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Fuzzy modely vícekriteriálního hodnocení a fuzzy klasifikace

Katedra matematické analýzy a aplikací matematiky
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DISSERTATION THESIS

Fuzzy models of multiple-criteria evaluation and fuzzy classification

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Abstrakt: Hlavní část této dizertační práce se zabývá uceleným systémem fuzzy metod vícekriteriálního hodnocení. Společnou vlastností těchto metod je použitý typ hodnocení – hodnocení variant představuje (fuzzy) stupeň naplnění daného cíle. Úloha vícekriteriálního hodnocení je popsána pomocí struktury zvané strom dílčích cílů. Podporována jsou kvalitativní i kvantitativní kritéria. Pro agregaci dílčích hodnocení v rámci stromu dílčích cílů lze použít více metod – lze využít fuzzifikované verze známých agregačních operátorů (fuzzy vážený průměr, fuzzy OWA operátor, fuzzifikovaný WOWA operátor, fuzzy Choquetův integrál) nebo fuzzy expertní systém. Druhá část práce se zabývá fuzzy klasifikací, kdy rozdělení objektů do jednotlivých tříd je popsáno pomocí báze fuzzy pravidel. Na rozdíl od většiny publikací na toto téma, které se soustředí zejména na odvození báze pravidel z dat, tato práce se zabývá situací, kdy pravidla jsou již známá (byla zadána expertem, nebo odvozena z dat) a je třeba podle nich přiřadit objektům odpovídající třídu. Bude definováno několik typů úloh fuzzy klasifikace a pro každý z nich budou rozebrány vhodné postupy řešení. Součástí této práce je i software FuzzME, který implementuje systém metod popsaný v této dizertační práci. Pomocí FuzzME je možné navrhovat i poměrně složité modely vícekriteriálního hodnocení (a fuzzy klasifikace). Možnosti tohoto software byly otestovány na praktických aplikacích, které budou v této práci také popsány.

Klíčová slova: Vícekriteriální hodnocení, fuzzy množiny, fuzzy klasifikace, agregační operátory

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Abstract: The first part of this thesis is devoted to a compact system of fuzzy methods for multiple-criteria evaluation. The basic structure describing the evaluation process is called a goals tree. Both qualitative and quantitative criteria are supported in this system. Multiple methods can be used for aggregation of the partial evaluations within the goals tree. The aggregation of partial fuzzy evaluations is done either by one of the fuzzified aggregation operators (fuzzy weighted average, fuzzy OWA operator, fuzzified WOWA operator, or fuzzified Choquet integral) or by a fuzzy expert system. The common feature of all these methods is the used type of evaluation – the evaluations of alternatives represent the (fuzzy) degrees of fulfillment of the given goals. The second part of the thesis focuses on the topic of fuzzy classification where the division of the objects into the individual classes is described by a fuzzy rule base. The majority of the literature studies mainly various means of deriving the fuzzy rule base from the data. The thesis, however, deals with the situation when the fuzzy rule base is already known (it has been set either by experts or derived from data) and it is necessary to assign the classes to the individual objects accordingly. Different types of fuzzy classification problems will be described and the choice of suitable methods will be discussed for each of them. A part of this thesis is the FuzzME software (attached on a CD). The FuzzME represents the software implementation of the described methods. It makes it possible to design even very complex models of multiple-criteria evaluation (or fuzzy classification models). The software has been applied on real-world problems that showed its strengths and versatility. These applications will be also described in this thesis.

Key words: Multiple-criteria evaluation, fuzzy sets, fuzzy classification, aggregation operators

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Prohlášení

Prohlašuji, že jsem dizertační práci zpracoval samostatně pod vedením doc. RNDr. Jany Talašové, CSc. a všechny použité zdroje jsem uvedl v seznamu literatury.

V Olomouci dne ....

podpis

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Poděkování

Rád bych poděkoval doc. RNDr. Janě Talašové, CSc. za vedení na doktorském studiu a za cenné rady při přípravě této práce.

List of the appendices

Appendix 1	–	The user guide for the latest version of the FuzzME software
Appendix 2	–	The professional curriculum vitae of the thesis author
CD	–	The FuzzME software

Notation and symbols

$\emptyset$ ... empty set

$\Re$ ... set of real numbers

$[a, b]$ ... closed interval with endpoints a and b

(a, b) ... open interval with endpoints a and b

$[a, b]^n$ ... n-th Cartesian power of the interval $[a, b]$

$\wp(G)$ ... power set of the set G

$C(x)$ ... membership degree of x in the fuzzy set C

$(\underline{c}(\alpha), \overline{c}(\alpha))$ ... a pair of functions used to define the fuzzy number C (see Definition 2.7 for more information)

Ker C ... kernel of a fuzzy set C

Supp C ... support of a fuzzy set C

C_α ... α-cut of a fuzzy set C

hgt C ... height of a fuzzy set C

$\mathcal{F}(X)$ ... system of all fuzzy sets on the universal set X

$\mathcal{F}_N([a, b])$ ... system of all fuzzy numbers on the interval $[a, b]$

μ ... fuzzy measure

$\tilde{\mu}$ ... FNV-fuzzy measure (fuzzy number valued fuzzy measure)

$C = (c_1, c_2, c_3, c_4)$ ... C is a linear (trapezoidal) fuzzy number.

$C = (c_1, c_2, c_3)$ ... C is a triangular fuzzy number.

Chapter 1

Introduction

1.1. Goals of the thesis

Several goals have been set for this thesis. The thesis should provide a complete description of the system of fuzzy multiple-criteria evaluation methods based on the fuzzification and extension of the Partial Goals Method [81]. Although the individual methods from the system have been published in various papers, the coherent description was missing, or it existed only in a reduced form [38, 81] because of the space limitation. The thesis should provide a complete description of the system in its full depth.

In frame of this system of methods, the further theoretical development should be performed. This has been accomplished by introducing multiple new results – specifically, by introduction of the FWOWA (fuzzified WOWA) aggregation method, by the proposal of the new Sugeno-WOWA inference, and by introduction of two methods for transitions from a simpler to some more advanced aggregation method, which can be very useful when a model with interacting criteria is designed.

Another goal of the thesis was to create a software implementation of the mentioned system of methods. This led to the development of the FuzzME software. The FuzzME is a complex software tool equipped with variety of methods and algorithms that so far existed only on a theoretical level. The software enables the decision-makers to use these novel methods and it makes it possible for mathematicians to study their behavior on real-world applications.

The thesis aimed to focus also on the topic of fuzzy classification. Specifically, it dealt with the situation when the fuzzy rule base had already been set and the objects should be assigned to the matching classes accordingly. The thesis shows that an important role in this process is played by the type of the structure formed by the classes. It proposes also the form, in which the classification results should be presented. Again, the theoretical results have been implemented into the FuzzME software.

The FuzzME was applied to multiple real-world problems, either in direct cooperation with the author of this thesis, or by foreign authors. The thesis summarizes these applications so that the reader could see how the described methods can be used in the practice.

1.2. Structure of the thesis

The thesis deals with two seemingly different topics – fuzzy multiple-criteria evaluation and fuzzy classification. Therefore, it will be divided into two main parts accordingly. Later, it will be shown that these two topics have a lot in common.

First, a system of fuzzy methods for solving multiple-criteria evaluation problems will be discussed in the Chapter 2. The chapter starts with a brief introduction and it lists relevant software tools. It will be shown, that there is a lack of complex software tools for fuzzy multiple-criteria evaluation, which emphasizes the importance of the FuzzME software that has been created in frame of this thesis. Next, the system of methods based on the Partial Goals Method [81] will be described in detail. After this description, the software implementation of these methods, the FuzzME, will be presented together with the list of real-world applications of this software.

The Chapter 3 is devoted to the topic of the fuzzy classification. After a description of the problem of interest, the fuzzy classification problems will be divided into categories and each of them will be studied separately. Finally, the use of the FuzzME software for fuzzy classification will be discussed.

The thesis text ends with the list of the research accomplishments presented in the thesis and the summary of the main facts in the conclusion. The thesis also contains the documentation for the FuzzME software in the Appendix 1 and the author's curriculum vitae in the Appendix 2. The FuzzME software itself can be found on the CD attached to this thesis.

Chapter 2

Fuzzy models of multiple-criteria evaluation and decision-making

2.1. Introduction

Making decisions is one of important humans' skills. For simpler problems, the decision can be made just by an intuition. However, with increasing number of alternatives and criteria that should be taken into account, the problem can easily become too complex. For important problems, for example in the business, making a wrong decision can be very costly. Moreover, in many situations, the transparency is required – sometimes even by the law. For example, in case of public tenders, the reasons that led to choosing the particular winner cannot be concealed in the "black box" of the human brain. Formalized methods for finding the best decision are necessary.

Generally, a multiple-criteria decision-making (MCDM) problem has the following structure. A set of alternatives $A = \{A_1, \ldots, A_n\}$ is given. These alternatives are assessed with respect to a given set of criteria $C = \{C_1, \ldots, C_m\}$. The pursued task is to find the best alternative from A taking into the consideration the criteria values themselves and also the additional information about the criteria (the decision-maker's preferences related to the criteria values and to the criteria themselves, and potentially, the interactions among the criteria).

To find the optimum alternative, it is sufficient just to calculate the overall evaluations of the alternatives and to choose the alternative with the maximum evaluation. The multiple criteria evaluation (MCE) can be thus seen as the first step in solving the MCDM problem. Depending on the character of the obtained evaluations, we will be able to make different conclusions. We can distinguish the following types of evaluation:

- **Ordinal evaluation:** In this case, the evaluation expresses just the ordering of the alternatives. If we know the ordering of the alternatives, we can select the best one. However, an ordinal evaluation is not sufficient to quantify the differences in performances of the various alternatives and a stronger type of evaluation must be used in these cases.

- **Cardinal evaluation of relative type:** Such an evaluation contains more information. When a ratio scale is used, we can say from the proportion of evaluations of two alternatives that the first alternative is, e.g., twice as good as the second one. When an interval scale is used and we consider three alternatives A_1, A_2 and A_3 ordered from the best one to the worst one, we can make conclusions such as that the difference between the evaluations of A_1 and A_2 is twice as big as the difference between the evaluations of A_2 and A_3. But still we cannot determine if these alternatives fulfill the given goal enough for the decision-maker. For example, this evaluation type would be insufficient for making decision whether a bank should grant a credit to a particular client.

- **Cardinal evaluation of absolute type (with respect to a given goal):** This evaluation type provides us with the most extensive information. Not only that we can compare the alternatives, but we can also say for each of them how much it satisfies the decision-maker's needs. In case of the bank, the evaluation would express the creditworthiness of a client. Then, the bank can decide on a threshold and grant the credit only to the clients with evaluation above this threshold.

This thesis deals with the system of methods that produce cardinal evaluations of the absolute type (with respect to the given goal of the decision-maker), i.e. the strongest of the listed three evaluation types. This makes it possible to solve a broad range of MCDM problems.

The MCDM problems are usually divided according to various aspects. The set of alternatives can be either finite or infinite. The first case is called multiple-criteria decision-making and the alternatives are usually listed explicitly. The latter case is known as multiple-criteria optimization. In this case, the alternatives are given by a set of constraints.

Further division of the problems can be made according to the number of decision-makers. In a simpler case, only one decision-maker is involved. In more complex cases, the final decision depends on the opinion of multiple decision-makers (group decision-making).

The MCDM problems can be also divided according to the possible presence of randomness. If there is some randomness involved, we obtain decision-making under risk, or decision-making under uncertainty.

In the following text, we will study only the situations with one decision-maker and no randomness present.

2.2. Methods of multiple criteria decision-making

A vast number of decision-making approaches have been developed over the time – from simple methods to highly sophisticated ones.

A large group of the methods is based on combining of the partial evaluations with respect to the criteria into the total evaluation by some aggregation operator. The simplest and obviously the most popular aggregation operator is the weighted average. These methods then differ in the way how the partial evaluations and the weights were obtained and on their interpretation.

A well-known approach that belongs to this group of methods is the Multiple Attribute Utility Theory (MAUT), which is based on the principles published in [99, 23]. The MAUT is theoretically very elaborate. One of its major advantages is that it can address the risk. However, an obstacle in using this method in practice is the amount of information that has to be provided. The nature of the information can also present a problem. The decision-maker is required to compare imaginary alternatives that may not be meaningful in the real world. Nevertheless, the MAUT has been applied in many fields. According to [96], the common applications areas for MAUT are economics, financing, insurance industry, water and energy management, and agriculture.

Another very popular method from this group is the Analytical Hierarchy Process (AHP) [70] proposed by Thomas L. Saaty. An important feature of the AHP is that the weights and evaluations of the alternatives with respect to the criteria are obtained by pair-wise comparisons, which are expressed using linguistic terms. The intensities of preferences for the pairs are written in form of the Saaty's matrix. The matrix is reciprocal and it consists of the values from Saaty's scale (numbers $1/9, 1/7, \ldots, 1, \ldots, 7, 9$ corresponding to the selected linguistic descriptions) with 1 on the main diagonal. The original AHP is based on the eigenvector method. In practice, modifications of this original approach with different methods to obtain the weights vector (or the partial evaluations vectors) form the matrix can be encountered. A distinct feature of AHP is also the use of a hierarchical structure for the description of the decision-making process. The AHP can be very easily used on problems where only a few alternatives are considered. It provides an objective decision and it makes it possible to measure the amount of inconsistency of the decision-makers in comparisons they have provided. However, with increasing size of the alternatives set, the AHP quickly becomes cumbersome. For a larger number of alternatives, the decision-makers are required to provide a big amount of data. Moreover, it can be quite difficult to keep the consistency of those data at the acceptable level.

A generalized version of AHP is called Analytical Network Process (ANP) [71, 72]. The Analytical Network Process makes it possible to take into account also the interactions between the criteria. The hierarchical structure typical for AHP is replaced by a more general network structure in ANP. Both the AHP and ANP have been applied on many important real-world problems [73] by well-recognized organizations such as the Nuclear Regulatory Commission of the US, Xerox Corporation, British Airways, IBM and others. Summary of other AHP applications can be found, e.g., in [94].

Another large group of methods, which are called the outranking methods, is base on a different principle. They construct preference (outranking) relations for each of the criteria. Those relations are used to make the final decision. This group is quite diverse as it contains several methods and, for each of them, multiple versions exist. The best known representatives are the ELECTRE [68] and PROMETHEE methods [8]. The ELECTRE methods (ELimination Et Choix Traduisant la REalité; in English ELimination and Choice Expressing the REality) has been introduced by Roy in 1968 [68] The younger PROMETHEE method has been proposed by Brans at the beginning of 1980s. A good overview of these methods can be found for example in [22, 8].

Those are only a few of the best known methods. Naturally, many more of them exist. More information on MCDM methods can be found e.g. in [21].

With the development of the fuzzy sets theory, fuzzy MCDM methods were appearing. Multiple-criteria decision-making was even one of the earliest applications of fuzzy sets – Bellman and Zadeh defined fuzzy goals and fuzzy constrains as fuzzy sets in the space of alternatives in [7]. The fuzzy decision is then obtained as an intersection of the given fuzzy goals and fuzzy constraints.

Instead of devising a new fuzzy method, many authors tried to incorporate the fuzzy notions in the classical time-proved MCDM methods. Such fuzzifications range in their quality from simple naive ones to highly sophisticated methods. There are multiple ways how one can modify the existing methods to work with fuzzy numbers instead of real numbers. In simpler cases, the authors resorted just to replacing the arithmetical operations in the formulae with the corresponding operations of the fuzzy arithmetic. During the time, the mathematicians begun to realize that a mechanical application of this approach is not possible and that more sophisticated fuzzification according the extension principle (taking into the account also the relationships among the variables used in the formula in form of requisite constrains [40]) is necessary. The consequence is that for a single non-fuzzy MCDM method there can be found usually multiple different fuzzy counterparts in the literature. This can be illustrated for instance on the case of fuzzy weighted average. Multiple definitions (e.g.[1], [15], or [62]) appeared over quite a long time period. Often, different conditions on the structure used to represent the fuzzy weights have been used in these definitions. An overview of those diverse fuzzy weights definitions is given for example in [60].

As AHP represents one of the most popular MCDM methods, several attempts for its fuzzification have been made. This was a source of a critique by the founder of the method Thomas L. Saaty, who is a strong opponent of incorporating any fuzziness into AHP [74]. Nevertheless, many fuzzy AHP approaches (including related approaches that use e.g. geometric mean instead of the eigenvector method) appeared. Among the first of them, van Laarhoven and Pedrycz proposed a fuzzy method using triangular fuzzy numbers. Later, Chang [9] proposed extent analysis, which became quite popular in some areas and many simple applications using this method appeared. However, many flaws of the extent analysis method have been addressed later. They are summarized e.g. in [107]. Different approaches working with pair-wise comparison matrices with fuzzy elements have been devised over the time. For example, in [18], the authors use triangular fuzzy numbers in the pair-wise comparisons matrix and the resulting weights are obtained by a geometric mean using fuzzy numbers arithmetic (simplified for triangular fuzzy numbers) Among newer sophisticated approaches, the one in the paper [63] can be named. The paper dealt also with another issue – how to measure the inconsistency of the pairwise comparison matrices with fuzzy elements. A new inconsistency index is proposed in the paper. In [64], a method that can handle also the dependencies among the criteria is described.

Also the other methods have been subject to the fuzzification. For example fuzzy ELECTRE has been used in [67], or fuzzy PROMETHEE has been proposed in [28].

The first part of the thesis is devoted to a complex system of fuzzy MCE methods. The feature common to all of these methods is the used type of evaluation – cardinal evaluations of absolute type (with respect to a given goal) are employed. The foundation of the system has been laid by the Solver methodology introduced in the book [81]. Over the time, the system has been developed and extended rapidly. The described system of methods is quite powerful and relatively easy for the decision-maker at the same time. Its main advantages are:

1. It uses the cardinal evaluation of the absolute type with respect to a given goal. So in contrast to the most of the other MCDM methods, it provides also information how much an alternative satisfies our goal.

2. It is suitable for problems where many alternatives have to be considered (even hundreds or thousands of alternatives).

3. Because the evaluation of the used type is not dependent on the particular set of alternatives, the system of methods is applicable also for cases when the set of the alternatives is not known in advance. It can be specified later after the model has been designed.

4. It is convenient for problems where many criteria have to be taken into account.

5. It can deal with interactions among the criteria.

6. It is relatively easy to understand and to use for the decision-maker.

7. Multiple methods can be used in the same goals tree. Therefore, parts of the problem where there are no interactions can be solved with a simpler method (such as fuzzy weighted average) and parts of the problems with interactions can be treated by some more advanced method (e.g. by a fuzzy expert system).
8. An alternation of the alternatives set does not present a problem. If a new alternative should be added after the evaluation has already been performed, only the criteria values for this new alternative have to be provided. The same situation would cause a big problem e.g. in case of AHP – the whole pair-wise comparison matrix would need to be set by the decision-maker again.
9. A user-friendly software tool for this system of fuzzy methods is available.

The last point is crucial for any method to be applied in practice. Therefore, the next section will study the availability of the MCDM software.

2.3. Software tools for decision-making

Many multiple-criteria decision-making methods require a large number of mathematical operations to be performed and only minimum of the real-world problems can be solved without the help of computers. For the rest of them, a corresponding software tool is essential. This section lists such software tools.

It will be shown that there are many software tools for the classical (i.e. non-fuzzy, or in other words, crisp[1]) decision-making methods. However, if the decision-makers want to use some of the fuzzy methods, their options to select suitable software are much more limited.

2.3.1. Software for classical (non-fuzzy) MCDM

There is a variety of software for the classical MCDM methods. For example, the paper [2] selects 10 major software tools, analyses their capabilities and compares them. Most of them represent professional software products with a user-friendly interface and many important functions. Another overview is given by the report [55]. The authors identified more than 20 decision support systems. From the older reviews, we can name [100] which lists tenths of software tools for MCDM.

It seems that there is a sufficiency of the decision support software that uses (crisp) MCDM methods. The next section compares the situation for fuzzy MCDM software tools.

[1] In the literature related to the fuzzy MCDM, the term "crisp" can be encountered often. In this context, it is basically a synonym for the word "non-fuzzy", i.e. a crisp method is a non-fuzzy method, a crisp number is a real number, etc.

2.3.2. Software for fuzzy MCDM

Many MCDM methods require a large number of calculations. Moreover, their fuzzified versions are usually computationally much more demanding. There are plenty of software tools that support most of the classical MCDM methods. However, if the decision-makers want to employ some of the fuzzy MCDM methods, the variety of software tools that can be used for this task is quite limited.

In the 50 years of the existence of the fuzzy set theory, several fuzzy MCDM software products have been developed. However, the difference in their number is still huge compared to the software for classical MCDM. This section lists the MCDM software tools that utilize fuzzy modeling principles in some way.

FuzzyTECH

Currently, the most commonly used software for the fuzzy multiple-criteria evaluation and decision-making could be FuzzyTECH. The main application area of the software is the fuzzy control. However, its versatility makes it possible to employ it for fuzzy MCDM, too.

FuzzyTECH is a general software product that makes it possible to design and use fuzzy expert systems. It has an intuitive graphical user interface. The graphs of the evaluation functions can be plotted so the decision-maker can study the behavior of the designed fuzzy rule base. There is also a possibility to derive the fuzzy rule bases from the data by neural networks by the NeuroFuzzy module. The software is also able to generate the documentation for the designed project automatically.

In the literature, many interesting applications of the software are described. The book [97] explains its use in more than 30 case studies. Another book [98] describes its applications in the area of business and finance.

The FuzzyTECH is commercial software product. On its website [26], a demo version can be downloaded. However, the limitation of the demo version is that saving of the data is disabled.

NEFRIT

In 2000, a software product called NEFRIT [80] has been developed by the Czech software company TESCO SW Inc. The NEFRIT can work with expert fuzzy evaluations of alternatives according to qualitative criteria. The values of quantitative criteria can be either crisp or fuzzy and the evaluating functions for those criteria represent membership functions of partial fuzzy goals. Aggregation is done by a weighted average of partial fuzzy evaluations. The weights (crisp) express the shares of the particular partial evaluations in the total evaluation. Fuzzy evaluations on all levels of the goals tree express fuzzy degrees of fulfillment of the corresponding goals.

The software NEFRIT was originally developed for the Czech National Bank and was also used by the Czech Tennis Association, by the Czech Basketball Association, etc. It has been also tested by the Supreme Audit Office of the Czech Republic.

The demo version of the software is enclosed to the book [81].

FuzzME

The FuzzME (**Fuzz**y Methods of **M**ultiple-Criteria **E**valuation) software, whose development was one of the goals of this thesis, is based on the same theoretical conception as NEFRIT. But in contrast to NEFRIT, the range of the supported methods and functions is much wider.

The development of the software started around the year 2008. The first versions used a fuzzy weighted average (the weights could be expressed by fuzzy numbers in contrast to the NEFRIT, which worked only with real weights) and a simple fuzzy expert system. In the later years, the software has been extended rapidly. Many new methods have been added – fuzzy OWA, fuzzified WOWA and fuzzy Choquet integral and many user functions have been included. The software also includes analytical tools that make it possible to study the behavior of the designed evaluation functions.

The FuzzME was tested on a soft-fact rating problem of an Austrian bank [25, 83, 38]. It has also been applied in the area of HR (human resources) management [104], or agri-food buildings evaluation [3].

The demo version can be downloaded at `http://www.FuzzME.net`. It is fully functional for 5 days. After this period, the saving of the data is disabled.

Matlab with the Fuzzy Logic Toolbox

Matlab, with help of its Fuzzy Logic Toolbox, can be quite a powerful tool for fuzzy MCDM. This toolbox contains a graphical FIS (fuzzy inference system) editor that makes it possible to design all linguistic variables and the fuzzy rule bases in an easy way without any knowledge of programming.

The FIS editor supports the Mamdani inference and the Sugeno (or Takagi-Sugeno) inference. It has a few nice features that make it easier to understand the behavior of the designed fuzzy rule base. The graph of the resulting function can be plotted in 2D or in 3D. Moreover, the calculation of the result for particular values of the inputs is shown in an explanatory way so that it can be clearly seen which rules have been fired. The toolbox contains also tools for construction of the FIS from data.

The limitation of the FIS used in Matlab is that even though it can work with fuzzy numbers inside the fuzzy inference system (for example the values of the linguistic variables are modeled by fuzzy numbers), the inputs are required to be crisp and also the outputs are defuzzified with one of the supported methods and returned in form of crisp numbers. This is natural because the tool box was developed primarily for fuzzy controllers design. This feature, however, presents a slight limitation for its use in the area of multiple-criteria decision-making.

Besides the fuzzy inference system, the Fuzzy Logic Toolbox contains functions for using several types of fuzzy numbers and performing arithmetic operations with them. One can thus implement other fuzzy MCDM methods in Matlab with some programming skills.

Matlab is developed by the MathWorks company. One must keep in mind that to use the mentioned functions, not only Matlab itself but also the Fuzzy Logic Toolbox has to be bought.

Fuzzy AHP

A prototype of this application is mentioned in the literature [18, 17]. The application, called by their authors *Fuzzy AHP*, enables to calculate the weights vector from a pair-wise comparison matrix. The items of this matrix are triangular fuzzy numbers. The authors calculate the result by the geometric mean method using the fuzzy numbers arithmetic.

The application has been programmed in the MATLAB 7.0. The software was applied on a machine tool selection problem [18]. Later, a similar application of this software for a maintenance management system selection appeared in the literature [17].

FVK

The FVK [65] implements a method based on pair-wise comparisons. The elements of the pair-wise comparison matrix are expressed by triangular fuzzy numbers and the results of the evaluation are also triangular fuzzy numbers. An important feature is that it possible to take into account the feedback between the criteria.

This software is available as an add-in for Microsoft Excel. The main advantages of the software are its sophisticated mathematical background, ability to take into the account the criteria feedbacks, and also the price – the FVK is avalable for free.

Decider

Decider [50] is a software tool for multiple-criteria group decision-making in the fuzzy environment. The criteria are organized into a multi-level hierarchical structure called criteria tree. Similarly, the list of evaluators is organized into a tree structure (for example according to their hierarchy in the company).

The Decider can work with criteria values given by linguistic terms (from one of the predefined scales), by intervals, or Boolean values. All values are internally represented by fuzzy sets on the interval $[0, 1]$. For aggregation within the both of the trees, the fuzzy weighted average is used. However, its implementation is different compared to the fuzzy weighted average in the FuzzME software.

For comparison of the alternatives, the Decider calculates the distance of the alternatives evaluations to the ideal and to the worst possible alternative (crisp 1, or crisp 0, respectively). The decider can calculate the distances and compare the alternatives also for a selected part of the criteria tree or for the selected evaluators. The authors plan to develop a module which would implement a sensitivity analysis and a possibility to compare results from various models.

In the literature, two applications of the software are mentioned. In the first one, the Decider is used to evaluate possible scenarios for the Belgian energy policy by a group of experts [49]. In the latter one, a garment product design is evaluated by the Decider [69].

FLINTSTONES

The FLINTSTONES [20] is software for solving linguistic decision-making problems. It is based on 2-tuple linguistic model and its extensions. The evaluation is therefore performed by the computing with words paradigm.

What makes the software unique is its elaborated website [24]. The website contains several case-studies. They are accompanied by video tutorials, datasets and other sources of information. This makes it easy to learn the methods implemented in the software.

The software is available for both Windows and for Linux. Another advantage is its price. The FLINTSTONES are developed under GNU General Public license, so it is free.

Other possibilities for applying fuzzy MCDM methods

If there is no software available for the selected method, researchers can calculate the result with help of general mathematical software products (such as Matlab) and their fuzzy logic toolboxes. Finally, if there is a need to create a software support for a specified problem, specialized libraries implementing the fuzzy sets and the operations with them exist. For example, jFuzzyLogic [11] for the Java programming language is very popular.

Even though it is possible to create a software implementation for any fuzzy MCDM method either in Matlab or in some programming language, this approach has several drawbacks. The process is time-consuming and the person must have programming skills. Next, there is a danger of a mistake. The software suites were usually tested on multiple problems and therefore the risk that some of the methods were implemented incorrectly is minimal. On the other hand, if a mathematician implements the methods on his/her own just for a particular problem, the risk of a mistake in such a code is higher. Finally, the specialized fuzzy MCDM software products usually contain multiple functions for analysis and testing of the designed model. Because of these aspects, a professional software implementation is crucial for any fuzzy MCDM method in order to apply it in practice.

When the available software for fuzzy MCDM is compared with the software for the crisp MCDM, differences can be observed. The number of the software tools for fuzzy MCDM is significantly lower. Moreover, if there is a software tool for the selected fuzzy method, it is usually much simpler than the software for its crisp counterpart. These reasons are why the FuzzME software has been developed as a part of this thesis. The FuzzME implements several new fuzzy MCDM methods, it is applicable to a wide range of real-world problems; it provides also analytical tools that make it easier to check the correctness of the designed model and, finally, its use is easy and does not require any specialized technical skills.

In the following text, the complex system of fuzzy MCDM methods that are implemented in the FuzzME will be described in detail. The following chapter summarizes the basic notions from the fuzzy set theory that will be used throughout this thesis.

2.4. Introduction to fuzzy set theory

In the human language, vagueness is a very common feature. There are many expressions, such as a *big profit* or a *reasonable price* that cannot be clearly defined by stating a single boundary. Attempts to use (ordinary) sets to delimit them, e.g., to say which of all possible profits can be considered to be big, either fail completely or lead to various paradoxes. The answer to this problem was given in 1965 by Lotfi A. Zadeh. In [102], he introduced fuzzy sets and therefore laid the foundations of the fuzzy set theory. This section gives a brief summary of the notions relevant to this thesis. More detailed information on the fuzzy set theory can be found for example in [16].

Definition 2.1 *Let X be a nonempty set. A fuzzy set A on a universal set X is defined by a mapping $\mu_A : X \to [0, 1]$.*
The function μ_A is called a membership function of the fuzzy set A. For any $x \in X$, the value $\mu_A(x)$ is called a membership degree of the element x in the fuzzy set A.

In order to simplify the notation, the same symbol will be used for both the fuzzy set (e.g. A) and for its membership function ($A(\cdot)$). Then, for a fuzzy set A on a universal set X, $A(x)$ will denote the membership degree of the element $x \in X$ in the fuzzy set A.

In the following text, $\mathcal{F}(X)$ will denote the system of all fuzzy sets on the universal set X.

Definition 2.2 *Let A be a fuzzy set on the universal set X. Then:*

- *the kernel of the fuzzy set A is defined as*

$$\text{Ker } A = \{x \in X \mid A(x) = 1\},$$

- *the support of A is*

$$\text{Supp } A = \{x \in X \mid A(x) > 0\},$$

- *the α-cut of A is given for any $\alpha \in [0,1]$ by the following formula*

$$A_\alpha = \{x \in X \mid A(x) \geq \alpha\},$$

- *the height of the fuzzy set A is defined as follows*

$$\text{hgt } A = \sup\left\{A(x) \mid x \in X\right\}.$$

Remark 2.1 *To prevent possible confusion if an expression A_α is meant to be the α-cut the fuzzy set A, we will use the following convention – if it is not said explicitly otherwise, Greek letters in subscripts of fuzzy sets will be used for α-cuts of these fuzzy sets and other letters (and numbers) will be used to denote indices. This rule applies also in situations with more than one symbol in the subscript so, for example, $A_{i\alpha}$ denotes an α-cut of a fuzzy set A_i.*

Definition 2.3 *Let A and B be fuzzy sets on X. Then an intersection and a union of the fuzzy sets A and B are fuzzy sets on X whose membership functions are defined for all $x \in X$ by the following formulae:*

$$(A \cap B)(x) = \min\left\{A(x), B(x)\right\},$$

$$(A \cup B)(x) = \max\left\{A(x), B(x)\right\}.$$

Definition 2.4 *A fuzzy number is a fuzzy set C on the set of all real numbers $\Re$ that satisfies the following conditions:*

1. *The kernel of C is not empty.*

2. *The α-cuts of C are closed intervals for all $\alpha \in (0,1]$.*

3. *The support of C is bounded.*

In the literature, various definitions of fuzzy numbers can be found. The conditions may differ slightly (for example, some authors do not require the support to be bounded).

Definition 2.5 *A fuzzy number C is said to be defined on $[a, b]$, if*

$$\text{Supp } C \subseteq [a, b].$$

In the following text, $\mathcal{F}_N([a, b])$ will denote the set of all fuzzy numbers on $[a, b]$.

Definition 2.6 *Real numbers $c^1 \leq c^2 \leq c^3 \leq c^4$ are called significant values of the fuzzy number C if the following holds:*

$$[c^1, c^4] = \text{Cl}(\text{Supp } C),$$

$$[c^2, c^3] = \text{Ker } C,$$

where $\text{Cl}(\text{Supp } C)$ *denotes the closure of* $\text{Supp } C$.

Any fuzzy numbers can be described, beside its membership function, in an alternative way, which is often very advantageous for calculations. It can be characterized by a pair of functions defined on $[0, 1]$, which are given by the following definition.

Definition 2.7 *Let C be a fuzzy number. Then, C can be expressed as $C = \{[\underline{c}(\alpha), \overline{c}(\alpha)], \alpha \in [0, 1]\}$, where the functions $\underline{c} : [0, 1] \to \Re$ and $\overline{c} : [0, 1] \to \Re$ are defined by the following formulae*

$$C_\alpha = [\underline{c}(\alpha), \overline{c}(\alpha)] \quad \text{for all } \alpha \in (0, 1], \text{ and}$$

$$\text{Cl}(\text{Supp } C) = [\underline{c}(0), \overline{c}(0)].$$

Definition 2.8 *A fuzzy number C is said to be linear if its membership function has the following form:*

$$C(x) = \begin{cases} 0, & \text{for } x < c^1; \\ \frac{x - c^1}{c^2 - c^1}, & \text{for } c^1 \leq x < c^2; \\ 1, & \text{for } c^2 \leq x \leq c^3; \\ \frac{c^4 - x}{c^4 - c^3}, & \text{for } c^3 < x \leq c^4; \\ 0, & \text{for } c^4 < x; \end{cases}$$

providing that the significant values $c^1 \neq c^2$ and $c^3 \neq c^4$. If $c^1 = c^2$ then $C(c^1) = C(c^2) = 1$, and similarly if $c^3 = c^4$, then $C(c^3) = C(c^4) = 1$ with the rest of the formula remaining unchanged.

Definition 2.9 *A linear fuzzy number C is called triangular if $c^2 = c^3$, otherwise it is called trapezoidal.*

The Figures 2.1 and 2.2 show examples of a triangular and a trapezoidal linear fuzzy numbers.

Fuzzy numbers can be also used to express real numbers and closed intervals. Obviously, if for a fuzzy number C it holds that $c^1 = c^2 = c^3 = c^4 = c$, for some $c \in \Re$, this fuzzy number can be seen as a representation of a real number c. Such a fuzzy number will be called a fuzzy singleton containing the element c

Similarly, if it holds that $c^1 = c^2 = a$ and $c^3 = c^4 = b$, for $a, b \in \Re$, then C represents the interval $[a, b]$ in form of a fuzzy number.

Definition 2.10 *The fuzzy number $\tilde{c}$ is said to be a fuzzy singleton containing a single element c if its membership function is, for any $x \in \Re$, given as follows*

$$\tilde{c}(x) = \begin{cases} 1 & \text{for } x = c, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.2 *In the following text, the symbol $\tilde{0}$ will denote a fuzzy singleton containing a single element 0. Similarly, $\tilde{1}$ will denote a fuzzy singleton representing crisp 1.*

Because any linear fuzzy number is fully determined by its significant values, we will use the following notation. The expression $C = (c^1, c^2, c^3, c^4)$ will denote a linear (trapezoidal) fuzzy number with the significant values c^1, c^2, c^3, and c^4. For triangular fuzzy numbers, we can simply write $C = (c^1, c^2, c^4)$ since $c^2 = c^3$. This short notation will be used mostly in examples.

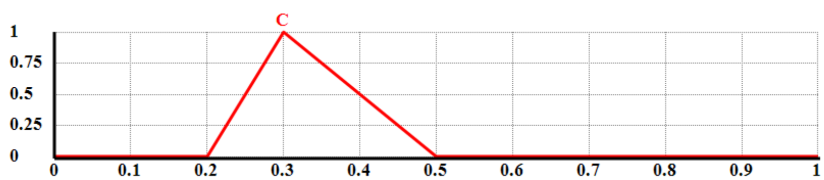


Figure 2.1: An example of a triangular fuzzy number

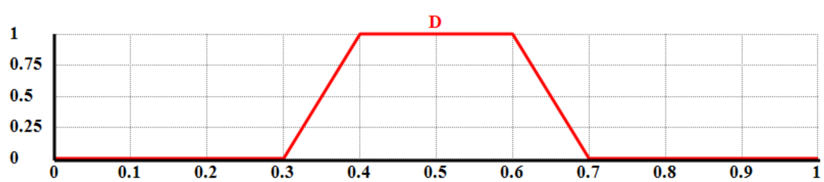


Figure 2.2: An example of a trapezoidal fuzzy number

Example 2.1 *Multiple ways for defining a specific fuzzy numbers have been mentioned so far. The triangular fuzzy number C in the Figure 2.1 can be described in the following ways:*

1. *By specifying its membership function:*

$$C(x) = \begin{cases} 0, & \text{for } x < 0.2; \\ \frac{x-0.2}{0.1}, & \text{for } 0.2 \le x < 0.3; \\ \frac{0.5-x}{0.2}, & \text{for } 0.3 \le x \le 0.5; \\ 0, & \text{for } 0.5 < x; \end{cases}$$

2. *In the form* $C = \{[\underline{c}(\alpha), \overline{c}(\alpha)], \alpha \in [0,1]\}$ *by defining the functions* $\underline{c}(\alpha)$ *and* $\overline{c}(\alpha)$ *for any* $\alpha \in [0,1]$ *as:*

$$\underline{c}(\alpha) = 0.2 + 0.1\alpha, \tag{2.1}$$

$$\overline{c}(\alpha) = 0.5 - 0.2\alpha. \tag{2.2}$$

We can also simply write $C = \big\{[0.2 + 0.1\alpha,\ 0.5 - 0.2\alpha],\ \alpha \in [0,1]\big\}$.

3. *Because* C *is a triangular fuzzy number, we can just provide its significant values:*

$$C = (0.2, 0.3, 0.5).$$

Analogically, those three ways can be used to describe a trapezoidal fuzzy number D*, which is depicted in the Figure 2.2:*

1. $$D(x) = \begin{cases} 0, & \text{for } x < 0.3; \\ \frac{x-0.3}{0.1}, & \text{for } 0.3 \le x < 0.4; \\ 1, & \text{for } 0.4 \le x \le 0.6; \\ \frac{0.7-x}{0.1}, & \text{for } 0.6 < x \le 0.7; \\ 0, & \text{for } 0.7 < x; \end{cases}$$

2. $D = \big\{[0.3 + 0.1\alpha,\ 0.7 - 0.1\alpha],\ \alpha \in [0,1]\big\}$,

3. $D = (0.3, 0.4, 0.6, 0.7)$.

Besides linear fuzzy numbers, there exist other special types of fuzzy numbers such as quadratic [81] (Figure 2.3) or piecewise linear fuzzy numbers [81] (Figure 2.4). The piecewise linear fuzzy numbers comprise a very useful class because they can be used to approximate more complex types of fuzzy numbers and they are very convenient for calculations. Therefore, in the FuzzME software, which will be described later, any fuzzy number is internally represented as a piecewise linear fuzzy number of a given degree according to the following definition.

Definition 2.11 *A piecewise linear fuzzy number of a degree* n, $n \in \{0, 1, \ldots\}$ *defined on an interval* $[a, b]$ *is a fuzzy number with a piecewise linear membership function defined by the following sequence of* $2n + 4$ *points:*

$$\{(x_1, 0),\ (x_2, \frac{1}{n+1}),\ \ldots,\ (x_{n+1},\ \frac{n}{n+1}),\ (x_{n+2}, 1),$$
$$(x_{n+3}, 1),\ (x_{n+4},\ \frac{n}{n+1}),\ \ldots,\ (x_{2n+3}, \frac{1}{n+1}),\ (x_{2n+4}, 0)\},$$

where $a \leq x_1 \leq x_2 \leq \ldots \leq x_{n+1} \leq x_{n+2} \leq x_{n+3} \leq x_{n+4} \leq \ldots \leq x_{2n+3} \leq x_{2n+4} \leq b$.

A higher degree means better accuracy of the approximation with the natural drawback of more time required for the calculations. More information on those different types can be found in [81].

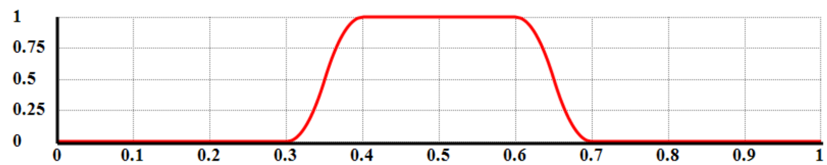


Figure 2.3: An example of a quadratic fuzzy number

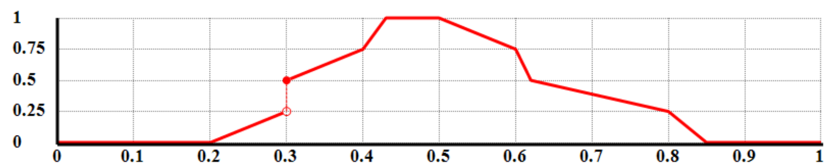


Figure 2.4: An example of a piecewise linear number

The following definition makes it possible to measure a (normalized) distance [81] of two fuzzy numbers.

Definition 2.12 *Let* C *and* D *be fuzzy numbers on the interval* $[a, b]$, $C = \{[\underline{c}(\alpha), \overline{c}(\alpha)],\ \alpha \in [0,1]\}$, $D = \{[\underline{d}(\alpha), \overline{d}(\alpha)],\ \alpha \in [0,1]\}$. *Their (normalized) distance is a real number on* $[0, 1]$ *defined by the following formula:*

$$d(C, D) = \frac{\int_0^1 \left(|\underline{c}(\alpha) - \underline{d}(\alpha)| + |\overline{c}(\alpha) - \overline{d}(\alpha)| \right) d\alpha}{2(b - a)} \tag{2.3}$$

According to the mentioned definition, the normalized distance of fuzzy numbers C and D is zero if their membership functions equal. However, the maximum distance (equal to one) is achieved by fuzzy singletons on the opposite parts of the interval $[a, b]$.

In many situations, we will need to order the fuzzy numbers. There are multiple ways. One of them is the ordering based on α-cuts, which is given by the following definition.

Definition 2.13 *An ordering of fuzzy numbers is defined as follows: a fuzzy number* C *is greater than or equal to a fuzzy number* D *(we write* $C \geq D$*) if* $C_\alpha \geq D_\alpha$ *for all* $\alpha \in (0, 1]$. *The inequality of the* α*-cuts* $C_\alpha \geq D_\alpha$ *is the inequality of intervals* $C_\alpha = [\underline{c}(\alpha), \overline{c}(\alpha)]$, $D_\alpha = [\underline{d}(\alpha), \overline{d}(\alpha)]$, *which is defined as*

$$[\underline{c}(\alpha), \overline{c}(\alpha)] \geq [\underline{d}(\alpha), \overline{d}(\alpha)] \text{ if, and only if, } \underline{c}(\alpha) \geq \underline{d}(\alpha) \text{ and } \overline{c}(\alpha) \geq \overline{d}(\alpha).$$

A disadvantage of the described ordering is that many fuzzy numbers are incomparable in this manner. The above relation is only a partial ordering. However, other approaches for ordering of the fuzzy numbers exist, for instance the ordering according to the centers of gravity [16].

Any real continuous function f of n real variables can be extended to a FNV-function (a fuzzy-number-valued function) of n FNV-variables. This is done by the so-called extension principle.

Definition 2.14 (Extension principle) *Let a mapping f, $f : X_1 \times X_2 \times \cdots \times X_n \to Y$, be given. Then its fuzzy extension according to the extension principle is a mapping $f_F : \mathcal{F}(X_1) \times \mathcal{F}(X_2) \times \cdots \times \mathcal{F}(X_n) \to \mathcal{F}(Y)$ assigning to any fuzzy sets $A_1, \ldots, A_n$, $A_i \in X_i$, $i = 1, \ldots, n$, a fuzzy set B whose membership function is defined for any $y \in Y$ as follows*

$$B(y) = \begin{cases} \sup\{\min\{A_1(x_1), \ldots, A_n(x_n)\} \mid \\ \qquad y = f(x_1, \ldots, x_n), x_i \in X_i, \\ \qquad i = 1, 2, \ldots, n\} & \text{if } f^{-1}(y) \neq \emptyset, \\ \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

In this thesis, fuzzy extensions of real-valued functions will be of interest, i.e. $f : \Re^n \to \Re$. The values of the variables for the corresponding fuzzy extension f_F will be usually expressed by fuzzy numbers. However, the result of f_F given by the extension principle need not to be a fuzzy number (generally, it is a only a fuzzy set on $\Re$). To guarantee that the result will be again a fuzzy number, the function f has to be continuous. More information can be found [81].

The following theorem shows that the fuzzification of a function according to the extension principle can be expressed in a much simpler way if certain conditions are satisfied.

Theorem 2.1 *Let $f : \Re^n \to \Re$ be a real continuous function of n variables, which is non-decreasing in all the variables, and let f_F be the fuzzy extension of f. Let $C_i = \{[\underline{c}_i(\alpha), \overline{c}_i(\alpha)], \alpha \in [0, 1]\}$ be fuzzy numbers, $i = 1, \ldots, n$. Then for the fuzzy number $D = f_F(C_1, \ldots, C_n)$, $D = \{[\underline{d}(\alpha), \overline{d}(\alpha)], \alpha \in [0, 1]\}$, the following holds*

$$\underline{d}(\alpha) = f(\underline{c}_1(\alpha), \underline{c}_2(\alpha), \ldots, \underline{c}_n(\alpha)), \tag{2.5}$$

$$\overline{d}(\alpha) = f(\overline{c}_1(\alpha), \overline{c}_2(\alpha), \ldots, \overline{c}_n(\alpha)). \tag{2.6}$$

Proof: The proof can be found in [59]. □

Basic arithmetic operations can be defined for fuzzy numbers (e.g. [27]). In this thesis, only addition of fuzzy numbers and their multiplication with a real number will be used.

Definition 2.15 *Let C and D be fuzzy numbers, $C = \{[\underline{c}(\alpha), \overline{c}(\alpha)], \alpha \in [0,1]\}$, $D = \{[\underline{d}(\alpha), \overline{d}(\alpha)], \alpha \in [0,1]\}$. Then their addition is a fuzzy number given as follows:*

$$C + D = \left\{ \left[\underline{c}(\alpha) + \underline{d}(\alpha), \quad \overline{c}(\alpha) + \overline{d}(\alpha) \right], \quad \alpha \in [0,1] \right\}, \tag{2.7}$$

Definition 2.16 *Let $c \in \Re$ be a real number and let D be a fuzzy number, $D = \{[\underline{d}(\alpha), \overline{d}(\alpha)], \alpha \in [0,1]\}$. Then the multiplication of the fuzzy number D and the real number c is defined as follows:*

$$c \cdot D = \begin{cases} \left\{ \left[c \cdot \underline{d}(\alpha), \quad c \cdot \overline{d}(\alpha) \right], \quad \alpha \in [0,1] \right\} \text{ for } c \geq 0, \\ \left\{ \left[c \cdot \overline{d}(\alpha), \quad c \cdot \underline{d}(\alpha) \right], \quad \alpha \in [0,1] \right\} \text{ for } c < 0. \end{cases} \tag{2.8}$$

Definition 2.17 *The fuzzy numbers $T_1, T_2, \ldots, T_s$ defined on $[a,b]$ are said to form a fuzzy scale on $[a,b]$, if they form a Ruspini fuzzy partition on the interval, i.e., for all $x \in [a,b]$ the following holds*

$$\sum_{i=1}^{s} T_i(x) = 1, \tag{2.9}$$

and if they are numbered according to their linear ordering.

A fuzzy scale makes it possible to represent a closed interval of real numbers by a finite set of fuzzy numbers. Since the fuzzy numbers form a fuzzy partition, they can always be linearly ordered in sense of the Definition 2.13 (see [81]).

In the fuzzy MCDM models described in this thesis, the instruments of the linguistic fuzzy modeling will be used very often. The essential notion is the linguistic variable introduced by the Zadeh [103].

Definition 2.18 *A linguistic variable is defined as a quintuple $(\mathcal{V}, \mathcal{T}(\mathcal{V}), X, G, M)$, where*

- *$\mathcal{V}$ is the name of the variable,*
- *$\mathcal{T}(\mathcal{V})$ is a set of its linguistic values,*
- *X is a universal set on which the meanings of the linguistic values are defined,*

- *G is a syntactic rule for generating values in $\mathcal{T}(\mathcal{V})$, and*
- *M is a semantic rule which maps each linguistic value $\mathcal{A} \in \mathcal{T}(\mathcal{V})$ to its mathematical meaning, $A = M(\mathcal{A})$, which is a fuzzy set on X.*

In this thesis, the linguistic term $\mathcal{A}$ will be distinguished from its mathematical meaning A, which is a fuzzy set, by means of a different font. Usually, the meanings of the linguistic terms are modeled by fuzzy numbers.

A linguistic scale [81] offers simplified description of a continuous real variable with values on $[a, b]$ by specifying a finite number of linguistic values modeled by fuzzy numbers on $[a, b]$.

Definition 2.19 *A linguistic scale on $[a, b]$ is a special case of the linguistic variable $(\mathcal{V}, \mathcal{T}(\mathcal{V}), X, G, M)$, where $X = [a, b]$, $\mathcal{T}(\mathcal{V}) = \{\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s\}$, and the meanings of the linguistic values $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s$ are modeled by fuzzy numbers $T_1, T_2, \ldots, T_s$, which form a fuzzy scale on $[a, b]$.*

Other structures have also been derived from the linguistic scale [80, 81]. For example, an extended linguistic scale or a linguistic scale with intermediate values, which will be described in the Section 2.9.1.

By linguistic variables, e.g. linguistic scales, we can map the linguistic terms to the corresponding fuzzy numbers. Sometimes, an opposite process is required. We would like to map a fuzzy value to the best fitting term from a given linguistic scale. This can be achieved with a linguistic approximation. It can be used for example to describe the resulting fuzzy evaluation verbally (see Section 2.11.1 for more details). Two methods will be considered here – one using the fuzzy numbers distance [81], which has already been defined in this section, and another one based on a fuzzy sets similarity [81].

Definition 2.20 *Let A and B be fuzzy sets on the interval $[a, b]$ and let their membership functions be Borel measurable. Then, their similarity is a real number on $[0, 1]$ defined by the following formula:*

$$S(A, B) = 1 - \frac{\int_a^b |A(x) - B(x)| \; dx}{\int_a^b (A(x) + B(x)) \, dx}. \tag{2.10}$$

The fuzzy sets A and B have similarity equal to one when their membership functions are equal. And, conversely, their similarity is zero if their supports are disjoint. If A and B are fuzzy numbers, we can also measure their distance by the previously mentioned definition. The linguistic approximation (i.e. the best fitting term) of a fuzzy set A on $\Re$, or a fuzzy number C, using a given linguistic scale, can be determined, e.g., by the following approaches.

Definition 2.21 *Let* $(\mathcal{V}, \mathcal{T}(\mathcal{V}), [a,b], G, M)$ *be a linguistic scale with linguistic terms* $\mathcal{T}(\mathcal{V}) = \{\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s\}$ *whose mathematical meanings are denoted by* $T_i = M(\mathcal{T}_i)$ *for* $i = 1, \ldots, s$*. Let* A *be a fuzzy set on* $[a,b]$ *with a Borel measurable membership function. Then the linguistic approximation (based on fuzzy sets similarities) of* A *by the linguistic scale* $\mathcal{V}$ *is such a linguistic term* $\mathcal{T}_{i^*}$ *from that scale whose mathematical meaning satisfies the following condition:*

$$S(A, T_{i^*}) = \max_{i=1,\ldots,s} \{S(A, T_i)\}. \quad (2.11)$$

If we want to approximate linguistically a fuzzy number, then, besides the mentioned approach, the following method, which uses the normalized distance from the Definition 2.12, can be also used.

Definition 2.22 *Let* $(\mathcal{V}, \mathcal{T}(\mathcal{V}), [a,b], G, M)$ *be a linguistic scale with linguistic terms* $\mathcal{T}(\mathcal{V}) = \{\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s\}$ *whose mathematical meanings are fuzzy numbers on* $[a,b]$ *denoted by* $T_i = M(\mathcal{T}_i)$*,* $i = 1, \ldots, s$*. Let* C *be a fuzzy number defined on* $[a,b]$*. Then the linguistic approximation (based on fuzzy numbers distances) of* C *by the linguistic scale* $\mathcal{V}$ *is such a term* $\mathcal{T}_{i^*}$ *from that linguistic scale whose mathematical meaning satisfies the following condition:*

$$d(C, T_{i^*}) = \min_{i=1,\ldots,s} \{d(C, T_i)\}. \quad (2.12)$$

It is often necessary to combine more values into a single one by a process called aggregation (for example to calculate a partial evaluation of the higher level from the lower-level partial evaluations). To accomplish that, we can use one of aggregation operators. Different definitions of the aggregation operators exist. We can mention for example the following one [57].

Definition 2.23 *An aggregation operator* A *is a non-decreasing mapping* $A : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ *that satisfy the following conditions:*

- $A(x) = x$ *for any* $x \in [0,1]$*,*
- $A(0,0,\ldots,0) = 0$ *and* $A(1,1,\ldots,1) = 1$*,*
- $A(x_1,\ldots,x_n) \leq A(y_1,\ldots,y_n)$ *whenever* $x_i \leq y_i$ *for all* $i = 1,2,\ldots,n$*, where* $n = 2,3,\ldots,$ *and* $x_1,\ldots,x_n \in [0,1]$*,* $y_1,\ldots,y_n \in [0,1]$*.*

The well-known aggregation operators, e.g. a weighted average, can be used to aggregate values that are real numbers. In the fuzzy MCDM models, where both the partial evaluations and the information on the decision-makers' preferences concerning the criteria are expressed by fuzzy numbers, it is natural to work with fuzzified versions of the aggregation operators. The fuzzification process can be divided into two steps [4]. When we speak about the *first-level fuzzification* of an aggregation operator, we mean that the aggregated values can be fuzzy but the parameters of the operator remain crisp (for example the first-level fuzzification of weighted average can aggregate fuzzy numbers, but the used weights are still real numbers). A more advanced type is the *second-level fuzzification*. In this case, not only the aggregated values but also the operator parameters are fuzzy.

2.5. Specification of the multiple-criteria evaluation problem

The problem that will be studied in this part of the thesis is to construct a complex mathematical model for evaluating alternatives of certain type with respect to a given goal. The overall goal can be divided step by step into partial goals of a lower level. The degrees of fulfillment of the partial goals on the lowest level can be then assessed by corresponding characteristics of alternatives – criteria.

Various requirements of the evaluator on the behavior of evaluating function should be met. The model of multiple-criteria evaluation that will be described in the following sections is able to process uncertain, expertly-defined data and to utilize expert knowledge related to the evaluation process. The number of the used criteria can be high and interactions among them can be present.

The set of evaluated alternatives is not required to be known in advance. Therefore, an evaluation model can be designed first and then it can be applied to the individual incoming alternatives.

The evaluation results will serve as a support to decision-making. Therefore it must have a form that is easy to understand for a human decision-maker. That is why a linguistic description of the resulting evaluation is also provided.

2.6. The type of the used evaluation

Because we do not only compare alternatives in a given set but we also assess how much do the alternatives, which enter the system progressively one by one, meet our requirements, an evaluation of the relative type cannot be used, and an evaluation of the absolute type with respect to a given goal must be utilized.

Instead of simply comparing two alternatives and stating which one is better with respect to a pursued goal (relative evaluation on an ordinal scale), or instead of saying how much larger is the difference in evaluation with one pair of alternatives in contrast to this difference with another pair (relative evaluation on a cardinal, interval scale), what we needed is some sort of assessment to what extent does the alternative meet the pursued goal. An evaluation of a bank client requesting a credit may be taken as an example.

An appropriate scale of evaluation for this kind of assessment is the interval $[0, 1]$, where 0 means that the alternative is completely unsatisfactory, while 1 means that it fully satisfies the given goal. An evaluation in the interval $[0, 1]$ then denotes the degree to which the goal has been fulfilled (for example, evaluation of 0.75 can be interpreted as reaching the goal at 75%). Such an evaluation of an alternative can be then conceived of as its degree of membership to the fuzzy set of alternatives fulfilling the given goal. This is the way how Bellman and Zadeh [7] interpreted the evaluation of alternatives in their classic paper.

In the evaluation models described below, the evaluations are not only real numbers from $[0, 1]$ but fuzzy numbers defined on this interval. These fuzzy numbers then express uncertain degrees of fulfillment of a given goal by respective alternatives [80, 81]. For example, a fuzzy evaluation in form of a triangular fuzzy number $(0.6, 0.75, 0.8)$ expresses that the alternative is most likely to reach the given goal at 75%, however, the degree of the fulfillment is admitted to range from 60% to 80%.

