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PARAMETER INVARIANT LAGRANGIAN FORMULATION OF KAWAGUCHI GEOMETRY

Ph.D. Thesis

ERICO TANAKA

Program: P1102 Mathematics,

Global Analysis and Mathematical Physics

Supervisor: PROF. RNDR. DEMETER KRUPKA, DRSC.

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I declare that this dissertation is my own work.
All sources used for the text is either acknowledged or listed in the references.

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Erico Tanaka

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Abstract

This Ph.D. thesis is devoted to the construction of reparameterisation invariant theory of Lagrangian formulation, based on Finsler and Kawaguchi geometry. Since we want the theory to be applicable to the problems of physics, we will use a less restrictive definition of Finsler geometry, compared to the standard definitions. We will define Kawaguchi geometry as a natural extension of this less restrictive Finsler geometry, in such way that the k-dimensional subset of the n-dimensional manifold would be given an invariant k-area, with respect to reparameterisation. By these settings, we obtain a reparameterisation invariant Lagrangian and Euler-Lagrange equations. The relations to the conventional theories that are parameter invariant are discussed.

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Abstrakt

Tato Ph.D práce je věnována konstrukci reparametrizace invariantní teorie Lagrangeovy formulace založené na Finslerově a Kawaguchiho geometrii. Protože chceme, aby teorie byla aplikovatelná do fyzikálních problémů, budeme používat méně restriktivní definici ve srovnání se standardními definicemi Finslerovy geometrie. Kawaguchiho geometrii budeme definovat jako přirozené rozšíření této méně restriktivní Finslerovy geometrie takovým způsobem, že k-rozměrná podmnožina n-rozměrné variety by měla být invariantní k-plocha s ohledem na reparametrizaci. Tímto způsobem získáme reparametrizaci invariantního lagrangiánu a Eulerových-Lagrangeových rovnic. Relace s konvenční teorií, které jsou parametricky invariantní, jsou diskutovány.

Preface

This thesis was submitted to Palacký university, Faculty of Science, in partial fulfillment of the requirements for the degree of Ph.D. in Mathematics. Related publications are [9, 14–16].

The work is devoted to the constructions of Lagrangian formulation, which has reparameterisation invariant property. In the basic courses of analytical mechanics, the time is taken as the parameter, and the motion of a particle is described by a trajectory in a n-dimensional configuration space. Therefore, position and velocity (or momentum) of the particle is expressed as a function of time, and if the equations of motion which determines this trajectory could be derived from a Lagrangian, this Lagrangian was given as a function of these variables; namely time, position and velocity. Such view point matches our intuition well, and indeed covers wide range of physical phenomena we experience in our everyday life. However, since our concept on time and space changed drastically after the emerging of the theory of relativity, it became a major movement in theoretical and mathematical physics to reconstruct the existing theory in such a way that does not distinguish time as a special coordinate among the others of the same spacetime. The theory which treats time and space equally is said to have the property of *covariance*. In case of mechanics, the motion of a particle will be realised as a trajectory on a $(n+1)$-dimensional manifold M, and each point on the trajectory is the position of the particle in M; *i.e*, its local coordinate expression is given by the coordinate functions on M, without any preferrence to a specific one as time. In the case of relativity theory, such case was considered with the aid of Riemannian geometry. In this thesis, we will try to consider such situation with Finsler and Kawaguchi geometry, which is a generalisation of Riemannian geometry. While Finsler is viable for first order (velocity) Lagrangian mechanics, Kawaguchi is considered for higher order case. These geometries have further possibilities to express more complicated physical theories such as irreversible systems or hysterisis, etc. However, these problems are out of the scope of this thesis, and would be left for future research.

There are further issues similar to the case of mechanics, in the case of field theory. Modern theoritical physics especially in particle physics has sought a theory, which does not depend on the choice of k-dimensional spacetime M. Now our target is shifted from position and velocity to field configurations and its derivative with respect to the spacetime. If we are to consider the "covariant" version of such theory in the analogy of mechanics, we should now consider a spacetime M lying in a greater manifold; the total space, which is also made from the fields. This way of thinking is not possible with

Riemannian geometry, or Finsler geometry, and we have to use the Kawaguchi geometry, which is still not well-established. In this thesis, we will treat the case only where spacetime lying in the total space is diffeomorphic to closed k-rectangle of $\mathbb{R}^k$. For the case of higher order, we will also restrict ourselves to where the total space is $\mathbb{R}^n$, $n \geq k$. Such approach essentialy means, that we do not distinguish between the fields and spacetime. We will call such property as *generalised covariance*.

We will provide the mathematical foundations for constructing such theories, and introduce the Lagrange formulation in the above context. Some elementary examples are given to make comparison to the conventional theory. Though the theory is far from complete, we hope that further research will supplement the imperfection, and our foundations will become useful in constructing more precise and viable theory for both mathematics and physics.

I would like to thank Prof. Demeter Krupka for his generous and enduring help for this work, also for giving me the chance to study at Palacký university. I also thank Prof. Lorenzo Fatibene and Prof. Mauro Francaviglia for both scientific discussions and support during my stay at Torino University. I also found the discussion with Prof. David Saunders, Prof. Takayoshi Ootsuka, invaluable and inspiring. Thanks also goes a lot to Prof. Zbyňek Urban and Ph.D. students at Torino University, who encouraged me a lot and gave me impressive questions. This thesis would not have finished without the help of Prof. Jan Brajerčik and Prof. Milan Demko. Also I thank them for the hospitality they gave me during my stay at Prešov University.

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Basic symbol list (unless otherwise stated):

$\mathbb{R}$: real numbers
$\mathbb{R}^n$: real n-dimensional Cartesian vector space with its natural topology
V: vector space over $\mathbb{R}$
V^*: dual vector space of V
M: C^∞-differentiable manifold
TM: tangent bundle of M
$\Lambda^k TM$: all k-multivectors over M or k-multivector bundle of M
$C^\infty(M)$: module of C^∞-functions
$\mathfrak{X}(M)$: $C^\infty(M)$-module of all vector fields over M
$\mathfrak{X}^k(M)$: $C^\infty(M)$-module of all k-multivector fields over M
$\mathfrak{X}^{\wedge k}(M)$: $C^\infty(M)$-module of all decomposable k-multivector fields over M
$\tilde{\mathfrak{X}}^{\wedge k}(M)$: $C^\infty(M)$-module of all locally decomposable k-multivector fields over M
$\Omega^k(M)$: $C^\infty(M)$-module of all differential k-forms over M
$J^r Y$: r-th jet-prolongation of fibred manifold Y

pr_k : Cartesian product projection onto the k-th set of product
${}_nC_k$: Combination ${}_nC_k := \dfrac{n!}{(n-k)!k!}$

Contents

Chapter 1

Introduction

In this thesis, we will discuss the parameterisation invariant theory of Lagrangian formulation in terms of Finsler and Kawaguchi geometry. By setting up a clear and simple mathematical construction, we hope the theory to be viable for considering and extending the basic theories of physics, such as mechanics and field theory. The Finsler geometry is the foundation we will use for the first order mechanics, while Kawaguchi geometry is considered for the higher order mechanics and field theories. However, we would like to emphasise that in this thesis, these geometries are not the direct objects of our research, in contrary, we will only use their basic properties to build the structures we need for Lagrangian formulation. For instance, no fundamental tensor or connection will be discussed. Also it must be noted that our definition of Finsler geometry is much looser than those introduced in standard textbooks [1, 2], for the aim to make it more applicable to the problems of physics. The only crucial condition we require for the Finsler function is the homogeneity condition. We also consider the Hilbert form as a fundamental structure rather than the Finsler function, which we will take as our Lagrangian.

Kawaguchi geometry, which is the generalisation of Finsler geometry, is still in its developing state, and there are no standard definitions written in modern mathematical language. There are two directions of generalisation of Finsler geometry, one to higher order and another to multi-dimensional parameter space. In this thesis, we propose a new definition of Kawaguchi geometry, especially for the second order 1-dimensional parameter space, and first order k-dimensional parameter space, using multivector bundle and a global differential form, which we call as Kawaguchi form. The Kawaguchi form is constructed in a way such that it satisfies similar properties as the Hilbert form in the case of Finsler geometry. We will take this Kawaguchi form as our Lagrangian. We will also consider the structure of second order k-dimensional parameter space, but only locally.

Using these structures, we will consider the Lagrange formulation, and obtain the Euler-Lagrange equations that are reparameterisation invariant. Examples on simple case as Newton dynamics and De Broglie field is presented, and the results are compared with the standard formulation, which is parameter dependent.

The reason that we expect Finsler and Kawaguchi geometry (in the above context) and the Lagrange formulation considered on these setting to be important in constructing viable theories in physics is because of the parameterisation invariance and its extendability compared to, for example, Riemannian geometry. Ootsuka, Tanaka and Yahagi proposed concrete examples of such application in [8, 9, 16]. This thesis will provide the mathematical background for these discussions. Especially, we intended to prepare a foundation that can provide a classical field theory a geometrically natural extension, which unifies the spacetime and field, in language of physics. Mathematically, this means we will consider the spacetime as submanifold embedded in a higher dimensional space, without any fibration over the parameter space. In this thesis, we will only consider the case where spacetime is diffeomorphic to a closed k-dimensional rectangle in $\mathbb{R}^k$.

The parameter invariant theories of calculus of variations are also considered in different mathematical settings, notably by Grigore, D. Krupka, M. Krupka, Saunders, Urban, in terms of Jets and Grassmannian fibrations [3, 6, 11, 17].

The structure of this thesis will be as the following. In the following chapter, we will begin with setting up the basic structures and definitions used in the theory. Basics definitions such as bundles, multivector bundles, induced charts, multi-tangent maps, integration on a submanifold are given. In Chapter 3, we will introduce the Finsler geometry and its basic properties. Some historical concepts are described briefly. Curves, arc segment, its parameterisation and length are described. We will also discuss the reason why the standard definition of Finsler geometry is too strict for the application to physics. The relation between Hilbert form and Cartan form is also presented here. In Chapter 4, we will introduce the Kawaguchi geometry and its basic properties. We propose a global definition of Kawaguchi geometry, such that for the higher order case, the length of an arc segment will be invariant with respect to reparameterisation. For the multi dimensional case, we will also introduce k-curve, k-patches and its k-area. Similarly as in the previous case, we propose a definition of Kawaguchi geometry, such that the k-area of a k-patch remain unchanged by reparameterisation. We especially construct the second order 1-dimensional parameter case, first order k-dimensional parameter case globally, and finally the second order k-dimensional parameter case locally (namely, the total space is $\mathbb{R}^n$, with $n \geq k$).

In Chapter 5 we finally discuss on Lagrange formulation, using the structures introduced in the previous chapters. First for the Finsler case, then Kawaguchi case for second order 1-dimensional parameter case, and first order k-dimensional case. The obtained results are compared for concrete example such as Newton dynamics and De Broglie field. We will summarise our results in Chapter 6.

About the references:

For the basic structures as manifolds, coordinate charts, tangent vectors, vector fields, we referred to the text by D. Krupka, "Advanced Analysis on Manifolds (to be published)", Y. Matsumoto, "Foundation of Manifolds(多様体の基礎)", B. O'Neill, "Semi-Riemannian Geometry With Applications to Relativity". The book "Metrical differential geometry(計量微分幾何学)" by M. Matsumoto is one of the basic references for Finsler and Kawaguchi geometry, which unfortunately is not translated to English. Other references will be stated when appeared.

About the notations:

Unless otherwise stated, the double occurrence of indices in the formula means summation, following the standard convention of Einstein. The symmetrisation of indices is denoted by round parenthesis, for example,

$$A_{(i}B_{j)} := \frac{1}{2!}\left(A_i B_j + A_j B_i\right) \tag{1.0.1}$$

The anti-symmetrisation of indices is denoted by square parenthesis, for example,

$$A_{[i}B_{j]} := \frac{1}{2!}\left(A_i B_j - A_j B_i\right) \tag{1.0.2}$$

The bases of k-multivector at a point $p \in M$ is often expressed $\left(\frac{\partial}{\partial x^{\mu_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{\mu_k}}\right)_p$, which abbreviates $\left(\frac{\partial}{\partial x^{\mu_1}}\right)_p\wedge\cdots\wedge\left(\frac{\partial}{\partial x^{\mu_k}}\right)_p$. Throughout this thesis, we will consider a real smooth manifold, which has Hausdorff, second-countable and connected topology.

About the conventions:

The differential forms (cotangent vectors) are related to the tensor product by the following convention,

$$\alpha\wedge\beta := \alpha\otimes\beta - \beta\otimes\alpha. \tag{1.0.3}$$

α, β are 1-forms(cotangent vectors). The k-covector in the form $\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k \in \Lambda^k V^*$, maps k vectors $X_1, X_2, \cdots, X_k \in V$ to a number by
$\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k (X_1, X_2, \cdots, X_k) := \det(\alpha_i(X_j))$. The general k-covector maps k vectors $X_1, X_2, \cdots, X_k \in V$ to a number by its linear extension. In coordinate basis

$$\alpha = \frac{1}{k!} \alpha_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \quad X_i = X_i{}^j \frac{\partial}{\partial x^j}, \tag{1.0.4}$$

$$\alpha(X_1, X_2, \cdots, X_k) := \alpha_{i_i \cdots i_k} X_1{}^{j_1} \cdots X_k^{j_k} \delta_{i_1}^{[j_1} \cdots \delta_{i_k}^{j_k]} = \alpha_{i_i \cdots i_k} X_1{}^{i_1} \cdots X_k^{i_k}. \tag{1.0.5}$$

Chapter 2

Basic structures

Our aim here is to set up the space where Finsler and Kawaguchi structure will be endowed, and the calculus of variation could be carried out. We begin by defining standard vector bundle structures and the products of them, and introduce the k-multivector bundle which is a k-fold antisymmetric tensor product bundle. This bundle will be the stage for considering the theory of calculus of variation for first order k parameter space, i.e., first order field theory, where parameter space corresponds to the spacetime. For constructing the stage for second order field theory, we will extend this concept to second order k-antisymmetric tensor product space by standard manipulations on vector bundles. We will begin by introducing the fibred manifold and fibred coordinates, trivialisation and concept of bundles, then additionally the properties of a vector space to consider vector bundles and its product spaces. Then we will further consider the sub-bundles of them, by certain bundle isomorphism. The obtained final bundle will be the stage for our parameterisation invariant theory of calculus of variation, which we apply to the field theory. This chapter mostly refers to the textbook by D. J. Saunders, ”The Geometry of Jet Bundles”, with slight changes in notations.

2.1 Bundles

Definition 2.1. Fibred manifold
A *fibred manifold* is a triple (E, π, M) where E and M are manifolds, and $\pi : E \to M$ is a surjective submersion. E is called the *total space*, M the *base space*, and π is a *projection*. The subset $\pi^{-1}(p) \subset E$ over each point $p \in M$ is called a *fibre*, and is usually denoted by E_p.

We occasionally use the projection π to denote the total fibred manifold, instead of writing the triple.

Definition 2.2. Adapted chart of a fibred manifold
Let (E, π, M) be a fibred manifold such that $\dim M = n, \dim E = n + m$. The *adapted chart* of an open set $V \subset E$ is a chart $(V, \psi), \psi = (y^\nu), \nu = 1, \cdots, n + m$, such that for any points $a, b \in V$, $\pi(a) = \pi(b) = p$, $p \in M$, then $pr_1(\psi(a)) = pr_1(\psi(b))$, where $pr_1 := \mathbb{R}^{n+m} \to \mathbb{R}^n$.

The existence of such adapted chart is guaranteed by the following lemma.

Lemma 2.3. Let $f : E \to M$ be a $C^r(1 \leqslant r \leqslant \infty)$ map, and $\dim M = n,\ \dim E = n + m$. If f is a submersion at $q_0 \in E$, that is, if for the neighbourhood V of q_0, the tangent map $T_q f : T_q E \to T_{f(q)} M$ at $\forall q \in V$ is surjective and has constant rank, then there exists a chart $(V, \psi), \psi = (y^\nu)$ on E such that, for any $a, b \in V$, $f(a) = f(b) = p$, $p \in M$, then $pr_1(\psi(a)) = pr_1(\psi(b))$, where $pr_1 := \mathbb{R}^{n+m} \to \mathbb{R}^n$.

For the proof, we refer to [5], [10].

The adapted chart (V, ψ) on the total space E induces a chart on the base space M by π. This induced chart can be denoted as $(\pi(V), \varphi)$, where the coordinate function $\varphi : \pi(V) \to \mathbb{R}^n$ is given by setting $\varphi(\pi(a)) = pr_1(\psi(a)), a \in V$. It is convenient to use the same notation for the first n coordinate functions for both ψ and φ, so that, (V, ψ), $\psi = (x^1, \cdots, x^n, y^1, \cdots, y^m)$ and $(\pi(V), \varphi)$, $\varphi = (x^1, \cdots, x^n)$.

Definition 2.4. Local trivialisation
Let (E, π, M) a fibred manifold and $p \in M$. A *local trivialisation of* π *around* p is a triple (U_p, F_p, t_p), where U_p is a neighbourhood of p, F_p is a manifold, and

$$t_p : \pi^{-1}(U_p) \to U_p \times F_p \tag{2.1.1}$$

is a diffeomorphism satisfying the condition $pr_1 \circ t_p = \pi|_{\pi^{-1}(U_p)}$, with pr_1 denoting the Cartesian product projection onto the first set.

Definition 2.5. Bundle
A fibred manifold which has at least one local trivialisation around each point on M is called a *bundle*.

We occasionally use the projection π to denote the bundle itself, instead of writing the triple.

Example 2.6. Tangent bundle
For the case $E = TM$, a tangent space of M, the triple (TM, τ, M) with τ being a natural projection which sends each tangent vector $v_p \in T_pM$ at point $p \in M$ to $p \in M$ becomes a bundle, and is called the tangent bundle.

Each chart on M induces a local trivialisation of TM. The local trivialisation of the tangent bundle around $p \in M$ could be introduced in the following way. Let (U_p, φ), $\varphi = (x^\mu)$, $\mu = 1, \cdots, n$, such that $p \in U_p$ be a chart on the base space M. For any element ξ of $\tau^{-1}(U_p) \subset TM$ at p have the coordinate expression, $\xi = \xi^\mu \left(\frac{\partial}{\partial x^\mu}\right)_p$. Then the trivialisation (U_p, F_p, t_p) induced by the chart (U_p, φ) on the base space M is $(U_p, \mathbb{R}^n, t_p)$, where t_p is given by $t_p(\xi) = (\tau(\xi), (\xi^1, \cdots, \xi^n))$. Let (U, φ) be a chart on M. In general, the set $\pi^{-1}(U)$, $U \subset M$ of a bundle (E, π, M) cannot be covered by a single chart. H owever, for the case of a tangent bundle this is possible. The charts on M induces such specific charts on TM via local trivialisation.

Definition 2.7. Induced charts of a tangent bundle
Let (TM, τ, M) be a tangent bundle with $\dim M = n$, and (U, φ), $\varphi = (x^\mu)$ a chart on $U \subset M$. Since for any $p \in M$ we have the local trivialisation $(U_p, \mathbb{R}^n, t_p)$, $t_p(\tau^{-1}(U_p)) = U_p \times \mathbb{R}^n$, we define the *induced chart of a tangent bundle* on $\tau^{-1}(U_p) \subset TM$ for any $p \in M$ by $(\tau^{-1}(U_p), \psi)$, $\psi(\xi) = (\varphi(\tau(\xi)), (\xi^1, \cdots, \xi^n))$, where $\xi \in \tau^{-1}(U_p)$.
Especially, we may use the convenient expressions as $\psi = (x^1, x^2, \cdots, x^n, y^1, y^2, \cdots, y^n)$, where $y^\mu(\xi) = \xi^\mu, \mu = 1, \cdots, n$.

These induced charts define on TM the structure of C^∞-manifold of dimension $2n$.

Definition 2.8. Global trivialisation
Let (E, π, M) be a fibred manifold. A *global trivialisation of* π is a triple (M, F, t), where F is a manifold, and $t : E \to M \times F$ is a diffeomorphism satisfying the condition $pr_1 \circ t = \pi$, with pr_1 denoting the Cartesian product projection onto the first set. F is called a *typical fibre of* π.

Definition 2.9. Trivial fibre bundle
A fibred manifold which has a global trivialisation is called a *trivial fibre bundle*.

We show in Figure 2.1, three diagrams of fibred manifolds : general, bundle, and trivial.

Definition 2.10. Fibred product bundles
Let (E, π, M) and (H, ρ, M) be bundles over the same base manifold M. The *fibred*

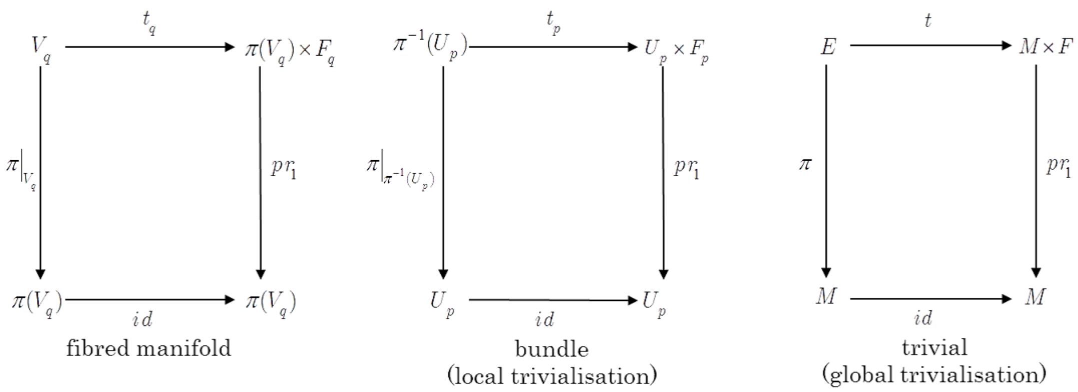

Figure 2.1: Fibre bundles

product bundle is a triple $(E\times_M H, \pi\times_M \rho, M)$, where the total space $E\times_M H$ is defined by

$$E \times_M H := \{(p,q) \in E \times H : \pi(p) = \rho(q)\} \tag{2.1.2}$$

and the projection map $\pi\times_M\rho$ is defined by

$$(\pi\times_M\rho)(p,q) = \pi(p) = \rho(q). \tag{2.1.3}$$

Bellow we will check that the triple $(E\times_M H, \pi\times_M\rho, M)$ indeed has a bundle structure.

First, the total space $E\times_M H$ is a submanifold of $E\times H$, since $E\times_M H = (\pi\times\rho)^{-1}\Delta_M$, where Δ_M is the diagonal set

$$\Delta_M = \{(p,q) \in M \times M \,|\, p,q \in M,\, p = q\}. \tag{2.1.4}$$

Suppose the adapted charts on $V_E \subset E$, $V_H \subset H$, such that $\pi(V_E) \cap \rho(V_H) \neq \emptyset$ are given by (V_E, ψ_E), $\psi_E = (x^\mu, y^a)$, $(V_H, \psi_H), \psi_H = (x^\mu, z^A)$, then it induces a chart on $U = (\pi(V_E) \cap \rho(V_H)) \subset M$, namely, (U, φ), $\varphi = (x^\mu)$. Let the adapted chart on $V_{E\times H} \subset E \times H$ be $(V_{E\times H},\ \psi_{E\times H})$, with $V_{E\times H} = (\pi\times\rho)^{-1}U$, $\psi_{E\times H}(p,q) = \big(\varphi(\pi(p)), y^a(p), \varphi(\rho(q)), z^A(q)\big)$, where $(p,q) \in E \times H$. Then by considering the coordinate expressions of equations of a submanifold $\varphi(\pi(p)) = \varphi(\rho(q))$, the total space

$E \times_M H$ has an adapted chart $(V_{E\times_M H}, \psi_{E\times_M H})$, where

$$\begin{aligned} V_{E\times_M H} &= (E\times_M H) \cap V_{E\times H}, \\ \psi_{E\times_M H}(p,q) &= \big(\varphi(\pi(p)), y^a(p), z^A(q)\big) \,, \end{aligned} \tag{2.1.5}$$

with $(p,q) \in E \times H, \pi(p) = \rho(q)$.

These charts define on $E\times_M H$ the structures of a C^∞-manifold.

Next, we will obtain the local trivialisation (U_r, F_r, t_r) on every point of $r \in M$ in the following way.

Since (E,π,M) and (H,ρ,M) are both bundles, they have a local trivialisation around each point $r \in M$. Denote these as (U_r, E_r, s_r), (V_r, H_r, u_r), then $s_r : \pi^{-1}(U_r) \to U_r \times E_r$, $u_r : \rho^{-1}(V_r) \to V_r \times H_r$. Then for any $r \in M$, we can obtain the local trivialisation of $(E\times_M H, \pi\times_M\rho, M)$ around r by $t_r : (\pi\times_M\rho)^{-1}(U_r \cap V_r) \to (U_r \cap V_r) \times F_r$, where $F_r = E_r \times H_r$, and $t_r(p,q) = (\pi(p), y^a(p), z^A(q)) = (\rho(q), y^a(p), z^A(q))$. Accordingly, the triple $(E\times_M H, \pi\times_M\rho, M)$ becomes a bundle.

Definition 2.11. Bundle morphism
If (E,π,M) and (H,ρ,N) are bundles, then a *bundle morphism from* π *to* ρ is a pair $(f,\bar f)$ where $f : E \to H$, $\bar f : M \to N$ and $\rho\circ f = \bar f\circ\pi$. The map $\bar f$ is called a *projection* of f.

Example 2.12. Let (E,π,M) be a bundle, and $(TE, T\pi, TM)$ its tangent bundle. There is a bundle morphism from $T\pi$ to π, which is a pair of tangent bundle projections (τ_E, τ_M).

Example 2.13. Let (E,π,M) be a bundle, and $(X,\bar X)$ a bundle morphism to $(TE, T\pi, TM)$. X is called a *projectable vector field*, and its projection $\bar X$ satisfies the relation, $\bar X\circ\pi = T\pi\circ X$.

Definition 2.14. Pull-back bundle
Let (E,π,M) be a bundle, and $f : N \to M$ a smooth map. The pull-back of the bundle π by f is denoted by $(f^*E, f^*\pi, N)$, where the total space f^*E is defined by,

$$f^*E = \{(u,x) \in E \times N \mid \pi(u) = f(x)\}, \tag{2.1.6}$$

that is, $f^*E = E\times_M N$, and the projection map $f^*\pi$ is defined by $f^*\pi(u,x) = x$. The bundle $f^*\pi$ is called a *pull-back bundle* of π by f.

Definition 2.15. Sub-bundle
If (E,π,M) is a bundle and $E' \subset E$ is a submanifold such that the triple $(E', \pi|_{E'}, \pi(E'))$ is itself a bundle, then $\pi|_{E'}$ is called a *sub-bundle* of π.

Definition 2.16. Vector bundle
A *vector bundle* is a quintuple (E, π, M, σ, μ) with the following structures,
1. (E, π, M) is a bundle
2. Denote the fibre over $p \in M$ as E_p. Namely, $E_p = \pi^{-1}(p)$. Then,
(a) $\sigma : E\times_M E \to E$ satisfies, for each $p \in M$, $\sigma(E_p \times E_p) \subset E_p$
(b) $\mu : \mathbb{R} \times E \to E$ satisfies, for each $p \in M$, $\mu(\mathbb{R} \times E_p) \subset E_p$
(c) $(E_p, \sigma|_{E_p\times E_p}, \mu|_{\mathbb{R}\times E_p})$ is a real vector space for each $p \in M$
3. for each $p \in M$, there is a local trivialization $(W_p, \mathbb{R}^n, t_p)$ called a *linear local trivialisation*, satisfying the condition that, for $q \in W_p$, the map $pr_2\circ\, t_p|_{E_p} : E_p \to \mathbb{R}^n$ where $t_p|_{E_p} : E_p \to \{q\} \times \mathbb{R}^n$ and $pr_2 : \{q\} \times \mathbb{R}^n \to \mathbb{R}^n$, is a linear isomorphism.

