&a \ge b > -a, && 2-(2\Theta-1)ah < -|2c+(2\Theta-1)bh|;\\
&|a|-|b| > 0, && \operatorname{sgn}(a)\ (2-(2\Theta-1)ah) = \operatorname{sgn}(a)\,(-1)^{k+1}\,(2c+(2\Theta-1)bh) < 0;\\
&|a|-|b| < 0, && -\operatorname{sgn}(b)(2-(2\Theta-1)ah) > \ |2c+(2\Theta-1)bh|, \quad \tau < \tilde{\tau}_1^\Theta(h); \quad (3.69)\\
&|a|-|b| > 0, && |2-(2\Theta-1)ah| < \operatorname{sgn}(a)\,(-1)^k\,(2c+(2\Theta-1)bh)\,, \quad \tau < \tilde{\tau}_2^\Theta(h)\,.
\end{aligned}$$

The proof of both these assertions is a technical modification of that of Theorem 3.17 and is omitted.

Remark 3.23. If $\frac{1}{2} < \Theta \leq 1$ then $\Sigma_\tau^\Theta(h)$ involves a subregion defined by the conditions (3.45) and (3.46) independently of τ and h. It particularly implies that, contrary to the trapezoidal rule case, if (a, b) is a real couple satisfying $a + |b| < 0$, then $(a, b, 1) \in \Sigma_\tau^\Theta(h)$ as well as $(a, b, -1) \in \Sigma_\tau^\Theta(h)$ regardless of parity of k. In other words, (3.54), (3.55) is asymptotically stable in the critical case $|c| = 1$ for all possible stepsizes h provided $a + |b| < 0$ and $\frac{1}{2} < \Theta \leq 1$.

In the asymptotically critical case $|c| = 1$, the stability properties of the Θ-method (3.54), (3.55) are more favourable for $\Theta > 1/2$ than for $\Theta = 1/2$ (see Remark 3.23). If $|c| < 1$, the situation is different in the sense that the condition (3.56), defining $N\tau(0)$-stability of Θ-methods, holds for $\Theta = 1/2$, but not for $\Theta > 1/2$. More precisely, a deeper analysis of the behaviour of transcendental curves forming a part of the true and numerical stability boundary reveals that there exist triplets $(a, b, c) \in \Sigma_\tau^*$ with c close to -1 such that $(a, b, c) \notin \Sigma_\tau^\Theta(\tau/2)$ for any $1/2 < \Theta < 1$ (see Theorem 4.3 of [21] and a related discussion).

As a consequence of Theorem 3.22, we can extend this result and specify such a neighbourhood of $c = -1$ with respect to the values of Θ, h and τ. To make next steps as clear as possible, we use a simple geometrical argumentation. In particular, we avoid an analysis of the transcendental boundary curve $\tau = \tilde{\tau}_1^\Theta(h)$ and consider instead the first two inequalities of the condition (3.69). These inequalities guarantee the domain of $\tilde{\tau}_1^\Theta$, but also determine an area, where the corresponding curve $\tau = \tilde{\tau}_1^\Theta(h)$ is located.

Let $\frac{1}{2} < \Theta \leq 1$. Since the delay-independent part of Σ_τ^* is involved in $\Sigma_\tau^\Theta(h)$ for any stepsize h (see Remark 3.23), we analyse the delay-dependent part. For a fixed $c \in (-1, 1)$ and a fixed $\tau > 0$, we consider the $(a,\, b)$-plane, where the delay-dependent part of Σ_τ^* is bounded above by the lines $a + b = 0$, $a - b = 0$ and below by the transcendental curve

$$\tau \left(\frac{b^2 - a^2}{1 - c^2} \right)^{1/2} - \arccos \frac{a - bc}{ac - b} = 0 \,. \tag{3.70}$$

Moreover, $P_1 = ((1-c)/\tau, (c-1)/\tau)$ is a double point for this stability boundary, i.e. the point, where the line $a + b = 0$ and the curve (3.70) intersect.

Now let Ω be a part of the $(a,\, b)$-plane bounded above by the lines $a+b = 0$, $a-b = 0$ and consider the delay-dependent part of $\Sigma_\tau^\Theta(h)$ restricted to Ω. The analytical description of this area is given by (3.69). In particular, the second condition of (3.69) implies that such a delay-dependent part of $\Sigma_\tau^\Theta(h)$ is bounded below by the line

$$a - b - \frac{2(1+c)}{(2\Theta - 1)h} = 0 \,, \tag{3.71}$$

which is parallel to $a - b = 0$ and orthogonal to $a + b = 0$. The lines (3.71) and $a + b = 0$ intersect at

$$P_2 = \left(\frac{(1+c)}{(2\Theta - 1)h}, -\frac{(1+c)}{(2\Theta - 1)h} \right) .$$

Comparing locations of P_1 and P_2 at the line $a + b = 0$, one can obtain an obvious geometrical conclusion: If $\Sigma_\tau^\Theta(h) \supset \Sigma_\tau^*$ then P_1 is located above P_2 or coincides with P_2 (equivalently, $h(1-c) \leq (1+c)\tau/(2\Theta-1)$). In the opposite case, when P_1 is located below

P_2, we can introduce a non-empty region $\tilde{\Sigma}_\tau^\Theta(h)$, bounded by the lines (3.71), $a+b=0$ above and by the curve (3.70) below. Obviously, this region satisfies the property

$$\tilde{\Sigma}_\tau^\Theta(h) = \Sigma_\tau^* \setminus \Sigma_\tau^\Theta(h)\,. \tag{3.72}$$

Both these cases are depicted in Figure 3.5 and Figure 3.6, where Figure 3.5 illustrates the case when P_1 is located below P_2 and Figure 3.6 the opposite one. The dashed curve in both the figures indicates a part of the stability boundary of the underlying differential equation (see (3.70)) and the line (3.71) is denoted here as p.

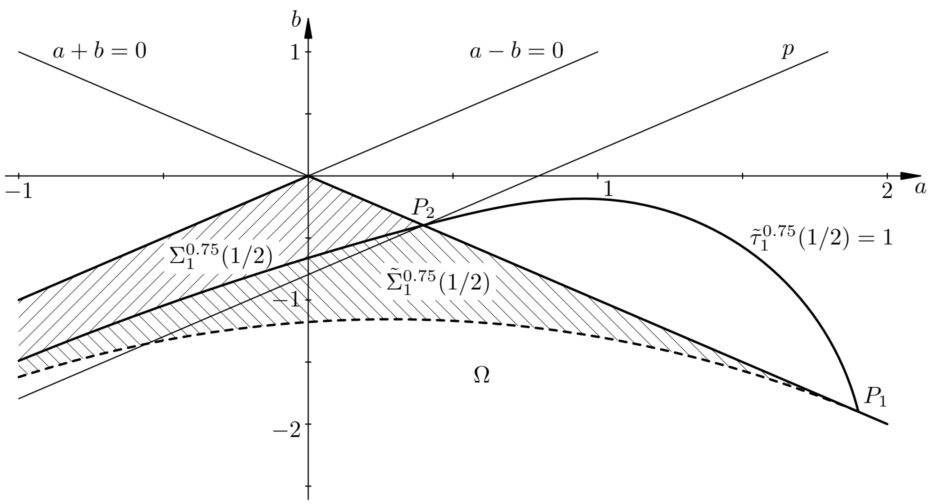

Figure 3.5: Delay-dependent parts of the stability regions $\Sigma_1^{0.75}(1/2)$ and Σ_1^* for $c=-0.9$

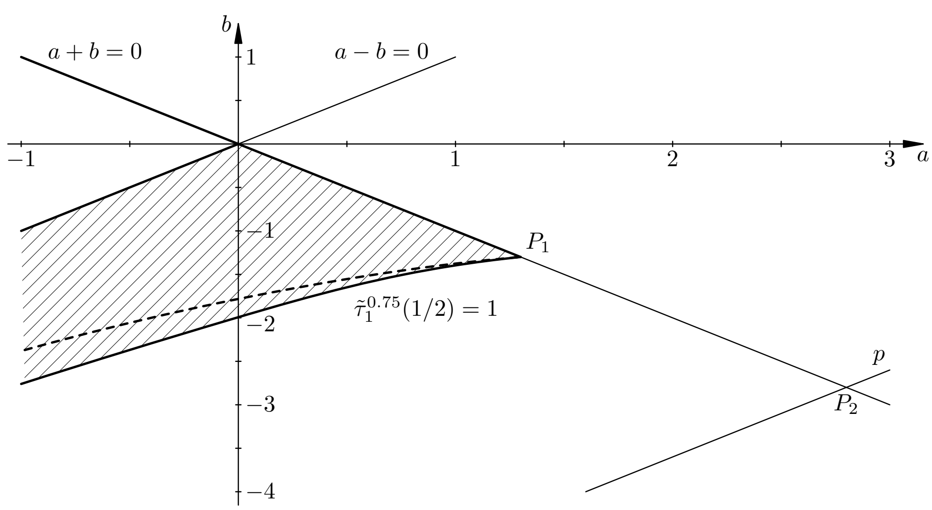

Figure 3.6: Delay-dependent parts of the stability regions $\Sigma_1^{0.75}(1/2)$ and Σ_1^* for $c=-0.3$

Previous considerations can be summarized in the following

Corollary 3.24. *Let* $\frac{1}{2} < \Theta \le 1$ *and*

$$c < \frac{(1-2\Theta)h+\tau}{(1-2\Theta)h-\tau} \tag{3.73}$$

for a given h and τ. Then the set $\tilde{\Sigma}_\tau^\Theta(h)$ is non-empty and, by (3.72), for any $(a,b,c) \in \tilde{\Sigma}_\tau^\Theta(h)$ the exact equation (3.42) is asymptotically stable, whereas its Θ-method discretization (3.54), (3.55) is not asymptotically stable.

Remark 3.25. The condition (3.73) becomes more restrictive with increasing Θ as well as with increasing h. In particular, letting $h \to 0$ we can see that this critical value of c is tending to -1, which corresponds to the stability condition (3.47) for the exact equation. On the other hand, in the case of the backward Euler method ($\Theta = 1$), the inequality (3.73) can be read as

$$c < \frac{h-\tau}{h+\tau}\,,$$

which particularly implies that for any $c < 0$ one can find $(a,b,c) \in \Sigma_\tau^*$ and a stepsize h such that the corresponding backward Euler formula is not asymptotically stable, i.e. $(a,b,c) \notin \Sigma_\tau^1(h)$.