If these fuzzy evaluations are again interpreted as membership degrees to the fuzzy set of alternatives fulfilling the given goal, then this fuzzy set can be viewed as a type-2 fuzzy set [16]. Fuzzy evaluations expressing uncertain degrees of goal fulfillment enter the presented models on all levels of evaluation. This holds for the partial evaluations with respect to goals linked to particular criteria (both qualitative and quantitative), as well as for the evaluations with respect to the overall goal.

2.7. The basic structure of the evaluation model

The basic structure of the fuzzy model of multiple-criteria evaluation, which is considered in this thesis, is expressed by a goals tree. Its general structure can be seen in the Figure 2.5. The root of the tree represents the overall goal of evaluation and each other node corresponds to a partial goal. The goals at the ends of branches are associated with either quantitative or qualitative criteria.

When an alternative is evaluated, evaluations with respect to the criteria connected with the terminal branches are calculated first. Independently of the criterion type, each evaluation is described by a fuzzy number defined on $[0, 1]$. It thus expresses the fuzzy degree of fulfillment of the corresponding partial goal.

According to qualitative criteria, alternatives are evaluated verbally by means of values of linguistic variables of a special kind – linguistic evaluating scales. Mathematical meanings of the linguistic values are modeled by fuzzy numbers on $[0, 1]$, as mentioned above.

The evaluation according to a quantitative criterion is calculated from the measured value of the criterion (which can be crisp or fuzzy) by means of an evaluating function expertly defined for that criterion. The evaluating function is the membership function of the corresponding partial goal defined on the domain

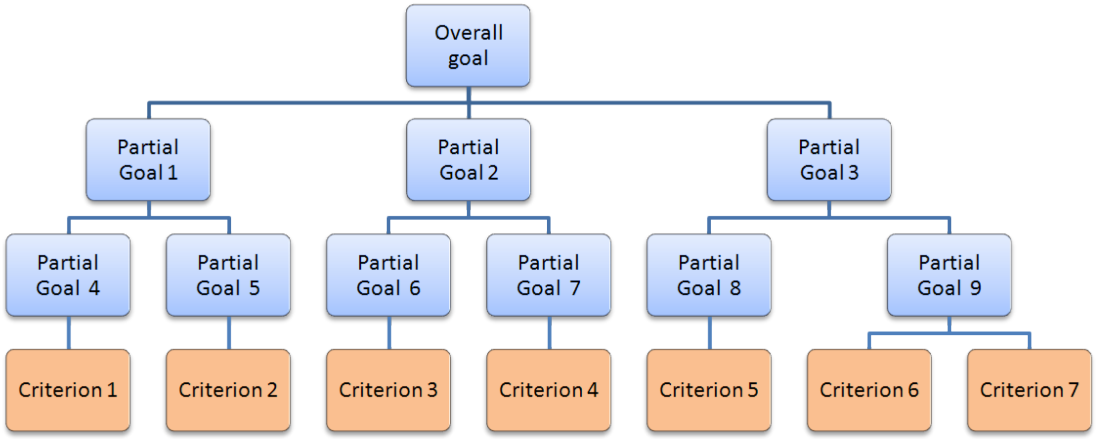


Figure 2.5: A goals tree

of the criterion of interest.

Then, the partial fuzzy evaluations are consecutively aggregated according to the structure of the goals tree by one of several supported methods (fuzzy weighted average, fuzzy OWA, fuzzified WOWA, fuzzified Choquet integral, or fuzzy expert system). The choice of the appropriate method depends on the evaluator's requirements and on the relationships among the evaluation criteria.

If the importances of the individual criteria are known and there are no interactions among them, the decision-maker can use the fuzzy weighted average (FuzzyWA, see [62]) for the aggregation. If different weights are assigned to the individual partial evaluations in dependence on their order, the fuzzy ordered weighted average (FuzzyOWA, see [82]) can be employed. If both of these aspects should be taken into the account, it can be accomplished by the fuzzified WOWA operator (FWOWA, see e.g. [38], for crisp WOWA operator see [89]).

When relationships of redundancy or complementarity that are stable over the whole domain of criteria are present, the fuzzified discrete Choquet integral is used (for fuzzy Choquet integral see [5]; for crisp Choquet integral see [10, 29], or [92]). In case of more complex interactions among the criteria, a fuzzy expert system has to be used. The fuzzy expert system can be applied under any complex relationship among criteria – if the expert knowledge of the evaluation rules is known. Generally, it holds that any continuous (even any Borel-measurable) function can be approximated to arbitrary precision by a fuzzy rule base with a finite number of rules and a suitable inference algorithm (more information can be found in [41]). For that reason, fuzzy expert systems with various approximate-reasoning algorithms (Mamdani, Sugeno) can be used to aggregate the partial evaluations under complex interactions. Fuzzy expert systems make it possible to utilize expert knowledge for modeling complex evaluating functions. In order to obtain required properties of the evaluating functions, it is possible to modify the usual fuzzy-inference algorithms by employing a less common aggregation

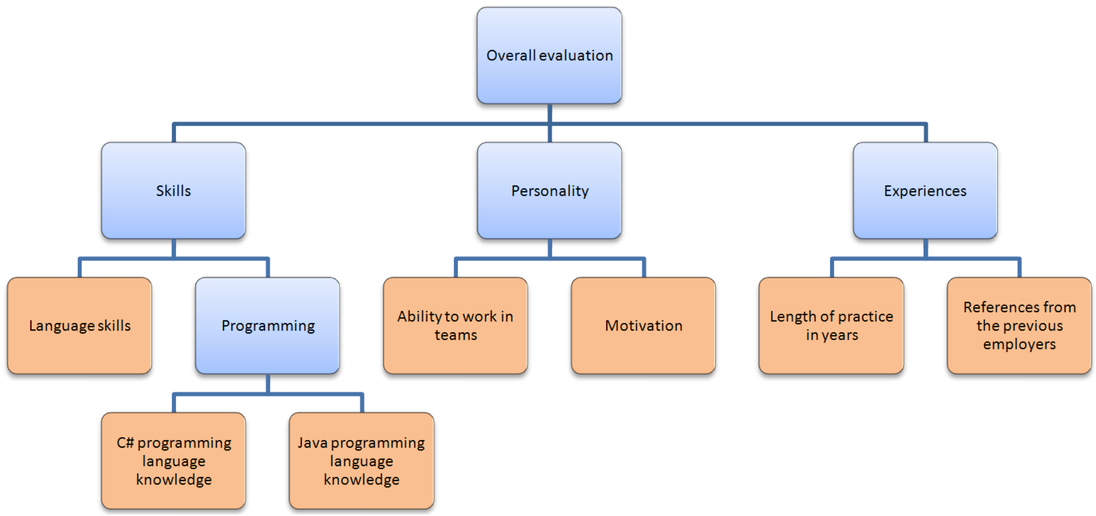


Figure 2.6: A sample goals tree for evaluation of the candidates applying for a new job in a software company

method. As an example of such a modified inference algorithms, the Sugeno-WOWA [34] can be named.

The final result of the consecutive aggregation of partial fuzzy evaluations is the overall fuzzy evaluation of the given alternative. The obtained overall fuzzy evaluation is again a fuzzy number on $[0, 1]$. It expresses the uncertain degree of fulfillment of the main goal by the particular alternative. The fuzzy evaluations are not easy to utilize directly, e.g., to decide which of the alternatives is the best one, to clearly order the alternatives, etc. Therefore, the resulting fuzzy evaluations are often approximated by real numbers (e.g. by the center of gravity method), or linguistically (using linguistic evaluating scales).

In the next sections, the individual parts of the evaluation model will be described in detail.

2.8. The goals tree

The considered fuzzy model of multiple-criteria evaluation has a hierarchical structure. A goals tree is constructed in such a way that, first, for the overall goal, a set of lower-level partial goals is specified with the property that the extent of their fulfillment suggests the extent of fulfillment of the overall goal, i.e. the higher-level goal. The process of division of the partial goals into partial goals of the lower level repeats until such goals are reached whose fulfillment can be assessed by means of some known characteristics of alternatives – by quantitative or qualitative criteria.

Example 2.2 *Let us assume that a software company wants to hire a new programmer. Many candidates applied for this vacancy. The company has to evaluate the candidates and choose the best of them.*

The HR-manager of the company can identify that the overall goal "the person must have a good aptitude for the given job" can be divided into the three partial goals of a lower level – "the person must be skilled", "the person has to have a suitable personality for the job", and "the person must have already some experiences in this field". The partial goals can be briefly called Skills, Personality, and Experiences.

These partial goals could be again divided into partial goals of a lower level. When the process of division stops, the following criteria are identified: "language skills", "C# programming language knowledge", "Java programming language knowledge", "ability to work in teams", "motivation", "length of practice in years", and the "references from the previous employers".

The final goals tree for this sample problem is depicted in the Figure 2.6.

The type of each node in the goals tree must be specified. For the nodes at the ends of branches, the expert makes explicit if the node is connected with a qualitative or a quantitative criterion. For the other nodes the expert chooses the type of aggregation – FuzzyWA, FuzzyOWA, fuzzified WOWA, fuzzified Choquet integral or a fuzzy expert system. Of course, non-fuzzified versions of the aggregation operators can be used as well, as they are special cases of their respective fuzzy versions.

2.9. Evaluation criteria

In the evaluation models, two types of criteria can be used – qualitative and quantitative. Both types can be combined arbitrarily within the same goals tree.

2.9.1. Qualitative criteria

Qualitative criteria are those criteria of evaluation whose values cannot be measured and should be evaluated expertly. In the considered model, alternatives are evaluated according to qualitative criteria verbally. The basic linguistic variable used for such an evaluation is a linguistic scale on $[0, 1]$. When an alternative is evaluated according to a qualitative criterion, the expert chooses the best fitting term from the set of terms of a linguistic variable. For example, the linguistic scale *Quality of Product* can contain linguistic values *poor*, *substandard*, *standard*, *above standard*, and *excellent* (see Figure 2.9). When an alternative is evaluated, the expert chooses one of those verbal descriptions.

Instead of a linguistic scale, a richer structure can be used. For example, an extended linguistic scale or a linguistic scale with intermediate values can be chosen. These two structures were introduced in [81] and they are defined as follows.

Definition 2.24 *Let* $(\mathcal{V}, \mathcal{T}(\mathcal{V}), [0,1], G, M)$ *be a linguistic fuzzy scale on* $[0,1]$. *Then an extended linguistic scale derived from this linguistic scale is a linguistic variable* $(\mathcal{V}', \mathcal{T}'(\mathcal{V}'), [0,1], G', M')$ *defined as follows:*

- *The variable* $\mathcal{V}'$ *contains all terms from the original linguistic scale, i.e. the terms* $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s \in \mathcal{T}(\mathcal{V})$. *Those terms are called to be elementary terms.*

- *Besides the elementary terms, the variable* $\mathcal{V}'$ *also contains derived terms in the form*
$$\mathcal{T}_i \text{ to } \mathcal{T}_j,$$
where $i < j, \;\; i, j \in \{1, 2, \ldots, s\}$.

- *The meanings of the elementary terms remain unchanged, i.e. for any* $i \in \{1, \ldots, s\}$ *it holds that*
$$M'(\mathcal{T}_i) = M(\mathcal{T}_i) = T_i.$$

- *The meanings of the derived terms are modeled by the Lukasiewicz union, i.e. for any* $i, j \in \{1, 2, \ldots, s\}, i < j$ *we define*
$$M'(\mathcal{T}_i \text{ to } \mathcal{T}_j) = M(\mathcal{T}_i) \cup_L M(\mathcal{T}_{i+1}) \cup_L \ldots \cup_L, M(\mathcal{T}_j),$$
where $\cup_L$ *denotes the union of fuzzy sets based on the Lukasiewicz disjunction defined by the following formula:*
$$(T_i \cup_L T_{i+1})(x) = \min\{1, T_i(x) + T_{i+1}(x)\}$$
for all $x \in \Re$ *and any* $i \in \{1, 2, \ldots, s-1\}$.

The extended linguistic scales provide the expert with more options because, besides the original elementary terms, the terms of the form $\mathcal{A}$ *to* $\mathcal{B}$ (for elementary terms $\mathcal{A}$ and $\mathcal{B}$ whose mathematical meanings satisfy $A < B$) are also included. The extended linguistic scale can be beneficial when the uncertainty of the expert's knowledge about the criterion value differs for the individual alternatives. For example if the expert's knowledge about the quality of one of the evaluated products is limited, it is natural that he/she can be hesitant to select a single term from the scale. The extended linguistic scale gives the expert possibility to express the verbal evaluation in form of a range. For example, the quality of a product can be assessed to be *substandard to standard*. Or if the only available information about the quality of the product is that it is definitely

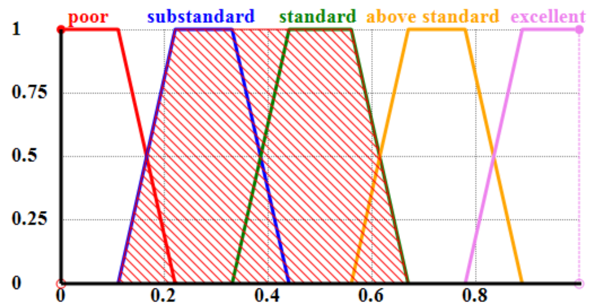


Figure 2.7: The mathematical meaning of the term *substandard to standard* (highlighted by the red hatching)

not worse than the standard, the expert can express this by the term *standard to excellent*.

On the other hand, for cases when the expert would like to use evaluations not only from the given scale but also the evaluations that lie in between them, a scale with intermediate values can be used.

Definition 2.25 *Let* $(\mathcal{V}, \mathcal{T}(\mathcal{V}), [0,1], G, M)$ *be a linguistic fuzzy scale on* $[0,1]$. *Then a linguistic scale with intermediate values derived from this linguistic scale is a linguistic variable* $(\mathcal{V}', \mathcal{T}'(\mathcal{V}'), [0,1], G', M')$ *defined as follows:*

- *The variable* $\mathcal{V}'$ *contains all terms from the original linguistic variable, i.e. the terms* $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_s \in \mathcal{T}(\mathcal{V})$. *Those terms are called elementary terms.*

- *Besides the elementary terms, the variable* $\mathcal{V}'$ *also contains derived terms in the form*
$$\text{between } \mathcal{T}_i \text{ and } \mathcal{T}_{i+1},$$
where $i \in \{1, 2, \dots, s-1\}$.

- *The meanings of the elementary terms remain the same, i.e. it holds that*
$$M'(\mathcal{T}_i) = M(\mathcal{T}_i) = T_i$$
for any $i \in \{1, 2, \dots, s\}$.

- *The meanings of the derived terms are modeled by the arithmetic mean of fuzzy numbers, i.e., for any* $i \in \{1, 2, \dots, s-1\}$, *we define*
$$M'(\text{between } \mathcal{T}_i \text{ and } \mathcal{T}_{i+1}) = \frac{1}{2} \cdot (T_i + T_{i+1}).$$
The value is calculated using the Definitions 2.15 and 2.16.

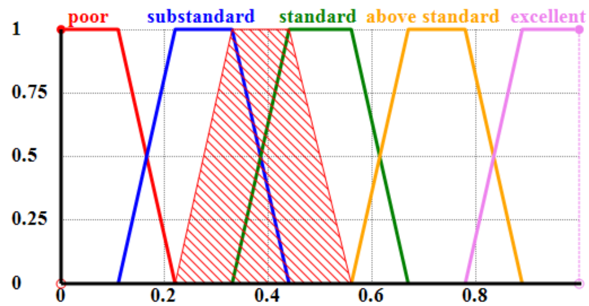


Figure 2.8: The mathematical meaning of the term *between substandard and standard* (highlighted by the red hatching)

Using the linguistic scale with intermediate values, the expert can, besides the original terms, use expressions in the form *between* $\mathcal{A}$ *and* $\mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are neighboring values from the linguistic scale. The quality of a product can be therefore rated for example as *between substandard and standard*.

There is an important difference between the terms $\mathcal{A}$ *to* $\mathcal{B}$ (extended linguistic scale) and *between* $\mathcal{A}$ *and* $\mathcal{B}$ (linguistic scale with intermediate values). The first one means that the expert does not know the precise criterion value; it can be either $\mathcal{A}$ or $\mathcal{B}$ (Figure 2.7). In the latter case, the expert expresses that the criterion value is neither $\mathcal{A}$ nor $\mathcal{B}$ but something in between (Figure 2.8).

It is possible that for particular alternatives the values of some of the criteria are unknown. The described system of methods can deal even with this situation. Regardless the selected scale, the expert has an option to select a special term called *unknown* as the criterion value. Its mathematical meaning is modeled by the fuzzy number $(0, 0, 1, 1)$. By this term, the expert expresses that no knowledge about the criterion value for this alternative is available and therefore any evaluation from $[0, 1]$ is fully possible. It can be noted that this values is also the mathematical meaning corresponding to one of the values from the extended linguistic scales – $\mathcal{T}_1$ *to* $\mathcal{T}_s$.

When an alternative is evaluated according to a qualitative criterion, the expert selects the best fitting term from the given linguistic scale or from one of the derived structures (i.e. the extended linguistic scale, or the linguistic scale with intermediate values). The evaluation of such a criterion for the particular alternative is then given by the mathematical meaning of the selected term (a fuzzy numbers on $[0, 1]$). The Table 2.1 contains a few examples of possible linguistic values and their mathematical meanings.

2.9.2. Quantitative criteria

Besides qualitative criteria, which are evaluated by means of linguistic terms, quantitative criteria can be also used in the model. For this type of criteria, their values are measured. If the value can be determined accurately enough, it is

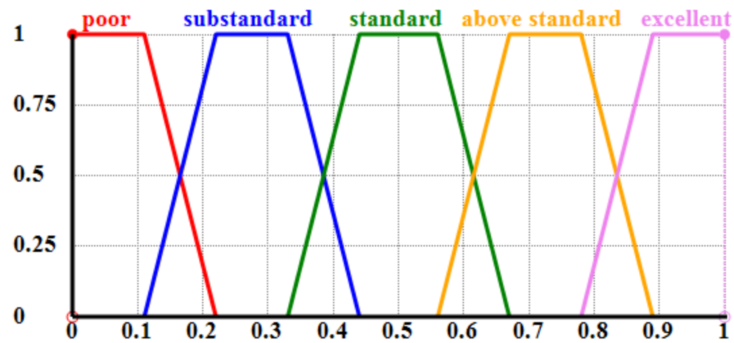


Figure 2.9: The linguistic scale for the criterion *Quality of Product*

The term selected by the expert	The corresponding fuzzy evaluation
Subtandard	(0.11, 0.22, 0.33, 0.44)
Standard	(0.33, 0.44, 0.56, 0.67)
Between substandard and standard	(0.22, 0.33, 0.44, 0.56)
Substandard to standard	(0.11, 0.22, 0.56, 0.67)
Standard to excellent	(0.33, 0.44, 1, 1)
Unknown	(0, 0, 1, 1)

Table 2.1: Examples of the linguistic values and their mathematical meanings for the criterion *Quality of Product*

expressed by a real number. Otherwise, if the value is not known with sufficient accuracy (e.g. only an estimate given by an expert is available), it is modeled by a fuzzy number. Even the values measured by some tool (a thermometer, weights, etc.) can be uncertain due to the limited accuracy of that tool.

For each quantitative criterion, the expert defines an evaluation function $u : [a, b] \to [0, 1]$, where $[a, b]$ represents the interval of possible values of this criterion. The evaluation function expresses how the particular value fulfills the particular partial goal. If the value of the evaluation function is zero, $u(x) = 0$, for some measured value $x \in [a, b]$, it means that this value is completely unsatisfactory with respect to the given partial goal. On the other hand, if $u(x) = 1$, the value x is considered to be fully satisfactory by the expert.

This evaluation function can be perceived as a membership function of a fuzzy set representing the corresponding goal (analogy to the notion of a fuzzy goal introduced in [7] can be seen). Specifically, in the described model, this membership function can be set by means of a fuzzy number – the values in its kernel represent the fully satisfactory values, i.e. the perfect fulfillment of the goal. The values in its support represent at least partial fulfillment of the goal, and, vice versa, the values in $[a, b]$ that are outside the support represent the total lack of fulfillment of the given goal. The expert can thus specify the evaluation function by providing the significant values for this fuzzy number.

When an alternative is evaluated, the evaluation with respect to the particular quantitative criterion is calculated as follows. The expert specifies the criterion value. If the measured criterion value is crisp, then it can be simply inserted into the evaluation function. If the criterion value is fuzzy, then the resulting fuzzy evaluation of the alternative with respect to this criterion is calculated by the extension principle (see Definition 2.14).

Similarly to the qualitative criteria, it is admissible that some quantitative criteria values are unknown. In this case, the expert sets the fuzzy number (a, a, b, b) as the criterion value. The expert is thus expressing that any value from the domain $[a, b]$ is fully possible.

Example 2.3 *If a company is evaluating a set of projects, one of the criteria that can be used for such an evaluation is the profit (in millions of CZK). For this criterion, the expert can define an evaluating function that is shown in the Figure 2.10. If the domain for this criterion has been specified to be, e.g.,* $[0, 20]$ *(representing the minimum and the maximum theoretically possible values for this criterion according to the company), this evaluation function can be expressed by a fuzzy number* $(1, 5, 20, 20)$.

According to the given evaluation function, if the profit for a project was 3 million CZK last year, then the project satisfies the corresponding partial goal in the degree 0.5 (i.e. the company's satisfaction with this profit would be 50 %). If a project whose profit has been only estimated and expressed, e.g., by a triangular fuzzy number $(2, 3, 4)$ *should be evaluated, then the criterion evaluation is calculated according to the extension principle as* $(0.25, 0.5, 0.75)$.

Generally, the resulting fuzzy evaluation need not to be linear even if the criterion value is expressed by a linear fuzzy number. Let us consider the profit expressed for example by the triangular fuzzy number $(2, 7, 8)$. *Then, the resulting fuzzy evaluation will not be a linear fuzzy number. Its membership function is shown in the Figure 2.11.*

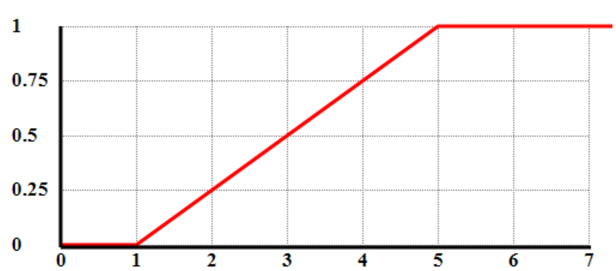


Figure 2.10: The evaluating function for the criterion *Profit (in millions of CZK)*

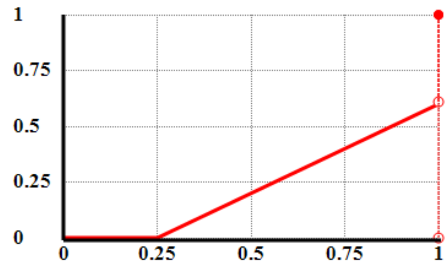


Figure 2.11: The resulting fuzzy evaluation for a project with the profit estimated by the fuzzy number $(2, 7, 8)$ in millions CZK.

2.10. Methods of aggregation of partial evaluations

So far, we have described how the partial evaluations of alternatives with respect to individual criteria are calculated. From these partial evaluations, it is necessary to come to the overall evaluation. This is achieved by an aggregation, which is performed according to the structure of the goals tree. For each node of the tree, the required aggregation mode has to be selected. Its choice is determined by both the evaluator's preferences and the type of interactions among criteria of evaluation, if there are any. The key requirement is that the evaluation at each level of aggregation should express (uncertain) extent of fulfillment of the corresponding goal.

In order to utilize the expert knowledge to its full extent, the mentioned methods are able to aggregate the evaluations in form of fuzzy numbers and also all the parameters for this aggregation are fuzzy (with a sole exception of the fuzzified WOWA operator, which aggregates fuzzy evaluations but the weights are crisp).

To avoid unnecessary repetition, we will make the following convention. In this section, the fuzzy evaluations that should be aggregated by some of the fuzzy methods will be denoted by the symbols $U_1, \ldots, U_m$ and the result of such an aggregation will be denoted by a fuzzy number U. For these fuzzy numbers, the form described in the Definition 2.7 will be also used. Specifically, for $i = 1, \ldots, m$:

$$U_i = \{[\underline{u}_i(\alpha), \overline{u}_i(\alpha)], \alpha \in [0, 1]\}, \text{ and}$$

$$U = \{[\underline{u}(\alpha), \overline{u}(\alpha)], \alpha \in [0, 1]\}.$$

It is obvious that to define the final result of the aggregation, i.e., the fuzzy number U, it is sufficient to define the two functions $\underline{u}(\alpha)$ and $\overline{u}(\alpha)$ for $\alpha \in [0, 1]$.

For fuzzy methods requiring additional information in form of normalized fuzzy weights, the symbols $W_1, \ldots, W_m$ will be used for such weights. Again, we will be using the following form for these fuzzy numbers for any $i = 1, \ldots, m$:

$$W_i = \{[\underline{w}_i(\alpha), \overline{w}_i(\alpha)], \alpha \in [0,1]\}.$$

In the following text, five different aggregation methods will be discussed – FuzzyWA, FuzzyOWA, fuzzified WOWA, fuzzified Choquet integral, and a fuzzy expert system.

2.10.1. Aggregation by the fuzzy weighted average (FuzzyWA)

The FuzzyWA is the simplest of the fuzzy methods discussed in this text. It is a fuzzification of the standard weighted average. Despite its relative simplicity from the decision-maker's point of view, it is sufficient for the majority of fuzzy MCDM problems.

This method can be used if the goal corresponding to the node of interest is fully decomposed into disjunctive goals of a lower level. In other words, significance defined on a system of subsets of these partial goals represents a standard additive normalized measure (probability measure) on the set of partial goals, whose evaluations should be aggregated.

Before the FuzzyWA is described, the (crisp) weighted average will be mentioned first.

Weighted average in the crisp case

In order to aggregate partial evaluations $u_1, \ldots, u_m$, $u_i \in [0,1]$, $i = 1, \ldots, m$ by means of the weighted average, normalized weights of partial goals must be set.

Definition 2.26 *Real numbers $w_1, \ldots, w_m$ are said to be normalized weights if $w_i \in [0,1]$ for all $i = 1, \ldots, m$ and $\sum_{i=1}^{m} w_i = 1$.*

Definition 2.27 *A weighted average (WA) of (real) numbers $u_1, \ldots, u_m$ using the vector of normalized weights $\mathbf{w} = (w_1, \ldots, w_m)$ is defined as*

$$WA_{\mathbf{w}}(u_1, \ldots, u_m) = \sum_{i=1}^{m} w_i \cdot u_i. \tag{2.13}$$

The weighted average is a simple and widely used aggregation operator. It the next section, its fuzzification will be described. This will make it possible to use fuzzy numbers instead of real numbers as the aggregated values and also as the weights. It will be demonstrated how such a generalization can be beneficial for the decision-maker.

Fuzzy weighted average

In reality, the weights are seldom known exactly. They usually represent expertly-set parameters to the model. That is why the use of a fuzzy weighted average (with fuzzy weights) is appropriate here. In FuzzyWA, both the weights and the partial evaluations will be represented by fuzzy numbers. To define uncertain weights consistently, a special structure of fuzzy numbers – normalized fuzzy weights – must be used.

Definition 2.28 *Fuzzy numbers $W_1, \ldots, W_m$ defined on $[0,1]$ form normalized fuzzy weights if for any $i \in \{1, \ldots, m\}$ and any $\alpha \in (0,1]$ it holds that for any $w_i \in W_{i\alpha}$ there exist $w_j \in W_{j\alpha}$, $j = 1, \ldots, m$, $j \neq i$, such that*

$$w_i + \sum_{j=1, j \neq i}^{m} w_j = 1. \tag{2.14}$$

Normalized fuzzy weights are generally used to model an uncertain division of the whole into m parts. When the normalized fuzzy weights have been set, the aggregated evaluation can be calculated from the evaluations with respect to the partial goals of the lower level according to the following definition.

Definition 2.29 *The FuzzyWA of the partial fuzzy evaluations, i.e., of fuzzy numbers $U_1, \ldots, U_m$ defined on $[0,1]$, with the normalized fuzzy weights $W_1, \ldots, W_m$, is a fuzzy number U on $[0,1]$, whose membership function is defined for any $u \in [0,1]$ as follows*

$$\begin{aligned} U(u) = \max\{&\min\{W_1(w_1), \ldots, W_m(w_m), U_1(u_1), \ldots, U_m(u_m)\} \\ &\mid \sum_{i=1}^{m} w_i \cdot u_i = u,\ \sum_{i=1}^{m} w_i = 1,\ w_i, u_i \in [0,1], i = 1, \ldots, m\}. \end{aligned} \tag{2.15}$$

The definition is based on the constrained fuzzy arithmetic [40]. In case of FuzzyWA, it stems from the extension principle setting an addition requirement that only those $w_i \in [0,1]$, $i = 1, \ldots, m$, that satisfy the condition $\sum_{i=1}^{m} w_i = 1$ are taken into the consideration. Let us note, that the idea of constraining the calculations has been already used by Zadeh in [103]. He showed that for calculations with interacting fuzzy numbers, the extension principle should be used in a form restricted by a given relation.

Two problems may arise if we want to use FuzzyWA for aggregation of the partial evaluations:

1. It is difficult to set normalized fuzzy weights that satisfy the above-mentioned condition in the Definition 2.28.

2. An effective algorithm for FuzzyWA calculation is necessary.

Both problems were solved in [59, 61, 62], and the results will be summarized in the following text.

Setting the normalized fuzzy weights

It can be quite difficult for the expert to set normalized fuzzy weights directly without breaching the condition given by the Formula 2.14. That is why multiple methods for constructing normalized fuzzy weights have been developed. Two main methods will be mentioned. The first one creates normalized fuzzy weights from normalized real weights. The latter more advanced method makes it possible to start with some estimations of the normalized fuzzy weights, which are fuzzy numbers. These estimations are then used to derive normalized fuzzy weights.

It was proved in Pavlačka [59] that symmetric triangular fuzzy numbers with the same spans form normalized fuzzy weights if elements of their kernels form normalized weights. Therefore, normalized fuzzy weights can be obtained easily from normalized real weights using the following algorithm.

Algorithm 2.1 *First, the expert is expected to provide normalized (real) weights* $w_1, \ldots, w_m$ *and a span* $\delta \in (0, 1)$*, which expresses how much uncertainty should be added. The parameter* δ *must be chosen so that the following conditions would be satisfied:*

$$\min\{w_1, \ldots, w_m\} \geq \delta, \tag{2.16}$$

$$\max\{w_1, \ldots, w_m\} \leq 1 - \delta. \tag{2.17}$$

Breaching these conditions would cause the resulting normalized fuzzy weights to be outside the interval $[0, 1]$*. In that case, a lower value can be chosen for* δ*.*

If the condition is met, we can then construct the normalized fuzzy weights $W_1, \ldots, W_m$ *as triangular fuzzy numbers so that* $W_i = (w_i - \delta,\ w_i,\ w_i + \delta)$*,* $i = 1, \ldots, m$*.*

For example, let the following normalized weights represent crisp estimates of significance of three partial goals: $w_1 = 0.2$, $w_2 = 0.5$ and $w_3 = 0.3$. Then, for $\delta = 0.1$, the following triangular fuzzy numbers are obtained as normalized fuzzy weights: $W_1 = (0.1, 0.2, 0.3)$, $W_2 = (0.4, 0.5, 0.6)$ and $W_3 = (0.2, 0.3, 0.4)$. The great advantage of this method is its simplicity. No advanced calculations are required.

In more complex cases, it is possible to let the expert set rough estimates of the uncertain weights. These estimates, which are fuzzy numbers, are then transformed into normalized fuzzy weights [62]. The transformation removes only the inconsistencies in the expert's formulation without losing any information. The normalized fuzzy weights can be derived from the expert's estimates using the following algorithm [62].

Algorithm 2.2 *First, the expert gives estimates of the uncertain normalized weights by means of fuzzy numbers* $V_1, \ldots, V_m$ *defined on the interval* $[0, 1]$*,* $V_i =$

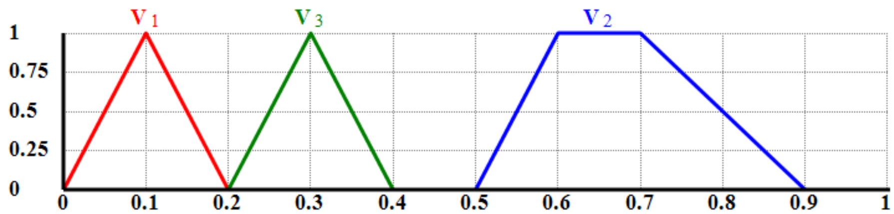


Figure 2.12: Estimates of the criteria fuzzy importances given by the expert

$\{[\underline{v}_i(\alpha), \overline{v}_i(\alpha)], \alpha \in [0,1]\}$, $i = 1\ldots, m$. *Regardless of the condition in the Formula 2.14, these estimates are only required to satisfy the following weaker condition:*

$$\exists v_1, \ldots, v_m,\ v_i \in \mathrm{Ker}(V_i),\ i = 1, \ldots, m : \sum_{i=1}^{m} v_i = 1. \tag{2.18}$$

In the next step, normalized fuzzy weights $W_1, \ldots, W_m$ are calculated by the formulae

$$\underline{w}_i(\alpha) = \max\{\underline{v}_i(\alpha),\ 1 - \sum_{j=1, j\neq i}^{m} \overline{v}_j(\alpha)\}, \tag{2.19}$$

$$\overline{w}_i(\alpha) = \min\{\overline{v}_i(\alpha),\ 1 - \sum_{j=1, j\neq i}^{m} \underline{v}_j(\alpha)\}. \tag{2.20}$$

This method is also implemented in the FuzzME software, which will be described later in the Section 2.13, and it makes it very easy to set normalized fuzzy weights.

As an example, let us assume that there are three criteria and the expert gave us rough estimates of their importances in form of the following linear fuzzy numbers: $V_1 = (0, 0.1, 0.2)$, $V_2 = (0.5, 0.6, 0.7, 0.9)$ and $V_3 = (0.2, 0.3, 0.4)$. These estimates are depicted in the Figure 2.12. We can see that expert had difficulty to assess the importance of the second criterion and the given estimate is much more uncertain than the others.

To obtain normalized fuzzy weights, we have to check if the given fuzzy numbers satisfy the condition given by the Formula 2.18, i.e. if their kernels contain at least one vector of normalized (real) weights. This is true, because the normalized (real) weights 0.1, 0.6, and 0.3 can be found in the kernels. We may thus proceed and the Algorithm 2.2 gives us the following normalized fuzzy weights: $W_1 = (0, 0.1, 0.2)$, $W_2 = (0.5, 0.6, 0.8)$ and $W_3 = (0.2, 0.3, 0.4)$. They are shown in the Figure 2.13. We can see that the second weight has been modified and its uncertainty decreased.

Effective algorithm for FuzzyWA calculation

Looking at the definition of the Fuzzy Weighted Average, we can see that the calculation of its value is not a trivial task. However, an effective algorithm

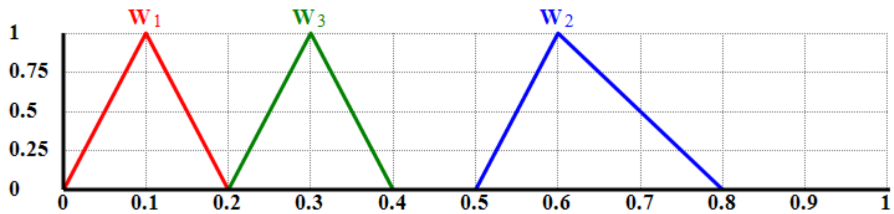


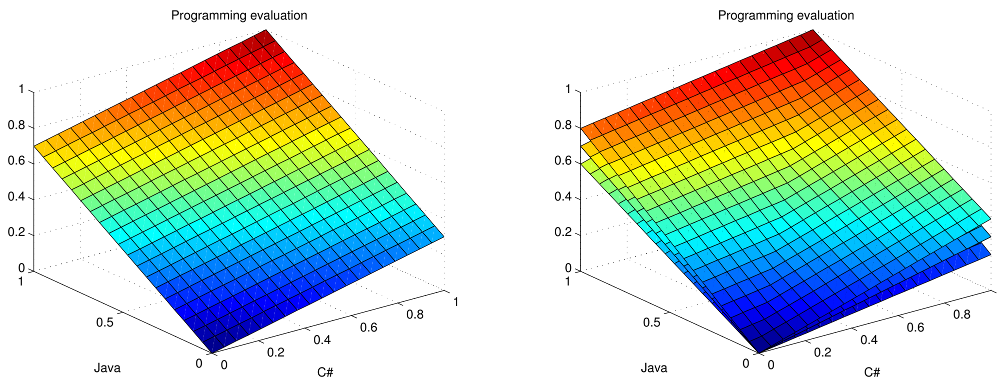
Figure 2.13: The resulting normalized fuzzy weights

suitable for FuzzyWA calculation exists [62]. The result of FuzzyWA aggregation of fuzzy numbers $U_1, \ldots, U_m$ with normalized fuzzy weights $W_1, \ldots, W_m$ is a fuzzy number U obtained through the following steps.

Algorithm 2.3 *For any $\alpha \in [0, 1]$ we calculate the result as follows:*

1. *Let σ and τ be permutations of the set of indices $\{1, \ldots, m\}$ such that $\underline{u}_{\sigma(1)}(\alpha) \leq \ldots \leq \underline{u}_{\sigma(m)}(\alpha)$ and $\overline{u}_{\tau(1)}(\alpha) \geq \ldots \geq \overline{u}_{\tau(m)}(\alpha)$.*
2. *Let for $k \in \{1, \ldots, m\}$ the values $w_k^L(\alpha)$ and $w_k^R(\alpha)$ be given as*
$$w_k^L(\alpha) = 1 - \sum_{i=1}^{k-1} \overline{w}_{\sigma(i)}(\alpha) - \sum_{i=k+1}^{m} \underline{w}_{\sigma(i)}(\alpha),$$
$$w_k^R(\alpha) = 1 - \sum_{i=1}^{k-1} \overline{w}_{\tau(i)}(\alpha) - \sum_{i=k+1}^{m} \underline{w}_{\tau(i)}(\alpha).$$
3. *Let k^* and k^{**} denote such indices that the following inequalities hold:*
$$\underline{w}_{\sigma(k^*)}(\alpha) \leq w_{k^*}^L(\alpha) \leq \overline{w}_{\sigma(k^*)}(\alpha),$$
$$\underline{w}_{\tau(k^{**})}(\alpha) \leq w_{k^{**}}^R(\alpha) \leq \overline{w}_{\tau(k^{**})}(\alpha),$$
where $k^*, k^{**} \in \{1, 2, \ldots, m\}$.
4. *The resulting values of the functions $\underline{u}(\alpha)$ and $\overline{u}(\alpha)$ are then obtained by the following formulae.*
$$\underline{u}(\alpha) = \sum_{i=1}^{k^*-1} \overline{w}_{\sigma(i)}(\alpha) \cdot \underline{u}_{\sigma(i)}(\alpha) + w_{k^*}^L(\alpha) \cdot \underline{u}_{\sigma(k^*)}(\alpha) + \sum_{i=k^*+1}^{m} \underline{w}_{\sigma(i)}(\alpha) \cdot \underline{u}_{\sigma(i)}(\alpha),$$
$$\overline{u}(\alpha) = \sum_{i=1}^{k^{**}-1} \overline{w}_{\tau(i)}(\alpha) \cdot \overline{u}_{\tau(i)}(\alpha) + w_{k^{**}}^R(\alpha) \cdot \overline{u}_{\tau(k^{**})}(\alpha) + \sum_{i=k^{**}+1}^{m} \underline{w}_{\tau(i)}(\alpha) \cdot \overline{u}_{\tau(i)}(\alpha).$$

In reality, the calculations are performed only for some $\alpha_i \in [0, 1]$, $i = 1, \ldots, n$, where $n \in \mathbb{N}$. The higher the number of α-cuts n is chosen, the better the approximation of the result is obtained. In practice, $n = 60$ turned out to be a good trade-off between the accuracy and the time complexity of the calculations.



(a) In the crisp case

(b) In the fuzzy case

Figure 2.14: Evaluation function using FuzzyWA from the Example 2.4

Example

Following the Example 2.2, the fuzzy weighted average could be used in the following way.

Example 2.4 *If the company from the Example 2.2 prefers the candidates to have knowledge of Java, they could use the fuzzy weighted average with the following normalized fuzzy weights for both of the criteria:* $W_{C\#} = (0.2, 0.3, 0.4)$, $W_{Java} = (0.6, 0.7, 0.8)$.

The Figure 2.14b shows the graph of the evaluation function. The results will be, on our case, approximated by triangular fuzzy numbers so three surfaces are plotted in the graph. Each of them represents one significant value of the resulting fuzzy number. This way, we are able to visualize almost entire information about the result, and not just its single characteristic (such as the center of gravity). This will help to understand the behavior of the evaluation function better. On the x-axe, there is the evaluation of C# knowledge, on the y-axe, there is the evaluation of the Java knowledge. In order to be able to construct the graph, we assume only crisp values of these two partial evaluations (i.e. fuzzy singletons on [0, 1]). For the comparison, Figure 2.14a shows the result with the crisp weights $W_{C\#} = 0.3$, $W_{Java} = 0.7$.

Features of FuzzyWA

The weighted average is a special case of the fuzzy weighted average where the aggregated fuzzy evaluations and the used normalized fuzzy weights are fuzzy singletons; it can be easily verified that normalized real weights are a special case of normalized fuzzy weights [62].

Generally, the FuzzyWA does not preserve the linearity [62]. It means that even though the partial fuzzy evaluations and the normalized fuzzy weights are

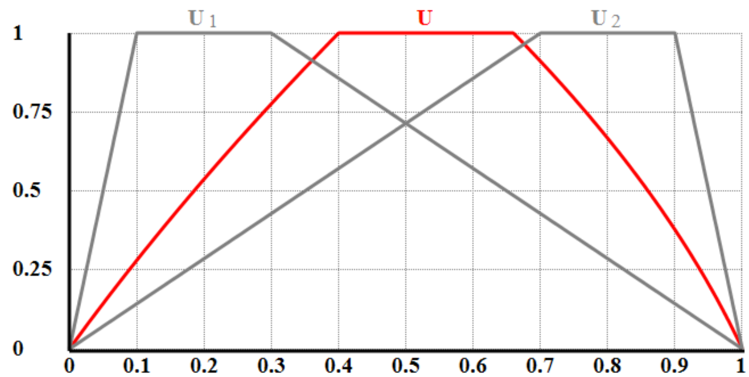
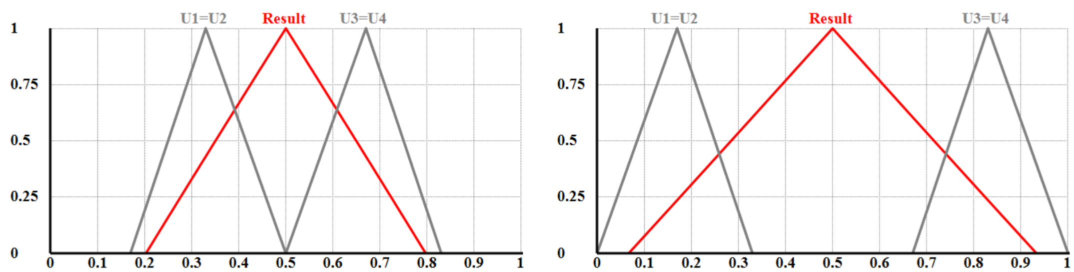


Figure 2.15: A sample FuzzyWA result for evaluations and weights expressed by linear fuzzy numbers; the result is, however, not a linear fuzzy number.

expressed by linear fuzzy numbers, generally, the result need not to be a linear fuzzy number. For example, for $U_1 = (0, 0.1, 0.3, 1)$, $U_2 = (0, 0.7, 0.9, 1)$, $W_1 = (0.2, 0.4, 0.5, 0.6)$, and $W_2 = (0.4, 0.5, 0.6, 0.8)$, the result is shown in the Figure 2.15. Although it very close to a linear fuzzy number, a slight curvature of the resulting membership function is apparent. However, if the normalized fuzzy weights are crisp (or more accurately, set by means of fuzzy singletons), the linearity is preserved [81, 4].

An interesting feature of FuzzyWA is how the dispersion of the evaluations with respect to the individual partial goals is reflected in the uncertainty of the resulting fuzzy evaluation [62]. To make this behavior clearer, let us consider a fuzzy weighted average of 4 partial evaluations with uniform normalized fuzzy weights $W_1 = W_2 = W_3 = W_4 = (0.05, 0.25, 0.45)$. The partial evaluations can represent evaluations of various aspects of a bank client. Let us consider the situation when the client evaluation is average according to all of the four aspects, and another situations, when a client is evaluated as excellent according the half of the aspects and completely unsatisfactory according to the rest of them. Then evaluation with the (crisp) weighted average would make no difference between these two cases and both clients would be rated to be average (0.5). If the fuzzy weighted average is used the two fuzzy evaluations will have the same centers of gravity, however, the latter one will be much more uncertain. The fuzzy weighted average takes into consideration also the dispersion of the aggregated values. This is shown in the Figures 2.16a and 2.16b. In the Figure 2.16a, the aggregated values are closer to each other and therefore the result is less uncertain. On the other hand, in the Figure 2.16b, the aggregated values differ more so the resulting fuzzy evaluation is more uncertain. This is an important property of the fuzzy weighted average, which is very beneficial for the multiple-criteria decision-making as it will be be shown later on a real-world application in the Section 2.14.1.

More details on this aggregation method can be found in [59]. It should be



(a) (b)

Figure 2.16: Comparison of results of the fuzzy weighted average when the aggregated values are (a) close to each other, (b) further apart

also noted that the described FuzzyWA is not the only approach to fuzzification of the weighted average. Other approaches have been proposed (e.g. [1, 15]). The mentioned method has the described property of reflecting the information on the dispersion of the aggregated values into the aggregation result. The FuzzyWA uses also a special structure to represent the weights, provides methods for setting them and, moreover, an algorithm for efficient calculation is available.

2.10.2. Aggregation by the ordered fuzzy weighted average (FuzzyOWA)

The FuzzyOWA [82] is a fuzzification of the OWA operator introduced by Yager [101]. Similarly to FuzzyWA, FuzzyOWA also uses the structure of normalized fuzzy weights. Contrary to FuzzyWA, the weights are not linked to particular partial goals. They express the evaluator's requirements concerning the structure of partial evaluation. In particular, the i-th weight is linked to the i-th largest evaluation with respect to the partial goals. With FuzzyOWA, various preferences can be modeled by weights of special form as it will be shown on the examples later.

OWA operator in the crisp case

For the crisp case, the OWA operator is defined as follows.

Definition 2.30 *Ordered Weighted Average of (real) numbers $u_1, \ldots, u_m$ using the vector of normalized weights $\mathbf{w} = (w_1, \ldots, w_m)$ is defined by the following formula*

$$\mathrm{OWA}_{\mathbf{w}}(u_1, \ldots, u_m) = \sum_{i=1}^{m} w_i \cdot u_{\phi(i)}, \tag{2.21}$$

where ϕ denotes such a permutation of the set of indices $\{1, \ldots, m\}$ that $u_{\phi(1)} \geq u_{\phi(2)} \geq \ldots \geq u_{\phi(m)}$.

According to the choice of the weights, various aggregation operators ranging from minimum to maximum can be obtained. For example, we can get the following operators if the weights are selected in a specific way:

- **maximum** for the weights $w_1 = 1$ and $w_i = 0$, $i = 2, 3, \ldots, m$;
- **minimum** for $w_m = 1$ and $w_i = 0$, $i = 1, 2, \ldots, m-1$;
- **arithmetic mean** using the weights $w_i = \frac{1}{m}$, $i = 1, 2, \ldots, m$;
- **median** can be obtained using the following weights, for $i = 1, \ldots, m$:
  1. If m is odd:
  $$w_i = \begin{cases} 1 & \text{for } i = \frac{m+1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$
  2. If m is even:
  $$w_i = \begin{cases} 0.5 & \text{for } i = \frac{m}{2}, \\ 0.5 & \text{for } i = \frac{m}{2} + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly as in case of the fuzzy weighted average, the importances of the criteria are seldom known preciously in the practice. In this case, it can be advantageous to use a fuzzified version of the operator, which makes it possible to express the weights and the aggregated values in form of fuzzy numbers.

Fuzzy OWA operator

FuzzyOWA represents a fuzzification of the crisp OWA operator according to the extension principle where the condition of normalized weights is respected. Uncertain weights are modeled by normalized fuzzy weights, as in the case of FuzzyWA. Again, partial evaluations are modeled by fuzzy numbers on $[0, 1]$.

Definition 2.31 *The FuzzyOWA of the partial fuzzy evaluations, i.e., of fuzzy numbers $U_1, \ldots, U_m$ defined on $[0, 1]$, with normalized fuzzy weights $W_1, \ldots, W_m$ is a fuzzy number U on $[0, 1]$ whose membership function is defined for any $u \in [0, 1]$ as follows*

$$U(u) = \max\{\min\{W_1(w_1), \ldots, W_m(w_m), U_1(u_1), \ldots, U_m(u_m)\} \mid \sum_{i=1}^{m} w_i \cdot u_{\phi(i)} = u,\ \sum_{i=1}^{m} w_i = 1,\ w_i, u_i \in [0, 1], i = 1, \ldots, m\}, \tag{2.22}$$

where ϕ denotes such a permutation of the set of indices $\{1, \ldots, m\}$ that $u_{\phi(1)} \geq u_{\phi(2)} \geq \ldots \geq u_{\phi(m)}$.

Generally, the same questions as for the FuzzyWA should be asked before the FuzzyOWA can be successfully used in the practice: a) How should we set the normalized fuzzy weights? and b) How can we calculate FuzzyOWA efficiently? As in the case of FuzzyWA, these questions have been answered – for setting of the normalized fuzzy weights, the procedures described in the Section 2.10.1 related to the FuzzyWA can be used. For calculations of the resulting evaluation, an efficient algorithm exists.

Efficient algorithm for FuzzyOWA calculation

Similarly to FuzzyWA, the definition of FuzzyOWA is not well-suited for computation. However, an effective algorithm has been proposed in [82]. The algorithm is analogous to the one used for FuzzyWA. The resulting fuzzy number U is calculated in the following way.

Algorithm 2.4 *For any $\alpha \in [0,1]$:*

1. *Let σ and τ be permutations of the set of indices $\{1,\ldots,m\}$ such that $\underline{u}_{\sigma(1)}(\alpha) \leq \ldots \leq \underline{u}_{\sigma(m)}(\alpha)$ and $\overline{u}_{\tau(1)}(\alpha) \geq \ldots \geq \overline{u}_{\tau(m)}(\alpha)$.*

2. *Let for $k \in \{1,\ldots,m\}$ the values $w_k^L(\alpha)$ and $w_k^R(\alpha)$ be given as*

$$w_k^L(\alpha) = 1 - \sum_{i=1}^{k-1} \underline{w}_i(\alpha) - \sum_{i=k+1}^{m} \overline{w}_i(\alpha),$$

$$w_k^R(\alpha) = 1 - \sum_{i=1}^{k-1} \overline{w}_i(\alpha) - \sum_{i=k+1}^{m} \underline{w}_i(\alpha).$$

3. *Let k^* and k^{**} denote such indices that the following holds:*

$$\begin{aligned} \underline{w}_{k^*}(\alpha) &\leq w_{k^*}^L(\alpha) \leq \overline{w}_{k^*}(\alpha), \\ \underline{w}_{k^{**}}(\alpha) &\leq w_{k^{**}}^R(\alpha) \leq \overline{w}_{k^{**}}(\alpha), \end{aligned}$$

   where $k^*, k^{**} \in \{1,2,\ldots,m\}$.

4. *The values of the functions $\underline{u}(\alpha)$ and $\overline{u}(\alpha)$ are then obtained as follows.*

$$\underline{u}(\alpha) = \sum_{i=1}^{m-k^*} \overline{w}_{m-i+1}(\alpha) \cdot \underline{u}_{\sigma(i)}(\alpha) + w_{k^*}^L(\alpha) \cdot \underline{u}_{\sigma(m-k^*+1)}(\alpha) + \sum_{i=m-k^*+2}^{m} \underline{w}_{m-i+1}(\alpha) \cdot \underline{u}_{\sigma(i)}(\alpha),$$

$$\overline{u}(\alpha) = \sum_{i=1}^{k^{**}-1} \overline{w}_i(\alpha) \cdot \overline{u}_{\tau(i)}(\alpha) + w_{k^{**}}^R(\alpha) \cdot \overline{u}_{\tau(k^{**})}(\alpha) + \sum_{i=k^{**}+1}^{m} \underline{w}_i(\alpha) \cdot \overline{u}_{\tau(i)}(\alpha).$$

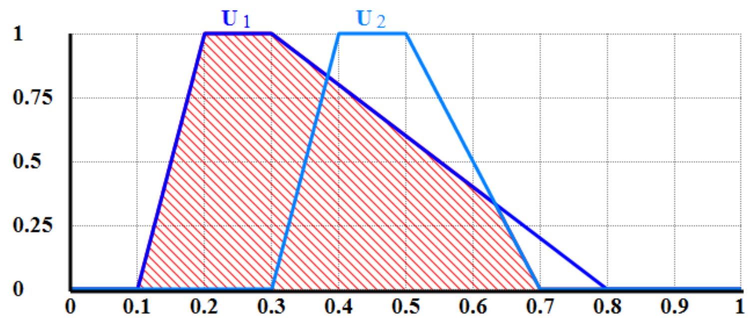


Figure 2.17: A fuzzy minimum of the fuzzy numbers $U_1 = (0.1, 0.2, 0.3, 0.8)$ and $U_2 = (0.3, 0.4, 0.5, 0.7)$.

Examples

The possibilities and the behavior of the FuzzyOWA will be demonstrated on three examples. The first one shows how to obtain a fuzzy minimum using FuzzyOWA.

Example 2.5 *Let us consider an HR agent wishing to employ new workers whose partial fuzzy evaluations are known at the job interview. In his/her view, only those candidates who are not significantly bad according to any of the considered criteria can be hired. A fuzzy equivalent of minimum would be appropriate for such an evaluation.*

The evaluation can be calculated by the FuzzyOWA with the weight W_m (corresponding to the minimum of the partial evaluation) set to $\tilde{1}$. For the rest of the weights $W_1, \ldots, W_{m-1}$, the value $\tilde{0}$ is used.

The aggregated fuzzy evaluations then represent the guaranteed fuzzy degrees of fulfillment of all the partial goals (the FuzzyMin method). For example, let us consider a simple case when only two evaluations are to be aggregated: $U_1 = (0.1, 0.2, 0.3, 0.8)$ and $U_2 = (0.3, 0.4, 0.5, 0.7)$. The result of such an evaluation can be seen in the Figure 2.17.

Example 2.6 *Another example of using the FuzzyOWA operator is the evaluation of workers by their colleagues. Because, in every team, there are friends and foes, we will ignore the best and the worst partial evaluations. The other partial evaluations have the same importance so their weights will be uniform. If we have, for example, the evaluation from five people, then the weights for the crisp OWA would be $w_1 = w_5 = 0$ and $w_i = \frac{1}{3}$ for $i = 2, 3, 4$.*

In the fuzzy case, we could use FuzzyOWA. The normalized fuzzy weights represent a fuzzification of the mentioned normalized crisp weights. For example, the normalized fuzzy weights could be $W_1 = W_5 = \tilde{0}$ and $W_i = (\frac{1}{4}, \frac{1}{3}, \frac{1}{2})$ for $i = 2, 3, 4$.

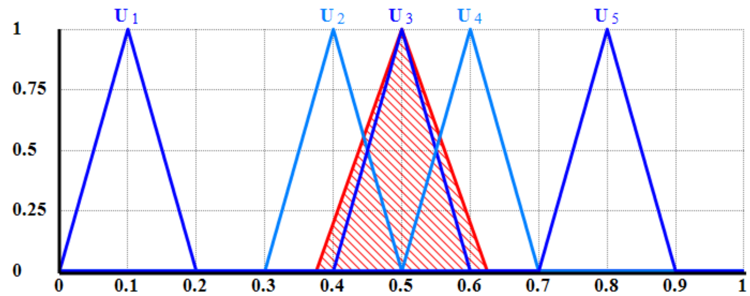


Figure 2.18: The result of the FuzzyOWA from the Example 2.6

The Figure 2.18 compares sample partial evaluations and the overall evaluation. It can be seen that the extreme evaluations (U_1 and U_5) are ignored.