Under the linear local trivialization, the maps σ and μ correspond to addition of vectors on $\mathbb{R}^n$, and scalar multiplication of vectors on $\mathbb{R}^n$, respectively.

Example 2.17. The tangent bundle (TM, τ, M) is a vector bundle. The linear local trivialisation around each point $p \in M$ is given by $(U_p, \mathbb{R}^n, t_p)$, for each $p \in M$, the fibre T_pM has the property of vector space, therefore $av_1 + bv_2 \in T_pM,\ a, b \in \mathbb{R},\ v_1, v_2 \in T_pM.$

Definition 2.18. Vector bundle adapted charts
Let $m = \dim M$, $n = \dim E$, and $(\pi^{-1}(W), \psi), \psi = (u^1, \cdots, u^n)$ the adapted coordinates on (E, π, M) induced by the chart $(W, \varphi), \varphi = (x^1, \cdots, x^m)$ on $W \subset M$, chosen to be linear on π. Such charts are called *vector bundle adapted charts*.

In such charts, the elements of E can be expressed by $\xi = \xi^\alpha e_\alpha$, $\alpha = 1, \cdots, m$, where the base $e_\alpha \in \Gamma_W(\pi)$ is a family of local section defined by $u^\beta(e_\alpha(p)) = \delta^\beta_\alpha$ for all $p \in W$. In this way, linear operations on sections could be defined pointwise.

Example 2.19. The vector bundle adapted chart of the tangent bundle (TM, τ, M), is the induced chart $(\tau^{-1}(U), \psi)$,$\psi(\xi) = (x^\mu(\tau(\xi)), \xi^\mu)$ given in (2.7), and the corresponding local sections are the vector fields $\dfrac{\partial}{\partial x^\mu}$.

Example 2.20. The vector bundle adapted chart of the cotangent bundle (T^*M, τ^*, M), is the induced chart $((\tau^*)^{-1}(U), \psi^*)$,$\psi^*(\alpha) = (x^\mu(\tau^*(\alpha)), \alpha_\mu)$, where $\alpha \in T^*M$, and the corresponding local sections are the 1-forms dx^μ.

Definition 2.21. Tensor product
Let (E, π, M) and (F, ρ, M) be vector bundles with fibres E_p, F_p respectively. The *tensor product* of π and ρ is the vector bundle with fibres $E_p \otimes F_p$ and is denoted $(E \otimes F, \pi \otimes \rho, M)$.

Definition 2.22. : Antisymmetric/symmetric tensor product
Let $(E_1, \pi_1, M), \cdots, (E_k, \pi_k, M)$ be vector bundles. The *k-fold antisymmetric tensor product* is a vector bundle with completely antisymmetric properties denoted by $(E_1 \wedge E_2 \wedge \cdots \wedge \mathrm{E}_k, \pi_1 \wedge \pi_2 \wedge \cdots \wedge \pi_k, M)$. For $k = 2$, every fibre is defined by the antisymmetric product

$$E_{1,p} \wedge E_{2,p} = E_{1,p} \otimes E_{2,p} - E_{2,p} \otimes E_{1,p} \tag{2.1.7}$$

for all $p \in M$. Similarly, the *k-fold symmetric product* is a vector bundle with completely symmetric properties denoted by $(E_1 \odot E_2 \odot \cdots \odot \mathrm{E}_k, \pi_1 \odot \pi_2 \odot \cdots \odot \pi_k, M)$. For $k = 2$, every fibre is defined by the symmetric product

$$E_{1,p} \odot E_{2,p} = E_{1,p} \otimes E_{2,p} + E_{2,p} \otimes E_{1,p} \tag{2.1.8}$$

for all $p \in M$.

Any tensor product of tangent and cotangent bundles which has the same base space M become a tensor product bundle over M.

2.1.1 Second order tangent bundle

Here we will review the definitions of T^2M and its basic properties which will be used for considering the theory of second order mechanics. Higher order theory could be constructed by similar induction, and is introduced briefly in the following section. These method of construction will be also used in order to create the basic structures for the higher order field theory, which later will be introduced in this chapter.

Definition 2.23. Second order tangent bundle T^2M over TM
Let (TM, τ_M, M) be the tangent bundle with base space M and (TTM, τ_{TM}, TM) the tangent bundle with the base space TM. The tangent map $T\tau_M$ of τ_M also induces the bundle $(TTM, T\tau_M, TM)$. Denote the subset of elements $\xi \in TTM$ which satisfy the equations of submanifold

$$\tau_{TM}(\xi) = T\tau_M(\xi) \tag{2.1.9}$$

as T^2M, and a map from T^2M to TM by $\tau_M^{2,1} := \tau_{TM}|_{T^2M}$. The triple $(T^2M, \tau_M^{2,1}, TM)$ becomes a bundle, which is a sub-bundle of (TTM, τ_{TM}, TM). We will call this a *second order tangent bundle over TM*.

To check $(T^2M, \tau_M^{2,1}, TM)$ is a bundle, we first introduce manifold structure on T^2M, namely we will introduce charts, and prove that these charts define a smooth structure on T^2M. Let $\dim M = n$, and the chart on $U \subset M$ be $(U, \varphi), \varphi = (x^\mu)$. The induced chart on TM and TTM is (V, ψ), $V = \tau_M^{-1}(U)$, $\psi = (x^\mu, \dot{x}^\mu)$, and $(\tilde{V}^2, \tilde{\psi}^2)$, $\tilde{V}^2 = \tau_{TM}^{-1}(V)$, $\tilde{\psi}^2 = (x^\mu, \dot{x}^\mu, y^\mu, \dot{y}^\mu)$ respectively. The elements of T^2M have a specific form, namely they satisfy the condition $\tau_{TM}(\xi) = T\tau_M(\xi)$, $\xi \in TTM$. Let the local expression of $\xi_q \in T_qTM, q \in V \subset TM$ be

$$\xi_q = \xi_1^\mu \left(\frac{\partial}{\partial x^\mu}\right)_q + \xi_2^\mu \left(\frac{\partial}{\partial \dot{x}^\mu}\right)_q, \tag{2.1.10}$$

then $\tau_{TM}(\xi_q) = q$ and $T\tau_M(\xi_q) = \xi_1^\mu \left(\frac{\partial}{\partial x^\mu}\right)_{\tau_M(q)}$. In coordinate expressions,

$$(x^\mu(q), \dot{x}^\mu(q)) = \left(x^\mu \left(\xi_1^\nu \left(\frac{\partial}{\partial x^\nu}\right)_{\tau_M(q)}\right), \dot{x}^\mu \left(\xi_1^\nu \left(\frac{\partial}{\partial x^\nu}\right)_{\tau_M(q)}\right)\right) = ((x^\mu \circ \tau_M)(q), \xi_1^\mu). \tag{2.1.11}$$

Therefore, the elements of T^2U have the form

$$v_q = \dot{x}^\mu(q) \left(\frac{\partial}{\partial x^\mu}\right)_q + v^\mu \left(\frac{\partial}{\partial \dot{x}^\mu}\right)_q, \tag{2.1.12}$$

for $q \in V$. Furthermore, since every elements of T^2U in chart expression are in the form,

$$(x^\mu(v_q), \dot{x}^\mu(v_q), y^\mu(v_q), \dot{y}^\mu(v_q)) = ((x^\mu \circ \tau_M(q), \dot{x}^\mu(q), \dot{x}^\mu(q), v^\mu), \tag{2.1.13}$$

we can choose a chart (V^2, ψ^2), $\psi^2 = (x^\mu, \dot{x}^\mu, \ddot{x}^\mu)$, $V^2 = \tilde{V}^2 \cap T^2M$, $\mu = 1, \ldots, n$ on T^2M.

Clearly the fibres of $\tau_M^{2,1}$ are not vector spaces, since in general the scalar multiplication of v_q does not belong to the same fibre.

Now, let (V^2, ψ^2), $\psi^2 = (x^\mu, y^\mu, z^\mu)$ and $(\bar{V}^2, \bar{\psi}^2)$, $\bar{\psi}^2 = (\bar{x}^\mu, \bar{y}^\mu, \bar{z}^\mu)$ be two charts on T^2M, with $V^2 \cap \bar{V}^2 \neq \emptyset$. Then express the element $\xi_q \in V^2 \cap \bar{V}^2$, $q \in TM$, by these charts,

$$\xi_q = y^\mu(q) \left(\frac{\partial}{\partial x^\mu}\right)_q + \xi^\mu \left(\frac{\partial}{\partial y^\mu}\right)_q = \bar{y}^\mu(q) \left(\frac{\partial}{\partial \bar{x}^\mu}\right)_q + \bar{\xi}^\mu \left(\frac{\partial}{\partial \bar{y}^\mu}\right)_q, \tag{2.1.14}$$

with $z^\mu(\xi_q) = \xi^\mu,\ \bar{z}^\mu(\xi_q) = \bar{\xi}^\mu$. The bases of TM will transform as

$$\begin{aligned}\frac{\partial}{\partial \bar{x}^\mu} &= \frac{\partial x^\nu}{\partial \bar{x}^\mu}\frac{\partial}{\partial x^\nu} + \frac{\partial y^\nu}{\partial \bar{x}^\mu}\frac{\partial}{\partial y^\nu},\\ \frac{\partial}{\partial \bar{y}^\mu} &= \frac{\partial y^\nu}{\partial \bar{y}^\mu}\frac{\partial}{\partial y^\nu} = \frac{\partial x^\nu}{\partial \bar{x}^\mu}\frac{\partial}{\partial y^\nu},\end{aligned} \tag{2.1.15}$$

and we have the transformation rule for the new coordinate z^μ by

$$\bar{z}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^\nu}z^\nu + \frac{\partial \bar{y}^\mu}{\partial x^\nu}y^\nu = \frac{\partial \bar{x}^\mu}{\partial x^\nu}z^\nu + \frac{\partial^2 \bar{x}^\mu}{\partial x^\nu \partial x^\rho}y^\nu y^\rho. \tag{2.1.16}$$

These transformations are smooth, and the charts form a smooth atlas on T^2M. The natural inclusion $\iota : T^2M \to TTM$ has a coordinate expression,

$$x^\mu \circ \iota = x^\mu,\ \dot{x}^\mu \circ \iota = y^\mu,\ y^\mu \circ \iota = y^\mu,\ \dot{y}^\mu \circ \iota = z^\mu. \tag{2.1.17}$$

The it l.h.s. denotes coordinates on TTM while the *r.h.s.* denotes coordinates on T^2M. This shows that T^2M is a submanifold of TTM. Now $\tau_M^{2,1}$ is a surjective submersion by definition, so it remains to check the local trivialisation. The local trivialisation of $(T^2M, \tau_M^{2,1}, TM)$ around any point $p \in TM$ is given by $(V_p, \mathbb{R}^n, t_p)$,

$$t_p : (\tau_M^{2,1})^{-1}(V_p) \to V_p \times \mathbb{R}^n, \quad p \in V_p, \tag{2.1.18}$$

where V_p is an open set of TM, which in chart expression for any $\xi \in (\tau_M^{2,1})^{-1}(V_p)$ is

$$t_p(\xi) = (\tau_M^{2,1}(\xi), \ddot{x}^\mu(\xi)). \tag{2.1.19}$$

Therefore, $(T^2M, \tau_M^{2,1}, TM)$ is indeed a bundle.

Definition 2.24. Second order tangent bundle
The triple $(T^2M, \tau_M^{2,0}, M)$ with $\tau_M^{2,0} = \tau_M \circ \tau_M^{2,1}$ is also a bundle with the trivialisation $(U_p, \mathbb{R}^{2n}, t_p)$, $t_p : (\tau_M^{2,0})^{-1}(U_p) \to U_p \times \mathbb{R}^{2n}$, around any $p \in M$, which in chart expression for any $\xi \in (\tau_M^{2,0})^{-1}(U_p)$ is

$$t_p(\xi) = (\tau_M^{2,0}(\xi), \dot{x}^\mu(\xi), \ddot{x}^\mu(\xi)). \tag{2.1.20}$$

We will call this a *second order tangent bundle over M*, or simply, *second order tangent bundle*.

In the theory of second order mechanics, the dynamical variables are the section of the second order tangent bundle.

2.1.2 Higher order tangent bundle

Now we will briefly introduce the higher order tangent bundles. This concept will be used to define the total derivatives of the higher order (Chapter 5), when deriving the reparameterisation invariant Euler-Lagrange equations. We will especially give the construction of $(T^3M, \tau_M^{3,2}, T^2M)$, and the r-th order $(T^rM, \tau_M^{r,r-1}, T^rM)$ could be obtained iteratively.

Definition 2.25. Third order tangent bundle T^3M over T^2M
Consider the bundle morphism $(T\tau_M^{2,1}, \tau_M^{2,1})$ from $(TT^2M, \tau_{T^2M}, T^2M)$ to (TTM, τ_{TM}, TM). We define the set T^3M by,

$$T^3M := \{u \in TT^2M \mid \iota\circ\tau_{T^2M}(u) = T\tau_M^{2,1}(u)\}, \tag{2.1.21}$$

where ι is the inclusion map $\iota : T^2M \to TTM$, and its coordinate expression given by (2.1.17). Let $(\tilde{V}^2, \tilde{\psi}^2)$, $\tilde{\psi}^2 = (x^\mu, y^\mu, \dot{x}^\mu, \dot{y}^\mu)$ be the induced chart on TTM and $(\tilde{V}^3, \tilde{\psi}^3)$, $\tilde{\psi}^3 = (x^\mu, y^\mu, z^\mu, \dot{x}^\mu, \dot{y}^\mu, \dot{z}^\mu)$ be the induced chart on TT^2M. The elements of TT^2M have the local expressions

$$u_q = u_1^\mu\left(\frac{\partial}{\partial x^\mu}\right)_q + u_2^\mu\left(\frac{\partial}{\partial y^\mu}\right)_q + u_3^\mu\left(\frac{\partial}{\partial z^\mu}\right)_q, \quad q \in T^2M \tag{2.1.22}$$

and the submanifold equations will give

$$\begin{aligned}
x^\mu(T_q\tau_M^{2,1}(u)) &= x^\mu\circ\iota(q),\\
y^\mu(T_q\tau_M^{2,1}(u)) &= y^\mu\circ\iota(q),\\
\dot{x}^\mu(T_q\tau_M^{2,1}(u)) &= u_1^\mu = y^\mu\circ\iota(q),\\
\dot{y}^\mu(T_q\tau_M^{2,1}(u)) &= u_2^\mu = z^\mu\circ\iota(q),
\end{aligned} \tag{2.1.23}$$

where the coordinate functions in the *l.h.s.* are on TTM, while the *r.h.s* represents the coordinate functions of T^2M. Then we will have for coordinate functions on TT^2M,

$$\begin{aligned}
x^\mu(u) &= x^\mu\circ\iota(q),\\
y^\mu(u) &= y^\mu\circ\iota(q),\\
\dot{x}^\mu(u) &= u_1^\mu = y^\mu\circ\iota(q),\\
\dot{y}^\mu(u) &= u_2^\mu = z^\mu\circ\iota(q),
\end{aligned} \tag{2.1.24}$$

therefore, the elements of T^3M will have the form

$$w_q = y^\mu(q)\left(\frac{\partial}{\partial x^\mu}\right)_q + z^\mu(q)\left(\frac{\partial}{\partial y^\mu}\right)_q + w^\mu\left(\frac{\partial}{\partial z^\mu}\right)_q, \tag{2.1.25}$$

and we will take the induced chart on T^3M as (V^3, ψ^3), $\psi^3 = (x^\mu, y^\mu, z^\mu, w^\mu)$. We can check the set $(T^3M, \tau_M^{3,2}, T^2M)$ with $\tau_M^{3,2} := \tau_{T^2M}|_{T^3M}$ is a bundle in the similar way we did for the case of $(T^2M, \tau_M^{2,1}, TM)$. Clearly, $\tau_M^{3,2}$ is a sub-bundle of $(TT^2M, \tau_{T^2M}, T^2M)$. We will call the set $(T^3M, \tau_M^{3,2}, T^2M)$ the *third order tangent bundle over* T^2M.

We can similarly construct the r-th order tangent bundle over $T^{r-1}M$; namely $(T^rM, \tau_M^{r,r-1}, T^{r-1}M)$ by induction. The bundle projection is defined by,

$$\tau_M^{r,r-1} := \tau_{T^{r-1}M}|_{T^rM}. \tag{2.1.26}$$

Consider the bundle morphism $(T\tau_M^{r-1,r-2}, \tau_M^{r-1,r-2})$ from $(TT^{r-1}M, \tau_{T^{r-1}M}, T^{r-1}M)$ to $(TT^{r-2}M, \tau_{T^{r-2}M}, T^{r-2}M)$. Then we will define the total space T^rM by

$$T^rM := \{u \in TT^{r-1}M | T\tau_M^{r-1,r-2}(u) = \iota_{r-1}\circ\tau_{T^{r-1}M}(u)\}, \tag{2.1.27}$$

where $\iota_{r-1} : T^{r-1}M \to TT^{r-2}M$ is the inclusion map. $(T^rM, \tau_M^{r,r-1}, T^{r-1}M)$ is a sub-bundle of $(TT^{r-1}M, \tau_{T^{r-1}M}, T^{r-1}M)$.

2.1.3 Multivector bundle

The completely antisymmetric tensor product bundles are the structures we need for further discussions on calculus of variation, especially when the dimension of parameter space is greater than one, namely for the consideration of a field theory. It is related to the concept of multivectors and multivector fields, which is introduced below.

Definition 2.26. k-multivectors
A k-*multivector* with $k \leqslant n$ is an element of exterior algebra over a vector space V denoted by $\Lambda^k(V)$. It is a linear combination of the multivectors of the form $v_1\wedge\cdots\wedge v_k$, $v_1, \cdots, v_k \in V$. When the k-multivector is in the form $v_1\wedge\cdots\wedge v_k$, it is called a *decomposable multivector*.

This space $\Lambda^k(V)$ has a natural space of the vector space.

One geometric way to understand the vector field was to see it as a section of a tangent bundle, (TM, τ, M). Similarly, the multivector fields could be understood as a section of the k-fold antisymmetric tensor product bundle of $(\Lambda^k TM, \Lambda^k \tau_M, M)$. We denote

$$\Lambda^k TM := \bigcup_{p \in M} \Lambda^k T_p M \tag{2.1.28}$$

and $\Lambda^k \tau_M := \tau_{M \wedge \cdots \wedge} \tau_M$ (k-alternating products). Below we will give the definition, and see that this is indeed a vector bundle.

Definition 2.27. k-Multivector bundle
The triple $(\Lambda^k TM, \pi, M)$ where $\pi = \Lambda^k \tau_M$ is a projection $\pi(v) = p$ for $v \in \Lambda^k TM$, $p \in M$, has a vector bundle structure, and is called the *k-multivector bundle*.

First, we will introduce a smooth structure on $\Lambda^k TM$. Let the chart on the base space M be $(U, \varphi), \varphi = (x^\mu)$, and $v \in \pi^{-1}(p), p \in U$. The bases of k-multivector constructed from natural bases $\left(\left(\frac{\partial}{\partial x^1} \right)_p, \cdots, \left(\frac{\partial}{\partial x^n} \right)_p \right)$ on U are in the form $\left(\frac{\partial}{\partial x^{i_1}} \right)_p \wedge \cdots \wedge \left(\frac{\partial}{\partial x^{i_k}} \right)_p$, where $i_1, \cdots, i_k$ are integers taken from $1, \cdots, n$ without overlapping, and we denote this by the abbreviation $\left(\frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}} \right)_p$. The coordinate expression of v by these bases is

$$v = \frac{1}{k!} v^{i_1 \cdots i_k} \left(\frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}} \right)_p, \tag{2.1.29}$$

where $v^{i_1 \cdots i_k}$ are real numbers with alternating superscripts. We may also use the local expression of v using ordered bases, i.e.,

$$v = \sum_{i_1 < i_2 < \cdots < i_k} v^{i_1 \cdots i_k} \left(\frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}} \right)_p. \tag{2.1.30}$$

Define the functions $y^{\mu_1 \cdots \mu_k}$ on $\pi^{-1}(U)$ by $y^{\mu_1 \cdots \mu_k}(v) = v^{\mu_1 \cdots \mu_k}$, then we can obtain the induced chart on $V \subset \Lambda^k TM$, by (V, ψ), $V = \pi^{-1}(U)$, $\psi = (x^\mu, y^{\mu_1 \cdots \mu_k})$. When considering an exact value, we assume the superscripts are ordered.

To see the coordinate transformations, let (V, ψ), $V = \pi^{-1}(U)$, $\psi = (x^\mu, y^{\mu_1 \cdots \mu_k})$ and $(\bar{V}, \bar{\psi})$, $\bar{V} = \pi^{-1}(\bar{U})$, $\bar{\psi} = (\bar{x}^\mu, \bar{y}^{\mu_1 \cdots \mu_k})$ be two charts on $\Lambda^k TM$, with $V \cap \bar{V} \neq \emptyset$. Then express the element $v_p \in V \cap \bar{V}$, $p \in U \cap \bar{U} \subset M$ by these charts,

$$v_p = \frac{1}{k!} y^{\mu_1 \cdots \mu_k}(p) \left(\frac{\partial}{\partial x^{\mu_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_k}} \right)_p = \frac{1}{k!} \bar{y}^{\mu_1 \cdots \mu_k}(p) \left(\frac{\partial}{\partial \bar{x}^{\mu_1}} \wedge \cdots \wedge \frac{\partial}{\partial \bar{x}^{\mu_k}} \right)_p. \tag{2.1.31}$$

The bases of $\Lambda^k TM$ will transform as

$$\frac{\partial}{\partial \bar{x}^{\mu_1}}\wedge\cdots\wedge\frac{\partial}{\partial \bar{x}^{\mu_k}} = \frac{\partial x^{\nu_1}}{\partial \bar{x}^{\mu_1}}\cdots\frac{\partial x^{\nu_k}}{\partial \bar{x}^{\mu_k}}\frac{\partial}{\partial x^{\nu_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{\nu_k}}, \tag{2.1.32}$$

so the transformation equation for the coordinate functions on $\Lambda^k TM$ are

$$\bar{x}^{\mu} = \bar{x}^{\mu}(x^{\nu}), \tag{2.1.33}$$

$$\bar{y}^{\nu_1\cdots\nu_k} = \frac{\partial \bar{x}^{\nu_1}}{\partial x^{\mu_1}}\cdots\frac{\partial \bar{x}^{\nu_k}}{\partial x^{\mu_k}} y^{\mu_1\cdots\mu_k}. \tag{2.1.34}$$

These transformations are smooth. Such induced charts define on $\Lambda^k TM$ the structure of C^∞-manifold of dimension $n + {}_nC_k$.

Now, $(\Lambda^k TM, \pi, M)$ naturally becomes a bundle by the projection, $\pi(v) = p$ for $v \in \Lambda^k T_p M, p \in M$, since for any $v \in \Lambda^k TM$, there always exist a unique $p \in M$, such that $v \in \Lambda^k T_p M$, and we assumed $\dim M$ and therefore $\dim \Lambda^k TM$ are both constant.

This bundle structure can be also constructed by taking a *k-fold alternating product of* (TM, τ, M), which is denoted by $(\Lambda^k TM, \Lambda^k\tau, M)$, and is a sub-bundle of the tensor product bundle $(\otimes^k TM, \otimes^k\tau, M)$.

Definition 2.28. k-multivector field
k-multivector field is a section of $(\Lambda^k TM, \pi, M)$. We denote all sections of π by $\Gamma(\Lambda^k TM)$, or equivalently $\mathfrak{X}^k(M)$.

Definition 2.29. Local coordinate expression of k-multivector field
Let $(U, \varphi), \varphi = (x^\mu)$ be a chart on M, the local expression of $Y \in \Gamma(\Lambda^k TU)$ is

$$Y = \frac{1}{n!} f^{i_1\cdots i_k}\frac{\partial}{\partial x^{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_k}} \tag{2.1.35}$$

with $f^{i_1\cdots i_k} \in C^\infty(U)$ alternating in all the superscripts.

Definition 2.30. Decomposable k-multivector field
Let X be a k-multivector field. X is a *decomposable k-multivector field* iff there exists $X_1, \cdots, X_k \in \mathfrak{X}(M)$ such that $X = X_1\wedge\cdots\wedge X_k$. We denote all decomposable k-multivector fields by $\mathfrak{X}^{\wedge k}(M)$.

Definition 2.31. Locally decomposable k-multivector field
Let X be a k-multivector field. We say that X is *decomposable at* $p \in M$, if there exists a neighbourhood $U_p \subset M$ of p, and $X_1, \cdots, X_k \in \mathfrak{X}(U_p)$, such that $X = X_1\wedge\cdots\wedge X_k$ on U_p. X is called *locally decomposable k-multivector field* iff for every $p \in M$ there exists a neighbourhood $U_p \subset M$ and $X_1, \cdots, X_k \in \mathfrak{X}(U_p)$ such that $X = X_1\wedge\cdots\wedge X_k$ on each U_p.

We denote all locally decomposable k-multivector fields by $\widetilde{\mathfrak{X}}^{\wedge k}(M)$. k-multivector fields forms a $C^\infty(M)$-module, and it is a dual concept to the k-form. We will show this in the following lemma 2.33, and for this purpose, we introduce the theorem by Morita.

Theorem 2.32. Correspondence of alternating map and Forms (Morita)
Let M be a n-dimensional C^∞-manifold, and $\Omega^k(M)$ a module of all k-forms over M. $\Omega^k(M)$ can be naturally identified with the module of $C^\infty(M)$ multi-linear, alternating mapping $\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M)$.

Proof. Suppose we have a map ω which satisfies the above properties. We first prove that the value $\omega(X_1, \cdots, X_k)(p)$, is determined only by the values of $X_1, \cdots, X_k \in \mathfrak{X}(M)$ at point $p \in M$. By multi-linearity, it is sufficient to prove that for certain X_i, with $1 \leqslant i \leqslant k$, $\omega(X_1, \cdots, X_k)(p) = 0$ for $X_i(p) = 0$. Take $i = 1$, and local coordinates around $p : (U; x_1, \cdots, x_n)$. On U, the vector field X_1 is expressed by $X_1 = f_j \dfrac{\partial}{\partial x^j}$, with $f_i(p) = 0$. Consider an open set V which $\bar{V} \subset U$, and a function $h \in C^\infty(M)$ such that,

$$h(q) = \begin{cases} 1 \ldots q \in V \\ \\ 0 \ldots q \notin U \end{cases} . \tag{2.1.36}$$

Let $Y_j = h\dfrac{\partial}{\partial x^j}$, $Y_j \in \mathfrak{X}(M)$, and $\tilde{f}_j := hf_j$, then, $\tilde{f}_j \in C^\infty(M)$.