Our previous observations extend the discussion performed by Guglielmi in [21]. We recall that we did not employ here analysis of the transcendental curve $\tau = \tilde{\tau}_1^\Theta(h)$, but only a region, where this curve is situated. On this account, the condition (3.73) is sufficient for the existence of a non-empty set $\tilde{\Sigma}_\tau^\Theta(h)$, characterized by the property (3.72), but not necessary. Some additional calculations show the necessity of (3.73) when $h = \tau$. More precisely, if we restrict to the delay-dependent case $|a| + b < 0$, $|c| < 1$, then $(a,b,c) \in \Sigma_\tau^\Theta(\tau)$ if and only if

$$-2(1+c) + (2\Theta - 1)(a-b)\tau < 0 < 1 - c + b\tau - \Theta(a+b)\tau. \tag{3.74}$$

This condition follows either from (3.69) with $h = \tau$, or it can be derived directly (the characteristic polynomial is now quadratic). To prove the necessity of (3.73) when $h = \tau$, we assume that $c \geq (\Theta - 1)/\Theta$, i.e. P_1 is above P_2 (see our geometrical argumentation in the (a,b)-plane). In this case, the delay-dependent part of $\Sigma_\tau^\Theta(\tau)$ is bounded above by the lines $a + b = 0$, $a - b = 0$, below by the line (3.71) and right by the line

$$1 - c + b\tau - \Theta(a+b)\tau = 0 \tag{3.75}$$

(see (3.74)). We recall that the exact stability set Σ_τ^* is bounded below by the curve (3.70), hence it remains to compare the locations of this curve and the lines (3.71), (3.75). Straightforward calculations based on derivatives of the curve (3.70) and both the lines show that (3.70) is located above (3.71) and (3.75) in the investigated area (the only intersection of (3.70) and (3.75) is the point P_1 on the stability boundary). It implies

Corollary 3.26. *Let $\frac{1}{2} < \Theta \leq 1$. Then*

$$\Sigma_\tau^* \subset \Sigma_\tau^\Theta(\tau) \qquad \Longleftrightarrow \qquad c \geq (\Theta - 1)/\Theta.$$

In the last part of this section, we deal with the forward Euler discretization for a particular case of (3.42), namely

$$x'(t) = b\,x(t-\tau) + c\,x'(t-\tau), \qquad t > 0\,. \tag{3.76}$$

We provide the necessary and sufficient conditions describing its stability region and based on them we investigate its monotony property with respect to changing stepsize h.

Let us describe its asymptotic stability region via necessary and sufficient conditions on b, c, τ and h. They can be derived from Theorem 3.22, but they are also a direct consequence of Corollary 2.5 with $\beta = -c$ and $\gamma = c - bh$.

Corollary 3.27 (Theorem 2.3 in [28])**.** *A real triplet* $(0,b,c)$ *belongs to* $\Sigma_\tau^0(h)$ *if and only if*

$$b<0, \quad |bh-2c|<2, \quad \tau \arccos \frac{2(1-c^2)+2bch-b^2h^2}{2(1+bch-c^2)} < h \arccos \frac{2c-bh}{2}.$$

In Theorem 3.2 we have shown the monotony property of stability intervals of forward Euler discretization for the purely delayed equation (3.1). The analysis of $\Sigma_\tau^0(h)$ denies such a conclusion for the discretization of the neutral equation (3.76).

Theorem 3.28 (Theorem 3.1 in [28])**.** *Let* $k_1<k_2$ *be arbitrary positive integers and let* $h_1=\tau/k_1>\tau/k_2=h_2$ *be corresponding stepsizes. Then there exist real triplets* $(0,b_1,c_1)$*,* $(0,b_2,c_2)$*,* $(0,b_3,c_3)$ *such that*

(i) $(0,b_1,c_1)\notin\Sigma_\tau^0(h_1)$ *and* $(0,b_1,c_1)\in\Sigma_\tau^0(h_2)$

(ii) $(0,b_2,c_2)\in\Sigma_\tau^0(h_1)$ *and* $(0,b_2,c_2)\in\Sigma_\tau^0(h_2)$

(iii) $(0,b_3,c_3)\in\Sigma_\tau^0(h_1)$ *and* $(0,b_3,c_3)\notin\Sigma_\tau^0(h_2)$

Proof. First, we recall the form of the stability interval $I_\tau^0(h)$ of (3.1) (i.e. the case $c=0$)

$$I_\tau^0(h)=\left\{b\in\mathbb{R}: -\frac{2}{h}\cos\frac{\tau\pi}{2\tau+h}<b<0\right\},$$

for which we have shown that the function defining its left endpoint is increasing and therefore $I_\tau^0(h_1)\subset I_\tau^0(h_2)\subset I_\tau^*$. Moreover, both the inclusions are sharp (see Theorem 3.2). Let $\hat{b}$ be the left endpoint of $I_\tau^0(h_1)$, i.e.

$$\hat{b}=-\frac{2}{h_1}\cos\frac{\tau\pi}{2\tau+h_1}.$$

Then $\hat{b}\in I_\tau^0(h_2)$ and $\hat{b}\notin I_\tau^0(h_1)$. Consequently, $(0,\hat{b},0)\notin\Sigma_\tau^0(h_1)$ and $(0,\hat{b},0)\in\Sigma_\tau^0(h_2)$. Thus, we have proved (i), where

$$b_1=-\frac{2}{h_1}\cos\frac{\tau\pi}{2\tau+h_1} \qquad \text{and} \qquad c_1=0.$$

Setting $h=\tau$ we get

$$I_\tau^0(\tau)=\left\{b\in\mathbb{R}: -\frac{1}{\tau}<b<0\right\}.$$

Because of the monotony property of $I_\tau^0(h)$, it holds $(0,\tilde{b},0)\in\Sigma_\tau^0(h_1)$ and $(0,\tilde{b},0)\in\Sigma_\tau^0(h_2)$ for each $\tilde{b}\in I_\tau^0(\tau)$. Consequently, it proves (ii) where $b_2\in(-1/\tau,0)$ and $c_2=0$.

Further, let us consider $c=-1$. Then the inequality

$$\tau \arccos \frac{2(1-c^2)+2bch-b^2h^2}{2(1+bch-c^2)} < h \arccos \frac{2c-bh}{2}$$

can be simplify into

$$-\frac{4}{h}\sin^2\frac{\pi h}{2(\tau+h)}<b.$$

We define

$$f(h) = -\frac{4}{h}\sin^2\frac{\pi h}{2(\tau+h)}, \qquad 0 < h \le \tau.$$

In order to analyse the monotony of $f(h)$, we drop the constraint $h = \tau/k$, $k \in \mathbb{Z}^+$ and consider $f(h)$ to be a function of a continuous argument. Then

$$f'(h) = -\frac{2}{h}\left(-\frac{2}{h}\sin^2\frac{\pi h}{2(\tau+h)} + \frac{\pi\tau}{(\tau+h)^2}\sin\frac{\pi h}{\tau+h}\right).$$

Obviously, $f'(h) < 0$ when

$$-2\sin^2\frac{\pi h}{2(\tau+h)} + \frac{\pi\tau h}{(\tau+h)^2}\sin\frac{\pi h}{\tau+h} > 0, \qquad 0 < h \le \tau,$$

or equivalently

$$\tan\frac{\pi h}{2(\tau+h)} < \frac{\pi\tau h}{(\tau+h)^2}, \qquad 0 < h \le \tau.$$

Let

$$g_1(h) = \tan\frac{\pi h}{2(\tau+h)}, \qquad g_2(h) = \frac{\pi\tau h}{(\tau+h)^2}.$$

We show that $g_1(h) < g_2(h)$ for some $h > 0$. Doing this, we investigate their derivatives

$$g_1'(h) = \frac{\pi\tau}{2(\tau+h)^2}\cos^{-2}\frac{\pi h}{2(\tau+h)}, \qquad g_2'(h) = \frac{\pi\tau(\tau-h)}{(\tau+h)^3}.$$

It holds $g_1(0) = g_2(0)$ and

$$g_1'(h) < \frac{2\pi\tau}{3(\tau+h)^2}, \qquad 0 \le h < \frac{\tau}{2}.$$

It implies that $g_1'(h) < g_2'(h)$ for all $h < \tau/5$. Hence, $f(h)$ is decreasing for $0 < h < \tau/5$. The remaining issue is to investigate the behaviour of $f(h)$ for $h = \tau/\bar{k}$, $\bar{k} = 1, 2, 3, 4, 5$. The values of $f(h)$ for such h are computed in the following table.

h	τ	$\tau/2$	$\tau/3$	$\tau/4$	$\tau/5$
$f(h)$	-2/τ	$-2/\tau$	$-1.757/\tau$	$-1.528/\tau$	$-1.400/\tau$

Table 3.3: The values of function $f(h)$ for some particular h

We may conclude that, with the exception of $h = \tau$ and $h = \tau/2$, $f(h)$ is decreasing in $h = \tau/k$, $k \in \mathbb{Z}^+$. Therefore, by setting

$$\bar{b} = -\frac{4}{h_2}\sin^2\frac{\pi h_2}{2(\tau+h_2)}, \qquad k_2 = \frac{\tau}{h_2} > 2$$

we get $(0, \bar{b}, -1) \in \Sigma_\tau^0(h_1)$ and $(0, \bar{b}, -1) \notin \Sigma_\tau^0(h_2)$ for $k_2 > 2$.

To complete the proof we have to find a triplet $(0, \bar{\bar{b}}, \bar{c})$ such that $(0, \bar{\bar{b}}, \bar{c}) \in \Sigma_\tau^0(\tau)$ and $(0, \bar{\bar{b}}, \bar{c}) \notin \Sigma_\tau^0(\tau/2)$. To this purpose we investigate the stability boundary given by

$$\tau\arccos\frac{2(1-c^2) + 2bch - b^2h^2}{2(1+bch-c^2)} = h\arccos\frac{2c-bh}{2}$$

in the neighbourhood of $(0, -2/\tau, -1)$, because this point is a common boundary point for both $\Sigma^0_\tau(\tau)$ and $\Sigma^0_\tau(\tau/2)$. Using the implicit differentiation formula we get

$$b'(c) = \frac{\tau b(bh-2c) + 2(1-c^2) + 2bch}{\tau[2(1-c^2)+bch] + h(1+bch-c^2)}, \quad 4-(2c-bh)^2 \neq 0,\ 1+bch-c^2 \neq 0.$$

Setting $h = \tau$ we obtain $b'(-1) = 1/\tau$, while for $h = \tau/2$ it holds $b'(-1) = 0$. Hence, there exists a point $\bar{\bar{c}}$ in a left neighbourhood of the value $c = -1$ such that $(0, -2/\tau, \bar{\bar{c}}) \in \Sigma^0_\tau(\tau)$ and $(0, -2/\tau, \bar{\bar{c}}) \notin \Sigma^0_\tau(\tau/2)$. Thus, we have proved (iii), where for $k_2 > 2$

$$b_3 = -\frac{4}{h_2}\sin^2\frac{\pi h_2}{2(\tau+h_2)} \qquad \text{and} \qquad c_3 = -1$$

while for $k_2 = 2$ we have

$$b_3 = -2/\tau \qquad \text{and} \qquad c_3 = -1-\epsilon,$$

where ϵ is a sufficiently small positive number. $\square$

The behaviour of $\Sigma^0_1(h)$ is illustrated in the $(b,\, c)$-plane in Figure 3.7. The depicted curves are the asymptotic stability boundaries for different values of the stepsize h. In all the cases, the stability area is bounded from above by the c-axis.