Example 2.7 *Following the Example 2.2, if the software company prefers neither of the programming languages but wants the candidate to be good at least in one of them (but at least basic knowledge of the other one is also favorable by the company), the FuzzyOWA with the following normalized fuzzy weights can be used: $W_1 = (0.7, 0.8, 0.9)$, $W_2 = (0.1, 0.2, 0.3)$. This way, the higher importance will be assigned to the programming language that the candidate knows better. However, the evaluation of knowledge of the other programming language will be taken into account, too.*

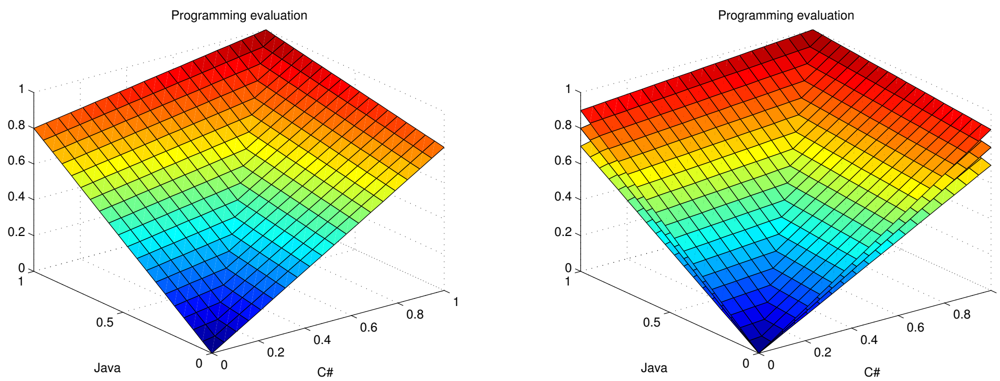
The behavior of the evaluation function can be seen in the Figure 2.19b. It can be compared with the crisp OWA operator (using weights $w_1 = 0.8$, $w_2 = 0.2$) whose graph is depicted in the Figure 2.19a. It can be seen that the behavior of the FuzzyOWA with normalized fuzzy weights is similar as in case of the fuzzy weighted average. The more dispersion between the aggregated partial evaluations, the more uncertain result.

Features of the FuzzyOWA

The main features of the FuzzyOWA are similar to those for FuzzyWA. FuzzyOWA also does not preserve the linearity [4]. Contrary to the FuzzyWA, the linearity is not preserved even though fuzzy singleton weights are used. This could be seen for instance in the Example 2.5.

Similarly to FuzzyWA, the FuzzyOWA reflects the dispersion of the aggregated evaluations into the uncertainty of the overall evaluation. This property can be observed in the Example 2.7.

Again, FuzzyOWA does not represent the only way of fuzzification of the OWA operator. For instance, the type-1 OWA proposed by Zhou et al. [106] is a well-known mean of the OWA fuzzification. Type-1 OWA aggregates (type-1) fuzzy sets using fuzzy weights.



(a) In the crisp case

(b) In the fuzzy case

Figure 2.19: Evaluation function using FuzzyOWA from the Example 2.7

To sum up this part, FuzzyOWA described in this thesis makes it possible to define preferences of partial evaluations according to their order for a particular alternative. However, with FuzzyOWA it is impossible to express the preferences connected with the partial goals themselves.

2.10.3. Aggregation by the fuzzified WOWA operator

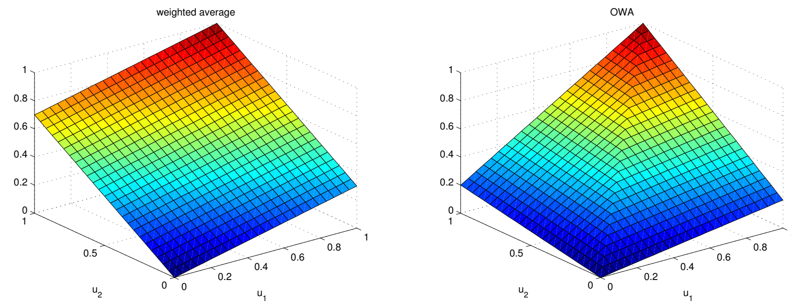
In many situations, the weighted average or the OWA operator are sufficient. As already mentioned, the weighted average is used when the significances of partial goals are given. On the other hand, the OWA is used when the importances of evaluations with respect to partial goals are given by ordering of these evaluations. If the expert needs to take into account both aspects, one of the possible solutions is the WOWA (weighted OWA) operator introduced by Torra in [89].

WOWA operator in the crisp case

The WOWA operator uses two m-tuples of normalized weights – the first of them $\mathbf{p} = (p_1, p_2, \ldots, p_m)$, is connected to the individual partial goals; the latter one $\mathbf{w} = (w_1, w_2, \ldots, w_m)$, is related to the decreasing order of partial evaluations. The aggregation with the (crisp) WOWA operator is performed according to the following definition.

Definition 2.32 *Weighted Ordered Weighted Average (WOWA) of the values* $u_1, \ldots, u_m$ *using the vectors of normalized weights* $\mathbf{p} = (p_1, \ldots, p_m)$ *and* $\mathbf{w} = (w_1, \ldots, w_m)$ *is defined as*

$$\mathrm{WOWA}^{\mathbf{p}}_{\mathbf{w}}(u_1, \ldots, u_m) = \sum_{i=1}^{m} \omega_i \cdot u_{\phi(i)}, \tag{2.23}$$



(a) Weighted average with $\mathbf{p} = (0.3, 0.7)$

(b) OWA with the weights $\mathbf{w} = (0.2, 0.8)$

Figure 2.20: Weighted average and OWA

where ϕ denotes such a permutation of the set of indices $\{1, \ldots, m\}$ that $u_{\phi(1)} \geq u_{\phi(2)} \geq \ldots \geq u_{\phi(m)}$. The weight ω_i is defined as

$$\omega_i = z(\sum_{j \leq i} p_{\phi(j)}) - z(\sum_{j<i} p_{\phi(j)}), \tag{2.24}$$

for $i = 1, \ldots, m$, and z is a nondecreasing function interpolating the following points

$$\{(0,0)\} \ \cup \ \{(i/m, \ \sum_{j \leq i} w_j)\}_{i=1,\ldots,m}. \tag{2.25}$$

The function z is required to be a straight line when the points can be interpolated in that way.

Although several ways of constructing the interpolation function z are discussed in the literature (e.g. [91]), the simplest one will be used in this theses – z will be a a piecewise linear function connecting the individual points.

In the following text, $\boldsymbol{\eta}$ will denote a vector of uniform weights of the length m, i.e. $\boldsymbol{\eta} = (\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m})$. It can be easily shown that the WOWA is a generalization of both weighted average and OWA [89, 92]. If the weights $\mathbf{w}$ are uniform, i.e. $\mathbf{w} = \boldsymbol{\eta}$, the result of the WOWA equals to the weighted average with the weights $\mathbf{p}$. Vice versa, using a uniform weights $\mathbf{p}$ is equivalent to the aggregation by OWA with the weights $\mathbf{w}$.

Initially, the way how the mixture weights are calculated in the WOWA definition can seem complicated. A graphical insight can be obtained by comparing the graphs for the weighted average (Figure 2.20a), the OWA (Figure 2.20b), and the WOWA using the same weights (Figure 2.21). Looking at the figures, we can see that the graph of the WOWA is composed of the OWA graph "rotated" according to the weights used for the weighted average.

It is worth noting that the WOWA does not represent the only way of generalizing the weighted average and OWA operator. Multiple methods for combining

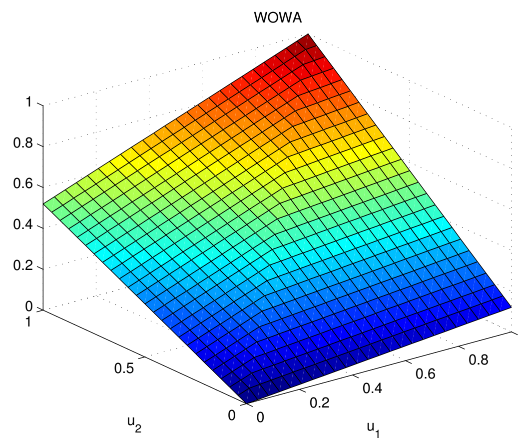


Figure 2.21: WOWA with $\mathbf{p} = (0.3, 0.7)$ and $\mathbf{w} = (0.2, 0.8)$

the weights for both of the aggregation operators can be found in the literature. Besides the WOWA, this task can be accomplished by the operator proposed by Engemann et al. [19]. As noted in [48], the same operator appeared over the time with different names in the literature such as hybrid weighted averaging operator (HWAA) in [45], or immediate probability OWA (IP-OWA) in [52]. The operator calculates new composite weights basically by multiplying the weights for WA and OWA corresponding to the same criteria. As these new weights generally do not sum to 1, they are normalized. A big drawback of this operator is the lack of monotony, which makes its relevance for the multiple-criteria evaluation to be very limited.

Another interesting approach are SUOWA operators introduced in [47]. For given two vectors of weights, the SUOWA operators do not aggregate them into a single composite weighting vector like the WOWA. Instead, they derive a capacity (fuzzy measure) for the Choquet integral. As SUOWA operators are based on the Choquet integral, they retain its good properties (including the monotony).

The advantage of the WOWA operator is that it has many important properties. In [89], it has been proven that it is idempotent, monotone in relation to the input variables $u_1, \ldots, u_m$, and that it satisfies a boundary condition (its result remains between the minimum and the maximum of the arguments). It has been also shown that the WOWA is a special case of the Choquet integral with a particular fuzzy measure [90].

The drawback of the WOWA is that its behavior is reported to be counter-intuitive in some cases. Specifically, the result of the WOWA needs not to lie between the values returned by the weighted average (WA) and the OWA with the corresponding weights. Even though both the WA and OWA agree on the same result, the WOWA result can be, generally, different. The practical behavior of the WOWA operator is studied in [46]. The decision-makers should be familiar with the behavior of the WOWA in these cases. Before applying WOWA, they should make sure that it does not present a difficulty for the particular problem that is being solved.

In the next part, the fuzzification of the WOWA operator will be described.

Fuzzified WOWA operator

The fuzzified WOWA operator (first-level fuzzification) [38], considered in this text, is able to aggregate the fuzzy partial evaluations $U_1, \ldots, U_m$ that are fuzzy numbers. However, the weights $p_1, p_2, \ldots, p_m$ and $w_1, w_2, \ldots, w_m$ must be crisp. Identically to the crisp case, the first vector of weights, $\mathbf{p} = (p_1, p_2, \ldots, p_m)$, is connected to individual partial goals and the second one, $\mathbf{w} = (w_1, w_2, \ldots, w_m)$, is related to the decreasing order of partial evaluations. The fuzzified WOWA is defined according to the extension principle as follows.

Definition 2.33 *Let $U_1, \ldots, U_m$ be fuzzy numbers defined on $[0, 1]$ and let $\mathbf{p} = (p_1, \ldots, p_m)$ and $\mathbf{w} = (w_1, \ldots, w_m)$ be two vectors of normalized (real) weights. Then the result of the aggregation by a fuzzified WOWA operator is a fuzzy number U with the membership function defined for any $y \in [0, 1]$ as follows:*

$$U(y) = \max \big\{ \min\{U_1(u_1), \ldots, U_m(u_m)\} \mid u_i \in [0,1], i = 1, \ldots, m, \quad y = \mathrm{WOWA}^{\mathbf{p}}_{\mathbf{w}}(u_1, u_2, \ldots, u_m) \big\}. \tag{2.26}$$

The result of the fuzzified WOWA of $U_1, \ldots, U_m$ with the weights $\mathbf{p}$ and $\mathbf{w}$ will be denoted as $\mathrm{FWOWA}^{\mathbf{p}}_{\mathbf{w}}(U_1, \ldots, U_m)$.

The definition is not very suitable for direct calculations. However, the following theorem shows how we can compute the fuzzified WOWA straightforwardly.

Theorem 2.2 *The result of the fuzzified WOWA of the fuzzy numbers $U_1, \ldots, U_m$ defined on $[0, 1]$ with the weights $\mathbf{p} = (p_1, \ldots, p_m)$ and $\mathbf{w} = (w_1, \ldots, w_m)$ is a fuzzy number U defined for any $\alpha \in [0, 1]$ as follows:*

$$\underline{u}(\alpha) = \mathrm{WOWA}^{\mathbf{p}}_{\mathbf{w}}(\underline{u}_1(\alpha), \underline{u}_2(\alpha), \ldots, \underline{u}_m(\alpha)), \tag{2.27}$$

$$\overline{u}(\alpha) = \mathrm{WOWA}^{\mathbf{p}}_{\mathbf{w}}(\overline{u}_1(\alpha), \overline{u}_2(\alpha), \ldots, \overline{u}_m(\alpha)). \tag{2.28}$$

.

Proof: Applying the Theorem 2.1 to the Definition 2.33, we obtain the Formulae 2.27 and 2.28. The required monotonicity has been proven in [89] and continuousness is derived from the fact that the WOWA is a special case of the Choquet integral with a particular fuzzy measure [90], which is continuous [30]. □

Properties of the fuzzified WOWA operator

In this section, some properties of the fuzzified WOWA operator will be studied. The following theorem confirms that the presented fuzzified WOWA generalizes the WOWA operator.

Theorem 2.3 *Let $U_1, \ldots, U_m$ be fuzzy singletons containing elements $u_i \in [0, 1]$, $i = 1, \ldots, m$. Then for any vectors of normalized weights $\mathbf{p}$ and $\mathbf{w}$, it holds that*

$$\text{FWOWA}_{\mathbf{w}}^{\mathbf{p}}(U_1, \ldots, U_m) = \text{WOWA}_{\mathbf{w}}^{\mathbf{p}}(u_1, \ldots, u_m). \tag{2.29}$$

Proof: If the fuzzy numbers U_i are fuzzy singletons with the elements $u_i \in \Re$, $i = 1, \ldots, m$, then $\underline{u}_i(\alpha) = \overline{u}_i(\alpha)$, for all $\alpha \in [0, 1]$ and any $i = 1, \ldots, m$. Using the Formulae 2.27 and 2.28, it is apparent that

$$\underline{u}(\alpha) = \overline{u}(\alpha) = \text{WOWA}_{\mathbf{w}}^{\mathbf{p}}(u_1, \ldots, u_m). \tag{2.30}$$

□

Next, we will see that the fuzzified WOWA has some basic properties such as idempotency and boundary conditions.

Theorem 2.4 *The fuzzified WOWA is idempotent, i.e. if U is a fuzzy number and $\mathbf{p}$ and $\mathbf{w}$ are two vectors of normalized weights, then it holds that*

$$U = \text{FWOWA}_{\mathbf{w}}^{\mathbf{p}}(U, \ldots, U). \tag{2.31}$$

Proof: The theorem is a direct result of the Theorem 2.2 and the fact that the crisp WOWA is idempotent. □

Theorem 2.5 *Let $U_1, \ldots, U_m$ be fuzzy numbers and $\mathbf{p}$ and $\mathbf{w}$ be two vectors of normalized weights. For the fuzzy number $U = \text{FWOWA}_{\mathbf{w}}^{\mathbf{p}}(U_1, \ldots, U_m)$, it holds that for any $\alpha \in [0, 1]$*

$$\underline{u}(\alpha) \geq \min\{\underline{u}_1(\alpha), \underline{u}_2(\alpha), \ldots, \underline{u}_m(\alpha)\}, \tag{2.32}$$

$$\overline{u}(\alpha) \leq \max\{\overline{u}_1(\alpha), \overline{u}_2(\alpha), \ldots, \overline{u}_m(\alpha)\}. \tag{2.33}$$

Proof: The theorem follows from the fact that $\underline{u}(\alpha)$ and $\overline{u}(\alpha)$ can be calculated using the Formulae 2.27 and 2.28 by a crisp WOWA operator and, moreover, that for the WOWA operator of values $u_1, \ldots, u_m \in \Re$, it holds that

$$\min\{u_1, \ldots, u_m\} \leq \text{WOWA}_{\mathbf{w}}^{\mathbf{p}}(u_1, \ldots, u_m) \leq \max\{u_1, \ldots, u_m\},$$

which has been proven in [89]. □

Relation of the fuzzified WOWA to FuzzyWA and FuzzyOWA operators

It is well-know that in the crisp case, if one of the weights vectors is uniform, the WOWA operator reduces either to the weighted average, or to the OWA [89]. In the following text, it will be proven that a similar relation is true also in the fuzzy case. Because the first-level fuzzification of the WOWA operator is presented in the thesis (the aggregated values are fuzzy numbers, but the weights are crisp), we will have to start with the definition of the first-level fuzzy weighted average and the first-level fuzzy OWA (for more details on these two operators, see [82, 4]). They are just simpler cases of the FuzzyWA and FuzzyOWA when normalized real weights are used instead of normalized fuzzy weights [4].

Definition 2.34 *Let $U_1, \ldots, U_m$ be fuzzy numbers defined on $[0,1]$ and let $p_1, \ldots, p_m$ be normalized (real) weights. The first-level fuzzy weighted average [4] of the values $U_1, \ldots, U_m$ with the weights $p_1, \ldots, p_m$ is a fuzzy number U with the membership function given for any $y \in [0,1]$ as follows:*

$$U(y) = \max \Big\{ \min\{U_1(u_1), \ldots, U_m(u_m)\} \mid u_i \in [0,1], i = 1, \ldots, m, \quad y = \sum_{i=1}^{m} u_i p_i \Big\}. \tag{2.34}$$

Definition 2.35 *Let $U_1, \ldots, U_m$ be fuzzy numbers defined on $[0,1]$ and let $w_1, \ldots, w_m$ be normalized (real) weights. Then the first-level fuzzy OWA [4] of the values $U_1, \ldots, U_m$ with the weights $w_1, \ldots, w_m$ is a fuzzy number U with the membership function given for any $y \in [0,1]$ as follows:*

$$U(y) = \max \Big\{ \min\{U_1(u_1), \ldots, U_m(u_m)\} \mid u_i \in [0,1],\ i = 1, \ldots, m, \quad y = \sum_{i=1}^{m} u_i w_{\phi(i)} \Big\}, \tag{2.35}$$

where ϕ denotes such a permutation of the set of indices $\{1, \ldots, m\}$ that $u_{\phi(1)} \geq u_{\phi(2)} \geq \ldots \geq u_{\phi(m)}$.

In [82], the following two theorems for easier calculation of these two operators can be found.

Theorem 2.6 *The (first-level) fuzzy weighted average of the fuzzy numbers $U_1, \ldots, U_m$ defined on $[0,1]$ with normalized weights $p_1, \ldots, p_m$ is a fuzzy number U that can be calculated, for any $\alpha \in [0,1]$, as follows:*

$$\underline{u}(\alpha) = \sum_{i=1}^{m} p_i \cdot \underline{u}_i(\alpha), \tag{2.36}$$

$$\overline{u}(\alpha) = \sum_{i=1}^{m} p_i \cdot \overline{u}_i(\alpha). \tag{2.37}$$

Proof: The proof can be found in [4]. □

Theorem 2.7 *The (first-level) fuzzy OWA of the fuzzy numbers $U_1, \ldots, U_m$ with normalized weights $w_1, \ldots, w_m$ is a fuzzy number U, which can be obtained by the following formulae for any $\alpha \in [0, 1]$:*

$$\underline{u}(\alpha) = \sum_{i=1}^{m} w_i \cdot \underline{u}_{\sigma(i)}(\alpha), \tag{2.38}$$

$$\overline{u}(\alpha) = \sum_{i=1}^{m} w_i \cdot \overline{u}_{\tau(i)}(\alpha), \tag{2.39}$$

where σ and τ are such permutations of the set of indices $\{1, \ldots, m\}$ that $\underline{u}_{\sigma(1)} \geq \underline{u}_{\sigma(2)} \geq \ldots \geq \underline{u}_{\sigma(m)}$ and $\overline{u}_{\tau(1)} \geq \overline{u}_{\tau(2)} \geq \ldots \geq \overline{u}_{\tau(m)}$.

Proof: The proof can be found in [4]. □

Now we can study the relationship between the first-level fuzzy weighted average, first-level fuzzy OWA and the presented (first-level) fuzzified WOWA. The following theorems show that the relationship is identical as in the crisp case.

Theorem 2.8 *Let $U_1, \ldots, U_m$ be fuzzy numbers and $\mathbf{p}$ be a vector of normalized weights. Further, let $\mathbf{w}$ be a vector of uniform real weights, $\mathbf{w} = \boldsymbol{\eta}$. Then the result of the fuzzified WOWA of $U_1, \ldots, U_m$ with the weights $\mathbf{p}$ and $\mathbf{w}$ is identical to the result of the first-level fuzzy weighted average of $U_1, \ldots, U_m$ with the weights $\mathbf{p}$.*

Proof: As the weighted average is a special case of the WOWA, it holds that $\text{WOWA}_{\boldsymbol{\eta}}^{\mathbf{p}}(u_1, \ldots, u_m) = \text{WA}_{\mathbf{p}}(u_1, \ldots u_m)$, for any $u_1, \ldots u_m \in \Re$ and therefore

$$\begin{aligned}\underline{u}(\alpha) &= \text{WOWA}_{\boldsymbol{\eta}}^{\mathbf{p}}(\underline{u}_1(\alpha), \underline{u}_2(\alpha), \ldots, \underline{u}_m(\alpha)) \\ &= \text{WA}_{\mathbf{p}}(\underline{u}_1(\alpha), \underline{u}_2(\alpha), \ldots, \underline{u}_m(\alpha)) \\ &= \sum_{i=1}^{m} p_i \cdot \underline{u}_i(\alpha), \end{aligned} \tag{2.40}$$

$$\begin{aligned}\overline{u}(\alpha) &= \text{WOWA}_{\boldsymbol{\eta}}^{\mathbf{p}}(\overline{u}_1(\alpha), \overline{u}_2(\alpha), \ldots, \overline{u}_m(\alpha)) \\ &= \text{WA}_{\mathbf{p}}(\overline{u}_1(\alpha), \overline{u}_2(\alpha), \ldots, \overline{u}_m(\alpha)) \\ &= \sum_{i=1}^{m} p_i \cdot \overline{u}_i(\alpha). \end{aligned} \tag{2.41}$$

These two formulae correspond to the first-level fuzzification of the weighted average from the Theorem 2.6. □

Similarly, it can be shown that the fuzzified WOWA is also a generalization of the first-level fuzzy OWA operator.

Theorem 2.9 *Let $U_1, \ldots, U_m$ be fuzzy numbers, $\mathbf{w}$ be a vector of normalized weights and $\mathbf{p}$ be a vector of uniform real weights, $\mathbf{p} = \boldsymbol{\eta}$. Then the result of the fuzzified WOWA of $U_1, \ldots, U_m$ with the weights $\mathbf{p}$ and $\mathbf{w}$ is identical to the first-level fuzzy OWA of $U_1, \ldots, U_m$ with the weights $\mathbf{w}$.*

Proof: Because the OWA is a special case of the WOWA, it holds that $\mathrm{WOWA}_{\mathbf{w}}^{\boldsymbol{\eta}}(u_1, \ldots, u_m) = \mathrm{OWA}_{\mathbf{w}}(u_1, \ldots u_m)$, for any $u_1, \ldots u_m \in \Re$. We can thus write:

$$\begin{aligned} \underline{u}(\alpha) &= \mathrm{WOWA}_{\mathbf{w}}^{\boldsymbol{\eta}}(\underline{u}_1(\alpha), \underline{u}_2(\alpha), \ldots, \underline{u}_m(\alpha)) \\ &= \mathrm{OWA}_{\mathbf{w}}(\underline{u}_1(\alpha), \underline{u}_2(\alpha), \ldots, \underline{u}_m(\alpha)) \\ &= \sum_{i=1}^{m} w_i \cdot \underline{u}_{\sigma(i)}(\alpha), \end{aligned} \tag{2.42}$$

$$\begin{aligned} \overline{u}(\alpha) &= \mathrm{WOWA}_{\mathbf{w}}^{\boldsymbol{\eta}}(\overline{u}_1(\alpha), \overline{u}_2(\alpha), \ldots, \overline{u}_m(\alpha)) \\ &= \mathrm{OWA}_{\mathbf{w}}(\overline{u}_1(\alpha), \overline{u}_2(\alpha), \ldots, \overline{u}_m(\alpha)) \\ &= \sum_{i=1}^{m} w_i \cdot \overline{u}_{\tau(i)}(\alpha). \end{aligned} \tag{2.43}$$

where σ and τ are such permutations of the set of indices $\{1, \ldots, m\}$ that $\underline{u}_{\sigma(1)} \geq \underline{u}_{\sigma(2)} \geq \ldots \geq \underline{u}_{\sigma(m)}$ and $\overline{u}_{\tau(1)} \geq \overline{u}_{\tau(2)} \geq \ldots \geq \overline{u}_{\tau(m)}$. Again, it can be seen that these two formulae are identical to those from the Theorem 2.7. □

When we consider the fact that the first-level fuzzy weighted average and the first-level fuzzy OWA are special cases of FuzzyWA and FuzzyOWA [4], the consequence of the Theorems 2.3, 2.8, and 2.9 is that FuzzyWA and FuzzyOWA are special cases of the fuzzified WOWA if the used normalized fuzzy weights are fuzzy singletons.

Examples

In this example, we will modify the Examples 2.4 and 2.7, where the software company is evaluating candidates for a new vacancy, to utilize the fuzzified WOWA operator.

Example 2.8 *As it has been mentioned, the fuzzified WOWA requires two vectors of normalized (real) weights – one, which is connected to the individual partial goals as in case of the weighted average, and another one, which is connected to*

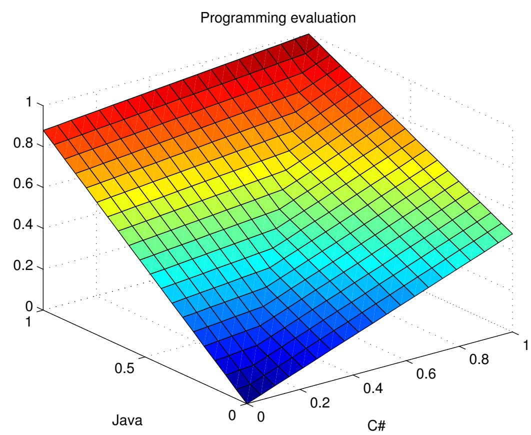


Figure 2.22: Evaluation function using FWOWA from the Example 2.8

the order of their evaluations as in the case of OWA. The company could therefore use the following weights to combine the previous two approaches: $p_{C\#} = 0.3$, $p_{Java} = 0.7$, *and* $w_1 = 0.8$, $w_2 = 0.2$.

The evaluating function is depicted in the Figure 2.22. As this version of fuzzified WOWA uses crisp weights only (and therefore, if evaluation of the C# and Java are crisp, the result will be also crisp), the graph coincides with the graph of the crisp WOWA.

Features of the fuzzified WOWA

The advantage of the fuzzified WOWA operator is that both the importances of partial goals and the importances with respect to the ordering of the partial evaluations are taken into account. This makes it possible to model various preference systems of the decision-maker.

Similarly as a FuzzyOWA, it does not preserve the linearity. Fuzzy WOWA of linear fuzzy numbers need not to be, generally, a linear fuzzy number. This can be easily shown on the following example. Let us consider $U_1 = (0.3, 0.4, 0.5)$, $U_2 = (0.1, 0.8, 0.9)$, $p_1 = 0.3$, $p_2 = 0.7$, $w_1 = 0.8$ and $w_2 = 0.2$. The result of the fuzzified WOWA is shown in the Figure 2.23. Although U_1 and U_2 are linear fuzzy numbers, the result of the fuzzified WOWA is not linear.

Again, there are other approaches for combining fuzzy weighted average and fuzzy OWA into a single operator. As an example, the UIWOWA or the UIOWAWA operators [53] could be named. The first one uses the standard WOWA definition replacing the arithmetic operations with the corresponding operations of the fuzzy numbers arithmetic; the latter one creates the compound weights by a weighted average of the original corresponding weights. As the fuzzy numbers can be generally incomparable, both of these approaches require the order of the

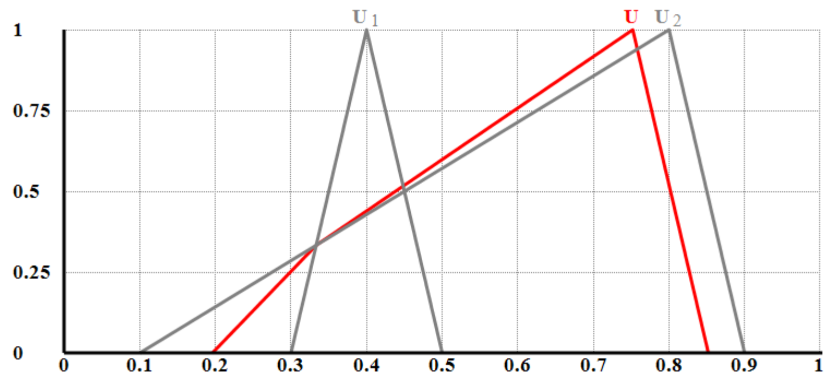


Figure 2.23: An example of the fuzzified WOWA of linear fuzzy numbers; the result, however, is not a linear fuzzy number.

fuzzy numbers to be specified explicitly (by means of auxiliary order-inducing variables). The advantage of the presented fuzzified WOWA is that it does not require the decision-maker to provide any additional information.

The fuzzified WOWA enables to combine the aggregation by a weighted average and by an OWA for fuzzy numbers. However, if interactions among the criteria should be taken into the consideration, more complex mean of aggregation has to be used. For a specific type of interactions, the fuzzified Choquet integral is suitable.

2.10.4. Aggregation by the fuzzified Choquet integral

If there are interactions among the criteria (or among corresponding partial goals), the fuzzified discrete Choquet integral [10] should be considered for aggregation of the partial evaluations. Generally, the Choquet integral can be used for such interactions among criteria that are stable over the whole domain of criteria. There are two types of interaction among criteria (or partial goals) that can be modeled by the Choquet integral – redundancy and complementarity.

In case of redundancy, partial goals are overlapping – they have something in common. Therefore, the significance of this set of overlapping goals is lower than the sum of weights of individual goals. In this case, the weighted average cannot be used for aggregation of partial evaluations because the evaluation of the overlapping part would be included several times.

The opposite type of interaction is complementarity (also called a support between criteria or partial goals in [92]). The cases when all these partial goals are fulfilled are especially valuable for the evaluator. We can say that fulfilling all these partial goals brings some "additional value". The total significance of a considered group of partial goals is then greater than the sum of weights of the individual goals. Again, the weighted average is not suitable for this case because this "additional value" would not be incorporated at all.

In the following text, only a discrete Choquet integral will be considered. First, its non-fuzzy version will be mentioned, then its fuzzy version will be described.

Choquet integral in the crisp case

If the Choquet integral is used for the aggregation, significance is no longer a standard additive normalized measure (probability measure) defined on a set of partial goals whose evaluations should be aggregated. A more general normalized measure must be used here to express significancies of all subsets in this set of partial goals. For these cases, a fuzzy measure introduced by Sugeno in [77] is suitable.

Definition 2.36 *A (real-valued) fuzzy measure on a finite nonempty set G is a set function $\mu : \wp(G) \rightarrow [0, 1]$ satisfying the following axioms:*

- $\mu(\emptyset) = 1, \quad \mu(G) = 1,$
- $C \subseteq D$ *implies* $\mu(C) \leq \mu(D)$ *for any* $C, D \in \wp(G)$.

This real-valued fuzzy measure generalizes a classic normalized measure, with additivity replaced by monotonicity.

Remark 2.3 *Despite its name, the values of a fuzzy measure are real numbers. A generalization, which uses fuzzy numbers as the measure values, will be called FNV-fuzzy measure (fuzzy-number-valued fuzzy measure) [4] and will be presented later in this text.*

Remark 2.4 *The obvious obstacle in using the Choquet integral in multiple-criteria evaluation is that $2^m - 2$ values of the fuzzy measure must be given. Therefore, the fuzzy measure can be directly set only for some reasonably low number of criteria in the practice. One of the solutions is to use k-additive fuzzy measure, which can reduce the number of the values that have to be set significantly (see e.g. [29]).*

To define the Choquet integral, the following notation will be used. For any m-tuple of real numbers $(u_1, \ldots, u_m)$, ρ will denote such a permutation of the set of indices $\{1, \ldots, m\}$ that $u_{\rho(1)} \leq u_{\rho(2)} \leq \ldots \leq u_{\rho(m)}$. Moreover, let us denote $B_{\rho(i)} = \{G_{\rho(i)}, \ldots, G_{\rho(m)}\}$. By definition, we will set $B_{\rho(m+1)} = \emptyset$. In the crisp case, the Choquet integral is used for aggregating partial evaluations in the following way.

Definition 2.37 *Let real numbers $u_1, \ldots, u_m$, $u_i \in [0,1]$, $i = 1, \ldots, m$ be partial evaluations with respect to the goals $G_1, \ldots, G_m$. Let the importance of the partial-goal sets be defined by a fuzzy measure μ on G. Then overall evaluation u is given as the following value of the Choquet integral:*

$$(C)\!\int_G f d\mu = \sum_{i=1}^{m} f(G_{\rho(i)}) \cdot \left[\mu(B_{\rho(i)}) - \mu(B_{\rho(i+1)})\right], \tag{2.44}$$

where $f(G_i) = u_i$.

The (discrete) Choquet integral can be written in several forms (the others can be found e.g. in [92, 29]). However, the alternative forms will not be needed for this thesis.

Fuzzified Choquet integral

In the fuzzy case, the weights of subsets of partial goals are defined by a FNV-fuzzy measure (fuzzy-number-valued fuzzy measure).

Definition 2.38 *Let $G = \{G_1, \ldots, G_m\}$ be a nonempty finite set, $\wp(G)$ be the family of all its subsets. Then, a FNV-fuzzy measure [4] on G is a set function $\tilde{\mu} : \wp(G) \to \mathcal{F}_N([0,1])$ satisfying the following conditions:*

- *$\tilde{\mu}(\emptyset) = \tilde{0}$, $\tilde{\mu}(G) = \tilde{1}$, and*
- *$C \subseteq D$ implies $\tilde{\mu}(C) \leq \tilde{\mu}(D)$ for any $C, D \in \wp(G)$.*

The comparison $\tilde{\mu}(C) \leq \tilde{\mu}(D)$ is performed by the Definition 2.13.

Let $\tilde{\mu}$ be a FNV-fuzzy measure on G, and $F : G \to \mathcal{F}_N([0,1])$, $F(G_i) = U_i$, $i = 1, \ldots, m$, be a FNV-function (fuzzy-number-valued function). In our case, $G_1, \ldots, G_m$ represent the partial goals, $U_1, \ldots, U_m$ are the fuzzy evaluations according to the partial goals, and $\tilde{\mu}(K)$, $K \subseteq G$, expresses the weight of a partial-goal subset K. Analogically to the crisp case, let us denote $B_{\rho(i)} = \{G_{\rho(i)}, \ldots, G_{\rho(m)}\}$. By definition, we will set $B_{\rho(m+1)} = \emptyset$. The meaning of ρ will be also the same as in the crisp case – for a given m-tuple of real numbers $(u_1, \ldots, u_m)$, it is such a permutation of the set of indices $\{1, \ldots, m\}$ that it holds $u_{\rho(1)} \leq u_{\rho(2)} \leq \ldots \leq u_{\rho(m)}$.

Definition 2.39 *The fuzzified Choquet integral [5, 58] of a FNV-function F with respect to the FNV-fuzzy measure $\tilde{\mu}$ is defined as a fuzzy number U with the membership function given for any $u \in [0,1]$ by*

$$U(u) = \max\Big\{ \min\{U_1(u_1), \dots, U_m(u_m), \tilde{\mu}(B_{\rho(1)})(\mu_1), \dots, \tilde{\mu}(B_{\rho(m)})(\mu_m)\} \mid$$
$$u = (C)\!\int_G f d\mu, \text{ where } f : G \to [0,1] \text{ such that } f(G_i) = u_i, i = 1, \dots, m,$$
$$\mu \text{ is a fuzzy measure on } G \text{ such that } \mu(B_{\rho(i)}) = \mu_i, i = 1, \dots, m\Big\}. \quad (2.45)$$

Again, there are multiple forms in which the definition of the fuzzified Choquet integral can be written (they are listed in [4]). In this text, the form used in [58] slightly modified for better clarity has been used.

Although the definition looks quite complex, an efficient algorithm for calculating the fuzzified Choquet integral exists [5].

Efficient algorithm for the fuzzified Choquet integral calculation

Calculation of the fuzzified Choquet integral according to its definition would be very complicated. Bebčáková et al. [5] has proposed an algorithm that makes it possible to calculate the result directly. In this text, the formulae used in her algorithm were rewritten into a sightly different form for better clarity. It can be thus seen that the problem of fuzzified Choquet integral calculation can be transformed into the problem of multiple (crisp) Choquet integrals calculations.

Algorithm 2.5 *Let us denote $U_i = \{[\underline{u}_i(\alpha), \overline{u}_i(\alpha)], \alpha \in [0,1]\}$, $i = 1, \dots, m$. Let a FNV-fuzzy measure $\tilde{\mu}$ on G be given, $\tilde{\mu}(K) = \{[\underline{\tilde{\mu}}(K)(\alpha), \overline{\tilde{\mu}}(K)(\alpha)], \alpha \in [0,1]\}$ for any $K \subseteq G$. Then fuzzified Choquet integral expressing the overall fuzzy evaluation U, $U = \{[\underline{u}(\alpha), \overline{u}(\alpha)], \alpha \in [0,1]\}$, can be calculated for any $\alpha \in [0,1]$ as follows:*

$$\underline{u}(\alpha) = (C)\!\int_G f_L \, d\mu_L, \quad (2.46)$$

where $f_L : G \to [0,1]$ is a function such that $f_L(G_i) = \underline{u}_i(\alpha)$, $i = 1, \dots, m$, and $\mu_L : \wp(G) \to [0,1]$ is such a fuzzy measure that $\mu_L(K) = \underline{\tilde{\mu}}(K)(\alpha)$ for any $K \subseteq G$.

$$\overline{u}(\alpha) = (C)\!\int_G f_R \, d\mu_R, \quad (2.47)$$

where $f_R : G \to [0,1]$ is a function such that $f_R(G_i) = \overline{u}_i(\alpha)$, $i = 1, \dots, m$, and $\mu_R : \wp(G) \to [0,1]$ is such a fuzzy measure that $\mu_R(K) = \overline{\tilde{\mu}}(K)(\alpha)$ for any $K \subseteq G$.

Examples

To illustrate application of the fuzzified Choquet integral in the multiple-criteria evaluation, two examples are given below. As already mentioned, the Choquet integral is appropriate under complementarity or redundancy among criteria. First, let us assume the complementarity. The fuzzy measure of the considered group of criteria is then greater than the sum of fuzzy measures of the individual criteria.

Example 2.9 *As an example, we would like to evaluate career perspective of young mathematicians according to three criteria – math knowledge, English knowledge, and communication skills. We can set $\mu(Math) = 0.7$, $\mu(English) = 0.1$, and $\mu(Communication) = 0.05$. The knowledge of the math is the most important to us but without the other skills the mathematician will not be able to publish and present his/her results on conferences, which is a necessity in science. For mathematicians with limited knowledge of mathematics but with excellent English, the career perspective is rather dim. Very low evaluation is assigned to mathematicians with communication skills but no knowledge of math and English. Now we can consider the fuzzy measures of pairs of criteria – $\mu(Math, English) = 0.85$, $\mu(Math, Communication) = 0.75$, and $\mu(English, Communication) = 0.2$. The fuzzy measure of the set containing all three criteria equals 1; this is also the evaluation of a mathematician who fulfills perfectly all the three goals – the synergic effect is present. The fuzzy measure of an empty set $\mu(\emptyset) = 0$.*

We replace the real numbers in the values of the fuzzy measure by fuzzy numbers. This way we could define for example the following FNV-fuzzy measure:

- $\tilde{\mu}(\emptyset) = \tilde{0}$,
- $\tilde{\mu}(Math) = (0.6, 0.7, 0.8)$,
- $\tilde{\mu}(English) = (0, 0.1, 0.2)$,
- $\tilde{\mu}(Communication) = (0, 0.05, 0.1)$.
- $\tilde{\mu}(Math, English) = (0.7, 0.85, 1)$,
- $\tilde{\mu}(Math, Communication) = (0.6, 0.75, 0.8)$,
- $\tilde{\mu}(English, Communication) = (0.05, 0.2, 0.35)$,
- $\tilde{\mu}(Math, English, Communication) = \tilde{1}$.

If the partial evaluations are for example $U_{Math} = (0.8, 0.9, 1)$, $U_{English} = (0.6, 0.7, 0.8)$, and $U_{Communication} = (0.4, 0.5, 0.6)$, then the overall evaluation by the fuzzified Choquet integral will be the fuzzy number shown in the Figure 2.24.

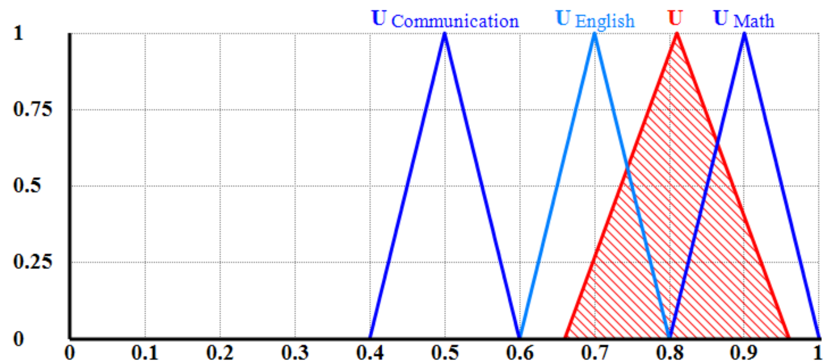


Figure 2.24: The evaluation by the Choquet integral from the Example 2.9

Similarly, the Choquet integral can be used in case of partial redundancy among criteria.

Example 2.10 *This type of interaction occurs, for instance, if students' aptitude for study of science should be evaluated according to the test results in Mathematics, Physics, and Chemistry. The measure of the partial goals could be defined as follows:* $\mu(Mathematics) = 0.5$*,* $\mu(Physics) = 0.4$ *and* $\mu(Chemistry) = 0.3$*. Then we will use the following reasoning to set the rest of the fuzzy measure values. Students who are good at Math are usually also good at Physics. The reason is that these two subjects have a lot in common. Therefore, we set the measure* $\mu(Mathematics, Physics) = 0.7$*, which is less than* $\mu(Mathematics) + \mu(Physics)$*. Similarly,* $\mu(Mathematics, Chemistry) = 0.6$ *and* $\mu(Physics, Chemistry) = 0.6$*. The measure* $\mu(Mathematics, Physics, Chemistry) = 1$*, and* $\mu(\emptyset) = 0$*.*

From this fuzzy measure, we can derive a FNV-fuzzy measure whose values will be fuzzy numbers given as follows

- $\tilde{\mu}(\emptyset) = \tilde{0}$,
- $\tilde{\mu}(Mathematics) = (0.4, 0.5, 0.6)$,
- $\tilde{\mu}(Physics) = (0.3, 0.4, 0.5)$,
- $\tilde{\mu}(Chemistry) = (0.2, 0.3, 0.4)$.
- $\tilde{\mu}(Mathematics, Physics) = (0.6, 0.7, 0.8)$,
- $\tilde{\mu}(Mathematics, Chemistry) = (0.5, 0.6, 0.7)$,
- $\tilde{\mu}(Physics, Chemistry) = (0.5, 0.6, 0.7)$,
- $\tilde{\mu}(Mathematics, Physics, Chemistry) = \tilde{1}$.

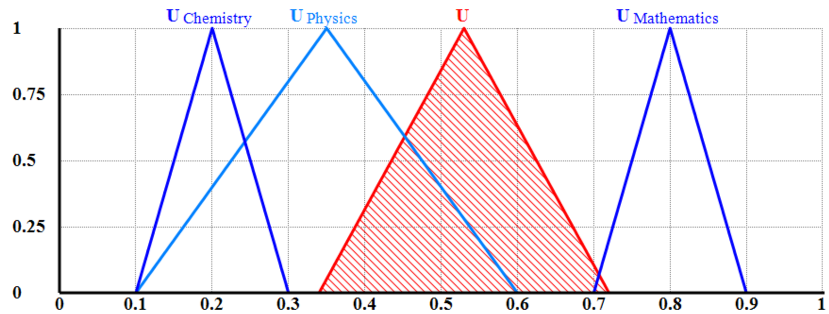


Figure 2.25: The evaluation by the Choquet integral from the Example 2.10

Let us assume that some student's evaluations of the individual subjects are expressed by triangular fuzzy numbers $U_{Mathematics} = (0.7, 0.8, 0.9)$, $U_{Physics} = (0.1, 0.35, 0.6)$, and $U_{Chemistry} = (0.1, 0.2, 0.3)$. Then the result obtained by the fuzzified Choquet integral is shown in the Figure 2.25.

Example 2.11 *Again, if we consider the Example 2.2, we could employ the fuzzified Choquet integral.*

The C# and Java programming languages have lots in common. Many techniques are the same in both languages. Therefore, if an amount of the skills should be evaluated, we could take this overlapping part of skills into account and use the fuzzified Choquet integral for the evaluation.

The following FNV-fuzzy measure can be used for this case:

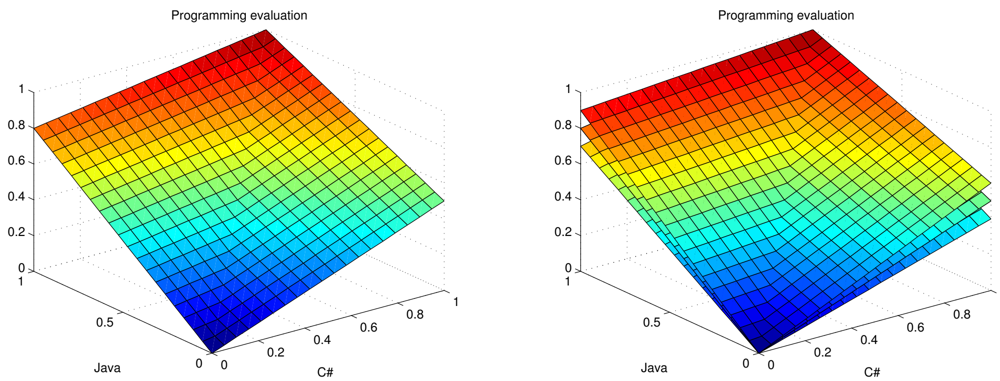
- $\tilde{\mu}(\emptyset) = \tilde{0}$,
- $\tilde{\mu}(C\#) = (0.4, 0.5, 0.6)$,
- $\tilde{\mu}(Java) = (0.7, 0.8, 0.9)$,
- $\tilde{\mu}(C\#,\ Java) = \tilde{1}$.

The setting of the FNV-fuzzy measure is easy in this case. The company has to provide just two fuzzy values expressing how valuable is a programmer who excels in one of the programming languages but does not know the other one at all. The other two FNV-fuzzy measure values are set by the definition.

The graph of the resulting evaluation function can be seen in the Figure 2.26b. For comparison, the graph of the crisp Choquet integral is in the Figure 2.26a (the used fuzzy measure is $\mu(\emptyset) = 0$, $\mu(C\#) = 0.5$, $\mu(Java) = 0.8$, $\mu(C\#,\ Java) = 1$).

Features of the fuzzified Choquet integral

The fuzzified Choquet integral is a generalization of all of the already mentioned fuzzy aggregation operators. Its advantage rests in handling certain interactions among criteria. However, it can be quite challenging to set the fuzzy



(a) The crisp case (a fuzzy measure is used)

(b) The fuzzy case (a FNV-fuzzy measure is used)

Figure 2.26: Evaluation function using by the Choquet integral

measure or FNV-fuzzy measure. This problem can be partly overcome by using only 2-additive fuzzy measure that takes into account interactions only between pairs of criteria. Moreover, a common evaluator may find the method rather difficult to comprehend. If the relationship among the criteria is even more complex, or the comprehensibility is the main issue, a fuzzy expert system can be used to define the evaluation function linguistically.

2.10.5. Aggregation by the fuzzy expert system

The fuzzy expert system can be used in the multiple-criteria evaluation even if interactions among the criteria are very complex, e.g., if the intensity of complementarity or redundancy among the criteria varies within the criterion space, or if these two types of interaction interchange. However, it is necessary to have an expert knowledge about the evaluating function to be able to create the fuzzy rule base representing the multiple-criteria evaluation function.

Theoretically, it is possible to model, with an arbitrary precision, any Borel-measurable function by means of a fuzzy rule base (properties of the Mamdani and Sugeno fuzzy controllers, see e.g. Kosko [41]). In reality, the quality of the approximation is limited by the expert's knowledge of the relationship.

The fuzzy-rule base, which models the relationship between the partial evaluations of lower level and the aggregated evaluation, has the following form:

$$
\begin{aligned}
&\text{If } \mathcal{E}_1 \text{ is } \mathcal{A}_{1,1} \text{ and } \dots \text{ and } \mathcal{E}_m \text{ is } \mathcal{A}_{1,m}, \text{then } \mathcal{E} \text{ is } \mathcal{B}_1,\\
&\text{If } \mathcal{E}_1 \text{ is } \mathcal{A}_{2,1} \text{ and } \dots \text{ and } \mathcal{E}_m \text{ is } \mathcal{A}_{2,m}, \text{then } \mathcal{E} \text{ is } \mathcal{B}_2,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\text{If } \mathcal{E}_1 \text{ is } \mathcal{A}_{n,1} \text{ and } \dots \text{ and } \mathcal{E}_m \text{ is } \mathcal{A}_{n,m}, \text{then } \mathcal{E} \text{ is } \mathcal{B}_n,
\end{aligned}
$$

where for $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$:

- $(\mathcal{E}_j, \mathcal{T}(\mathcal{E}_j), [0,1], M_j, G_j)$ are linguistic scales representing partial evaluations,
- $\mathcal{A}_{ij} \in \mathcal{T}(\mathcal{E}_j)$ are linguistic values from these scales, and $A_{ij} = M_j(\mathcal{A}_{ij})$ are fuzzy numbers on $[0,1]$ representing their meanings,
- $(\mathcal{E}, \mathcal{T}(\mathcal{E}), [0,1], M, G)$ is a linguistic scale representing the overall evaluation,
- $\mathcal{B}_i \in \mathcal{T}(\mathcal{E})$ are linguistic values from this scale, and $B_i = M(\mathcal{B}_i)$ are fuzzy numbers on $[0,1]$ representing their meanings.

First, a note on the interpretation of the fuzzy rules should be made. Traditionally, the words *if* and *then* are used in the rules, which suggests that each rule should behave as an implication. However the interpretation depends on the choice of the inference algorithm. In case of the Mamdani inference, we obtain so-called conjunctive rules. The individual rules does not represent constraints but they act more like a pieces-of data, or examples showing the possible results for the given combination of inputs. In contrast to that, there are also implicative rules that behave like restrictions of the possible resulting values. More information on these types of rules can be found e.g. in [93]. In this thesis, the words *if* and *then* will be used in the rules, even though there is no implication in the mathematical sense, because this way the fuzzy rules are traditionally formulated in the literature.

Many inference algorithms exist and can be used to calculate the final evaluation. In this text, three of them will be discussed – Mamdani [51], Sugeno-WA [81], and Sugeno-WOWA [34].

The Mamdani inference [51] is a well-known and widely used inference method proposed by the Mamdani and Assilian in 1975. However, its drawback in using it for multiple-criteria evaluation is that since we assume that all of the evaluations are fuzzy number on $[0,1]$ in this system, the result obtained by the Mamdani inference need not to be a fuzzy number. Therefore it cannot be used directly and it must be approximated by a fuzzy number first.

The Sugeno-WA is a generalization of the classic Sugeno algorithm [78]. In the Sugeno inference, there are real numbers on the right-hand sides of the rules. In the Sugeno-WA, these real numbers are replaced with fuzzy numbers. The result of such an inference is also a fuzzy number. Sugeno-WA turned out to be very suitable for the multiple-criteria evaluation. Originally, this inference method was proposed under the name Generalized Sugeno [81]. In the later publications (e.g. [38]) and also in this thesis, the name Sugeno-WA has been adopted to distinguish this method from another presented algorithm, which is the Sugeno-WOWA.

The Sugeno-WOWA inference method can be appropriate in some special cases. It is similar to Sugeno-WA but uses the fuzzified WOWA operator for the calculations instead of the weighted average of fuzzy numbers.

Mamdani inference

In case of the Mamdani fuzzy inference [51], the calculation can be divided into the following steps.

1. First, for all $i = 1, \ldots, n$, the degree of correspondence between the given m-tuple of fuzzy values $(U_1, U_2, \ldots, U_m)$ of partial evaluations and the mathematical meaning of the left-hand side of the i-th rule is calculated in the following way:
$$h_i = \min\{\mathrm{hgt}(U_1 \cap A_{i,1}), \ldots, \mathrm{hgt}(U_m \cap A_{i,m})\}. \tag{2.48}$$
2. For each of the rules, the output fuzzy value U'_i, $i = 1, \ldots, n$, for the given inputs, is a fuzzy set on $[0, 1]$ with the membership function defined as follows:
$$\forall y \in [0, 1] : U'_i(y) = \min\{h_i, B_i(y)\}. \tag{2.49}$$
3. The result is a fuzzy set given as a union of all the fuzzy values calculated for the individual rules in the previous step, i.e.:
$$U'' = \bigcup_{i=1}^{n} U'_i. \tag{2.50}$$

Generally, the result U'' obtained by the Mamdani inference algorithm need not to be a fuzzy number. So, for further calculations within the fuzzy model, it must be approximated by a fuzzy number U. One of the possible methods was proposed in [81] – all fuzzy numbers that model the mathematical meanings of the terms from the extended scale derived from $\mathcal{E}$ are considered and the one that is the most similar to the fuzzy set U'' provided by the Mamdani inference is the final fuzzy evaluation U. The similarity is calculated by the Definition 2.20 in this method.

This way, the result from the Mamdani inference U'', which is a fuzzy set, can be approximated by a fuzzy number U, and therefore it can be used as the fuzzy evaluation in the considered fuzzy multiple-criteria evaluation system. However, the whole process can be simplified by using a more appropriate inference algorithm that produces the results in form of fuzzy numbers. One of them is the Sugeno-WA inference.

Sugeno-WA inference

The Sugeno-WA [81] inference algorithm is a generalization of the classic Sugeno inference [78] where the real numbers on the right-hand sides of the rules are replaced by fuzzy numbers. These fuzzy numbers can represent meanings of linguistic terms, and therefore this inference algorithm can be also used with the fuzzy rule base mentioned above.

The result of the Sugeno-WA inference is obtained as follows.

1. In the first step, the degrees of correspondence h_i, $i = 1, \dots, n$, are calculated for all rules in the same way as in the Mamdani fuzzy inference algorithm.

2. The resulting fuzzy evaluation U is then computed as the weighted average of the fuzzy evaluations B_i, $i = 1, \dots, n$, which model mathematical meanings of linguistic evaluations on the right-hand sides of the rules, with the weights h_i. This is done by the formula

$$U = \frac{\sum_{i=1}^{n} h_i \cdot B_i}{\sum_{i=1}^{n} h_i}. \tag{2.51}$$

   The value is calculated using the fuzzy numbers arithmetic, i.e. by the Formulae 2.7 and 2.8.

The result of the Sugeno-WA is therefore the weighted average of the fuzzy numbers that model the meanings of the linguistic values on the right-hand sides of the rules where the degrees of correspondence h_i, $i = 1, \dots, n$, are in role of the weights.

Sugeno-WOWA inference

For more complex cases, Sugeno-WOWA inference can be used [34]. This method requires, besides a fuzzy rule base, normalized weights $w_1, w_2, \dots, w_s$. These normalized weights are assigned to individual values of the linguistic scale representing the output variable $\mathcal{E}$. By these weights, the expert can express his/her optimism or pessimism (a pessimist assigns larger weights to bad evaluations, while an optimist to good evaluations). This can be utilized, for example, when the risk of a bank client is evaluated by a fuzzy expert system. The scale for the resulting evaluations can consist of the following terms – *very high risk*, *high risk*, *medium risk*, and *no risk recognized*. The expert can assign, for example, a weight 0.45 to the term *very high risk*, 0.35 to *high risk*, 0.15 to *medium risk*, and 0.05 to *no risk recognized*.

To define Sugeno-WOWA inference, let us rewrite the formula for Sugeno-WA into an alternative form first:

Let us recall that $(\mathcal{E}, \mathcal{T}(\mathcal{E}), [0,1], M, G)$ represents the linguistic variable for the right-hand side of the rules. Then, let $\mathcal{T}_i \in \mathcal{T}(\mathcal{E})$, $i = 1, \dots, s$, be all its linguistic terms and let T_i, $i = 1, \dots, s$, denote the fuzzy numbers that model the meanings of these terms, i.e

$$T_i = M(\mathcal{T}_i), \text{ where } \mathcal{T}_i \in \mathcal{T}(\mathcal{E}),\ i \in \{1 \dots, s\}. \tag{2.52}$$

For each term $\mathcal{T}_i$, $i = 1, \dots, s$, we can calculate the value p'_i, $i = 1, \dots, s$, which expresses the sum of the degrees of correspondence of all rules that have the term $\mathcal{T}_i$ on their right-hand side:

$$p'_i = \sum_{\substack{j \in \{1,\dots,n\}: \\ \mathcal{B}_j = \mathcal{T}_i}} h_j. \tag{2.53}$$

For further calculations, the values are normalized:

$$p_i = \frac{p'_i}{\sum_{j=1}^{s} p'_j}. \tag{2.54}$$

The Sugeno-WA inference algorithm can be then expressed as:

$$U = \sum_{i=1}^{s} p_i T_i. \tag{2.55}$$

In the original Formula 2.51, the inference result is calculated as a weighted average of n fuzzy numbers (one for each rule) whereas the Formula 2.55 calculates it as a weighted average of s fuzzy numbers (one for each value that can appear on the right-hand side of the rules). This can be exploited for calculations since s is usually much lower than n and the number of time-consuming arithmetic operations with fuzzy numbers is therefore reduced.