By simple modification; $X_1 = f_j Y_j + (1 - h)X_1$, and by the multi-linearity of the map ω,

$$\begin{aligned} &\omega(X_1, \cdots, X_k)(p) = \omega(f_j Y_j + (1 - h)X_1, X_2, \cdots, X_k)(p) \\ &= f_j(p)\omega(Y_j, X_2, \cdots, X_k)(p) + (1 - h(p))\omega(X_1, \cdots, X_k)(p) = 0. \end{aligned} \tag{2.1.37}$$

Having this, we can obtain a k-form as the following. For every $p \in M$, and tangent vectors $\tilde{X}_1, \cdots, \tilde{X}_k \in T_pM$, choose vector fields X_i, $(i = 0, \cdots, k)$, such that $X_i(p) = \tilde{X}_i$. Put $\tilde{\omega}_p(\tilde{X}_1, \cdots, \tilde{X}_k) = \omega(X_1, \cdots, X_k)(p)$. By the previous discussion, this does not depend on the choice of vector field, and it is obvious that $\tilde{\omega}_p$ is C^∞. Then, $\tilde{\omega}$ is the differential form identified with ω. □

We can prove a similar result regarding multivector fields in the following lemma.

Lemma 2.33. Duality of locally decomposable k-vector fields and k-form
1. Suppose we have a map $\omega : \mathfrak{X}^k(M) \to C^\infty(M)$ which is $C^\infty(M)$-linear. Then there

exists a unique k-form Ω on M such that, on each chart $(U,\varphi), \varphi = (x^\mu)$ of M is related to ω by

$$\omega(X) = \frac{1}{k!} X^{\mu_1 \cdots \mu_k} \Omega \left(\frac{\partial}{\partial x^{\mu_1}}, \cdots, \frac{\partial}{\partial x^{\mu_k}} \right) \tag{2.1.38}$$

for $X \in \mathfrak{X}^k(M)$, $X^{\mu_1 \cdots \mu_k} \in C^\infty(M)$.
2. Let $\{(U_\iota, \varphi_\iota)\}$ be the set of charts covering M. The restriction $\omega|_{L.D.} : \widetilde{\mathfrak{X}}^{\wedge k}(M) \to C^\infty(M)$ given by $\omega|_{L.D.}(X) = \omega(X)$ where $X \in \widetilde{\mathfrak{X}}^{\wedge k}(M)$, is related to Ω by

$$\omega(X) = \Omega(X_1, \cdots, X_k) \tag{2.1.39}$$

where $X_1, \cdots, X_k \in \mathfrak{X}(U_\iota)$ are the arbitrary decomposition of X on each U_ι.

Proof. Suppose we have a k-form Ω. Let $X_1, \cdots, X_k$ be vector fields on M, and let $X_i = X_i^\mu \dfrac{\partial}{\partial x^\mu}$, $i = 1, \ldots, k$ be the chart expression of X_i in the chart $(U,\varphi), \varphi = (x^\mu)$. Then by multi-linearity of a k-form,

$$\Omega(X_1, \cdots, X_k) = X_1^{\mu_1} \cdots X_k^{\mu_k} \Omega \left(\frac{\partial}{\partial x^{\mu_1}}, \cdots, \frac{\partial}{\partial x^{\mu_k}} \right). \tag{2.1.40}$$

On the other hand, the k-alternating product of $X_1, \cdots, X_k$ forms a decomposable vector fields, $X_1 \wedge \cdots \wedge X_k = X_1^{\mu_1} \cdots X_k^{\mu_k} \left(\dfrac{\partial}{\partial x^{\mu_1}} \wedge \cdots \wedge \dfrac{\partial}{\partial x^{\mu_k}} \right)$, where $X_1^{\mu_1}, \cdots, X_k^{\mu_k} \in C^\infty(M)$. Set

$$\omega \left(\frac{\partial}{\partial x^{\mu_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_k}} \right) = \Omega \left(\frac{\partial}{\partial x^{\mu_1}}, \cdots, \frac{\partial}{\partial x^{\mu_k}} \right). \tag{2.1.41}$$

Now we extend ω on $\mathfrak{X}^k(M)$ by linearity. The chart $(U,\varphi), \varphi = (x^\mu)$ induces the bases of multivectors in the form $\dfrac{\partial}{\partial x^{\mu_1}} \wedge \cdots \wedge \dfrac{\partial}{\partial x^{\mu_k}}$, and any multivector field $Y \in \mathfrak{X}^k(M)$ has a coordinate expression

$$Y = \frac{1}{k!} Y^{\mu_1 \cdots \mu_k} \frac{\partial}{\partial x^{\mu_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_k}} \tag{2.1.42}$$

with $Y^{\mu_1 \cdots \mu_k} \in C^\infty(U)$. Set

$$\omega(Y) = \frac{1}{k!} Y^{\mu_1 \cdots \mu_k} \omega \left(\frac{\partial}{\partial x^{\mu_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_k}} \right) = \frac{1}{k!} Y^{\mu_1 \cdots \mu_k} \Omega \left(\frac{\partial}{\partial x^{\mu_1}}, \cdots, \frac{\partial}{\partial x^{\mu_k}} \right). \tag{2.1.43}$$

We will show that this expression is independent of charts.

Let $(\bar{U}, \bar{\varphi}), \bar{\varphi} = (\bar{x}^\mu)$ be another chart on M such that $U \cap \bar{U} \neq \emptyset$. On $U \cap \bar{U}$, we have also

$$\omega(Y) = \frac{1}{k!}\bar{Y}^{\mu_1 \cdots \mu_k}\Omega\left(\frac{\partial}{\partial \bar{x}^{\mu_1}}, \cdots, \frac{\partial}{\partial \bar{x}^{\mu_k}}\right). \tag{2.1.44}$$

But since we have

$$Y = \frac{1}{k!}Y^{\mu_1 \cdots \mu_k}\frac{\partial}{\partial x^{\mu_1}}\wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_k}} = \frac{1}{k!}Y^{\nu_1 \cdots \nu_k}\frac{\partial \bar{x}^{\mu_1}}{\partial x^{\nu_1}}\cdots\frac{\partial \bar{x}^{\mu_k}}{\partial x^{\nu_k}}\frac{\partial}{\partial \bar{x}^{\mu_1}}\wedge \cdots \wedge \frac{\partial}{\partial \bar{x}^{\mu_k}} \tag{2.1.45}$$

we get

$$\bar{Y}^{\mu_1 \cdots \mu_k} = Y^{\nu_1 \cdots \nu_k}\frac{\partial \bar{x}^{\mu_1}}{\partial x^{\nu_1}}\cdots\frac{\partial \bar{x}^{\mu_k}}{\partial x^{\nu_k}}, \tag{2.1.46}$$

and since

$$\Omega\left(\frac{\partial}{\partial \bar{x}^{\mu_1}}, \cdots, \frac{\partial}{\partial \bar{x}^{\mu_k}}\right) = \frac{\partial x^{\nu_1}}{\partial \bar{x}^{\mu_1}}\cdots\frac{\partial x^{\nu_k}}{\partial \bar{x}^{\mu_k}}\Omega\left(\frac{\partial}{\partial x^{\nu_1}}, \cdots, \frac{\partial}{\partial x^{\nu_k}}\right), \tag{2.1.47}$$

we have

$$\omega(Y) = \frac{1}{k!}Y^{\mu_1 \cdots \mu_k}\Omega\left(\frac{\partial}{\partial x^{\mu_1}}, \cdots, \frac{\partial}{\partial x^{\mu_k}}\right) = \frac{1}{k!}\bar{Y}^{\mu_1 \cdots \mu_k}\Omega\left(\frac{\partial}{\partial \bar{x}^{\mu_1}}, \cdots, \frac{\partial}{\partial \bar{x}^{\mu_k}}\right). \tag{2.1.48}$$

Therefore, this expression does not depend on the chart, and Ω is uniquely determined on M.

Conversely, suppose we have a $C^\infty(M)$-linear map $\omega : \mathfrak{X}^k(M) \to C^\infty(M)$. By the linearity of ω, for any multivector field $Y \in \mathfrak{X}^k(M)$ we have

$$\omega(Y) = \frac{1}{k!}Y^{\mu_1 \cdots \mu_k}\omega\left(\frac{\partial}{\partial x^{\mu_1}}\wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_k}}\right) \tag{2.1.49}$$

on each chart $(U, \varphi), \varphi = (x^\mu)$ of M. Set

$$\Omega\left(\frac{\partial}{\partial x^{\mu_1}}, \cdots, \frac{\partial}{\partial x^{\mu_k}}\right) = \omega\left(\frac{\partial}{\partial x^{\mu_1}}\wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_k}}\right), \tag{2.1.50}$$

then since the multivectors are multi-linear and skew symmetric in the vector fields $\dfrac{\partial}{\partial x^\mu}$, by the Theorem 2.32, Ω is identified as a k-form on U. Then for the general

$$\omega(Y) = \frac{1}{k!}Y^{\mu_1 \cdots \mu_k}\Omega\left(\frac{\partial}{\partial x^{\mu_1}}, \cdots, \frac{\partial}{\partial x^{\mu_k}}\right), \tag{2.1.51}$$

$\frac{1}{k!} Y^{\mu_1 \cdots \mu_k} \Omega$ is also a k-form on U, since $Y^{\mu_1 \cdots \mu_k}$ is a function on U. Globalisation is carried out similarly. Thus we have proved the first part of the Lemma.

Now consider the special case where $X \in \widetilde{\mathfrak{X}}^{\wedge k}(M)$ is locally decomposable. Without any loss of generality, it is always possible to choose an open covering $\{U_\iota\}_{\iota \in I}$ of M such that for each U_ι, X is decomposable, namely $X = X_{\iota,1} \wedge \cdots \wedge X_{\iota,k}$, with $X_{\iota,1}, \cdots, X_{\iota,k} \in \mathfrak{X}(U_\iota)$. $\iota \in I$ is an index taken from countable index set I. Let $\mathcal{A} = \{(U_\iota, \varphi_\iota)\}_{\iota \in I}$ be an atlas of M, then on each U_ι we have $\omega(X) = \omega(X_{\iota,1} \wedge \cdots \wedge X_{\iota,k})$. We will prove that this does not depend on the decomposition of X. Suppose X has another decomposition, $X = \tilde{X}_{\iota,1} \wedge \cdots \wedge \tilde{X}_{\iota,k}$, $\tilde{X}_{\iota,1}, \cdots, \tilde{X}_{\iota,k} \in \mathfrak{X}(U_\iota)$ on U_ι. Let the coordinates on U_ι be $\varphi = (x^\mu)$, then the local expression of X becomes,

$$X = X_{\iota,1}^{\mu_1} \cdots X_{\iota,k}^{\mu_k} \frac{\partial}{\partial x^{\mu_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_k}} = \tilde{X}_{\iota,1}^{\mu_1} \cdots \tilde{X}_{\iota,k}^{\mu_k} \frac{\partial}{\partial x^{\mu_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_k}}, \tag{2.1.52}$$

implying $X_{\iota,1}^{[\mu_1} \cdots X_{\iota,k}^{\mu_k]} = \tilde{X}_{\iota,1}^{[\mu_1} \cdots \tilde{X}_{\iota,k}^{\mu_k]}$. Then by the first part of the lemma, we have

$$\begin{aligned} &\omega(X_{\iota,1} \wedge \cdots \wedge X_{\iota,k}) \\ &= X_{\iota,1}^{\mu_1} \cdots X_{\iota,k}^{\mu_k} \Omega \left(\frac{\partial}{\partial x^{\mu_1}}, \cdots, \frac{\partial}{\partial x^{\mu_k}} \right) = X_{\iota,1}^{[\mu_1} \cdots X_{\iota,k}^{\mu_k]} \Omega \left(\frac{\partial}{\partial x^{\mu_1}}, \cdots, \frac{\partial}{\partial x^{\mu_k}} \right) \\ &= \tilde{X}_{\iota,1}^{[\mu_1} \cdots \tilde{X}_{\iota,k}^{\mu_k]} \Omega \left(\frac{\partial}{\partial x^{\mu_1}}, \cdots, \frac{\partial}{\partial x^{\mu_k}} \right) = \omega(\tilde{X}_{\iota,1} \wedge \cdots \wedge \tilde{X}_{\iota,k}) \end{aligned} \tag{2.1.53}$$

Therefore, this map does not depend on the choice of decomposition, and also implies $\omega(X_{\iota,1} \wedge \cdots \wedge X_{\iota,k}) = \Omega(X_{\iota,1}, \cdots, X_{\iota,k})$ on each U_ι. □

Definition 2.34. Tangent mapping of multivectors
Let M, N be smooth manifolds and $f : M \to N$ a C^∞-map. Let $(U, \varphi), \varphi = (x^\mu)$ be a chart on M and $(V, \psi), \psi = (y^\nu)$ a chart on N, such that $f(U) \subset V$. We extend the bundle morphism (Tf, f) of tangent bundles $Tf : TM \to TN$ and $f : M \to N$ to bundle morphism of multivector bundles, $(\Lambda^k TM, \Lambda^k \tau_M, M)$ to $(\Lambda^k TN, \Lambda^k \tau_N, N)$, similarly as in the case of contravariant tensor bundles. Let $v \in \Lambda^k T_p M$, and let the coordinate expression be $v = \frac{1}{k!} v^{\mu_1 \cdots \mu_k} \left(\frac{\partial}{\partial x^{\mu_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_k}} \right)_p$. Then we define the image of this multivector by the tangent map by

$$\Lambda^k Tf(v) = \frac{1}{k!} v^{\mu_1 \cdots \mu_k} \left(Tf \left(\frac{\partial}{\partial x^{\mu_1}} \right) \wedge \cdots \wedge Tf \left(\frac{\partial}{\partial x^{\mu_k}} \right) \right)_{f(p)}. \tag{2.1.54}$$

To distinguish between the usual tangent mappings, we used the notation $\Lambda^k Tf(v)$ to state that the map is acting on a k-multivector, and occasionally call them *multi-tangent map*.

2.1.4 Second order multivector bundle

In the previous sections, we have prepared the basics of multivectors and second order tangent bundle. With these foundations, we can now construct the underlying structures required for the second order field theory, which we call the bundle of *second order multivectors*, and denote by $((\Lambda^k T)^2 M, \Lambda^k \tau_M^{2,1}, \Lambda^k TM)$. The section of the bundle of second order multivectors is called *second order multivector field*, which corresponds to the physical fields as we shall see later.

Before constructing the bundle of second order multivectors, we begin with some basic observations.

Proposition 2.35. If (E, π, M) is a vector bundle, then $(TE, T\pi, TM)$ is a vector bundle.

Proof. Let $m = \dim M$, $n = \dim E$, and $(\pi^{-1}(W), \psi), \psi = (x^1, \cdots x^m, u^1, \cdots, u^n)$ a vector bundle adapted chart on E, induced by the chart $(W, \varphi), \varphi = (x^1, \cdots, x^m)$ on $W \subset M$. The element of the fibre E_p would then have a chart expression $\phi = \phi^i(e_i)_p$, $i = 1, \cdots, n$, where e_i are the local sections defined from the chart as in Definition 2.18. Let $p \in U \subset W$, and denote the local trivialisation of (E, π, M) at p by $(U_p, \mathbb{R}^n, s_p)$. The chart on M and induces the local trivialisation on the bundles (TM, τ_M, M) and (TE, τ_E, E), by $(U_p, \mathbb{R}^m, t_{M,p})$ and $(s_p(\pi^{-1}(U_p)) = U_p \times \mathbb{R}^n, \mathbb{R}^{m+n}, t_{E,q})$, $q \in E$, $\pi(q) = p$, respectively. We take induced charts on TM and TE associated to these trivialisation as $({\tau_M}^{-1}(U_p), \varphi^1), \varphi^1 = (x^\mu, y^\mu)$ and $(\pi^{-1}({\tau_M}^{-1}(U_p)), \psi^1), \psi^1 = (x^\mu, u^i, \dot{x}^\mu, \dot{u}^i)$. Furthermore, these trivialisations gives rise to the local trivialisation $(\hat{U}_p, F_p, \hat{t}_p)$ on $(TE, T\pi, TM)$, by

$$\hat{t}_p : T\pi^{-1}(U_p \times \mathbb{R}^m) \to U_p \times \mathbb{R}^m \times \mathbb{R}^{2n}, \tag{2.1.55}$$

where we denoted $U_p \times \mathbb{R}^m = \hat{U}_p$, and $\mathbb{R}^{2n} = F_p$. Here we used the isomorphism of the trivialisation,

$$S : U_p \times \mathbb{R}^n \times \mathbb{R}^{m+n} \to U_p \times \mathbb{R}^m \times \mathbb{R}^{2n}, \tag{2.1.56}$$

which in local coordinates could be expressed as $S(x^\mu, u^\alpha, \dot{x}^\mu, \dot{u}^\alpha) = (x^\mu, \dot{x}^\mu, u^\alpha, \dot{u}^\alpha)$. S is called a *swap map*. Let us see how this work. Let $\xi_\phi \in T_\phi E$, $\phi = \phi^i e_i(p) \in E_p$. The local coordinate expression of ξ_ϕ is given by $\xi_\phi = \xi^\mu \left(\frac{\partial}{\partial x^\mu}\right)_\phi + \bar{\xi}^i \left(\frac{\partial}{\partial u^i}\right)_\phi$ or equivalently, $\psi^1(\xi_\phi) = (x^\mu(\phi), \phi^i, \xi^\mu, \bar{\xi}^i)$. The swap map will take this to

$$S\psi^1(\xi_\phi) = S(x^\mu(\phi), \phi^i, \xi^\mu, \bar{\xi}^i) = (x^\mu(\phi), \xi^\mu, \phi^i, \bar{\xi}^i), \tag{2.1.57}$$

which by the local section e_i, $\dot{e}_i$ of $T\pi$, defined by $u^i(e_j(z)) = \delta^i_j$, $\dot{u}^i(\dot{e}_j(z)) = \delta^i_j$ for all $z \in {\tau_M}^{-1}(U_p)$, could be expressed by $\xi_q = \phi^i(e_i)_q + \bar{\xi}^i(\dot{e}_i)_q \in (TE)_q$, with $q = (x^\mu(p), \xi^\mu) \in {\tau_M}^{-1}(U_p)$. It is easy to see that this point q is consistent with the projection $T_\phi\pi(\xi_\phi)$. The swap map $S(\xi_\phi) = \xi_q$ simply changes the base point of the vector in TE. We can now see the fibres of $T\pi$ becomes a vector space over each point of $z \in {\tau_M}^{-1}(U_p) \subset TM$. Considering for every $p \in M$, these process for each trivialisation, we can conclude that $T\pi$ is a vector bundle. □

Proposition 2.35 tells that the k-multivector bundle $(\Lambda^k TM, \Lambda^k\tau_M, M)$ induces a vector bundle $(T(\Lambda^k TM), T\Lambda^k\tau_M, TM)$ by the tangent map $T\Lambda^k\tau_M : T(\Lambda^k TM) \to TM$. The element of $T(\Lambda^k TM)$ is a 1-vector at a point on $\Lambda^k TM$, and if we choose a vector bundle adapted chart (V, ψ), $V = (\Lambda^k TM)^{-1}(U_p)$, $\psi = (x^\mu, y^{\mu_1\cdots\mu_k})$ on $\Lambda^k TM$, induced by the chart (U, φ), $\varphi = (x^1, \cdots, x^m)$ on M, it has a coordinate expression

$$\xi_v = \xi^\mu\left(\frac{\partial}{\partial x^\mu}\right)_v + \frac{1}{k!}\xi^{\mu_1\cdots\mu_k}\left(\frac{\partial}{\partial y^{\mu_1\cdots\mu_k}}\right)_v, \tag{2.1.58}$$

with $\xi_v \in T_v(\Lambda^k TM)$, $v \in (\Lambda^k TM)_p$. or similarly,

$$\psi^1(\xi_v) = (x^\mu(p), v^{\mu_1\cdots\mu_k}, \xi^\mu, \xi^{\mu_1\cdots\mu_k}), \tag{2.1.59}$$

by the induced chart (V^1, ψ^1), $V^1 = (\tau_{\Lambda^k TM})^{-1}(V)$, $\psi^1 = (x^\mu, y^{\mu_1\cdots\mu_k}, \dot{x}^\mu, \dot{y}^{\mu_1\cdots\mu_k})$ on $T(\Lambda^k TM)$. The swap map

$$S\psi^1(\xi_v) = S\left(x^\mu(p), v^{\mu_1\cdots\mu_k}, \xi^\mu, \xi^{\mu_1\cdots\mu_k}\right) = \left(x^\mu(p), \xi^\mu, v^{\mu_1\cdots\mu_k}, \xi^{\mu_1\cdots\mu_k}\right) \tag{2.1.60}$$

will give the multivector the expression

$$\xi_q = \frac{1}{k!}v^{\mu_1\cdots\mu_k}(\frac{\partial}{\partial y^{\mu_1\cdots\mu_k}})_q + \frac{1}{k!}\xi^{\mu_1\cdots\mu_k}(\frac{\partial}{\partial \dot{y}^{\mu_1\cdots\mu_k}})_q \in (T\Lambda^k TM)_q, \tag{2.1.61}$$

with $q = (x^\mu(p), \xi^\mu) \in {\tau_M}^{-1}(U_p)$, where the local section

$$e_{\mu_1\cdots\mu_k} = \frac{\partial}{\partial y^{\mu_1\cdots\mu_k}}, \ \dot{e}_{\mu_1\cdots\mu_k} = \frac{\partial}{\partial \dot{y}^{\mu_1\cdots\mu_k}} \tag{2.1.62}$$

of $T\Lambda^k\tau_M$ was defined by $y^{\mu_1\cdots\mu_k}(e_{\nu_1\cdots\nu_k}(z)) = k!\delta^{[\mu_1}_{\nu_1}\cdots\delta^{\mu_k]}_{\nu_k}$, $\dot{y}^{\mu_1\cdots\mu_k}(\dot{e}_{\nu_1\cdots\nu_k}(z)) = k!\delta^{[\mu_1}_{\nu_1}\cdots\delta^{\mu_k]}_{\nu_k}$, for all $z \in \Lambda^k\tau_M{}^{-1}(U_p)$. The map $T\Lambda^k\tau_M$ sends ξ_v to $T_{\Lambda^k\tau_M(v)}M$ by

$$\begin{aligned} T\Lambda^k\tau_M(\xi_v) &= \left.\frac{\partial x^\mu \Lambda^k\tau_M\psi^{-1}}{\partial x^\nu}\right|_{\psi(v)} \xi^\nu \left(\frac{\partial}{\partial x^\mu}\right)_p \\ &+ \frac{1}{k!}\left.\frac{\partial x^\mu \Lambda^k\tau_M\psi^{-1}}{\partial y^{\nu_1\cdots\nu_k}}\right|_{\psi(v)} \xi^{\nu_1\cdots\nu_k}\left(\frac{\partial}{\partial x^\mu}\right)_p = \xi^\mu\left(\frac{\partial}{\partial x^\mu}\right)_p, \end{aligned} \tag{2.1.63}$$

which is indeed the base point q of ξ_q, and we see that $\left(T\Lambda^k\tau_M\right)_q$ becomes a vector space.

The k-multivector bundle $(\Lambda^kTM, \Lambda^k\tau_M, M)$ also induces a second order bundle $(\Lambda^kT\Lambda^kTM, \Lambda^k\tau_{\Lambda^kTM}, \Lambda^kTM)$ by iteration. We could introduce the induced charts for $\Lambda^kT\Lambda^kTM$ similarly as we did in Definition 2.27. Let (V,ψ), $\psi = (x^\mu, y^{\mu_1\cdots\mu_k})$ be a chart on Λ^kTM, then the natural bases of $\Lambda^kT\Lambda^kTM$ on V are

$$\left(\frac{\partial}{\partial x^\mu}, \quad \frac{\partial}{\partial y^{\mu_1\cdots\mu_k}}\right), \tag{2.1.64}$$

where $\mu_1, \cdots, \mu_k = 1, \cdots, n$ are alternating indices. The element of $\Lambda^kT\Lambda^kTM$ at a point $p \in \Lambda^kTM$ will have the form

$$\begin{aligned} w = &\frac{1}{k!}w^{\mu_1\cdots\mu_k}\left(\frac{\partial}{\partial x^{\mu_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{\mu_k}}\right)_p + \frac{1}{(k-1)!}w^{I\mu_1\cdots\mu_{k-1}}\left(\frac{\partial}{\partial y^I}\wedge\frac{\partial}{\partial x^{\mu_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{\mu_{k-1}}}\right)_p \\ &+ \frac{1}{(k-2)!2!}w^{I_1I_2\mu_1\cdots\mu_{k-2}}\left(\frac{\partial}{\partial y^{I_1}}\wedge\frac{\partial}{\partial y^{I_2}}\wedge\frac{\partial}{\partial x^{\mu_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{\mu_{k-2}}}\right)_p + \cdots \\ &+ \frac{1}{(k-1)!}w^{I_1I_2\cdots I_{k-1}\mu}\left(\frac{\partial}{\partial y^{I_1}}\wedge\cdots\wedge\frac{\partial}{\partial y^{I_{k-1}}}\wedge\frac{\partial}{\partial x^\mu}\right)_p \\ &+ \frac{1}{k!}w^{I_1I_2\cdots I_k}\left(\frac{\partial}{\partial y^{I_1}}\wedge\frac{\partial}{\partial y^{I_2}}\wedge\cdots\wedge\frac{\partial}{\partial y^{I_k}}\right)_p. \end{aligned} \tag{2.1.65}$$