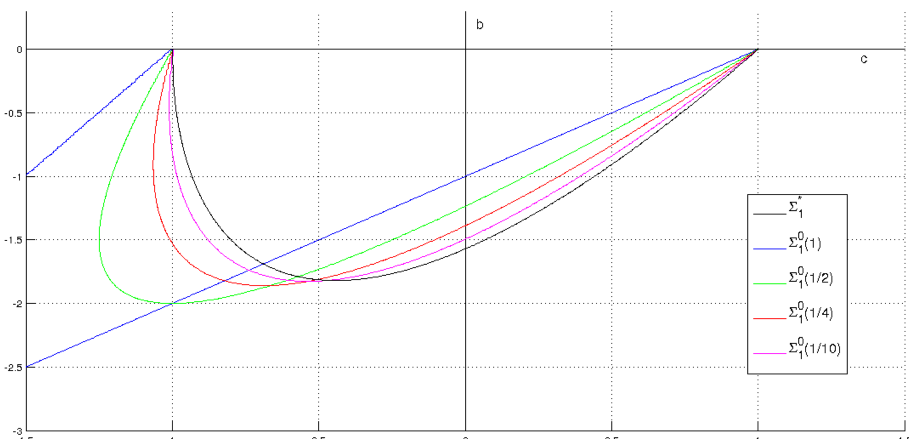

Figure 3.7: The asymptotic stability regions $\Sigma^0_1(h)$ for $a = 0$

Remark 3.29. Using formula (3.65), the last relation in Theorem 3.27 can be equivalently expressed in the form

$$\tau \arctan\left(\frac{b^2h^2}{(2(1-c)+bh)\,(2(1+c)-bh)}\right)^{1/2} < h \arctan\left(\frac{2(1-c)+bh}{2(1+c)-bh}\right)^{1/2},$$

which can be simplified for $h = \tau$ as

$$b > \frac{c-1}{\tau}.$$

Considering also the remaining conditions of Theorem 3.27 we get the following necessary and sufficient asymptotic stability condition describing $\Sigma^0_\tau(\tau)$ in the form

$$c - 1 < b\tau < \min(0, 2(1+c)).$$

3.4. The equation $x'(t) = a\,x(t) + b_1\,x(t-\tau_1) + b_2\,x(t-\tau_2)$

In this section, we provide an overview of the asymptotic stability results for a delay differential equation with two constant lags.

3.4.1. Asymptotic stability of the differential equation

We consider the delay differential equation

$$x'(t) = a\,x(t) + b_1\,x(t-\tau_1) + b_2\,x(t-\tau_2), \qquad t > 0\,, \tag{3.77}$$

where a, b_1, b_2, $\tau_1 > 0$, $\tau_2 > 0$ are real scalars. The asymptotic stability region $\mathbf{S}^*_{\tau_1,\tau_2}$ for (3.77) is then defined as the set of all real triplets (a, b_1, b_2) for which any solution $x(t)$ of (3.77) tends to zero as $t \to \infty$.

The study of the asymptotic stability of (3.77) is based on analysis of zero locations of its characteristic quasi-polynomial

$$P(\lambda) = \lambda + a + b_1 \mathrm{e}^{-\lambda\tau_1} + b_2 \mathrm{e}^{-\lambda\tau_2}\,. \tag{3.78}$$

The equation (3.77) is asymptotically stable if all the zeros λ of $P(\lambda)$ satisfy

$$\Re(\lambda) \le \delta < 0 \qquad \text{for a real scalar } \delta\,. \tag{3.79}$$

The widely used method to determine the coefficient space in which (3.79) holds, is the D-partition. It consists in the determination of hypersurfaces, the points of which correspond to (3.78) with at least one zero on the imaginary axis. Then, since the zeros of (3.78) are continuous functions of its coefficients, the points in each region of D-partition correspond to (3.78) with the same number of zeros with positive real part. Then, the region containing only zeros with negative real parts is selected and forms $\mathbf{S}^*_{\tau_1,\tau_2}$.

Such an approach has been used by Levitskaya [42] who considered the case $a = 0$. However, because of the complexity of hypersurface description (in this case curves), the stability domains have been constructed just by means of numerical and graphical experiments.

A variant of the D-partition method is the τ-decomposition, which involves first decomposing the delay τ-axis into intervals such that within each interval the same stability character prevails, and then studying the change of stability character as the boundary points of the intervals are crossed [52].

The combination of the D-partition and τ-decomposition has been used by Ruan and Wei [52] who investigated (3.77) with $a < 0$ and $b_1 = b_2$. For each $a < 0$, they defined intervals of coefficients b_1 such that (3.78) has all zeros with negative real part. However, the determination of the intervals requires to solve of an auxiliary nonlinear equation, which makes it inconvenient for the asymptotic stability analysis.

The same method have been used also by Mahaffy and Busken [48], who studied (3.77) with $\tau_1 = 1$ and $\tau_2 \in (0, 1)$. They introduced the minimum region of stability - i.e region for which all the zeros of (3.78) have a negative real part independently on τ_2. It is given by the following condition

$$|b_1| + |b_2| < -a\,.$$

Further, they observed that the stability regions enlarges significantly for $\tau_2 = 1/\ell$, where $\ell \in \mathbb{Z}^+$. More precisely, it has been shown by graphical and some analytical results, that location of $\mathbf{S}^*_{\tau_1,\tau_2}$ enlargement depends on parity of ℓ. Further, it seems that if $\tau_2 \neq 1/\ell$, $\ell \in \mathbb{Z}^+$, the enlargement is very small comparing to $\tau_2 = 1/\ell$, $\ell \in \mathbb{Z}^+$. However, this result is based only on graphical experiments.

Summarizing the preceding overview, we may conclude that the asymptotic stability of (3.77) is an unsolved problem, whose analysis seems to be very difficult.

3.4.2. Discretization of the differential equation

In order to obtain the simplest difference equation as numerical approximation of (3.77), we use the Euler method as a tool of discretization. It yields

$$y_{n+1} + \alpha y_n + \beta y_{n-k_1} + \gamma y_{n-k_2} = 0, \tag{3.80}$$

where α, β, γ are real coefficients depending on a, b_1, b_2 and $k_2 > k_1$ are suitable positive integers. A particular case $\alpha = -1$, which corresponds to Euler discretization of (3.77) with $a = 0$, has been investigated by Kipnis and Levitskaya [34] who used the boundary locus technique, which is a discrete analogue of the D-partition. We recall that (3.80) with $\alpha = -1$ is asymptotically stable if and only if its characteristic polynomial

$$P(\lambda) = \lambda^{k_2+1} - \lambda^{k_2} + \beta\lambda^{k_2-k_1} + \gamma$$

is of a Schur type, i.e. all its zeros are located inside the open unit circle. The boundary locus technique is based on finding all curves in (β, γ)-plane, such that their corresponding $P(\lambda)$ has at least one zero with modulus equal to one. Such curves divide the plane into regions which contain same number of zeros with modulus greater than one. The part of the (β, γ)-plane to which only zeros with modulus less than one belong, is then the stability region for (3.80). However, Kipnis and Levitskaya performed only the numerical and graphical experiments, because of the complexity of the resulting curves. The stability of the same particular case has been investigated also by Györi et al. [24], but under some specific and restrictive assumptions.

Analogously to the continuous case, the asymptotic stability of the discretized equation is a difficult task, which has not been answered yet.

4. Delay differential equation with infinite lag

In this chapter, we study the delay differential equations with infinite lag. In the first section, we recall known results for the pantograph equation. The second section provides the generalization of these results for a delay differential equation with a general lag. Finally, in the third section we investigate the delay differential equations with several lags. The results of the second and third section originates from the paper which is now in a preparation.

4.1. The equation $x'(t) = a\,x(t) + b\,x(qt)$

We begin this chapter with the study of the pantograph equation, which is a typical representative of the delay differential equations with infinite lag. We provide a survey of asymptotic stability results of the differential equation involving also an asymptotic estimate of its solution. Unlike for the constant lag case, here we introduce a discretization on a constrained mesh with a non-constant stepsize which simplifies the qualitative analysis of corresponding numerical schemes. The recent results describing the asymptotic stability region as well as asymptotic behaviour of the numerical solution are presented.

4.1.1. Asymptotic stability of the differential equation

We consider the delay differential equation

$$x'(t) = a\,x(t) + b\,x(qt), \qquad t > 0, \tag{4.1}$$

where a, b and $0 < q < 1$ are real scalars. The asymptotic stability region S_q^* for (4.1) is then defined as the set of all real couples (a, b) for which any solution $x(t)$ of (4.1) tends to zero as $t \to \infty$. It is known (see e.g. Kato and McLeod [33]) that S_q^* for (4.1) is given by

$$S_q^* = \left\{(a, b) \in \mathbb{R}^2 : |b| < -a\right\},$$

which yields the necessary and sufficient condition for the asymptotic stability of (4.1). Furthermore, the analysis of qualitative behaviour performed by Kato and McLeod [33] and Iserles [31] showed that the solution of (4.1) satisfies

$$x(t) = O\left(t^{-\log_q |b/a|}\right) \qquad \text{as } t \to \infty, \tag{4.2}$$

where the constant $-\log_q |b/a|$ is not improvable.

4.1.2. Discretization of the differential equation

In the previous chapter, we have used discretizations on a uniform mesh with a constraint on its stepsize given by $h = \tau/k$, where $\tau > 0$ was a constant lag of the exact equation and k a positive integer. By this choice, we have avoided the interpolation of delayed terms and we obtained the constant coefficients difference equation of the order $k + 1$. Unfortunately, this approach is not suitable for the delay differential equations with infinite lag among which (4.1) also belongs. Considering the uniform grid, the application of the Θ-method to (4.1) yields a difference equation of variable order with non-constant coefficients. Moreover, the evaluation of the delayed terms requires a supplementary interpolation procedures. Therefore, we use a different discretization scheme, which was proposed and analysed by Liu [44] and Bellen at al. [3] and results in a difference equation of a fixed order.

Let T_0 be a fixed positive number. We divide the interval $\langle 0, T_0\rangle$ by p grid points

$$0 = t_0 < t_1 < t_2 < \cdots < t_p = T_0 \,.$$

Further, we build a primary mesh based on the following relation

$$T_{k+1} = \frac{T_k}{q}, \qquad k = 0, 1, \ldots$$

In this way, we define the primary intervals

$$H_k = T_{k+1} - T_k = T_0 \frac{1-q}{q^{k+1}}, \qquad k = 0, 1, \ldots$$

Observe that the primary intervals H_k increases exponentially. Furthermore, we define the global mesh by partitioning every primary interval into a fixed number m of subintervals of the same size. We set

$$h_n = \frac{H_{\lfloor (n-p)/m \rfloor}}{m} = \frac{T_0}{m} \frac{1-q}{q^{\lfloor (n-p)/m \rfloor + 1}} \qquad n = p, p+1, \ldots,$$

where $\lfloor (n-p)/m \rfloor$ denotes the integer part of the ratio $(n-p)/m$.