Using this alternative view on the Sugeno-WA, its modification, the Sugeno-WOWA, can be introduced easily. As $\mathcal{E}$ represents a linguistic scale, the mathematical meanings of its terms form a fuzzy scale and they are ordered in sense of the Definition 2.13, i.e. it holds that $T_i < T_{i+1}$, $i = 1, \dots, s-1$. Then, the expert provides a vector of the normalized weights $\mathbf{w} = (w_1, w_2, \dots, w_s)$ with a similar interpretation as the weights used for the OWA. The weight w_i, $i = 1, \dots, s$, provided by the expert, corresponds to the i-th greatest of those fuzzy numbers, i.e. to the T_{s-i+1}. Then, if the weighted average of fuzzy numbers in Sugeno-WA is replaced with the fuzzified WOWA to take into account the weights $\mathbf{w}$, the inference result U, which is a fuzzy number on $[0, 1]$, is calculated as follows:

$$U = FWOWA^{\mathbf{p}}_{\mathbf{w}}(T_1, T_2, \dots T_s). \tag{2.56}$$

The weights $\mathbf{p} = (p_1, \dots, p_s)$ are calculated in the same way as for the Sugeno-WA inference (i.e., by the Formula 2.54).

Using the weights $\mathbf{w}$, the expert's optimism or pessimism can be taken into the account.

		Java knowledge		
		Bad	Average	Good
C# knowledge	Bad	Bad	Bad	Bad
	Average	Bad	Average	Good
	Good	Bad	Good	Good

Figure 2.27: Sample fuzzy rule base written in form of a table

Example

The company from the Example 2.2 can alternatively use a fuzzy expert system in the following way.

Example 2.12 *The relationship between the two aggregated partial evaluations and the resulting evaluation could be described by a set of if-then rules. An example of such a rule might be:*

If Java knowledge is good and C# knowledge is very good, then programming knowledge is very good.

Because there are only two partial evaluations to be aggregated, the rule base can be written in form of a table, which can be seen in the Figure 2.27. Let us assume, that a linguistic scale that is depicted in the Figure 2.28 has been used for evaluation of the both programming languages and also for the overall evaluation. Then the resulting evaluation can be calculated using the well-known Mamdani inference, by a Sugeno-WA inference algorithm, or alternatively Sugeno-WOWA.

The Figure 2.29a shows the resulting evaluation function obtained by the Sugeno-WA (only centers of gravity of the resulting fuzzy evaluations have been plotted). For comparison, evaluations functions for the Sugeno-WOWA are also shown. In the Figure 2.29b, the Sugeno-WOWA with the weights $w_1 = 0.6$ (corresponding to the best of the possible outcomes, i.e. Good*), $w_2 = 0.3$ (corresponding to* Average*), and $w_3 = 0.1$ (corresponding to* Bad*) has been used. Alternatively, in the Figure 2.29c, the evaluation function obtained by the Sugeno-WOWA with the weights $\mathbf{w} = (0.1, 0.3, 0.6)$ is shown.*

An increase or decrease of significance of rules with a particular consequent could be achieved by assigning a weight to each of these rules (the results would differ in dependence on the mechanism how that weight is incorporated into the calculation of the overall result). However, the use of the Sugeno-WOWA has the big advantage in its simplicity for the decision-maker as much lower number of parameters have to be provided (the number of values in the linguistic scale is usually much lower than the number of rules). The attitude of the decision-maker (e.g. his/her optimism, or pessimism) can be also seen from the weights vector $\mathbf{w}$ immediately. If the weights for the individual rules would be used, it could be quite difficult to make a simple conclusion on the decision-maker's attitude from them.

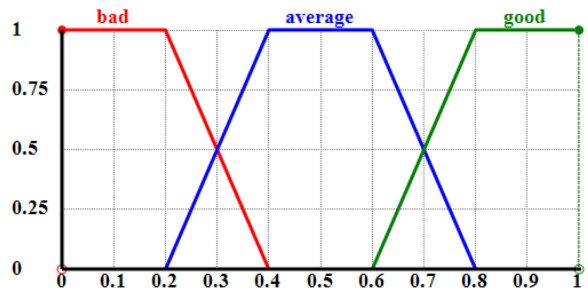
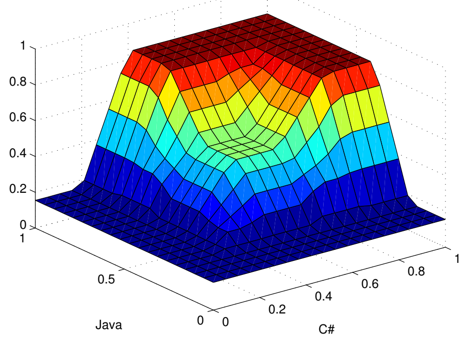
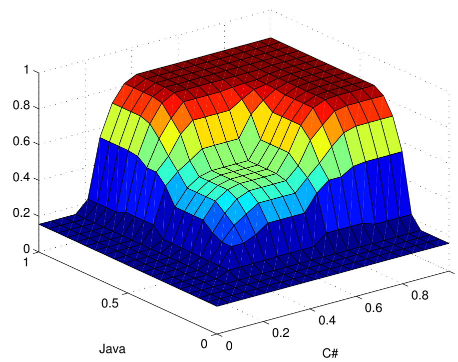


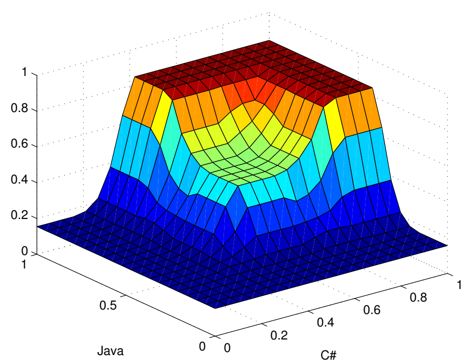
Figure 2.28: The linguistic scale used in the fuzzy expert system in the Example 2.12



(a) An evaluation function obtained with Sugeno-WA



(b) An evaluation function obtained with Sugeno-WOWA with the weights $\mathbf{w} = (0.6, 0.3, 0.1)$



(c) An evaluation function obtained with Sugeno-WOWA with the weights $\mathbf{w} = (0.1, 0.3, 0.6)$

Figure 2.29: Comparison of the evaluation functions obtained by fuzzy expert system

Features of the aggregation by a fuzzy expert system

The fuzzy expert system is the most general of the described aggregation methods. Its great advantage is that it can handle complex interactions. Moreover, the evaluating function described by fuzzy rules is comprehensible to the evaluator (contrary to the WOWA or Choquet integral) and it can be easily adjusted. The natural drawback is the amount of data (fuzzy rules) that have to be provided by the expert.

If it is possible, aggregation methods such as FuzzyWA or FuzzyOWA are preferable because of their simplicity from the decision-maker's point of view. However, if the relationship among the partial evaluations is complicated, or a linguistic description of the used evaluation function (by means of the if-then rules) is needed, then the fuzzy expert system can be utilized.

2.11. Using the evaluation results in the decision-making

The process of aggregation is repeated until the root of the goals tree is reached. The evaluation in this node (which corresponds to the main goal) is the overall evaluation of the alternative. This evaluation is expressed by a fuzzy number. The decision-maker is therefore provided with richer information, which could be otherwise lost if some of the crisp methods were used. The benefits were pointed out when the features of the FuzzyWA were described (Section 2.10.1).

However, the evaluation in form of a fuzzy number brings besides advantages also new questions. The first one is how to present the evaluation in a form easily comprehensible to the decision-maker. Another one is how to compare the alternatives and how to select the best of them. The following sections discuss the possible solutions that can be used in the described system of methods.

2.11.1. Presentation of the final fuzzy evaluation

For any MCDM method, it is essential that its results are presented to the decision-maker in a comprehensible form so that they could be interpreted correctly. This issue is especially important if the evaluation results are expressed by fuzzy numbers. Several forms can be used – graphical, numerical, or verbal descriptions are the most common ones.

Graphical representation

A plot of the membership function of the resulting fuzzy evaluation represents a good way to present the full information about the alternative evaluation to the decision-maker. The decision-maker can see easily how the alternatives perform and with how much uncertainty is the alternative evaluation afflicted.

Numeric characteristics

For a brief overview, some numeric characteristics of the evaluation can be provided. For example, the decision-maker can be provided with:

- the center of gravity of the fuzzy evaluation,
- its relative cardinality,
- its support,
- its kernel.

The center of gravity is a well-known defuzzification method. It can be used to replace the fuzzy evaluation with a crisp evaluation. Generally, the center of gravity (e.g. [16]) of an fuzzy number is defined as follows.

Definition 2.40 *A center of gravity of a fuzzy number C defined on $[a, b]$ that is not a fuzzy singleton is defined as follows*

$$t_C = \frac{\int_a^b C(x) \cdot x \, dx}{\int_a^b C(x) \, dx}. \tag{2.57}$$

Otherwise, if C is a fuzzy singleton containing a single element c, $c \in \Re$, then $t_C = c$.

In case of the described system of methods, all fuzzy evaluations are fuzzy numbers on the interval $[0, 1]$, and we therefore set $a = 0$ and $b = 1$.

While the center of gravity of a fuzzy evaluation expresses how good the alternative is, the relative cardinality [81] informs the decision-maker how much uncertainty is contained in this fuzzy evaluation.

Definition 2.41 *A relative cardinality of a fuzzy number C defined on $[a, b]$ is defined as follows*

$$f_C = \frac{\int_a^b C(x) \, dx}{b - a}. \tag{2.58}$$

The relative cardinality ranges from 0 for fuzzy singletons to 1 for absolutely uncertain fuzzy evaluations (fuzzy numbers that contain the entire interval $[0, 1]$ with the membership degree 1).

The kernel and the support of the fuzzy evaluation can supplement the information for the decision-maker.

Linguistic approximation of the fuzzy evaluation

Often, the most natural way of presenting information is describing it by words. The verbal description of the alternative fuzzy evaluation can be therefore provided. Using a linguistic approximation by means of a predefined linguistic scale, the best-fitting description of the alternative is determined. There are multiple ways how to measure the suitability of each of the available terms for the fuzzy evaluation. Two examples of such approaches are the linguistic approximation using the fuzzy sets similarity from the Definition 2.21, and the linguistic approximation using the fuzzy numbers distance from the Definition 2.22. The following examples compare the results obtained by both of the methods.

Example 2.13 *Let us assume that we have a linguistic scale with five terms –* poor, substandard, standard, above standard, *and* excellent. *Let us assume that the evaluation of an alternative is given by the trapezoidal fuzzy number* $C = (0.5, 0.8, 0.85, 0.9)$. *This setting is depicted in the Figure 2.30. The Table 2.2 shows the similarities and the distances between the fuzzy number* C *and each of the terms meanings calculated according to the Definitions 2.20 and 2.12. The maximum similarity and the minimum distance is in the bold font. It can be seen that both of the linguistic approximation methods would describe the evaluation* C *as* above standard.

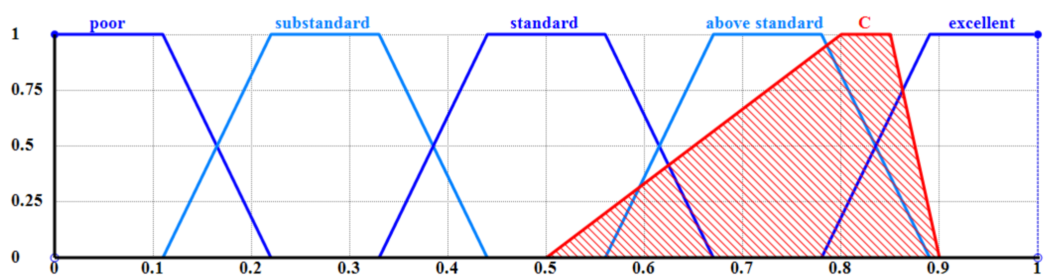


Figure 2.30: The linguistic scale and the fuzzy number C (hatched red), whose linguistic approximation is calculated in the Example 2.13

Besides the original linguistic scale, some richer structures can be also used – for example an extended linguistic scale or a linguistic scale with intermediate values can be employed.

Example 2.14 *Let us assume that we are measuring the temperature of an ill person, but we are equipped only with a mediocre thermometer. The measured temperature will be model by the triangular fuzzy number* C *(see Figure 2.31). In this case, the linguistic approximation given by the Definition 2.22 based on the fuzzy numbers distance would describe the temperature* C *by the term* Raised. *Looking at the Figure 2.31, we can see that this is incorrect because the patient*

Term	Similarity	Distance
poor	0	0.68
substandard	0	0.49
standard	0.15	0.26
above standard	**0.8**	**0.05**
excellent	0.23	0.16

Table 2.2: The similarities and distances calculated for the Example 2.13

has clearly a fever. The reason why this linguistic approximation method fails on this example is the significant difference in the uncertainty of the fuzzy numbers modeling the meanings of the terms from the given scale. The linguistic approximation from the Definition 2.21 based on the fuzzy sets similarities returns the correct result Fever*.*

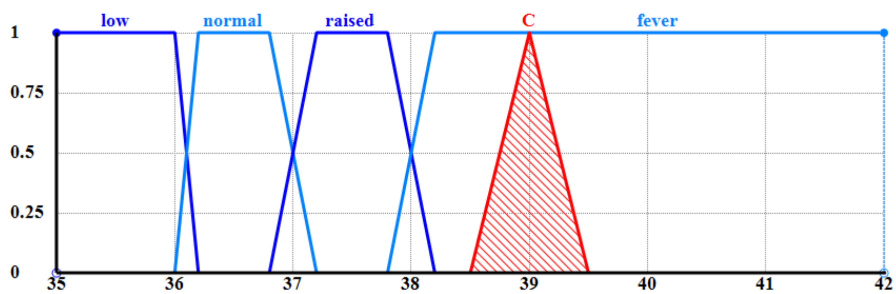


Figure 2.31: Linguistic approximation of a fuzzy number C (red) by a linguistic scale from the Example 2.14.

The conclusion that can be made from the previous example is that one should try the behavior of the selected linguistic approximation method in advance and the decision-maker should be sure that the selected method is suitable for the given problem. Two approaches have been chosen. Of course, many more linguistic approximation methods have been developed. Some of them are summarized e.g. in [14].

2.11.2. Comparison of the fuzzy evaluations

Another obstacle in using of the fuzzy numbers is that, generally, they can be incomparable and it is harder to use them for ordering of the alternatives. However, the fuzzy numbers can be always ordered by their centers of gravity [16].

Definition 2.42 *Let A and B be two fuzzy numbers and let t_A and t_B be their* *centers of gravity. Then we say that A is greater than or equal to B according to their centers of gravity, $A \geq_t B$, iff $t_a \geq t_b$.*

Of course, the center of gravity represents only one of many possible methods for comparison of fuzzy numbers. Generally, any defuzzification method can be used to obtain crisp evaluations, which can be then ordered easily. The list of various defuzzification methods can be found in [95].

2.12. Adjustment of the model – the transitions between different aggregation methods

The design of the evaluation model is seldom a one-step process. Once the model is created, it should be tested and adjusted according to the test results. There is always a trade-off between the precision and the complexity. Building a complex model places considerable demands on the expert, who is required to provide a great amount of information (i.e. fuzzy rules, FNV-fuzzy measure values, etc.). It is therefore desirable to design the simplest possible model that reflects the reality sufficiently according to the expert.

That is why it is often better to start with a simple model that presents just a rough approximation of the final model. In the next step, the model is tested and its parts that should be improved are identified. This improved model is tested again and the process is repeated until the expert is satisfied with the model performance. During this process, it can be often found out that the aggregation method used for a particular goal in frame of the goals tree in the original simple model has to be replaced by a more complex one. In the next text, this situation will be studied. Two algorithms will be proposed to make the transition to the new more complex aggregation function (the fuzzified Choquet integral or the fuzzy expert system) as simple as possible. The algorithms make it possible to derive the parameters of the new aggregation function from the parameters of the original one. The expert can then adjust only some parameters for the new more complex method instead of setting all of them. This way, a significant amount of time and effort can be saved. This work has been published in [35].

Two cases will be studied. In the first case, the original aggregation method should be replaced by a fuzzified Choquet integral. In the latter case, a fuzzy expert system should be used as the new aggregation method.

In concordance with the previous text, $m \in \mathbb{N}$ will denote the number of the evaluations that should be aggregated (i.e. the number of subgoals). Moreover, the FNV-function (fuzzy-number-valued function), $\tilde{f} : \mathcal{F}_N([0,1])^m \to \mathcal{F}_N([0,1])$, will be the original function that has been used for the aggregation, i.e. the FuzzyWA, FuzzyOWA, or fuzzified WOWA.

2.12.1. Transition to the fuzzifiled Choquet integral

In this section, we will study the situation when a fuzzy weighted average, a fuzzy OWA or a fuzzified WOWA has originally been used for the aggregation and the expert would like to use the fuzzified Choquet integral instead. The fuzzified Choquet integral requires $2^m - 2$ values of the FNV-fuzzy measure to be set by the expert (the last two values of the FNV-fuzzy measure are given by its definition). The following algorithm makes it possible to propose the corresponding FNV-fuzzy measure automatically. After the FNV-fuzzy measure is created, the expert can modify some of its values according to his/her requirements (when these modifications are performed by the expert, the monotonicity condition required for FNV-fuzzy measures should be verified). The algorithm is based on a method of setting a fuzzy measure by an expert in the crisp case, which is presented in [29].

Algorithm 2.6 *Let a FNV-function $\tilde{f} : \mathcal{F}_N([0,1])^m \to \mathcal{F}_N([0,1])$ represent a FuzzyWA, or a FuzzyOWA with normalized fuzzy weights $W_1, \ldots, W_m$, or a fuzzified WOWA with normalized real weights $w_1, \ldots, w_m$ and $p_1, \ldots, p_m$. Let $G = \{G_1, \ldots, G_m\}$ be the set of individual partial goals of interest. Then, the FNV-fuzzy measure $\tilde{\mu}$ on G is derived from $\tilde{f}$ as follows:*

1. *The value $\tilde{\mu}(\emptyset) = \tilde{0}$ is set by the definition.*
2. *Similarly, the value $\tilde{\mu}(G) = \tilde{1}$ is set by the definition.*
3. *For the rest of the $2^m - 2$ values, the FNV normalized measure of K, $K \subset G$, is calculated as $\tilde{\mu}(K) = \tilde{f}(C_1, \ldots, C_m)$, where the fuzzy numbers C_i, $i = 1, \ldots, m$, are defined as follows*

$$C_i = \begin{cases} \tilde{1} & \text{if } G_i \in K, \\ \tilde{0} & \text{otherwise.} \end{cases}$$

In the following text, it will be checked that the mapping $\tilde{\mu}$ obtained by the Algorithm 2.6 is really a FNV-fuzzy measure, providing that the FNV-function $\tilde{f}$ represents one of the permitted aggregation methods (FuzzyWA, FuzzyOWA, or fuzzified WOWA).

In case that a FuzzyWA with m-tuple of normalized fuzzy weights $W_1, \ldots, W_m$ is used as the function $\tilde{f}$, the corresponding mapping $\tilde{\mu}$ obtained by the above-mentioned algorithm is known in the literature as a fuzzy probability in case of a probability space with a finite set of elementary events [84]. The authors of the paper showed that this fuzzy probability has properties that represent a natural generalization (fuzzification) of the classical probability axioms. In this thesis, it will be shown, that it also satisfies the conditions required for a FNV-fuzzy measure (i.e. boundary conditions and monotonicity). First, an alternative way of FuzzyWA calculation will be described by the following theorem.

Theorem 2.10 *Let $U_i = \{[\underline{u}_i(\alpha), \overline{u}_i(\alpha)], \alpha \in [0,1]\}$, $i = 1, \dots, m$, be fuzzy numbers defined on $[0,1]$ and let $W_i = \{[\underline{w}_i(\alpha), \overline{w}_i(\alpha)], \alpha \in [0,1]\}$, $i = 1, \dots, m$, be normalized fuzzy weights. Then, the FuzzyWA of the values $U_1, \dots, U_m$ with the normalized fuzzy weights $W_1, \dots, W_m$ is a fuzzy number U, $U = \{[\underline{u}(\alpha), \overline{u}(\alpha)], \alpha \in [0,1]\}$ defined on $[0,1]$ that can be calculated as follows:*

$$\underline{u}(\alpha) = \min\{\sum_{i=1}^{m} w_i \underline{u}_i(\alpha) \mid w_i \in [\underline{w}_i(\alpha), \overline{w}_i(\alpha)], \sum_{i=1}^{m} w_i = 1, i = 1, \dots, m\}, \quad (2.59)$$

$$\overline{u}(\alpha) = \max\{\sum_{i=1}^{m} w_i \overline{u}_i(\alpha) \mid w_i \in [\underline{w}_i(\alpha), \overline{w}_i(\alpha)], \sum_{i=1}^{m} w_i = 1, i = 1, \dots, m\}. \quad (2.60)$$

Proof: *See [59].* □

Theorem 2.11 *Let $G = \{G_1, \dots, G_m\}$ be the set of partial goals. Let $W_1, \dots, W_m$ be an m-tuple of normalized fuzzy weights representing the importances of these partial goals, $W_i = \{[\underline{w}_i(\alpha), \overline{w}_i(\alpha)], \alpha \in [0,1]\}$, $i = 1, \dots, m$. Moreover, let a FNV-function $\tilde{f} : \mathcal{F}_N([0,1])^m \to \mathcal{F}_N([0,1])$ represent the FuzzyWA with the normalized fuzzy weights $W_1, \dots, W_m$. Then the mapping $\tilde{\mu}$ obtained by the Algorithm 2.6 represents a FNV-fuzzy measure on G.*

Proof: It is necessary to verify that $\tilde{\mu}$ satisfies the boundary conditions and the monotonicity. The boundary conditions $\tilde{\mu}(\emptyset) = \tilde{0}$, and $\tilde{\mu}(G) = \tilde{1}$ are obviously satisfied; these values are set directly in the algorithm. It is therefore sufficient to verify only the monotonicity, i.e. that $A \subseteq B$ implies $\tilde{\mu}(A) \leq \tilde{\mu}(B)$ for any $A, B \in \wp(G)$.

For any $K \in \wp(G)$, the value $\tilde{\mu}(K)$, $\tilde{\mu}(K) = \{[\underline{\tilde{\mu}}(K)(\alpha), \overline{\tilde{\mu}}(K)(\alpha)], \alpha \in [0,1]\}$, is a fuzzy number. Let $A, B \in \wp(G)$ be sets satisfying $A \subseteq B$. We need to prove that, for any $\alpha \in [0,1]$, it holds that

$$\underline{\tilde{\mu}}(A)(\alpha) \leq \underline{\tilde{\mu}}(B)(\alpha) \quad \text{and} \quad (2.61)$$

$$\overline{\tilde{\mu}}(A)(\alpha) \leq \overline{\tilde{\mu}}(B)(\alpha). \quad (2.62)$$

This obviously holds if A (or alternatively both A and B) is an empty set (Step 1 of the algorithm), since $\tilde{\mu}(\emptyset) = \tilde{0}$ and all the other values are fuzzy numbers on $[0,1]$. Similarly, the condition holds also if B (or alternatively both A and B) is G (Step 2 of the algorithm), since $\tilde{\mu}(G) = \tilde{1}$. Let us check the condition for the rest of the values (Step 3 of the algorithm) – in the following text, we will assume that $A \subseteq B$ and $A \neq \emptyset$ and $B \neq G$.

Taking into account the fact that the values $\tilde{\mu}(A)$ and $\tilde{\mu}(B)$ are calculated as FuzzyWA of values that consist only from $\tilde{0}$ or $\tilde{1}$ according to the Algorithm 2.6, and moreover that the FuzzyWA result can be expressed using the formulae given by the Theorem 2.10, we can write for any $\alpha \in [0,1]$:

$$\begin{aligned}
\underline{\tilde{\mu}}(A)(\alpha) &= \min\Big\{ \sum_{\substack{i\in\{1,\dots,m\}:\\ G_i\in A}} w_i \mid w_i \in [\underline{w}_i(\alpha), \overline{w}_i(\alpha)], \sum_{i=1}^{m} w_i = 1, \\
&\qquad i = 1,\dots,m \Big\}, \qquad (2.63)\\
\underline{\tilde{\mu}}(B)(\alpha) &= \min\Big\{ \sum_{\substack{i\in\{1,\dots,m\}:\\ G_i\in B}} w_i \mid w_i \in [\underline{w}_i(\alpha), \overline{w}_i(\alpha)], \sum_{i=1}^{m} w_i = 1, i = 1,\dots,m \Big\} \\
&= \min\Big\{ \sum_{\substack{i\in\{1,\dots,m\}:\\ G_i\in A}} w_i + \sum_{\substack{i\in\{1,\dots,m\}:\\ G_i\in B\setminus A}} w_i \mid w_i \in [\underline{w}_i(\alpha), \overline{w}_i(\alpha)], \qquad (2.64)\\
&\qquad \sum_{i=1}^{m} w_i = 1, i = 1,\dots,m \Big\}.
\end{aligned}$$

Comparing values of the Formulae 2.63 and 2.64, it can be seen that the first one is lesser than or equal to the latter one and therefore $\underline{\tilde{\mu}}(A)(\alpha) \leq \underline{\tilde{\mu}}(B)(\alpha)$ for any $\alpha \in [0,1]$.

The inequality given by the Formula 2.62 could be verified in same way. Therefore it holds that $\tilde{\mu}(A) \leq \tilde{\mu}(B)$. □

Theorem 2.12 *Let $G = \{G_1,\dots,G_m\}$ be the set of partial goals. Let $W_1,\dots,W_m$ be an m-tuple of normalized fuzzy weights representing the importances of these partial goals. Let a FNV-function $\tilde{f} : \mathcal{F}_N([0,1])^m \to \mathcal{F}_N([0,1])$ represent the FuzzyOWA with the normalized fuzzy weights $W_1,\dots,W_m$. Then the mapping $\tilde{\mu}$ obtained by the Algorithm 2.6 represents a FNV-fuzzy measure on G.*

Proof: The proof is an analogy to the previous one. In [4], a theorem for the FuzzyOWA similar to the Theorem 2.10 for the FuzzyWA is presented. The only difference is the presence of a permutation, which however does not affect the next steps. Then, the same reasoning as in the proof of the Theorem 2.11 can be used. □

Theorem 2.13 *Let $\mathbf{p} = (p_1,\dots,p_m)$ and $\mathbf{w} = (w_1,\dots,w_m)$ be two vectors of normalized real weights and $G = \{G_1,\dots,G_m\}$ be individual partial goals. Let a FNV-function $\tilde{f} : \mathcal{F}_N([0,1])^m \to \mathcal{F}_N([0,1])$ represent the fuzzified WOWA with the normalized weights $\mathbf{p}$ and $\mathbf{w}$. Then the mapping $\tilde{\mu}$ obtained by the Algorithm 2.6 is a FNV-fuzzy measure on G.*

Proof: Similarly as in the proof of the Theorem 2.11, only the monotonicity condition has to be checked. Let $A, B \in \wp(G)$, $A \subseteq B$. If A (or both A and B) is an empty set (Step 1 of the algorithm), the monotonicity condition holds, since $\tilde{\mu}(\emptyset) = \tilde{0}$ and all the other values are fuzzy numbers on $[0, 1]$. The condition obviously holds also if B (or both A and B) is G (Step 2 of the algorithm).

Let us consider the rest of the values (Step 3 of the algorithm), i.e. $A, B \neq \emptyset$ and $A, B \neq G$, $A \subseteq B$. The value $\tilde{\mu}(A)$ is calculated as follows:

$$\tilde{\mu}(A) = \mathrm{FWOWA}_{\mathbf{w}}^{\mathbf{p}}(C_1, \ldots, C_m),$$

where, for $i = 1, \ldots, m$, C_i is $\tilde{1}$ if $G_i \in A$, and $\tilde{0}$ otherwise. Similarly, value $\tilde{\mu}(B)$ is calculated as

$$\tilde{\mu}(B) = \mathrm{FWOWA}_{\mathbf{w}}^{\mathbf{p}}(D_1, \ldots, D_m),$$

where again, for $i = 1, \ldots, m$, D_i, is $\tilde{1}$ if $G_i \in B$, and $\tilde{0}$ otherwise. Therefore, if $A \subseteq B$, then it holds that $C_i \leq D_i$ for all $i = 1, \ldots, m$.

Using the Theorem 2.2, the fuzzified WOWA result can be calculated by multiple (non-fuzzy) WOWA calculations. Taking into the account the monotonicity of the WOWA operator [89] and the fact that $C_i \leq D_i$ for all $i = 1, \ldots, m$, it can be deduced that

$$\mathrm{FWOWA}_{\mathbf{w}}^{\mathbf{p}}(C_1, \ldots, C_m) \leq \mathrm{FWOWA}_{\mathbf{w}}^{\mathbf{p}}(D_1, \ldots, D_m)$$

Therefore, $\tilde{\mu}(A) \leq \tilde{\mu}(B)$ and the monotonicity condition is satisfied. The mapping $\tilde{\mu}$ produced by the Algorithm 2.6 for the fuzzified WOWA is thus a FNV-fuzzy measure on G (specifically, in case of the fuzzified WOWA, which uses weights expressed by real numbers, the values of the resulting FNV-fuzzy measure consist of fuzzy singletons). □

The benefits of the presented algorithm will be demonstrated on an example.

Example 2.15 *Let us suppose that a university wants to evaluate the high-school students applying for the study according to their results in the Math, Physics, and English. However, the knowledge that is necessary for the good grades in Math and Physics is overlapping – there is a lot of common skills that are required in order to succeed in either of them. Therefore, there is a relationship of redundancy between them, and the Choquet integral (or its fuzzified version in our case) should be used.*

The use the fuzzified Choquet integral for the aggregation implies that the expert should provide 6 values of the FNV-fuzzy measure. It could be a difficult task for the expert to determine the values of the FNV-fuzzy measure and to remain consistent at the same time. The task would be even more complex if more subjects were used for the evaluation. However, because there are interactions just between the two of the criteria, we can simplify this task first.

Let us consider a simpler model where the importances of the subjects are expressed by normalized fuzzy weights and the fuzzy weighted average is used for the aggregation. The university could assign the following normalized fuzzy weights to the particular subjects: $W_{Math} = (0.3, 0.4, 0.5)$*,* $W_{Physics} = (0.25, 0.35, 0.45)$*, and* $W_{English} = (0.15, 0.25, 0.35)$*.*

In the next step, we will replace the simple model with a more complex one that uses the fuzzified Choquet integral for the aggregation. Applying the Algorithm 2.6, we obtain the following FNV-fuzzy measure:

- $\tilde{\mu}(\emptyset) = \tilde{0}$
- $\tilde{\mu}(Math) = (0.3, 0.4, 0.5)$
- $\tilde{\mu}(Physics) = (0.25, 0.35, 0.45)$
- $\tilde{\mu}(English) = (0.15, 0.25, 0.35)$
- $\tilde{\mu}(Math, Physics) = (0.65, 0.75, 0.85)$
- $\tilde{\mu}(Math, English) = (0.55, 0.65, 0.75)$
- $\tilde{\mu}(Physics, English) = (0.5, 0.6, 0.7)$
- $\tilde{\mu}(Math, Physics, English) = \tilde{1}$

Finally, the expert has to modify just a single value of the FNV-fuzzy measure $\tilde{\mu}$ *in order to reflect the redundancy between the Math and Physics and to obtain the final model – the value of* $\tilde{\mu}(Math,\ Physics)$ *could be simply decreased to* $\tilde{\mu}(Math,\ Physics) = (0.5, 0.6, 0.7)$*.*

It can be seen that the use of the algorithm made the problem much simpler for the expert. Instead of setting all 6 of the FNV-fuzzy measure values, the FNV-fuzzy measure is generated automatically and the expert has to modify just one of its values.

The provided example is very simple and it has been chosen only for illustration. The true benefits of the proposed approach can be seen on problems with interactions among larger groups of criteria.

For more complex cases, a fuzzy expert system could be used for the aggregation. The next section shows how to replace a simpler method (from the described system of MCE methods) by a fuzzy expert system.

2.12.2. Transition to the fuzzy expert system

The fuzzified Choquet integral can handle only certain types of interactions among the partial goals (a complementarity or a redundancy). If the relationship is more complex, a fuzzy expert system can be used. In this section we assume that a fuzzy weighted average, a fuzzy OWA, a fuzzified WOWA or a fuzzified Choquet integral is used for the aggregation and the expert would like to use a fuzzy expert system instead. The fuzzy expert system requires a fuzzy rule base to be defined. The following algorithm makes it possible to create the fuzzy rule base automatically so that the aggregation result would be as similar as possible to the result obtained with the original aggregation method (in this case, the similarity will be assessed using the method from the Definition 2.20). For the fuzzy rule base and the individual linguistic scales, the notation from the Section 2.10.5 will be used.

Algorithm 2.7 *Let a FNV-function $\tilde{f} : \mathcal{F}_N([0,1])^m \to \mathcal{F}_N([0,1])$ be a FuzzyWA, FuzzyOWA or fuzzified WOWA with some weights, or the fuzzified Choquet integral with some FNV-fuzzy measure.*

First, the expert defines the linguistic scales $(\mathcal{E}_i, \mathcal{T}(\mathcal{E}_i), [0,1], M_i, G_i)$, $i = 1, \ldots, m$, for the partial evaluations to be aggregated, and the linguistic scale $(\mathcal{E}, \mathcal{T}(\mathcal{E}), [0,1], M, G)$ for the overall evaluation. For any possible combination of the criteria values (terms of the corresponding linguistic scales), a rule is created as follows. Let s_i, $i = 1, \ldots, m$, denote the number of the terms of the linguistic scale $\mathcal{E}_i$ and let s denote the number of the terms of the linguistic scale $\mathcal{E}$. Then $n = s_1 \cdot s_2 \cdot \cdots \cdot s_m$ denotes the total number of the rules that should be created. The following steps are performed for each of them. Let the antecedent (the left-hand part) of such an i-th rule, $i = 1, \ldots, n$, be

$$\textit{If } \mathcal{E}_1 \textit{ is } \mathcal{A}_{i,1} \textit{ and } \ldots \textit{ and } \mathcal{E}_n \textit{ is } \mathcal{A}_{i,m}.$$

The consequent (right-hand part) $\mathcal{B}_i$ for this rule is determined in the following way:

- *A fuzzy number C_i is calculated as $C_i = \tilde{f}(A_{i1}, \ldots, A_{im})$, where $A_{ij} = M_j(\mathcal{A}_{i,j})$, $j = 1, \ldots, m$.*
- *The linguistic term $\mathcal{B}_i \in \mathcal{T}(\mathcal{E})$ is then found by the linguistic approximation of C_i using the linguistic scale $(\mathcal{E}, \mathcal{T}(\mathcal{E}), [0,1], M, G)$. Specifically, $\mathcal{B}_i$ is such a linguistic term for whose mathematical meaning, the fuzzy number B_i, $B_i = M(\mathcal{B}_i)$, it holds that*

  $$\forall \mathcal{D} \in \mathcal{T}(\mathcal{E}) : S(C_i, B_i) \geq S(C_i, D),$$

  where $D = M(\mathcal{D})$, and S denotes the similarity from the Definition 2.20.

Because the algorithm creates a fuzzy rule for each of the criteria values combinations, it is applicable only when the number of criteria to be aggregated with the fuzzy expert system is low (e.g. two or three). If there are more criteria it is recommended to decompose the problem and solve it with multiple fuzzy expert systems.

It is also worth mentioning that the quality of the approximation of the original aggregation method given by the FNV-function $\tilde{f}$ depends on the number of terms in the linguistic scales $\mathcal{E}_1, \ldots, \mathcal{E}_m$ and $\mathcal{E}$. There is a trade-off between the resulting number of generated rules and the precision of the approximation.

Example 2.16 *In this example, the university wants to evaluate the applying high-school students according to their grades and their results of the entrance exam. The university specified that the entrance test results are important. The overall evaluation should consist from about 70 % of the entrance exam evaluation and only from about 30 % of the grades evaluation. However, the university requires two exceptions – evaluation of the students who failed the entrance exam completely will be* very bad *no matter what their grades were. And vice versa, the students with the excellent results in the entrance exams should be rated as* excellent *regardless their grades.*

The problem with such relationships between the two criteria should be modeled by a fuzzy rule base and a fuzzy expert system should be used for the evaluation. For the entrance exam evaluation, grades evaluation, and overall evaluation, the linguistic scale depicted in the Figure 2.32 is used.

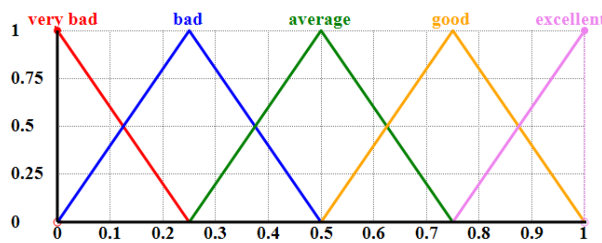


Figure 2.32: The linguistic scale used for the grades, entrance exam results and the student's overall evaluation.

Normally, the expert would be required to set 25 fuzzy rules. However, because we can see that the desired evaluation function could be very close to the fuzzy weighted average, we will again break the creation of the model into two steps. In the first one, a simple model based on the fuzzy weighted average will be designed. This simple model can be perceived as a rough approximation of the final model which neglects the additional two requirements of the university. In the next step, it will be used to create a more complex model that uses a fuzzy rule base.

In the first simple model, the normalized fuzzy weights are set as follows: $W_{Exam} = (0.6, 0.7, 0.8)$ *and* $W_{Grades} = (0.2, 0.3, 0.4)$, *and the fuzzy weighted average is used for the aggregation. Then, the Algorithm 2.7 is used to generate the fuzzy rule base automatically. The proposed fuzzy rule base consists of the following 25 rules that are summarized in the Table 2.3.*

In order to reflect the two conditions given by the university, the expert should modify the consequents of just a few rules. Specifically, the consequent parts of the rules number 3, 4, and 5 will be changed to very bad *and the consequents of the rules number 21, 22, and 23 will be modified to* excellent.

Again, it can be seen that the use of the algorithm saves the time and effort of the expert. Instead of setting all 25 fuzzy rules from the beginning, the expert had to modify just 6 rules.

	If		Then
	Entrance exam	Grades	Overall evaluation
1	very bad	very bad	very bad
2	very bad	bad	very bad
3	very bad	average	bad
4	very bad	good	bad
5	very bad	excellent	bad
6	bad	very bad	bad
7	bad	bad	bad
8	bad	average	bad
9	bad	good	average
10	bad	excellent	average
11	average	very bad	bad
12	average	bad	average
13	average	average	average
14	average	good	average
15	average	excellent	good
16	good	very bad	average
17	good	bad	average
18	good	average	good
19	good	good	good
20	good	excellent	good
21	excellent	very bad	good
22	excellent	bad	good
23	excellent	average	good
24	excellent	good	excellent
25	excellent	excellent	excellent

Table 2.3: The rule base generated by the Algorithm 2.7.

It has been shown on the examples, that both of the algorithms can simplify the designing of a new fuzzy evaluation model significantly. For comfortable usage of these models, the whole system of the fuzzy multiple-criteria evaluation methods described so far in this thesis has been implemented into a software tool called FuzzME.

2.13. The FuzzME software

The first foundations of the described system of methods were laid by the book [81]. The book describes a methodology called the Solver. Over more than ten years, this methodology has been extended and improved rapidly. A coherent complex system of fuzzy multiple-criteria evaluation methods, which are described in the thesis, has been formed. One of the major goals of this thesis was to create a software implementation of this whole system of methods. The resulting software is called FuzzME. Its name is an acronym of **Fuzz**y Methods of **M**ultiple-Criteria **E**valuation.

Transferring such a large system of mathematical methods, which moreover can be arbitrarily combined in a single fuzzy MCDM model, into a form of software presents many challenges. First, the methods have to be implemented in an effective way. Efficiency is necessary in order to be able to use the software on large complex problems. This goal has been achieved in the FuzzME well. The evaluations are calculated in the real time. As soon as any parameter of the model is changed, the evaluations are recalculated immediately so the expert can see the impact of the performed changes at once.

The next requirement, which is no less important, was that the software has to be intuitive and user-friendly. The FuzzME accompanies the numeric results with a graphical output and the evaluations can be also described verbally.

The first step of creating a model in the FuzzME is to design a goals tree for the given problem. The user creates the structure of the tree and then determines the type of each node. For nodes at the ends of branches, the user selects between qualitative and quantitative type of criteria. For the rest of the nodes, an appropriate aggregation method is chosen. All the aggregation methods can be arbitrarily combined within the same goals tree.

The FuzzME takes maximum advantage of the linguistic approximation. The user can design a linguistic scale for each node. This step is not necessary but it is recommended. If the linguistic scale is created, the user can then see also the linguistic evaluation of a particular partial goal. The scale is designed in the Linguistic scale editor (Figure 2.33). The process is simplified for the user as much as possible. The user is asked how many terms should be in the scale and what type of fuzzy numbers should be used to model them. The FuzzME then creates a uniform scale and the user just types appropriate names for the terms, or adjusts the fuzzy numbers representing their meanings if necessary. The

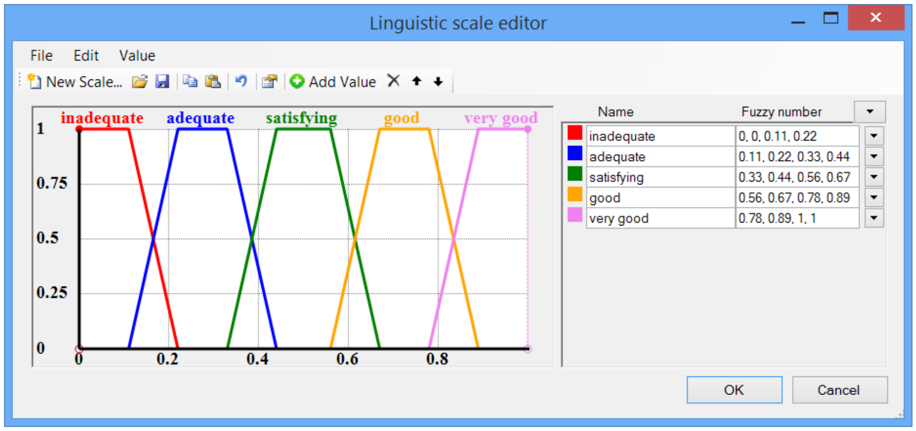


Figure 2.33: Linguistic scale editor in the FuzzME

frequently used scales can be saved into files and reused later easily.

The next step is to fill in the necessary information for all goals tree nodes representing the partial goals. The required information depends on the selected method. For FuzzyWA and FuzzyOWA, normalized fuzzy weights are defined. For the fuzzified WOWA operator, two sets of normalized (crisp) weights are set. The fuzzified Choquet integral requires a FNV-fuzzy measure to be defined. And finally, a fuzzy rule base must be designed if a fuzzy expert system is used. Concerning the criteria, for each quantitative criterion, the user must specify its domain and define the evaluating function. For qualitative criteria, it suffices to define the linguistic evaluating scales.

The FuzzME strives to make the process of setting various parameters of the model as simple as possible. For example, when normalized fuzzy weights should be set, the FuzzME checks if the condition from the Definition 2.28 holds for the fuzzy numbers provided by the user. If it does not hold, the FuzzME offers a remedy and the normalized fuzzy weights can be derived using the Algorithm 2.2 by a single click (Figure 2.34).

Similarly, when the user has to set a FNV-fuzzy measure, the monotonicity condition is checked. If the condition is broken, the FuzzME also reports, which values have to be modified. The FuzzME offers a diagram view where the red lines highlight the partial goals sets whose values have to be modified in order to satisfy the monotonicity (Figure 2.35).

When all the above mentioned steps are completed, the model is finished and ready for the evaluation process. The alternatives can be inserted manually by the user, but import from outside sources, such as Microsoft Excel, is also supported. The resulting evaluations can be exported, e.g. to Excel, for further analysis and processing.

The alternatives can be displayed in the list (Figure 2.36) showing the membership functions graphs of the resulting fuzzy evaluations and also the linguistic

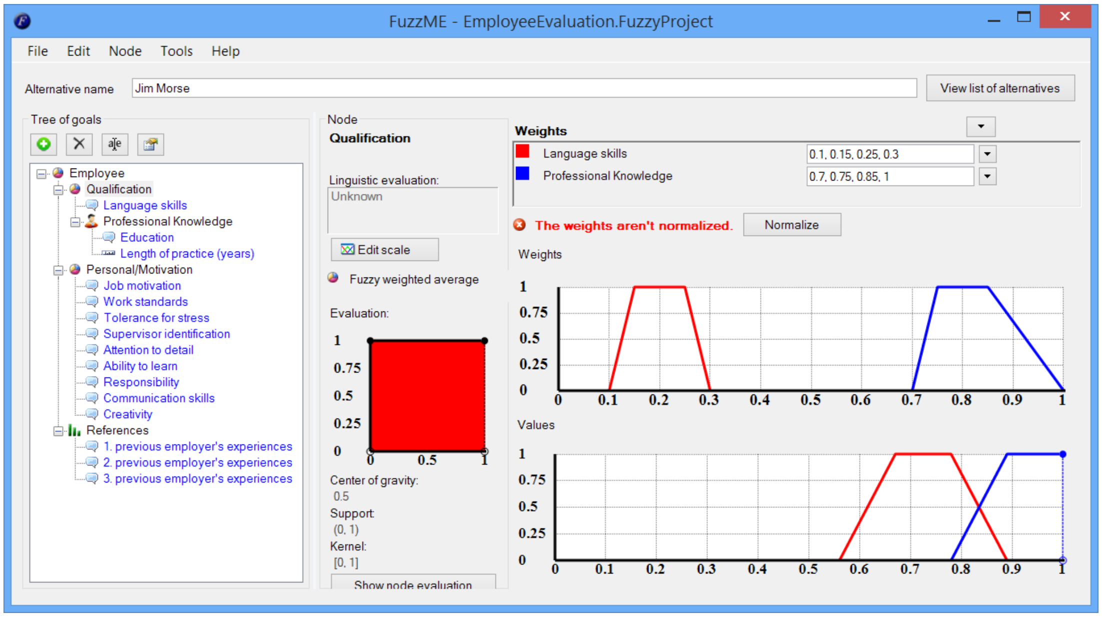


Figure 2.34: Setting the weights in the FuzzME

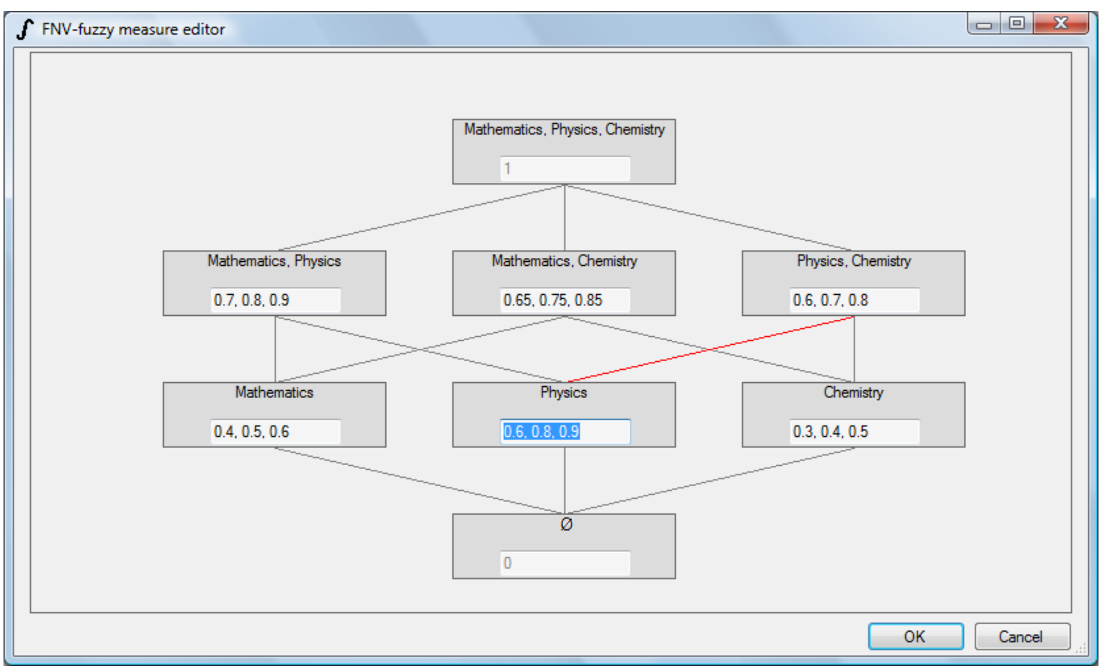


Figure 2.35: Designing the FNV-fuzzy measure in the FuzzME

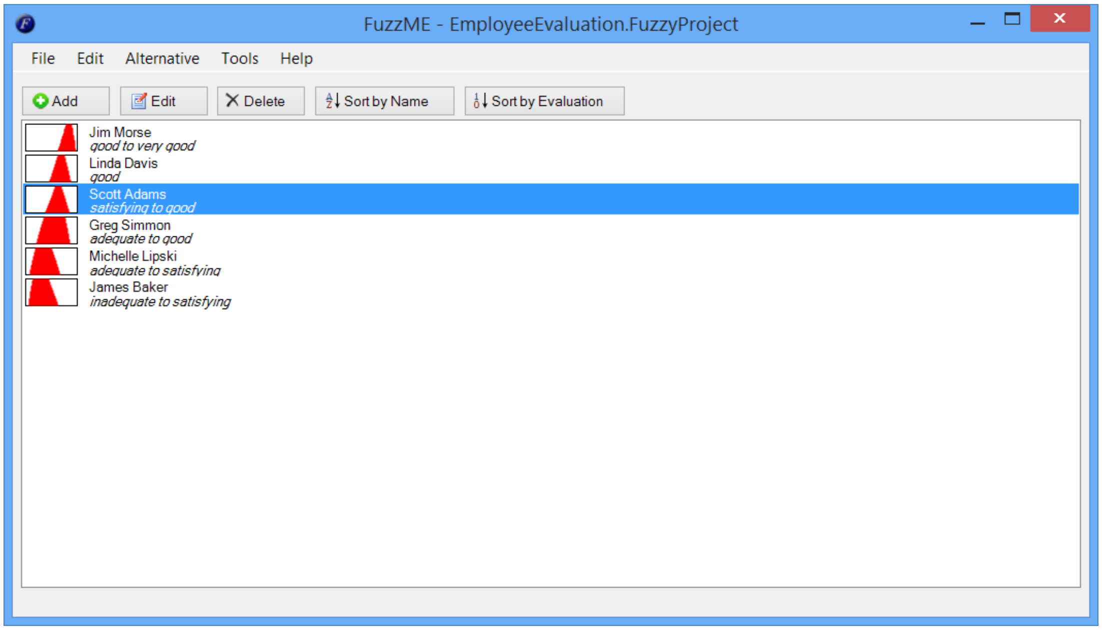


Figure 2.36: The list of alternatives in the FuzzME

descriptions of the final fuzzy evaluations. The alternatives can be ordered with respect to the centers of gravity of their fuzzy evaluations. For all alternatives, the user can view the evaluation according to any partial goal.

The FuzzME offers a simple analytical tool to study the behavior of the designed evaluation function or to plot its graph (for 3D visualization, a connection to the MATLAB is used). Moreover, any relevant graphics in the FuzzME can be saved as an image, which makes the documentation of the designed model and its publication much easier.

The FuzzME has been written in the C# programming language. It requires .NET framework 2.0 (this library is a standard part of Windows and it is usually not necessary to install it). The software is multi-platform. It can run on both Windows and Linux. For Linux, a special implementation of the .NET framework, which is called Project Mono, has to be installed.

A demo version of the FuzzME can be downloaded from http://www.FuzzME.net. The FuzzME is also included on the CD enclosed to this thesis.

2.14. Applications of the FuzzME software on real-world problems

As soon as the first version of the FuzzME was released, its applications on real-world problems begun to emerge. Examples of such applications will be shown in this section. The solved problems come from diverse areas. This fact shows the versatility of the FuzzME software as well as the described system of fuzzy multiple-criteria evaluation methods.

2.14.1. Soft-fact rating of bank clients

The FuzzME software was tested on a soft-fact-rating problem of one of the Austrian banks [25]. The problem was solved in co-operation with the Technical University in Vienna. The fuzzy model of evaluation represented one part of the creditability evaluation of companies carried out by the bank – evaluation according to soft (qualitative) data that complemented evaluation according to hard (quantitative) data.

The rating system used by the bank utilized, at that time, self-organizing maps and neural networks to analyze hard-fact data from the company's balance sheet and to calculate 17 ratios from them. These ratios were then analyzed by four expert systems and the result was the hard-fact rating of the company. The applied rating system was however able to take into account also a soft-fact rating to calculate the overall evaluation. The weights assigned to the hard-fact and soft-fact ratings were 80% and 20%, respectively. In cooperation with colleagues from the Technical University in Vienna, we have focused on soft-fact rating procedures and tried to enhance them by means of instruments of the fuzzy set theory.

The original soft-fact rating system worked as follows: The input data for a given company were obtained from a questionnaire filled in by the bank experts. The questionnaire consisted of 28 questions. The answers took the form of linguistic terms, whose meanings were modeled by integers 1 to 5. For instance, one of the questions could be: "*How would you rate the company's experience?*" As an answer, one of the linguistic terms ranging from *very good* (1) to *inadequate* (5) could be used. All the questions in the questionnaire were grouped into 10 sections: *quality of management*, *accounting and reporting*, *balancing behavior*, *organizational structure*, *ownership structure*, *production*, *market and market position*, *dependencies*, *location*, and *miscellaneous*. A crisp weight was assigned to each section and to each question within the section. The rating score for every section was calculated as a weighted average of the questions' scores. Again, the final soft-fact rating score was obtained from section scores by the weighted average operator. Finally, the hard-fact and soft-fact rating scores were aggregated and the system yielded the final rating.

The original model was fuzzified by means of the FuzzME. First, the items of the original discrete scales were replaced by fuzzy numbers. It turned out that, in some cases, the correspondence between the linguistic and numerical values of the original scales was not ideal. Therefore, two alternative mathematical structures (equal fuzzy scales and unequal fuzzy scales) were employed in modeling the expert evaluation. The equal (i.e. uniform) fuzzy scales represent a straightforward fuzzification of the original numeric scales. The unequal fuzzy scales work with fuzzy values that strive to model, as closely as possible, the linguistic values used in the original evaluation model.

As a next step, the normalized crisp weights were replaced by the normalized fuzzy weights. The fuzzy weights are more suitable in situations when the criteria weights are set expertly.

The newly formed soft-fact rating fuzzy model was tested on real data from 62 companies. Four alternatives of the soft-fact rating model were tested and compared – they differed in using either crisp or fuzzy weights, and in two possible mathematical interpretations of the linguistically defined values in the questionnaire (equal or unequal fuzzy scales). The alternative of the model with fuzzy weights and unequal fuzzy scales was evaluated as the most promising. The reason was that the fuzzy weights correspond better to the vague expertly defined information and the unequal (i.e. non-uniform) scales model better the real meaning of the used linguistic terms.

The results of the proposed fuzzy model were analyzed and compared to the original results. An advantage of the fuzzy model was that a different uncertainty of the obtained results could be taken into account. For instance, let us suppose that the weights of two criteria of interest are the same. In the original model, a company with one criterion graded as 1 (*very good*) and the other as 5 (*inadequate*) would get the same overall evaluation as a company with both criteria graded as 3 (*average*). In reality, the former company is considered more risky as regards loan granting. With the new fuzzy model, the evaluations of both companies would have the same centers of gravity but their uncertainties would differ. For the former company, worse values of overall evaluation are possible, denoting the company as more risky. This behavior has been claimed to be desirable for the bank and it represents one of the advantages of the using fuzzy methods over the crisp ones in this case.

However, it became apparent through testing and discussion of the results that the condition of the criteria independence, which is necessary for the weighted average, was not fully met in all cases. It was therefore proposed that, besides the above evaluation based on the fuzzy weighted average (*average rating*), another rating should be obtained by a fuzzy expert system (*a risk rate of the company*). This new rating indicates dangerous combinations of criteria values. Both these ratings were then aggregated with the FuzzyMin aggregation method (a special case of FuzzyOWA). The structure of the new goals tree can be seen in the Figure 2.37.