Here we have introduced a multi-index notation for visibility. The upper case latin index $I_1, I_2, \cdots, I_k$ is a combination of alternating k indices, such that $I = \mu_1\cdots\mu_k$, and has its own unique label, and we suppressed the factorial coefficient which we will understand as already included in the summation over I appearing consequently. For example, in the case $n = 4$ and $k = 2$, $I = \mathbf{1}, \mathbf{2}, \cdots, {}_4C_2 = \mathbf{6}$, and

$$\mathbf{1} := (1\,2), \mathbf{2} := (1\,3), \mathbf{3} := (1\,4), \mathbf{4} := (2\,3), \mathbf{5} := (2\,4), \mathbf{6} := (3\,4), \tag{2.1.66}$$

we are able to write explicitly

$$
\begin{aligned}
&\frac{1}{2}w^{IJ}\frac{\partial}{\partial y^I}\wedge\frac{\partial}{\partial y^J}\\
&= w^{\mathbf{12}}\frac{\partial}{\partial y^{\mathbf{1}}}\wedge\frac{\partial}{\partial y^{\mathbf{2}}} + w^{\mathbf{13}}\frac{\partial}{\partial y^{\mathbf{1}}}\wedge\frac{\partial}{\partial y^{\mathbf{3}}} + w^{\mathbf{14}}\frac{\partial}{\partial y^{\mathbf{1}}}\wedge\frac{\partial}{\partial y^{\mathbf{4}}} + w^{\mathbf{15}}\frac{\partial}{\partial y^{\mathbf{1}}}\wedge\frac{\partial}{\partial y^{\mathbf{5}}} + w^{\mathbf{16}}\frac{\partial}{\partial y^{\mathbf{1}}}\wedge\frac{\partial}{\partial y^{\mathbf{6}}}\\
&+ w^{\mathbf{23}}\frac{\partial}{\partial y^{\mathbf{2}}}\wedge\frac{\partial}{\partial y^{\mathbf{3}}} + w^{\mathbf{24}}\frac{\partial}{\partial y^{\mathbf{2}}}\wedge\frac{\partial}{\partial y^{\mathbf{4}}} + w^{\mathbf{25}}\frac{\partial}{\partial y^{\mathbf{2}}}\wedge\frac{\partial}{\partial y^{\mathbf{5}}} + w^{\mathbf{26}}\frac{\partial}{\partial y^{\mathbf{2}}}\wedge\frac{\partial}{\partial y^{\mathbf{6}}}\\
&+ w^{\mathbf{34}}\frac{\partial}{\partial y^{\mathbf{3}}}\wedge\frac{\partial}{\partial y^{\mathbf{4}}} + w^{\mathbf{35}}\frac{\partial}{\partial y^{\mathbf{3}}}\wedge\frac{\partial}{\partial y^{\mathbf{5}}} + w^{\mathbf{36}}\frac{\partial}{\partial y^{\mathbf{3}}}\wedge\frac{\partial}{\partial y^{\mathbf{6}}}\\
&+ w^{\mathbf{45}}\frac{\partial}{\partial y^{\mathbf{4}}}\wedge\frac{\partial}{\partial y^{\mathbf{5}}} + w^{\mathbf{46}}\frac{\partial}{\partial y^{\mathbf{4}}}\wedge\frac{\partial}{\partial y^{\mathbf{6}}}\\
&+ w^{\mathbf{56}}\frac{\partial}{\partial y^{\mathbf{5}}}\wedge\frac{\partial}{\partial y^{\mathbf{6}}}\\
&= w^{(12)(13)}\frac{\partial}{\partial y^{(12)}}\wedge\frac{\partial}{\partial y^{(13)}} + w^{(12)(14)}\frac{\partial}{\partial y^{(12)}}\wedge\frac{\partial}{\partial y^{(14)}} + w^{(12)(23)}\frac{\partial}{\partial y^{(12)}}\wedge\frac{\partial}{\partial y^{(23)}}\\
&+ w^{(12)(24)}\frac{\partial}{\partial y^{(12)}}\wedge\frac{\partial}{\partial y^{(24)}} + w^{(12)(34)}\frac{\partial}{\partial y^{(12)}}\wedge\frac{\partial}{\partial y^{(34)}} + w^{(13)(14)}\frac{\partial}{\partial y^{(13)}}\wedge\frac{\partial}{\partial y^{(14)}}\\
&+ w^{(13)(23)}\frac{\partial}{\partial y^{(13)}}\wedge\frac{\partial}{\partial y^{(23)}} + w^{(13)(24)}\frac{\partial}{\partial y^{(13)}}\wedge\frac{\partial}{\partial y^{(24)}} + w^{(13)(34)}\frac{\partial}{\partial y^{(13)}}\wedge\frac{\partial}{\partial y^{(34)}}\\
&+ w^{(14)(23)}\frac{\partial}{\partial y^{(14)}}\wedge\frac{\partial}{\partial y^{(23)}} + w^{(14)(24)}\frac{\partial}{\partial y^{(14)}}\wedge\frac{\partial}{\partial y^{(24)}} + w^{(14)(34)}\frac{\partial}{\partial y^{(14)}}\wedge\frac{\partial}{\partial y^{(24)}}\\
&+ w^{(23)(24)}\frac{\partial}{\partial y^{(23)}}\wedge\frac{\partial}{\partial y^{(24)}} + w^{(23)(34)}\frac{\partial}{\partial y^{(23)}}\wedge\frac{\partial}{\partial y^{(34)}} + w^{(24)(34)}\frac{\partial}{\partial y^{(24)}}\wedge\frac{\partial}{\partial y^{(34)}}.
\end{aligned}
\tag{2.1.67}
$$

Therefore in this notation, coordinate functions are labelled as

$$y^I := y^{\mu_1\cdots\mu_k}, \quad z^{I_1 I_2 \nu_3\cdots\nu_k} := z^{(\mu_1^1\cdots\mu_k^1)(\mu_1^2\cdots\mu_k^2)\nu_3\cdots\nu_k}, \tag{2.1.68}$$

etc. The summation conventions between ordered and non-ordered indices are,

$$\frac{\partial K}{\partial y^I}y^I := \sum_{\mu_1<\mu_2<\cdots<\mu_k}\frac{\partial K}{\partial y^{\mu_1\cdots\mu_k}}y^{\mu_1\cdots\mu_k} = \frac{1}{k!}\frac{\partial K}{\partial y^{\mu_1\cdots\mu_k}}y^{\mu_1\cdots\mu_k}, \tag{2.1.69}$$

$$
\begin{aligned}
&\frac{1}{l!(k-l)!}\frac{\partial K}{\partial z^{I_1\cdots I_l\nu_{l+1}\cdots\nu_k}}z^{I_1\cdots I_l\nu_{l+1}\cdots\nu_k} := \sum_{I_1<I_2<\cdots<I_l}\ \sum_{\nu_{l+1}<\nu_{l+2}<\cdots<\nu_k}\frac{\partial K}{\partial z^{I_1\cdots I_l\nu_{l+1}\cdots\nu_k}}z^{I_1\cdots I_l\nu_{l+1}\cdots\nu_k}\\
&= \sum_{\text{ordered by } I}\ \sum_{\mu_1^1<\cdots<\mu_k^1}\cdots\sum_{\mu_1^l<\cdots<\mu_k^l}\ \sum_{\nu_{l+1}<\cdots<\nu_k}\frac{\partial K}{\partial z^{(\mu_1^1\cdots\mu_k^1)\cdots(\mu_1^l\cdots\mu_k^l)\nu_{l+1}\cdots\nu_k}}z^{(\mu_1^1\cdots\mu_k^1)\cdots(\mu_1^l\cdots\mu_k^l)\nu_{l+1}\cdots\nu_k},\\
&\qquad 1\leqslant l\leqslant k,
\end{aligned}
\tag{2.1.70}
$$

and the induced chart of the space $\Lambda^k T\Lambda^k TM$ could be introduced as $(\tilde{W}^2, \tilde{\psi}^2)$, $\tilde{W}^2 = (\Lambda^k \tau_{\Lambda^k TM})^{-1}(V)$, $\tilde{\psi}^2 = (x^\mu, y^{\mu_1\cdots\mu_k}, z^{\mu_1\cdots\mu_k}, z^{\mathrm{I}\mu_1\cdots\mu_{k-1}}, z^{I_1 I_2\mu_1\cdots\mu_{k-2}}, ..., z^{I_1\cdots I_k})$.

Now, consider the extended tangent map $\Lambda^k T\Lambda^k\tau_M : \Lambda^k T\Lambda^k TM \to \Lambda^k TM$. This map sends k-multivector at $p \in \Lambda^k TM$ to k-multivector at $\Lambda^k\tau_M(p) \in M$, and induces a vector bundle $(\Lambda^k T\Lambda^k TM, \Lambda^k T\Lambda^k\tau_M, \Lambda^k TM)$. As we did in the case for constructing the bundle $(T^2M, \tau_{TM}|_{T^2M}, TM)$, we can use the bundle isomorphism between the two bundles, $(\Lambda^k T\Lambda^k TM, \Lambda^k\tau_{\Lambda^k TM}, \Lambda^k TM)$ and $(\Lambda^k T\Lambda^k TM, \Lambda^k T\Lambda^k\tau_M, \Lambda^k TM)$, to construct the bundle of second order multivectors. Let $w \in \Lambda^k T\Lambda^k TM$ be the k-multivector at point $p \in \Lambda^k TM$. Then by the previous definition,

$$\begin{aligned}
\Lambda^k T\Lambda^k\tau_M(w) &= \frac{1}{k!}w^{\mu_1\cdots\mu_k}\left(T\Lambda^k\tau_M\left(\frac{\partial}{\partial x^{\mu_1}}\right)\wedge T\Lambda^k\tau_M\left(\frac{\partial}{\partial x^{\mu_2}}\right)\wedge\cdots\wedge T\Lambda^k\tau_M\left(\frac{\partial}{\partial x^{\mu_k}}\right)\right)_p \\
&+ \frac{1}{k!}w^{I\mu_1\cdots\mu_{k-1}}\left(T\Lambda^k\tau_M\left(\frac{\partial}{\partial y^I}\right)\wedge T\Lambda^k\tau_M\left(\frac{\partial}{\partial x^{\mu_1}}\right)\wedge\cdots\wedge T\Lambda^k\tau_M\left(\frac{\partial}{\partial x^{\mu_{k-1}}}\right)\right)_p \\
&+\cdots+ \frac{1}{k!}w^{I_1 I_2\cdots I_{k-1}\mu}\left(T\Lambda^k\tau_M\left(\frac{\partial}{\partial y^{I_1}}\right)\wedge\cdots\wedge T\Lambda^k\tau_M\left(\frac{\partial}{\partial y^{I_{k-1}}}\right)\wedge T\Lambda^k\tau_M\left(\frac{\partial}{\partial x^{\mu}}\right)\right)_p \\
&+ \frac{1}{k!}w^{I_1 I_2\cdots I_k}\left(T\Lambda^k\tau_M\left(\frac{\partial}{\partial y^{I_1}}\right)\wedge T\Lambda^k\tau_M\left(\frac{\partial}{\partial y^{I_2}}\right)\wedge\cdots\wedge T\Lambda^k\tau_M\left(\frac{\partial}{\partial y^{I_k}}\right)\right)_p \\
&= \frac{1}{k!}w^{\mu_1\cdots\mu_k}\left(\frac{\partial}{\partial x^{\mu_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{\mu_k}}\right)_{\Lambda^k\tau_M(p)} \in \Lambda^k T_{\Lambda^k\tau_M(p)}M. \qquad (2.1.71)
\end{aligned}$$

On the other hand, $\Lambda^k\tau_{\Lambda^k TM}(w) = p$, and the equation for the isomorphism is

$$\frac{1}{k!}w^{\mu_1\cdots\mu_k}\left(\frac{\partial}{\partial x^{\mu_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{\mu_k}}\right)_{\Lambda^k\tau_M(p)} = p, \qquad (2.1.72)$$

and in coordinate expression becomes,

$$x^\mu(p) = x^\mu(\Lambda^k\tau_M(p)), \quad y^{\mu_1\cdots\mu_k}(p) = w^{\mu_1\cdots\mu_k} = z^{\mu_1\cdots\mu_k}(w). \qquad (2.1.73)$$

Therefore, we can take as the chart on $(\Lambda^k T)^2 M$, (W^2, ψ^2), $W^2 = (\Lambda^k\tau_{\Lambda^k TM}\big|_{(\Lambda^k T)^2 M})^{-1}(V)$, $\psi^2 = (x^\mu, y^{\mu_1\cdots\mu_k}, z^{\mathrm{I}\mu_1\cdots\mu_{k-1}}, z^{I_1 I_2\mu_1\cdots\mu_{k-2}}..., z^{I_1\cdots I_k})$. The coordinate $z^{\mu_1\cdots\mu_k}$ is not present from the original chart on $\Lambda^k T\Lambda^k TM$ because of the above equation of submanifolds.

Definition 2.36. Second order k-multivector bundle $(\Lambda^k T)^2 M$ over $\Lambda^k TM$
Let $(\Lambda^k TM, \Lambda^k\tau_M, M)$ be the k-multivector bundle with the base space M and $(\Lambda^k T\Lambda^k TM, \Lambda^k\tau_{\Lambda^k TM}, \Lambda^k TM)$ the k-multivector bundle with the base space $\Lambda^k TM$. Denote the sub-

set of elements $w \in \Lambda^k T \Lambda^k TM$ which satisfy the equations of submanifold

$$\Lambda^k \tau_{\Lambda^k TM}(w) = \Lambda^k T \Lambda^k \tau_M(w) \tag{2.1.74}$$

as $(\Lambda^k T)^2 M$, and a map from $(\Lambda^k T)^2 M$ to $\Lambda^k TM$ by

$$\Lambda^k \tau_M^{2,1} := \Lambda^k \tau_{\Lambda^k TM}\big|_{(\Lambda^k T)^2 M}. \tag{2.1.75}$$

The triple $((\Lambda^k T)^2 M, \Lambda^k \tau_M^{2,1}, \Lambda^k TM)$ becomes a bundle, and we call them *second order k-multi-vector bundle over $\Lambda^k TM$*.

Again this is not a vector bundle, i.e. the fibres of $\Lambda^k \tau_M^{2,1}$ are not vector spaces, since in general, the scalar multiplication of $w_p \in (\Lambda^k T)^2 M$, $p \in \Lambda^k TM$ does not belong to the same fibre. As in Section 2.1.1, we can similarly check that $((\Lambda^k T)^2 M, \Lambda^k \tau_M^{2,1}, \Lambda^k TM)$ is a bundle. Namely we first introduce manifold structure on $(\Lambda^k T)^2 M$, and then consider the local trivialisation. The charts on $(\Lambda^k T)^2 M$ are already introduced, so let us check the coordinate transformations.
Let (W^2, ψ^2), $\psi^2 = (x^\mu, y^{\mu_1 \cdots \mu_k}, z^{\mathrm{I}\,\mu_1 \cdots \mu_{k-1}}, z^{I_1 I_2 \mu_1 \cdots \mu_{k-2} \cdots}, z^{I_1 \cdots I_k})$ and $(\bar{W}^2, \bar{\psi}^2)$, $\bar{\psi}^2 = \big(\bar{x}^\mu, \bar{y}^{\mu_1 \cdots \mu_k}, \bar{z}^{\mathrm{I}\,\mu_1 \cdots \mu_{k-1}}, \bar{z}^{I_1 I_2 \mu_1 \cdots \mu_{k-2}}, \dots, \bar{z}^{I_1 \cdots I_k}\big)$ be two charts on $(\Lambda^k T)^2 M$, with $W^2 \cap \bar{W}^2 \neq \emptyset$. Then express the element $w_q \in W^2 \cap \bar{W}^2$, $q \in \Lambda^k TM$, by these charts,

$$\begin{aligned}
w_q &= \frac{1}{k!} y^{\mu_1 \cdots \mu_k}(q) \left(\frac{\partial}{\partial x^{\mu_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_k}} \right)_q + \frac{1}{(k-1)!} w^{I_1 \mu_2 \cdots \mu_k} \left(\frac{\partial}{\partial y^{I_1}} \wedge \frac{\partial}{\partial x^{\mu_2}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_k}} \right)_q \\
&\quad + \frac{1}{(k-2)!2!} w^{I_1 I_2 \mu_3 \cdots \mu_k} \left(\frac{\partial}{\partial y^{I_1}} \wedge \frac{\partial}{\partial y^{I_2}} \wedge \frac{\partial}{\partial x^{\mu_3}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_k}} \right)_q \\
&\quad + \cdots + \frac{1}{(k-l)!l!} w^{I_1 \cdots I_l \mu_{l+1} \cdots \mu_k} \left(\frac{\partial}{\partial y^{I_1}} \wedge \cdots \wedge \frac{\partial}{\partial y^{I_l}} \wedge \frac{\partial}{\partial x^{\mu_{l+1}}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_k}} \right)_q \\
&\quad + \cdots + \frac{1}{k!} w^{I_1 I_2 \cdots I_k} \left(\frac{\partial}{\partial y^{I_1}} \wedge \frac{\partial}{\partial y^{I_2}} \wedge \cdots \wedge \frac{\partial}{\partial y^{I_k}} \right)_q \\
&= \frac{1}{k!} \bar{y}^{\mu_1 \cdots \mu_k}(q) \left(\frac{\partial}{\partial \bar{x}^{\mu_1}} \wedge \cdots \wedge \frac{\partial}{\partial \bar{x}^{\mu_k}} \right)_q + \frac{1}{(k-1)!} \bar{w}^{I_1 \mu_2 \cdots \mu_k} \left(\frac{\partial}{\partial \bar{y}^{I_1}} \wedge \frac{\partial}{\partial \bar{x}^{\mu_2}} \wedge \cdots \wedge \frac{\partial}{\partial \bar{x}^{\mu_k}} \right)_q \\
&\quad + \frac{1}{(k-2)!2!} \bar{w}^{I_1 I_2 \mu_3 \cdots \mu_k} \left(\frac{\partial}{\partial \bar{y}^{I_1}} \wedge \frac{\partial}{\partial \bar{y}^{I_2}} \wedge \frac{\partial}{\partial \bar{x}^{\mu_3}} \wedge \cdots \wedge \frac{\partial}{\partial \bar{x}^{\mu_k}} \right)_q \\
&\quad + \cdots + \frac{1}{(k-l)!l!} \bar{w}^{I_1 \cdots I_l \mu_{l+1} \cdots \mu_k} \left(\frac{\partial}{\partial \bar{y}^{I_1}} \wedge \cdots \wedge \frac{\partial}{\partial \bar{y}^{I_l}} \wedge \frac{\partial}{\partial \bar{x}^{\mu_{l+1}}} \wedge \cdots \wedge \frac{\partial}{\partial \bar{x}^{\mu_k}} \right)_q \\
&\quad + \cdots + \frac{1}{k!} \bar{w}^{I_1 I_2 \cdots I_k} \left(\frac{\partial}{\partial \bar{y}^{I_1}} \wedge \frac{\partial}{\partial \bar{y}^{I_2}} \wedge \cdots \wedge \frac{\partial}{\partial \bar{y}^{I_k}} \right)_q,
\end{aligned} \tag{2.1.76}$$

with

$$
\begin{aligned}
&z^{I_1\mu_2\cdots\mu_k}(w_q)=w^{I_1\mu_2\cdots\mu_k},z^{I_1I_2\mu_3\cdots\mu_k}(w_q)=w^{I_1I_2\mu_3\cdots\mu_k},\cdots,z^{I_1I_2\cdots I_k}(w_q)=w^{I_1I_2\cdots I_k},\\
&\bar{z}^{I_1\mu_2\cdots\mu_k}(w_q)=\bar{w}^{I_1\mu_2\cdots\mu_k},\bar{z}^{I_1I_2\mu_3\cdots\mu_k}(w_q)=\bar{w}^{I_1I_2\mu_3\cdots\mu_k},\cdots,\bar{z}^{I_1I_2\cdots I_k}(w_q)=w^{I_1I_2\cdots I_k}.
\end{aligned}
\tag{2.1.77}
$$

Since the base of one vectors transform as

$$
\frac{\partial}{\partial\bar{x}^\mu}=\frac{\partial x^\nu}{\partial\bar{x}^\mu}\frac{\partial}{\partial x^\nu}+\frac{\partial y^I}{\partial\bar{x}^\mu}\frac{\partial}{\partial y^I},\frac{\partial}{\partial\bar{y}^I}=\frac{\partial y^J}{\partial\bar{y}^I}\frac{\partial}{\partial y^J}=k!\frac{\partial x^J}{\partial\bar{x}^I}\frac{\partial}{\partial y^J},\tag{2.1.78}
$$

the bases of $\Lambda^kT\Lambda^kTM$ will transform as

$$
\begin{aligned}
&\frac{\partial}{\partial\bar{x}^{\mu_1}}\wedge\cdots\wedge\frac{\partial}{\partial\bar{x}^{\mu_k}}=\left(\frac{\partial x^{\nu_1}}{\partial\bar{x}^{\mu_1}}\frac{\partial}{\partial x^{\nu_1}}+\frac{\partial y^{I_1}}{\partial\bar{x}^{\mu_1}}\frac{\partial}{\partial y^{I_1}}\right)\wedge\cdots\wedge\left(\frac{\partial x^{\nu_k}}{\partial\bar{x}^{\mu_k}}\frac{\partial}{\partial x^{\nu_k}}+\frac{\partial y^{I_k}}{\partial\bar{x}^{\mu_k}}\frac{\partial}{\partial y^{I_k}}\right)\\
&=\frac{\partial x^{\nu_1}}{\partial\bar{x}^{\mu_1}}\cdots\frac{\partial x^{\nu_k}}{\partial\bar{x}^{\mu_k}}\frac{\partial}{\partial x^{\nu_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{\nu_k}}+{}_kC_1\frac{\partial y^{I_1}}{\partial\bar{x}^{\mu_1}}\frac{\partial x^{\nu_2}}{\partial\bar{x}^{\mu_2}}\cdots\frac{\partial x^{\nu_k}}{\partial\bar{x}^{\mu_k}}\frac{\partial}{\partial y^{I_1}}\wedge\frac{\partial}{\partial x^{\nu_2}}\wedge\cdots\wedge\frac{\partial}{\partial x^{\nu_k}}\\
&+{}_kC_2\frac{\partial y^{I_1}}{\partial\bar{x}^{\mu_1}}\frac{\partial y^{I_2}}{\partial\bar{x}^{\mu_2}}\frac{\partial x^{\nu_3}}{\partial\bar{x}^{\mu_3}}\cdots\frac{\partial x^{\nu_k}}{\partial\bar{x}^{\mu_k}}\frac{\partial}{\partial y^{I_1}}\wedge\frac{\partial}{\partial y^{I_2}}\wedge\frac{\partial}{\partial x^{\nu_3}}\wedge\cdots\wedge\frac{\partial}{\partial x^{\nu_k}}\\
&+\cdots+\frac{\partial y^{I_1}}{\partial\bar{x}^{\mu_1}}\frac{\partial y^{I_2}}{\partial\bar{x}^{\mu_2}}\cdots\frac{\partial y^{I_k}}{\partial\bar{x}^{\mu_k}}\frac{\partial}{\partial y^{I_1}}\wedge\frac{\partial}{\partial y^{I_2}}\wedge\cdots\wedge\frac{\partial}{\partial y^{I_k}},
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial\bar{y}^{I_1}}\wedge\frac{\partial}{\partial\bar{x}^{\mu_2}}\wedge\cdots\wedge\frac{\partial}{\partial\bar{x}^{\mu_k}}\\
&=k!\frac{\partial x^{J_1}}{\partial\bar{x}^{I_1}}\frac{\partial}{\partial y^{J_1}}\wedge\left(\frac{\partial x^{\nu_2}}{\partial\bar{x}^{\mu_2}}\frac{\partial}{\partial x^{\nu_2}}+\frac{\partial y^{I_2}}{\partial\bar{x}^{\mu_2}}\frac{\partial}{\partial y^{I_2}}\right)\wedge\cdots\wedge\left(\frac{\partial x^{\nu_k}}{\partial\bar{x}^{\mu_k}}\frac{\partial}{\partial x^{\nu_k}}+\frac{\partial y^{I_k}}{\partial\bar{x}^{\mu_k}}\frac{\partial}{\partial y^{I_k}}\right)\\
&=k!\left(\frac{\partial x^{J_1}}{\partial\bar{x}^{I_1}}\frac{\partial x^{\nu_2}}{\partial\bar{x}^{\mu_2}}\cdots\frac{\partial x^{\nu_k}}{\partial\bar{x}^{\mu_k}}\frac{\partial}{\partial y^{J_1}}\wedge\frac{\partial}{\partial x^{\nu_2}}\wedge\cdots\wedge\frac{\partial}{\partial x^{\nu_k}}\right.\\
&+{}_{k-1}C_1\frac{\partial x^{J_1}}{\partial\bar{x}^{I_1}}\frac{\partial y^{J_2}}{\partial\bar{x}^{\mu_2}}\frac{\partial x^{\nu_3}}{\partial\bar{x}^{\mu_3}}\cdots\frac{\partial x^{\nu_k}}{\partial\bar{x}^{\mu_k}}\frac{\partial}{\partial y^{J_1}}\wedge\frac{\partial}{\partial y^{J_2}}\wedge\frac{\partial}{\partial x^{\nu_3}}\wedge\cdots\wedge\frac{\partial}{\partial x^{\nu_k}}\\
&\left.+\cdots+\frac{\partial x^{J_1}}{\partial\bar{x}^{I_1}}\frac{\partial y^{J_2}}{\partial\bar{x}^{\mu_2}}\cdots\frac{\partial y^{I_k}}{\partial\bar{x}^{\mu_k}}\frac{\partial}{\partial y^{I_1}}\wedge\frac{\partial}{\partial y^{I_2}}\wedge\cdots\wedge\frac{\partial}{\partial y^{I_k}}\right),
\end{aligned}
$$

$$\vdots$$

$$
\frac{\partial}{\partial\bar{y}^{I_1}}\wedge\cdots\wedge\frac{\partial}{\partial\bar{y}^{\mu_k}}=(k!)^k\frac{\partial x^{J_1}}{\partial\bar{x}^{I_1}}\frac{\partial}{\partial y^{J_1}}\wedge\cdots\wedge\frac{\partial x^{J_k}}{\partial\bar{x}^{I_k}}\frac{\partial}{\partial y^{J_k}}
$$

$$= (k!)^k \frac{\partial x^{J_1}}{\partial \bar{x}^{I_1}} \cdots \frac{\partial x^{J_k}}{\partial \bar{x}^{I_k}} \frac{\partial}{\partial y^{J_1}} \wedge \cdots \wedge \frac{\partial}{\partial y^{J_k}}, \tag{2.1.79}$$

and the coordinate transformations of $w \in \Lambda^k T \Lambda^k TM$ with base point on $\Lambda^k TM$ will be

$$\begin{aligned}
w &= \frac{1}{k!} \bar{y}^{\mu_1 \cdots \mu_k} \frac{\partial x^{\nu_1}}{\partial \bar{x}^{\mu_1}} \cdots \frac{\partial x^{\nu_k}}{\partial \bar{x}^{\mu_k}} \frac{\partial}{\partial x^{\nu_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\nu_k}} \\
&+ \left(\frac{1}{k!} \bar{y}^{\mu_1 \cdots \mu_k} {}_k C_1 \frac{\partial y^{J_1}}{\partial \bar{x}^{\mu_1}} \frac{\partial x^{\nu_2}}{\partial \bar{x}^{\mu_2}} \cdots \frac{\partial x^{\nu_k}}{\partial \bar{x}^{\mu_k}} + \frac{k!}{(k-1)!} \bar{w}^{I_1 \mu_2 \cdots \mu_k} \frac{\partial x^{J_1}}{\partial \bar{x}^{I_1}} \frac{\partial x^{\nu_2}}{\partial \bar{x}^{\mu_2}} \cdots \frac{\partial x^{\nu_k}}{\partial \bar{x}^{\mu_k}} \right) \\
&\qquad \times \frac{\partial}{\partial y^{J_1}} \wedge \frac{\partial}{\partial x^{\nu_2}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\nu_k}} \\
&+ \left(\frac{1}{k!} \bar{y}^{\mu_1 \cdots \mu_k} {}_k C_2 \frac{\partial y^{J_1}}{\partial \bar{x}^{\mu_1}} \frac{\partial y^{J_2}}{\partial \bar{x}^{\mu_2}} \frac{\partial x^{\nu_3}}{\partial \bar{x}^{\mu_3}} \cdots \frac{\partial x^{\nu_k}}{\partial \bar{x}^{\mu_k}} + \frac{k!}{(k-1)!} \bar{w}^{I_1 \mu_2 \cdots \mu_k} {}_{k-1} C_1 \frac{\partial x^{J_1}}{\partial \bar{x}^{I_1}} \frac{\partial y^{J_2}}{\partial \bar{x}^{\mu_2}} \frac{\partial x^{\nu_3}}{\partial \bar{x}^{\mu_3}} \cdots \frac{\partial x^{\nu_k}}{\partial \bar{x}^{\mu_k}} \right. \\
&\left. + \frac{(k!)^2}{(k-2)!2!} \bar{w}^{I_1 I_2 \mu_3 \cdots \mu_k} {}_{k-2} C_0 \frac{\partial x^{J_1}}{\partial \bar{x}^{I_1}} \frac{\partial x^{J_2}}{\partial \bar{x}^{I_2}} \frac{\partial x^{\nu_3}}{\partial \bar{x}^{\mu_3}} \cdots \frac{\partial x^{\nu_k}}{\partial \bar{x}^{\mu_k}} \right) \frac{\partial}{\partial y^{J_1}} \wedge \frac{\partial}{\partial y^{J_2}} \wedge \frac{\partial}{\partial x^{\nu_3}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\nu_k}} \\
&+ \cdots + \left(\frac{1}{k!} \bar{y}^{\mu_1 \cdots \mu_k} {}_k C_l \frac{\partial y^{J_1}}{\partial \bar{x}^{\mu_1}} \cdots \frac{\partial y^{J_l}}{\partial \bar{x}^{\mu_l}} \frac{\partial x^{\nu_{l+1}}}{\partial \bar{x}^{\mu_{l+1}}} \cdots \frac{\partial x^{\nu_k}}{\partial \bar{x}^{\mu_k}} \right. \\
&+ \frac{k!}{(k-1)!} \bar{w}^{I_1 \mu_2 \cdots \mu_k} {}_{k-1} C_{l-1} \frac{\partial x^{J_1}}{\partial \bar{x}^{I_1}} \frac{\partial y^{J_2}}{\partial \bar{x}^{\mu_2}} \cdots \frac{\partial y^{J_l}}{\partial \bar{x}^{\mu_l}} \frac{\partial x^{\nu_{l+1}}}{\partial \bar{x}^{\mu_{l+1}}} \cdots \frac{\partial x^{\nu_k}}{\partial \bar{x}^{\mu_k}} \\
&+ \frac{(k!)^2}{(k-2)!2!} \bar{w}^{I_1 I_2 \mu_3 \cdots \mu_k} {}_{k-2} C_{l-2} \frac{\partial x^{J_1}}{\partial \bar{x}^{I_1}} \frac{\partial x^{J_2}}{\partial \bar{x}^{I_2}} \frac{\partial y^{J_3}}{\partial \bar{x}^{\mu_3}} \cdots \frac{\partial y^{J_l}}{\partial \bar{x}^{\mu_l}} \frac{\partial x^{\nu_{l+1}}}{\partial \bar{x}^{\mu_{l+1}}} \cdots \frac{\partial x^{\nu_k}}{\partial \bar{x}^{\mu_k}} \\
&\left. + \cdots + \frac{(k!)^l}{(k-l)!l!} \bar{w}^{I_1 \cdots I_l \mu_{l+1} \cdots \mu_k} {}_{k-l} C_0 \frac{\partial x^{J_1}}{\partial \bar{x}^{I_1}} \cdots \frac{\partial x^{J_l}}{\partial \bar{x}^{I_l}} \frac{\partial x^{\nu_{l+1}}}{\partial \bar{x}^{\mu_{l+1}}} \cdots \frac{\partial x^{\nu_k}}{\partial \bar{x}^{\mu_k}} \right) \\
&\qquad \times \frac{\partial}{\partial y^{J_1}} \wedge \cdots \wedge \frac{\partial}{\partial y^{J_l}} \wedge \frac{\partial}{\partial x^{\nu_{l+1}}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\nu_k}} \\
&+ \cdots + \left(\frac{1}{k!} \bar{y}^{\mu_1 \cdots \mu_k} \frac{\partial y^{J_1}}{\partial \bar{x}^{\mu_1}} \frac{\partial y^{J_2}}{\partial \bar{x}^{\mu_2}} \cdots \frac{\partial y^{J_k}}{\partial \bar{x}^{\mu_k}} + \frac{k!}{(k-1)!} \bar{w}^{I_1 \mu_2 \cdots \mu_k} \frac{\partial x^{J_1}}{\partial \bar{x}^{I_1}} \frac{\partial y^{J_2}}{\partial \bar{x}^{\mu_2}} \cdots \frac{\partial y^{I_k}}{\partial \bar{x}^{\mu_k}} \right. \\
&+ \cdots + \frac{(k!)^l}{(k-l)!l!} \bar{w}^{I_1 \cdots I_l \mu_{l+1} \cdots \mu_k} \frac{\partial x^{J_1}}{\partial \bar{x}^{I_1}} \cdots \frac{\partial x^{J_l}}{\partial \bar{x}^{I_l}} \frac{\partial y^{J_{l+1}}}{\partial \bar{x}^{\mu_{l+1}}} \cdots \frac{\partial y^{I_k}}{\partial \bar{x}^{\mu_k}} \\
&\qquad \left. + \cdots + \frac{(k!)^k}{k!} \bar{w}^{I_1 I_2 \cdots I_k} \frac{\partial x^{J_1}}{\partial \bar{x}^{I_1}} \cdots \frac{\partial x^{J_k}}{\partial \bar{x}^{I_k}} \right) \times \frac{\partial}{\partial y^{J_1}} \wedge \cdots \wedge \frac{\partial}{\partial y^{J_k}}.
\end{aligned} \tag{2.1.80}$$