Setting $\ell = (n-p) \mod m$, we define the grid points of the resulting quasi-geometric mesh as

$$t_n := T_{\lfloor (n-p)/m \rfloor} + \ell h_{n-1} \qquad n = p, p+1, \ldots$$

An example of this mesh for $q = 2$ is depicted in Figure 4.1 (we set the parameters $p = 8$, $m = 4$ and $T_0 = 1$).

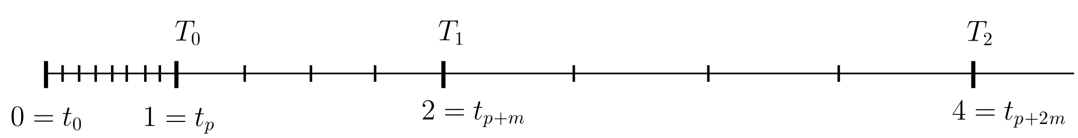

Figure 4.1: The scheme of quasi-geometric mesh

Note that one of the advantages of this grid consists in the avoidance of the interpolation of the delayed terms due to

$$q t_n = t_{n-m} \qquad n = p+m, p+m+1, \ldots$$

Using the quasi-geometric mesh, the Θ-method applied to (4.1) yields the recurrence

$$y_{n+1} + \alpha y_n + \beta y_{n-m+1} + \gamma y_{n-m} = 0, \qquad n = p+m, p+m+1, \ldots \tag{4.3}$$

with

$$\alpha = -\frac{1 + (1-\Theta) a h_n}{1 - \Theta a h_n}, \quad \beta = -\frac{\Theta b h_n}{1 - \Theta a h_n}, \quad \gamma = -\frac{(1-\Theta) b h_n}{1 - \Theta a h_n} \,. \tag{4.4}$$

We assume here $1 - \Theta a h_n \neq 0$.

Note that the difference equation (4.3), (4.4) is of a fixed order but its coefficients are not constant, because we have employed the mesh with a non-constant stepsize. Therefore, the analysis of the Θ-method discretization for (4.1) cannot be performed in the same way as for the differential equation with a constant delay (3.12), even though the resulting difference equations are seemingly similar (compare (3.15), (3.16) and (4.3), (4.4)).

By the asymptotic stability region $S_{q,p}^{\Theta}(m)$ of the Θ-method discretization of (4.1) we understand the set of real couples (a, b) for which any solution y_n of (4.3), (4.4) tends to zero as $n \to \infty$. We say that the Θ-method for (4.1) is asymptotically stable if it holds

$$S_q^* \subset \bigcap_{m=1}^{\infty} S_{q,p}^{\Theta}(m)\,.$$

4.1.3. Numerical stability of the Θ-methods and related issues

First, we recall the basic result of the Θ-method applied to (4.1) concerning its asymptotic stability.

Theorem 4.1. *The Θ-method applied to (4.1) on the quasi-geometric mesh is asymptotically stable if and only if $1/2 < \Theta \leq 1$.*

This theorem has been proved by Bellen et al. in [3]. Their result has been extended by Čermák in [6] who provided the necessary and sufficient conditions describing $S_{q,p}^{\Theta}(m)$ and derived the following asymptotic estimates of (4.3), (4.4).

Theorem 4.2. *Let y_n be a solution of (4.3), (4.4), where $a,\, b \neq 0$ and $0 < \Theta \leq 1$. Then we distinguish the following cases:*

(a) *Let $|b|\Theta^m \geq |a|(1-\Theta)^m$. If $b\Theta^m + a(\Theta - 1)^m \neq 0$, then*

$$y_n = O\left(|b/a|^{\frac{n}{m}}\right) \qquad \text{as } n \to \infty\,. \tag{4.5}$$

If $b\Theta^m + a(\Theta - 1)^m = 0$, then

$$y_n = O\left(n|b/a|^{\frac{n}{m}}\right) \qquad \text{as } n \to \infty\,.$$

(b) *Let $|b|\Theta^m < |a|(1-\Theta)^m$. Then there exists a constant η (depending on y_n) such that*

$$y_n = (\eta + o(1))\left(\frac{\Theta - 1}{\Theta}\right)^n \qquad \text{as } n \to \infty\,. \tag{4.6}$$

Theorem 4.3. *Let y_n be a solution of (4.3), (4.4), where $a,\, b \neq 0$, $1 + a h_n \neq 0$ for all $n \in \mathbb{Z}^+$ and $\Theta = 0$. Then there exists a constant ν (depending on y_n) such that*

$$y_n = (\nu + o(1)) \prod_{j=0}^{n-1} (1 + a h_j) \qquad \textit{as } n \to \infty\,.$$

The consequence of Theorem 4.2, Theorem 4.3 and their consecutive discussion in [6] is a description of the $S_{q,p}^{\Theta}(m)$.

Corollary 4.4. *Let $0 < q < 1$. A real couple (a, b) belongs to $S^{\Theta}_{q,p}(m)$ if and only if*

$$|b| < |a|, \qquad \Theta > 1/2\,.$$

We point out that the stability region for the trapezoidal rule is an empty set, which is in contrast with its application to differential equations with a constant lag, for which we observed the closest resemblance, among the Θ-methods, with the stability regions of underlying equations. However, it was shown in Remark 3.1 of [3] that any solution y_n of (4.3), (4.4) is bounded if $\Theta = 1/2$ and $|b| < |a|$. Furthermore, from (4.6) follows that any solution of (4.3), (4.4) tends to a finite constant provided $\Theta = 1/2$ and $|b| < |a|$. In addition, Corollary 4.3 in [6] provides the following long-time behaviour of trapezoidal rule on the stability boundary $|b| = -a$ of the exact equation (4.1).

Corollary 4.5. *Let $\Theta = 1/2$ and $|a| = |b|$. Then all solutions y_n of (4.3), (4.4) are bounded if and only if*

$$b + (-1)^m a \neq 0\,.$$

In particular, let a, b be non-zero real scalars and let $b = a$ ($b = -a$). Then all solutions of (4.3), (4.4) are bounded if and only if m is even (m is odd), respectively.

Finally, Čermák in [6] investigated the potential of (4.3), (4.4) to retain the asymptotic estimate (4.2) of the exact solution of (4.1). It was shown, that (4.5) presents exactly the same asymptotic estimate as it holds for the exact solution of the differential equation (4.1).

4.2. The equation $x'(t) = a\,x(t) + b\,x(\xi(t))$

Our next aim is to extend the results of the previous section to equations where the delayed argument is given by a general function $\xi(t) \in C^1(\langle t_0, \infty))$ satisfying

$$\begin{aligned}
&\xi(t_0) = t_0\,, &&\xi(t) < t, &&\text{for } t > t_0\,, &&\lim_{t\to\infty} \xi(t) = \infty,\\
&\xi'(t_0) < 1\,, &&\xi'(t) > 0, &&\text{for } t \geq t_0\,, &&\xi'(t) \text{ is non-increasing for } t \geq t_0\,.
\end{aligned} \tag{4.7}$$

We introduce a constrained mesh suitable for discretization of such an equation and determine the asymptotic stability of both the exact and discretized equations. Further, we provide and compare asymptotic estimates of their solutions.

4.2.1. Asymptotic stability of the differential equation

We consider the delay differential equation

$$x'(t) = a\,x(t) + b\,x(\xi(t)), \qquad t \in I = (t_0, \infty), \tag{4.8}$$

where a, b are real scalars and $\xi(t)$ is a continuously differentiable function on I satisfying (4.7). The asymptotic stability region S^*_ξ for (4.8) is defined as the set of all real couples (a, b) for which any solution $x(t)$ of (4.8) tends to zero as $t \to \infty$. To obtain its description, we first recall the result of Heard [26] who studied the asymptotic behaviour of (4.8) using the Schröder's equation. The Schröder's equation is a functional equation

$$\varphi(\xi(t)) = q\varphi(t), \qquad t \in \langle t_0, \infty), \tag{4.9}$$

where $q \in \mathbb{R}$ and $\xi(t)$ is a given function. A study of this equation is given in the book Kuczma, Choczewski and Ger [38]. Here, we state the result relevant to our further analysis.

Proposition 4.6. *Let $\xi(t)$ be a continuously differentiable function on $\langle t_0, \infty)$ satisfying (4.7) and let $q = \xi'(t_0)$. Then there exists a positive solution $\varphi(t) \in C^1(I)$ of (4.9) with a positive and bounded derivative on I such that $\lim_{t\to\infty} \varphi(t) = \infty$.*

Proof. The proof of this assertion is covered by Proposition 1 of Čermák [5]. □

The asymptotic estimate derived by Heard [26] is as follows.

Theorem 4.7. *Let $a < 0$ and $q = \xi'(t_0)$. Then the solution $x(t)$ of (4.8) satisfies*

$$x(t) = O\left((\varphi(t))^{-\log_q |b/a|}\right) \qquad \text{as } t \to \infty, \tag{4.10}$$

where $\varphi(t)$ is a solution of the Schröder's equation (4.9) with the properties described in Proposition 4.6.

Moreover, the analysis provided by Heard [26] implies the following necessary and sufficient condition describing S^*_ξ.

Theorem 4.8. *Let $\xi(t)$ be a continuously differentiable function on I satisfying (4.7). A real couple (a, b) belongs to S^*_ξ if and only if*

$$|b| < -a\,.$$

4.2.2. Discretization of the differential equation

Similarly to the previous section, the Θ-method discretization of (4.8) on the uniform grid results in a difference equation of variable order with non-constant coefficients. Since analysis of such a difference equation is a difficult task, we introduce the following constrained grid proposed by Guglielmi and Zennaro [22], which is a generalization of a quasi-geometric mesh used for the discretization of the pantograph equation (4.1) and ensures a fixed order of the resulting recurrence.

Let T_0 be a fixed positive number. We divide the interval $\langle t_0, T_0\rangle$ by p grid points

$$t_0 < t_1 < t_2 < \cdots < t_p = T_0\,.$$

Then we build a primary mesh based on the following relation

$$T_{k+1} = \xi^{-1}(T_k), \qquad k = 0, 1, \ldots$$

Further, we evenly divide the first primary interval $\langle T_0, T_1\rangle$ into fixed number of m subintervals. The division of the subsequent primary intervals is then given by the relation

$$t_n = \xi^{-1}(t_{n-m}), \qquad n = p+m, p+m+1, \ldots$$

Since $\xi'(t) \le \tilde{q} = \xi'(t_0) < 1$ on I, then the stepsize satisfies

$$h_n \ge \frac{h_{n-m}}{\tilde{q}}, \qquad n = p+m, p+m+1, \ldots \tag{4.11}$$

Therefore

$$\lim_{n\to\infty} h_n = \infty$$

and, moreover, h_n increases exponentially. The resulting mesh is called an almost-geometric mesh, which is due to property (4.11).

The Θ-method applied to (4.8) on the almost-geometric mesh is formally the same as for the proportional delay, i.e.

$$y_{n+1} + \alpha y_n + \beta y_{n-m+1} + \gamma y_{n-m} = 0, \qquad n = p+m, p+m+1, \ldots \tag{4.12}$$

with

$$\alpha = -\frac{1+(1-\Theta)ah_n}{1-\Theta ah_n}, \quad \beta = -\frac{\Theta bh_n}{1-\Theta ah_n}, \quad \gamma = -\frac{(1-\Theta)bh_n}{1-\Theta ah_n}. \tag{4.13}$$

We assume $1-\Theta ah_n \neq 0$, too. The difference with respect to the proportional case (4.3), (4.4) consists in a distinct growth of h_n.