Fuzzy rules of the fuzzy expert system for the risk rate calculation are e.g. of the following form: "*If equipment is outdated and market position is bad, then risk rate is very high*". The linguistic scale for the company's risk rate contains the following terms: *very high risk*, *high risk*, *medium risk*, and *no risk recognized*. Their meanings are modeled by fuzzy numbers on $[0, 1]$, where 0 means the completely unsatisfactory rating and 1 means fully satisfactory rating. A weight has been assigned to each of these terms (the bigger the risk, the greater

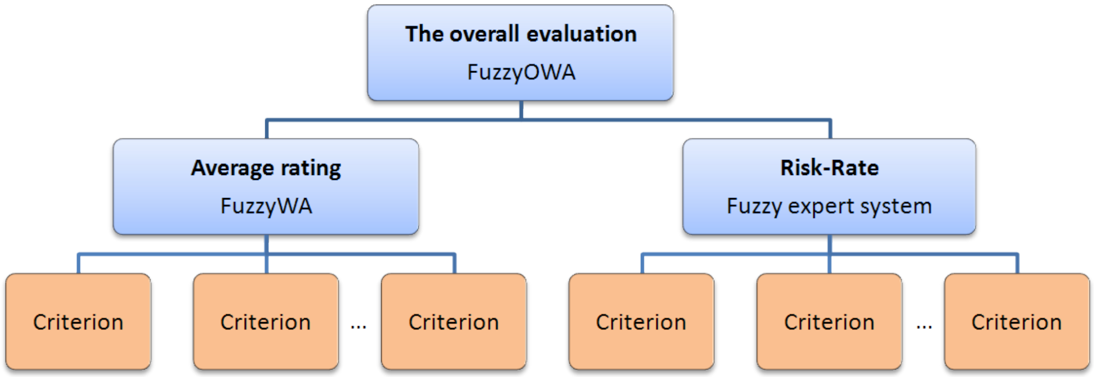


Figure 2.37: The simplified structure of the used goals tree for soft-fact rating of companies

the weight). Then, the Sugeno-WOWA inference algorithm was used to calculate the risk rate. It allowed putting a higher significance on the rules suggesting a significant risk in contrast to those that suggested that the risk is low. Again, this behavior proved to be desirable during the testing.

Fürst summarizes the results of the testing from the bank's point of view in [25], and concludes that the instruments of the fuzzy set theory in the soft-fact rating model are a step in the right direction. In particular, the ability to visualize the uncertainty of ratings gives a better view of the company's status. In her report, she also recommends to take into account the uncertainty of the final evaluation. If the uncertainty is low then it suggests that the evaluator who filled the questionnaire was very confident regarding the situation of the company. In these cases, a greater weight can be assigned to the soft-fact rating in the system of the bank. On the other hand, if the uncertainty is high, the author of the paper recommends visualizing the final evaluation before making the final decision. This is a clear advantage of the fuzzy models over the standard crisp methods. The additional information on uncertainty reflects the reliability of the final evaluation and can be used to adjust the weight given to this evaluation compared to the others, or it can be treated as a warning that more detailed inspection for the particular company is required.

2.14.2. Employees evaluation in an IT company

The FuzzME was used in the area of HR management for periodic evaluation of employees in the IT company AXIOM SW Ltd [104]. The company has taken the evaluation methodology of Microsoft as a base and adjusted it to fit their needs.

The evaluation is based on the so-called competency model. Competencies are, in this context, summary of knowledge, abilities and other skills. Examples of such competencies are, for instance, *language skills*, *willingness to learn*, *quality* *of work*, or *leadership*. They act as criteria in the model. The competences are grouped into three main categories: input (representing knowledge, or skills), output (assessing the results), and process (rating the behavior).

The following working roles were identified in the company: senior executive, head of the project, analyst, consultant, software engineer, dealer, and marketing agent. As there are different requirements on the competencies for a particular role, each of the working roles has different (fuzzy) weights assigned to the particular competencies.

The quantitative criteria are assessed by the linguistic scales. The evaluation is therefore verbal. The used linguistic scale contains six terms assessing to what extend does the evaluated person possess the required level of a given competency. The values range from "Does not meet at all" to "Meets very well". The linguistic scale is not uniform. The reason is that people tend to use an average evaluation and they often avoid using the extreme terms ("Does not meet at all", or "Meets very well"). Therefore, the fuzzy numbers modeling those two terms were designed to be much less uncertain than the fuzzy numbers modeling the others. If evaluators select one of the two extreme values, it is considered that they are very confident in their assessment. Concerning the quantitative criteria, their values are filled according to the employee's personal record.

The competencies of individual employees are evaluated by several evaluators – by the evaluated employees themselves, by their direct supervisor, their subordinates (only in case of mangers), and their colleagues working on the same project.

First, the evaluations of a particular competency by the individual evaluators are aggregated by a fuzzy weighted average. Then, a fuzzy weighted average is used again to obtain the final evaluation from the assessments of the individual competencies.

The results from the FuzzME need not to be used only for a direct assessment of employees. The evaluations of specific competencies groups (input, output, and process) can be moreover used to determine the type of the employee [104]. For example, employees with a high performance in the input and process groups but low performance in the output group are classified by the type "Promising". For each of the types, a motivation strategy exists. For instance, in case of the "Promising" type, the recommended strategy is to give the person more support and to stimulate his/her self-confidence. Therefore, the results from the FuzzME need not to be used only for a direct evaluation of the employee, but they can be also used for classification of the employees and for choosing the proper motivation strategies.

More details on the designed mode are described by her author in the paper [104].

2.14.3. Assessment of safety in agri-food buildings

In the third application, its authors from the Università degli studi Mediterranea, Italy, performed an assessment of the safety in agri-food buildings [3]. They were motivated by the recent EU policy focusing on the food-safety. The authors state that the environment where the food has been produced plays an essential role in this topic. Therefore they focused on the assessment of the buildings where the food is produced. For this evaluation they have used the instruments of the fuzzy set theory and the FuzzME software.

In the paper, a global safety building index (GSBI) is proposed. Two main aspects of agri-food buildings are assessed – the hygienic safety and the workers' safety. The authors emphasize that the latter aspect is also very important. Besides the ethical reasons, the workers' well-fare influences their performance and therefore its impact is also economical. The buildings are, for purpose of the assessment, divided into following functional areas: 1) receiving, 2) processing, 3) packaging, and 4) support. The evaluation is performed separately for each of them. This division helps to identify the parts of the building where an improvement should be made.

Both qualitative and quantitative criteria are used. The qualitative criteria are assessed by an expert. The values of quantitative criteria are measured by convenient instruments. For example, one indicator of the workers' safety is the slipperiness of the floor. In the paper, the authors used a Tertus digital tribometer to measure the slipperiness. Overall, more than 40 criteria have been identified.

The resulting model had a form of a five-level hierarchical structure. On the first level, there were the two aspects of the interest – the hygienic safety and the operators (workers) safety. On the second level, the tree branches according to the individual areas of the building. The rest of the levels are used to group the criteria into the related categories.

The FuzzyWA has been applied for the aggregation. However, for some groups of interacting criteria, a fuzzy expert system has been used. In case of the fuzzy expert system, the Sugeno-WA inference method has been employed.

The model has been tested on the manufacturing area of a dairy farm located in Calabria, Italy. The authors conclude that this application confirmed the appropriateness of the model and the easiness of its use.

2.14.4. Photovoltaic power plant location selection

Recently, a series of experiments have been conducted by Dr. Burak Ömer Saraçoglu. The FuzzME has been used for selection of the photovoltaic power plant location. Different models have been designed and tested. The results were presented at the 18$^{\text{th}}$ Online World Conference on Soft-Computing in Industrial Applications (WSC18) under the following topics:

1. An Experimental Fuzzy Weighted Average (Fuzzy WA) Aggregated Location Selection Model For The Very Large Photovoltaic Power Plants In The Globalgrid Concept In The Very Early Engineering Design Process Stages;
2. An Experimental Ordered Fuzzy Weighted Average (Fuzzy OWA) Aggregated Location Selection Model For The Very Large Concentrated Photovoltaic Power Plants In The Middle East and North Africa Region In The Very Early Engineering Design Process Stages;
3. An Experimental Fuzzy Expert System Based Application For The Go/No-Go Decisions To The Geospatial Investigation Studies Of The Regions Of The Very Large Concentrated Solar Power Plants In The European Supergrid Concept;
4. A Fuzzy Expert System Proposal For The Commercial & Participation Banks In The Power Plant Projects Financing (Loan) Suitability Evaluations In Turkey.

In the first study, the author used the FuzzME to propose suitable places for very large photovoltaic power plants (VLPVPPs). In his study, VLPVPPs are considered to be the power plants with the peak power of 1 000 MWp or more. The author proposed multiple criteria taking into the account both the technical aspects and the political stability of a given region.

The second study presents an alternative model to the first one. It focuses on the Middle East and the North Africa regions. These regions are specific because, while they present ideal locations from the performance point of view, they are politically very unstable and the threat of wars and other conflicts is very high. When the political conditions make the building of a power plant to be impossible, the other technical factors lose their importance. Therefore, the model uses the FuzzyOWA operator to take this fact into the account.

The last two studies present sample models based on the use of a fuzzy expert system. The Mamdani inference is used to make decision on building the particular photovoltaic power plant and on the appropriateness of its financing from the point of the view of a bank. The presentations for these studies are available online on Dr. Saraçoglu's profile on the ResearchGate (http://www.researchgate.net/profile/Burak_Saracoglu/publications).

Chapter 3

Fuzzy Classification

Classification problems can be encountered very often in the real world. Because the practical classification problems contain elements of uncertainty, it is natural to study the classification methods that make use of the fuzzy sets theory.

Many papers have been written on the fuzzy classification. For example, the book by Kuncheva [42] gives a broad overview of the topic. The vast majority of authors focus mainly on deriving fuzzy rules for the fuzzy classification from given data (e.g. in [43], [44], [66], [105], [13], and [56]). Various techniques from evolutionary algorithms to clustering are used to obtain a fuzzy rule base. Nevertheless, this is just the first step in tackling the problem.

When the fuzzy rule base has been determined (either derived from the data or defined expertly), it is necessary to use a proper method that would assign a class to the classified object according to this fuzzy rule base. Generally, this second step is often neglected in the literature. However, there are some authors who studied also this particular phase of the fuzzy classification. For example, Ishibuchi et al. [39] compared performance of various voting schemes for selection of the resulting class for the classified objects. They studied the voting schemes for both a single fuzzy rule-based classification system and for multiple fuzzy rule-based classification systems. In [12], the authors state that since the commonly used fuzzy reasoning method selects the resulting class for the given object only by taking into account the fuzzy rule with the greatest degree of association, the information given by the other fuzzy rules is lost. Therefore, the authors of the paper proposed several new fuzzy reasoning methods and tested their performance. The usage of different aggregation methods in the fuzzy classification is studied by Mesiarová-Zemánková and Ahmad [54] – specifically, the multi-polar OWA operators and multi-polar Choquet integral are considered for the fuzzy classification.

This thesis is dealing with the phase of solving a fuzzy classification problem when the fuzzy rule base is already known and it is necessary to assign the best-fitting classes to the classified objects. Various fuzzy classification scenarios can be encountered in the practice. Besides the classification in the common sense of the word, a classification whose purpose is an evaluation will be also considered in the thesis. For instance, assigning a country to one of the Moody's rating classes (Aaa, Aa, A, Baa, Ba, ..., C) can be perceived as an evaluation of this country. This thesis will provide a systematic study of the different fuzzy classification scenarios. The fuzzy classification problems will be divided according to the possible existence of relationships among the given classes, and according to the nature of this relationship. The conclusions have already been published in [37, 36].

The theory will be accompanied by the examples from the area of human resources management (HR management). The examples were chosen as simple as possible for a better clarity. An extension of the examples to more complex applications would be quite straightforward.

First, a few basic terms should be defined. More details can be found e.g. in the book [42].

Definition 3.1 *Let $\Re^m$ be the space of m-dimensional real vectors. These vectors describe the objects that should be classified. Let $\Omega = \{\omega_1, \dots, \omega_k\}$ be a set of class labels. Then, a crisp classifier is defined by a mapping*

$$D : \Re^m \to \Omega. \tag{3.1}$$

In case of the crisp classifier, every object is assigned to exactly one class and this classification is unambiguous. A more general case is represented by a possibilistic classifier [42].

Definition 3.2 *A possibilistic classifier, which classifies objects described by m-tuples of real numbers into k classes, is defined as a mapping*

$$D^p : \Re^m \to \Re^k \setminus \{\mathbf{0}\}, \tag{3.2}$$

where $\mathbf{0}$ denotes a k-dimensional vector of zeros.

For any object $\mathbf{x}$ that is described by m values of its characteristics, its membership degrees $D_1^p(\mathbf{x}), \dots, D_k^p(\mathbf{x})$ to each of the classes $\omega_1, \dots, \omega_k$ are obtained by the possibilistic classifier. The case that all of these membership degrees are zero, i.e. that the object would not belong to any of the classes, is excluded by this definition.

Although the definitions of crisp and possibilistic classifiers require all objects to be classifiable, in this thesis we also consider objects that cannot be classified well enough. A method for this case will be described. The classifiers of this type will be able to refuse the classification if its result could be misleading.

Concerning fuzzy classifiers, different interpretations of this notion exist. In a broad sense, a fuzzy classifier is any classifier that uses fuzzy sets either during its initial training (i.e. during deriving the fuzzy rule base from the data) or during the object classification itself [42]. In the thesis, we consider fuzzy classifiers in this broad sense. They share the common feature that linguistically defined fuzzy rule bases are used to describe the classes. However, it makes no difference whether crisp or fuzzy values of the characteristics are used for the description of classified objects, and whether the classification of the objects is unambiguous or ambiguous.

3.1. Specification of the problem of interest

In both science and in the real life it is common to classify objects into classes which are defined rather vaguely – by verbally specified values of the objects' characteristics. The pursued task is to assign an object, described either by crisp or by vaguely given values of its characteristics, to some of these classes; or more generally, to determine its location in relation to these classes.

In this text, we assume that two values are available for each of the classes – its numeric identifier and its verbal label. The numeric identifiers will be used for the calculations while the classes labels (the names of the classes) are necessary in order to be able to set the fuzzy classification model and present its results verbally. Both pieces of information should be unique for the class.

The classes will be described by means of a fuzzy rule base. On the left-hand side of each rule, there is a combination of linguistic variables values that defines a particular class. On the right-hand side of each rule, there is the label (name) of the class.

The output of a fuzzy classification system depends on whether we are solving the basic problem of object identification or whether we are classifying objects for the purpose of their evaluation. In the former case, the result is a single class for the object or information that the object cannot be classified well enough. The set of the used class identifiers can be viewed as a nominal scale. In the latter case, where classification is used as a certain kind of evaluation, the class identifiers can form either an ordinal scale or a cardinal scale and that affects the form of the classification results. In case of the ordinal scale, several neighboring classes (together with the membership degrees of the classified object to these classes) can be the fuzzy output of the classification. In case of the cardinal scale, the location of the object in relation to the classes can be calculated. Since the definition of the classes and potentially also of the object itself involves uncertainty, the idea of an uncertain classification of objects into classes is meaningful.

Three real-world applications of fuzzy classification will be shown. All of them originate from the area of human resources management on universities. In the first one, academic staff members are classified according the area on which they focus, i.e. if they achieve significant result in the area of pedagogical activities, in the research, or if neither of these two areas prevail significantly. Three classes are used in this fuzzy classification mode: *Researcher*, *Teacher*, and

Non-specific. The result of this classification can be used in the HR management – the superordinates can offer the academic staff members an option to engage in that area in which they show the best aptitude.

In the second application, we are trying to determine if the performance and composition of activities of a particular academic staff member corresponds to the position of an assistant professor, an associate professor, or a professor. The position determined in this way is then compared to the actual position of this particular academic staff member. The information can be used in the HR management to find promising academic staff members who are aspiring to a higher academic rank. Contrary to the previous example, the classes (*Assistant professor*, *Associate professor*, and *Professor*) do not form a nominal but an ordinal evaluation scale and therefore a different model of the objects classification is chosen.

In the last example, the academic staff members are divided into classes according to their overall performance [85]. The overall performance is calculated from their performance in the areas of pedagogical activities and R&D (research and development). This way an academic staff member is assigned to one of the performance classes: *Unsatisfactory*, *Substandard*, *Standard*, *Very Good*, or *Excellent*. For the classes the following numeric identifiers were chosen: 0, 0.5, 1, 1.5, and 2. These numeric identifiers express multiples of the actual performance of an academic staff member in comparison to the standard performance for his/her position. For example, the corresponding numeric identifier for the class *Substandard* is 0.5, which means that the typical representative of this class has just half of the performance expected for his/her position. Therefore, the class identifiers form a cardinal scale. This model has been implemented into an information system called IS HAP [75, 76, 36], which is currently successfully applied at faculties of 6 universities in the Czech Republic.

3.2. The fuzzy rule base used for the classification

The fuzzy rule base for the purpose of a fuzzy classification can have multiple forms. The most common ones are mentioned for instance in [12]. In this text, the following form will be used. The classes will be described by fuzzy rules. On the left-hand side of each rule, there are linguistic variables together with their linguistic values that specify the class of interest. On the right-hand side, there is a label of the class. It is possible to describe one class by multiple rules.

Let C be a set of numeric identifiers of the classes of interest, usually $C = \{1, \ldots, k\}$, $k \in N$. (Formally, it could be perceived that also on the right-hand sides of the rules, there is a linguistic variable *Class* – the labels of the individual classes comprise the linguistic values of this variable, and the numeric class identifiers then represent the mathematical meanings of these linguistic values.)

A fuzzy classification system can then be described by means of a fuzzy rule base in the following form:

$$\begin{array}{l}\text{If } \mathcal{E}_1 \text{ is } \mathcal{A}_{1,1} \text{ and } \dots \text{ and } \mathcal{E}_m \text{ is } \mathcal{A}_{1,m}, \text{ then the class is } \mathcal{D}_1\\ \text{If } \mathcal{E}_1 \text{ is } \mathcal{A}_{2,1} \text{ and } \dots \text{ and } \mathcal{E}_m \text{ is } \mathcal{A}_{2,m}, \text{ then the class is } \mathcal{D}_2\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \text{If } \mathcal{E}_1 \text{ is } \mathcal{A}_{n,1} \text{ and } \dots \text{ and } \mathcal{E}_m \text{ is } \mathcal{A}_{n,m}, \text{ then the class is } \mathcal{D}_n,\end{array}$$

where for $i = 1, 2, \dots, n, j = 1, 2, \dots, m$:

- $(\mathcal{E}_j, \mathcal{T}(\mathcal{E}_j), [p_j, q_j], M_j, G_j)$ are linguistic variables, usually linguistic scales, for the individual features of the objects;
- $\mathcal{A}_{i,j} \in \mathcal{T}(\mathcal{E}_j)$ are linguistic values of these variables, and $A_{ij} = M_j(\mathcal{A}_{i,j})$ are fuzzy numbers on $[p_j, q_j]$ modeling their meanings;
- $\mathcal{D}_i$ are the class labels and D_i are the corresponding numeric class identifiers, $D_i \in C$.

If the goal of classification is just an identification of the object as a member of one of the classes and if there are no relationships among the classes, or their relationships are not related to the problem being solved, then the scale formed by the numeric identifiers of the classes is considered to be the nominal one. In the next sections, fuzzy classification algorithms appropriate for this situation will be described. The result of the classification will be the number of the best fitting class for the classified object, or the information that the object cannot be classified unambiguously. This type of fuzzy classification will be illustrated on the example of determining the type of an academic staff member (see Section 3.3.1).

If the goal of classification is the evaluation of objects, it makes sense to assume that the numbers identifying the classes form an ordinal, or even cardinal scale. It is meaningful to permit also the case that the objects lie between two neighboring classes. Moreover, in the case of a cardinal evaluation scale, it is also reasonable to calculate the particular location of the objects between these classes. If we require a natural verbal description of both the evaluation process and a fuzzy classification results, it is suitable to use the Sugeno-Yasukawa model [79].

The great advantage of using the tools of linguistic fuzzy modeling in all of the mentioned cases is that the fuzzy classification rules and the final results are described in the most natural way for humans, i.e. verbally. This may seem important only for interpretation of the rules that have been generated automatically from data. However, the verbal description of fuzzy rule base that has been designed by an expert is no less important.

Specifically, in applications such as the academic staff performance evaluation, the model should be understandable for both the university management and the evaluated academic staff members themselves. A fuzzy rule base summarized for example as in the Figure 3.8 can be comprehended easily without any requirements on higher mathematical skills. The comprehension by the management and by the evaluated academic staff members themselves is also essential for wide acceptance of the model.

3.3. A real world application of the fuzzy classification models - IS HAP

The various types of fuzzy classification mentioned in the previous section will be illustrated by examples originating from the area of academic staff performance evaluation. Part of these methods (specifically, the method described in the Section 3.3.3) has been implemented in an information system called IS HAP. In the last year, the system has become widely used at the universities in the Czech Republic. Currently, it is applied at faculties of 6 universities for the academic staff performance evaluation. The system is designed so that it would not be limited only to the specifics of the Czech tertiary education system. Its adjustability makes it possible to be used also at the universities in different countries. The rest of the models described in the following sections could be implemented into this information system in the future.

Within the IS HAP system, the performance of each member of academic staff is evaluated in both pedagogical, and research and development (R&D) areas of activities. Input data are acquired from a form filled in by the staff where particular activities are assigned scores according to their importance and time-consumption. Three areas are taken into consideration for pedagogical performance evaluation: (a) lecturing, (b) supervising students, and (c) work associated with the development of fields of study. The evaluation of research and development activities is based on the R&D methodology of evaluation valid in the Czech Republic (papers in important journals, and patents are valued very highly) but other important activities (grant project management, editorial board memberships, etc.) are also included.

Both pedagogical and R&D areas are assigned standard scores – different for senior assistant professors, associate professors, and professors. The number representing a partial evaluation of a particular academic staff member in a certain area is determined as a multiple of the respective standard for his or her position.

For better clarity and easier interpretation, linguistic fuzzy scales are defined on the domains of the partial evaluations (see Figures 3.1 and 3.2). If, for example, the performance of an academic staff member in R&D is 1.25 times of the standard, using the scale in the Figure 3.2, it can be linguistically interpreted that the performance is *75 % standard and 25 % high*.

In the pedagogical area (see Figure 3.1), the evaluation of the activities is based namely on their time consumption; so the double of the standard performance is already considered to be an extreme performance. In R&D (see Figure 3.2), the mentioned methodology is used. Within this methodology, the evaluation of journals grows sharply with their importance; so the triple of the standard score is still achievable for the academic staff members.

In the following sections we are going to apply different types of fuzzy classification to answer the following questions that are important from the point of view of the HR management:

1. Is an academic staff member more teacher or researcher? In which area does he/she perform particularly well? Where should an additional space be granted to him/her for his/her further development?

2. Which working position corresponds to the behavior of the academic staff member – assistant professor, associate professor, or professor? Is he/she a promising academic worker whose behavior corresponds better to the higher academic rank than his/her real one?

3. What is the overall performance (i.e. the total performance according to the evaluations in the areas of pedagogical activities and R&D)?

These questions were deliberately chosen so that the different types of fuzzy classification scenarios could be shown in an illustrative way.

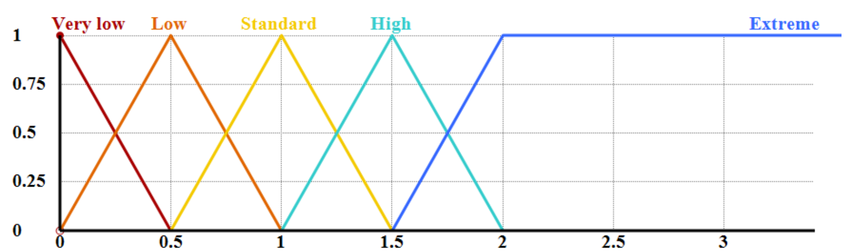


Figure 3.1: A linguistic scale used for the evaluation of academic staff members in the area of pedagogical activities

3.3.1. Is the academic staff member more teacher or researcher?

In this section, we will use a fuzzy classifier to solve an identification problem. The academic staff members will be classified according to the area where they

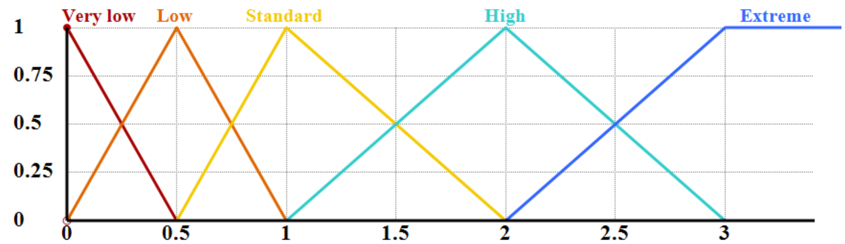


Figure 3.2: A linguistic scale used for the evaluation of academic staff members in the area of R&D

perform better. The possible classes are *Teacher*, *Researcher*, and *Non-specific*. The third class represents the academic staff members that have balanced evaluation in both of the evaluated areas. The knowledge of the type of an academic staff member can be used in human resource management at the university. If academic staff members perform significantly well in one area and have not-so-good results in the other, then their supervisor can give them more space to focus on that area of activities for which they are better suited.

This classification is based on evaluation of academic staff members in the areas of pedagogical activities and R&D. The designed fuzzy rule base is shown in Table 3.3.

		Research and Development Performance				
		Very low	Low	Standard	High	Extreme
Pedagogical Activities Performance	Very low	Non-specific	Non-specific	Non-specific	Researcher	Researcher
	Low	Non-specific	Non-specific	Non-specific	Researcher	Researcher
	Standard	Non-specific	Non-specific	Non-specific	Non-specific	Researcher
	High	Teacher	Teacher	Non-specific	Non-specific	Non-specific
	Extreme	Teacher	Teacher	Teacher	Non-specific	Non-specific

Figure 3.3: Fuzzy rule base used for determining the type of academic staff members

To these three classes *Teacher*, *Researcher*, and *Non-specific*, numeric identifiers 1, 2 and 3 are assigned. It is obvious that the class identifiers form only a nominal scale. Two classification algorithms suitable for this type of classification were tested — *Single Winner* and *Voting by Multiple Fuzzy Rules* [39]. Of course, there are also other methods that could be suitable for this type of problem (e.g. [12]).

In this section, it will be assumed that the fuzzy rule base has been designed so that for any object at least one of the fuzzy rules applies. In other words, the fuzzy rule base is designed so that it would cover the entire input space.

Single Winner

In the *Single Winner* method [39], the classification of objects is done as follows. Let us suppose that an object is described by the values of its characteristics, i.e. by real numbers $a_1, \ldots, a_m$. Moreover, we expect that a fuzzy rule base is given and that it is in the form described in Section 3.2. Then, the classification by *Single Winner* is done by the following procedure.

First, the degrees of correspondence h_i, $i = 1, \ldots, n$, between the inputs and the left-hand sides of the rules are calculated

$$h_i = A_{i1}(a_1) \cdot A_{i2}(a_2) \cdot \ldots \cdot A_{im}(a_m), \quad i = 1, \ldots, n. \tag{3.3}$$

The membership degrees in the Formula 3.3 express the fulfillment of the individual conditions on the left-hand sides of the rules. In [39], multiplication was applied for their aggregation. However, it is possible to use another t-norm (e.g. minimum) or even an averaging aggregating operator (e.g. a weighted arithmetic mean) instead. The choice of a suitable operator depends on the nature of the problem to be solved.

Let us note that the Formula 3.3 can be generalized for the case when the objects to be classified are described by fuzzy values of their characteristics. If an object is described by fuzzy numbers $A_1^{'}, \ldots, A_m^{'}$, then the degrees of correspondence h_i, $i = 1, \ldots, n$, can be calculated by the following formula

$$h_i = \mathrm{hgt}(A_1^{'} \cap A_{i1}) \cdot \mathrm{hgt}(A_2^{'} \cap A_{i2}) \cdot \ldots \cdot \mathrm{hgt}(A_m^{'} \cap A_{im}), \quad i = 1, \ldots, n. \tag{3.4}$$

In the next step, the so-called *number of votes* is calculated for each class as follows:

$$v_T = \max_{\substack{i \in \{1,\ldots,n\}: \\ D_i = T}} h_i, \quad T \in C. \tag{3.5}$$

The resulting class T^* for a given object is the one with the maximum value of v_T, i.e. the one for which it holds that

$$v_{T^*} = \max_{T \in C} v_T. \tag{3.6}$$

If there are more classes with the maximum number of votes, these ties are resolved usually randomly in practice. In the one of the following sections, a method for objects that cannot be classified unambiguously is proposed. This way, the result of the classification will be the information that this particular object cannot be assigned reliably to a single class.

Voting by Multiple Fuzzy Rules

In case of the *Voting by Multiple Fuzzy Rules* method [39], the degrees of correspondence h_i are calculated in the same way as with the *Single Winner*:

$$h_i = A_{i1}(a_1) \cdot A_{i2}(a_2) \cdot \ldots \cdot A_{im}(a_m), \quad i = 1, \ldots, n. \tag{3.7}$$

Next, the number of votes is calculated for each class as the sum of the correspondence degrees of those rules that voted for that particular class:

$$v_T = \sum_{\substack{i \in \{1,\dots,n\}: \\ D_i = T}} h_i, \quad T \in C. \tag{3.8}$$

The resulting class T^* is again the one with the maximum value of v_T.

Comparison on the given example

Figure 3.4 compares the results that were obtained by the two mentioned methods for the described example. Each dot in the figure can be perceived as an academic staff member with a particular value of performance in the area of pedagogical activities (x axis) and in the area of R&D (y axis). The resulting classes are differentiated by the color of the dots - black for *Teachers*, gray for *Researchers*, and white for the *Non-specific* academic staff members. It can be seen that the border between the classes is smoother for *Voting by Multiple Fuzzy Rules* method.

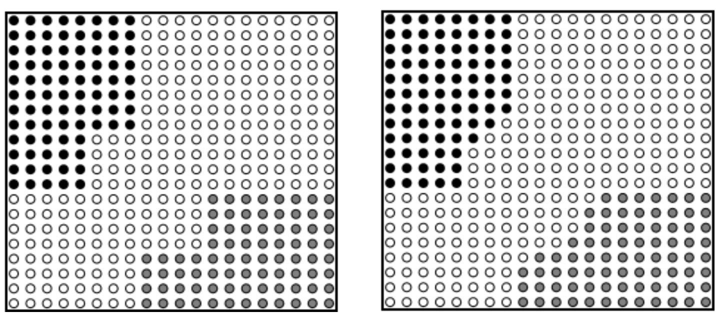


Figure 3.4: Results obtained by *Single Winner* (left) and *Voting by Multiple Fuzzy Rules* (right)

Objects that cannot be classified

In some cases, an object cannot be classified unambiguously, i.e., if it were classified, then its membership degree to the chosen class would not be significantly higher than its membership degree to some other class or classes. In the above example, academic staff members were classified into three classes. In case of the *Single Winner* method, the optimum class for an academic staff member was determined by the largest number of votes; in case of a draw, it would be possible to select any of them as the resulting class for the object. So far, we have not studied how reliable the assignment of the class to an academic staff member was. This will be discussed in the following text and a method for identifying unclassifiable objects will be described [37].

The boundary between classifiable and unclassifiable objects can be set by choosing a minimal required *distinctiveness of the winner*. The distinctiveness of the winner DW is a real number on $[0, 1]$ defined as

$$DW = 1 - (V'/V), \tag{3.9}$$

where

$$V = \max_{T \in C} v_T = v_{T^*}, \tag{3.10}$$

$$V' = \max_{\substack{T \in C: \\ T \neq T^*}} v_T. \tag{3.11}$$

In these formulae, V represent the number of votes for the winning class T^* and V' is the number of votes for the second best fitting class. According to the assumptions, the fuzzy rule base has to be defined so that the fuzzy rules would cover the entire input space. Therefore V is never zero.

If the distinctiveness of the winner is lower than the selected one, it means that the classification is ambiguous and the object belongs to more than one class in similar degrees.

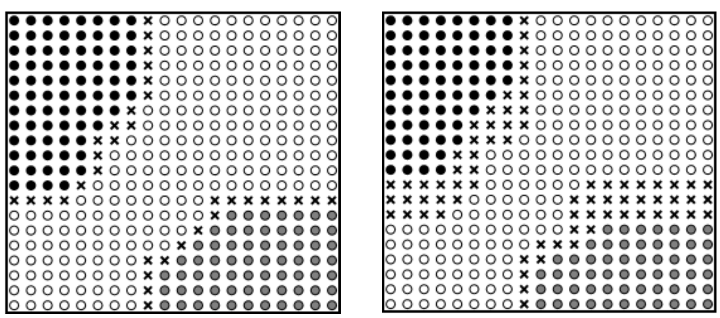


Figure 3.5: Results for $DW_{min} = 0.4$ (left) and $DW_{min} = 0.7$ (right)

Figure 3.5 shows the results for two different values of the minimal required distinctiveness DW_{min} that were obtained for the case of the *Single Winner* algorithm. The unclassifiable objects are denoted by crosses. For example, let us assume four academic staff members with different evaluations in the area of pedagogical activities and R&D. Tables 3.1 and 3.2 compare the results of fuzzy classification for these academic staff members. The proposed class for the first three is the same. However, for the last one, each of the algorithms proposed a different class. Notice that in this case the distinctiveness of the winner is significantly lower.

Pedagogical activities	R&D	Proposed class	Distinctiveness
0.9 (Standard)	2.8 (Extreme)	Researcher	0.75
2.1 (Extreme)	0.5 (Low)	Teacher	1
1.3 (High)	2.2 (High)	Non-specific	0.83
0.8 (Standard)	2.3 (High)	Non-specific	0.33

Table 3.1: Sample results of the academic staff members' fuzzy classification by the *Single Winner* method

Pedagogical activities	R&D	Proposed class	Distinctiveness
0.9 (Standard)	2.8 (Extreme)	Researcher	0.81
2.1 (Extreme)	0.5 (Low)	Teacher	1
1.3 (High)	2.2 (High)	Non-specific	0.92
0.8 (Standard)	2.3 (High)	Researcher	0.28

Table 3.2: Sample results of the academic staff members' fuzzy classification by the *Voting by Multiple Fuzzy Rules* method

3.3.2. Which working position corresponds to the behavior of the academic staff member – assistant professor, associate professor or professor?

In this example, another classification of academic staff members will be done. The result of this classification will be that working role (*assistant professor*, *associate professor*, or *professor*) which corresponds the best with the academic staff member's performance during the year. This information can be valuable for the HR management, especially if the performance of an academic staff member corresponds to a higher academic rank than his/her current one. The results can be used to find promising academic staff members who are aspiring to a higher rank. For example, if an assistant professor has the performance typical for the associate professor, the head of department should give him/her time to prepare for the higher academic rank (e.g. in form of a sabbatical).

For the classification, a fuzzy classifier based on a fuzzy rule base is again used. The classifier is applied only to those academic staff members whose performance in both evaluating areas (pedagogy and R&D) is at least acceptable. For each of the working roles (*assistant professor*, *associate professor*, or *professor*), one rule is present in the fuzzy rule base. The rules reflect the typical behavior of the representatives of these working positions. With a higher academic rank, a significant increase of the performance in R&D is expected. In the pedagogical area, with an increase of the academic rank, the focus is moved from lecturing to students supervising (diploma students, doctoral students) or to the work associated with the development of fields of study.

The classification of academic staff members into the above-mentioned classes represents a certain type of evaluation. To the individual classes (*assistant professor*, *associate professor*, or *professor*), numeric identifiers 1, 2, and 3 are assigned. Although the same numeric identifiers have been chosen as in the previous example, the situation is different. It is obvious that if an academic staff member is assigned to the class *professor*, it represents better results for him/her than if he/she would be assigned to the class *associate professor*. On the other hand, on the basis of the two characteristics used for the classification, it is not possible to quantify the distances between neighboring classes in the evaluation. The classes can only be ordered but their distances cannot be measured. This case represents the evaluation on an ordinal scale.

In the academic staff performance evaluation model implemented at the Palacky University in Olomouc, the standard score in R&D for professors was set as a double of the standard for associate professors, which is again the double of the standard score for assistant professors.

Concerning the evaluation of the pedagogical area, the standard scores for all three positions are the same, but they are acquired from different type of activities. The academic rank affects the ratio between supervising diploma and doctoral students and work associated with the development of fields of study on one side, and frontal teaching on the other side.

Two input variables will be used for the fuzzy classification model – R&D outcomes (shortly *R&D*) and prevailing pedagogical activities (shortly *pedagogics*). The first real-valued input variable is defined as the ratio between the achieved score in R&D and the standard score for the lowest of the three mentioned academic ranks. The second real-valued input variable is defined as the ratio between scores acquired by the particular academic staff member for supervising students and the work associated with the development of fields of study on one hand, and the lecturing on the other hand. The linguistic fuzzy scales for both variables are depicted in Figures 3.6 and 3.7. The final fuzzy rule base then looks as follows:

1. If (*R&D* is *low*) and (*pedagogics* is *teaching*), then class is *assistant professor*.
2. If (*R&D* is *medium*) and (*pedagogics* is *balanced*), then class is *associate professor*.
3. If (*R&D* is *high*) and (*pedagogics* is *students supervising*), then class is *professor*.

From the fuzzy rule base and the Figure 3.6, it can be seen that an associate professor is expected to have approximately twice higher R&D outcomes than an assistant professor. Similarly, a professor should have at least four times higher performance in R&D compared to an assistant professor according to the used linguistic variable.

In the area of pedagogical activities, professors typically focus more on supervising students. According to the linguistic variable in the Figure 3.7, this is interpreted as having at least twice as much scores for the supervising students compared to the other pedagogical activities And, vice versa, assistant professors are not expected to have more than 1/2 of their pedagogical activities scores gained for supervising of the students. Both of the used linguistic variables could be adjusted so that the meanings of the linguistic terms would be tailored to the particular university's needs.

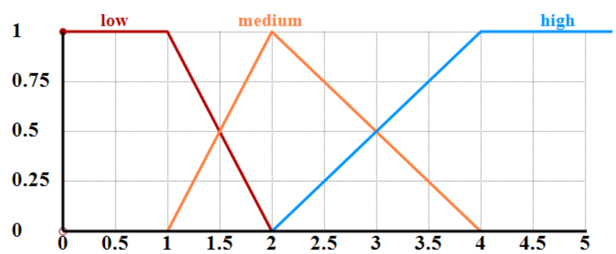


Figure 3.6: Linguistic variable for the evaluation of R&D outcomes

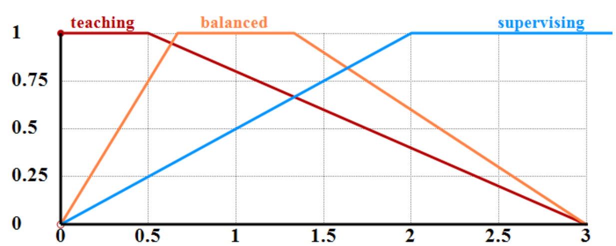


Figure 3.7: Linguistic variable for the prevailing pedagogical activities

It is obvious that the evaluating function defined by the fuzzy rule base is non-decreasing in both variables. Contrary to the previous classification problem, the numeric identifiers of the classes (1 – *assistant professor*, 2 – *associate professor*, and 3 – *professor*) form not just a nominal scale but an ordinal one. Therefore, the resulting information that we can obtain will be different. If the fuzzy rule base expresses an evaluation on an ordinal scale, then it is meaningful to expect that this linguistic evaluating function is non-decreasing in all its variables (similarly as in our example), providing that the input variables represent evaluations in the individual areas. Then it is obvious that an object can be (partially) assigned either to one class or to a sequence of mutually neighboring classes. The membership degrees to these classes are given by the Formula 3.5. The assignment of objects into such a sequence of several neighboring classes is meaningful especially in case of objects described by fuzzy values of their characteristics.

Concerning the linguistic description of the classification results for an object, two cases can occur. The object can have a non-zero membership degree just to one of the classes. Then the result of the evaluation will be the name of this class (e.g. *professor*). Alternatively, the object can have non-zero membership degrees to multiple neighboring classes $i, i+1, \ldots, j-1, j$. In this case, we say that this object belongs to the classes $\mathcal{A}_i$ to $\mathcal{A}_j$, where $\mathcal{A}_i$ and $\mathcal{A}_j$ are the labels of the i-th and the j-th class (the result of the fuzzy classification for a particular academic staff member would be, e.g., *associate professor to professor*). Alternatively, instead of considering non-zero membership degrees, a threshold could be set and only values of the membership degrees higher than this threshold would be assumed.

Similarly as in the previous section, the fuzzy classifier should be able to refuse to classify a particular object if the result of this classification would not be reliable enough. In this case, if the number of votes given by the Formula 3.5 is zero (or lower than a given threshold) for all of the classes, the result of the classification would be the information that this particular object cannot be classified well enough.

In the following section, the use of the fuzzy classification for an evaluation on a cardinal scale will be studied. It will be shown that in this case it is possible to apply the Sugeno or Sugeno-Yasukawa inference algorithms.

3.3.3. What is the overall performance of the academic staff member?

This application is the main part of the academic staff performance evaluation model that was developed at the Palacky University in Olomouc, Czech Republic [75, 85, 88]. In this application, several performance classes are defined for academic staff members. The classification is based on their evaluations in the areas of pedagogical activities and R&D, whose calculation is described in the introduction to Section 3.3. Since the evaluating scales used for these two areas differ in their character, the aggregation of these two partial evaluations is difficult. That is why a model that uses a fuzzy rule base (Figure 3.8) was designed. Specifically, the fuzzy classification based on the Sugeno-Yasukawa approach [79] was applied.

Concerning the calculation of the results, the Sugeno-Yasukawa approach is analogous to the Sugeno approach. However, on the right-hand sides of the rules, there are linguistic values. For the calculation of the fuzzy weighted average in the Sugeno-Yasukawa fuzzy inference algorithm real numbers are used. In the original Sugeno-Yasukawa approach, these numbers are the centers of gravity of fuzzy numbers that model the mathematical meanings of those linguistic values. In the modified version of this approach [75] that has been used for this application, the elements in the kernels of the triangular fuzzy numbers are used instead of the centers of gravity.

		Research and Development Performance				
		Very low	Low	Standard	High	Extreme
Pedagogical Activities Performance	Very low	Unsatisfactory	Unsatisfactory	Substandard	Standard	Very Good
	Low	Unsatisfactory	Unsatisfactory	Substandard	Very Good	Excellent
	Standard	Substandard	Substandard	Standard	Very Good	Excellent
	High	Standard	Very Good	Very Good	Excellent	Excellent
	Extreme	Very Good	Excellent	Excellent	Excellent	Excellent

Figure 3.8: Fuzzy rule base used for classification according to the overall performance of academic staff members

In this application, the numeric identifiers of the classes were defined as significant values on a continuous cardinal evaluating scale. In this case, to make interpretation easier, the class identifiers are not integers; numeric values 0, 0.5, 1, 1.5, and 2 were used. By fuzzification of these values, elements of the fuzzy scale were obtained. These elements were subsequently described by the linguistic terms *unsatisfactory*, *substandard*, *standard*, *very good*, and *excellent*. The original numeric values lie in the kernels of the triangular fuzzy numbers that form the fuzzy scale (see Figure 3.9).

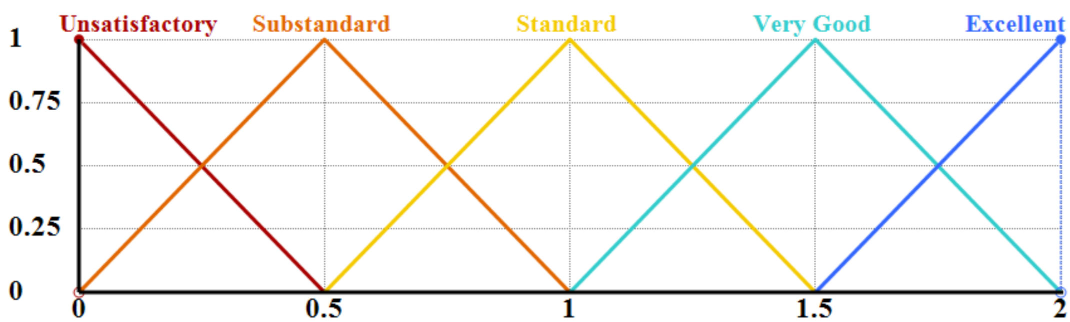


Figure 3.9: The linguistic fuzzy scale used for performance classes

The overall evaluation of academic staff members is calculated as follows. First, the rule base described in Figure 3.8 is modified so that instead of the linguistic terms stated in the table, there are only the significant values of the classes, i.e. the above-mentioned real numbers 0, 0.5, 1, 1.5, and 2, on the right-hand sides of the rules. Then, the Sugeno inference algorithm [78] is applied to the crisp evaluations of a given academic staff member in the area of pedagogical activities (pa) and R&D (rd). In this way, a crisp value of the overall evaluation ($eval(pa, rd)$) will be calculated. This procedure can be expressed by the following formula [75]:

$$eval(pa, rd) = \frac{\sum_{i=1}^{n} A_{i1}(pa) \cdot A_{i2}(rd) \cdot ev_i}{\sum_{i=1}^{n} A_{i1}(pa) \cdot A_{i2}(rd)}, \tag{3.12}$$

where for $i = 1, \dots, n$:

- A_{i1} is the fuzzy number representing the meaning of the linguistic term describing the evaluation in the pedagogical area in the i-th rule;
- A_{i2} is the fuzzy number representing the meaning of the linguistic term describing the evaluation in the area of R&D in the i-th rule;
- ev_i is the real number representing the most typical value of the linguistic term describing the resulting class $\mathcal{D}_i$ in the i-th rule (ev_i is the single element in the kernel of the respective triangular fuzzy number).

In [75], it has been proven for this fuzzy rule base that the denominator in the Formula 3.12 always equals to one, so a simplified formula can be used to calculate the overall evaluation of the academic staff member:

$$eval(pa, rd) = \sum_{i=1}^{n} A_{i1}(pa) \cdot A_{i2}(rd) \cdot ev_i. \tag{3.13}$$

From the numeric evaluation $eval(pa, rd)$ we proceed to the linguistic description of the result, which is more suitable in the context of HR management. For that purpose, we make use of the linguistic scale in Figure 3.9. If $eval(pa, rd) = ev_i$, for some $i = 1, \ldots, n$, then the academic staff member fully belongs to the class with the characteristic value ev_i and the linguistic interpretation of the result is clearly given by the corresponding term (e.g. *standard*). Otherwise, it belongs to the two nearest neighboring classes, i.e. to which the value $eval(pa, rd)$ belongs with a non-zero membership degree. Membership degrees of $eval(pa, rd)$ to these two classes are used for the linguistic description of the resulting evaluation. The sum of these membership degrees will be always one (this can be easily seen in the Figure 3.9), so they can be presented in form of percents. For example, a possible result can be that the overall performance of a given academic staff member is *70 % standard and 30 % very good*. Both the linguistically defined evaluation function and the real evaluating function given by the Formula 3.13 are non-decreasing in both variables. Because of this property of the fuzzy rule base, only a single class or several neighboring classes can have a non-zero weight in the Sugeno inference algorithm.

More information on this particular model can be found in [75, 85, 88].

3.4. Using FuzzME for fuzzy classification

The fuzzy classification problems can be also solved in the FuzzME software. The expert describes the classes by a fuzzy rule base first. Then, the *Single Winner* or *Voting by Multiple Fuzzy Rules* method can be applied to determine the class. The Figure 3.10 shows the model from the Example 3.3.1 designed in the FuzzME. The *Single Winner* algorithm is used for proposal of the academic staff member type. For a particular academic staff member, the numbers of

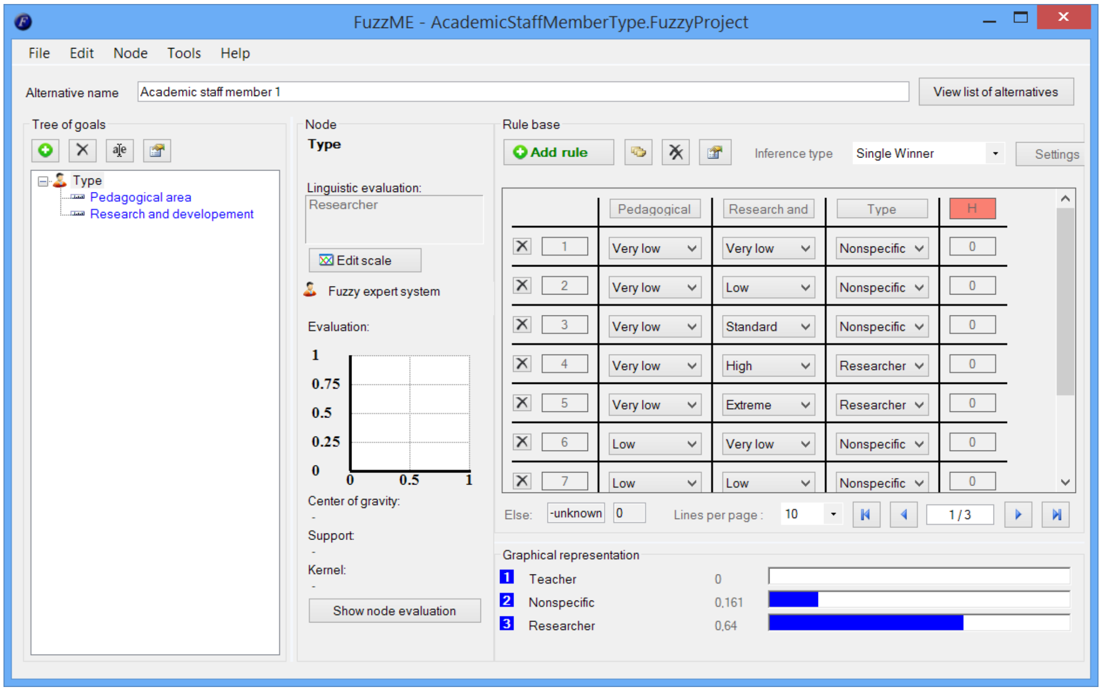


Figure 3.10: Fuzzy classification using the Single Winner algorithm in the FuzzME

"votes" for each of the classes are compared in the graphical form as it can be seen in the Figure 3.10. The minimal required distinctiveness of the winner can be also set in the FuzzME.

The classified objects are displayed in the FuzzME together with the proposed class (Figure 3.11). Besides its name, its numeric identifier is displayed. If the distinctiveness for the best fitting class is lower than the threshold selected by the user, the FuzzME displays *Unknown* as the resulting class, signalizing that, in case of the particular object, no reliable choice of a single class can be made.

If the classification has a form of an evaluation, then the Sugeno-WA (described in the Section 2.10.5) can be employed since it constitutes a special case of the classic Sugeno inference algorithm.

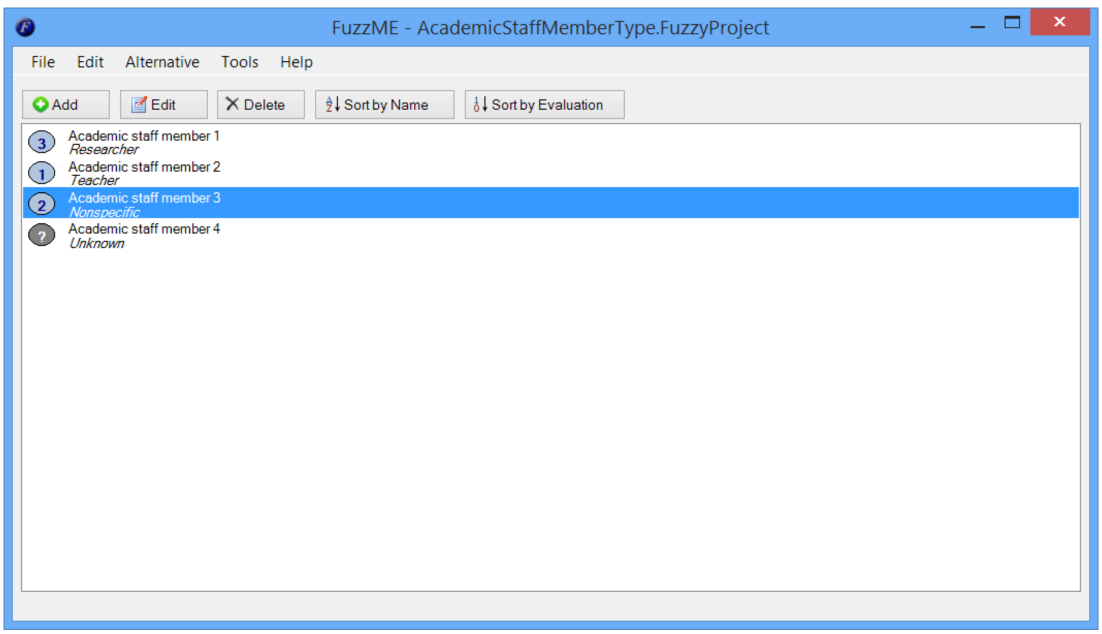


Figure 3.11: List of classified objects in the FuzzME

Chapter 4

A summary of the research accomplishments presented in the thesis

The thesis presents multiple original results achieved by the author during his doctoral studies of the Applied Mathematics at the Palacky University, Olomouc. These results were presented at international conferences and published in multiple peer-reviewed journals and conference proceedings. A summary of publication outputs can be found in the enclosed curriculum vitae (see Appendix 2); the author's contribution to them is specified at the end of this chapter.

These results are implemented in the FuzzME software developed by the author of this thesis that constitutes a universal multiple-criteria fuzzy evaluation tool. This software has been frequently used for research purposes by the research team focusing on the fuzzy MCDM methods at Palacky University, Olomouc (e.g. [25, 104]. Moreover, it has been applied also by foreign researchers (e.g. [3]). The software implements a complex system of fuzzy multiple-criteria evaluation methods. Even though the evaluation is its main application area, it can be also used for the fuzzy classification. According to the research of the resources available on the Internet, no other comparable software for the fuzzy multiple-criteria decision-making has been found (see Section 2.3.2 for more details). The extensive system of methods, as well as the number of functions for the users, make this software a unique tool in the area of the fuzzy MCDM.

The main original methods and results presented in the thesis are the following:

- the FuzzME software (see Section 2.13), which is a multiple-criteria fuzzy evaluation tool developed by the author suitable for research purposes as well as practical applications (`http://www.FuzzME.net`) – the software and its gradual development have been described in [31, 83, 34, 33, 32, 38], the practical applications of the software (see Section 2.14) have been published for example in [25, 104, 3];

- the fuzzification of the WOWA operator and study of its basic properties – specifically, the Definition 2.33 and the Theorems 2.2, 2.3, 2.4, 2.5, and 2.8 [38];
- the new Sugeno-WOWA inference algorithm (described in the Section 2.10.5, a real-world application of the algorithm can be found in the Section 2.14.1) [34, 38];
- the Algorithms 2.6 and 2.7 for transitions from a simpler to a more general aggregation method (the fuzzified Choquet integral or a fuzzy expert system) [35] including the Theorems 2.11, 2.12, and 2.13 verifying that the mapping obtained by the algorithms is a FNV-fuzzy measure;
- the proposal of division of the fuzzy classification methods according to the structure formed by the class identifiers and the discussion of the use of these methods in various applications contexts, especially with a focus on the use of the fuzzy classification for the purpose of evaluation [37, 36].

Many of the original methods presented in this thesis were created because the need for such a method has been identified during the solution of a practical problem – practical applications often required comprehensive study of the state-of-the-art methods and algorithms and inspired the author to develop new ones, hence creating new theoretical results. For example, the soft-fact rating problem described in the Section 2.14.1 was the impulse for proposal of the new Sugeno-WOWA inference, which turned out to be suitable for this application area.

The contribution of the author in the publications presenting the above-mentioned results that are listed in the enclosed curriculum vitae (see Appendix 2) can be summarized in the following way.

The author has co-operated on the writing of the paper [75]. He is the author of the academic staff performance evaluation model based on the use of the WOWA (weighted ordered weighted average) aggregation operator, which represents the most advanced model from the first class of models described in the paper. The second class of the models, which is currently used in the IS HAP information system (see Chapter 3), is based on the fuzzy rule bases and the author of these models is Jan Stoklasa. Pavel Holeček is the author of the software implementation for this second class of models.

He has also co-authored another paper in a journal with non-zero impact factor [6] dealing with the fuzzified Choquet integral. He has implemented the methods proposed in this paper and designed an effective software tool for multiple-criteria fuzzy evaluation based on the Choquet integration.

He is also the main author of another three papers in reviewed journals, which contain his original results. Specifically, the paper [31] introduces the first version of the FuzzME software and describes the methods used in the software. In the paper [34], the author proposes the Sugeno-WOWA inference algorithm. This inference algorithm makes it possible to derive the fuzzy evaluation by means of fuzzy rule bases but it also takes into account a vector of weights, which represent the optimism or pessimism of the decision-maker. In the paper [36], he studies systematically various fuzzy classification scenarios.

Pavel Holeček is the main author of the book chapter [38], which contains a broad overview of the topic of his Ph.D. thesis. In this book chapter, he has also introduced a new fuzzified WOWA aggregation operator. This fuzzified aggregation operator has been implemented into the FuzzME software. He is also a co-author of 2 other book chapters [87, 86] that, however, are outside the scope of this thesis.

He is the main author of 5 papers in peer-reviewed conference proceedings. They also contain original research results that comprise the basis for his Ph.D. thesis. For example, the paper [35] introduces a new group of methods and algorithms for transitions between different types of aggregation methods. They can simplify the whole process of designing the evaluation models as they make it possible to start with a simple mean of aggregation and they derive the settings for a more complex aggregation method. He is also a co-author of 4 other papers in conference proceedings.