To reduce the space, we made an abbreviation,

$$\frac{\partial x^{J_1}}{\partial \bar{x}^{I_1}} := \frac{\partial x^{j_1}}{\partial \bar{x}^{i_1}} \cdots \frac{\partial x^{j_k}}{\partial \bar{x}^{i_k}}. \tag{2.1.81}$$

Now we will get the transformation rule for the coordinates as,

$$
\begin{aligned}
&\bar{y}^{\nu_1\cdots\nu_k} = \frac{\partial \bar{x}^{\nu_1}}{\partial x^{\mu_1}}\cdots\frac{\partial \bar{x}^{\nu_k}}{\partial x^{\mu_k}} y^{\mu_1\cdots\mu_k},\\
&\bar{z}^{J_1\nu_2\cdots\nu_k} = k! z^{I_1\mu_2\cdots\mu_k}\frac{\partial \bar{x}^{J_1}}{\partial x^{I_1}}\frac{\partial \bar{x}^{\nu_2}}{\partial x^{\mu_2}}\cdots\frac{\partial \bar{x}^{\nu_k}}{\partial x^{\mu_k}} + y^{\mu_1\cdots\mu_k}\frac{\partial \bar{y}^{J_1}}{\partial x^{\mu_1}}\frac{\partial \bar{x}^{\nu_2}}{\partial x^{\mu_2}}\cdots\frac{\partial \bar{x}^{\nu_k}}{\partial x^{\mu_k}},\\
&\bar{z}^{J_1J_2\nu_3\cdots\nu_k} = (k!)^2 z^{I_1I_2\mu_3\cdots\mu_k}\frac{\partial \bar{x}^{J_1}}{\partial x^{I_1}}\frac{\partial \bar{x}^{J_2}}{\partial x^{I_2}}\frac{\partial \bar{x}^{\nu_3}}{\partial x^{\mu_3}}\cdots\frac{\partial \bar{x}^{\nu_k}}{\partial x^{\mu_k}}\\
&\quad + 2!k! z^{I_1\mu_2\cdots\mu_k}\frac{\partial \bar{x}^{J_1}}{\partial x^{I_1}}\frac{\partial \bar{y}^{J_2}}{\partial x^{\mu_2}}\frac{\partial \bar{x}^{\nu_3}}{\partial x^{\mu_3}}\cdots\frac{\partial \bar{x}^{\nu_k}}{\partial x^{\mu_k}} + y^{\mu_1\cdots\mu_k}\frac{\partial \bar{y}^{J_1}}{\partial x^{\mu_1}}\frac{\partial \bar{y}^{J_2}}{\partial x^{\mu_2}}\frac{\partial \bar{x}^{\nu_3}}{\partial x^{\mu_3}}\cdots\frac{\partial \bar{x}^{\nu_k}}{\partial x^{\mu_k}},\\
&\vdots\\
&\bar{z}^{J_1\cdots J_l\nu_{l+1}\cdots\nu_k} = (k!)^l z^{I_1\cdots I_l\mu_{l+1}\cdots\mu_k}\frac{\partial \bar{x}^{J_1}}{\partial x^{I_1}}\cdots\frac{\partial \bar{x}^{J_l}}{\partial x^{I_l}}\frac{\partial \bar{x}^{\nu_{l+1}}}{\partial x^{\mu_{l+1}}}\cdots\frac{\partial \bar{x}^{\nu_k}}{\partial x^{\mu_k}}\\
&\quad + {}_lC_{l-1}(k!)^{l-1} z^{I_1\cdots I_{l-1}\mu_l\cdots\mu_k}\frac{\partial \bar{x}^{J_1}}{\partial x^{I_1}}\cdots\frac{\partial \bar{x}^{J_{l-1}}}{\partial x^{I_{l-1}}}\frac{\partial \bar{y}^{J_l}}{\partial x^{\mu_l}}\frac{\partial \bar{x}^{\nu_{l+1}}}{\partial x^{\mu_{l+1}}}\cdots\frac{\partial \bar{x}^{\nu_k}}{\partial x^{\mu_k}}\\
&\quad + {}_lC_{l-2}(k!)^{l-2} z^{I_1\cdots I_{l-2}\mu_{l-1}\cdots\mu_k}\frac{\partial \bar{x}^{J_1}}{\partial x^{I_1}}\cdots\frac{\partial \bar{x}^{J_{l-2}}}{\partial x^{I_{l-2}}}\frac{\partial \bar{y}^{J_{l-1}}}{\partial x^{\mu_{l-1}}}\frac{\partial \bar{y}^{J_l}}{\partial x^{\mu_l}}\frac{\partial \bar{x}^{\nu_{l+1}}}{\partial x^{\mu_{l+1}}}\cdots\frac{\partial \bar{x}^{\nu_k}}{\partial x^{\mu_k}}\\
&\quad + \cdots + {}_lC_1(k!)\, z^{I_1\mu_2\cdots\mu_k}\frac{\partial \bar{x}^{J_1}}{\partial x^{I_1}}\frac{\partial \bar{y}^{J_2}}{\partial x^{\mu_2}}\cdots\frac{\partial \bar{y}^{J_l}}{\partial x^{\mu_l}}\frac{\partial \bar{x}^{\nu_{l+1}}}{\partial x^{\mu_{l+1}}}\cdots\frac{\partial \bar{x}^{\nu_k}}{\partial x^{\mu_k}}\\
&\quad + y^{\mu_1\cdots\mu_k}\frac{\partial \bar{y}^{J_1}}{\partial x^{\mu_1}}\cdots\frac{\partial \bar{y}^{J_l}}{\partial x^{\mu_l}}\frac{\partial \bar{x}^{\nu_{l+1}}}{\partial x^{\mu_{l+1}}}\cdots\frac{\partial \bar{x}^{\nu_k}}{\partial x^{\mu_k}},\\
&\vdots\\
&\bar{z}^{J_1\cdots J_k} = (k!)^k z^{I_1\cdots I_k}\frac{\partial \bar{x}^{J_1}}{\partial x^{I_1}}\cdots\frac{\partial \bar{x}^{J_k}}{\partial x^{I_k}} + (k!)^{k-1}{}_kC_{k-1} z^{I_1\cdots I_{k-1}\mu_k}\frac{\partial \bar{x}^{J_1}}{\partial x^{I_1}}\cdots\frac{\partial \bar{x}^{J_{k-1}}}{\partial x^{I_{k-1}}}\frac{\partial \bar{y}^{J_k}}{\partial x^{\mu_k}}\\
&\quad + \cdots + (k!)^{k-(k-l)}{}_kC_l z^{I_1\cdots I_l\mu_{l+1}\cdots\mu_k}\frac{\partial \bar{x}^{J_1}}{\partial x^{I_1}}\cdots\frac{\partial \bar{x}^{J_l}}{\partial x^{I_l}}\frac{\partial \bar{y}^{J_{l+1}}}{\partial x^{\mu_{l+1}}}\cdots\frac{\partial \bar{y}^{I_k}}{\partial x^{\mu_k}}\\
&\quad + \cdots + k!{}_kC_1 z^{I_1\mu_2\cdots\mu_k}\frac{\partial \bar{x}^{J_1}}{\partial x^{I_1}}\frac{\partial \bar{y}^{J_2}}{\partial x^{\mu_2}}\cdots\frac{\partial \bar{y}^{J_k}}{\partial x^{\mu_k}} + y^{\mu_1\cdots\mu_k}\frac{\partial \bar{y}^{J_1}}{\partial x^{\mu_1}}\cdots\frac{\partial \bar{y}^{J_k}}{\partial x^{\mu_k}}.
\end{aligned}
\tag{2.1.82}
$$

These transformations are smooth, and the charts form a smooth atlas on $(\Lambda^k T)^2 M$. The natural embedding

$$
\begin{aligned}
&(x^\mu, y^{\mu_1\cdots\mu_k}, z^{I_1\mu_2\cdots\mu_k}, z^{I_1I_2\mu_3\cdots\mu_k}, \cdots, z^{I_1\cdots I_k})\\
&\quad\to (x^\mu, y^{\mu_1\cdots\mu_k}, y^{\mu_1\cdots\mu_k}, z^{I_1\mu_2\cdots\mu_k}, z^{I_1I_2\mu_3\cdots\mu_k}, \cdots, z^{I_1\cdots I_k})
\end{aligned}
\tag{2.1.83}
$$

shows that $(\Lambda^k T)^2 M$ is a submanifold of $\Lambda^k T \Lambda^k T M$. To save the space, we will frequently use the abbreviation such as $\bar{y}^I = k!\dfrac{\partial \bar{x}^I}{\partial x^J} y^J$, instead of $\bar{y}^{\nu_1\cdots\nu_k} = \dfrac{\partial \bar{x}^{\nu_1}}{\partial x^{\mu_1}}\cdots\dfrac{\partial \bar{x}^{\nu_k}}{\partial x^{\mu_k}} y^{\mu_1\cdots\mu_k}$.

In case of $k = 2$,

$$
\begin{aligned}
w &= \frac{1}{2}y^{\mu_1\mu_2}\frac{\partial}{\partial x^{\mu_1}}\wedge\frac{\partial}{\partial x^{\mu_2}} + w^{I_1\mu_2}\frac{\partial}{\partial y^{I_1}}\wedge\frac{\partial}{\partial x^{\mu_2}} + \frac{1}{2}w^{I_1I_2}\frac{\partial}{\partial y^{I_1}}\wedge\frac{\partial}{\partial y^{I_2}} \\
&= \frac{1}{2}\bar{y}^{\mu_1\mu_2}\frac{\partial x^{\nu_1}}{\partial \bar{x}^{\mu_1}}\frac{\partial x^{\nu_2}}{\partial \bar{x}^{\mu_2}}\frac{\partial}{\partial x^{\nu_1}}\wedge\frac{\partial}{\partial x^{\nu_2}} + \left(\bar{y}^{\mu_1\mu_2}\frac{\partial y^{J_1}}{\partial \bar{x}^{\mu_1}}\frac{\partial x^{\nu_2}}{\partial \bar{x}^{\mu_2}} + 2\bar{w}^{I_1\mu_2}\frac{\partial x^{J_1}}{\partial \bar{x}^{I_1}}\frac{\partial x^{\nu_2}}{\partial \bar{x}^{\mu_2}}\right)\frac{\partial}{\partial y^{J_1}}\wedge\frac{\partial}{\partial x^{\nu_2}} \\
&\quad + \frac{1}{2}\left(\bar{y}^{\mu_1\mu_2}\frac{\partial y^{J_1}}{\partial \bar{x}^{\mu_1}}\frac{\partial y^{J_2}}{\partial \bar{x}^{\mu_2}} + 4\bar{w}^{I_1\mu_2}\frac{\partial x^{J_1}}{\partial \bar{x}^{I_1}}\frac{\partial y^{J_2}}{\partial \bar{x}^{\mu_2}} + 4\bar{w}^{I_1I_2}\frac{\partial x^{J_1}}{\partial \bar{x}^{I_1}}\frac{\partial x^{J_2}}{\partial \bar{x}^{I_2}}\right)\frac{\partial}{\partial y^{J_1}}\wedge\frac{\partial}{\partial y^{J_2}} \\
\bar{y}^{\nu_1\nu_2} &= \frac{\partial \bar{x}^{\nu_1}}{\partial x^{\mu_1}}\frac{\partial \bar{x}^{\nu_2}}{\partial x^{\mu_2}}y^{\mu_1\mu_2}, \\
\bar{z}^{J_1\nu_2} &= 2z^{I_1\mu_2}\frac{\partial \bar{x}^{J_1}}{\partial x^{I_1}}\frac{\partial \bar{x}^{\nu_2}}{\partial x^{\mu_2}} + y^{\mu_1\mu_2}\frac{\partial \bar{y}^{J_1}}{\partial x^{\mu_1}}\frac{\partial \bar{x}^{\nu_2}}{\partial x^{\mu_2}}, \\
\bar{z}^{J_1J_2} &= 4z^{I_1I_2}\frac{\partial \bar{x}^{J_1}}{\partial x^{I_1}}\frac{\partial \bar{x}^{J_2}}{\partial x^{I_2}} + 4z^{I_1\mu_2}\frac{\partial \bar{x}^{J_1}}{\partial x^{I_1}}\frac{\partial \bar{y}^{J_2}}{\partial x^{\mu_2}} + y^{\mu_1\mu_2}\frac{\partial \bar{y}^{J_1}}{\partial x^{\mu_1}}\frac{\partial \bar{y}^{J_2}}{\partial x^{\mu_2}}.
\end{aligned}
\tag{2.1.84}
$$

Now $\Lambda^k\tau_M^{2,1}$ is a surjective submersion by definition, so it remains to check the local trivialisation. The local trivialisation of $((\Lambda^kT)^2M, \Lambda^k\tau_M^{2,1}, \Lambda^kTM)$ around any point $p \in \Lambda^kTM$ is given by $(V_p, \mathbb{R}^n, t_p)$, $t_p : (\Lambda^k\tau_M^{2,1})^{-1}(V_p) \to V_p \times \mathbb{R}^l$, $l = {}_nC_k$, $p \in V_p$, where V_p is an open set of Λ^kTM, which in chart expression for any $\xi \in (\Lambda^k\tau_M^{2,1})^{-1}(V_p) \subset (\Lambda^kT)^2M$ is

$$
t_p(\xi) = (\Lambda^k\tau_M^{2,1}(\xi), z^{I_1\mu_2\cdots\mu_k}(\xi), z^{I_1I_2\mu_3\cdots\mu_k}(\xi), \cdots, z^{I_1\cdots I_k}(\xi)). \tag{2.1.85}
$$

Therefore, $((\Lambda^kT)^2M, \Lambda^k\tau_M^{2,1}, \Lambda^kTM)$ is indeed a bundle.

Definition 2.37. Second order k-multivector bundle
Similarly as in the case of mechanics ($k = 1$), the triple $((\Lambda^kT)^2M, \Lambda^k\tau_M^{2,0}, M)$ with $\Lambda^k\tau_M^{2.0} = \Lambda^k\tau_M\circ\Lambda^k\tau_M^{2,1}$, $\Lambda^k\tau_M^{2,1} := \Lambda^k\tau_{\Lambda^kTM}\big|_{(\Lambda^kT)^2M}$ is also a bundle with the trivialisation $(U_p, \mathbb{R}^{N-n}, t_p)$, $t_p : (\Lambda^k\tau_M^{2,0})^{-1}(U_p) \to U_p \times \mathbb{R}^N$, where U_p is an open set of M, and

$$
N = {}_{(n+l)}C_k = l + {}_nC_{k-1} \times {}_lC_1 + {}_nC_{k-2} \times {}_lC_2 + \cdots + {}_lC_k, \tag{2.1.86}
$$

where $l = {}_nC_k$, around any $p \in U_p \subset M$. We call this a *Second order k-multivector bundle over M* or simply, *Second order k-multivector bundle*.

In the second order field theory, the dynamical variables are the section of the second order k-multivector bundle, and $(\Lambda^kT)^2M$ will be the space where the Lagrangian should be defined.

2.1.5 Higher order k-multivector bundle

Here we will briefly introduce the higher order k-multivector bundles. The construction is the same as the Section 2.1.2, and the r-th order k-multivector bundle over $(\Lambda^k T)^{r-1}M$; namely, $((\Lambda^k T)^r M, \Lambda^k \tau_M^{r,r-1}, (\Lambda^k T)^{r-1}M)$ can be constructed by induction. The projection map is defined by

$$\Lambda^k \tau_M^{r,r-1} := \Lambda^k \tau_{(\Lambda^k T)^{r-1}M}\Big|_{(\Lambda^k T)^r M}. \tag{2.1.87}$$

Consider the bundle morphism $((\Lambda^k T)\Lambda^k \tau_M^{r-1,r-2}, \Lambda^k \tau_M^{r-1,r-2})$ from $(\Lambda^k T((\Lambda^k T)^{r-1}M),$ $\Lambda^k \tau_{(\Lambda^k T)^{r-1}M}, (\Lambda^k T)^{r-1}M)$ to $(\Lambda^k T((\Lambda^k T)^{r-2}M), \Lambda^k \tau_{(\Lambda^k T)^{r-2}M}, (\Lambda^k T)^{r-2}M)$. Then we will define the total space $(\Lambda^k T)^r M$ by

$$(\Lambda^k T)^r M := \{u \in \Lambda^k T((\Lambda^k T)^{r-1}M) \,|\, (\Lambda^k T)\Lambda^k \tau_M^{r-1,r-2}(u) = \iota_{r-1} \circ \Lambda^k \tau_{(\Lambda^k T)^{r-1}M}(u)\}, \tag{2.1.88}$$

where $\iota_{r-1} : (\Lambda^k T)^{r-1}M \to \Lambda^k T((\Lambda^k T)^{r-2}M)$ is the inclusion map. $((\Lambda^k T)^r M, \Lambda^k \tau_M^{r,r-1},$ $(\Lambda^k T)^{r-1}M)$ is a sub-bundle of $(\Lambda^k T((\Lambda^k T)^{r-1}M), \Lambda^k \tau_{(\Lambda^k T)^{r-1}M}, (\Lambda^k T)^{r-1}M)$.

2.2 Integration of differential forms

For the calculus of variations on the Kawaguchi manifold, we need to integrate a Lagrangian which is a k-form on a k-dimensional submanifold. In this section, we will describe how to implement such integrals, and begin by introducing the method to define the integration of a n-form on a n-dimensional compact oriented manifold M. The integration of k-form $(k < n)$ on a k-dimensional compact oriented submanifold would then be given, first in the case where there is an immersion map (which is called parameterisation) from the parameter space to the total space M, and second for the case where there is no such map. We will begin with basic definitions and theorems.

Definition 2.38. Orientation of two charts
Let $(U, \varphi), \varphi = (x^i)$ and $(\bar{U}, \bar{\varphi}), \bar{\varphi} = (y^j)$ be two charts on M such that $U \cap \bar{U} \neq \emptyset$. We say that (U, φ) *and* $(\bar{U}, \bar{\varphi})$ *has the same orientation*, when

$$\det\left(\frac{\partial y^i}{\partial x^j}\right) > 0. \tag{2.2.1}$$

Definition 2.39. Orientable manifold
We say that a manifold M *is orientable*, if there exist an atlas $\mathcal{A} = \{(U_\iota, \varphi_\iota)\}_{\iota \in I}$ such that for any pair of intersecting charts from $\mathcal{A}$, has the same orientation. The manifold which is given such an atlas is called an *oriented manifold*, and it has the *orientation* associated to this atlas.

Suppose we have an oriented manifold with an atlas $\mathcal{A}$, then by definition, this manifold has a specific orientation associated to $\mathcal{A}$. Transfer one axis in opposite direction for every chart in $\mathcal{A}$, and denote this new atlas by $\tilde{\mathcal{A}}$. Every chart in $\tilde{\mathcal{A}}$ still has the same orientation, but it is opposite to the orientation of $\mathcal{A}$. In this way, orientable manifold can always have two orientations.

Definition 2.40. Orientation preserving (reversing) map
Let $M$and N be two smooth n-dimensional oriented manifolds, $\alpha : M \to N$ a diffeomorphism. In particular, the tangent mapping $T_p\alpha : T_pM \to T_{\alpha(p)}N$ has constant rank n for every $p \in M$. Choose a chart (U, φ) on M and a chart (V, ψ) on N such that $\alpha(U) \subset V$. We define a number ε_α, equal to 1 or -1, by the chart expressions

$$|\det D\psi\alpha\varphi^{-1}| = \varepsilon_\alpha \cdot \det D\psi\alpha\varphi^{-1}. \tag{2.2.2}$$

This number is independent of the choice of charts that has the same orientation. To see this, let $(\bar{U}, \bar{\varphi})$ be a chart on M which has the same orientation as (U, φ), and $(\bar{V}, \bar{\psi})$ a chart on N which has the same orientation as (V, ψ). Then,

$$\begin{aligned}
\varepsilon_\alpha &= \operatorname{sgn}\det D\psi\alpha\varphi^{-1} \\
&= \operatorname{sgn}(\det D\psi\bar{\psi}^{-1} \cdot \det D\bar{\psi}\alpha\bar{\varphi}^{-1} \cdot \det D\bar{\varphi}\varphi^{-1}) \\
&= \operatorname{sgn}(\det D\psi\bar{\psi}^{-1} \cdot \det D\bar{\varphi}\varphi^{-1}) \cdot \operatorname{sgn}\det D\bar{\psi}\alpha\bar{\varphi}^{-1} \\
&= \operatorname{sgn}\det D\bar{\psi}\alpha\bar{\varphi}^{-1}.
\end{aligned} \tag{2.2.3}$$

Thus, the number ε_α is independent of the charts with the same orientation. We say that α is *orientation preserving map* (resp. *orientation-reversing map*), if $\varepsilon_\alpha = 1$ ($\varepsilon_\alpha = -1$).

Example 2.41. The special case is when $M = U$, $N = \mathbb{R}^n$ and $\alpha = \varphi$, where $\varphi = (x^\mu)$ are the coordinate functions of the domain U. We assume that we always choose the map φ as to be orientation preserving with respect to the canonical coordinates on $\mathbb{R}^n$.

Definition 2.42. support of k-form$(k \leqslant n)$ α
The *support* of α is the closure of $\{p \in M | \alpha_p \neq 0\}$. We denote it by $\operatorname{supp}(\alpha)$

Definition 2.43. σ-compact
We say that the topological space X is σ-compact when X is a union of countable number of compact subsets, X_i, $(i = 1, 2, 3, \cdots)$. Namely, $X = \bigcup_{i=1}^{\infty} X_i$.

Theorem 2.44. Partition of Unity
Let M be a σ-compact C^∞-manifold, and $\{U^\iota\}_{\iota \in I}$ an arbitrary open covering of M. $\iota \in I$ is an index taken from countable index set I. Then, there exists countable number of C^∞-functions $h_j : M \to \mathbb{R},\ j = 1, 2, 3, \cdots$, such that satisfies the following conditions:
1. $0 \leqslant h_j \leqslant 1$, for $\forall j = 1, 2, 3, \cdots$
2. The family $\{\mathrm{supp}(h_j)\}_{j \in I}$ is a locally finite open covering of M and also a refinement of $\{U^\iota\}_{\iota \in I}$
3. $\sum_{j \in I} h_j = 1$.

The sum in 3. is over finite functions, since $\{\mathrm{supp}(h_j)\}_{j \in I}$ is a locally finite set. The proof could be found in the standard textbooks, e.g., Spivak [12].

Definition 2.45. The family $\{h_i\}_{i \in I}$ satisfying the conditions 1., 2., 3. in Theorem 2.44 is called a *partition of unity subordinate to* $\{U^\iota\}_{\iota \in I}$.

Definition 2.46. Rectangular region
Let V be an open subset of U. We call V *a rectangular region*, when there exists a chart (U, φ), such that by appropriate shrinking of the domain, induces a chart $(V, \tilde{\varphi})$, $\tilde{\varphi}(V) = \{p \in \mathbb{R}^n | \tilde{\varphi}^i(p) \in (-a^i, a^i),\ a^i \in \mathbb{R},\ i = 1, \cdots, n\}$. We call $(V, \tilde{\varphi})$ a *rectangle chart*.

Now we will introduce the integral of an n-form on a n dimensional manifold M.