By the asymptotic stability region $S^{\Theta}_{\xi,p}(m)$ of the Θ-method discretization of (4.8) we understand the set of real couples (a,b) for which any solution y_n of (4.12), (4.13) tends to zero as $n\to\infty$.

We say that the Θ-method for (4.8) is asymptotically stable if it holds

$$S^*_\xi \subset \bigcap_{m=1}^{\infty} S^{\Theta}_{\xi,p}(m)\,.$$

4.2.3. Numerical stability of the Θ-methods and related issues

The asymptotic stability of Θ-methods for (4.8) follows directly from the analysis of more general equations provided by Guglielmi (see [22] and [23]) and it can be summarized as follows.

Theorem 4.9. *The Θ-method applied to (4.8) on the almost-geometric mesh is asymptotically stable if and only if $1/2 < \Theta \leq 1$.*

However, the precise description of the stability region $S^{\Theta}_{\xi,p}(m)$ as well as the asymptotic properties of numerical solution have remained an unsolved problem. We provide the answer for both these issues. Firstly, we deal with the asymptotics of (4.12), (4.13).

Theorem 4.10. *Let y_n be a solution of (4.12), (4.13), where $a, b \neq 0$ and $0 < \Theta \leq 1$. Then we distinguish the following cases:*

(a) *Let $|b|\Theta^m \geq |a|(1-\Theta)^m$. If $b\Theta^m + a(\Theta-1)^m \neq 0$, then*

$$y_n = O\left(|b/a|^{\frac{n}{m}}\right) \qquad \text{as } n\to\infty\,. \tag{4.14}$$

If $b\Theta^m + a(\Theta-1)^m = 0$, then

$$y_n = O\left(n|b/a|^{\frac{n}{m}}\right) \qquad \text{as } n\to\infty\,.$$

(b) *Let $|b|\Theta^m < |a|(1-\Theta)^m$. Then there exists a constant η (depending on y_n) such that*

$$y_n = (\eta + o(1))\left(\frac{\Theta-1}{\Theta}\right)^n \qquad \text{as } n\to\infty\,.$$

Proof. Since (4.12), (4.13) differs from (4.3), (4.4) only in the behaviour of the stepsize h_n, the proof of this theorem is analogous to the proof of Theorem 4.2 provided by Čermák in [6]. The analysis is based on the application of theory of Poincaré difference equations. More precisely, it utilizes Theorem 2.11 and Theorem 2.12 in the case (a), while Theorem 2.13 is employed in the case (b). The key assumption for both cases is validity of (2.20), which follows from convergence of the sum $\sum_{n=1}^{\infty} 1/h_n$ ensured due to (4.11). □

Theorem 4.11. *Let y_n be a solution of (4.12), (4.13), where a, $b \neq 0$, $1 + ah_n \neq 0$ for all $n \in \mathbb{Z}^+$ and $\Theta = 0$. Then there exists a constant ν (depending on y_n) such that*

$$y_n = (\nu + o(1)) \prod_{j=0}^{n-1} (1 + ah_j) \qquad \text{as } n \to \infty\,.$$

Proof. Analogously to the proof of Theorem 4.10, the proof this theorem is only a simple modification of the proof for the proportional delay provided in [6] and therefore it is omitted. □

The following description of the asymptotic stability region $S^{\Theta}_{\xi,p}(m)$ follows from Theorem 4.10 and Theorem 4.11 due to fact that their asymptotic estimates cannot be improved. More precisely, there exists solutions of (4.12), (4.13) asymptotically equivalent to $|b/a|^{\frac{n}{m}}$, $n|b/a|^{\frac{n}{m}}$, $(\Theta - 1)/\Theta$ and $\prod_{j=0}^{n-1}(1 + ah_j)$.

Corollary 4.12. *Let $\xi(t)$ be a continuously differentiable function on I satisfying (4.7). Then a real couple (a, b) belongs to $S^{\Theta}_{\xi,p}(m)$ if and only if*

$$|b| < |a|, \qquad \Theta > 1/2\,.$$

The remaining issue is the comparison of the asymptotic estimates of the exact and discretized equation. First, we observe that

$$\varphi(t_n) = q^{-\lfloor (n-p)/m \rfloor} \varphi(\xi^{(-\lfloor (n-p)/m \rfloor)}(t_n)) = q^{-\lfloor (n-p)/m \rfloor} \varphi(t_{p+j})$$

for some $j = 0, 1, \ldots, m - 1$. Thus,

$$q^{\frac{p-m}{m}}\, \varphi(t_{p+j})\, q^{-\frac{n}{m}} \le \varphi(t_n) \le q^{\frac{p}{m}}\, \varphi(t_{p+j})\, q^{-\frac{n}{m}}\,.$$

Then, we can rewrite (4.14) as

$$y_n = O\left(\left(q^{\log_q(|b/a|)} \right)^{\frac{n}{m}} \right) = O\left(\left(q^{-\frac{n}{m}} \right)^{-\log_q(|b/a|)} \right) = O\left((\varphi(t_n))^{-\log_q(|b/a|)} \right) \quad \text{as } n \to \infty.$$

We summarize previous considerations in the following

Corollary 4.13. *Let a, $b \neq 0$, $0 < \Theta \le 1$, $|b|\Theta^m \ge |a|(1-\Theta)^m$ and $b\Theta^m + a(\Theta - 1)^m \neq 0$. Then the solution y_n of (4.12), (4.13) satisfies*

$$y_n = O\left((\varphi(t_n))^{-\log_q(|b/a|)} \right) \quad \text{as } n \to \infty, \tag{4.15}$$

which presents exactly the same estimate of the numerical solution as (4.10) yields for the exact solution.

4.3. The equation $x'(t) = a\,x(t) + \sum_{i=1}^{r} b_i\,x(\xi_i(t))$

In this section, we deal with a delay differential equation which involves several terms with unbounded lags. A problem of this type has been studied by Liu and Li [43] for the multipantograph equation, i.e. equation with several proportional delays. We assume the case of general unbounded lags and derive the conditions ensuring the asymptotic stability of the exact as well as discretized equation. Moreover, we provide an asymptotic estimate of the exact solution, too. Finally, we present necessary and sufficient conditions describing the stability region of a discretized equation with two (iterated) delays, including the asymptotic estimates of its solution for a specific sign variant of coefficients of the differential equation.

4.3.1. Asymptotic stability of the differential equation

We consider the delay differential equation

$$x'(t) = a\,x(t) + \sum_{i=1}^{r} b_i\,x(\xi_i(t)), \qquad t \in I = (t_0, \infty), \tag{4.16}$$

where a, b_i are real scalars, r is a positive integer and $\xi_i(t)$ are functions from the set $\mathcal{F}_\xi$ given by

$$\mathcal{F}_\xi = \{\tilde{\xi}(t):\ \tilde{\xi}(t) \equiv \xi^j(t),\ j = 1, 2, \dots\},$$

where $\xi(t)$ is a continuously differentiable function on I satisfying (4.7) and the symbol $\xi^j(t)$ means the j-th iterate of $\xi(t)$.

Analogously to the case $r = 1$, we define the asymptotic stability region $\mathbf{S}^*_\xi$ for (4.16) as the set of all real $r+1$-tuples $(a, b_1, b_2, \dots, b_r)$ for which any solution $x(t)$ of (4.16) tends to zero as $t \to \infty$. To determine $\mathbf{S}^*_\xi$, we firstly provide the asymptotic estimate of the solution. Similarly to the case $r = 1$, we introduce the system of Schröder's equations and recall assertion discussing existence of its simultaneous solution which is due to Čermák [5].

Proposition 4.14. *Let $\xi_i(t) \in \mathcal{F}_\xi$ and $q_i = \xi_i'(t_0)$. Then the system of Schröder's equations*

$$\varphi(\xi_i(t)) = q_i\varphi(t), \qquad i = 1, 2, \dots, r \tag{4.17}$$

has a positive solution $\varphi(t) \in C^1(I)$ with a positive and bounded derivative on I such that $\lim_{t\to\infty} \varphi(t) = \infty$.

Remark 4.15. Let $\xi'(t_0) = q$. Since $\xi_i(t) \in \mathcal{F}_\xi$, then $q_i = q^{w_i}$ where $w_i \in \mathbb{Z}^+$ denotes the number of iterations required to obtain $\xi_i(t)$ from the generating function $\xi(t)$.

Theorem 4.16. *Let $a < 0$ and $q_i = \xi_i'(t_0)$. Then the solution $x(t)$ of (4.16) satisfies*

$$x(t) = O\left((\varphi(t))^\kappa\right) \qquad \text{as } t \to \infty, \tag{4.18}$$

where $\varphi(t)$ is a solution of the system of Schröder's equations (4.17) with the properties described in Proposition 4.14 and $\kappa \in \mathbb{R}$ is a solution of the auxiliary equation

$$\sum_{i=1}^{r} |b_i| {q_i}^\kappa = -a\,. \tag{4.19}$$

Remark 4.17. Note that $f(t) \equiv \sum_{i=1}^{r} |b_i|(q_i)^t$, $q_i = \xi_i'(t_0)$ is a continuous bijection of $(-\infty, \infty)$ onto $(0, \infty)$. Therefore, the auxiliary equation (4.19) has exactly one solution κ. In particular, for $r = 1$ (4.19) becomes $|b|q^\kappa = -a$, solution of which is $\kappa = -\log_q |b/a|$. This coincides with the result given by Theorem 4.7.

The proof procedure of Theorem 4.16 originates from that in [5]. We present it here in a full version.