The results of the thesis were presented on multiple conferences – specifically, 18 presentations on international conferences and 1 presentation on a national conference. The author has also been involved in several research projects and research internships. The detailed list of the publication outputs as well as other research activities can be found in the author's professional curriculum vitae, which is enclosed as the Appendix 2 to this thesis.

Chapter 5

Conclusion

The thesis dealt with a system of fuzzy methods for multiple-criteria evaluation and decision-making. The theoretical approach to evaluation is common to all the mentioned methods – the leading idea is that the evaluation of an alternative can be viewed as a (fuzzy) degree of fulfillment of a given goal. The individual steps in creating an evaluation model were described in detail and demonstrated on illustrative examples. The thesis provided exhaustive description of existing methods supplemented with several original methods. The software implementation of the whole system, the FuzzME software, has been described. Moreover, real-world applications of the FuzzME software and the mentioned system of fuzzy MCE methods have been presented. The applications, in the first place, showed the power and versatility of the described system as well as the FuzzME itself.

The second part of the thesis focuses on the topic of fuzzy classification. Three fuzzy classification problems were described. All of them originate from the same area – academic human resource management. Each of the problems represents a different fuzzy classification scenario. The first case can be perceived as an identification problem when it is necessary to decide to which of the classes a given object belongs (or alternatively, to decide that the object cannot be classified satisfactorily). No relationships among the classes were considered in this case; their identifiers formed a nominal scale. In the second and third cases, fuzzy classification was applied to solving evaluation problems. An ordinal evaluating scale was used in the second case, whereas a cardinal evaluating scale was applied in the third one. Suitable mathematical models were described for both of them. The identification and the evaluation represent two typical problems where fuzzy classification is applied in practice.

A substantial part of the work is also the FuzzME software enclosed to this thesis on a CD. It represents a unique tool for fuzzy MCDM that made it possible to apply these methods in the practice and to study their behavior and performance on real-world applications.

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Appendix 1

The user guide for the FuzzME 2.3

Contents

Software description

FuzzME is a tool for creating fuzzy models of multiple-criteria evaluation and decision making. It was developed at the Faculty of Science at Palacký University Olomouc by Mgr. P. Holeček, doc. RNDr J. Talašová, CSc., RNDr. O. Pavlačka Ph.D. and Mgr. I. Bebčáková, Ph.D.

In the FuzzME software, both quantitative and qualitative criteria can be used. For the aggregation of partial evaluations, any of the following methods can be utilized:

- Fuzzy weighted average,
- Fuzzy OWA operator,
- Fuzzified WOWA operator,
- Fuzzy Choquet integral,

- Fuzzy expert system.

The software makes it possible to evaluate a set of alternatives and, subsequently, compare them according to the centers of gravity of their evaluations.

The name of the software itself is an abbreviation of "**Fuz**zy Methods of **M**ultiple **C**riteria **E**valuation". The FuzzME is available in Czech and English versions.

Apart from multiple criteria evaluation, the FuzzME can be also used to solve fuzzy classification problems. Such a problem is then described by a fuzzy rule base and it is solved by one of two available methods - Single Winner or Voting by Multiple Fuzzy Rules.

In the following text, it is expected that the user is familiar with the theoretical background and the methods that are used in FuzzME.

Installation

Hardware and software requirements

The FuzzME has the following software requirements:

- Microsoft Windows XP, or newer,
- Microsoft .NET framework 2.0, or newer.

The Microsoft .NET framework is required by FuzzME. It is pre-installed on all computers with Windows Vista, Windows 7, and Windows 8. However, it is possible that users with the older Windows XP will have to install this component manually. In this case, the .NET framework can be downloaded from the Microsoft website.

The minimal hardware requirements are as follows:

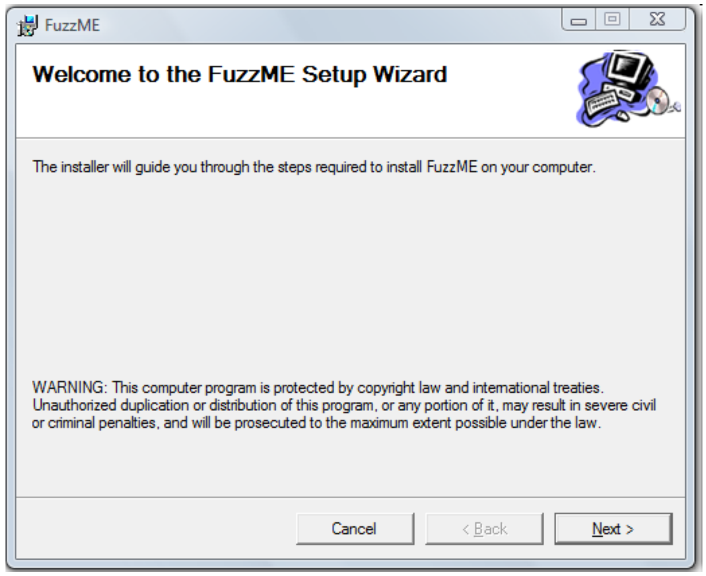
- screen resolution at least 1024 x 768 pixels,
- the other hardware requirements are the same as the hardware requirements of Windows XP.

Installation

The installation should be started automatically after the CD is inserted. If this function is disabled in the Windows, you can start the installation manually. To do that, open the CD in Windows Explorer and double-click on the file *setup.exe*.

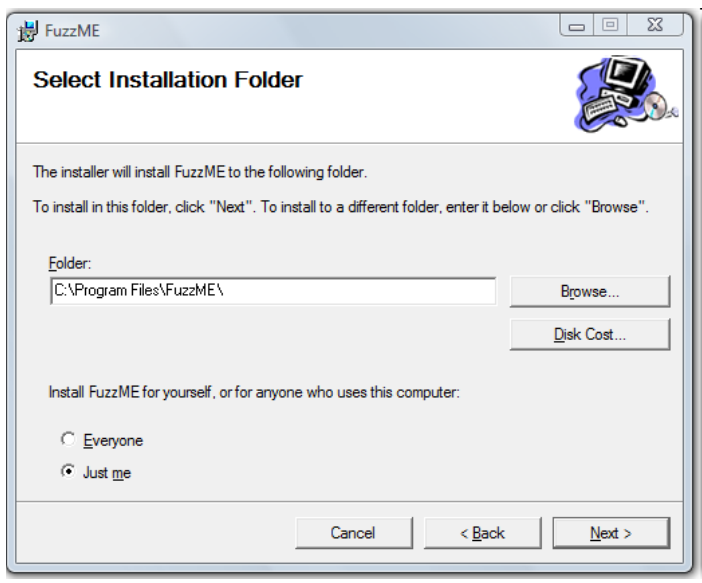
The installer first checks if the .NET framework is installed on the computer. If this component is missing, the installer will attempt to download it and install it from the web. For the installation of the .NET framework, administrator rights and internet access are required. If the installer will not be able to download the component automatically, you should download .NET framework manually from the Microsoft website, or find and install it from Windows Update service. However, the .NET framework is packed together with Windows Vista and the newer versions, so it has already been installed on most of the computers.

On the first screen, the installer displays the basic information. Continue by clicking on *Next*.



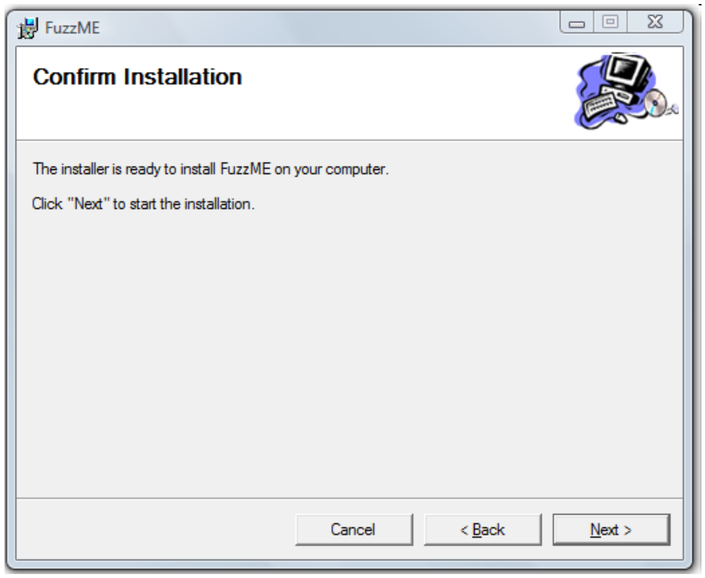
Installation – step 1

On the second screen, you can change the location where FuzzME should be installed. Proceed by clicking on *Next*.



Installation – step 2

Start the installation by clicking on *Next* button. After successful installation, you should see the FuzzME shortcut on the desktop and in the Start menu.



Installation – step 3

Note: It is also possible to use FuzzME without installing it your computer. In this case, simple double-click on the *FuzzME.exe* file in the Windows Explorer. On Windows XP, it is possible that .NET framework is missing on the computer. In this case, an error message will appear when the FuzzME is started. To solve this problem, you have to install the .NET framework first.

Note: FuzzME is developed and tested on the Windows operating system. To use FuzzME on Linux, you have to install .NET framework for Linux by Project Mono. Since the FuzzME is developed for Windows, it cannot be guaranteed that all function will work properly on Linux.

Uninstallation

If the software was installed properly, it can be uninstalled from the Windows in the following way:

1. Click on *Start*, *Control Panel*, and then double-click on *Add or Remove Programs* (the name differs depending on the operation system version).
2. Select FuzzME from the list.
3. Click on Remove.

Note: It is also possible to use FuzzME without installing it on the computer. In this case, simply delete the program folder to remove the software.

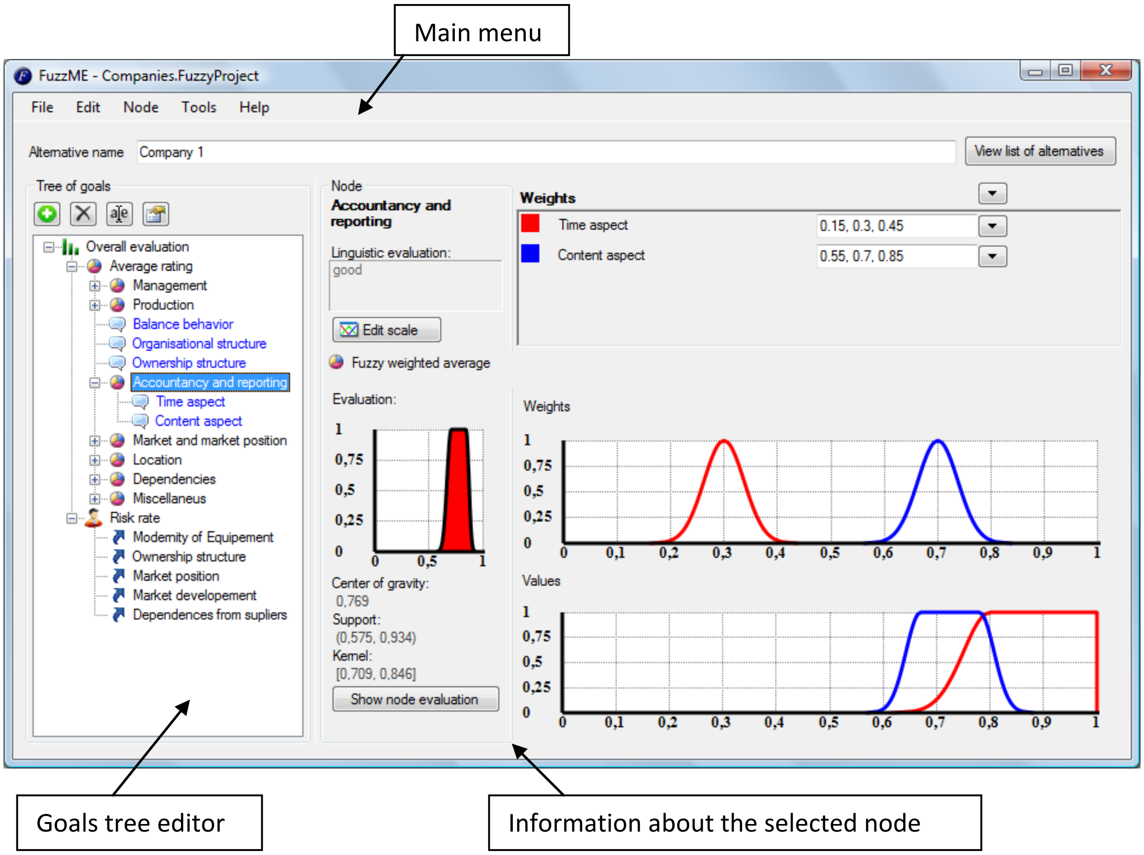
Launching the FuzzME

After the installation, the FuzzME can be launched by double-clicking on its icon on the desktop or in the Start menu. FuzzME can be also launched by double-clicking on the file *FuzzME.exe* in Windows Explorer.

Main window

Main window

The main window of the program is divided into several parts.



On the left side of the window, there is a goals tree editor. In this editor, the structure of the goals tree is displayed and can be modified. The user can select a node of the tree by clicking on it.

On the right side of the windows, the information about the selected node is displayed. The user can see the node name, evaluation of the partial goal corresponding to the node, and other node parameters. The parameters differ according to the type of the selected node.

The top part of the main window is occupied by the main menu and the name of the selected alternative.

Main menu

The main menu contains the following items.

File

- **New** – It creates a new empty project.
- **Open** – It opens a project.
- **Save** – The edited project is saved into a file.
- **Save As** –The edited project is saved into a file under the name selected by the user.

- **Recent Projects** – Contains the list of recently opened projects. They can be opened again by clicking on them.
- **Import** – This item shows a dialogue for import of the alternatives and their criteria values into FuzzME.
- **Export** – This menu item contains dialogues for export of the data from a FuzzME project. It makes it possible to export alternatives (their names together with their criteria values) or the final evaluations of the alternatives. If a fuzzy expert system node is selected then there is a possibility to export the fuzzy rule base into format used by Matlab.
- **Exit** –The program is closed by clicking on this item.

Edit

- **Cut Node** –The selected node is copied into the clipboard and then removed from the goals tree.
- **Copy Node** – The selected node is copied into the clipboard so that it could be pasted later into another node of the tree, or into a different project.
- **Paste Node** – A node is pasted from the clipboard. It will be pasted as a child node of the selected node.

Note: When the list of alternatives is opened, the Edit menu contains items for clipboard operations with the alternatives instead. The names and meaning of these items are similar and therefore will not be mentioned here.

Node

This menu is displayed only when the goals tree is visible.

- **Add Subnode** – A new node will be created (as a child node of the selected node).
- **Delete** –The selected node will be deleted.
- **Rename** – It makes it possible to change the name of the node. The user can type a new name. The editing is ended by pressing *Enter* or canceled by pressing *Esc*.
- **Change type** – It makes it possible to change the type of the node. A dialog with all supported node types is displayed to the user.
- **Other node operations** – This menu contains some less frequently used operations. They are described in the sections Goals Tree Editor and Link to another criterion.
- **Edit Linguistic Scale** – It displays the linguistic scale editor. The user can create a new scale for the selected node or edit the existing one.
- **Scale Type** – In this menu, user can choose one of the linguistic scale types. This choice will be used to obtain the linguistic description for the evaluation of the node.
- **Weights** – The menu contains items specific for nodes that require weights as a parameter. The items of this menu allow saving the weights into a file, and vice versa, loading them from a file. There are also items for creating the weights.
- **FNV-fuzzy measure** – The menu is specific for the fuzzy Choquet integral node. See Choquet integral for more information.
- **Rule Base** – The menu is specific for the fuzzy expert system node. See Fuzzy expert system for more information.
- **Show Node Evaluation** – It opens a window with details on the evaluation of the selected node.

- **Compare inputs and result** – It opens a window where all input values and the output value of the selected node are displayed.
- **Compare results for various inputs** – For the selected node, it allows to visualize the changes of its evaluation depending on various input values.

Alternative

This menu is displayed only when the alternative list is visible.

- **Add** – A new alternative is created.
- **Edit** – The selected alternative is edited.
- **Delete** – The selected alternative is deleted.
- **Delete All** – All alternatives in the project are deleted.
- **Show Alternative Evaluation** – It opens a window with details on the evaluation of the selected alternative.
- **Recompute All Evaluations** – It clears all caches for evaluations and it performs the evaluation of all the alternatives in the project again.
- **Find** – The function can be used to find an alternative by its name.
- **Sort by Name** – The alternatives list will be ordered by the alternative names.
- **Sort by Evaluation** – The alternatives list will be ordered by the alternative evaluation (according to the centers of gravity).

Tools

- **Mode** – This item switches between the goals tree view and the alternative list view. It can be also accomplished by pressing *F7* key.
- **Project Statistics** – It opens a window with basic information about the opened project (e.g. the number of nodes and the number of alternatives).
- **Options** – It opens a window with the program settings.

Help

- **User Guide** – It opens this user guide.
- **Mathematical Methods used in FuzzME** – It opens a paper with some basic information on the mathematical model and methods used in FuzzME.
- **About** – It shows a window with basic information about the FuzzME (e.g. current version number).

How to create a model in FuzzME

A model in FuzzME is described by a goals tree. The tree has to be designed and all required parameters for the nodes have to be set.

Steps of creating the model

A model is created in the following steps:

1. **Designing the structure of the goals tree** – First, the expert defines the structure of the goals tree. All the nodes are created and named properly.

2. **Designing the linguistic scales for tree nodes** – A linguistic scale is required for qualitative criteria and for the fuzzy expert system. This step is not mandatory for other node types, however, it is highly recommended.
3. **Defining type of each node in the goals tree** – The expert determines the type of each node in the goals tree. The nodes at the end of the branches are criteria. The expert chooses if the criterion is qualitative or quantitative (or alternatively a link to another criterion can be defined). The rest of the nodes are aggregating nodes. The expert has to choose the aggregation method that will be used.
4. **Setting parameters of the node** – Finally the expert sets parameters for each node. The parameters depend on the node type, which was chosen in the previous step. For example, normalized fuzzy weights have to be set for fuzzy weighted average.

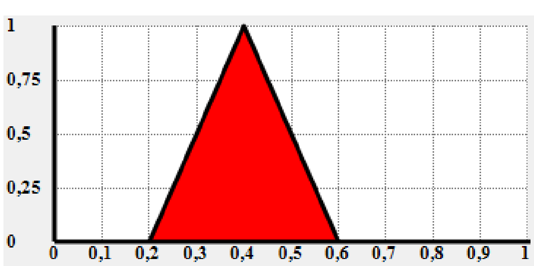
When those steps are performed, the model is created. The expert can proceed to evaluation of the alternatives. This process is done by the following steps:

1. **Adding or importing the alternatives** – The alternatives can be added either manually or they can be imported, e.g., from Excel.
2. **Evaluation of the alternatives** – When a new alternative is added, its evaluation is calculated automatically. The expert can have the alternatives ordered according their evaluations.
3. **Export of the results** – It is possible to export the evaluation results, e.g. into Excel, for their further analyze, or for their presentation.

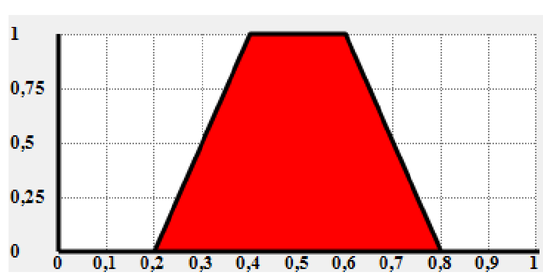
Using fuzzy numbers in FuzzME

Different mathematical software products use different notations for fuzzy numbers. For example in Matlab, a space is used to separate the significant points of fuzzy numbers. The fuzzy numbers in FuzzME are written by the user as a **list of the significant points separated by comas**. This notation is used in the whole software. There is always possibility to use a real number or an interval, as they are the special cases of fuzzy numbers. The notation should be clearer from the following examples.

Notation	Meaning
0.8	A real number 0.8.
0.2, 0.4	An interval 0.2 to 0.4.
0.2, 0.4, 0.6	A triangular fuzzy number
0.2, 0.4, 0.5, 0.6	A trapezoidal fuzzy number, whose support is (0.2, 0.6) and whose kernel is [0.4, 0.5].



A triangular fuzzy number.

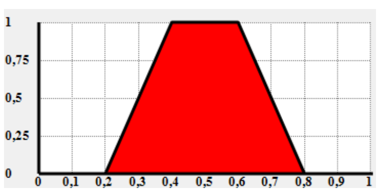


A trapezoidal fuzzy number.

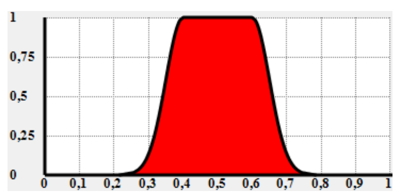
Note: Because the numbers are the significant values, they are always in ascending order. Otherwise, the fuzzy number is not valid and a warning can appear. An example of such an invalid notation of fuzzy number can be "0.1, 0.3, 0.2".

Input box for fuzzy numbers

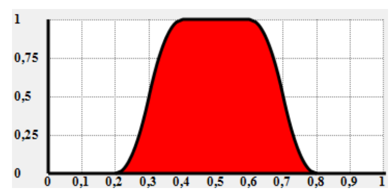
There are several membership functions that can be used. The membership function type can be chosen from the drop-down box on the right-hand side. The chosen type influences the shape of the membership function. The different membership function types can be seen in the following figures.



Linar fuzzy number.



Gaussian fuzzy number.



Quadratic fuzzy number.

Note: The FuzzME works with piecewise linear fuzzy numbers. They are used to approximate more complex fuzzy numbers (e.g. Gaussian or Quadratic). The preciseness of this approximation (and other calculations with fuzzy numbers) depends on the number of used α-cuts, which can be set in *Options*.

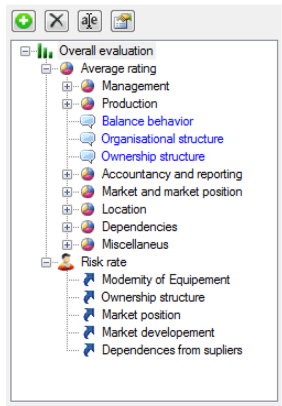
Note: The user can set which character should be used to separate significant points in the *Options*. It is possible to use either a comma (default), or a space (as in the Matlab).

A fuzzy number can be also edited in the Fuzzy number editor in FuzzME. This can be done by clicking with the right mouse button on an input box for fuzzy numbers and selecting *Show in Fuzzy Number Editor*. Now you can edit the individual significant points and see the characteristics of the edited fuzzy number (e.g. center of gravity, uncertainty, etc.). You can also choose the number of α-cuts that will be displayed (from the menu *View*). Another interesting feature is creating the fuzzy number by drawing it by mouse. This can be done by choosing *Edit | Design the Fuzzy Number by Drawing* from the main menu. Then press the left mouse button and start drawing the fuzzy number into the graph. The FuzzME checks the condition common for all fuzzy numbers, so the left part of its membership function has to be non-decreasing and its right-part has to be non-increasing. Release the mouse button when you have finished the drawing. The original fuzzy number will be replaced with the drawn one.

Designing the goals tree

Creating the structure of the goals tree

The structure of the goals tree is displayed on the left side of the main window. In this step, the structure is designed and the names are given to the nodes.



The user can do the following operation:

- **Select node** – The node is selected simply by clicking on it. In the right-hand part of the main windows, the node details are displayed.
- **Add node** – When a new node should be added, the user select one of the nodes in the goals tree first. The new node will be a child node of the selected one. Then, the user can either click on the corresponding icon, or choose *Node | Add subnode* from the main menu. After that, it is possible to type name of the node. The editing of the node name is ended by pressing *Enter*.
  Note: It is not possible to add a child node to a criterion since the criteria should always be the terminal nodes.
  Note: Initially, the new nodes have undefined type (which is signalized by a red question mark icon). The type can be selected in the right-hand part of the main windows. The type can be also changed later.
- **Delete node** – The user can delete a node by selecting it and clicking on the corresponding icon or choosing *Node | Delete* from the main menu. If the node has any child nodes, they will be deleted, too.
- **Rename node** – To rename a node, the user selects it and then clicks on the corresponding icon or chooses *Node | Rename* from the main menu (or presses *F2* key). The new name can be typed and subsequently confirmed by pressing *Enter*.
- **Moving the node** – The node can by copied or moved to another place in the goals tree. This can be performed by standard drag & drop technique (i.e. left mouse button is held, the mouse pointer is moved to the new location and then the mouse button is released). The node can be also copied through clipboard. The user selects the node of interest, chooses *Edit | Copy node* from the menu, selects place, where the node should be copied and, finally, chooses *Edit | Paste node* from the menu.

- **Choosing another node type for the node** – The type of each node can be changed any time. To do that, user selects the node and then clicks on the corresponding icon or chooses *Node | Change type* from the main menu.
- **Making a node to be a root of the goals tree** – Sometimes the expert needs to choose another node to be the root of the goals tree. This can be accomplished by selecting the node and choosing *Node | Other Node Operations | Make Selected Node to Be Root* from the main menu. This function can be also handy, when the expert wants to extend the goals tree and add a node above the current root node. In this case a new node is added anywhere and then selected as the new root of the tree.

Designing a linguistic scale for the nodes

Linguistic scales and linguistic approximation in FuzzME

Linguistic scales are used for the linguistic approximation in FuzzME. The scale has to be defined for all qualitative criteria (since they are evaluated verbally). It also has to be defined for fuzzy expert system and its child nodes (the fuzzy rule base used in the fuzzy expert system is defined verbally). For the other nodes, it is not necessary to define such a scale, however it is highly recommended. If the scale is defined, the user can see also the linguistic evaluation of the node.

On the right-hand side of the window, just under the node name, the linguistic evaluation of this node is displayed. Under this evaluation, there is an *Edit scale* button, which opens the Linguistic Scale Editor and makes it possible to create or edit the linguistic scale for the selected node.

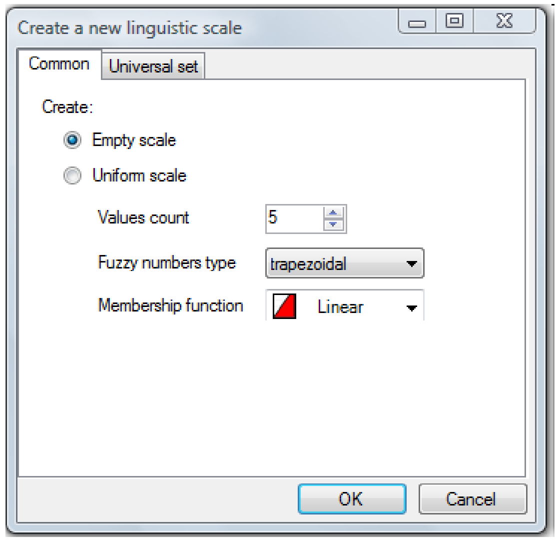
After the scale is defined, the user can set the mode of the linguistic approximation for the given node. The mode can be selected in the *Node | Scale Type* menu. There are three possibilities:

1. **Simple** – Only terms from the scale will be used.
2. **Extended** – The scale is extended by terms in form “A to B”, where A and B are the elementary terms from the original scale.
3. **Scale with Intermediate Values** - The scale is extended by terms in form “between A and B”, where A and B are the elementary terms from the original scale.

Linguistic Scale Editor

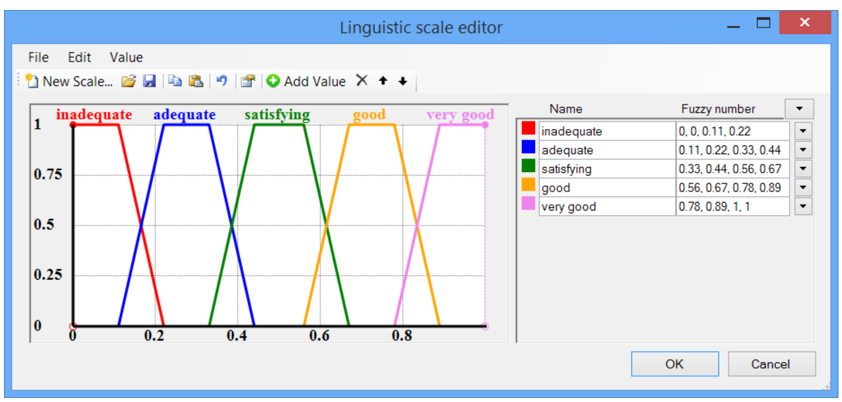
The linguistic scale for the selected node can be edited in the Linguistic Scale Editor. The editor can be opened by clicking on *Edit scale* button or by choosing *Node | Edit Linguistic Scale* from the main menu.

If the scale has not been created yet, a dialog for creating a new scale appears. The user can create a uniform scale with a minimal effort. It is necessary to fill in just the number of scale items. The user can also select the required shape of the fuzzy numbers which model the scale items. The dialog is then confirmed by clicking on *OK*.



Note: The scale is defined on the interval 0 to 1 by default. For more advanced users, there is a possibility to change this universe. On the *Universe* tab in the previous dialog, they can modify the minimal value, the maximal value and the scale type.

After the scale is created, the main window of the Linguistic Scale Editor appears.



On the left side of the window there is a graphical representation of the scale items. On the right side, there is a list of the scale items – their names and fuzzy numbers, which model them. When a new scale is created, its items are named “a”, “b”, “c”, etc. by default. The user should edit this texts and give them proper names (e.g. “inadequate”, “average”, “good”, etc.).

The user can perform the following operation with the linguistic fuzzy scale and its items. All of them are accessible through the main menu:

- **Create a new scale** – A new scale for the selected node is created. This can be done from the main menu by selecting *File | New scale*.
- **Open the scale from a file** – The scale for the selected node is imported (loaded) from a file. This can be done from the main menu by selecting *File | Open*.

- **Save the scale into a file** – The edited scale is exported (saved) into a file by clicking on *File | Save*.
  Note: This option can be handy if you want to use the same (or very similar) linguistic scale for multiple nodes of the goals tree. You just have to design the scale once and save it to a file. Then, open the Linguistic Scale Editor for the other nodes and load this scale.
  Note: This function serves for exporting the fuzzy scale, not saving the changes made in the edited fuzzy scale. The changes for the currently edited scales are saved simply by clicking on the *OK* button of the Linguistic Scale Editor.
- **Edit scale universal set** – This function opens a dialog for setting the scale universal set (which is the interval 0 to 1 by default). All the values of the scale will be recalculated so that they would fit the new universal set.
- **Undo** – This function undo the last modification of the scale made by the user (it returns one step back)
- **Copy scale into the clipboard** – The edited scale is copied into the clipboard (*Edit | Copy scale*). This makes it possible to use the same scale for different nodes without saving it into a file.
- **Paste scale from the clipboard** – The fuzzy scale is pasted from the clipboard (*Edit | Paste scale*).
- **Add a new value to the scale** – A new value is added to the linguistic scale (*Value | Add value*, or the corresponding button on the toolbar). The value appears in the list on the right-hand side of the window. The user has to fill the name of the value and the fuzzy number that models the value.
- **Delete the value** – The selected (edited) value of the linguistic scale is deleted (*Value | Delete value).*
- **Move the value up or down in the list** – The value is moved one position up or down in the list (*Value | Move up*, or *Value | Move down).*

The editor can be closed and the changes are confirmed by clicking on OK. By clicking on Cancel, the editor is closed and the changes are not saved. If one of the fuzzy numbers is not set by the user properly, a warning appears. See Using fuzzy numbers in FuzzME for more information on the correct notation of the fuzzy numbers in FuzzME.

Specifying the criteria

Setting type of a criterion

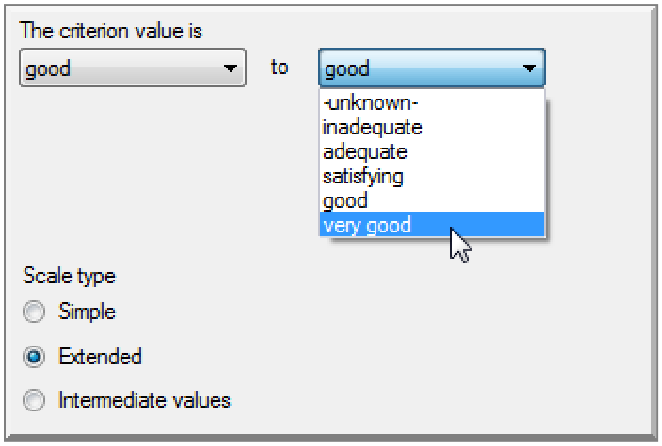
After the structure of the goals tree is designed and the linguistic fuzzy scales are defined, the user can set the type for all the criteria. The criteria are at the ends of the goals tree branches. There are two possible types – qualitative and quantitative criteria. Besides that, the user can also create a link to another already defined criterion.

First, select the node in the goals tree. If the type of the node was not chosen yet, a dialog with the supported types appears in the right-hand part of the main windows. Choose qualitative criterion, quantitative criterion or link to another node by clicking on the corresponding button in the dialog. The type of the criterion can be changed in the future.

Then, set the parameters specific for the selected criterion type.

Qualitative criterion

According to qualitative criteria, the alternatives are evaluated verbally. The expert chooses the best fitting term from the linguistic scale. Make sure that the linguistic scale was defined for the

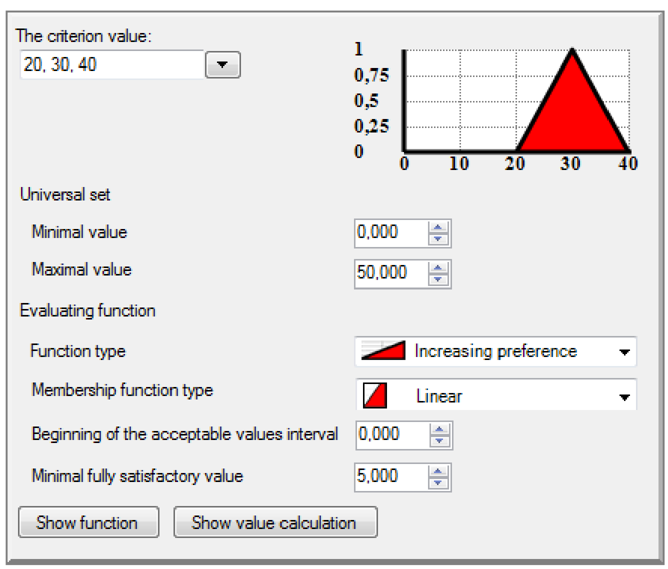


qualitative criterion.

The dialog for a qualitative criterion contains drop-down box (or boxes) where the value of the criterion for the selected alternative is chosen. Underneath them, the user can set the type of the used scale – simple, extended, or a scale with intermediate values. For more details, see Linguistic fuzzy scales and linguistic approximation in FuzzME.

Quantitative criterion

For a quantitative criterion, its evaluating function has to be defined.



The dialog contains the following items:

- **Criteria value** – The value of the criterion for the selected alternative. The value can be fuzzy or it can be a real number.
- **Universal set** – The universal set for the selected criterion, which determines the minimal and the maximal possible values of this criterion.
- **Evaluating function** – To define the evaluating function, the user chooses the type of the function first. The type can be
  - **Increasing preference** - The greater values of the criterion, the better.
  - **Decreasing preference** - The lower values of the criterion, the better.
  - **Preference of selected values** – The user can specify a value (or interval of values) which are fully satisfactory and the values which are not satisfactory at all.
  - **Other** – The shape of the function is set manually (in form of a fuzzy number). For advanced users only.

  Next, the user sets the parameters which differ according to the selected type of the evaluating function (for example "Minimal fully satisfactory value of the criterion"). The user can also choose type of the evaluating function (Linear, Gauss and Quadratic).

There are two buttons at the bottom of the dialog

- **Show function** – The graph of the evaluating function is displayed.
- **Show value calculation** – The graphical representation of the criteria evaluation calculation is displayed. The image depicts the calculation of the value according to the extension principle.

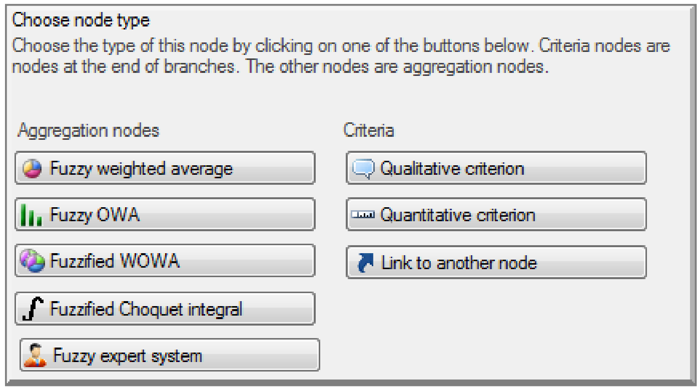
Link to another criterion

On some rare occasions, it is necessary to use one criterion multiple times in the same goals tree. In this case, the criterion is defined just once and then a link to this criterion is used. The user selects the linked criterion from the drop-down list. The node will always have the same evaluation as the linked criterion.

The aggregation nodes

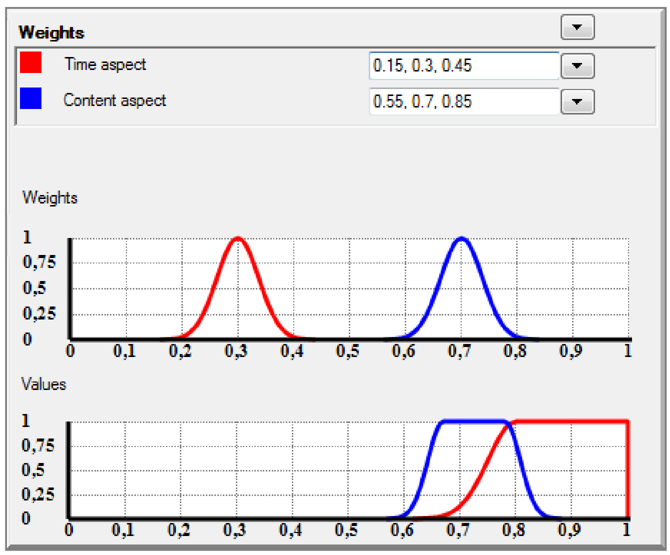
Setting type of an aggregation node

Setting the type of an aggregation node is done in the same way as setting the type of a criterion. First, select the node in the goals tree. If the type of the node was not chosen yet, a dialog with the list of supported types appears in the right-hand part of the main windows. The user chooses the appropriate node type by clicking on the corresponding button in the dialog. The supported aggregation types are: fuzzy weighted average, fuzzy OWA operator, fuzzified WOWA operator, fuzzy Choquet integral and fuzzy expert system. The type of any node can be changed in the future.



Fuzzy weighted average

For fuzzy weighted average, the normalized fuzzy weights have to be set. Each weight is expressed by a fuzzy number (a real number as a special case of a fuzzy number can be also used).



The FuzzME checks all the conditions that the fuzzy weights have to satisfy. If they do not do that, one of the following warnings appears:

- **Weights are not set** – This warning is displayed when the node has just been created and the weights have not been set yet.
- **The weights are not correct** – One or more of the weights are not correct fuzzy numbers. Check if they are written in the correct format. See Using fuzzy numbers in FuzzME for more information.
- **The weights are not normalizable** – The given estimations of the fuzzy weights cannot be used to derive normalized fuzzy weights. They are not on the interval [0, 1] or their kernels do not satisfy the condition for normalization (it must be possible to select a real normalized weights within the kernels of the fuzzy weights).

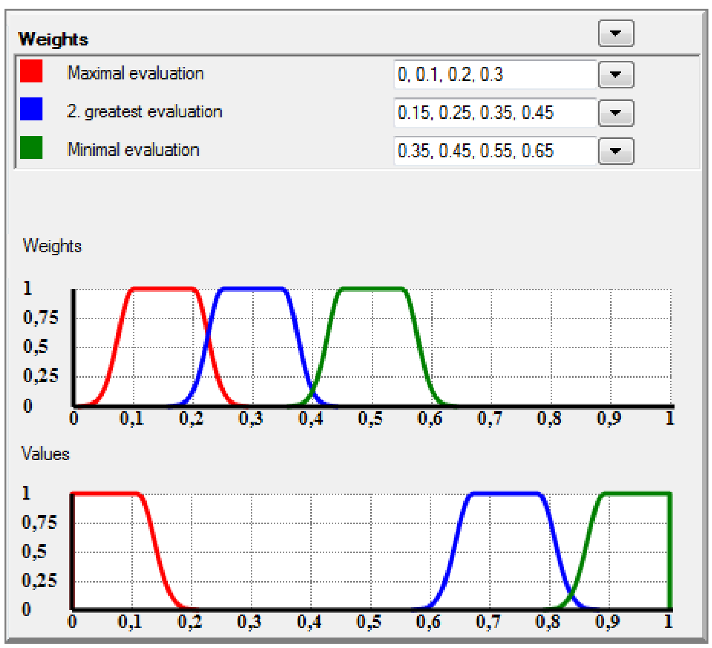
- **The weights are not normalized** – The given estimations of the fuzzy weights were set correctly, but they do not form normalized fuzzy weights, yet. To derive normalized fuzzy weights from these estimations, click on the *Derive normalized fuzzy weights* button.

The weights and the input values for the fuzzy weighted average are depicted at the bottom of the dialog. The following operations can be done with the normalized fuzzy weights:

- **Load weights from a file** – The weights can be loaded from a file by selecting *Node | Weights | Open* from the main menu.
- **Save weights into a file** – The weights can be saved into a file so that they could be used for another node or in a different goals tree (*Node | Weights |Save as)*.
- **Create uniform weights** – A real uniform weights are created (*Node | Weights | Create uniform weights*).
- **Edit support and kernel length** – This function can be used to fuzzify existing real weights. First, real weights have to be set. Then this dialog is opened (*Node | Weights | Edit support and kernel length*) and length of the support and kernel of the weights is set.

Fuzzy OWA operator

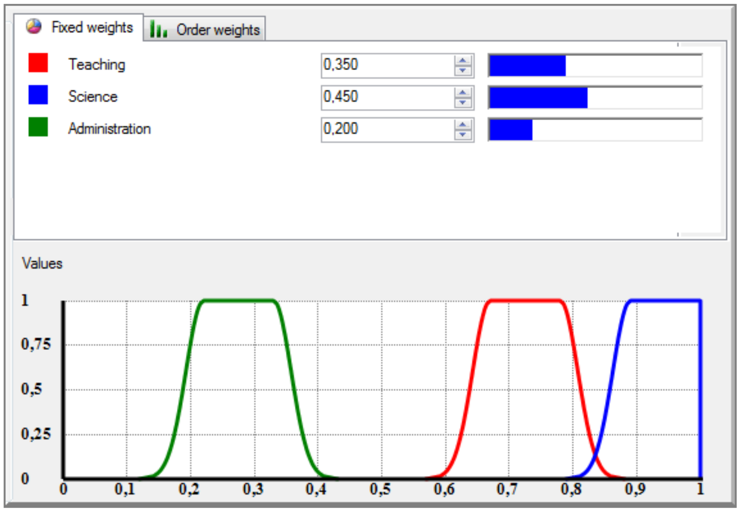
For fuzzy OWA operator, the normalized fuzzy weights are required. The process of their setting is identical as in the case of fuzzy weighted average. Therefore the description of the dialog will be



skipped (it differs only in labels for the weights). See Fuzzy weighted average for more information.

Fuzzified WOWA operator

Fuzzified WOWA operator utilizes two sets of weights – fixed weights and order weights. For each of them, there is one tab.



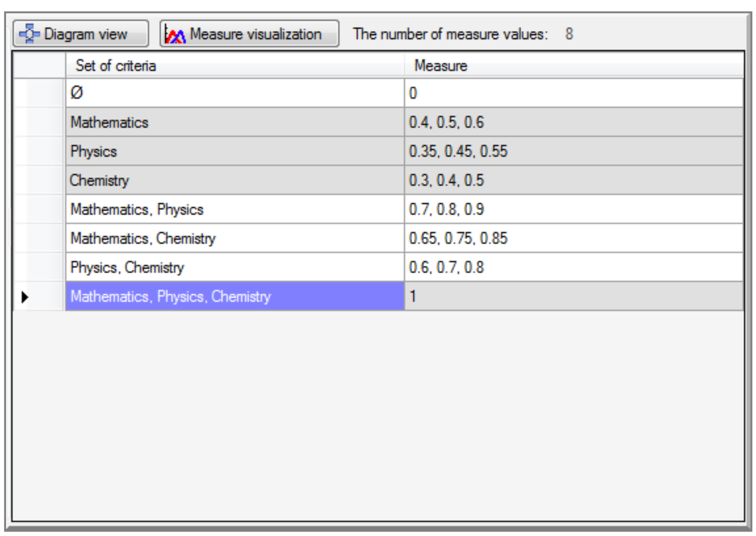
The weights are real number and their sum must equal one for both of the sets. If the sum is not equal one, a warning appears. The user can click on *Normalize* button which modifies the value of the last weight so that the sum would be equal to one.

It is possible to save the weights into a file, load them from a file, or create uniform weights. All these operations with weights are described in the section Fuzzy weighted average.

Fuzzy Choquet integral

For fuzzy Choquet integral, a FNV-fuzzy measure must be defined. The user can see all subsets of criteria (or child nodes) and their measure. Each row in the table represents one value of the FNV-fuzzy measure. The value is a fuzzy number (a real number, as a special case of a fuzzy number, can be also used).

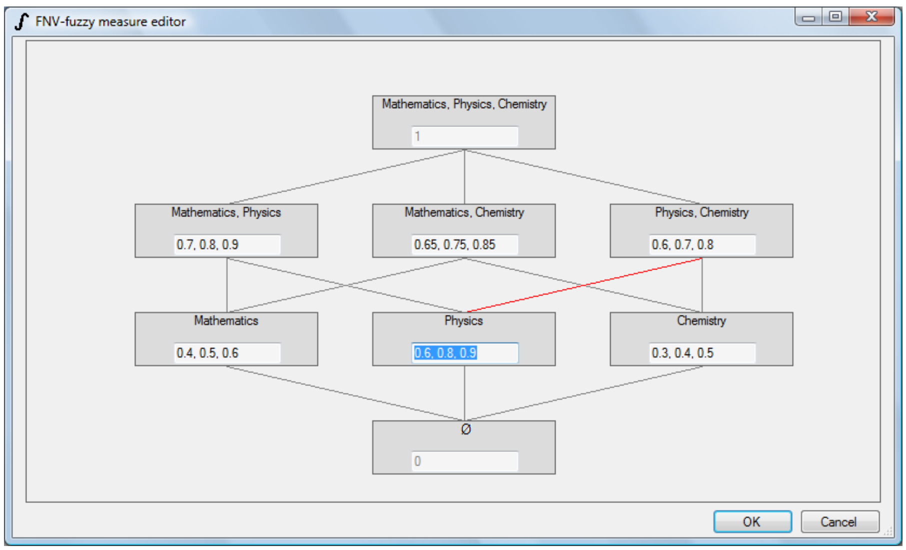
Note: For *n* child nodes, the FNV-fuzzy measure has 2^n values. If the number of the child nodes of the selected node is high, the software can be slow.



The FNV-fuzzy measure is required to satisfy the monotonicity condition. Otherwise, an icon of an exclamation mark is shown next to the measure values that do not meet the condition.

Besides the table view, there are two more ways of displaying the FNV- fuzzy measure in FuzzME:

- **Diagram view** – The measure is displayed in form of a diagram. This view is convenient for small number of measure values (otherwise it can become too big and therefore slow and difficult to interpret). If the condition of monotonicity is not satisfied, the user can see easily where exactly it is broken. This is signalized by a red line. The diagram view can be opened by clicking on the *Diagram view* button or by selecting *Node | FNV-fuzzy measure | Diagram view*.
- **Measure visualization** – All measure values are presented as a set of images. The visualization can be opened by clicking on the *Measure visualization* button or by selecting *Node | FNV-fuzzy measure | Measure visualization* from the main menu.

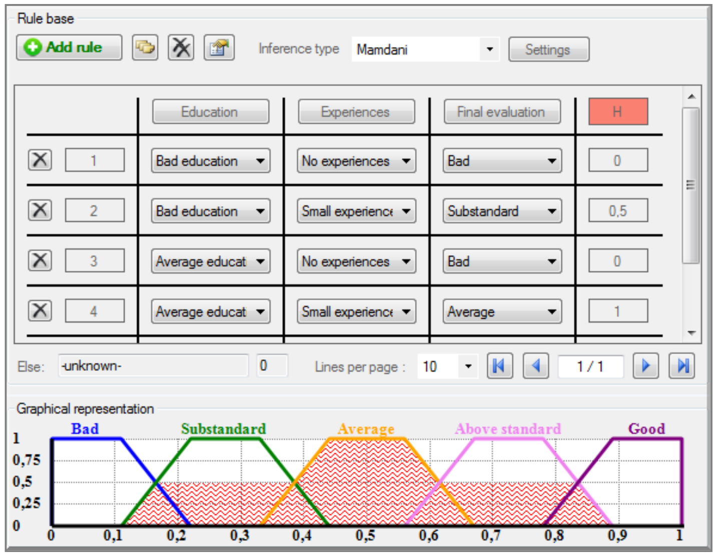


Fuzzy expert system

The fuzzy expert system in the FuzzME can be used either for a fuzzy evaluation or for a fuzzy classification. Multiple inference algorithms are supported. The user can choose the Mamdani, Sugeno-WA, or Sugeno-WOWA inference algorithms for the purposes of fuzzy evaluation. For fuzzy classification, Single Winner or Voting by Multiple Fuzzy Rules algorithms can be applied.

First, the expert has to define a fuzzy rule base for the fuzzy expert system. Then an appropriate inference algorithm is chosen. Because the fuzzy rule base is defined linguistically, a linguistic fuzzy scale has to be defined for the fuzzy expert system node and all its child nodes first.

The fuzzy rule base is presented in the form of a table. Each row of the table represents one rule from the base. The rules are in form of “if - then”. In each column, except for the last one, there is a value of a criterion or a child node (if-part). In the last column, there is the result (then-part of the rule). The number next to the rule (denoted in the software as *H*) is the degree in which was the given rule fired. If the number of rules is high, they are divided into the several pages. The arrow icons under the table can be used to move to the next or to the previous page.



The user can use the following function:

- **Add rule** – By clicking on the *Add rule* button, a new rule is added at the end of the base. The user then selects the values for of the child nodes and the value for the result from drop-down boxes.
- **Delete rule** – Clicking on the black cross icon on the left-hand side of any rule will cause that the rule will be deleted.
- **Edit rule** – Any values of the rule can be modified directly by choosing a new one from the drop-down box. The values in each drop-down box are the linguistic values of the fuzzy scale defined for the corresponding node.
- **Edit linguistic scale** – The linguistic scales can be edited quickly by clicking on the button with the criterion name in the header of the table.
- **Clear base** – All rules will be deleted. This function is available from the menu *Node | Rule base | Clear base*.
- **Criteria combinations** – This function offers a quick way of creating the fuzzy rule base (*Node | Rule base | Criteria combinations*). A new rule base will be created. Each rule in this base represents each possible combination of the input values. Then, the expert just has to select the result (output) value for each rule, or to delete the rule (e.g. if such a combination of input values cannot occur).
- **Adjust the fuzzy expert system settings** – A dialog with the fuzzy expert system settings is opened (*Node | Rule base | Fuzzy Expert System Settings*). The items in this dialog are described in following text of this section.
- **Show fuzzy expert system details** – A window with additional information is opened (*Node | Rule base | Show Fuzzy Expert System Details*). The user can see there, e.g., the total number of rules and the number of fired rules.
- **Open** – The rule base can be loaded from a file (*Node | Rule base | Open*).

- **Save as** – The rule base is saved into the file so that it can be used in another fuzzy expert system (*Node | Rule base | Save As*).
- **Export into Matlab** – The fuzzy expert system created in FuzzME can be exported into Matlab. Select *File | Export | Export Expert System into Matlab*.

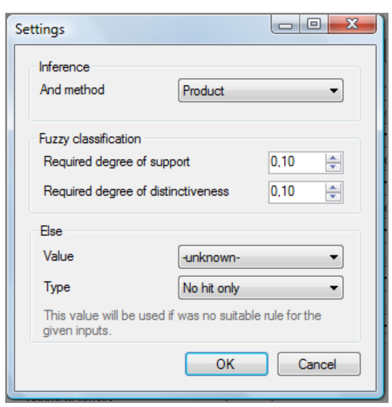
The fuzzy rule base can be created either by adding the rules one-by-one (*Add rule* button), or by generating the whole fuzzy rule base, so that it would contain all possible combination of the input values (*Create combinations* functions) and choosing the output values for the new rules.

After the fuzzy rule base is created, the expert chooses the inference type from a drop-down box at the top of the dialog. If Sugeno-WOWA algorithm is selected, a *Weights* button appears. Clicking on the button opens a window where the weights for the Sugeno-WOWA can be set.

At the bottom of the dialog, there is a graphical representation of the output linguistic fuzzy scale. The displayed information depends on the selected inference algorithm. In case of Sugeno and Sugeno-WOWA, the numbers above the scale values express the resulting weights for the Sugeno inference. If the Mamdani inference is selected, the result of the inference (fuzzy set) is highlighted in the image. In case of Single Winner or Voting by Multiple Fuzzy Rules classification algorithms, the number of votes for each of the classes is displayed.

The user can adjust the behavior of the fuzzy expert system by clicking on the *Settings* button (or by choosing *Node | Rule base | Fuzzy Expert System Settings*). In the window, the user can adjust the following settings:

- **And-Method** – The t-norm that will be used for modeling of the "and". The supported methods are the *minimum* and the *product*.
- **Fuzzy Classification** – The parameters concerning the fuzzy classification. For more information see the section **Fuzzy classification in FuzzME**.
- **Else** – The else-value is the value that will be used if no rule can be applied for the given input. By default, the *unknown* value (i.e. the whole interval [0, 1]) is used but the user can choose another one. The *type* determines in which situation this value will be used. If "no hit only" is selected, the value will be used only if no rule was fired at all. If "rule" is selected, it will be added as a separate rule that will be fired in the degree one minus the maximal of the degrees in which the rest of the rules was fired.



Analyze the aggregation behavior

In the FuzzME, there are functions that make it easy to understand the behavior of the selected aggregation method and to set the proper parameters:

Compare inputs and results

This function displays all the input values and the aggregation result in the same image. Select the node of interest and choose *Node | Compare inputs and result* from the menu.

Compare results for various inputs

This function demonstrates how the result of the aggregation will be affected if some of the input values change. Select the node of interest in the goals tree and choose *Node | Compare results for various inputs* from the main menu. The user can select up to two criteria of interest for which the analysis should be performed.

For the two selected criteria, the user sets their minimum value, their maximum value and the number of values (this will represent the number of rows or columns in the table). The table below then shows the evaluation of this partial goal when the values of the two selected criteria are changing from the selected minimum value to the maximum one and the values of the other criteria remain unchanged.

The table rows represent the different (crisp) values of the first criterion and the columns represent the different values of the second one. In the corresponding cell, there is the resulting evaluation. The user can select which characteristic of the resulting evaluation should be displayed. One of the following options can be chosen:

- Center of gravity;
- Uncertainty amount - crisp numbers have uncertainty equal to zero, if the evaluation represents the value *unknown* (i.e. the whole interval [0, 1]), its uncertainty will be one;
- Middle of kernel - the center of the fuzzy evaluation kernel;
- Lowest of kernel - the lowest value of the fuzzy evaluation kernel;
- Highest of kernel - the highest value of the fuzzy evaluation kernel;
- Middle of support - the center of the fuzzy evaluation support;
- Lowest of support - the lowest value of the fuzzy evaluation support;
- Highest of support - the highest value of the fuzzy evaluation support;
- All significant points - all four main significant points of the fuzzy evaluation, delimited by the coma, will be displayed.

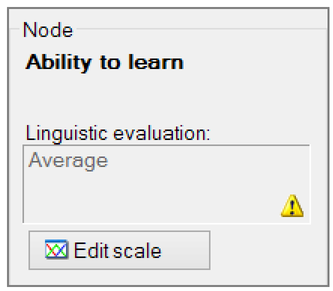
The table can be copied, i.e., to Excel and saved for purpose of the further analysis.

Clicking on the *Plot in Matlab* button makes it possible to plot the content of the table in the Matlab. The evaluation function can be visualized this way, which can be very valuable for the expert in the analysis of the model. When the user clicks on the button, a dialogue appears. The user selects in the dialogue where should be the M-file generated by the FuzzME saved. The FuzzME opens this M-file in the Matlab automatically.

For generating the graph of the evaluation function, it is recommended to set a reasonable number of values for the first and the second criterion. It is recommended to use more than 10 values for each criterion; otherwise the graph is too rough. On the other hand, more than 20 values for each criterion involve a larger number of calculations and it could take some time.

Analyze the linguistic approximation behavior

In the middle part of the main window, the linguistic description of the evaluation in the selected goals tree node is displayed. It can look as in the following image.