Definition 2.47. Integration of a top form
Let ω be a n-form on M, and $(U, \varphi), \varphi = (x^1, \cdots, x^n)$ a chart on M. First, suppose that $\mathrm{supp}(\omega) \subset V$, where V is a rectangular region of U. Let the local coordinate expression of ω be $\omega = f dx^1 \wedge \cdots \wedge dx^n$, with $f \in C^\infty(M)$. The integration of ω is defined by,

$$\int_M \omega = \int_V \omega = \int_{\tilde{\varphi}(V)} \tilde{\varphi}^*(f dx^1 \wedge \cdots \wedge dx^n) := \int_{-a_1}^{a_1} \cdots \int_{-a_n}^{a_n} f(x^1, \cdots, x^n) dx^1 \cdots dx^n, \tag{2.2.4}$$

where for simplicity, we denoted the pull back of the coordinates on U to $\mathbb{R}^n$ also as (x^μ). The right hand side is the standard multiple integral.

Lemma 2.48. The Right hand side of (2.2.4) does not depend on the choice of the rectangular region which contains the support of ω, provided that it has the same orientation.

Proof. Suppose there is another chart $(\bar{U}, \bar{\varphi}), \bar{\varphi} = (y^1, \cdots, y^n)$, such that $U \cap \bar{U} \neq \emptyset$, and $V \subset U \cap \bar{U}$. By definition, the rectangle chart of V on the first chart is

$$\tilde{\varphi}(V) = \{p \in \mathbb{R}^n | \tilde{\varphi}^i(p) \in (-a^i, a^i),\ a^i \in \mathbb{R},\ i = 1, \cdots, n\}.$$

The second chart, $\bar{\varphi}(V)$ is not a rectangle chart in general, however in $\mathbb{R}^n$ we can always choose a open rectangle such that contains $\bar{\varphi}(V)$. For instance, choose

$$L := \{(-b^1, b^1) \times \cdots \times (-b^n, b^n) | b^i = \sup\{\left|\bar{\varphi}^i(p)\right|, p \in V\} \in \mathbb{R}\},$$

then since $\bar{\varphi}$ is a diffeomorphism, we can always choose a $\phi \in \mathit{Diff}(\mathbb{R}^n)$ such that $\phi \circ \bar{\varphi}(V) = L$. Let $\varphi' = \phi \circ \bar{\varphi}$, and the local coordinate expression of ω in this chart be $\omega = g dy^1 \wedge \cdots \wedge dy^n$, with $g \in C^\infty(M)$. Since

$$\begin{aligned}
dy^1 \wedge \cdots \wedge dy^n &= \frac{1}{k!} \varepsilon_{i_1 \cdots i_n} dy^{i_1} \wedge \cdots \wedge dy^{i_n} = \frac{1}{k!} \varepsilon_{i_1 \cdots i_n} \frac{\partial y^{i_1}}{\partial x^{j_1}} \cdots \frac{\partial y^{i_n}}{\partial x^{j_n}} dx^{j_1} \wedge \cdots \wedge dx^{j_n} \\
&= \frac{1}{k!} \varepsilon_{i_1 \cdots i_n} \varepsilon^{j_1 \cdots j_n} \frac{\partial y^{i_1}}{\partial x^{j_1}} \cdots \frac{\partial y^{i_n}}{\partial x^{j_n}} dx^1 \wedge \cdots \wedge dx^n = \varepsilon^{j_1 \cdots j_n} \frac{\partial y^1}{\partial x^{j_1}} \cdots \frac{\partial y^n}{\partial x^{j_n}} dx^1 \wedge \cdots \wedge dx^n \\
&= \det\left(\frac{\partial y^i}{\partial x^j}\right) dx^1 \wedge \cdots \wedge dx^n,
\end{aligned} \tag{2.2.5}$$

we will have

$$f = g \det\left(\frac{\partial y^i}{\partial x^j}\right). \tag{2.2.6}$$

We suppose our manifold M is orientable, then it is always possible to choose two charts such that $\det\left(\frac{\partial y^i}{\partial x^j}\right) > 0$ on $U \cap \bar{U}$. Then,

$$\begin{aligned}
\int_V \omega &= \int_{\tilde{\varphi}(V)} \tilde{\varphi}^*(f dx^1 \wedge \cdots \wedge dx^n) = \int_{-a_1}^{a_1} \cdots \int_{-a_n}^{a_n} f(x^1, \cdots, x^n) dx^1 \cdots dx^n \\
&= \int_{-a_1}^{a_1} \cdots \int_{-a_n}^{a_n} g(y^1, \cdots, y^n) \det\left(\frac{\partial y^i}{\partial x^j}\right) dx^1 \cdots dx^n \\
&= \int_{-a_1}^{a_1} \cdots \int_{-a_n}^{a_n} g(y^1, \cdots, y^n) \left|\det\left(\frac{\partial y^i}{\partial x^j}\right)\right| dx^1 \cdots dx^n \\
&= \int_{-b_1}^{b_1} \cdots \int_{-b_n}^{b_n} g(y^1, \cdots, y^n) dy^1 \cdots dy^n.
\end{aligned} \tag{2.2.7}$$

Accordingly, the value does not depend on the choice of charts. □

Now, suppose M is orientable and compact. Since every point $p \in M$ is in some rectangle region, M could be covered by finite number of rectangle regions $\{V^1, \cdots, V^s\}$. Then, choose the atlas of M as $\mathcal{A} = \{(V^1, \varphi^1), \cdots, (V^s, \varphi^s)\}$, where all charts in $\mathcal{A}$ have the same orientation. Take a partition of unity $\{h_i\}_{i\in I}$, subordinate to $\{V^1, \cdots, V^s\}$. Then the integral of n-form ω on M can be calculated by

$$\int_M \omega := \int_M \sum_{i=1}^{\infty} h_i\omega = \sum_{i=1}^{\infty} \int_M h_i\omega = \sum_{i=1}^{\infty} \int_{\mathrm{supp}(h_i)} h_i\omega = \sum_{j=1}^{s} \sum_{i:\mathrm{supp}(h_i)\subset V^j} \int_{V^j} h_i\omega, \tag{2.2.8}$$

since $\mathrm{supp}(h_i\omega) \subseteq \mathrm{supp}(h_i)$, and $\{\mathrm{supp}(h_i)\}_{i\in N}$ is a refinement of $\{V^1, \cdots, V^s\}$, for any $i \in \mathbb{N}$, $\mathrm{supp}(h_i\omega) \subset V^j$ for some j, $1 \leqslant j \leqslant s$. We can calculate the *r.h.s.* of (2.2.8) by (2.2.7),

$$\begin{aligned}\sum_{j=1}^{s} \sum_{i:\mathrm{supp}(h_i)\subset V^j} \int_{V^j} h_i\omega &= \sum_{j=1}^{s} \sum_{i:\mathrm{supp}(h_i)\subset V^j} \int_{\tilde{\varphi}_j(V^j)} \tilde{\varphi}_j^*(h_i f_j dx_j^1\wedge\cdots\wedge dx_j^n) \\ &= \sum_{j=1}^{s} \sum_{i:\mathrm{supp}(h_i)\subset V^j} \int_{-a^j{}_1}^{a^j{}_1} \cdots \int_{-a^j{}_n}^{a^j{}_n} (h_i\circ\tilde{\varphi}_j)(f_j\circ\tilde{\varphi}_j)(x^1, \cdots, x^n) \det\left(\frac{\partial x_j^b}{\partial x^a}\right) dx^1\cdots dx^n,\end{aligned} \tag{2.2.9}$$

where

$$f_j dx_j^1\wedge\cdots\wedge dx_j^n \tag{2.2.10}$$

is a local expression of ω on the rectangle chart $(V^j, \tilde{\varphi}^j)$, $\tilde{\varphi}^j = (x_j^1, \cdots, x_j^n)$, induced by the chart (V^j, φ^j). This expression does not depend on the choice of partition of unity. To see this, consider just two coordinate patches V_1, V_2.

$$\int_{V^1\cup V^2} \omega := \int_{V^1\cup V^2} \sum_{i\in I} h_i\omega \tag{2.2.11}$$

For $V^1 \cap V^2 = \emptyset$,

$$\int_{V^1\cup V^2} \sum_{i=1}^{\infty} h_i\omega = \sum_{i:\mathrm{supp}(h_i)\subset V^1} \int_{V^1} h_i\omega + \sum_{i:\mathrm{supp}(h_i)\subset V^2} \int_{V^2} h_i\omega$$

$$= \sum_{i\in I}\int_{V^1} h_i\omega + \sum_{i\in I}\int_{V^2} h_i\omega = \int_{V^1}\sum_{i\in I} h_i\omega + \int_{V^2}\sum_{i\in I} h_i\omega = \int_{V^1}\omega + \int_{V^2}\omega. \tag{2.2.12}$$

For $V^1 \cap V^2 \neq \emptyset$,

$$\begin{aligned}
\int_{V^1\cup V^2}\omega &:= \int_{V^1\cup V^2}\sum_{i=1}^{\infty} h_i\omega \\
&= \sum_{i:\mathrm{supp}(h_i)\subset V^1}\int_{V^1\setminus(V^1\cap V^2)} h_i\omega + \sum_{i:\mathrm{supp}(h_i)\subset V^2}\int_{V^2\setminus(V^1\cap V^2)} h_i\omega + \sum_{i\in I} h_i\int_{V^1\cap V^2}\omega \\
&= \sum_{i\in I}\int_{V^1\setminus(V^1\cap V^2)} h_i\omega + \sum_{i\in I}\int_{V^2\setminus(V^1\cap V^2)} h_i\omega + \sum_{i\in I} h_i\int_{V^1\cap V^2}\omega \\
&= \int_{V^1\setminus(V^1\cap V^2)}\sum_{i\in I} h_i\omega + \int_{V^2\setminus(V^1\cap V^2)}\sum_{i\in I} h_i\omega + \int_{(V^1\cap V^2)}\sum_{i\in I} h_i\omega \\
&= \int_{V^1\setminus(V^1\cap V^2)}\omega + \int_{V^2\setminus(V^1\cap V^2)}\omega + \int_{(V^1\cap V^2)}\omega. \qquad (2.2.13)
\end{aligned}$$

The extension to the arbitrary number of covers is apparent.

Now we introduce two ways of defining an integration of a k-form on a k-dimensional compact subset S of M. The first is when S is given by an inclusion (injective immersion) of k-dimensional manifold P into M, namely S is an immersed submanifold of M, and the second is when S is an embedded submanifold of M. Though in the further discussion we always consider the case when we have the inclusion map, we will also introduce the definition for the second case as well.

Definition 2.49. Integration of k-form on n-dimensional manifold
Let S be a immersed submanifold of M, given by $S = \iota(P) \subset M$, where P is a compact k-dimensional manifold and ι an inclusion map. Suppose we have a k-form ω on M. The *integration of k-form on M* is given by,

$$\int_{\iota(P)}\omega = \int_P \iota^*\omega. \tag{2.2.14}$$

Then since the *r.h.s.* is an integration of a k-form over a k-dimensional manifold, the previous definition could be applied to calculate the integral.

S will be diffeomorphic to P, and will inherit the topology of P by ι. In general this is not the same as the subset topology. In the later chapters, we will consider especially when P a closed k-rectangle. Then P is called a *parameter space* and the map ι is called a *parameterisation*.

Now, suppose we have a submanifold S of M, we can still define the integration of k-form on S in terms of pieces and adapted charts of a submanifold.

Definition 2.50. Submanifold and adapted charts
Let M be a n-dimensional C^∞-manifold and choose a non-empty subset $S \subset M$. S is said to be a k-dimensional submanifold of M, if to every point $p_0 \in S$ there exists a chart $(U, \varphi), \varphi = (x^\mu)$ on M at p_0 such that the set $S \cap U$ is given by the equation,

$$x^{k+1}(p) = 0, x^{k+2}(p) = 0, \cdots, x^n(p) = 0. \tag{2.2.15}$$

for $\forall p \in S \cap U$. Coordinate system $(U, \varphi), \varphi = (x^\mu)$ with the above properties is said to be *adapted to the submanifold S at* p_0.

Definition 2.51. Half space
Let $t^1, \cdots, t^n$ be the canonical coordinates of $\mathbb{R}^n$, and $\mathbb{R}^n_- = \{y \in \mathbb{R}^n | t^1(y) \geqslant 0\}$ and $\partial\mathbb{R}^n_- = \{y \in \mathbb{R}^n_- | t^n(y) = 0\}$. The subset $\mathbb{R}^n_-$ of $\mathbb{R}^n$, considered with the subspace topology, is called the *half space* of$\mathbb{R}^n$, and $\partial\mathbb{R}^n_-$ considered as the topological subspace of $\mathbb{R}^n_-$, is canonically isomorphic with $\mathbb{R}^{n-1}$, and is called the *boundary* of $\mathbb{R}^n_-$.

Definition 2.52. Open sets in $\mathbb{R}^n_-$
Let U be a subset of $\mathbb{R}^n_-$. We say that U is open in $\mathbb{R}^n_-$ when $U = \Omega \cap \mathbb{R}^n_-$, where Ω is some open subset of $\mathbb{R}^n$.

Let M be a n-dimensional C^∞-differentiable manifold, and Ω be a nonempty compact subset of M. Let $p_0 \in \Omega$ be a fixed point, and $(U, \varphi), \varphi = (x^\mu)$ a chart on M such that $p_0 \in U$. We say that the chart (U, φ) is adapted to Ω at p_0, if $\varphi(\Omega \cap U)$ is open in $\mathbb{R}^n_-$. If the chart (U, φ) is adapted to Ω at p_0, it is adapted at every point of $p \in \Omega \cap U$, and we say a chart (U, φ) is adapted to Ω. Clearly, since Ω is supposed to be compact, there exist finitely many points $p_1, p_2, \cdots, p_N$ of Ω and adapted charts $(U_1, \varphi_1), \cdots, (U_N, \varphi_N)$, such that $p_1 \in U_1, p_2 \in U_2, \cdots, p_N \in U_N$, and $\Omega \subset \bigcup_\iota U_\iota$.

Definition 2.53. Pieces of a manifold
$\Omega \subset M$ is called a *piece* of M, if it is compact and to each point $p \in \Omega$ there exists a chart at p adapted to Ω.

Let $\operatorname{int}\Omega$ be the set of interior points of Ω, and set $\partial\Omega = \Omega \backslash \operatorname{int}\Omega$. Let $q_0 \in \partial\Omega$, and let $(U, \varphi), \varphi = (x^\mu)$ be a chart adapted to Ω at q_0. Then for every $q \in \partial\Omega \cap U$, $x^n(q) = 0$, and the set $\partial\Omega$ has on $\partial\Omega \cap U$ the equation $x^n = 0$. Thus by the definition $\partial\Omega$

is a submanifold of Ω, of dimension $\dim \partial\Omega = n-1$. The submanifold $\partial\Omega$ is called the *boundary* of Ω. It is easily seen that $\partial\Omega$ is compact.

Example 2.54. A ball $B_1 = \{(x,y) \in \mathbb{R}^2 | x^2+y^2 \leqslant 1\}$ is a piece of $\mathbb{R}^2$. B_1 is compact so it suffices to show that to each point $p \in B_1$ there exists a chart at p adapted to B_1. For the points $p \in \operatorname{int} B_1$, it is apparent we can find an open chart on $\mathbb{R}^2$ which is adapted to B_1. For the points on the boundary, $p \in \partial B_1$, let $(U,\phi), \phi = (r,\theta)$ be a chart on $\mathbb{R}^2$, with $-1 < r < R$, $c < \varphi < c+\pi$ with R some constant greater than 0, and c a constant such that $p \in U$. Then $\phi(B_1 \cap U) = (-1,0] \times (c, c+\pi) = \{(-1,R)\times(c,c+\pi)\} \cap \mathbb{R}^2_-$ is open in $\mathbb{R}^n_-$. By choosing appropriate c, we can always find such adapted chart for any $p \in \partial B_1$.

One can also show that if Ω is orientable, $\partial\Omega$ is also orientable. Let Ω be a piece of M, $p_0 \in \partial\Omega$ a point. We say that a vector $\xi \in T_{p_0}M$ is oriented outwards Ω, if there exists a chart $(U,\varphi), \varphi = (x^\mu)$ on M adapted to Ω at p_0, such that the chart expression $\xi = \xi^i \left(\frac{\partial}{\partial x^i}\right)_{p_0}$ satisfies the condition $\xi^n > 0$. We show that this definition is independent of the choice of adapted chart which has the same orientation. Let $(\bar{U},\bar{\varphi}), \bar{\varphi} = (\bar{x}^\mu)$ be the second adapted chart which has the same orientation as $(U,\varphi), \varphi = (x^\mu)$. Then on this second chart,

$$\xi = \bar{\xi}^i \left(\frac{\partial}{\partial \bar{x}^i}\right)_{p_0}, \quad \bar{\xi}^n = \left(\frac{\partial \bar{x}^n}{\partial x^i}\right)_{p_0} \xi^i, \tag{2.2.16}$$

but on $U \cap \bar{U} \cap \partial\Omega$, $\bar{x}^n(x^1, x^2, \cdots, x^{n-1}, 0) = 0$.
Hence, $\frac{\partial \bar{x}^n}{\partial x^1} = 0, \frac{\partial \bar{x}^n}{\partial x^2} = 0, \cdots, \frac{\partial \bar{x}^n}{\partial x^{n-1}} = 0$ at $\varphi(p_0)$, and we have

$$\bar{\xi}^n = \frac{\partial \bar{x}^n}{\partial x^n}\xi^n. \tag{2.2.17}$$

However, since both charts are adapted and has the same orientation, $\frac{\partial \bar{x}^n}{\partial x^n} > 0$, and therefore $\bar{\xi}^n > 0$. In particular if $(U,\varphi), \varphi = (x^i)$ is a chart adapted to Ω at $p_0 \in \partial\Omega$, then the vector $\left(\frac{\partial}{\partial x^n}\right)_{p_0}$ is oriented outwards Ω. Now suppose Ω is oriented with the orientation S. Then by the compactness of Ω, we have finitely many charts $(U_k,\varphi_k), \varphi_k = (x^i_k)$, $1 \leqslant k \leqslant N$, adapted to Ω with the same orientation, and $\Omega \subset \bigcup_k U_k$. We set $V_k = U_k \cap \partial\Omega, \quad \psi_k = (x^1_k, x^2_k, \cdots, x^{n-1}_k)$, where the coordinate functions $x^1_k, \cdots, x^{n-1}_k$ are considered to be restricted to V_k. The pairs $(V_k,\psi_k), \psi_k = (x^i_k)$, $1 \leqslant k \leqslant N$ form a smooth atlas on $\partial\Omega$. Then by definition and from $\dfrac{\partial \bar{x}^n}{\partial x^1} = 0, \dfrac{\partial \bar{x}^n}{\partial x^2} = 0, \cdots, \dfrac{\partial \bar{x}^n}{\partial x^{n-1}} = 0$, for any pair of $1 \leqslant k, l \leqslant N$, $\det D\varphi_k {\varphi_l}^{-1} = \dfrac{\partial x_k^n}{\partial x_l^n} \cdot \det D\psi_k {\psi_l}^{-1}$. But since $\det D\varphi_k {\varphi_l}^{-1} > 0$ and $\dfrac{\partial x_k^n}{\partial x_l^n} > 0$, $\det D\psi_k {\psi_l}^{-1} > 0$. Therefore by definition, $\partial\Omega$ is an orientable manifold. The orientation of $\partial\Omega$ defined by the atlas (V_k, ψ_k), $1 \leqslant k \leqslant N$ is said to be associated with the given orientation of M, defined by the charts (U_k, φ_k), $1 \leqslant k \leqslant N$.

Definition 2.55. Integration of a k-form on n-dimensional manifold (by submanifold chart)
Consider a manifold M and an orientable submanifold $S \subset M$ with $\dim S = k$. Suppose we have a k-form ρ on the neighbourhood of S. We will now introduce the integration of this form on S by means of adapted charts of a submanifold. Let $\Omega \subset S$ be a compact piece (compact submanifold with boundary) which is covered by the adapted chart (U, φ), $\varphi = (x^1, \cdots, x^k, 0, \cdots, 0)$. In this chart, the local expression of ρ is

$$\rho = \frac{1}{k!} \rho_{i_1 \cdots i_k} dx^{i_i} \wedge \cdots \wedge dx^{i_k}, \tag{2.2.18}$$

where $\rho_{i_1 \cdots i_k}$ is a function of $x^1, \cdots, x^k$. Then we can define the integral of ρ on the piece Ω by

$$\int_\Omega \rho = \int_{\varphi(\Omega)} (\varphi^{-1})^* \rho. \tag{2.2.19}$$

If we choose Ω and φ appropriately, this can be expressed as

$$\int_\Omega \rho = \int_{\varphi(\Omega)} (\varphi^{-1})^* \rho = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_k}^{b_k} \frac{1}{k!} \varepsilon^{i_1 \cdots i_k} \rho_{i_1 \cdots i_k} \circ \varphi^{-1} \, dx^1 \cdots dx^k, \tag{2.2.20}$$

since we assumed that the orientation are preserved by the mapping by φ.

To consider the integral over whole S, consider the finite family of submanifold charts on X, $\{(U_1, \varphi_1), (U_2, \varphi_2), \ldots, (U_N, \varphi_N)\}$, such that the family of open sets (in S) $\{U_1 \cap S, U_2 \cap S, \ldots, U_N \cap S\}$ covers S. Let $\{\chi_1, \chi_2, \ldots, \chi_N\}$ be a partition of unity, subordinate to this covering. Since by definition $\operatorname{supp} \chi_j \cap S$ is a closed subset of the compact set S, it must be compact; on the other hand, $\operatorname{supp} \chi_j \cap S$ is a subset of the set $U_j \cap S$, the chart neighbourhood of the induced chart by (U_j, φ_j) on S. In particular, the integral

$$\int_{\operatorname{supp} \chi_j \cap S} \chi_j \eta \tag{2.2.21}$$

is defined by formula (2.2.20) for each j. We set

$$\int_S \eta = \sum_{j=1}^{N} \int_{\operatorname{supp} \chi_j \cap S} \chi_j \eta. \tag{2.2.22}$$

The real number given by the formula (2.2.22), is called the *integral of the form η on S*.

We show that the right-hand side of (2.2.22) is independent of the choice of the family of submanifold charts $\{(U_1, \varphi_1), (U_2, \varphi_2), \ldots, (U_N, \varphi_N)\}$ and the partition of unity $\{\chi_1, \chi_2, \ldots, \chi_N\}$. Let $\{(\bar{U}_1, \bar{\varphi}_1), (\bar{U}_2, \bar{\varphi}_2), \ldots, (\bar{U}_M, \bar{\varphi}_M)\}$ be another family of charts and $\{\bar{\chi}_1, \bar{\chi}_2, \ldots, \bar{\chi}_M\}$ the corresponding partition of unity. By (2.2.22), for each j,

$$\int_{\operatorname{supp} \chi_j \cap S} \chi_j \eta = \sum_{i=1}^{M} \int_{\operatorname{supp} \bar{\chi}_i \cap \operatorname{supp} \chi_j \cap S} \bar{\chi}_i \chi_j \eta, \tag{2.2.23}$$

and similarly for each i,

$$\int_{\operatorname{supp} \bar{\chi}_i \cap S} \bar{\chi}_i \eta = \sum_{j=1}^{N} \int_{\operatorname{supp} \chi_j \cap \operatorname{supp} \bar{\chi}_i \cap S} \chi_j \bar{\chi}_i \eta. \tag{2.2.24}$$

Thus

$$\begin{aligned}\sum_{j=1}^{N} \int_{\operatorname{supp} \chi_j \cap S} \chi_j \eta &= \sum_{j=1}^{N} \sum_{i=1}^{M} \int_{\operatorname{supp} \bar{\chi}_i \cap \operatorname{supp} \chi_j \cap S} \bar{\chi}_i \chi_j \eta = \sum_{i=1}^{M} \sum_{j=1}^{N} \int_{\operatorname{supp} \bar{\chi}_i \cap \operatorname{supp} \chi_j \cap S} \chi_j \bar{\chi}_i \eta \\ &= \sum_{i=1}^{M} \int_{\operatorname{supp} \bar{\chi}_i \cap S} \bar{\chi}_i \eta \end{aligned} \tag{2.2.25}$$

as required.

Chapter 3

Basics of Finsler geometry and parameterisation

In this chapter 3, we will briefly introduce the properties of Finsler geometry and some related structures that we will use for the considerations of calculus of variations. Since our motivation is to construct a theory applicable to concrete models of physics, there are certain aspects that may differ from the standard approach of a geometer, whose main interest lies on the construction or understanding of the geometrical structure itself. In particular, some of the standard definitions that allows further inquiries into problems of geometry may simply be unsuitable for tackling problems of physics. Therefore, in such cases, we have to modify or loosen some conditions. For instance, if we require strong convexity for the definition of Finsler manifold, most of the standard physical problems would be out of the scope. Also, if we require convexity (or "regularity" in some references), no gauge theories can be handled. We therefore propose to use only the minimal definitions, and use the name *Finsler manifold* in such broad sense. Nevertheless, for the construction of the theory of calculus of variation, the minimal definitions turn out to be sufficient, and no additional structures such as connections and curvature are required. We will begin with a very short historical review on how Finsler geometry was introduced, and then give the basic structures of Finsler geometry in the more modern terms, namely the tangent bundle and Finsler function, and introduce the Finsler length and its parameterisation. Then we will introduce the important concept of Finsler-Hilbert form, which is directly connected with Finsler length. These objects are the main tools for the calculus of variation, discussed in chapter 5.

3.1 Introduction to Finsler geometry

Historically, in his inaugural lecture, Riemann already mentioned on the special case of Finsler metric by stating, '$\cdots$*the line element can be an arbitrary homogeneous function of first degree in the quantities* dx *which remains the same when all the quantities* dx *change the sign, and in which the arbitrary constants are function of the quantities* x.' [12], where he referred to dx as an "infinitesimal displacement" from the position x. This statement can be translated to the formula

$$F(x^1, x^2, \cdots, x^n, \lambda dx^1, \lambda dx^2, \cdots, \lambda dx^n) = |\lambda|\, F(x^1, x^2, \cdots, x^n, dx^1, dx^2, \cdots, dx^n), \quad \lambda \in \mathbb{R}. \tag{3.1.1}$$

Then in the subsequent discussion, '...*and consequently* ds *equals the square root of an everywhere positive homogeneous function of the second degree in the quantities* dx*, in which the coefficients are continuous functions of the quantities* x.' Therefore, by this statement he restricted this function $F = ds$ to a more special case where it is given by, $F(x^i, dx^i) = \sqrt{g_{ij}(x)dx^i dx^j}$, which is the infinitesimal length of a curve on a Riemannian manifold. After the development of tensor analysis and exterior differential calculus, the theory was reformed in such a way that infinitesimal displacements dx was replaced by one forms, therefore allowing the concept to be treated in the realm of linear algebra, and the square of the structure ds was replaced by a symmetric tensor $g = g_{ij}(x)dx^i \odot dx^j$, which gives an inner product of tangent vectors at each point p on M. In other words, the concept of infinitesimal displacement was in a sense abandoned; instead of considering a length of infinitesimal piece of a curve, the new structure g defines the length of any finite sized vector at a point p on M. Also, since g is an inner product, it will give not only the length but also defines the angle between the two vectors.

In this modern view, the geometry is described by considering the tangent bundle, the arena of "vectors", not only by its base manifold M and the curve C, as in the original idea of Riemann. The additional structure at each point of M is called a fibre in mathematics, and in physics it is frequently called the "internal space".