Proof. The function $\varphi(t)$ is positive for all $t > t_0$. Then the substitution

$$s = \log \varphi(t), \qquad z(s) = (\varphi(t))^{-\kappa} x(t), \tag{4.20}$$

where $t > t_0$, converts (4.16) into the form

$$z'(s) = (av'(s) - \kappa)z(s) + \sum_{i=1}^{r} b_i {q_i}^\kappa v'(s) z(s - u_i)\,,$$

where $s \in J = \langle s_0, \infty)$, $s_0 > \log \varphi(t_0)$, $v(s) \equiv \varphi^{-1}(\mathrm{e}^s)$ on J and $u_i = \log q_i^{-1}$ for $i = 1, 2, \ldots, r$. Then

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\mathrm{e}^{\kappa s - av(s)} z(s)\right] = \sum_{i=1}^{r} b_i {q_i}^\kappa v'(s) \mathrm{e}^{\kappa s - av(s)} z(s - u_i)\,. \tag{4.21}$$

Due to the boundedness of $\varphi'(t)$ on I,

$$\frac{1}{v'(s)} = \frac{\varphi'(v(s))}{\varphi(v(s))} = O\left(\mathrm{e}^{-s}\right) \qquad \text{as } s \to \infty. \tag{4.22}$$

Now we can choose $d_0 \geq s_0$ such that $\kappa - av'(s) > 0$ for every $s \geq d_0$. Put $u = \min(u_i,\ i = 1, 2, \ldots, r)$ and divide J into intervals $J_\ell = \langle d_{\ell-1}, d_\ell\rangle$, where $d_\ell = d_0 + \ell u$. Further, set $Z_\ell = \max\{|z(s)|,\ s \in \cup_{k=1}^{\ell} J_k\}$, $\ell = 1, 2, \ldots$. If we choose any $s^* \in J_{\ell+1}$, then we can integrate (4.21) over $\langle d_\ell, s^*\rangle$ to obtain

$$\mathrm{e}^{\kappa s - av(s)} z(s)|_{d_\ell}^{s^*} = \sum_{i=1}^{r} \int_{d_\ell}^{s^*} b_i {q_i}^\kappa v'(s) \mathrm{e}^{\kappa s - av(s)} z(s - u_i)\,\mathrm{d}s\,.$$

Then

$$z(s^*) = \mathrm{e}^{\kappa(d_\ell - s^*) + a(v(s^*) - v(d_\ell))} z(d_\ell) + \mathrm{e}^{-\kappa s^* + av(s^*)} \sum_{i=1}^{r} \int_{d_\ell}^{s^*} b_i {q_i}^\kappa v'(s) \mathrm{e}^{\kappa s - av(s)} z(s - u_i)\,\mathrm{d}s.$$

Consequently,

$$\begin{aligned} |z(s^*)| &\leq Z_\ell\, \mathrm{e}^{\kappa(d_\ell - s^*) + a(v(s^*) - v(d_\ell))} + Z_\ell\, \mathrm{e}^{-\kappa s^* + av(s^*)} \int_{d_\ell}^{s^*} \sum_{i=1}^{r} |b_i| {q_i}^\kappa v'(s) \mathrm{e}^{\kappa s - av(s)}\,\mathrm{d}s \\ &= Z_\ell\, \mathrm{e}^{\kappa(d_\ell - s^*) + a(v(s^*) - v(d_\ell))} - Z_\ell\, \mathrm{e}^{-\kappa s^* + av(s^*)} \int_{d_\ell}^{s^*} av'(s) \mathrm{e}^{\kappa s - av(s)}\,\mathrm{d}s \end{aligned} \tag{4.23}$$

due to (4.19). We estimate the last integral as

$$\int_{d_\ell}^{s^*}(\kappa - av'(s))\mathrm{e}^{\kappa s - av(s)}\mathrm{d}s - \int_{d_\ell}^{s^*}\kappa \mathrm{e}^{\kappa s - av(s)}\mathrm{d}s \le \mathrm{e}^{\kappa s - av(s)}|_{d_\ell}^{s^*} + |\kappa|\int_{d_\ell}^{s^*}\mathrm{e}^{\kappa s - av(s)}\mathrm{d}s. \quad (4.24)$$

Rewrite the last term of (4.24) as

$$|\kappa|\int_{d_\ell}^{s^*}\mathrm{e}^{\kappa s - av(s)}\mathrm{d}s = \int_{d_\ell}^{s^*}\frac{|\kappa|}{\kappa - av'(s)}\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{\kappa s - av(s)}\mathrm{d}s\,.$$

Notice that due to (4.22)

$$\frac{|\kappa|}{\kappa - av'(s)} = O\left(\mathrm{e}^{-s}\right) \qquad \text{as } s \to \infty.$$

Then

$$\int_{d_\ell}^{s^*}\frac{|\kappa|}{\kappa - av'(s)}\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{\kappa s - av(s)}\mathrm{d}s \le N\int_{d_\ell}^{s^*}\mathrm{e}^{-s}\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{\kappa s - av(s)}\mathrm{d}s \le N\mathrm{e}^{-d_\ell}\,\mathrm{e}^{\kappa s - av(s)}|_{d_\ell}^{s^*}$$

for a suitable $N > 0$. Consequently,

$$-\int_{d_\ell}^{s^*}av'(s)\mathrm{e}^{\kappa s - av(s)}\mathrm{d}s \le \mathrm{e}^{\kappa s - av(s)}|_{d_\ell}^{s^*}(1 + N\mathrm{e}^{-d_\ell})\,.$$

Substituting this back into (4.23) we obtain

$$|z(s^*)| \le Z_\ell \mathrm{e}^{\kappa(d_\ell - s^*) + a(v(s^*) - v(d_\ell))} + Z_\ell \mathrm{e}^{av(s^*) - \kappa s^*}\,\mathrm{e}^{\kappa s - av(s)}|_{d_\ell}^{s^*}(1 + N\mathrm{e}^{-d_\ell}) \le Z_\ell(1 + N\mathrm{e}^{-d_\ell})\,.$$

Since s^* has been chosen arbitrary it holds

$$Z_{\ell+1} \le Z_\ell(1 + N\mathrm{e}^{-d_\ell}) \le Z_1\prod_{k=1}^{\ell}(1 + N\mathrm{e}^{-d_k}), \qquad \ell = 1, 2, \ldots$$

Letting $\ell \to \infty$ we can see that the infinite product

$$\prod_{k=1}^{\infty}(1 + N\mathrm{e}^{-d_k})$$

converges. This implies that Z_ℓ is bounded as $\ell \to \infty$, hence $z(s)$ is bounded as $s \to \infty$. Substituting this back into (4.20) we obtain the asymptotic property (4.18). This completes the proof. □

Obviously, the rate of decay κ is negative whenever $\sum_{i=1}^{r}|b_i| < -a$. Consequently, we can state the following condition for the asymptotic stability of (4.16).

Corollary 4.18. *Let $a < 0$ and $\xi(t)$ be a continuously differentiable function on I satisfying (4.7). Then a real $r+1$-tuple $(a, b_1, b_2, \ldots, b_r)$ belongs to $\mathbf{S}_\xi^*$ if*

$$\sum_{i=1}^{r}|b_i| < -a\,. \tag{4.25}$$

This result is in agreement with results of Liu and Li [43], who derived the sufficient conditions for (4.16) with $\xi_i(t) = q_i t$, $0 < q_i < 1$, $i = 1, 2, \ldots, r$ using the Dirichlet series solution. However, their approach is not suitable for a general lags $\xi(t)$. Moreover, the result of Liu and Li [43] does not provide the asymptotic estimate of the solution.

4.3.2. Discretization of the differential equation

Analogously to the previous section, we use the almost-geometric mesh generated by the function $\xi(t)$. Then, by application of the Θ-method discretization, we obtain the difference equation

$$y_{n+1} + \alpha\, y_n + \sum_{i=1}^{r} (\beta_i\, y_{n-w_i m+1} + \gamma_i\, y_{n-w_i m}) = 0, \qquad n = p + w_r m, p + w_r m + 1, \ldots \quad (4.26)$$

where

$$\alpha = -\frac{1 + (1-\Theta) a h_n}{1 - \Theta a h_n}, \quad \beta_i = -\frac{\Theta b_i h_n}{1 - \Theta a h_n}, \quad \gamma_i = -\frac{(1-\Theta) b_i h_n}{1 - \Theta a h_n} \quad (4.27)$$

and $w_i \in \mathbb{Z}^+$ denotes the number of iterations required to obtain $\xi_i(t)$ from the generating function $\xi(t)$. We assume $w_r = \max(w_i, i = 1, 2, \ldots, r)$ and $1 - \Theta a h_n \neq 0$.

By the asymptotic stability region $\mathbf{S}^{\Theta}_{\xi,p}(m)$ of the Θ-method discretization of (4.8) we understand the set of real $r+1$-tuples $(a, b_1, b_2, \ldots, b_r)$ for which any solution y_n of (4.26), (4.27) tends to zero as $n \to \infty$.

We define the asymptotic stability of the Θ-method for (4.16) with respect to stability condition (4.25) derived in Corollary 4.18 for (4.16). Denote

$$\mathbf{S} = \left\{ (a, b_1, b_2, \ldots, b_r) \in \mathbb{R}^{r+1} : \sum_{i=1}^{r} |b_i| < -a \right\}.$$

We say that the Θ-method for (4.16) is asymptotically stable if it satisfies

$$\mathbf{S} \subset \bigcap_{m=1}^{\infty} \mathbf{S}^{\Theta}_{\xi,p}(m)\,.$$

4.3.3. Numerical stability of the Θ-methods and related issues

In this section, we first provide the conditions ensuring the asymptotic stability of (4.26), (4.27). Based on them, we determine the asymptotic stability of the Θ-methods for (4.16).

Theorem 4.19. *Let $\xi(t)$ be a continuously differentiable function on I satisfying (4.7). Then a real $r+1$-tuple $(a, b_1, b_2, \ldots, b_r)$ belongs to $\mathbf{S}^{\Theta}_{\xi,p}(m)$ if*

$$\sum_{i=1}^{r} |b_i| < |a|, \qquad \Theta > 1/2\,. \quad (4.28)$$

Proof. The recurrence (4.26), (4.27) is of a Poincaré type, which means that it is a linear difference equation whose non-constant coefficients have finite limits. Then, by Theorem 2.14, we have that (4.26), (4.27) is asymptotically stable if the limiting equation

$$y_{n+1} + \frac{1-\Theta}{\Theta} y_n + \sum_{i=1}^{r} \left(\frac{b_i}{a} y_{n-w_i m+1} + \frac{b_i(1-\Theta)}{a\Theta} y_{n-w_i m} \right) = 0$$

is asymptotically stable. This is equivalent to the problem whether all zeros of the corresponding characteristic polynomial

$$\begin{aligned} P(\lambda) &= \lambda^{w_r m+1} + \frac{1-\Theta}{\Theta}\lambda^{w_r m} + \sum_{i=1}^{r}\left(\frac{b_i}{a}\lambda^{(w_r-w_i)m+1} + \frac{b_i(1-\Theta)}{a\Theta}\lambda^{(w_r-w_i)m}\right) \\ &= \left(\lambda + \frac{1-\Theta}{\Theta}\right)\left(\lambda^{w_r m} + \sum_{i=1}^{r}\frac{b_i}{a}\lambda^{(w_r-w_i)m}\right) \end{aligned} \tag{4.29}$$

are located inside the unit circle. Obviously, $P(\lambda)$ has $w_r m+1$ zeros, where $\lambda_1 = \frac{\Theta-1}{\Theta}$ and $\lambda_2, \lambda_3, \ldots, \lambda_{w_r m+1}$ are the zeros of the polynomial

$$\tilde{P}(\lambda) = \lambda^{w_r m} + \sum_{i=1}^{r}\frac{b_i}{a}\lambda^{(w_r-w_i)m}.$$

Using Theorem 2.2 we get that $|\lambda_j| < 1$, $j = 2, 3, \ldots, w_r m+1$ if

$$\sum_{i=1}^{r}\left|\frac{b_i}{a}\right| < 1.$$

Since $|\lambda_1| < 1$ for $\Theta > 1/2$, we may conclude that (4.28) ensures the asymptotic stability of (4.26), (4.27). □

Theorem 4.19 immediately implies the asymptotic stability of Θ-methods for (4.16).