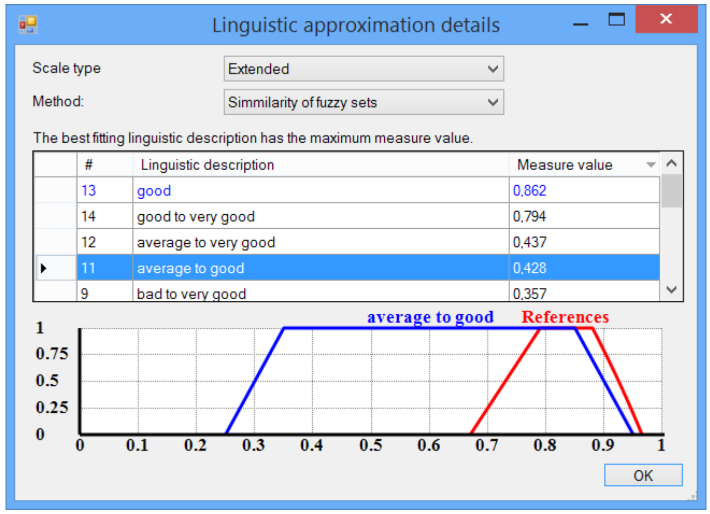


In the image, an exclamation mark icon is dispayed. It signalizes that there are also other linguistic descriptions that are as closed to the fuzzy evaluation as the displayed linguistic description. The decision maker should therefore examine also these alternative linguistic descriptions.

By clicking on the linguistic description, a new dialogue is opened showing the details on the linguistic approximation for this partial evaluations.

The dialogue makes it possible to select the used linguistic scale type and also the linguistic approximation method. Then all possible linguistic descriptions are listed together with the value that measures how well does the linguistic description fit the partial evaluation (it is the similarity or distance - depending on the selected method). The best-fitting term is highlighted by the blue color. By clicking on the header of the table, the values can be ordered (e.g. if you click on the header of the last colemn, the values are ordered according to how well they fit they the particular partial evaluation).

By clicking on any of the values in the table, the meaning of selected term (blue) and the actual value of the partial evaluation (red) are dispayed graphically. This makes it possible to check how well does the meaning of the linguistic term correspond to the partial evaluation of interest.

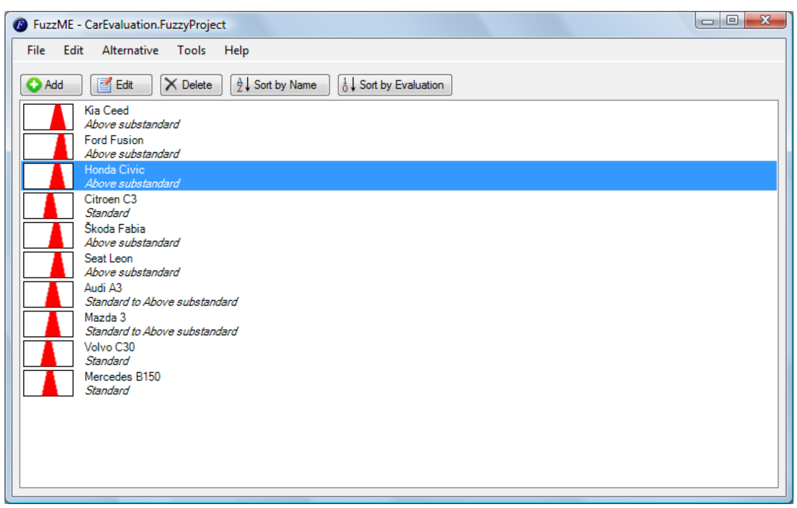


List of alternatives

Managing the alternatives

When the goals tree has been designed, the expert can proceed to evaluation of the alternatives. The user can open the list of the alternatives, by clicking on the button *View list of alternatives* or from the menu *Tools | Mode | List of alternatives*.

For each alternative in the list, there is its name and the graphical representation of its evaluation (which is a fuzzy number). If a linguistic scale has been defined for the root node of the goals tree, a linguistic evaluation of the alternative is displayed under the alternative name.



The user can do the following operations with alternatives:

- **Add alternative** – To create a new alternative, the user clicks on *Add* button or selects *Alternative | Add* from the menu. A new alternative is created and the window is switched to the alternative editing mode. In this mode, the user should fill in the alternative name first.

Then it is necessary to click on each criteria node in the goals tree and fill in the value of the criterion for this alternative. If some of these values are not known, the user can leave there the default value, which is "- *unknown* -". When the alternative name and the criteria values have been filled, the user can return back to the list of alternatives by pressing *View list of alternatives* button.

- **Edit alternative** – The alternative can be edited by double-clicking on it (or by pressing *Edit* button or selecting *Alternative | Edit*). The window is switched to the alternative editing mode. Subsequently, the user can modify the alternative name or the values of its criteria. This is done in the same way as in the case of adding an alternative. Finally, the user can return back to the list of alternatives by pressing *View list of alternatives* button.
- **Delete alternative** – The user selects the alternative in the list by clicking on it. Then the selected alternative is deleted by pressing *Delete* button, *Del* key on the keyboard, or by selecting *Alternative | Delete* from the menu.
- **Delete all alternatives** – All alternatives can be deleted from the menu *Alternative | Delete All*. At least one alternative always has to be in the list. That is why there is one default alternative after the deletion.
- **Show alternative evaluation** – The basic information about the alternative evaluation is displayed directly in the list of alternatives. To obtain more detailed information, select *Alternative | Show alternative evaluation*.
- **Recompute all evaluations** – The evaluations of the alternatives are calculated automatically after the change of the goals tree or values of an alternative. These evaluations are cached for better speed. However, it is possible to clear this cache and calculate the evaluations again by selecting *Alternative | Recompute All Evaluations*.
- **Find** – This function searches for an alternative by its name (*Alternative | Find* from the menu or *Ctrl+F7*).
- **Sort alternatives by name** – The alternatives are ordered according to the alphabetical order of their names (click on *Sort by Name* button or select *Alternative | Sort by Name*).
- **Sort alternatives by evaluation** – The alternatives are ordered according to the centers of gravity of their evaluations (click on *Sort by Evaluation* button or select *Alternative | Sort by Evaluation*). The alternatives with the best evaluations are displayed at the top of the list.
- **Copy, Cut, Paste** – The alternative can be copied to the clipboard and pasted into another project. Click on the alternative with the right mouse button. Then select the operation from the context menu. Keep on mind that if the alternative is copied into another project with different criteria, the criteria values need not to be transferred properly.
- **Import of the alternatives** – The alternative names and their criteria values can be imported, e.g., from Excel. See section Import of the alternatives.
- **Export of the alternatives or their evaluations** – See section Export for more information.

Import of the alternatives

It is not necessary to add all of the alternatives manually. They can be imported from another software product such as Microsoft Excel or Microsoft Access. The FuzzME supports CSV (comma separated values) files for the import.

To import data from the Excel, the Excel table should have the following format:

- The first row is the header with the criteria names (starting from the second column because the first one is occupied by the alternative names).
- The other rows represent the alternatives. In the first column of each row, there is a name of the alternative. In the rest of the columns, there are the criteria values for this alternative (each value in a separate cell).

Example of an Excel file in the suitable format can be seen in the following picture.

	A	B	C	D	E	F	G	H	I
1	Alternative name	Operating costs	Spare parts costs	Additional safety elements	Steering booster	ABS	Luggage capacity	Air-conditioning	Design
2	Kia Ceed	19000, 31000, 44000	High	Yes	Yes	Yes	340	Yes	above standard-good
3	Ford Fusion	20000, 33000, 35000	Low	Yes	Yes	Yes	337	Yes	average-above standard
4	Honda Civic	20000, 32000, 44000	Medium	Yes	Yes	Yes	415	Yes	good-excellent
5	Citroen C3	17000, 31000, 43000	Medium	No	Yes	Yes	305	Yes	average
6	Škoda Fabia	13000, 31000, 37000	High	Yes	Yes	No	300	No	above standard-good
7	Seat Leon	21000, 34000, 47000	Medium	Yes	Yes	Yes	292	Yes	good-excellent
8	Audi A3	26000, 39000, 52000	High	Yes	Yes	Yes	360	Yes	good-excellent
9	Mazda 3	19000, 33000, 45000	High	Yes	Yes	Yes	346	Yes	average-above standard
10	Volvo C30	24000, 38000, 52000	High	Yes	Yes	Yes	233	Yes	excellent
11	Mercedes B150	19000, 33000, 40000	High	Yes	Yes	Yes	554	Yes	excellent

To import data from the Excel into the FuzzME, the CSV file have to be created first:

1. Open a spreadsheet file of interest in Excel
2. Choose *Save As* from the *File menu* in the Excel.
3. There is the *Save As Type* drop-down list at the bottom of the dialog box. Choose the *CSV (Semicolon delimited)* option.
4. Close the Excel

Note: The CSV files contain basically only the data (text in the cells) without any additional information about colors, formatting, etc. That is why they are easy to process and a lot of software products (such as Matlab) support export of the data into this format.

Finally, import this CSV file into the FuzzME:

1. Open a project in FuzzME where the alternatives should be imported.
2. Select *File | Import | Import alternatives* form the main menu.
3. In the dialog, click on *Browse* and select the CSV file you have created in Excel. Keep on mind that there are two formats – standard (data are separated with a comma) and Excel (data are separated with a semicolon). You should select the correct format from the drop-down list in this dialog. Confirm the dialog by clicking on *Open*. Move to the next step by clicking on *Next* button.
4. In the next step, you have to link each criterion (on the left) with the column in the CSV file (on the right). You have to choose format in which the criteria values are written:
    a) **Value is a fuzzy number** – Values of all the criteria were written as fuzzy numbers (i.e. the list of significant points separated by commas) in Excel (for example "0, 0.1, 0.2").
    b) **Value is a scale item index** – Value of the qualitative criteria were written as an index of the scale item in Excel. For example, let us consider that the scale for a criterion has items "poor", "average" and "good". Then the value "2" denotes the second item of the scale ("average"). The value "2-3" denotes "average to good" and the value "2/3" means "between average and good". The value "0" denotes the "unknown"

value. The algorithm in FuzzME is very flexible. It is possible to use the scale item names (e.g. “poor”) instead of their indices (e.g. “1”) in the Excel.

5. Click on *Done* to start the import. The number of imported alternatives will be displayed and the dialog can be closed.

Export of the data

Export of the evaluations

The evaluations can be exported into a CSV file. This file can be then opened in Excel by double-clicking on it in Windows Explorer. To export the evaluations, select *File | Export | Export Evaluations* from the main menu.

Choose a name for the new CSV file, and specify its type. The type can be either standard CSV (values separated by a comma), or Excel CSV (values separated by a semicolon). Click on *Save* to start the export.

Export of the alternatives

The alternatives names and their criteria values can be exported into a CSV file. Select *File | Export | Export alternatives* from the main menu.

Choose a name for the new CSV file, and its type. The type can be either standard CSV (values separated by a comma), or Excel CSV (values separated by a semicolon). Click on *Save*. In the following dialog, the parameters for the export can be specified. The appropriate values of these parameters depend on the software that you want to use later for processing the CSV file. Click on *OK* to start exporting.

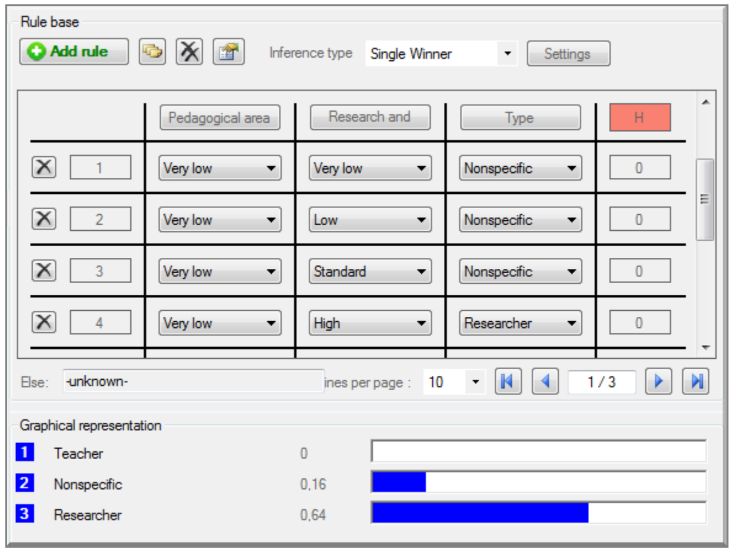
Fuzzy classification in FuzzME

Although the main function of the FuzzME is multiple-criteria fuzzy evaluation, it can be used also for fuzzy classification. Two fuzzy classification algorithms are available – Single Winner and Voting by Multiple Fuzzy Rules. The result will be the best fitting class for the object. The classification is described by a fuzzy rule base.

First, the goals tree is designed in the same way as it was described in the section **Designing the goals tree**. The root node of the tree will be a fuzzy expert system, since a fuzzy rule base will be used for description of the classification.

Next, define a linguistic fuzzy scale for the root node (the one with the fuzzy expert system). Create for example a uniform scale. Each item of the scale represents one class. Select suitable name for each of the items (classes). The fuzzy numbers used for modeling of the scale items are unimportant since the scale is nominal one. They serve just as numeric identifiers of the classes. You can use any mutually different numbers (or fuzzy numbers).

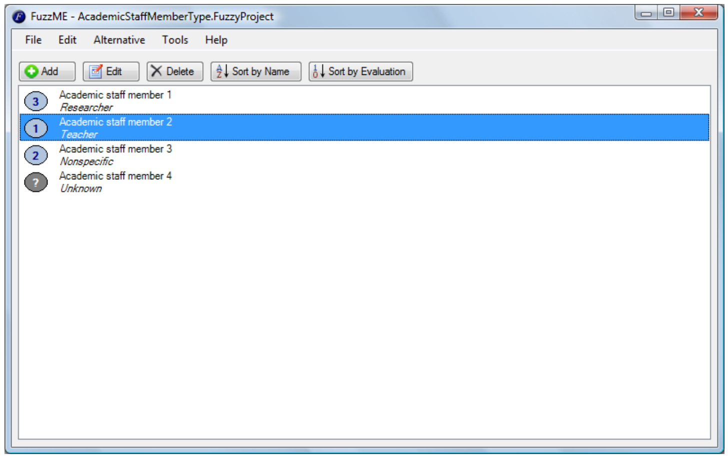
In the following step, define the fuzzy rule base. For each of the rules, select the corresponding class on the right-hand of the rule. Finally, instead of the Mamdani or Sugeno inference, select one of the mentioned classification algorithms. When one of them is selected, FuzzME will treat the project as a fuzzy classification project.



The FuzzME also supports the cases where the object cannot be classified. You can adjust the parameters for determining which objects cannot be classified. To do so, click on the *Settings* button, or select *Node | Rule Base | Fuzzy Expert System Settings* from the menu. There are two parameters:

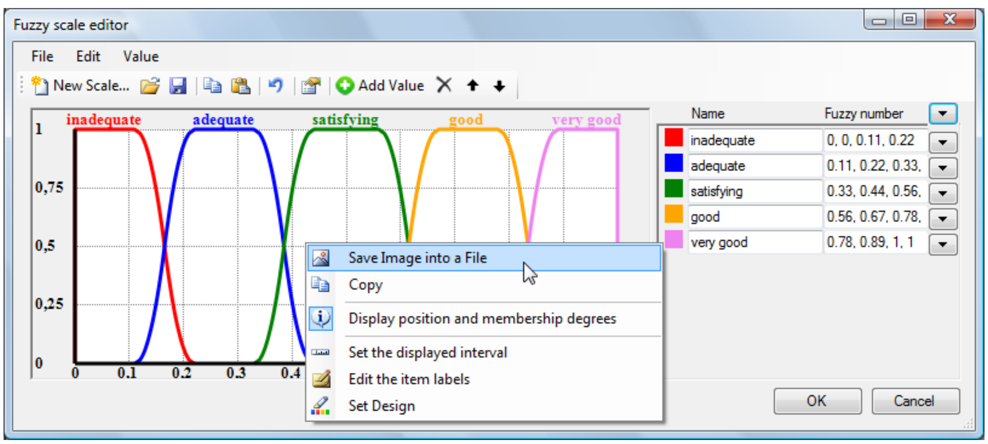
- **Required degree of support** – This parameter represent the minimal required number of votes for the class, i.e. one of the rules should be fired at least in this degree. Otherwise, the object will be marked as unclassifiable. The higher value, the more unclassifiable objects – value 1 means that the rule that proposed the winner class must be fully fired
- **Required degree of distinctiveness** – The greater value of this parameter, the greater difference in the number of votes for the best fitting class and for the second best fitting one will be required. The value 1 means, that the no other classes than the winner one can be proposed.

Notice a few changes when the FuzzME is used for the fuzzy classification. In the fuzzy rule base editor, the output fuzzy scale is no more displayed. Instead, there is a list of the classes and the "numbers of votes", which they received. In the alternatives list, there is the number of the class in front of each object, or a question mark if the given object cannot be classified.



Graphics in FuzzME

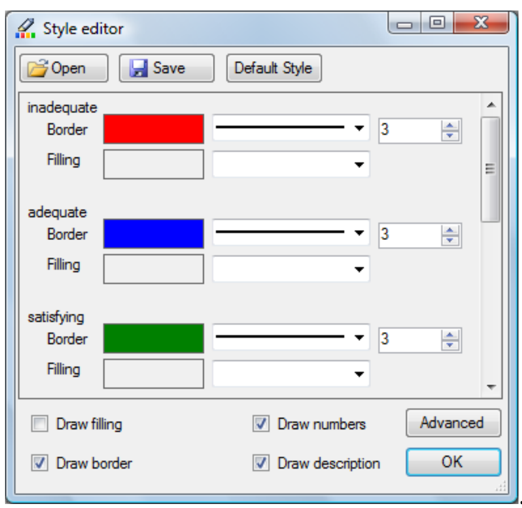
All the graphics created in FuzzME can be customized and saved as an image into a file. The parameters for the graphics can be set in the *Options* on the *Graphics* tab (see Tab Graphics).



For any image displayed in FuzzME, the user can use the following functions. They are available from the context menu (click on the image with the right mouse button to open it):

- **Save the image into a file** – Any image displayed in the FuzzME can be saved into a file. Click on the image with the right mouse button. A context menu appears. Select "*Save Image into a File*" from the context menu. Choose the name of the new file and the format of the image. The supported bitmap formats are png, gif, bmp and jpg. It is recommended to use the png format.
  The FuzzME can save the image also as a vector graphics either in the svg format, or in the Metapost. Free converters available on the Internet can be used to convert the svg image to eps, or pdf. Because of the characteristics of the vector graphics, the image need not to look exactly as in the screen (different font can be used, the texts might be slightly misplaced). The current version does not support some advanced graphical settings for the vector graphics (e.g. hatching in the Metapost).
  Save the image by clicking on the *Save* button.

- **Copy image to the clipboard** - By selecting *Copy* from the context menu, you can copy any image created in the FuzzME to the clipboard. The copied image can be then pasted e.g. to the Word or another software.
- **Turn on/off the displaying of the membership degrees** – If this option is turned on, a small box is displayed when the user moves the mouse over the image. In the box, there is the *x* (corresponding to the mouse position) and the membership degrees of the *x* for all of the fuzzy numbers displayed in the image. The option can be turned on or off by clicking on *Display position and membership degrees* in the context menu.
- **Set displayed interval** – Before the image is saved into a file or copied to the clipboard it can be useful to specify which interval will be displayed in the image. This can be done by selecting *Set the displayed interval* from the context menu.
- **Edit the fuzzy number labels** – Sometimes the labels of the fuzzy numbers can be overlapping because they are too long. This can be fixed by selecting new (shorter) labels for the fuzzy numbers (choose *Edit the Item Labels* in the context menu). The purpose of this function is just to make the image more appealing before it is saved or copied. The new labels are not persistent, the original labels will be reloaded when the image is redrawn again (e.g. when another node is selected).
- **Modify the colors and style of the image** – Click on the image with the right mouse button and select *Set Design* from the context menu. For each fuzzy number that is displayed, you can set the border (color, line style and line width) and the filling (color and the filling style). You can enable or disable drawing of the numbers at the axes, or descriptions. You can also save the style into a file and open it later. If you click on *Advanced*, a window with additional options appears. In the window, you can set the font for the descriptions (its style, size and color) and the font for the numbers. You can also set the background color or modify the properties of the axes in this dialog. The styles are saved into a configuration file so they will be used until they are changed. The default style can be restored later by selecting *Tools | Options* from the main menu, choosing *Graphics* tab and clicking on the *Set Default Style* button.



.

Note: The node names can contain upper and lower indices. They are written it the same notation as in TeX:

- For a lower index use “_”. If the index should be more than one character long, it must be enclosed into curly brackets. So a node with the name “A_1” will be drawn as an A with an index 1. Another example of this notation can be: “W_{something}“.

- For an upper index use “^“. Again, if the index has more than one character, it must be enclosed into curly brackets. Examples: “A^1“, “A^{something}“.

This behavior is advantageous if you want to use the images generated by FuzzME in a scientific paper, where the names in the image must correspond with the rest of the text. The same notation for upper and lower indices can be used in the function *Edit the fuzzy number labels.*

Options

The settings can be opened by selecting *Tools | Options* from the main menu. There are three tabs – *Common*, *Fuzzy Numbers*, and *Graphics*.

Tab Common

On the *General* tab, the following options can be set:

- **Language** – The software is available in the Czech and English versions. After the language is changed, the FuzzME needs to be restarted. If the *Autodetect language* checkbox is checked, the preferred language will be detected automatically from the language of the operating system.
- **List of alternatives** – The user can switch between the simple and advanced view. In the simple view, only alternative names are displayed without any additional information so it is convenient for slower computers.

Tab Fuzzy Numbers

On the *Fuzzy Numbers* tab, the following settings can be adjusted:

- **Format** – This option specifies which character will be used to delimit the significant points of fuzzy numbers. The default value is a comma (then a fuzzy number can be written by the user e.g. as “0.1, 0.2, 0.3”). However, the user can select also a space as a delimiter instead (e.g. “0.1 0.2 0.3”). This can be handy especially for users which are used to Matlab syntax.
- **Preciseness** – The user can choose the number of α-cuts that will be used for the calculations. The more α-cuts, the better preciseness. The fewer α-cuts, the better calculations speed.

Tab Graphics

On this tab, the user can set the parameters for the graphics in FuzzME:

- **Saved image size** – The resolution of the graphics saved into a file (see **Graphics in FuzzME**). If *Default* is selected, the images will have the same resolution as the graphics on the screen.

- **Styles** – The style (e.g. colors) of the graphics can be modified in the Style Editor (see **Graphics in FuzzME**). In this dialog, the user can select whether the modified style should be saved or if the default style should be always used.
- **Painting** - The user can select the graphics quality. If the *standard quality* is selected, the fuzzy numbers are drawn by a series of lines (as in the FuzzME 2.1 and former versions). Selecting the *high quality* causes that the membership functions of fuzzy numbers will be drawn in the usual mathematical style. This is especially suitable if the images should be used in scientific publications. The difference can be seen best on functions that are not continuous.
  The next checkbox in this group of settings allows the user to select whether the entire membership function should be drawn or if only parts where it is non-zero should be plotted. The first way is recommended if the image is to be published in a paper. However, if the image contains more than one fuzzy number, the membership functions overlays in the parts where they are zero. This may be rather confusing and the latter way of drawing is recommended in these cases.

Frequently Asked Questions (FAQ)

The program could not be started. An error occurs during its start.

Probably, .NET framework is not installed on the computer. This component is required by the FuzzME. See Hardware and software requirements and Installation for more information.

Which format should be used when setting a fuzzy number?

The fuzzy numbers are described by their significant values. The significant values are **separated by comas**. There is also a possibility to select another character as a delimiter of the significant points. In the numbers, the decimal point is used. See Using fuzzy numbers in FuzzME for more information.

Where can I download the latest version?

The latest version can be downloaded from http://www.fuzzme.net/.

Appendix 2

The professional curriculum vitae of the thesis author

CURRICULUM VITAE

MGR. PAVEL HOLEČEK

Akátová 2 / 213, Samotišky

772 00, Česká republika

Date of birth: 12. 7. 1983

Phone: +420 585 634 028

E-mail: pavel.holecek@upol.cz

EDUCATION

2008 - Now	**Palacký University in Olomouc**, Faculty of Science PhD study program: *Applied mathematics* PhD. thesis topic: Fuzzy models of multiple criteria evaluation and fuzzy classification.
2007 - 2008	**Palacký University in Olomouc**, Faculty of Science Master study program: *Computer Science* Thesis: Software processing of fuzzy evaluation methods.
2003 - 2007	**Palacký University in Olomouc**, Faculty of Science Bachelor study program: *Computer Science*

PRACTICE:

January 2013 - Now	**Palacký University in Olomouc**, Centre of the Region Haná Development and administration of the information system for academic staff members evaluation
October 2010 - Oct. 2013	**Palacký University in Olomouc**, Faculty of science, Department of Mathematical Analysis and Applications of Mathematics Post: expert for e-learning deployment and administration

COMPLETED INTERNSHIPS:

April 2012	**Institute for research and applications of fuzzy modeling** IRAFM, Ostrava, Czech Republic (1 week internship)
March 2011 - June 2011	**Technical Unversity in Vienna**, Vienna, Austria

PROFESSIONAL ORGANISATIONS MEMBERSHIP:

2014 – present	Member of the Czech Society for Operations Research

RESEARCH AREAS:

- Fuzzy models of multiple-criteria evaluation, fuzzy classification and their applications.

- Cooperation in the detailed mathematical analysis of the first version of Public Universities Financing Guidelines after 2010.

RESEARCH AND APPLICATIONS:

- Participation in the pilot project testing the possible use of a fuzzy controller for traffic regulation in the Austrian highways. The project was realized in 2011 by Nast Consulting ZT GmbH with cooperation of the Technical University in Vienna.
- Participation in the subsequent project, which studied the possibility of regulating traffic by a fuzzy controller on both highways and secondary roads. The project was carried out by Nast Consulting ZT GmbH in 2013 and 2014.
- Software implementation of the information system for academic staff performance evaluation (IS HAP). The information system is used at faculties of 6 Czech universities.
- Development of the FuzzME software. The software is a software implementation of the multiple-criteria fuzzy evaluation and fuzzy classification methods.
- Application of the FuzzME software for bank clients assessment (in co-operation with the Technical University in Vienna).

PUBLICATIONS

BOOK CHAPTERS:

- HOLEČEK P., TALAŠOVÁ, J., MÜLLER, I.: Fuzzy Methods of Multiple-Criteria Evaluation and Their Software Implementation. Cross-Disciplinary Applications of Artificial Intelligence and Pattern Recognition: Advancing Technologies (editors Mago V. K., Bhatia N.), IGI Global 2012, ISBN13: 9781613504291 , DOI: 10.4018/978-1-61350-429-1**.**
- TALAŠOVÁ, J., STOKLASA, J., HOLEČEK, P.: Registr uměleckých výkonů a hodnocení výsledků tvůrčí činnosti [Registry of artistic results and evaluation of creative work outcomes] *in Registr uměleckých výstupů 2* (editor Zelinský, M.), NAMU, Praha, 2012, ISBN 978-80-7331-231-2.
- TALAŠOVÁ, J., STOKLASA, J., HOLEČEK, P.: Hodnocení akademických pracovníků v kontextu organizačního klimatu univerzit [Academic staff evaluation in the context of the organization climate of universities]. In GRECMANOVÁ, H., DOPITA, M. ET AL.: *Determinanty organizačního klimatu vysokých škol a fakult*, Olomouc: Univerzita Palackého v Olomouci, 2013, s. 101-114. ISBN 978-80-244-3808-5.

PAPERS IN JOURNALS (with non-zero impact factor):

- STOKLASA, J., TALAŠOVÁ, J., HOLEČEK, P.: Academic staff performance evaluation – variants of models, *Acta Polytechnica Hungarica* 8 (3), 2011, p. 91 – 111, ISSN 1785-8860.
- BEBČÁKOVÁ, I., HOLEČEK, P, TALAŠOVÁ, J.: On the application of the fuzzified Choquet integral to multiple criteria evaluation. Acta Polytechnica Hungarica 8 (3), 2011, p. 65-78, ISSN 1785-8860.

PAPERS IN JOURNALS (others):

- HOLEČEK P., STOKLASA, J., TALAŠOVÁ J. (in press): Human resources management at universities - a fuzzy classification approach. International Journal of Mathematics in Operational Research.

- HOLEČEK P., TALAŠOVÁ J.: FuzzME: A New Software for Multiple-Criteria Fuzzy Evaluation. Acta Universitatis Matthiae Belii, series Mathematics; No. 16 (2010), pp. 35–51

PAPERS IN CONFERENCE PROCEEDINGS:

- HOLEČEK P., TALAŠOVÁ, J.: Multiple-Criteria Fuzzy Evaluation in FuzzME - Transitions Between Different Aggregation Operators. *Proceedings of the 32nd International conference on Mathematical Methods in Economics MME 2014*. Olomouc, Czech Republic, pp. 305-310. ISBN 978-80-244-4209-9.
- J. TALAŠOVÁ, J. STOKLASA, P. HOLEČEK: HR management through linguistic fuzzy rule bases - a versatile and safe tool? *Proceedings of the 32nd International conference on Mathematical Methods in Economics MME 2014*. Olomouc, Czech Republic, pp. 1027-1032. ISBN 978-80-244-4209-9.
- STOKLASA, J., HOLEČEK, P., TALAŠOVÁ, J.: A holistic approach to academic staff performance evaluation – a way to the fuzzy logic based evaluation, *Peer Reviewed Full Papers of the 8th International Conference on Evaluation for Practice "Evaluation as a Tool for Research, Learning and Making Things Better" A Conference for Experts of Education, Human Services and Policy*, 18 – 20 June 2012, Pori, Finland, Tampub, Pori, 2012, ISBN 978-951-44-8859-7
- HOLEČEK P., TALAŠOVÁ, J., STOKLASA J.: Fuzzy classification systems and their applications, *Proceedings of the 29th International Conference on Mathematical Methods in Economics 2011 – part I*, 2011, Praha, Czech Republic, p. 266 – 271, ISBN 978-80-7431-058-4
- ZEMKOVÁ B., TALAŠOVÁ, J., HOLEČEK, P.: Fuzzy Model for Determining the Type of Worker. *Proceedings of the 29th International Conference on Mathematical Methods in Economics 2011 – part II*. 2011 , Praha, Czech Republic, p. 768-773, ISBN 978-80-7431-059-1
- HOLEČEK, P., TALAŠOVÁ, J.: The Software Support for Multiple-Criteria Evaluation – Various Types of Partial Evaluations Aggregation. Lecture Notes in Management Science. Proceedings of the 2nd International Conference on Applied Operational Research (Mikael Collan ,Ed), Vol 2, 2010, pp. 1-5, ISSN 2008-0050
- HOLEČEK P., TALAŠOVÁ, J.: Designing Fuzzy Models of Multiple-Criteria Evaluation in FuzzME Software. Proceedings of the 28th International Conference on Mathematical Methods in Economics 2010. Vol 1., pp. 250 – 256, ISBN 978-80-7394-218-2
- TALAŠOVÁ J, HOLEČEK P.: Multiple-Criteria Fuzzy Evaluation: The FuzzME Software Package. Proceedings of 2009 IFSA World Congress, July 20-24, 2009, p. 681-686, ISBN 978-989-95079-6-8
- HOLEČEK P., TALAŠOVÁ J.: A New Software for Fuzzy Methods of Multiple Criteria Evaluation. Proceedings of 8th International Conference of Applied Mathematics APLIMAT 2009, February 3-6, 2009, p. 387-398, ISBN 978-80-89313-31-0.
- HOLEČEK P., TALAŠOVÁ J.: A New Software for Fuzzy Methods of Multiple Criteria Evaluation. Journal of Applied Mathematics; Vol. 2, Number 1, 2009, p. 103-116, ISSN 1337-6365.

CONFERENCES

- HOLEČEK P.: *Linguistically oriented fuzzy modeling in the FuzzMe software*. International Student Conference on Applied Mathematics and Informatics ISCAMI 2015, Kočovce, Slovakia (23.4. - 26.4. 2015).

- HOLEČEK P., TALAŠOVÁ, J.: *Multiple-Criteria Fuzzy Evaluation in FuzzME - Transitions Between Different Aggregation Operators.* 32nd International conference on Mathematical Methods in Economics MME 2014, Olomouc (10. 9. - 12. 9. 2014)
- HOLEČEK P., TALAŠOVÁ, J.: *Multiple-Criteria Fuzzy Evaluation in FuzzME - Recent Development*. 20th Conference of the International Federation of Operational Research Societies IFORS 2014, Barcelona (13. 7. - 18. 7. 2014).
- HOLEČEK P., TALAŠOVÁ, J.: *Multiple-Criteria Fuzzy Evaluation in FuzzME - Possible Transitions Between Different Aggregation Types*. International Student Conference on Applied Mathematics and Informatics ISCAMI 2014, Malenovice (27. 3. - 30. 3. 2014).
- STOKLASA, J., TALAŠOVÁ, J., HOLEČEK, P.: *Academic faculty evaluation models: What approach is appropriate for European universities?*, 26th European Conference on Operational Research EURO 2013, Rome, Italy (1.7. – 4.7. 2013).
- HOLEČEK P., KREJČÍ, J., STOKLASA J., TALAŠOVÁ, J.: *Evaluation of R&D outcomes using fuzzified AHP*. International Student Conference on Applied Mathematics and Informatics ISCAMI 2013, Malenovice (2.5. - 5.5.2013).
- STOKLASA, J., TALAŠOVÁ, J., HOLEČEK, P.: *Linguistically oriented model for academic staff performance evaluation*, 25th European Conference on Operational Research EURO 2012, Vilnius, Lithuania (8.7 – 11.7 2012)
- TALAŠOVÁ, J., STOKLASA J., HOLEČEK P.: *Models of academic staff performance evaluation,* Efficiency and Responsibility in Education ERIE 2012 Prague (7.6. – 8.6.2012)
- TALAŠOVÁ, J., STOKLASA J., HOLEČEK P.: *Vícekriteriální model hodnocení akademických pracovníků,* 13. seminář Hodnocení kvality vysokých škol (3.5. - 4.5. 2012)
- TALAŠOVÁ J., PAVLAČKA O., BEBČÁKOVÁ I., HOLEČEK P.: A framework for fuzzy models of multiple-criteria *evaluation. International conference on fuzzy set theory and applications FSTA 2012*, Liptovský Ján, Slovak Republic (30.1. - 3.2. 2012)
- HOLEČEK P., TALAŠOVÁ, J., STOKLASA J.: Fuzzy classification systems and their applications, 29th International Conference on Mathematical Methods in Economics 2011 MME 2011, Janská Dolina, Slovakia (6.9. – 9.9.2011)
- HOLEČEK, P., STOKLASA, J., TALAŠOVÁ, J., ZEMKOVÁ, B.: Fuzzy classification and its applications in HR management, International Student Conference on Applied Mathematics and Informatics ISCAMI 2011 (6.5. – 8.5.2011)
- HOLEČEK P., TALAŠOVÁ, J.: The Software Support for Fuzzy Multiple-Criteria Evaluation. International Conference Olomoucian Days of Applied Mathematics - ODAM 2011 (26.1. – 28.1. 2011)
- HOLEČEK P., TALAŠOVÁ, J.: Designing Fuzzy Models of Multiple-Criteria Evaluation in FuzzME Software, MME 2010, České Budějovice (8.9.-10.9. 2010)
- HOLEČEK P., TALAŠOVÁ, J: The Software Support for Multiple-Criteria Evaluation – Various Types of Partial Evaluations Aggregation, ICAOR 2010, Turku, Finland (25.-27.8.2010)
- Holeček P., Talašová, J: Software for Multiple Criteria Evaluation Support - The Aggregation of Partial Evaluations by Choquet Integral, Informatics ISCAMI 2010, 20.5.-23.5.2010, Bratislava
- Talašová J, Holeček P.: Multiple-Criteria Fuzzy Evaluation: The FuzzME Software Package. 2009 IFSA Word Congress, 20.7. – 24.7.2009, Lisbon
- Holeček P., Talašová J.: FuzzME: A New Software for Multiple-Criteria Fuzzy Evaluation. ISCAMI 2009, 13. 5. – 15. 5. 2009, Malenovice

- HOLEČEK P., TALAŠOVÁ J.: A New Software for Fuzzy Methods of Multiple Criteria Evaluation, APLIMAT 2009, 3.2. – 4.2.2009, Bratislava

INVITED LECTURES

- HOLEČEK P.: FuzzME – the software for multiple-criteria fuzzy evaluation and fuzzy classification, University of Trento, Italy, 14. 3. 2012.

SEMINARS

- HOLEČEK P.: *Vícekriteriální hodnocení ve FuzzME - přechody mezi jednotlivými typy agregace*. Seminář projektu AMathNet Fuzzy modely a jejich aplikace. Olomouc (14. 3. 2014)
- HOLEČEK P.: *FuzzME - software pro vícekriteriální hodnocení*. Seminář projektu AMathNet Fuzzy modely a jejich aplikace. Olomouc (10. 5. 2013).
- STOKLASA, J., TALAŠOVÁ, J., HOLEČEK, P.: *Information system for academic faculty performance evaluation (IS HAP): an example of a linguistic approach used in the Czech Republic*, Università Ca' Foscari Venezia, Italy, 18. 4. 2013.
- STOKLASA, J., TALAŠOVÁ, J., HOLEČEK, P.: Approaches to (academic) staff performance evaluation: non fuzzy approaches vs. fuzzy approach. University of Trento, Italy, 14. 3. 2012

APPLIED RESEARCH RESULTS:

- FuzzME software (Holeček, P., Talašová, J., Pavlačka, O., Bebčáková, I.). Demo version can be downloaded at http://www.FuzzME.net
- Choquet software (Holeček, P., Talašová, J., O., Bebčáková, I.). The software makes it possible to calculate the fuzzied Choquet integral and use it for multiple-criteria evaluation. The software is freeware and can be downloaded at http://www.FuzzME.net
- Information system for academic staff performance evaluation IS HAP (software: Holeček, P., Doubrava, M., model: Talašová, J., Stoklasa, J.)

AWARDS:

- The first prize in the students contest "O cenu děkana": First place in the PhD section in the category of the mathematics and computer science (Bebčáková, I., Holeček, P.: Fuzzification of Choquet integral and its application in multiple criteria decision making).

TEACHING RESPONSIBILITIES:

Currently, I am teaching the following labs or seminars at the Palacký University in Olomouc:

- Fuzzy Sets and their Application 1
- Fuzzy Sets and their Application 2
- Theory and Methods of Decision Making 2
- Mathematical and Economical Software
- Statistical Software 1
- Statistical Software 2
- Statistical Software 3
- TeX for Beginners

PROJECTS INVOLVEMENT

- Grant agency of the Czech Republic project *Operations research methods for decision support under uncertainty.* [Metody operačního výzkumu pro podporu rozhodování v podmínkách neurčitosti.] (14-02424S) – research team member, 2014 - 2016.
- Individual national project *KREDO – Quality, relevance, efficiency, diversification and openness of higher education in the Czech Republic. Strategy of higher education until 2030.* [Kvalita, relevance, efektivita, diversifikace & otevřenost vysokého školství v ČR. Strategie vysokého školství do roku 2030.] – an expert of a key activity, 2014 - present.
- Several subsequent projects funded by the Czech government aiming on the development of the registry of artistic performances (including a mathematical evaluation model for creative work outcomes of Czech art colleges): *A pilot project of creative work outcomes evaluation to determine the VKM and B3 indicators for art colleges and faculties; C41 Evaluating Creative Work Outcomes Pilot Project*, 2010 – present.
- Several projects funded by the Internal grant agency of Palacký University, Olomouc: *Mathematical models and structures* [Matematické modely a struktury], 2009, 2010, 2011, 2012 and *Mathematical models* [Matematické modely] – research team member.
- Project *MAPLIMAT - Streamlining the Applied Mathematics Studies at Faculty of Science of Palacky University in Olomouc* [MAPLIMAT - Modernizace studia aplikované matematiky na PřF Univerzity Palackého v Olomouci] (CZ.1.07/2.2.00/15.0243) – expert for e-learning deployment and administration, 2010 - 2013.

OTHER SKILLS:

English and German language knowledge, experiences with the web applications and software development (.NET platform and PHP).

UNIVERZITA PALACKÉHO V OLOMOUCI
PŘÍRODOVĚDECKÁ FAKULTA

AUTOREFERÁT
DIZERTAČNÍ PRÁCE

Fuzzy models of multiple-criteria evaluation and fuzzy classification

Katedra matematické analýzy a aplikací matematiky
Vedoucí dizertační práce: **doc. RNDr. Jana Talašová, CSc.**
Vypracoval: **Mgr. Pavel Holeček**
Studijní program: P1104 Aplikovaná matematika
Studijní obor Aplikovaná matematika
Forma studia: prezenční
Rok odevzdání: 2015

Výsledky obsažené v dizertační práci byly získány během doktorského studia oboru Aplikovaná matematika na Katedře matematické analýzy a aplikací matematiky Přírodovědecké fakulty Univerzity Palackého v Olomouci.

Uchazeč: **Mgr. Pavel Holeček**
Univerzita Palackého v Olomouci
Přírodovědecká fakulta
Katedra matematické analýzy a aplikací matematiky

Školitel: **doc. RNDr. Jana Talašová, CSc.**
Univerzita Palackého v Olomouci
Přírodovědecká fakulta
Katedra matematické analýzy a aplikací matematiky

Oponenti: **prof. RNDr. Radko Mesiar, DrSc.**
Slovenská technická univerzita v Bratislavě
Stavebná fakulta
Katedra matematiky a deskriptívnej geometrie

prof. RNDr. Jaroslav Ramík, CSc.
Slezská univerzita v Opavě
Obchodně podnikatelská fakulta v Karviné
Katedra matematických metod v ekonomii

prof. RNDr. Karel Zimmermann, DrSc.
Univerzita Karlova v Praze
Matematicko-fyzikální fakulta
Katedra aplikované matematiky

Obhajoba se koná dne 24. 6. 2015 v 11:30, před komisí pro obhajobu dizertační práce vědního oboru Matematika, studijního oboru P1104 Aplikovaná matematika, v místnosti LP-5068 na Katedře matematické analýzy a aplikací matematiky Přírodovědecké fakulty UP na tř. 17. listopadu 12, Olomouc.

S dizertační prací je možné seznámit se v knihovně Přírodovědecké fakulty Univerzity Palackého v Olomouci.

Contents

Poděkování

Rád bych poděkoval doc. RNDr. Janě Talašové, CSc. za vedení na doktorském studiu a za cenné rady při přípravě této práce.

1. Introduction

1.1. Goals of the thesis

Several goals have been set for this thesis. The thesis should provide a complete description of the system of fuzzy multiple-criteria evaluation methods based on the fuzzification and extension of the Partial Goals Method [45]. Although the individual methods from the system have been published in various papers, the coherent description was missing, or it existed only in a reduced form [22, 45] because of the space limitation. The thesis should provide a complete description of the system in its full depth.

In frame of this system of methods, the further theoretical development should be performed. This has been accomplished by introducing multiple new results – specifically, by introduction of the FWOWA (fuzzified WOWA) aggregation method, by the proposal of the new Sugeno-WOWA inference, and by introduction of two methods for transitions from a simpler aggregation method to another more advanced one, which can be very useful when a model with interacting criteria is designed.

Another goal of the thesis was to create a software implementation of the mentioned system of methods. This led to the development of the FuzzME software. The FuzzME is a complex software tool equipped with variety of methods and algorithms that so far existed only on a theoretical level. The software enables the decision-makers to use these novel methods and it makes it possible for mathematicians to study the behavior of the methods on real-world applications.

The thesis aimed to focus also on the topic of fuzzy classification. Specifically, it dealt with the situation when the fuzzy rule base had already been set and the objects should be assigned to the matching classes accordingly. The thesis shows that an important role in this process is played by the type of the structure formed by the classes. It proposes also the form, in which the classification results should be presented. Again, the theoretical results have been implemented into the FuzzME software.

The FuzzME was applied to multiple real-world problems, either in direct cooperation with the author of this thesis, or by foreign authors. The thesis summarizes these applications so that the reader could see how the described methods can be used in the practice.

1.2. Structure of the thesis

The thesis starts with the description of its structure and its goals in the Section 1. Then, it deals with two topics – fuzzy multiple-criteria evaluation and fuzzy classification.

First, a system of fuzzy methods for solving multiple-criteria evaluation problems is discussed in the Chapter 2. The chapter begins with a brief introduction to this topic and it lists relevant software tools. It is shown, that there is a lack of complex software tools for fuzzy multiple-criteria evaluation, which emphasizes the importance of the FuzzME software that has been created in frame of this thesis. Next, the system of methods based on the Partial Goals Method [45] is described in detail.

In this thorough description, multiple new methods and results are introduced. The following sections present the software implementation of the whole system of fuzzy methods, the FuzzME software, and they list its real-world applications.

The Chapter 3 of the thesis is devoted to the topic of the fuzzy classification. After a description of the problem of interest, the fuzzy classification problems are divided into categories and each of them is studied separately. Finally, the use of the FuzzME software for fuzzy classification is discussed.

The thesis text ends with the list of the research accomplishments presented in the thesis and the summary of the main facts in the conclusion. The thesis also contains the documentation for the FuzzME software in the Appendix 1 and the author's curriculum vitae in the Appendix 2. The FuzzME software itself can be found on the CD attached to the thesis.

1.3. A summary of the research accomplishments presented in the thesis

The thesis presents multiple original results achieved by the author during his doctoral studies of the Applied Mathematics at the Palacky University, Olomouc. These results were presented at international conferences and published in multiple peer-reviewed journals and conference proceedings. A summary of publication outputs can be found in the enclosed curriculum vitae; the author's contribution to them is specified at the end of this section.

These results have been implemented in the FuzzME software developed by the author of this thesis that constitutes a universal multiple-criteria fuzzy evaluation tool. This software has been frequently used for research purposes by the research team focusing on the fuzzy MCDM (multiple-criteria decision-making) methods at the Palacky University, Olomouc (e.g. [12, 56]). Moreover, it has been applied also by foreign researchers (e.g. [1]). The software implements a complex system of fuzzy multiple-criteria evaluation methods. Even though the evaluation is its main application area, it can be also used for the fuzzy classification. According to the research of the resources available on the Internet, no other comparable software for the fuzzy multiple-criteria decision-making has been found (see Section 2.3.2 of the thesis for more details). The extensive system of methods, as well as the number of functions for the users, makes this software a unique tool in the area of the fuzzy MCDM.

The main original methods and results presented in the thesis are the following:

- the FuzzME software (Section 2.13 in the thesis), which is a multiple-criteria fuzzy evaluation tool developed by the author suitable for research purposes as well as practical applications (`http://www.FuzzME.net`) – the software and its gradual development have been described in [15, 47, 18, 17, 16, 22], the practical applications of the software (Section 2.14 in the thesis) have been published for example in [12, 56, 1];

- the fuzzification of the WOWA operator [22], study of its basic properties and relations to the other fuzzified aggregation operators – specifically, the

Definition 2.33 and the Theorems 2.2, 2.3, 2.4, 2.5, and 2.8 in the thesis;

- the new Sugeno-WOWA inference algorithm [18, 22];
- two methods for transition from a simpler aggregation method to another more general one (the fuzzified Choquet integral or a fuzzy expert system) [19] including the theorems verifying that the mapping obtained by the presented algorithm is a FNV-fuzzy measure;
- the proposal of division of the fuzzy classification methods according to the structure formed by the class identifiers and the discussion of the use of these methods in various applications contexts, especially with a focus on the use of the fuzzy classification for the purpose of evaluation [21, 20].

Many of the original methods presented in this thesis were created because the need for such a method has been identified during the solution of a practical problem – practical applications often required comprehensive study of the state-of-the-art methods and algorithms and inspired the author to develop new ones, hence creating new theoretical results. For example, the soft-fact rating problem described in the thesis was the impulse for proposal of the new Sugeno-WOWA inference, which turned out to be suitable for this application area.

The contribution of the author in the publications presenting the above-mentioned results that are listed in the enclosed curriculum vitae can be summarized in the following way.

The author has co-operated on the writing of the paper [40]. He is the author of the academic staff performance evaluation model based on the use of the WOWA (weighted ordered weighted average) aggregation operator, which represents the most advanced model from the first class of models described in the paper. The second class of the models, which is currently used in the IS HAP information system (see Section 3.3.3 in the thesis), is based on the fuzzy rule bases and the author of these models is Jan Stoklasa. Pavel Holeček is the author of the software implementation for this second class of models.

He has also co-authored another paper in a journal with non-zero impact factor [3] dealing with the fuzzified Choquet integral. He has implemented the methods proposed in this paper and designed an effective software tool for multiple-criteria fuzzy evaluation based on the Choquet integration.

He is also the main author of three papers in reviewed journals, which contain his original results. Specifically, the paper [15] introduces the first version of the FuzzME software and describes the methods used in the software. In the paper [18], the author proposes the Sugeno-WOWA inference algorithm. This inference algorithm makes it possible to derive the fuzzy evaluation by means of fuzzy rule bases but it also takes into account a vector of weights, which represent the optimism or pessimism of the decision-maker. In the paper [20], he studies systematically various fuzzy classification scenarios.

Pavel Holeček is the main author of the book chapter [22], which contains a broad overview of the topic of his Ph.D. thesis. In this book chapter, he has also introduced a new fuzzified WOWA aggregation operator. This fuzzified aggregation operator has been implemented into the FuzzME software. He is also a co-author of 2 other book chapters [50, 49].

He is the main author of 5 papers in peer-reviewed conference proceedings. They also contain original research results that comprise the basis for his Ph.D. thesis. For example, the paper [19] introduces a new group of methods and algorithms for transitions between different types of aggregation methods. They can simplify the whole process of designing the evaluation models as they make it possible to start with a simple mean of aggregation and they propose the settings for a more complex aggregation method. He is also a co-author of 4 other papers in conference proceedings.

The results of the thesis were presented on multiple conferences – specifically, 18 presentations on international conferences and 1 presentation on a national conference. The author has also been involved in several research projects and research internships. The detailed list of the publication outputs as well as other research activities can be found in the enclosed curriculum vitae.

2. Fuzzy models of multiple-criteria evaluation and decision-making

Making decisions is one of important humans' skills. For simpler problems, the decision can be made just by an intuition. However, with increasing number of alternatives and criteria that should be taken into account, the problem can easily become too complex. For important problems, for example in the business, making a wrong decision can be very costly. Moreover, in many situations, the transparency is required – sometimes even by the law. For example, in case of public tenders, the reasons that led to choosing the particular winner cannot be concealed in the "black box" of the human brain. Formalized methods for finding the best decision are necessary.

Generally, a multiple-criteria decision-making (MCDM) problem has the following structure. A set of alternatives $A = \{A_1, \ldots, A_n\}$ is given. These alternatives are assessed with respect to a given set of criteria $C = \{C_1, \ldots, C_m\}$. The pursued task is to find the best alternative from A taking into the consideration the criteria values themselves and also the additional information about the criteria (the decision-maker's preferences related to the criteria values and to the criteria themselves, and potentially, the interactions among the criteria).

To find the optimum alternative, it is sufficient just to calculate the overall evaluations of the alternatives and to choose the alternative with the maximum evaluation. The multiple criteria evaluation (MCE) can be thus seen as the first step in solving the MCDM problem. Depending on the character of the obtained evaluations, we are able to make different conclusions. We can distinguish the following types of evaluation:

- **Ordinal evaluation:** In this case, the evaluation expresses just the ordering of the alternatives. If we know the ordering of the alternatives, we can select the best one. However, an ordinal evaluation is not sufficient to quantify the differences in performances of the various alternatives and a stronger type of evaluation must be used in these cases.

- **Cardinal evaluation of relative type:** Such an evaluation contains more information. When a ratio scale is used, we can say from the proportion of evaluations of two alternatives that the first alternative is, e.g., twice as good as the second one. When an interval scale is used and we consider three alternatives A_1, A_2 and A_3 ordered from the best one to the worst one, we can make conclusions such as that the difference between the evaluations of A_1 and A_2 is twice as big as the difference between the evaluations of A_2 and A_3. But still we cannot determine if these alternatives fulfill the given goal enough for the decision-maker. For example, this evaluation type would be insufficient for making decision whether a bank should grant a credit to a particular client.

- **Cardinal evaluation of absolute type (with respect to a given goal):** This evaluation type provides us with the most extensive information. Not only that we can compare the alternatives, but we can also say for each of them how much it satisfies the decision-maker's needs. In case of the bank, the evaluation would express the creditworthiness of a client. Then, the bank can decide on a threshold and grant the credit only to the clients with evaluation above this threshold.

The thesis deals with a system of methods that produce cardinal evaluations of the absolute type (with respect to the given goal of the decision-maker), i.e. the strongest of the listed three evaluation types. This makes it possible to solve a broad range of MCDM problems.

A vast number of decision-making approaches have been developed over the time – from simple methods to highly sophisticated ones.

A large group of the methods is based on combining of the partial evaluations with respect to the criteria into the total evaluation by some aggregation operator. The simplest and obviously the most popular aggregation operator is the weighted average. These methods then differ in the way how the partial evaluations and the weights were obtained and on their interpretation.

A well-known approach that belongs to this group of methods is the Multiple Attribute Utility Theory (MAUT), which is based on the principles published in [55, 11]. The MAUT is theoretically very elaborate. One of its major advantages is that it can address the risk. However, an obstacle in using this method in practice is the amount of information that has to be provided. The nature of the information can also present a problem. The decision-maker is required to compare imaginary alternatives that may not be meaningful in the real world. Nevertheless, the MAUT has been applied in many fields.

Another very popular method from this group is the Analytical Hierarchy Process (AHP) [35] proposed by Thomas L. Saaty. An important feature of the AHP is that the weights and evaluations of the alternatives with respect to the criteria are obtained by pair-wise comparisons, which are expressed using linguistic terms. The intensities of preferences for the pairs are written in form of the Saaty's matrix. The matrix is reciprocal and it consists of the values from Saaty's scale (numbers

$1/9, 1/7, \ldots, 1, \ldots, 7, 9$ corresponding to the selected linguistic descriptions) with 1 on the main diagonal. The original AHP is based on the eigenvector method. In practice, modifications of this original approach with different methods to obtain the weights vector (or the partial evaluations vectors) form the matrix can be encountered. A distinct feature of AHP is also the use of a hierarchical structure for the description of the decision-making process.

A generalized version of AHP is called Analytical Network Process (ANP) [36, 37]. The Analytical Network Process makes it possible to take into account also the interactions between the criteria. The hierarchical structure typical for AHP is replaced by a more general network structure in ANP. Both the AHP and ANP have been applied on many important real-world problems [38] by well-recognized organizations such as the Nuclear Regulatory Commission of the US, Xerox Corporation, British Airways, IBM and others.

Another large group of methods, which are called the outranking methods, is base on a different principle. They construct preference (outranking) relations for each of the criteria. Those relations are used to make the final decision. This group is quite diverse as it contains several methods and, for each of them, multiple versions exist. The best known representatives are the ELECTRE [34] and PROMETHEE methods [5].

Those are only a few of the best known methods. Naturally, many more of them exist. More information on MCDM methods can be found e.g. in [10].

With the development of the fuzzy sets theory, fuzzy MCDM methods were appearing. Multiple-criteria decision-making was even one of the earliest applications of fuzzy sets – Bellman and Zadeh made a connection between these two areas by introducing the notion of a fuzzy goal [4].

Instead of devising a new fuzzy method, many authors tried to incorporate the fuzzy notions in the classical time-proved MCDM methods. Such fuzzifications range in their quality from simple naive ones to highly sophisticated methods.

As AHP represents one of the most popular MCDM methods, several attempts for its fuzzification have been made. This was a source of a critique by the founder of the method Thomas L. Saaty, who is a strong opponent of incorporating any fuzziness into AHP [39]. Nevertheless, many fuzzy AHP approaches (including related approaches that use e.g. geometric mean instead of the original eigenvector method) appeared. Among the first of them, van Laarhoven and Pedrycz [54] proposed a fuzzy method using triangular fuzzy numbers. Later, Chang [6] proposed extent analysis, which became quite popular in some areas; however, many flaws of the method have been addressed later [58]. Among newer sophisticated approaches, the one in the paper [30] can be named. The paper dealt also with another issue – how to measure the inconsistency of the pairwise comparison matrices with fuzzy elements. A new inconsistency index is proposed in the paper. In [31], a method that can handle also the dependencies among the criteria is described.

Also the other methods have been subject to the fuzzification. For example fuzzy ELECTRE has been used in [33], or fuzzy PROMETHEE has been proposed in [13].

The thesis deals a coherent system of fuzzy multiple-criteria evaluation methods. The feature common to all of these methods is the used type of evaluation – cardinal evaluations of absolute type (with respect to a given goal) are employed. The foundation of the system has been laid by the Solver methodology introduced in the book [45]. Over the time, the system has been developed and extended rapidly. The described system of methods is quite powerful and relatively easy for the decision-maker at the same time.

2.1. The system of fuzzy multiple-criteria evaluation methods used in the thesis

The problem studied in the thesis is to construct a complex mathematical model for evaluating alternatives of certain type with respect to a given goal. The overall goal can be divided step by step into partial goals of a lower level. The degrees of fulfillment of the partial goals on the lowest level can be then assessed by corresponding characteristics of alternatives – criteria.