Finsler geometry also has these two perspectives, one from the study on the properties of infinitesimal length on M, that is, a view as a geometry of calculus of variations (Finsler, Carathéodory), and then another view from the study of bundles and tensor analysis (Synge, Berwald, Cartan). Now it has become more standard to understand the Riemann geometry in this latter perspective, and likewise can be said for Finsler geometry. In this and the following chapter 4, we will also discuss on Finsler geometry and the further extensions of Kawaguchi geometry using this perspective. However, it is also very useful to remember the original Riemann's idea as well. Finsler geometry is simply a consideration of an arc length in more general setting than Riemann geometry. It has no inner product structure, and therefore more fundamental. The arc length of the line element ds is a homogeneous function of degree one, homogeneous with respect to the infinitesimal dislocation dx, where ds and dx are simply functions on M. There are researches on Finsler and Kawaguchi geometry in this direction as well, especially in cases where calculus of variations are important [8, 16], and indeed we take such works as an inspiring reference to our later discussions in chapter 5.

The main reference used in this section is Matsumoto [7], Chern, Chen and Lam [1], and Tamassy [13].

3.2 Basic definitions of Finsler geometry

The geometric structure that defines the Finsler manifold is a function on the total space of a tangent bundle (TM, τ_M, M). This structure is called Finsler function or Finsler metric in some references.

Let M be a C^∞-differentiable manifold, (TM, τ_M, M) its tangent bundle, $T^0M := TM\backslash 0$ the slit tangent bundle excluding the zero section from TM, and $(U, \varphi), \varphi = (x^\mu, y^\mu), \mu = 1, \cdots, n$ an induced chart on TM.

Definition 3.1. Finsler manifold
The *n-dimensional Finsler manifold* is a pair (M, F) where F is a C^0 function on TM and C^∞ function on T^0M, satisfying the following conditions.
(I) Homogeneity

$$F(\lambda v) = \lambda F(v), v \in T^0M, \lambda > 0 \tag{3.2.1}$$

or in coordinate expression,

$$F(x^\mu, \lambda v^\mu) = \lambda F(x^\mu, v^\mu), \lambda > 0. \tag{3.2.2}$$

This condition (I) also implies the condition of *Euler's homogeneous function theorem*,

$$\frac{\partial F}{\partial y^\mu} y^\mu = F. \tag{3.2.3}$$

Function with such properties is called a *Finsler function*.

Remark 3.2. Depending on the authors, usually there are several additional properties required for a Finsler manifold. Besides the above homogeneity condition, other requirements are such as,
(I') absolute homogeneity

$$F(\lambda v) = |\lambda|\, F(v), \lambda \in \mathbb{R} \tag{3.2.4}$$

This corresponds to the condition (3.1.1) suggested by Riemann, sometimes it is called *symmetric Finsler manifold*.
(II) non-negativity of F

$$F : TM \to \mathbb{R}^+. \tag{3.2.5}$$

(III) Convexity (Regularity)
The determinant of the matrix

$$g_{ij} := \frac{1}{2} \frac{\partial^2 F}{\partial y^i \partial y^j}, \tag{3.2.6}$$

$i = 1, \cdots, n$ is non-zero.
The structure $g_{ij} \in C^\infty(TM)$ is usually called a *fundamental tensor*, or *fundamental form*. (Refer to : [1, 2, 7, 13])
(IV) Strong Convexity

$$g_{ij}(v) v^i v^j := \frac{1}{2} \frac{\partial^2 F}{\partial y^i \partial y^j}\bigg|_v v^i v^j > 0, \tag{3.2.7}$$

$\forall v \neq 0 \in T_pM, \forall p \in M$. (Refer to : [1, 2, 13])
Together with (II) this equation is equivalent to the triangular inequality,

$$F(v) + F(w) > F(v + w), \tag{3.2.8}$$

for $\forall p \in M, v, w \in T_pM,\ v \neq w$. However, the condition (II), (III), (IV) is too restrictive for the application to physics. Therefore, we will only require the minimal condition (I). This condition is sufficient for our following discussions.

The Finsler function F is the main structure of Finsler geometry, and this definition gives the view to Finsler geometry as geometry of tangent bundle endowed with a specific feature. In the following, we will define the Finsler length of a curve on M, and show that the homogeneity condition is equivalent to the parameterisation invariant property of this arc length. In this way the two perspectives of Finsler geometry become connected.

3.3 Parameterisation and Finsler length

Here we introduce the curves, arc segments, their parameterisations and Finsler length. The notion of the Finsler length is naturally extended to 1-dimensional immersed submanifolds.

Definition 3.3. C^r-curves
Let M be a smooth manifold, and $\sigma : I \to M$ a C^r-mapping, where I is an open interval of a real line $\mathbb{R}$. We denote the image of I by C; $C := \sigma(I) \subset M$, and call C, the *C^r-curve* on M.

Definition 3.4. Lift of C^r-curves
Consider a tangent bundle (TM, τ_M, M) where τ_M is the natural projection. By differentiating the map $\sigma : I \to M$, we get a natural mapping $\hat{\sigma} : I \to TM$. Denote the image of I by $\hat{C} := \hat{\sigma}(I) \subset TM$, where $\tau_M(\hat{C}) = C$, and $\hat{\sigma}(t)$, $t \in I$ is a tangent vector at the point $\sigma(t) \in M$, namely

$$\hat{\sigma}(t) = \left.\frac{d(x^\mu \circ \sigma)}{dt}\right|_t \left(\frac{\partial}{\partial x^\mu}\right)_{\sigma(t)}. \tag{3.3.1}$$

The map $\hat{\sigma}$ and its image $\hat{C}$ is called the *lift*, or the *tangent lift* of σ (resp. C).

Definition 3.5. Regularity of σ
The C^r-map $\sigma : I \to M$ is called *regular*, if its lift $\hat{\sigma}$ is nowhere 0.

Definition 3.6. Parameterisation of an immersed curve
The C^r-map $\sigma : I \to M$ is called an *immersion*, if its lift $\hat{\sigma}$ is injective, and the image $C = \sigma(I)$ is called an *immersed curve*. The map σ is called a *parameterisation* of immersed curve C, and I is called a *parameter space*, when σ is an immersion, and preserves orientation.

Definition 3.7. Parameterisation
Let $\sigma : I \to M$ be a C^r-map, and $C = \sigma(I)$. The map σ is called a *parameterisation* of C, and I is called a *parameter space*, when σ is injective, and preserves orientation.

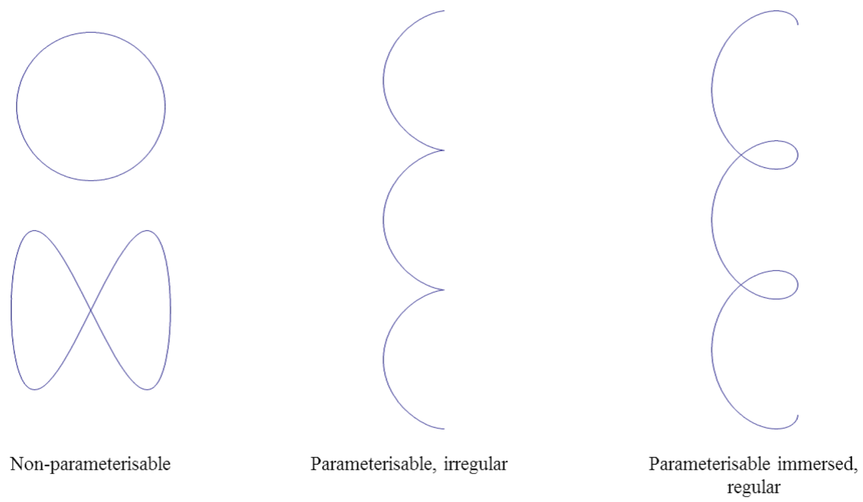

Figure 3.1: Example of curves in $\mathbb{R}^2$

Definition 3.8. Lift of a parameterisation
We call $\hat{\sigma}$ the *lift of parameterisation* σ, when σ is a parameterisation.

Given a curve C on M, more than one map and open interval may exist, namely for cases such as $C = \sigma(I) = \rho(J)$, where $\sigma,\ \rho$ are the maps, and $I,\ J$ are the open intervals in $\mathbb{R}$. We can classify the curves by considering the properties of these maps. Some example of the curves are shown in Figure 3.1.

Definition 3.9. Regular curve
Let C be a C^r-curve on M. C is called a *regular curve* on M, if there exists a regular C^r map σ and an open interval I of $\mathbb{R}$ such that $\sigma(I) = C$.

Definition 3.10. Parameterisable immersed curve
Let C be a C^r-curve on M. C is called a *parameterisable immersed curve* on M, if there exists an immersion σ and an open interval I of $\mathbb{R}$ such that $\sigma(I) = C$.

Definition 3.11. Parameterisable curve
Let C be a C^r-curve on M. C is called a *parameterisable curve* on M, if there exists an injective C^r-map σ and an open interval I of $\mathbb{R}$ such that $\sigma(I) = C$.

Example 3.12. S^1 embedded in $\mathbb{R}^2$ is neither a parameterisable curve nor a pararameterisable immersed curve. For instance, consider a map $\sigma : \mathbb{R} \to \mathbb{R}^2$, which in coordinates are given by $\sigma(t) := (\cos(t), \sin(t))$. The lift $\hat{\sigma}$ will be $\hat{\sigma} = (-\sin(t), \cos(t))$, so nowhere zero, meaning it is a regular curve, but neither σ nor $\hat{\sigma}$ are injective. Indeed we can consider different parameterisations, but since this is a closed curve, no injective map from an open subset of $\mathbb{R}$ exists. Furthermore, since it is also a smooth closed curve, no immersion from an open subset of $\mathbb{R}$ exists. Nevertheless, by restricting σ to finite interval $(0, 2\pi)$, we can have the parameterisation of the corresponding part of the circle.

Example 3.13. A curve "∞" in $\mathbb{R}^2$ is neither a parameterisable curve nor a pararameterisable immersed curve, since it is smooth and closed. Nevertheless, we may consider a map such as $\sigma(t) := (\cos(t), \sin(t))$, and by restricting the open interval to finite interval $(0, 2\pi)$, we can have the parameterisation of the corresponding part of the circle.

Example 3.14. A curve defined by a map $\sigma : \mathbb{R} \to \mathbb{R}^2$, which in coordinates are given by $\sigma(t) := (\cos(t), \sin(t) - t)$, is a parameterisable curve, but not regular. The lift $\hat{\sigma}$ will be $\hat{\sigma} = (-\sin(t), \cos(t) - 1)$, which becomes 0 at $t = 2n\pi$, for integer n.

Example 3.15. A spiral curve in $\mathbb{R}^2$ (on the right of Fig. 3.1) is a regular, and parameterisable immersed curve. There exists a regular parameterisation defined by a map $\sigma : \mathbb{R} \to \mathbb{R}^2$, which in coordinates are given by $\sigma(t) := (\cos(t), \sin(t) - t/2)$. The lift $\hat{\sigma}$ will be $\hat{\sigma} = (-\sin(t), \cos(t) - 1/2)$, σ is not injective since it gives the same point for t such that $t - \sin(t) = \pi$.

Example 3.16. In some cases, we can find a regular parameterisation of a curve that was originally given by a map which is not a regular parameterisation. Consider a curve defined by a map $\sigma : I \to \mathbb{R}^2$, $I = (0, 2\pi)$, which in coordinates are given by $\sigma(t) := (\sin(t), -\cos(2t))$. σ is not regular, since its lift $\hat{\sigma}$ becomes 0 at $t = \pi/2$. However, there exists a regular parameterisation of this curve $C = \sigma(I)$ by the map $\tilde{\sigma} : J \to \mathbb{R}^2$, $J = (-1, 1)$, which in coordinates are given by $\hat{\sigma}(s) := (s, 2s^2 - 1)$, and its lift is nowhere 0.

Occasionally, we implicitly refer to the pair (σ, I) by the parameterisation σ. In the following discussion, we will only consider regular, parameterisable curves.

Now we will introduce the concept of a length of a curve by integration on the parameter space. For simplicity, we will restrict ourselves to curves that are parameterisable, and consider its closed subset, which we define below.

Definition 3.17. arc segment
Let $\tilde{C}$ be a parameterisable curve on M with some parameterisation $\sigma : I \to M$. A subset of $\tilde{C}$ given by $C := \sigma([t_i, t_f]) \subset \tilde{C}$, where $[t_i, t_f] \subset I$ is called the *arc segment* on M, σ is called the *parameterisation of the arc segment* and the closed interval $[t_i, t_f]$ is called the *parameter space of an arc segment*.

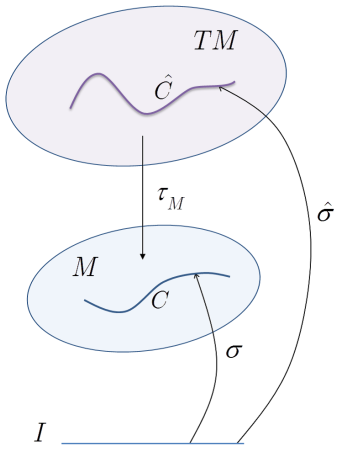

Figure 3.2: Parameterisation of a curve

The Finsler function defines a geometrical length of an arc segment C on M.

Definition 3.18. Finsler length
Let (M, F) be the n-dimensional Finsler manifold, and C the arc segment on M such that $C = \sigma([t_i, t_f])$. We assign to C the following integral

$$l^F(C) = \int_{t_i}^{t_f} F\left(\hat{\sigma}(t)\right) dt. \tag{3.3.2}$$

We call this number $l^F(C)$ the *Finsler length* of C.

Let (U, φ), $\varphi = (x^\mu, y^\mu)$, $\mu = 1, \cdots, n$ be the induced chart on TM. By chart expression, (3.3.2) is,

$$l^F(C) = \int_{t_i}^{t_f} F\left(x^\mu(\hat{\sigma}(t)), y^\mu(\hat{\sigma}(t))\right) dt = \int_{t_i}^{t_f} F\left(x^\mu(\sigma(t)), \frac{dx^\mu(\sigma(t))}{dt}\right) dt, \tag{3.3.3}$$

where we used the definition of $\hat{\sigma}$, and definition of induced coordinates of TM,

$$(x^\mu \circ \hat{\sigma})(t) = (x^\mu \circ \sigma)(t), \quad (y^\mu \circ \hat{\sigma})(t) = \left.\frac{d(x^\mu \circ \sigma)}{dt}\right|_t. \tag{3.3.4}$$

Let $\rho : J \to C$, $J \subset \mathbb{R}$ be another parameterisation of C. When there exists a diffeomorphism $\phi : J \to I$ such that $\rho = \sigma \circ \phi$, this gives an equivalence relation $\sigma \sim \rho$. We are able to find an important property of the Finsler length, which is the following lemma:

Lemma 3.19. Reparameterisation invariance of Finsler length
The Finsler length does not change by the reparameterisation $\rho = \sigma\circ\phi$, $\phi : J \to I$, where ϕ is a diffeomorphism such that preserves the orientation.

Proof. Dividing the interval $[t_i, t_f]$ if necessary into smaller closed sub-intervals we can suppose without loss of generality that the set $C = \sigma([t_i, t_f])$ lies in the coordinate neighbourhood of a chart (U, φ), $\varphi = (x^\mu)$. Then the lift of ρ becomes,

$$\hat{\rho}(s) = \left.\frac{d(x^\mu\circ\sigma\circ\phi)}{ds}\right|_s \left(\frac{\partial}{\partial x^\mu}\right)_{\sigma\circ\phi(s)} = \left.\frac{d(x^\mu\circ\sigma)}{dt}\right|_{\phi(s)} \left.\frac{d\phi}{ds}\right|_s \left(\frac{\partial}{\partial x^\mu}\right)_{\sigma\circ\phi(s)} = \left.\frac{d\phi}{ds}\right|_s \hat{\sigma}(\phi(s)), \tag{3.3.5}$$

for $s \in J$, and since ρ is a regular parameterisation that preserves orientation, $\dfrac{d\phi}{ds} > 0$. It is easy to see that the length of C is preserved by

$$\begin{aligned} l^F(C) &= \int_{s_i}^{s_f} F(\hat{\rho}(s))\, ds = \int_{s_i}^{s_f} F(\widehat{\sigma\circ\phi}(s))\, ds \\ &= \int_{s_i}^{s_f} F\left(\frac{d\phi}{ds}(s)\, \hat{\sigma}(\phi(s))\right) ds = \int_{\phi^{-1}(t_i)}^{\phi^{-1}(t_f)} F(\hat{\sigma}(\phi(s)))\, \frac{d\phi}{ds}(s) ds \\ &= \int_{t_i}^{t_f} F(\hat{\sigma}(t)))\, dt. \end{aligned} \tag{3.3.6}$$

$s_i,\ s_f$ are the pre-image of the boundary points $t_i,\ t_f$ by ϕ. □

In the second line of (3.3.6), we have used the homogeneity condition of F. The homogeneity of F and parameterisation invariance of Finsler length is an equivalent property.

Remark 3.20. The "Finsler length" does not have the properties of a "standard" length, considered by Euclid or Riemannian geometry, since we require only homogeneity condition of the Finsler function. For instance, when one changes the orientation of the curve, in general, it gives different values (not just signatures). However, in our following discussion of the calculus of variations, we can still use this concept to obtain extremals and equations of motion, and it maybe also an interesting tool for considering differential geometry of submanifolds, and possible generalisations of mechanics.

3.4 Finsler-Hilbert form

Given a Finsler manifold, we can obtain a important geometrical structure which is called a *Hilbert form* by some authors [1]. In this thesis, we sometimes call them *Finsler-Hilbert form*, just to stress it is for the first order mechanics. In chapter 4, we will generalise this concept to second order and higher dimensional parameter space.

Definition 3.21. Hilbert form
Let (V, ψ), $\psi = (x^\mu, y^\mu)$, $\mu = 1, \cdots, n$ be an induced chart on TM. Consider the following 1-form on T^0M, which in local coordinates are expressed by

$$\mathcal{F} = \frac{\partial F}{\partial y^\mu} dx^\mu. \tag{3.4.1}$$

This form is invariant with respect to the coordinate transformations by

$$x^\mu \to \tilde{x}^\mu = \tilde{x}^\mu(x^\nu), y^\mu \to \tilde{y}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} y^\nu, \tag{3.4.2}$$

therefore, it is a globally defined form on T^0M. We will call this global form with the local coordinate expression (3.4.1), *Hilbert form*.

Lemma 3.22. Let $\mathcal{F}$ be the Hilbert 1-form on T^0M, $C = \sigma(\bar{I})$ the arc segment on M, with $\bar{I} = [t_i, t_f]$ a closed interval in $\mathbb{R}$. Then,

$$\int_{\hat{C}} = l^F(C). \tag{3.4.3}$$

Proof. The simple calculation leads,

$$\begin{aligned}
\int_{\hat{C}} \mathcal{F} &= \int_{\hat{\sigma}(\bar{I})} \frac{\partial F}{\partial y^\mu} dx^\mu = \int_{t_i}^{t_f} \frac{\partial F}{\partial y^\mu} \circ \hat{\sigma}\, d(x^\mu \circ \hat{\sigma}) \\
&= \int_{t_i}^{t_f} \frac{\partial F}{\partial y^\mu}(\hat{\sigma}(t)) \frac{d(x^\mu(\sigma(t))}{dt} dt = \int_{t_i}^{t_f} \frac{\partial F}{\partial y^\mu}(\hat{\sigma}(t))\, y^\mu(\hat{\sigma}(t)) dt \\
&= \int_{t_i}^{t_f} F(\hat{\sigma}(t))\, dt = l^F(C)
\end{aligned} \tag{3.4.4}$$

where we used the pulled back homogeneity condition

$$\frac{\partial F}{\partial y^\mu} \circ \hat{\sigma} \cdot y^\mu \circ \hat{\sigma} = F \circ \hat{\sigma}. \tag{3.4.5}$$

□

Remark 3.23. This lemma extends the notion of Finsler length given by (3.3.2). Namely, since the Hilbert form can be integrated over *any* 1-dimensional submanifold of M, the identity (3.4.3) suggests that it is possible to extend the integration over arc segments to *arbitrary* 1-dimensional submanifold of M, by considering the Hilbert form.

Remark 3.24. Now that we showed that Finsler-Hilbert 1-form gives the Finsler length (and in a more general situation of a submanifold), we can redefine the pair $(M, \mathcal{F})$ as the Finsler manifold instead of taking (M, F). This is a more geometrical definition of a Finsler manifold, and also in close analogy to the case of Riemannian geometry, where the geometric structure is given by a tensor g, and not by a function. This observation is important also for the consideration of Kawaguchi geometry.

Remark 3.25. When given a Hilbert form, we can obtain the Cartan form, which is a one form defined on J^1Y, where Y is a $n+1$-dimensional manifold, and J^1Y is the prolongation of the bundle $(Y, \pi, \mathbb{R})$. (In most cases, $Y = \mathbb{R} \times Q$ is considered, and is called an extended configuration space. Q is a configuration space of dimension n. In such case, the bundle $(Y, pr_1, \mathbb{R})$ becomes a trivial bundle.) Let (U, ψ), $\psi = (t, q^i)$, $i = 1, \cdots, n$ be the adapted chart on Y, and the induced chart on $\mathbb{R}$ be $(\pi(U), t)$. We denote the induced chart on J^1Y by $((\pi^{1,0})^{-1}(U), \psi^1), \psi^1 = (t, q^i, \dot{q}^i)$, where $\pi^{1,0} : J^1Y \to Y$ is the prolongation of π. Suppose we have a Hilbert form on TY. Take the induced chart on TY as $((\tau_Y)^{-1}(U), \psi^1)$, $\psi^1 = (x^0, x^i, y^0, y^i)$, $i = 1, \cdots, n$. (In order to avoid confusion we use different symbols, but clearly $x^0 = t \circ \tau_Y, x^i = q^i \circ \tau_Y$.) Since both $J^1Y$and TY are bundles over Y, around every $p \in Y$, there exists a local trivialisation. Take $p \in U$, and let (U, F_p, t_p), $t_p : (\tau_Y)^{-1}(U) \to U \times F_p$ be the local trivialisation of TY and $(U, G_p, \tilde{t}_p)$, $\tilde{t}_p : (\pi^{1,0})^{-1}(U) \to U \times G_p$ be the local trivialisation of J^1Y where $F_p = \mathbb{R}^{2(n+1)}$ and $G_p = \mathbb{R}^{2n+1}$. Then there exists a natural inclusion $\iota : (\tilde{t}_p)^{-1}(U \times G_p) \hookrightarrow TM$, $\iota((\tilde{t}_p)^{-1}(U \times G_p)) = (t_p)^{-1}(U \times F_p)$, which in coordinate functions are given by

$$x^0 \circ \iota = t, \ x^i \circ \iota = q^i, \ y^0 \circ \iota = 1, \ y^i \circ \iota = \dot{q}^i,$$

so that the submanifold equation is $y^0 = 1$. The local coordinate expression of the Hilbert form is,

$$\mathcal{F} = \frac{\partial F}{\partial y^0} dx^0 + \frac{\partial F}{\partial y^i} dx^i = \frac{1}{y^0} \left(F - \frac{\partial F}{\partial y^i} y^i \right) dx^0 + \frac{\partial F}{\partial y^i} dx^i, \tag{3.4.6}$$

where the second equality holds by using the Euler's homogeneity condition. The pull back of this Hilbert form by ι is

$$\iota^*\mathcal{F} = \frac{1}{y^0}\circ\iota\left(F\circ\iota - \frac{\partial F}{\partial y^i}\circ\iota\cdot y^i\circ\iota\right)d(x^0\circ\iota) + \frac{\partial F}{\partial y^i}\circ\iota\cdot d(x^i\circ\iota) \tag{3.4.7}$$

However, by the submanifold equation $y^0 = 1$, this becomes exactly the Cartan form,

$$\Theta_C = \iota^*\mathcal{F} = \left(\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\dot{q}^i}\dot{q}^i\right)dt + \frac{\partial\mathcal{L}}{\partial\dot{q}^i}dx^i, \tag{3.4.8}$$

should be regarded as the "conventional" Lagrange function [1] defined on J^1Y. In fact, the inclusion map ι can be globalised for all J^1Y, and the relation (3.4.8) is global. In other words, Cartan form is a restriction of a Hilbert form to a submanifold J^1Y in TY. Also, the base manifold of the bundle $(Y,\pi,\mathbb{R})$ is naturally considered as a parameter space, and for Cartan form such structure was needed, while Hilbert form does not need such fibre bundle structure of Y. In this sense, Hilbert form is a generalisation of the Cartan form that does not depend on specific bundle structures.

In chapter 5, we will also consider the converse and discuss how to obtain the Hilbert form when a conventional Lagrangian is given.

[1] We simply use the term "conventional" to distinguish the Lagrangian function over J^1Y, from our Lagrangian which is the Hilbert 1-form over TY.

Chapter 4

Basics of Kawaguchi geometry and parameterisation

In this chapter 4, we will introduce a geometry which was originally considered by A. Kawaguchi as a extension of Finsler geometry. In contrast to Finsler geometry, Kawaguchi geometry still does not have a well-developed consensus yet, and it may be a bit early to be called as "geometry". Nevertheless, it follows the same line of thought that originates from Riemann, and with the hope of its future establishment, we will call so in this thesis.

4.1 Introduction to Kawaguchi geometry

A. Kawaguchi considered the generalisation of Finsler geometry in two directions, one for the case of higher order derivative, and another for the case of k-dimensional parameter space, from the viewpoint of calculus of variations [4]. The latter was referred to as Areal space. In either case, the theory was presented in means of local expressions. In this thesis, we will make an original exposition of Kawaguchi geometry by using multivector bundle and differential forms, and extend its validity to global expressions for the second order 1-dimensional parameter space and first order k-dimensional parameter space. The higher order k-dimensional parameter space is left for future research.

In the case of Finsler geometry, the definition for the Finsler-Hilbert form was such as it gives an invariant length of a parameterisable 1-dimensional submanifold. Namely, the homogeneity of the Finsler function and the parameterisation invariance of Finsler length was equivalent. For the Kawaguchi geometry, we can also consider a similar property as the main pillar for setting up the foundation.

4.2 Second order, 1-dimensional parameter space

Here in this section we will reconstruct the first direction of generalisation of Finsler geometry originally considered by Kawaguchi in a modern fashion, where the second order derivatives (acceleration) are considered. Kawaguchi considered the homogeneity condition of higher order functions by requiring the following invariance on integration under the change of parameterisation,

$$\begin{aligned}\int_{t_1}^{t_2} K\left(x^\mu, \frac{dx^\mu}{dt}, \frac{d^2x^\mu}{dt^2}\right) dt &= \int_{t_1}^{t_2} K\left(x^\mu, \frac{dx^\mu}{ds}\frac{ds}{dt}, \frac{d^2x^\mu}{ds^2}\left(\frac{ds}{dt}\right)^2 + \frac{dx^\mu}{ds}\frac{d^2s}{dt^2}\right) dt \\ &= \int_{t_1}^{t_2} K\left(x^\mu, \frac{dx^\mu}{ds}, \frac{d^2x^\mu}{ds^2}\right) ds. \qquad (4.2.1)\end{aligned}$$

This requirement gives a condition that the length of a curve defined by such function K becomes invariant, in other words it is a geometrical length. The above expression is in a single local chart, but we can prove that the condition could be extended globally. Below we will give a definition of the manifold with such properties.

4.2.1 Basic definitions of Finsler-Kawaguchi geometry

We will first define the geometric structure of the second order Finsler-Kawaguchi manifold by a function on the total space of a second order tangent bundle $(T^2M, \tau_M^{2,0}, M)$, such that gives a geometrical length to a curve on M. We will call this structure a second order Finsler-Kawaguchi function.