Corollary 4.20. *The Θ-method applied to (4.16) on the almost-geometric mesh is asymptotically stable if and only if* $1/2 < \Theta \le 1$.

Note, that in order to analyse the stability of (4.26), (4.27) we have to determine whether the characteristic polynomial (4.29) is of a Schur type. As it was mentioned in Chapter 2, to provide the explicit necessary and sufficient conditions guaranteeing this property is in general a difficult task. However, we can formulate the necessary and sufficient conditions describing $\mathbf{S}^{\Theta}_{\xi,p}(m)$ for some particular cases where the characteristic polynomial is of an appropriate type.

4.3.4. Some results on numerical stability of the equation $x'(t) = a\,x(t) + b_1\,x(\xi(t)) + b_2\,x(\xi^w(t))$

In the particular case with just two iterated delayed terms, we provide the necessary and sufficient conditions describing the appropriate asymptotic stability region $\mathbf{S}^{\Theta}_{\xi,p}(m)$. Moreover, in such a case we present some asymptotic estimates of the numerical solution and compare it with the corresponding estimate of the exact solution.

Let us consider the differential equation

$$x'(t) = a\,x(t) + b_1\,x(\xi(t)) + b_2\,x(\xi^w(t)), \qquad t \in I = (t_0, \infty), \tag{4.30}$$

where a, b_1, b_2 are real scalars, $\xi(t) \in C^1(I)$ satisfies (4.7) and w is a suitable integer. The Θ-method discretization of (4.30) on the almost-geometrical mesh yields the recurrence

$$y_{n+1} + \alpha\,y_n + \beta_1\,y_{n-m+1} + \gamma_1\,y_{n-m} + \beta_2\,y_{n-wm+1} + \gamma_2\,y_{n-wm} = 0, \tag{4.31}$$

$n = p + wm, p + wm + 1, \dots$, where

$$\alpha = -\frac{1 + (1-\Theta)ah_n}{1 - \Theta ah_n}, \quad \beta_i = -\frac{\Theta b_i h_n}{1 - \Theta ah_n}, \quad \gamma_i = -\frac{(1-\Theta)b_i h_n}{1 - \Theta ah_n} \tag{4.32}$$

for $i = 1, 2$.

To simplify the description of $\mathbf{S}^{\Theta}_{\xi,p}(m)$ for (4.31), (4.32), we introduce the symbols

$$W_1 = \arccos \frac{a^2 - b_1^2 - b_2^2}{2b_1 b_2} \Big/ \arccos \frac{b_2^2 - b_1^2 - a^2}{2ab_1},$$

$$W_2 = \arccos \frac{(-1)^w (a^2 - b_1^2 - b_2^2)}{2b_1 b_2} \Big/ \arccos \frac{a^2 + b_1^2 - b_2^2}{2ab_1},$$

where a, b_1, b_2 are given parameters.

Theorem 4.21. *Let $\xi(t)$ be a continuously differentiable function on I satisfying (4.7). A triplet (a, b_1, b_2) belongs to $\mathbf{S}^{\Theta}_{\xi,p}(m)$ for (4.31), (4.32) if and only if $\Theta > 1/2$ and one of the following conditions holds:*

$$|b_1| + |b_2| < |a|; \tag{4.33}$$

$$b_2 - b_1 = a, \qquad -|a| < \operatorname{sgn}(a)\, b_1 < 0; \tag{4.34}$$

$$b_1 + (-1)^w b_2 = a, \qquad 0 < \operatorname{sgn}(a)\, b_1 < |a|; \tag{4.35}$$

$$|a + b_1| < \operatorname{sgn}(a)\, b_2, \qquad \operatorname{sgn}(a)(b_1 + b_2) < |a|, \qquad (w-1) < W_1; \tag{4.36}$$

$$|a - b_1| < (-1)^{w+1} \operatorname{sgn}(a)\, b_2, \quad \operatorname{sgn}(a)((-1)^{w+1} b_2 - b_1) < |a|, \quad (w-1) < W_2. \tag{4.37}$$

Proof. Since (4.31), (4.32) is a Poincaré difference equation, it can be rewritten as

$$S_n + N_n = 0, \qquad n = p + m, p + m + 1, \dots$$

where S_n is the stationary part containing the limiting (constant coefficient) terms, namely

$$S_n = y_{n+1} + \frac{1-\Theta}{\Theta} y_n + \frac{b_1}{a} y_{n-m+1} \frac{b_1(1-\Theta)}{a\Theta} y_{n-m} + \frac{b_2}{a} y_{n-wm+1} + \frac{b_2(1-\Theta)}{a\Theta} y_{n-wm}$$

and N_n is the non-stationary part of (4.31), (4.32), i.e

$$\begin{aligned} N_n = &-\frac{1}{\Theta(1 - \Theta ah_n)} y_n - \frac{b_1}{a(1 - \Theta ah_n)} y_{n-m+1} - \frac{b_1(1-\Theta)}{a\Theta(1 - \Theta ah_n)} y_{n-m} \\ &- \frac{b_2}{a(1 - \Theta ah_n)} y_{n-wm+1} - \frac{b_2(1-\Theta)}{a\Theta(1 - \Theta ah_n)} y_{n-wm} \,. \end{aligned} \tag{4.38}$$

In order to analyse its asymptotic stability it is sufficient to do so for its limiting equation $S_n = 0$ (see Theorem 2.14). The problem of the asymptotic stability of $S_n = 0$ is equivalent to problem whether its characteristic polynomial

$$\begin{aligned} P(\lambda) &= \lambda^{wm+1} + \frac{1-\Theta}{\Theta}\lambda^{wm} + \frac{b_1}{a}\lambda^{(w-1)m+1} + \frac{b_1(1-\Theta)}{a\Theta}\lambda^{(w-1)m} + \frac{b_2}{a}\lambda + \frac{b_2(1-\Theta)}{a\Theta} \\ &= \left(\lambda + \frac{1-\Theta}{\Theta}\right)\left(\lambda^{wm} + \frac{b_1}{a}\lambda^{(w-1)m} + \frac{b_2}{a}\right) \end{aligned} \tag{4.39}$$

is of a Schur type, i.e. whether it has all the zeros inside the unit circle. Clearly, $\lambda_1 = \frac{\Theta-1}{\Theta}$ is a zero of $P(\lambda)$ and it implies the instability of (4.31), (4.32) for $\Theta \leq 1/2$. The remaining zeros of $P(\lambda)$ are zeros of

$$\tilde{P}(\lambda) = \lambda^{wm} + \frac{b_1}{a}\lambda^{(w-1)m} + \frac{b_2}{a}\,. \tag{4.40}$$

Substituting $\sigma = \lambda^m$ into (4.40), we arrive at

$$\bar{P}(\sigma) = \sigma^{w} + \frac{b_1}{a}\sigma^{w-1} + \frac{b_2}{a}\,. \tag{4.41}$$

To determine the necessary and sufficient conditions ensuring that $\bar{P}(\sigma)$ is of a Schur type we employ Theorem 2.3 with parameters α, β, γ and k given by

$$\alpha = \frac{b_1}{a}, \qquad \beta = 0, \qquad \gamma = \frac{b_2}{a}, \qquad k = w-1\,. \tag{4.42}$$

Substituting (4.42) into (C1) and (C2), we arrive at (4.33), (4.34), respectively. Conditions (C3) and (C4) can be jointly written as (4.35). The condition (2.5) with (4.42) yields

$$w - 1 < \arccos\frac{b_1^2 + b_2^2 - a^2}{2|b_1 b_2|} \Big/ \arccos\frac{a^2 + b_1^2 - b_2^2}{2|ab_1|}\,.$$

Then, it is a simple matter to show that (C5) and (C6), (C7) imply (4.36) and (4.37), respectively.

Therefore conditions (4.33)–(4.37) ensures that all the zeros $\bar{\sigma}$ of $\bar{P}(\sigma)$ satisfy $|\bar{\sigma}| < 1$. Then equivalently $|\lambda_j| < 1$ for all the zeros λ_j, $j = 2, 3, \ldots, wm+1$ of (4.39), which concludes the proof. □

In order to discuss the possible analogy of the asymptotic estimates of the exact and discretized equation, we first reformulate the auxiliary equation (4.19) for (4.30), i.e.

$$|b_1|q^{\kappa} + |b_2|q^{\kappa w} = -a, \qquad a < 0, \tag{4.43}$$

the solution κ of which provides a growth rate of the exact solution. Our aim is to obtain similar problem formulation as we have for the asymptotic stability of (4.31), (4.32). Substituting $p = q^{-\kappa}$ into (4.43) gives

$$R(p) \equiv p^{w} + \frac{|b_1|}{a}p^{w-1} + \frac{|b_2|}{a} = 0. \tag{4.44}$$

Since (4.43) has exactly one real zero (see Remark 4.17), then there exists exactly one positive zero $\tilde{p}$ of $R(p)$. Note, that $\kappa < 0$ (i.e. (4.30) is asymptotically stable) if $\tilde{p} < 1$. Thus, the sufficient condition for the asymptotic stability of (4.30) is equivalent to a condition ensuring that the polynomial $R(p)$ is of a Schur type. We recall that (4.31), (4.32) is asymptotically stable if $\Theta > 1/2$ and $\bar{P}(\sigma)$, given by (4.41), is of a Schur type. Comparing (4.44) and (4.41), we observe a close resemblance of the the asymptotic stability problem of the exact and discretized equation.