Various requirements of the evaluator on the behavior of evaluating function should be met. The model of multiple-criteria evaluation is able to process uncertain, expertly-defined data and to utilize expert knowledge related to the evaluation process. Moreover, the number of the used criteria can be high and interactions among them can be present. The set of evaluated alternatives is not required to be known in advance. Therefore, an evaluation model can be designed first and then it can be applied to the individual incoming alternatives.

Because we do not only compare alternatives in a given set but we also assess how much do the alternatives, which enter the system progressively one by one, meet our requirements, an evaluation of the relative type cannot be used, and an evaluation of the absolute type with respect to a given goal must be utilized.

The fuzzy evaluation used in the described system of methods expresses to what extent does the alternative meet the pursued goal. All the evaluations are expressed by fuzzy numbers defined on the interval [0, 1]. These fuzzy numbers then express uncertain degrees of fulfillment of a given goal by respective alternatives [44, 45]. For example, a fuzzy evaluation in form of a triangular fuzzy number $(0.6, 0.75, 0.8)$ expresses that the alternative is most likely to reach the given goal at 75%, however, the degree of the fulfillment is admitted to range from 60% to 80%.

The basic structure of the fuzzy model of multiple-criteria evaluation, which is considered in this thesis, is expressed by a goals tree. The root of the tree represents the overall goal of evaluation and each other node corresponds to a partial goal. The goals at the ends of the goals tree branches are associated with either quantitative or qualitative criteria.

First, when an alternative is evaluated, evaluations with respect to the criteria connected with the terminal branches are calculated first. Independently of the criterion type, each evaluation is described by a fuzzy number defined on $[0, 1]$ expressing the fuzzy degree of fulfillment of the corresponding partial goal.

According to qualitative criteria, alternatives are evaluated verbally by means of values of linguistic variables of a special kind – linguistic evaluating scales. Mathematical meanings of the linguistic values are modeled by fuzzy numbers on $[0, 1]$, as mentioned above.

The evaluation according to a quantitative criterion is calculated from the measured value of the criterion (which can be a real number or a fuzzy number) by means of an evaluating function expertly defined for that criterion. The evaluating function is the membership function of the corresponding partial goal defined on the domain of the criterion of interest.

In the next step, the partial fuzzy evaluations are consecutively aggregated according to the structure of the goals tree by one of several supported methods (fuzzy weighted average, fuzzy OWA, fuzzified WOWA, fuzzified Choquet integral, or fuzzy expert system). The choice of the appropriate method depends on the evaluator's requirements and on the relationships among the evaluation criteria.

If the importances of the individual criteria are known and there are no interactions among them, the decision-maker can use the fuzzy weighted average (FuzzyWA, see [29]) for the aggregation. If different weights are assigned to the individual partial evaluations in dependence on their order, the fuzzy ordered weighted average (FuzzyOWA, see [46]) can be employed. If both of these aspects should be taken into the account, it can be accomplished by the fuzzified WOWA operator (FWOWA, see e.g. [22], for crisp WOWA operator see [51]).

When relationships of redundancy or complementarity that are stable over the whole domain of criteria are present, the fuzzified discrete Choquet integral is used (for fuzzy Choquet integral see [2]; for crisp Choquet integral see [7, 14], or [53]). In case of more complex interactions among the criteria, a fuzzy expert system has to be used. The fuzzy expert system can be applied under any complex relationship among criteria – if the expert knowledge of the evaluation rules is known. Generally, it holds that any continuous (even any Borel-measurable) function can be approximated to arbitrary precision by a fuzzy rule base with a finite number of rules and a suitable inference algorithm (more information can be found in [24]). For that reason, fuzzy expert systems with various approximate-reasoning algorithms (Mamdani, Sugeno) can be used to aggregate the partial evaluations under complex interactions. Fuzzy expert systems make it possible to utilize expert knowledge for modeling complex evaluating functions. In order to obtain required properties of the evaluating functions, it is possible to modify the usual fuzzy-inference algorithms by employing a less common aggregation method. As an example of such a modified inference algorithms, the Sugeno-WOWA [18] can be named.

The final result of the consecutive aggregation of partial fuzzy evaluations (the fuzzy evaluation in the root of the goals tree) is the overall fuzzy evaluation of the given alternative. The obtained overall fuzzy evaluation is again a fuzzy number on $[0, 1]$. It expresses the uncertain degree of fulfillment of the main goal by the particular alternative.

The thesis describes the whole system of methods in detail. It also discusses the presentation of the results to the decision-maker. Various forms of presentation can be used. The system of methods considered in the thesis uses the principles of the linguistic fuzzy modeling to the maximum extend. Therefore, for the overall resulting fuzzy evaluations, their verbal descriptions are also available. This makes easier to interpret the evaluation results.

Because the most of the methods require non-trivial calculations, a suitable software tool is necessary in order to use these methods in practice. That is why, the FuzzME software, which implements the coherent system of fuzzy MCDM methods described in the thesis, has been developed.

2.2. The FuzzME software

The first foundations of the described system of methods were laid by the book [45]. The book describes a methodology called the Solver. Over more than ten years, this methodology has been extended and improved rapidly. A coherent complex system of fuzzy multiple-criteria evaluation methods, which are described in the thesis, has been formed. One of the major goals of this thesis was to create a software implementation of this whole system of methods. The resulting software is called FuzzME. Its name is an acronym of **Fuzz**y Methods of **M**ultiple-Criteria **E**valuation.

Transferring such a large system of mathematical methods, which moreover can be arbitrarily combined in a single fuzzy MCDM model, into a form of software presents many challenges. First, the methods have to be implemented in an effective way. Efficiency is necessary in order to be able to use the software on large complex problems. This goal has been achieved in the FuzzME well. The evaluations are calculated in the real time. As soon as any parameter of the model is changed, the evaluations are recalculated immediately so the expert can see the impact of the performed changes at once.

The next requirement, which is no less important, was that the software has to be intuitive and user-friendly. The FuzzME accompanies the numeric results with a graphical output and the evaluations can be also described verbally.

The creation of a model in the FuzzME is made as easy as possible for the user. The first step is to design a goals tree for the given problem. The user creates the structure of the goals tree and then he/she determines the type of each node. For nodes at the ends of the goals tree branches, the user selects between qualitative and quantitative type of criteria. For the rest of the nodes, an appropriate aggregation method is chosen. All the aggregation methods can be arbitrarily combined within the same goals tree.

The FuzzME takes the maximum advantage of the linguistic approximation. The user can design a linguistic scale for each partial goal. Then, the linguistic description for the evaluation of the particular partial goal is available. The linguistic scale is designed in the Linguistic scale editor. The process is simplified for the user as much as possible. The user is asked how many terms should be in the scale and what type of fuzzy numbers should be used to model them. The FuzzME then creates a uniform scale and the user just types appropriate names for the terms, or adjusts the fuzzy numbers representing their meanings if necessary. The frequently used scales can be saved into files and reused later easily.

As the next step when a model is created in the FuzzME, it is necessary to fill in information for all goals tree nodes representing the partial goals. The required information depends on the selected method. For the FuzzyWA and FuzzyOWA, normalized fuzzy weights are defined. For the fuzzified WOWA operator, two vectors of normalized (real) weights are set. The fuzzified Choquet integral requires a FNV-fuzzy measure to be defined. And finally, a fuzzy rule base must be designed if a fuzzy expert system is used. Concerning the criteria, for each quantitative criterion, the user must specify its domain and define the evaluating function. For qualitative criteria, it suffices to define the linguistic evaluating scales.

The FuzzME strives to make the process of setting various parameters of the model as simple as possible. For example, when normalized fuzzy weights should be set, the FuzzME checks if the fuzzy numbers provided by the expert form normalized fuzzy weights. If they do not, the FuzzME offers a remedy. This way, the normalized fuzzy weights can be derived using the methods described in the thesis by a single click. Similarly, when the user has to set a FNV-fuzzy measure, the monotonicity condition is checked automatically. If the condition is breached, the FuzzME also reports, which of the FNV-fuzzy measure values should be modified.

When all the above mentioned steps are completed, the model is finished and ready for the evaluation process. The alternatives can be inserted manually by the user, but import from outside sources, such as Microsoft Excel, is also supported. The resulting evaluations can be exported, e.g. to Excel, for further analysis and processing.

The FuzzME makes it possible to evaluate multiple alternatives. They are displayed to the decision-maker in a list showing the membership functions graphs of their resulting fuzzy evaluations and also the linguistic descriptions of the final fuzzy evaluations. This way, the information necessary for a qualified decision are summarized in a single list in the comprehensible form. The alternatives can be ordered with respect to the centers of gravity of their fuzzy evaluations. For each alternative, the decision-maker can view the evaluation according to any partial goal to get a deeper insight in the performance of the particular alternative according to the various partial goals.

The FuzzME also offers an analytical tool. It makes it possible to study the behavior of the designed evaluation function, or to plot its graph (for 3D visualization, a connection to the MATLAB is used).

Moreover, the graphics in the FuzzME can be saved as an image, which makes the documentation of the designed model and its publication much easier. The majority of the images in the thesis have been created in the FuzzME. Many options for the resulting images can be set. For example, it is possible to set colors, line width, line style, labels, and much more settings for any of the drawn objects. This way, color or black-and-white high-quality graphics suitable for publication in professional journals can be generated.

The FuzzME has been written in the C# programming language. It requires .NET framework 2.0 (this library is a standard part of Windows and it is usually not necessary to install it). The software is multi-platform. It can run on both Windows and Linux. For Linux, a special implementation of the .NET framework, which is called Project Mono, has to be installed.

The FuzzME can be downloaded from `http://www.FuzzME.net`. It is is also included on the CD enclosed to the thesis.

As soon as the first version of the FuzzME was released, its applications on real-world problems begun to emerge. The thesis presents the selected applications. The first one is the evaluation of the clients by one of the Austrian banks (Section 2.14.1 in the thesis). This evaluation can be then used by the bank in the decision-making whether the particular client should be granted a credit or not.

The second application presents an evaluation of the employees in an IT company. The results calculated in the FuzzME are used not only for a direct assessment of the employees but also for their classification into one of the predefined types. For each of the employees types, a different motivation strategy is used. More details can be found in the Section 2.14.2 in the thesis.

In the third application, a safety of agri-food buildings is assessed (Section 2.14.3 in the thesis). The authors of this application designed quite a sophisticated model using the methodology described in the thesis and the FuzzME software to evaluate various safety aspects in the buildings where the food is produced. They divide each building into several areas that are evaluated separately in order to identify the areas where an improvement should be made.

The last one of the applications described in the thesis is comprised of several smaller models concerning photovoltaic power plants (Section 2.14.4 in the thesis). In these models, their author tested several methods from the system of methods described in the thesis.

The solved problems come from diverse areas. This fact shows the versatility of the FuzzME software as well as the described system of fuzzy multiple-criteria evaluation methods.

2.3. Highlights of the new methods and results

In the frame of the thesis, the system of multiple-criteria evaluation methods described in the thesis has been extended, and several new methods and original results have been introduced. They will be briefly outlined in the following text. Because of the extensiveness of the topic and the limited space, only the basic idea for each of the results will be described and the reader is kindly asked to refer to the Chapter 2 of the thesis for the full description, related information, and illustrative examples.. The used notation and the basic notions are summarized in the Section 2.2 of the thesis.

In the this text, the basic principles of the following theoretical results from the area of fuzzy MCDM will be described: the fuzzified WOWA operator, the Sugeno-WOWA inference method, and two algorithms that can significantly simplify the design of a complex fuzzy multiple-criteria evaluation model – they can be applied when a simpler aggregation method is used and it should be replaced with a more complex one. Specifically, the first algorithm makes it possible to start with the fuzzy weighted average, FuzzyOWA, or fuzzified WOWA and replace them with the fuzzified Choquet integral. The corresponding FNV-fuzzy measure (fuzzy-number-valued fuzzy measure) is proposed by the algorithm. The latter algorithm makes it possible to use the fuzzy weighted average, FuzzyOWA, fuzzified WOWA, or fuzzified Choquet integral as the starting point and replace it by a fuzzy expert system. Again, the fuzzy rule base suitable for this purpose is proposed.

2.3.1. Fuzzified WOWA operator

In many situations, the weighted average or the OWA operator can be used for the aggregation of the partial evaluations. The weighted average is used when the significances of the individual partial goals are given. On the other hand, the OWA is used when the importances of evaluations with respect to the partial goals are given by the order of these evaluations. If the expert needs to take into account both aspects, one of the possible solutions is the WOWA (weighted OWA) operator introduced by Torra in [51].

The WOWA aggregates values expressed by real numbers. As the system of multiple-criteria evaluation methods studied in the thesis uses evaluations in form of fuzzy numbers, it is necessary to use a more general method that is able to work with partial evaluations expressed by fuzzy numbers – this new method, the fuzzified WOWA operator, is introduced in the thesis.

The WOWA operator uses two m-tuples of normalized weights – the first of them $\mathbf{p} = (p_1, p_2, \ldots, p_m)$, is connected to the individual partial goals; the latter one $\mathbf{w} = (w_1, w_2, \ldots, w_m)$, is related to the decreasing order of partial evaluations. The aggregation with the WOWA operator is performed according to the following definition.

Definition 2.1 *Weighted Ordered Weighted Average (WOWA) of the values $u_1, \ldots, u_m$ using the normalized weights $\mathbf{p} = (p_1, \ldots, p_m)$ and $\mathbf{w} = (w_1, \ldots, w_m)$ is defined as*

$$\mathrm{WOWA}_{\mathbf{w}}^{\mathbf{p}}(u_1, \ldots, u_m) = \sum_{i=1}^{m} \omega_i \cdot u_{\phi(i)}, \tag{1}$$

where ϕ denotes such a permutation of the set of indices $\{1, \ldots, m\}$ that $u_{\phi(1)} \geq u_{\phi(2)} \geq \ldots \geq u_{\phi(m)}$. The weight ω_i is defined as

$$\omega_i = z(\sum_{j \leq i} p_{\phi(j)}) - z(\sum_{j<i} p_{\phi(j)}), \tag{2}$$

for $i = 1, \ldots, m$, and z is a nondecreasing function interpolating the following points

$$\{(0,0)\} \ \cup \ \{(i/m, \ \sum_{j \leq i} w_j)\}_{i=1,\ldots,m}. \tag{3}$$

The function z is required to be a straight line when the points can be interpolated in that way.

Several ways of constructing the interpolation function z are discussed in the literature (e.g. [52]). In the thesis, z is a piecewise linear function connecting the individual points.

Contrary to the WOWA operator, which aggregates values expressed by real numbers, the fuzzified WOWA is able to process uncertain partial evaluations expressed by fuzzy numbers. It is defined as follows.

Definition 2.2 *Let $U_1, \ldots, U_m$ be fuzzy numbers defined on $[0,1]$ and let $\mathbf{p} = (p_1, \ldots, p_m)$ and $\mathbf{w} = (w_1, \ldots, w_m)$ be two vectors of normalized (real) weights. Then the result of the aggregation by a fuzzified WOWA operator is a fuzzy number U with the membership function defined for any $y \in [0,1]$ as follows:*

$$U(y) = \max\big\{ \min\{U_1(u_1), \ldots, U_m(u_m)\} \mid u_i \in [0,1], i = 1, \ldots, m, \\ y = \mathrm{WOWA}^{\mathbf{p}}_{\mathbf{w}}(u_1, u_2, \ldots, u_m)\big\}. \tag{4}$$

The thesis studies the properties and behavior of the new fuzzified WOWA. One of the results presented in the thesis, which is useful in the practice, is the following theorem. It shows an efficient and straightforward way how the fuzzified WOWA can be calculated.

Theorem 2.1 *The result of the fuzzified WOWA of the fuzzy numbers $U_1, \ldots, U_m$ defined on $[0,1]$ with the weights $\mathbf{p} = (p_1, \ldots, p_m)$ and $\mathbf{w} = (w_1, \ldots, w_m)$ is a fuzzy number U defined for any $\alpha \in [0,1]$ as follows:*

$$\underline{u}(\alpha) = \mathrm{WOWA}^{\mathbf{p}}_{\mathbf{w}}(\underline{u}_1(\alpha), \underline{u}_2(\alpha), \ldots, \underline{u}_m(\alpha)), \tag{5}$$

$$\overline{u}(\alpha) = \mathrm{WOWA}^{\mathbf{p}}_{\mathbf{w}}(\overline{u}_1(\alpha), \overline{u}_2(\alpha), \ldots, \overline{u}_m(\alpha)). \tag{6}$$

.

The thesis studies also the relation of the fuzzified WOWA to other fuzzified aggregation operators. Specifically, the following relations are proved in the thesis:

1. The fuzzified WOWA is a generalization of the non-fuzzy WOWA operator.
2. The fuzzified WOWA generalizes the first-level fuzzy weighted average.
3. The fuzzified WOWA generalizes the first-level fuzzy OWA operator.
4. The fuzzified WOWA is a special case of the FuzzyWA and FuzzyOWA, when normalized fuzzy weights used for these aggregation methods consist only of fuzzy singletons.

The presented fuzzified WOWA constitutes a significant part of the system of fuzzy methods considered in the thesis. It can be used in situations when the decision-maker requires to take into account both importances of the individual partial goals and the importances of the partial evaluations according to their order. The advantage of the fuzzified WOWA compared to more complex aggregation methods that could be also applicable in these cases (e.g. the fuzzified Choquet integral, or the fuzzy expert system) is a lower number of parameters that have to be set and an easier interpretation of these parameters.

2.3.2. Sugeno-WOWA inference method

A fuzzy expert system can be used for aggregation of the partial evaluations. It can be applied even in cases with complex interactions among the criteria. However, it is necessary to have an expert knowledge about the evaluating function to be able to create the fuzzy rule base representing the multiple-criteria evaluation function.

The fuzzy-rule base, which models the relationship between the partial evaluations of lower level and the aggregated evaluation, has the following form:

$$\begin{array}{l} \text{If } \mathcal{E}_1 \text{ is } \mathcal{A}_{1,1} \text{ and } \dots \text{ and } \mathcal{E}_m \text{ is } \mathcal{A}_{1,m}, \text{then } \mathcal{E} \text{ is } \mathcal{B}_1, \\ \text{If } \mathcal{E}_1 \text{ is } \mathcal{A}_{2,1} \text{ and } \dots \text{ and } \mathcal{E}_m \text{ is } \mathcal{A}_{2,m}, \text{then } \mathcal{E} \text{ is } \mathcal{B}_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \text{If } \mathcal{E}_1 \text{ is } \mathcal{A}_{n,1} \text{ and } \dots \text{ and } \mathcal{E}_m \text{ is } \mathcal{A}_{n,m}, \text{then } \mathcal{E} \text{ is } \mathcal{B}_n, \end{array}$$

where for $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$:

- $(\mathcal{E}_j, \mathcal{T}(\mathcal{E}_j), [0, 1], M_j, G_j)$ are linguistic scales representing partial evaluations,
- $\mathcal{A}_{ij} \in \mathcal{T}(\mathcal{E}_j)$ are linguistic values from these scales, and $A_{ij} = M_j(\mathcal{A}_{ij})$ are fuzzy numbers on $[0, 1]$ representing their meanings,
- $(\mathcal{E}, \mathcal{T}(\mathcal{E}), [0, 1], M, G)$ is a linguistic scale representing the overall evaluation,
- $\mathcal{B}_i \in \mathcal{T}(\mathcal{E})$ are linguistic values from this scale, and $B_i = M(\mathcal{B}_i)$ are fuzzy numbers on $[0, 1]$ representing their meanings.

Several inference methods can be then used to calculate the overall evaluation. The Mamdani inference [26] represents one of the best known of them. Another example of the inference method is Sugeno-WA [45], which generalizes the classical Sugeno approach [42].

For specific cases, Sugeno-WOWA inference can be used [18]. This method requires, besides a fuzzy rule base, normalized weights $w_1, w_2, \dots, w_s$. These normalized weights are assigned to the individual values of the linguistic scale representing the output variable $\mathcal{E}$. By these weights, the expert can express his/her optimism or pessimism (a pessimist assigns larger weights to bad evaluations, while an optimist to good ones). This can be utilized, for example, when the risk of a bank client is evaluated by a fuzzy expert system. The scale for the resulting evaluations can consist of the following terms – *very high risk*, *high risk*, *medium risk*, and *no risk recognized*. The expert can assign, for example, a weight 0.45 to the term *very high risk*, 0.35 to *high risk*, 0.15 to *medium risk*, and 0.05 to *no risk recognized*.

Let us recall that $(\mathcal{E}, \mathcal{T}(\mathcal{E}), [0, 1], M, G)$ represents the linguistic scale for the right-hand side of the rules. Then, let $\mathcal{T}_i \in \mathcal{T}(\mathcal{E})$, $i = 1, \dots, s$, be all its linguistic terms and let T_i, $i = 1, \dots, s$, denote the fuzzy numbers that model the meanings of these terms, i.e $T_i = M(\mathcal{T}_i)$, where $\mathcal{T}_i \in \mathcal{T}(\mathcal{E})$, $i \in \{1 \dots, s\}$.

For each term $\mathcal{T}_i$, $i = 1, \dots, s$, we can calculate the value p'_i, $i = 1, \dots, s$, which expresses the sum of the degrees of correspondence of all rules that have the term $\mathcal{T}_i$

on their right-hand side:

$$p'_i = \sum_{\substack{j \in \{1,\dots,n\}: \\ \mathcal{B}_j = \mathcal{T}_i}} h_j. \tag{7}$$

These values are normalized – normalized weights $p_1, \dots, p_s$ are obtained as follows:

$$p_i = \frac{p'_i}{\sum_{j=1}^{s} p'_j}, \tag{8}$$

for $i = 1, \dots, s$. As $\mathcal{E}$ represents a linguistic scale, the mathematical meanings of its terms form a fuzzy scale and they are ordered in sense of the Definition 2.13 in the thesis, i.e. it holds that $T_i < T_{i+1}$, $i = 1, \dots, s-1$. Then, the expert provides a vector of the normalized weights $\mathbf{w} = (w_1, w_2, \dots, w_s)$ with a similar interpretation as the weights used for the OWA. The weight w_i, $i = 1, \dots, s$, provided by the expert, corresponds to the i-th greatest of those fuzzy numbers, i.e. to the T_{s-i+1}. Then, the fuzzified WOWA is be used to calculate the aggregation result U, which is a fuzzy number on $[0, 1]$:

$$U = FWOWA^{\mathbf{p}}_{\mathbf{w}}(T_1, T_2, \dots, T_s). \tag{9}$$

Using the weights $\mathbf{w}$, the expert's optimism or pessimism can be taken into the account. The advantages, behavior, and the use of this method is discussed in the thesis. Specifically, the Sugeno-WOWA has been tested on an application from the area of banking; it showed that it is particular suitable for this area as it makes it possible to take into consideration also the expert's optimism or pessimism.

2.3.3. Transition from a simpler aggregation method to a more complex one

The design of the evaluation model is seldom a one-step process. Once the model is created, it should be tested and adjusted according to the test results. There is always a trade-off between the precision and the complexity. Building a complex model places considerable demands on the expert, who is required to provide a great amount of information (i.e. fuzzy rules, FNV-fuzzy measure values, etc.). It is therefore desirable to design the simplest possible model that reflects the reality sufficiently according to the expert.

That is why it is often better to start with a simple model that presents just a rough approximation of the final model. In the next step, the model is tested and its parts that should be improved are identified. This improved model is tested again and the process is repeated until the expert is satisfied with the model performance. During this process, it can be often found out that the aggregation method used for a particular goal in frame of the goals tree in the original simple model has to be replaced by a more complex one. The thesis studies this situation. Two algorithms have been proposed to make the transition to the new more complex aggregation function (specifically, to the fuzzified Choquet integral or to the fuzzy expert system) as simple as possible. The algorithms make it possible to derive the parameters of the new aggregation function from the parameters of the original one. The expert can then adjust only some parameters for the new more complex method instead of setting all of them. This way, a significant amount of time and effort can be saved. This work has been published in [19].

Two cases are studied in the thesis. In the first case, the original aggregation method should be replaced by a fuzzified Choquet integral. In the latter case, a fuzzy expert system should be used as the new aggregation method.

Transition to the fuzzified Choquet integral

Let us assume the situation when a fuzzy weighted average, a fuzzy OWA, or a fuzzified WOWA has originally been used for the aggregation and the expert would like to use the fuzzified Choquet integral instead. The fuzzified Choquet integral requires $2^m - 2$ values of the FNV-fuzzy measure to be set by the expert (the last two values of the FNV-fuzzy measure are given by its definition). The thesis presents an algorithm that proposes the suitable FNV-fuzzy measure. The expert then only modifies some values of this new FNV-fuzzy measure according to his/her requirements. This way a lot of time and effort of the expert can be saved; the benefits of this approach are demonstrated on an example in the thesis.

Algorithm 2.1 *Let a FNV-function $\tilde{f} : \mathcal{F}_N([0,1])^m \to \mathcal{F}_N([0,1])$ represent a FuzzyWA, or a FuzzyOWA with normalized fuzzy weights $W_1, \ldots, W_m$, or a fuzzified WOWA with normalized real weights $w_1, \ldots, w_m$ and $p_1, \ldots, p_m$. Let $G = \{G_1, \ldots, G_m\}$ be the set of individual partial goals of interest. Then, the FNV-fuzzy measure $\tilde{\mu}$ on G is derived from $\tilde{f}$ as follows:*

1. *The value $\tilde{\mu}(\emptyset) = \tilde{0}$ is set by the definition.*
2. *Similarly, the value $\tilde{\mu}(G) = \tilde{1}$ is set by the definition.*
3. *For the rest of the $2^m - 2$ values, the FNV normalized measure of K, $K \subset G$, is calculated as $\tilde{\mu}(K) = \tilde{f}(C_1, \ldots, C_m)$, where the fuzzy numbers C_i, $i = 1, \ldots, m$, are defined as follows:*

$$C_i = \begin{cases} \tilde{1} & \text{if } G_i \in K, \\ \tilde{0} & \text{otherwise.} \end{cases}$$

In the thesis, it is proved that the mapping $\tilde{\mu}$ obtained by the algorithm is really a FNV-fuzzy measure, providing that the FNV-function $\tilde{f}$ represents one of the permitted aggregation methods (FuzzyWA, FuzzyOWA, or fuzzified WOWA).

Transition to the fuzzy expert system

The fuzzified Choquet integral can handle only certain types of interactions among the partial goals (a complementarity or a redundancy). If the relationship is more complex, a fuzzy expert system can be used. Therefore, the thesis deals also with the situation when a fuzzy weighted average, a fuzzy OWA, a fuzzified WOWA or a fuzzified Choquet integral is used for the aggregation and the expert would like to use a fuzzy expert system instead. The fuzzy expert system requires a fuzzy rule base to be defined. The algorithm presented in the thesis makes it possible to create the fuzzy rule base automatically so that the aggregation result would be as similar as possible to the result obtained with the original aggregation method (in this case, the similarity will be assessed using the method from the Definition 2.20 in the thesis).

Algorithm 2.2 *Let a FNV-function* $\tilde{f} : \mathcal{F}_N([0,1])^m \to \mathcal{F}_N([0,1])$ *be a FuzzyWA, FuzzyOWA or fuzzified WOWA with some weights, or the fuzzified Choquet integral with some FNV-fuzzy measure.*

First, the expert defines the linguistic scales $(\mathcal{E}_i, \mathcal{T}(\mathcal{E}_i), [0,1], M_i, G_i)$, $i = 1, \ldots, m$, *for the partial evaluations to be aggregated, and the linguistic scale* $(\mathcal{E}, \mathcal{T}(\mathcal{E}), [0,1], M, G)$ *for the overall evaluation*

For any possible combination of the criteria values (terms of the corresponding linguistic scales), a rule is created as follows. Let s_i, $i = 1, \ldots, m$, *denote the number of the terms of the linguistic scale* $\mathcal{E}_i$ *and let* s *denote the number of the terms of the linguistic scale* $\mathcal{E}$. *Then* $n = s_1 \cdot s_2 \cdot \cdots \cdot s_m$ *denotes the total number of the rules that should be created. The following steps are performed for each of them. Let the antecedent (the left-hand part) of such an i-th rule,* $i = 1, \ldots, n$, *be*

$$\textit{If } \mathcal{E}_1 \textit{ is } \mathcal{A}_{i,1} \textit{ and } \ldots \textit{ and } \mathcal{E}_n \textit{ is } \mathcal{A}_{i,m}.$$

The consequent (right-hand part) $\mathcal{B}_i$ *for this rule is determined in the following way:*

- *A fuzzy number* C_i *is calculated as* $C_i = \tilde{f}(A_{i1}, \ldots, A_{im})$, *where* $A_{ij} = M_j(\mathcal{A}_{i,j})$, $j = 1, \ldots, m$.
- *The linguistic term* $\mathcal{B}_i \in \mathcal{T}(\mathcal{E})$ *is then found by the linguistic approximation of* C_i *using the linguistic scale* $(\mathcal{E}, \mathcal{T}(\mathcal{E}), [0,1], M, G)$. *Specifically,* $\mathcal{B}_i$ *is such a linguistic term for whose mathematical meaning, the fuzzy number* B_i, $B_i = M(\mathcal{B}_i)$, *it holds that*

  $$\forall \mathcal{D} \in \mathcal{T}(\mathcal{E}) : S(C_i, B_i) \geq S(C_i, D),$$

  where $D = M(\mathcal{D})$, *and* S *denotes the similarity from the Definition 2.20 in the thesis.*

Again, the benefits of this algorithm are demonstrated in the thesis on an example. This algorithm as well as the other new fuzzy multiple-criteria evaluation methods described in the thesis have been implemented in the FuzzME software so that they could be tested and applied in the practice. Besides the fuzzy multiple-criteria evaluation, the thesis dealt also with the topic of fuzzy classification.

3. Fuzzy classification

Classification problems can be encountered very often in the real world. Because the practical classification problems contain elements of uncertainty, it is natural to study the classification methods that make use of the fuzzy sets theory.

Many papers have been written on the fuzzy classification. For example, the book by Kuncheva [25] gives a broad overview of the topic. The vast majority of authors focus mainly on deriving fuzzy rules for the fuzzy classification from given data (e.g. in [32], [57], [9], and [28]). Various techniques from evolutionary algorithms to clustering are used to obtain a fuzzy rule base. Nevertheless, this is just the first step in tackling the problem.

When the fuzzy rule base has been determined (either derived from the data or defined expertly), it is necessary to use a proper method that would assign a class to the classified object according to this fuzzy rule base. Generally, this second step is often neglected in the literature. However, there are some authors who studied also this particular phase of the fuzzy classification. For example, Ishibuchi et al. [23] compared performance of various voting schemes for selection of the resulting class for the classified objects. They studied the voting schemes for both a single fuzzy rule-based classification system and for multiple fuzzy rule-based classification systems. In [8], the authors state that since the commonly used fuzzy reasoning method selects the resulting class for the given object only by taking into account the fuzzy rule with the greatest degree of association, the information given by the other fuzzy rules is lost. Therefore, the authors of the paper proposed several new fuzzy reasoning methods and tested their performance. The usage of different methods (specifically, the multi-polar OWA operators and multi-polar Choquet integral) in the fuzzy classification is studied also in [27].

The thesis deals with the phase of solving a fuzzy classification problem when the fuzzy rule base is already known and it is necessary to assign the best-fitting classes to the classified objects. Various fuzzy classification scenarios can be encountered in the practice. Besides the classification in the common sense of the word, a classification whose purpose is an evaluation is also considered in the thesis. For instance, assigning a country to one of the Moody's rating classes (Aaa, Aa, A, Baa, Ba, ..., C) can be perceived as an evaluation of this country. This thesis will provide a systematic study of the different fuzzy classification scenarios.

The fuzzy classification problems are divided in the thesis according to the possible existence of relationships among the given classes, and according to the nature of this relationship. The conclusions have already been published in [21, 20]. The theory is accompanied by examples from the area of human resources management (HR management).

3.1. Specification of the problem of interest

In both science and in the real life it is common to classify objects into classes which are defined rather vaguely – by verbally specified values of the objects' characteristics. The pursued task is to assign an object, described either by crisp (i.e. non-fuzzy) or by vaguely given values of its characteristics, to some of these classes; or more generally, to determine its location in relation to these classes.

In the thesis, we assume that two values are available for each of the classes – its numeric identifier and its verbal label. The numeric identifiers will be used for the calculations while the classes labels (the names of the classes) are necessary in order to be able to set the fuzzy classification model and present its results verbally. Both pieces of information should be unique for the class.

The classes are then described by means of a fuzzy rule base. On the left-hand side of each rule, there is a combination of linguistic variables values that defines a particular class. On the right-hand side of each rule, there is the label (name) of the class.

The output of a fuzzy classification system depends on whether we are solving the basic problem of object identification or whether we are classifying objects for the purpose of their evaluation. In the former case, the result is a single class for the object or information that the object cannot be classified well enough. The set of the used class identifiers can be viewed as a nominal scale. In the latter case, where classification is used as a certain kind of evaluation, the class identifiers can form either an ordinal scale or a cardinal scale and that affects the form of the classification results. In case of the ordinal scale, several neighboring classes (together with the membership degrees of the classified object to these classes) can be the fuzzy output of the classification. In case of the cardinal scale, the location of the object in relation to the classes can be calculated. Since the definition of the classes and potentially also of the object itself involves uncertainty, the idea of an uncertain classification of objects into classes is meaningful. Three main scenarios are studied in the thesis separately. Each of them is demonstrated on a real-world application (simplified for purpose of the better clarity) from the area of HR management.

If the goal of classification is just an identification of the object as a member of one of the classes and if there are no relationships among the classes, or their relationships are not related to the problem being solved, then the scale formed by the numeric identifiers of the classes is considered to be the nominal one. The thesis describes algorithms suitable for this case and deals with the situations when the objects cannot be classified reliably. The result of the classification is then the number (or label) of the best fitting class for the classified object, or the information that the object cannot be classified unambiguously.

If the goal of classification is the evaluation of objects, it makes sense to assume that the numbers identifying the classes form an ordinal, or even cardinal scale. It is meaningful to permit also the case that the objects lie between two neighboring classes. Moreover, in the case of a cardinal evaluation scale, it is also reasonable to calculate the particular location of the objects between these classes. If we require a natural verbal description of both the evaluation process and a fuzzy classification results, it is suitable to use the Sugeno-Yasukawa model [43]. Again, suitable methods and the forms of the results presentation are discussed in the thesis and the respective fuzzy classification scenarios are accompanied with case studies.

All the case studies in this part of the thesis originate from the area of HR management. For better clarity, the applications have been simplified but their extension to the more complex cases would be quite straightforward.

In the first application, the object identification problem is illustrated on the example of determining the type of an academic staff member. The academic staff members are classified according the area on which they focus, i.e. if they achieve significant results in the area of pedagogical activities, in the research, or if neither of these two areas prevails significantly. Three classes are used in this fuzzy classification mode: *Researcher*, *Teacher*, and *Non-specific*. The result of this classification can be used in the HR management – the superordinates can offer the academic staff members an option to engage in that area in which they show the best aptitude.

In the second application, we are trying to determine if the performance and composition of activities of a particular academic staff member corresponds to the position of an assistant professor, an associate professor, or a professor. The position determined in this way is then compared to the actual position of this particular academic staff member. The information can be used in the HR management to find promising academic staff members who are aspiring to a higher academic rank. Contrary to the previous example, the classes (*Assistant professor*, *Associate professor*, and *Professor*) do not form a nominal but an ordinal evaluation scale and therefore a different model of the objects classification is chosen.

In the last example, the academic staff members are divided into classes according to their overall performance [48]. The overall performance is calculated from their performance in the areas of pedagogical activities and R&D (research and development). This way an academic staff member is assigned to one of the performance classes: *Unsatisfactory*, *Substandard*, *Standard*, *Very Good*, or *Excellent*. For the classes the following numeric identifiers were chosen: 0, 0.5, 1, 1.5, and 2. These numeric identifiers express multiples of the actual performance of an academic staff member in comparison to the standard performance for his/her position. For example, the corresponding numeric identifier for the class *Substandard* is 0.5, which means that the typical representative of this class has just half of the performance expected for his/her position. Therefore, the class identifiers form a cardinal scale. This model has been implemented into an information system called IS HAP [40, 41, 20], which is currently being applied at faculties of 6 universities in the Czech Republic.

The great advantage of using the tools of linguistic fuzzy modeling in all of the mentioned cases is that the fuzzy classification rules and the final results are described in the most natural way for humans, i.e. verbally. This may seem important only for interpretation of the rules that have been generated automatically from data. However, the verbal description of fuzzy rule base that has been designed by an expert is no less important.

Specifically, in applications such as the academic staff performance evaluation, the model should be understandable for both the university management and the evaluated academic staff members themselves.

The fuzzy classification methods in the thesis have been implemented in the FuzzME software. The use of the FuzzME for purpose of fuzzy classification is also discussed in the thesis.

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4. Souhrn v českém jazyce

Dizertace je rozdělena do dvou hlavních částí. První část se věnuje ucelenému systému fuzzy metod vícekriteriálního hodnocení. Společnou vlastností těchto metod je použitý typ hodnocení – hodnocení variant představuje (fuzzy) stupeň naplnění daného cíle. Úloha vícekriteriálního hodnocení je popsána pomocí struktury zvané strom dílčích cílů. Podporována jsou kvalitativní i kvantitativní kritéria. Pro agregaci dílčích hodnocení v rámci stromu dílčích cílů lze použít více metod – lze využít fuzzifikované verze známých agregačních operátorů (fuzzy vážený průměr, fuzzy OWA operátor, fuzzifikovaný WOWA operátor, fuzzifikovaný Choquetův integrál) nebo fuzzy expertní systém. Tyto metody umožňují řešit širokou škálu úloh vícekriteriálního hodnocení.

Dizertační práce obsahuje ucelený popis tohoto systému metod. Zároveň také představuje nové metody a výsledky. Jedním z nových výsledků je fuzzifikace WOWA operátoru. Fuzzifikovaná WOWA umožňuje agregovat dílčí hodnocení, která jsou fuzzy čísla na intervalu $[0, 1]$. V dizertační práci je představen také alternativní zápis umožňující snadnější výpočet výsledného hodnocení a dále jsou zkoumány některé vlastnosti a vztahy mezi fuzzifikovaným WOWA operátorem a dalšími fuzzifikovanými agregačními operátory.

Pokud je vztah mezi dílčími hodnoceními a celkovým hodnocením složitý, lze ho vyjádřit pomocí báze fuzzy pravidel. Pro získání celkového hodnocení je pak použitá některá z inferenčních metod. Jednou z nových metod je Sugeno-WOWA. Při použití této metody expert zadává váhy, které umožňují vyjádřit jeho optimismus, nebo pesimismus. Tato metoda byla otestována na reálné aplikaci z oblasti bankovnictví. Ukázalo se, že oproti původně použité Mamdaniho inferenci se jedná o velice vhodnou metodu pro tento typ úloh.

Dizertační práce se zabývá i procesem návrhu modelu vícekriteriálního hodnocení a představuje dva algoritmy, které mohou tvorbu složitých modelů významně zjednodušit. Často je výhodnější vycházet z jednoduššího modelu, který představuje jen hrubou aproximací, a tento model postupně zpřesňovat. V dizertaci je nejprve zkoumána situace, kdy expert používá model založeny na agregaci pomocí fuzzy váženého průměru, fuzzy OWA operátoru, nebo fuzzifikovaného WOWA operátoru a kdy je třeba místo nich použít obecnější fuzzifikovaný Choquetův integrál. Je představen algoritmus, který navrhne FNV-fuzzy míru pro tento účel. Expert tak nemusí zadávat všechny hodnoty (pro m kritérií by bylo obecně nutné zadat $2^m - 2$ údajů), jen provede úpravy tam, kde je to třeba. Druhý algoritmus umožňuje nahradit agregaci pomocí fuzzy váženého průměru, fuzzy OWA operátoru, fuzzifikovaného WOWA operátoru, nebo fuzzifikovaného Choquetova integrálu fuzzy expertním systémem. Algoritmus navrhne příslušnou bázi pravidel. Expertovi tak opět významně šetří práci, protože místo zadávání celé báze stačí jen upravit část těchto pravidel.

Druhá část práce se zabývá fuzzy klasifikací, kdy rozdělení objektů do jednotlivých tříd je popsáno pomocí báze fuzzy pravidel. Na rozdíl od většiny publikací na toto téma, které se soustředí zejména na odvození báze pravidel z dat, tato práce se zabývá situací, kdy pravidla jsou již známá (byla zadána expertem, nebo odvozena z dat) a je třeba podle nich přiřadit objektům odpovídající třídu. Je definováno několik typů úloh fuzzy klasifikace a pro každý z nich jsou rozebrány vhodné postupy řešení a vhodná forma prezentace výsledků.

Součástí této práce je i software FuzzME, který implementuje systém metod popsaný v této dizertační práci. Pomocí FuzzME je možné navrhovat i poměrně složité modely vícekriteriálního hodnocení (a fuzzy klasifikace). Možnosti tohoto software byly otestovány na praktických aplikacích, které jsou rovněž popsány v dizertaci.

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29th International Conference on Mathematical Methods in Economics 2011, Part I, pages 266–271. Professional Publishing, 2011. ISBN 978-80-7431-058-4.

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CURRICULUM VITAE

MGR. PAVEL HOLEČEK

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EDUCATION

2008 - Now	**Palacký University in Olomouc**, Faculty of Science PhD study program: *Applied mathematics* PhD. thesis topic: Fuzzy models of multiple criteria evaluation and fuzzy classification.
2007 - 2008	**Palacký University in Olomouc**, Faculty of Science Master study program: *Computer Science* Thesis: Software processing of fuzzy evaluation methods.
2003 - 2007	**Palacký University in Olomouc**, Faculty of Science Bachelor study program: *Computer Science*

PRACTICE:

January 2013 - Now	**Palacký University in Olomouc**, Centre of the Region Haná Development and administration of the information system for academic staff members evaluation
October 2010 - Oct. 2013	**Palacký University in Olomouc**, Faculty of science, Department of Mathematical Analysis and Applications of Mathematics Post: expert for e-learning deployment and administration

COMPLETED INTERNSHIPS:

April 2012	**Institute for research and applications of fuzzy modeling** IRAFM, Ostrava, Czech Republic (1 week internship)
March 2011 - June 2011	**Technical Unversity in Vienna**, Vienna, Austria

PROFESSIONAL ORGANISATIONS MEMBERSHIP:

2014 – present	Member of the Czech Society for Operations Research

RESEARCH AREAS:

- Fuzzy models of multiple-criteria evaluation, fuzzy classification and their applications.
- Cooperation in the detailed mathematical analysis of the first version of Public Universities Financing Guidelines after 2010.

RESEARCH AND APPLICATIONS:

- Participation in the pilot project testing the possible use of a fuzzy controller for traffic regulation in the Austrian highways. The project was realized in 2011 by Nast Consulting ZT GmbH with cooperation of the Technical University in Vienna.
- Participation in the subsequent project, which studied the possibility of regulating traffic by a fuzzy controller on both highways and secondary roads. The project was carried out by Nast Consulting ZT GmbH in 2013 and 2014.
- Software implementation of the information system for academic staff performance evaluation (IS HAP). The information system is used at faculties of 6 Czech universities.
- Development of the FuzzME software. The software is a software implementation of the multiple-criteria fuzzy evaluation and fuzzy classification methods.
- Application of the FuzzME software for bank clients assessment (in co-operation with the Technical University in Vienna).

PUBLICATIONS

BOOK CHAPTERS:

- HOLEČEK P., TALAŠOVÁ, J., MÜLLER, I.: Fuzzy Methods of Multiple-Criteria Evaluation and Their Software Implementation. Cross-Disciplinary Applications of Artificial Intelligence and Pattern Recognition: Advancing Technologies (editors Mago V. K., Bhatia N.), IGI Global 2012, ISBN13: 9781613504291 , DOI: 10.4018/978-1-61350-429-1**.**
- TALAŠOVÁ, J., STOKLASA, J., HOLEČEK, P.: Registr uměleckých výkonů a hodnocení výsledků tvůrčí činnosti [Registry of artistic results and evaluation of creative work outcomes] *in Registr uměleckých výstupů 2* (editor Zelinský, M.), NAMU, Praha, 2012, ISBN 978-80-7331-231-2.
- TALAŠOVÁ, J., STOKLASA, J., HOLEČEK, P.: Hodnocení akademických pracovníků v kontextu organizačního klimatu univerzit [Academic staff evaluation in the context of the organization climate of universities]. In GRECMANOVÁ, H., DOPITA, M. ET AL.: *Determinanty organizačního klimatu vysokých škol a fakult*, Olomouc: Univerzita Palackého v Olomouci, 2013, s. 101-114. ISBN 978-80-244-3808-5.

PAPERS IN JOURNALS (with non-zero impact factor):

- STOKLASA, J., TALAŠOVÁ, J., HOLEČEK, P.: Academic staff performance evaluation – variants of models, *Acta Polytechnica Hungarica* 8 (3), 2011, p. 91 – 111, ISSN 1785-8860.
- BEBČÁKOVÁ, I., HOLEČEK, P, TALAŠOVÁ, J.: On the application of the fuzzified Choquet integral to multiple criteria evaluation. Acta Polytechnica Hungarica 8 (3), 2011, p. 65-78, ISSN 1785-8860.

PAPERS IN JOURNALS (others):

- HOLEČEK P., STOKLASA, J., TALAŠOVÁ J. (in press): Human resources management at universities - a fuzzy classification approach. International Journal of Mathematics in Operational Research.
- HOLEČEK P., TALAŠOVÁ J.: FuzzME: A New Software for Multiple-Criteria Fuzzy Evaluation. Acta Universitatis Matthiae Belii, series Mathematics; No. 16 (2010), pp. 35–51

PAPERS IN CONFERENCE PROCEEDINGS:

- HOLEČEK P., TALAŠOVÁ, J.: Multiple-Criteria Fuzzy Evaluation in FuzzME - Transitions Between Different Aggregation Operators. *Proceedings of the 32nd International conference on Mathematical Methods in Economics MME 2014*. Olomouc, Czech Republic, pp. 305-310. ISBN 978-80-244-4209-9.
- J. TALAŠOVÁ, J. STOKLASA, P. HOLEČEK: HR management through linguistic fuzzy rule bases - a versatile and safe tool? *Proceedings of the 32nd International conference on Mathematical Methods in Economics MME 2014*. Olomouc, Czech Republic, pp. 1027-1032. ISBN 978-80-244-4209-9.
- STOKLASA, J., HOLEČEK, P., TALAŠOVÁ, J.: A holistic approach to academic staff performance evaluation – a way to the fuzzy logic based evaluation, *Peer Reviewed Full Papers of the 8th International Conference on Evaluation for Practice "Evaluation as a Tool for Research, Learning and Making Things Better" A Conference for Experts of Education, Human Services and Policy*, 18 – 20 June 2012, Pori, Finland, Tampub, Pori, 2012, ISBN 978-951-44-8859-7
- HOLEČEK P., TALAŠOVÁ, J., STOKLASA J.: Fuzzy classification systems and their applications, *Proceedings of the 29th International Conference on Mathematical Methods in Economics 2011 – part I*, 2011, Praha, Czech Republic, p. 266 – 271, ISBN 978-80-7431-058-4
- ZEMKOVÁ B., TALAŠOVÁ, J., HOLEČEK, P.: Fuzzy Model for Determining the Type of Worker. *Proceedings of the 29th International Conference on Mathematical Methods in Economics 2011 – part II*. 2011 , Praha, Czech Republic, p. 768-773, ISBN 978-80-7431-059-1
- HOLEČEK, P., TALAŠOVÁ, J.: The Software Support for Multiple-Criteria Evaluation – Various Types of Partial Evaluations Aggregation. Lecture Notes in Management Science. Proceedings of the 2nd International Conference on Applied Operational Research (Mikael Collan ,Ed), Vol 2, 2010, pp. 1-5, ISSN 2008-0050
- HOLEČEK P., TALAŠOVÁ, J.: Designing Fuzzy Models of Multiple-Criteria Evaluation in FuzzME Software. Proceedings of the 28th International Conference on Mathematical Methods in Economics 2010. Vol 1., pp. 250 – 256, ISBN 978-80-7394-218-2
- TALAŠOVÁ J, HOLEČEK P.: Multiple-Criteria Fuzzy Evaluation: The FuzzME Software Package. Proceedings of 2009 IFSA World Congress, July 20-24, 2009, p. 681-686, ISBN 978-989-95079-6-8
- HOLEČEK P., TALAŠOVÁ J.: A New Software for Fuzzy Methods of Multiple Criteria Evaluation. Proceedings of 8th International Conference of Applied Mathematics APLIMAT 2009, February 3-6, 2009, p. 387-398, ISBN 978-80-89313-31-0.
- HOLEČEK P., TALAŠOVÁ J.: A New Software for Fuzzy Methods of Multiple Criteria Evaluation. Journal of Applied Mathematics; Vol. 2, Number 1, 2009, p. 103-116, ISSN 1337-6365.

CONFERENCES

- HOLEČEK P.: *Linguistically oriented fuzzy modeling in the FuzzMe software*. International Student Conference on Applied Mathematics and Informatics ISCAMI 2015, Kočovce, Slovakia (23.4. - 26.4. 2015).
- HOLEČEK P., TALAŠOVÁ, J.: *Multiple-Criteria Fuzzy Evaluation in FuzzME - Transitions Between Different Aggregation Operators*. 32nd International conference on Mathematical Methods in Economics MME 2014, Olomouc (10. 9. - 12. 9. 2014)
- HOLEČEK P., TALAŠOVÁ, J.: *Multiple-Criteria Fuzzy Evaluation in FuzzME - Recent Development*. 20th Conference of the International Federation of Operational Research Societies IFORS 2014, Barcelona (13. 7. - 18. 7. 2014).
- HOLEČEK P., TALAŠOVÁ, J.: *Multiple-Criteria Fuzzy Evaluation in FuzzME - Possible Transitions Between Different Aggregation Types*. International Student Conference on Applied Mathematics and Informatics ISCAMI 2014, Malenovice (27. 3. - 30. 3. 2014).
- STOKLASA, J., TALAŠOVÁ, J., HOLEČEK, P.: *Academic faculty evaluation models: What approach is appropriate for European universities?*, 26th European Conference on Operational Research EURO 2013, Rome, Italy (1.7. – 4.7. 2013).
- HOLEČEK P., KREJČÍ, J., STOKLASA J., TALAŠOVÁ, J.: *Evaluation of R&D outcomes using fuzzified AHP*. International Student Conference on Applied Mathematics and Informatics ISCAMI 2013, Malenovice (2.5. - 5.5.2013).
- STOKLASA, J., TALAŠOVÁ, J., HOLEČEK, P.: *Linguistically oriented model for academic staff performance evaluation*, 25th European Conference on Operational Research EURO 2012, Vilnius, Lithuania (8.7 – 11.7 2012)
- TALAŠOVÁ, J., STOKLASA J., HOLEČEK P.: *Models of academic staff performance evaluation*, Efficiency and Responsibility in Education ERIE 2012 Prague (7.6. – 8.6.2012)
- TALAŠOVÁ, J., STOKLASA J., HOLEČEK P.: *Vícekriteriální model hodnocení akademických pracovníků,* 13. seminář Hodnocení kvality vysokých škol (3.5. - 4.5. 2012)
- TALAŠOVÁ J., PAVLAČKA O., BEBČÁKOVÁ I., HOLEČEK P.: A framework for fuzzy models of multiple-criteria *evaluation. International conference on fuzzy set theory and applications FSTA 2012*, Liptovský Ján, Slovak Republic (30.1. - 3.2. 2012)
- HOLEČEK P., TALAŠOVÁ, J., STOKLASA J.: Fuzzy classification systems and their applications, 29th International Conference on Mathematical Methods in Economics 2011 MME 2011, Janská Dolina, Slovakia (6.9. – 9.9.2011)
- HOLEČEK, P., STOKLASA, J., TALAŠOVÁ, J., ZEMKOVÁ, B.: Fuzzy classification and its applications in HR management, International Student Conference on Applied Mathematics and Informatics ISCAMI 2011 (6.5. – 8.5.2011)
- HOLEČEK P., TALAŠOVÁ, J.: The Software Support for Fuzzy Multiple-Criteria Evaluation. International Conference Olomoucian Days of Applied Mathematics - ODAM 2011 (26.1. – 28.1. 2011)
- HOLEČEK P., TALAŠOVÁ, J.: Designing Fuzzy Models of Multiple-Criteria Evaluation in FuzzME Software, MME 2010, České Budějovice (8.9.-10.9. 2010)
- HOLEČEK P., TALAŠOVÁ, J: The Software Support for Multiple-Criteria Evaluation – Various Types of Partial Evaluations Aggregation, ICAOR 2010, Turku, Finland (25.-27.8.2010)
- Holeček P., Talašová, J: Software for Multiple Criteria Evaluation Support - The Aggregation of Partial Evaluations by Choquet Integral, Informatics ISCAMI 2010, 20.5.-23.5.2010, Bratislava

- Talašová J, Holeček P.: Multiple-Criteria Fuzzy Evaluation: The FuzzME Software Package. 2009 IFSA Word Congress, 20.7. – 24.7.2009, Lisbon
- Holeček P., Talašová J.: FuzzME: A New Software for Multiple-Criteria Fuzzy Evaluation. ISCAMI 2009, 13. 5. – 15. 5. 2009, Malenovice
- HOLEČEK P., TALAŠOVÁ J.: A New Software for Fuzzy Methods of Multiple Criteria Evaluation, APLIMAT 2009, 3.2. – 4.2.2009, Bratislava

INVITED LECTURES

- HOLEČEK P.: FuzzME – the software for multiple-criteria fuzzy evaluation and fuzzy classification, University of Trento, Italy, 14. 3. 2012.

SEMINARS

- HOLEČEK P.: *Vícekriteriální hodnocení ve FuzzME - přechody mezi jednotlivými typy agregace*. Seminář projektu AMathNet Fuzzy modely a jejich aplikace. Olomouc (14. 3. 2014)
- HOLEČEK P.: *FuzzME - software pro vícekriteriální hodnocení*. Seminář projektu AMathNet Fuzzy modely a jejich aplikace. Olomouc (10. 5. 2013).
- STOKLASA, J., TALAŠOVÁ, J., HOLEČEK, P.: *Information system for academic faculty performance evaluation (IS HAP): an example of a linguistic approach used in the Czech Republic*, Università Ca' Foscari Venezia, Italy, 18. 4. 2013.
- STOKLASA, J., TALAŠOVÁ, J., HOLEČEK, P.: *Approaches to (academic) staff performance evaluation: non fuzzy approaches vs. fuzzy approach*. University of Trento, Italy, 14. 3. 2012

APPLIED RESEARCH RESULTS:

- FuzzME software (Holeček, P., Talašová, J., Pavlačka, O., Bebčáková, I.). Demo version can be downloaded at http://www.FuzzME.net
- Choquet software (Holeček, P., Talašová, J., O., Bebčáková, I.). The software makes it possible to calculate the fuzzied Choquet integral and use it for multiple-criteria evaluation. The software is freeware and can be downloaded at http://www.FuzzME.net
- Information system for academic staff performance evaluation IS HAP (software: Holeček, P., Doubrava, M., model: Talašová, J., Stoklasa, J.)

AWARDS:

- The first prize in the students contest "O cenu děkana": First place in the PhD section in the category of the mathematics and computer science (Bebčáková, I., Holeček, P.: Fuzzification of Choquet integral and its application in multiple criteria decision making).

TEACHING RESPONSIBILITIES:

Currently, I am teaching the following labs or seminars at the Palacký University in Olomouc:

- Fuzzy Sets and their Application 1
- Fuzzy Sets and their Application 2
- Theory and Methods of Decision Making 2
- Mathematical and Economical Software
- Statistical Software 1

- Statistical Software 2
- Statistical Software 3
- TeX for Beginners

PROJECTS INVOLVEMENT

- Grant agency of the Czech Republic project *Operations research methods for decision support under uncertainty.* [Metody operačního výzkumu pro podporu rozhodování v podmínkách neurčitosti.] (14-02424S) – research team member, 2014 - 2016.
- Individual national project *KREDO – Quality, relevance, efficiency, diversification and openness of higher education in the Czech Republic. Strategy of higher education until 2030.* [Kvalita, relevance, efektivita, diversifikace & otevřenost vysokého školství v ČR. Strategie vysokého školství do roku 2030.] – an expert of a key activity, 2014 - present.
- Several subsequent projects funded by the Czech government aiming on the development of the registry of artistic performances (including a mathematical evaluation model for creative work outcomes of Czech art colleges): *A pilot project of creative work outcomes evaluation to determine the VKM and B3 indicators for art colleges and faculties; C41 Evaluating Creative Work Outcomes Pilot Project*, 2010 – present.
- Several projects funded by the Internal grant agency of Palacký University, Olomouc: *Mathematical models and structures* [Matematické modely a struktury], 2009, 2010, 2011, 2012 and *Mathematical models* [Matematické modely] – research team member.
- Project *MAPLIMAT - Streamlining the Applied Mathematics Studies at Faculty of Science of Palacky University in Olomouc* [MAPLIMAT - Modernizace studia aplikované matematiky na PřF Univerzity Palackého v Olomouci] (CZ.1.07/2.2.00/15.0243) – expert for e-learning deployment and administration, 2010 - 2013.

OTHER SKILLS:

English and German language knowledge, experiences with the web applications and software development (.NET platform and PHP).