Theorem 4.22. *Let y_n be a solution of (4.31), (4.32), where $a < 0$, $b_1 > 0$, $b_2 > 0$, $0 < \Theta \leq 1$, $q = \xi'(t_0)$, $\xi(t)$ is a function generating the almost-geometric mesh and κ is a solution of an auxiliary equation*

$$b_1\, q^{\kappa} + b_2\, q^{\kappa w} = -a\,. \tag{4.45}$$

Then we distinguish the following cases:
(a) *Let $q^{-\kappa}\Theta^m > (1-\Theta)^m$ or m is odd and $q^{-\kappa}\Theta^m = (1-\Theta)^m$. Then*

$$y_n = O\left(q^{\frac{-\kappa n}{m}}\right) \qquad \text{as } n \to \infty. \tag{4.46}$$

(b) *Let $q^{-\kappa}\Theta^m = (1-\Theta)^m$ for m even. Then*

$$y_n = O\left(nq^{\frac{-\kappa n}{m}}\right) \qquad \text{as } n \to \infty. \tag{4.47}$$

(c) *Let $q^{-\kappa}\Theta^m \geq (1-\Theta)^m$. Then there exists a constant η (depending on y_n) such that*

$$y_n = (\eta + o(1))\left(\frac{\Theta - 1}{\Theta}\right)^n \qquad \text{as } n \to \infty\,. \tag{4.48}$$

Proof. Since $a < 0$, $b_1 > 0$ and $b_2 > 0$, then (4.45) coincides with (4.43) and therefore (4.45) can be transformed into $R(p)$, which is identical to $\bar{P}(\sigma)$. Since (4.43) has exactly one real zero (see Remark 4.17), there exists exactly one real positive zero $\tilde{p} = q^{-\kappa}$ of $R(p)$. Consequently, there exists a real positive zero $\tilde{\sigma}$ of $\bar{P}(\sigma)$ such that $\tilde{\sigma} = q^{-\kappa}$. Further, we determine when this zero is the largest in modulus. To do so, we substitute $\mu = \sigma/q^{-\kappa}$ into (4.41) and get

$$\bar{P}(\mu) = q^{-\kappa w}\mu^w + \frac{b_1}{a}q^{-\kappa(w-1)}\mu^{w-1} + \frac{b_2}{a}\,.$$

The necessary and sufficient condition for all zeros σ_i of $\bar{P}(\sigma)$ to be $|\sigma_i| \leq q^{-\kappa}$ is that all zeros μ_i of $\bar{P}(\mu)$ are $|\mu_i| \leq 1$, which is due to Theorem 2.1 if and only if

$$b_1 q^{\kappa} + b_2 q^{\kappa w} \leq -a\,.$$

This is obviously satisfied. In order to describe a long time behaviour of solutions of (4.31), (4.32) we are also interested in multiplicity of the zeros of $\bar{P}(\sigma)$. We recall that a number $\bar{u}$ is a zero of multiplicity k of a polynomial $P(u)$ if $P(\bar{u}) = 0$, $P'(\bar{u}) = 0, \ldots, P^{(k-1)}(\bar{u}) = 0$. Therefore, all the zeros of $\bar{P}(\sigma)$ are distinct if

$$b_2 \neq -\left(\frac{b_1}{w}\right)^w \left(\frac{1-w}{a}\right)^{w-1}.$$

Note that since we assume $a < 0$, $b_1 > 0$ and $b_2 > 0$, this condition is always satisfied.

Therefore, the characteristic polynomial (4.39) of the limiting equation corresponding to (4.31), (4.32) has $wm + 1$ zeros such that $\lambda_1 = \frac{\Theta-1}{\Theta}$, $|\lambda_2| = \cdots = |\lambda_{m+1}| = q^{-\kappa/m}$ and $|\lambda_j| < q^{-\kappa/m}$ for $j = m+2, m+3, \ldots, wm+1$. In order to provide the asymptotic estimates of (4.31), (4.32) we distinguish the following cases:

(a) If $q^{-\kappa}\Theta^m > (1-\Theta)^m$ or m is odd and $q^{-\kappa}\Theta^m = (1-\Theta)^m$. Then $|\lambda_2| = \cdots = |\lambda_{m+1}| = q^{-\kappa/m} \geq |\lambda_1|$ and all the roots are distinct. Since $\sum_{n=1}^{\infty} 1/h_n < \infty$, then series given by the coefficients of (4.38) are absolutely convergent, i.e.

$$\sum_{n=1}^{\infty} \left| \frac{1}{\Theta(1 - ah_n\Theta)} \right| < \infty, \quad \sum_{n=1}^{\infty} \left| \frac{b_i}{a(1 - ah_n\Theta)} \right| < \infty, \quad \sum_{n=1}^{\infty} \left| \frac{b_i(1-\Theta)}{a\Theta(1 - ah_n\Theta)} \right| < \infty$$

for $i = 1, 2$. Thus, the assumption of Theorem 2.11 is fulfilled and we can conclude that (4.46) holds.

(b) If $q^{-\kappa}\Theta^m = (1-\Theta)^m$ for m even, then the zero λ_1 coincides with some λ_j, $j = 2, 3, \ldots m+1$ and $|\lambda_1| = |\lambda_2| = \cdots = |\lambda_{m+1}| = q^{-\kappa/m}$. By application of Theorem 2.12 we obtain the asymptotic estimate (4.47).

(c) If $q^{-\kappa}\Theta^m < (1-\Theta)^m$, then $|\lambda_1| > |\lambda_2| = \cdots = |\lambda_{m+1}|$. Therefore we apply Theorem 2.13 which implies (4.48). $\square$

Using the same line of arguments as for Corollary 4.13 we get

Corollary 4.23. *Let $a < 0$, $b_1 > 0$, $b_2 > 0$, $0 < \Theta \leq 1$, $q^{-\kappa}\Theta^m > (1-\Theta)^m$ or m is odd and $q^{-\kappa}\Theta^m = (1-\Theta)^m$, $q = \xi'(t_0)$, $\xi(t)$ be a function generating the almost-geometric mesh and κ be a solution of an auxiliary equation (4.45). Then the solution y_n of (4.31), (4.32) satisfies*

$$y_n = O\left((\varphi(t_n))^{-\kappa}\right) \qquad \text{as } n \to \infty,$$

which presents exactly the same estimate of the numerical solution as (4.18) yields for the exact solution.

5. Conclusion

The doctoral thesis concerns with the qualitative and numerical analysis of the linear delay differential equations with constant as well as infinite lag.

In the first part of the thesis, we investigated the linear neutral delay differential equation

$$x'(t) = a\,x(t) + b\,x(t-\tau) + c\,x'(t-\tau), \qquad t > 0\,, \tag{5.1}$$

where a, b, c and $\tau > 0$ are real scalars. We discretized (5.1) using the Θ-method and derived the necessary and sufficient conditions describing the stability region of both exact and discretized equations. Based on them, we concluded that the Θ-method is not $N\tau(0)$-stable for any $0 \le \Theta \le 1$. Some properties of the discretized stability regions were mentioned, too. Further, we dealt with the particular cases of (5.1) where $c = 0$ and $c = a = 0$. The explicit conditions for the asymptotic stability regions of the Θ-method discretization as well as the modified midpoint method were presented. We discussed some properties of the derived stability regions, mainly with respect to changing stepsize.

The second part of the thesis concerns the delay differential equations with infinite lag. We investigated the Θ-method discretization on the constrained mesh and provided the description of the stability regions together with asymptotic estimates for the exact and numerical solution. The asymptotic stability of the equation with several infinite lags was also analysed. We derived the asymptotic estimate of its solution as well as the sufficient condition under which this equation is asymptotically stable. It was shown that for $0 \le \Theta \le 1/2$ the asymptotic stability region of Θ-method discretization is an empty set, while for $1/2 < \Theta \le 1$ it contains the presented stability region of the differential equation. The necessary and sufficient conditions and some asymptotic estimates were provided for the discretized equation with two delayed terms.

Finally, we mention some open problems and general remarks. We analysed separately the numerical stability of equations with constant and infinite lag, however the analysis of the equations with the infinite lag seems to be less complicated. As it was shown for differential equations with two lags, we are able to provide the necessary and sufficient conditions for discrete asymptotic stability of the equation with infinite lags but the analysis of the constant lag case is still an unsolved problem. The key role in the analysis of discretization of both kinds of delay differential equations plays the analysis of the corresponding delay difference equations. The delay difference equations do not have many original applications and therefore they have not been widely studied. In fact, it is the numerical discretization which motivates further investigation of qualitative properties of different types of difference equations. In Table 5.1 we provide an overview of the difference equations which had to be analysed in order to discuss the asymptotic stability of numerical discretizations of the studied equations. We emphasize that our numerical analysis is based on results of Čermák et al. [9] and [10] who derived the necessary and sufficient conditions for the asymptotic stability of these difference equations in an explicit form.

The asymptotic stability analysis of the Θ-method for (5.1) is complete in the sense that the necessary and sufficient conditions describing its exact as well as discrete stability regions were derived. The open problem remains the investigation of the delay differential equation with two constant lags.

A numerical motivation	The resulting difference scheme
Θ-method discretization for $x'(t) = a\,x(t) + b\,x(t-\tau) + c\,x'(t-\tau)$	$y_{n+1} + \alpha y_n + \beta y_{n-k+1} + \gamma y_{n-k} = 0$
Θ-method discretization for $x'(t) = a\,x(t) + b_1\,x(\xi(t)) + b_2\,x(\xi^w(t))$	$y_{n+1} + \alpha y_n + \gamma y_{n-k} = 0$
trapezoidal rule for $x'(t) = a\,x(t) + b\,x(t-\tau)$	$y_{n+1} + \alpha y_n + \beta(y_{n-k+1} + y_{n-k}) = 0$
Θ-method discretization for $x'(t) = b\,x(t-\tau)$	$y_{n+1} - y_n + \beta y_{n-k+1} + \gamma y_{n-k} = 0$
trapezoidal rule for $x'(t) = b\,x(t-\tau)$	$y_{n+1} - y_n + \beta(y_{n-k+1} + y_{n-k}) = 0$
Euler methods for $x'(t) = b\,x(t-\tau)$	$y_{n+1} - y_n + \gamma y_{n-k} = 0$
modified midpoint method for $x'(t) = a\,x(t) + b\,x(t-\tau)$	$y_{n+2} + \mu y_n + \nu y_{n-k} = 0$
midpoint method for $x'(t) = b\,x(t-\tau)$	$y_{n+2} - y_n + \nu y_{n-k} = 0$

Table 5.1: The corresponding difference equations to the numerical methods for studied differential equations

The presented results for the delay differential equation with the infinite lags are partial results of our current research and their generalization for the neutral equation is one of its possible extensions. In particular, if we consider the neutral equation

$$x'(t) = a\,x(t) + b\,x(\xi(t)) + c\,x'(\xi(t))\,, \qquad t \in (t_0, \infty), \tag{5.2}$$

then our results on discretization of this equation in the pure delayed case ($c = 0$) can be easily extended to the neutral case ($c \neq 0$). However, appropriate stability and asymptotic results on the underlying differential equation (5.2) are not known (a possible generalization of Theorem 4.7 to neutral equation (5.2) can be the subject of the next research). Also, the asymptotic estimates for the delay differential equation with two infinite lags regardless of sign of its coefficient can be investigated.

The previous results, methods and problems become more complicated if we consider non-autonomous delay equations. In such a case, we probably cannot expect the optimal (i.e. necessary and sufficient) stability conditions, which is due to utilized techniques. For some related results on delay differential and difference equations we refer, e.g. to papers [4], [12], [13], [14] and [15], which may also serve as a motivation for future research.

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List of Symbols

$\mathbb{R}$	the set of real numbers
$\mathbb{R}^n$	the set of real n-tuples
$\mathbb{Z}$	the set of integers
$\mathbb{Z}^+$	the set of positive integers
$C^1(I)$	the set of continuously differentiable functions on I
$C^3(I)$	the set of functions with three continuous derivatives on I
$\langle a, b\rangle$	a closed interval of real numbers
$x'(t)$, $\frac{\mathrm{d}x(t)}{\mathrm{d}t}$	the first derivative of function $x(t)$ with respect to t
$\operatorname{sgn} a$	the signum function
$\|a\|$	the modulus of complex numbers
$\lfloor a \rfloor$	the interger part of a real number
$\Re(a)$	the real part of a complex number
$a \mod b$	remainder of division of a by b
i	an imaginary unit
$\xi^j(t)$	the j-th iterate of function $\xi(t)$
$\xi^{-1}(t)$	the inverse function to $\xi(t)$
$O(x(t))$	the big Omicron notation
$o(x(t))$	the little omicron notation