Definition 4.1. Second order Finsler-Kawaguchi manifold (Second order 1-dimensional parameter space)
Let (M, K) be a pair of n-dimensional C^∞-differentiable manifold M and a function $K \in C^\infty(T^2M)$, which for a adapted chart on T^2M, $(V^2, \psi^2), \psi^2 = (x^\mu, y^\mu, z^\mu)$, $\mu = 1, \cdots, n$, satisfies the *second order homogeneity condition*,

$$K(x^\mu, \lambda y^\mu, \lambda^2 z^\mu + \rho y^\mu) = \lambda K(x^\mu, y^\mu, z^\mu), \quad \lambda \in \mathbb{R}^+, \quad \rho \in \mathbb{R}. \qquad (4.2.2)$$

We will call the function with such properties, a *second order Finsler-Kawaguchi function*, and the pair (M, K) a *n-dimensional second order Finsler-Kawaguchi manifold.*

Compared to the case of first order Finsler, since T^2M is not a vector space, we do not have an expression such as λv, the vector multiplied by a constant. The condition (4.2.2) implies the following,

$$\begin{cases} y^\mu \dfrac{\partial K}{\partial y^\mu} + 2z^\mu \dfrac{\partial K}{\partial z^\mu} = K, \\ y^\mu \dfrac{\partial K}{\partial z^\mu} = 0. \end{cases} \tag{4.2.3}$$

which is called the *Zermelo's condition*. The proofs are given by Urban and Krupka [18].

These conditions (4.2.3) are coordinate independent. Take another chart $(\bar{U};\bar{\varphi})$, $\bar{\varphi} = (\bar{x}^\mu, \bar{y}^\mu, \bar{z}^\mu)$ on M, and then from the coordinate transformation rules, we have,

$$\begin{aligned} &\bar{y}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^\nu} y^\nu, \\ &\frac{\partial K}{\partial \bar{y}^\mu} = \frac{\partial K}{\partial y^\rho}\frac{\partial y^\rho}{\partial \bar{y}^\mu} + \frac{\partial K}{\partial z^\rho}\frac{\partial z^\rho}{\partial \bar{y}^\mu} = \frac{\partial K}{\partial y^\rho}\frac{\partial x^\rho}{\partial \bar{x}^\mu} + 2\frac{\partial K}{\partial z^\rho}\frac{\partial^2 x^\rho}{\partial \bar{x}^\alpha \partial \bar{x}^\mu}\bar{y}^\alpha, \\ &\bar{z}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^\nu} z^\nu + \frac{\partial^2 \bar{x}^\mu}{\partial x^\alpha \partial x^\beta} y^\alpha y^\beta, \\ &\frac{\partial K}{\partial \bar{z}^\mu} = \frac{\partial K}{\partial z^\rho}\frac{\partial z^\rho}{\partial \bar{z}^\mu}, \end{aligned} \tag{4.2.4}$$

and since

$$\frac{\partial \bar{x}^\mu}{\partial x^\nu}\frac{\partial^2 x^\rho}{\partial \bar{x}^\alpha \partial \bar{x}^\mu}\bar{y}^\alpha = \frac{\partial}{\partial \bar{x}^\alpha}\left(\frac{\partial \bar{x}^\mu}{\partial x^\nu}\frac{\partial x^\rho}{\partial \bar{x}^\mu}\bar{y}^\alpha\right) - \frac{\partial^2 \bar{x}^\mu}{\partial x^\sigma \partial x^\nu}\frac{\partial x^\sigma}{\partial \bar{x}^\alpha}\frac{\partial x^\rho}{\partial \bar{x}^\mu}\bar{y}^\alpha = -\frac{\partial^2 \bar{x}^\mu}{\partial x^\sigma \partial x^\nu}\frac{\partial x^\rho}{\partial \bar{x}^\mu} y^\sigma,$$

we obtain

$$\begin{aligned} &\bar{y}^\mu \frac{\partial K}{\partial \bar{y}^\mu} + 2\bar{z}^\mu \frac{\partial K}{\partial \bar{z}^\mu} = y^\mu \frac{\partial K}{\partial y^\mu} + 2z^\mu \frac{\partial K}{\partial z^\mu} = K, \\ &\bar{y}^\mu \frac{\partial K}{\partial \bar{z}^\mu} = y^\mu \frac{\partial K}{\partial z^\mu} = 0. \end{aligned} \tag{4.2.5}$$

4.2.2 Parameterisation invariant length of Finsler-Kawaguchi geometry

In Finsler geometry, the homogeneity condition of Finsler function implied the invariance of Finsler length and vice versa. We would similarly define the Finsler-Kawaguchi length, and then show that the condition (4.2.2) and the parameter independence of Finsler-Kawaguchi length is equivalent. We will begin by introducing the lift of a parameterisation.

Definition 4.2. Second order lift of parameterisation

$$\begin{tikzcd}
TTM \arrow[dr, "\tau_{TM}"'] & T^2M \arrow[l, hook'] \arrow[d, "\tau_M^{2,1}"] \\
& TM \arrow[d, "\tau_M"] \\
& M \supset C \\
I \arrow[uuu, bend left, "\hat{\hat{\sigma}}"] \arrow[ur, bend right, "\sigma"'] \arrow[uur, bend right=60, "\hat{\sigma}"'] \arrow[uuur, bend right=80, "\sigma^2"'] &
\end{tikzcd}$$

Figure 4.1: Second order lift of parameterisation

Consider a second order tangent bundle $(T^2M, \tau_M^{2,0}, M)$ defined in Section 2.1.1, and the induced chart (V^2, ψ^2), $\psi^2 = (x^\mu, y^\mu, z^\mu)$, $\mu = 1, \cdots, n$, on T^2M. Let σ be a parameterisation of C, namely $C = \sigma(I)$ defined in Section 3.3, and I be an open interval in $\mathbb{R}$. We call the map $\sigma^2 : I \to T^2M$, such that its local expression is given by

$$\sigma^2(t) = \left.\frac{d(x^\mu \circ \sigma)}{dt}\right|_t \left(\frac{\partial}{\partial x^\mu}\right)_{\hat{\sigma}(t)} + \left.\frac{d^2(x^\mu \circ \sigma)}{dt^2}\right|_t \left(\frac{\partial}{\partial y^\mu}\right)_{\hat{\sigma}(t)}, \tag{4.2.6}$$

the *second order lift of parameterisation* σ to T^2M. The image $C^2 = \sigma^2(I)$ is called the *second order lift of* C.

Clearly, $\tau_M^{2,0}(C^2) = C$. The second order lift of parameterisation σ is constructed by considering the subset of iterated tangent lift. Namely, construct the tangent lift $\hat{\hat{\sigma}} : I \to TTM$ of the parameterisation $\hat{\sigma} : I \to TM$, and then take its subset by $\sigma^2 := \{\hat{\hat{\sigma}} | T_{\hat{\sigma}(t)}\tau_M(\hat{\hat{\sigma}}(t)) = \tau_{TM}(\hat{\hat{\sigma}}(t)), t \in I\}$. The iterated tangent lift $\hat{\hat{\sigma}}(t)$ has the local coordinate expressions

$$\hat{\hat{\sigma}}(t) = \left.\frac{d(x^\mu \circ \hat{\sigma})}{dt}\right|_t \left(\frac{\partial}{\partial x^\mu}\right)_{\hat{\sigma}(t)} + \left.\frac{d(y^\mu \circ \hat{\sigma})}{dt}\right|_t \left(\frac{\partial}{\partial y^\mu}\right)_{\hat{\sigma}(t)},$$

and the condition for $\hat{\hat{\sigma}}(t)$ to be in T^2M by the Definition 2.23 will give us the coordinates of $\sigma^2(t)$,

$$(x^\mu \circ \sigma^2)(t) = (x^\mu \circ \hat{\sigma})(t) = (x^\mu \circ \sigma)(t),$$

$$
\begin{aligned}
(y^{\mu} \circ \sigma^2)(t) &= \left.\frac{d(x^{\mu} \circ \hat{\sigma})}{dt}\right|_t = \left.\frac{d(x^{\mu} \circ \sigma)}{dt}\right|_t = (y^{\mu} \circ \hat{\sigma})(t), \\
(z^{\mu} \circ \sigma^2)(t) &= \left.\frac{d(y^{\mu} \circ \hat{\sigma})}{dt}\right|_t = \left.\frac{d^2(x^{\mu} \circ \sigma)}{dt^2}\right|_t .
\end{aligned} \tag{4.2.7}
$$

From above we conclude $\sigma^2(t)$ has the expression (4.2.6).

The parameterisation σ where its second order lift σ^2 is nowhere 0 is called a *regular parameterisation of order* 2. In the discussions concerning second order Finsler-Kawaguchi geometry, we will only consider the regular parameterisation of order 2.

The r-th order parameterisation $\sigma^r : I \to T^r M$ can be obtained by iterative process. Namely, construct the lift $\widehat{(\sigma)^{r-1}} : I \to TT^{r-1}M$ of the parameterisation $\sigma^{r-1} : I \to T^{r-1}M$, and then regarding the construction on the higher-order tangent bundle (2.1.27), take its subset by

$$
\sigma^r := \{\widehat{(\sigma)^{r-1}} \mid T_{\sigma^{r-1}(t)} \tau_M^{r-1,r-2}(\widehat{(\sigma)^{r-1}}(t)) = \iota_{r-1} \circ \tau_{T^{r-1}M}(\widehat{(\sigma)^{r-1}}(t)),\, t \in I\}. \tag{4.2.8}
$$

Definition 4.3. r-th order parameterisation
Let σ be a parameterisation of the curve C on M. The map $\sigma^r : I \to T^r M$ given by (4.4.6) is called the *r-th order lift of parameterisation σ.*

where $\iota_{r-1} : T^{r-1}M \to TT^{r-2}M$ is the inclusion map.

Definition 4.4. Finsler-Kawaguchi length (second order)
The Finsler-Kawaguchi function defines a geometrical length for an arc segment $C = \sigma([t_i, t_f])$ on M by the lifted parameterisation σ^2 of order 2 as,

$$
l^K(C) = \int_{t_i}^{t_f} K\left(\sigma^2(t)\right) dt. \tag{4.2.9}
$$

By chart expression, this is,

$$
\begin{aligned}
l^K(C) &= \int_{t_i}^{t_f} K\left(x^{\mu}(\sigma^2(t)), y^{\mu}(\sigma^2(t)), z^{\mu}(\sigma^2(t))\right) dt \\
&= \int_{t_i}^{t_f} K\left(x^{\mu}(\sigma(t)), \frac{d(x^{\mu}(\sigma(t)))}{dt}, \frac{d^2(x^{\mu}(\sigma(t)))}{dt^2}\right) dt,
\end{aligned} \tag{4.2.10}
$$

where we used the definition of σ^2, and its coordinates expressed by the induced coordinates of T^2M,

$$
(x^{\mu} \circ \sigma^2)(t) = (x^{\mu} \circ \sigma)(t),
$$

$$
\begin{aligned}
(y^\mu \circ \sigma^2)(t) &= \left.\frac{d(x^\mu \circ \sigma)}{dt}\right|_t, \\
(z^\mu \circ \sigma^2)(t) &= \left.\frac{d^2(x^\mu \circ \sigma)}{dt^2}\right|_t.
\end{aligned} \tag{4.2.11}
$$

We call this $l^K(C)$, the (second order) *Finsler-Kawaguchi length* of the curve C.

For the case of first order Finsler geometry, there was an important property of parameterisation independence of the Finsler length (Section 3.3). We will show that the second order Finsler-Kawaguchi length also has the same property.

Lemma 4.5. Reparameterisation invariance of Finsler-Kawaguchi length
The second order Finsler-Kawaguchi length does not change by the reparameterisation $\rho = \sigma \circ \phi$, $\phi : J \to I$, where ϕ is a diffeomorphism such that preserves the orientation, and I, J are open intervals in $\mathbb{R}$.

Proof. The second order lift of ρ becomes,

$$
\begin{aligned}
\rho^2(s) = &\left.\frac{d(x^\mu \circ \sigma \circ \phi)}{ds}\right|_s \left(\frac{\partial}{\partial x^\mu}\right)_{\widehat{\sigma \circ \phi(s)}} + \left.\frac{d^2(x^\mu \circ \sigma \circ \phi)}{ds^2}\right|_s \left(\frac{\partial}{\partial y^\mu}\right)_{\widehat{\sigma \circ \phi(s)}} \\
= &\left.\frac{d(x^\mu \circ \sigma)}{dt}\right|_{\phi(s)} \left.\frac{d\phi}{ds}\right|_s \left(\frac{\partial}{\partial x^\mu}\right)_{\widehat{\sigma \circ \phi(s)}} + \frac{d}{ds}\left(\left.\frac{d(x^\mu \circ \sigma)}{dt}\right|_{\phi(s)} \left.\frac{d\phi}{ds}\right|_s\right) \left(\frac{\partial}{\partial y^\mu}\right)_{\widehat{\sigma \circ \phi(s)}} \\
= &\left.\frac{d(x^\mu \circ \sigma)}{dt}\right|_{\phi(s)} \left.\frac{d\phi}{ds}\right|_s \left(\frac{\partial}{\partial x^\mu}\right)_{\widehat{\sigma \circ \phi(s)}} + \left.\frac{d^2(x^\mu \circ \sigma)}{dt^2}\right|_{\phi(s)} \left(\left.\frac{d\phi}{ds}\right|_s\right)^2 \left(\frac{\partial}{\partial y^\mu}\right)_{\widehat{\sigma \circ \phi(s)}} \\
&+ \left.\frac{d(x^\mu \circ \sigma)}{dt}\right|_{\phi(s)} \left.\frac{d^2\phi}{ds^2}\right|_s \left(\frac{\partial}{\partial y^\mu}\right)_{\widehat{\sigma \circ \phi(s)}},
\end{aligned} \tag{4.2.12}
$$

for $s \in J$. Its coordinates in T^2M are,

$$
\begin{aligned}
(x^\mu \circ \rho^2)(s) &= (x^\mu \circ \sigma)(\phi(s)) = (x^\mu \circ \sigma^2)(\phi(s)), \\
(y^\mu \circ \rho^2)(s) &= \left(\left.\frac{d(x^\mu \circ \sigma)}{dt}\right|_{\phi(\cdot)} \frac{d\phi}{ds}\right)(s) = \left.\frac{d\phi}{ds}\right|_s \cdot (y^\mu \circ \sigma^2)(\phi(s)), \\
(z^\mu \circ \rho^2)(s) &= \left(\left.\frac{d^2(x^\mu \circ \sigma)}{dt^2}\right|_{\phi(\cdot)} \left(\frac{d\phi}{ds}\right)^2 + \left.\frac{d(x^\mu \circ \sigma)}{dt}\right|_{\phi(\cdot)} \frac{d^2\phi}{ds^2}\right)(s) \\
&= \left(\left.\frac{d\phi}{ds}\right|_s\right)^2 \cdot (z^\mu \circ \sigma^2)(\phi(s)) + \left.\frac{d^2\phi}{ds^2}\right|_s \cdot (y^\mu \circ \sigma^2)(\phi(s))
\end{aligned} \tag{4.2.13}
$$

and since ρ is a regular parameterisation that preserves orientation, $\dfrac{d\phi}{ds} > 0$.

Let $[s_i, s_f] \subset J$ and $[t_i, t_f] \subset I$ be closed intervals, where $\phi(s_i) = t_i,\ \phi(s_f) = t_f$. Now we can see that the length of $C = \sigma([t_i, t_f])$ is preserved by

$$
\begin{aligned}
& l^K(C) = \int_{s_i}^{s_f} K\left(\rho^2(s)\right) ds = \int_{s_i}^{s_f} K\left(x^\mu(\rho^2(s)), y^\mu(\rho^2(s)), z^\mu(\rho^2(s))\right) ds \\
& = \int_{s_i}^{s_f} K\left(x^\mu(\sigma^2(\phi(s))), \left.\frac{d\phi}{ds}\right|_s y^\mu(\sigma^2(\phi(s))), \left(\left.\frac{d\phi}{ds}\right|_s\right)^2 z^\mu(\sigma^2(\phi(s))) + \left.\frac{d^2\phi}{ds^2}\right|_s y^\mu(\sigma^2(\phi(s)))\right) ds \\
& = \int_{\phi^{-1}(t_i)}^{\phi^{-1}(t_f)} K(\sigma^2(\phi(s)))\,\frac{d\phi}{ds}(s) ds \\
& = \int_{t_i}^{t_f} K(\sigma^2(t)))\, dt. \qquad (4.2.14)
\end{aligned}
$$

□

In the second line of (4.2.14), we have used the pull-back of second order homogeneity condition (4.2.2). We can conclude that the homogeneity of K and parameterisation invariance of Finsler-Kawaguchi length is an equivalent property.

4.2.3 Finsler-Kawaguchi form

Given a Finsler-Kawaguchi manifold (M, K), we can obtain an important geometrical structure, which we call a *Finsler-Kawaguchi form*. It is a form which is constructed in accord to the second order homogeneity condition, and gives a conventional Lagrangian when pulled back to the parameter space by a certain parameterisation. As such as the Hilbert form reduced to Cartan form by choosing a specific fibration, the Finsler-Kawaguchi form is expected to give the Higher order Cartan form similarly by fixing the fibration.

Definition 4.6. Second order Finsler-Kawaguchi form
Let $(V^2, \psi^2), \psi^2 = (x^\mu, y^\mu, z^\mu),\ \mu = 1, \cdots, n$ be a chart on T^2M. The *second order Finsler-Kawaguchi form* $\mathcal{K}$ is a 1-form on T^2M, which in local coordinates are expressed by

$$
\mathcal{K} = \frac{\partial K}{\partial y^\mu} dx^\mu + 2\frac{\partial K}{\partial z^\mu} dy^\mu. \qquad (4.2.15)
$$

This corresponds to the first formula in (4.2.3).

The Finsler-Kawaguchi form is invariant with respect to the coordinate transformations by,

$$
\begin{aligned}
x^\mu &\to \bar{x}^\mu = \bar{x}^\mu(x^\nu), \\
y^\mu &\to \bar{y}^\mu(x^\mu, y^\mu) = \frac{\partial \bar{x}^\mu}{\partial x^\nu} y^\nu, \\
z^\mu &\to \bar{z}^\mu(x^\mu, y^\mu, z^\mu) = \frac{\partial \bar{x}^\mu}{\partial x^\nu} z^\nu + \frac{\partial^2 \bar{x}^\mu}{\partial x^\alpha \partial x^\beta} y^\alpha y^\beta
\end{aligned} \tag{4.2.16}
$$

Note that from the second formula of (4.2.2), we can similarly consider a form $\frac{\partial K}{\partial z^\mu} dx^\mu$, which is also coordinate independent. Adding such form to (4.2.15) does not contribute to the conventional Lagrangian or equation of motion which its only dynamical variable is the time t, since it becomes 0 when pulled back to the parameter space P. Still, it changes the Lagrangian and equation of motion before the pull-back, giving some ambiguity in the choice of such forms. Nevertheless, for the arbitrary order of derivatives, only one condition of Zermelo relates the derivatives of K in the formula to K, and we can always use this condition to construct the Finsler-Kawaguchi form.

Proposition 4.7. Let $\mathcal{K}$ be the second order Finsler-Kawaguchi 1-form on T^2M, I an open interval in $\mathbb{R}$, $[t_i, t_f] \subset I$, and σ^2 the second order lifted parameterisation $\sigma^2 : I \to T^2M$. The integration of $\mathcal{K}$ along $C^2 := \sigma^2([t_i, t_f])$ is given by the Finsler-Kawaguchi length $l^K(C)$.

Proof. The simple calculation leads,

$$
\begin{aligned}
\int_{C^2} \mathcal{K} &= \int_{\sigma^2(I)} \frac{\partial K}{\partial y^\mu} dx^\mu + 2\frac{\partial K}{\partial z^\mu} dy^\mu = \int_{t_i}^{t_f} \frac{\partial K}{\partial y^\mu} \circ \sigma^2 \, d(x^\mu \circ \sigma^2) + 2\frac{\partial K}{\partial z^\mu} \circ \sigma^2 \, d(y^\mu \circ \sigma^2) \\
&= \int_{t_i}^{t_f} \frac{\partial K}{\partial y^\mu}(\sigma^2(t)) \, \frac{d(x^\mu(\sigma^2(t))}{dt} + 2\frac{\partial K}{\partial z^\mu}(\sigma^2(t)) \, \frac{d(y^\mu(\sigma^2(t))}{dt} dt \\
&= \int_{t_i}^{t_f} \frac{\partial F}{\partial y^\mu}(\sigma^2(t)) \, y^\mu(\sigma^2(t)) + 2\frac{\partial K}{\partial z^\mu}(\sigma^2(t)) \, z^\mu(\sigma^2(t)) dt \\
&= \int_{t_i}^{t_f} K(\sigma^2(t)) \, dt = l^K(C),
\end{aligned} \tag{4.2.17}
$$

where we used the pull-back second order homogeneity condition

$$
\frac{\partial K}{\partial y^\mu} \circ \sigma^2 \cdot y^\mu \circ \sigma^2 + 2\frac{\partial K}{\partial z^\mu} \circ \sigma^2 \cdot z^\mu \circ \sigma^2 = K \circ \sigma^2. \tag{4.2.18}
$$

□

Remark 4.8. As we redefined the pair $(M, \mathcal{F})$ as the Finsler manifold instead of the pair (M, F), we can redefine the pair $(M, \mathcal{K})$ as the n-dimensional Finsler-Kawaguchi manifold instead of the pair (M, K).

Remark 4.9. We saw in the previous chapter, the relation between Hilbert form and the Cartan form (Remark 3.25). Similary, we can consider what the Finsler-Kawaguchi form corresponds to, when specifying the fibration. Let Y be a $n+1$-dimensional manifold, and fix the bundle $(Y, \pi, \mathbb{R})$. The second order prolongation of Y is denoted as J^2Y. Let (V, ψ), $\psi = (t, x^i)$, $i = 1, \cdots, n$ be the adapted chart on Y, and the induced chart on $\mathbb{R}$ be $(\pi(V), t)$. We denote the induced chart on J^2Y by $((\pi^{2,0})^{-1}(V), \psi^2), \psi^2 = (t, q^i, \dot{q}^i, \ddot{q}^i)$, where $\pi^{2,0} : J^2Y \to Y$ is the prolongation of π. Suppose we have a second order Finsler-Kawaguchi form $\mathcal{K}$ on $T^2Y \subset TTY$. Take the induced chart on T^2Y as $((\tau_Y^{2,0})^{-1}(V), \tilde{\psi}^2)$, $\tilde{\psi}^2 = (x^0, x^i, y^0, y^i, \dot{x}^0, \dot{x}^i, z^0, z^i)$, $i = 1, \cdots, n$. Consider an inclusion map $\iota : (\pi^{2,0})^{-1}(V) \hookrightarrow (\tau_Y^{2,0})^{-1}(V)$, which in coordinates are given by

$$\iota : (t, q^i, \dot{q}^i, \ddot{q}^i) \hookrightarrow (x^0 = t, x^i = q^i, y^0 = 1, y^i = \dot{q}^i, \dot{x}^0 = 1, \dot{x}^i = \dot{q}^i, z^0 = 0, z^i = \ddot{q}^i),$$

so that the submanifold equations are now given by $y^0 = \dot{x}^0 = 1$, $y^i = \dot{x}^i$, $z^0 = 0$. Expressing the Finsler-Kawaguchi form in these coordinates gives,

$$\mathcal{K} = \frac{\partial K}{\partial y^0} dx^0 + \frac{\partial K}{\partial y^i} dx^i + 2\frac{\partial K}{\partial z^0} dy^0 + 2\frac{\partial K}{\partial z^i} dy^i, \tag{4.2.19}$$

and using the homogeneity conditions:

$$\begin{aligned} \frac{\partial K}{\partial y^0} &= \left(K - \frac{\partial K}{\partial y^i} y^i - 2\frac{\partial K}{\partial z^0} z^0 - 2\frac{\partial K}{\partial z^i} z^i \right) \frac{1}{y^0}, \\ \frac{\partial K}{\partial z^0} &= -\frac{\partial K}{\partial z^i} z^i \frac{1}{y^0}, \end{aligned} \tag{4.2.20}$$

becomes

$$\mathcal{K} = \left(K - \frac{\partial K}{\partial y^i} y^i - 2\frac{\partial K}{\partial z^0} z^0 - 2\frac{\partial K}{\partial z^i} z^i \right) \frac{1}{y^0} dx^0 + \frac{\partial K}{\partial y^i} dx^i - \frac{\partial K}{\partial z^i} z^i \frac{1}{y^0} dy^0 + 2\frac{\partial K}{\partial z^i} dy^i. \tag{4.2.21}$$

The pull-back of this Finsler-Kawaguchi form by ι is

$$\iota^* \mathcal{K} = \frac{1}{y^0} \circ \iota \left(K \circ \iota - \frac{\partial K}{\partial y^i} \circ \iota \cdot y^i \circ \iota - 2\frac{\partial K}{\partial z^0} \circ \iota \cdot z^0 \circ \iota - 2\frac{\partial K}{\partial z^i} \circ \iota \cdot z^i \circ \iota \right) d(x^0 \circ \iota)$$

$$+\frac{\partial K}{\partial y^i}\circ\iota dx^i\circ\iota-\frac{1}{y^0}\circ\iota\cdot\frac{\partial K}{\partial z^i}\circ\iota\cdot z^i\circ\iota\, d(y^0\circ\iota)+2\frac{\partial K}{\partial z^i}\circ\iota\, d(y^i\circ\iota). \tag{4.2.22}$$

By the submanifold equations, this becomes

$$\begin{aligned}\Theta_K &:= \iota^*\mathcal{K}=\left(\mathcal{L}-\frac{\partial\mathcal{L}}{\partial\dot{q}^i}\dot{q}^i-2\frac{\partial\mathcal{L}}{\partial\ddot{q}^i}\ddot{q}^i\right)dt+\frac{\partial\mathcal{L}}{\partial\dot{q}^i}dq^i+2\frac{\partial\mathcal{L}}{\partial\ddot{q}^i}d\dot{q}^i\\ &=\mathcal{L}dt+\frac{\partial\mathcal{L}}{\partial\dot{q}^i}\omega^i+2\frac{\partial\mathcal{L}}{\partial\ddot{q}^i}\dot{\omega}^i, \end{aligned} \tag{4.2.23}$$

where we set $\mathcal{L}:=K\circ\iota$, and

$$\omega^i:=dq^i-\dot{q}^i dt,\ \dot{\omega}^i:=d\dot{q}^i-\ddot{q}^i dt. \tag{4.2.24}$$

The 1-forms $\omega^i,\ \dot{\omega}^i$ are called contact forms, and they disappear when pulled back to the base manifold of the bundle $(Y,\pi,\mathbb{R})$. Θ_K is the second order form which corresponds to the Cartan form in our context.

4.3 First order, k-dimensional parameter space

Here in this section we will consider the second direction of generalisation of Finsler geometry, to k-dimensional parameter space. We will begin with the first order case. For the construction, we will utilise the structure of multivector bundles, introduced in chapter 2 Section 2.1.3.

4.3.1 Basic definitions of Kawaguchi geometry (first order k-multivector bundle)

We will first define the geometric structure on the total space of a k-multivector bundle $(\Lambda^k TM,\Lambda^k\tau_M,M)$. We will call this structure a first order k-areal Kawaguchi function.

Definition 4.10. Kawaguchi manifold (First order k-dimensional parameter space)
Let (M,K) be a pair of n-dimensional C^∞-differentiable manifold M and a function $K\in C^\infty(\Lambda^k TM)$ with $k\leqslant n$ that satisfies the following homogeneity condition,

$$K(\lambda v)=\lambda K(v),\quad \lambda>0,\ \text{for } v\in\Lambda^k TM. \tag{4.3.1}$$

We will call the function with such properties, a *first order k-areal Kawaguchi function*, and the pair (M,K) a *n-dimensional k-areal Kawaguchi manifold*, or simply *Kawaguchi*

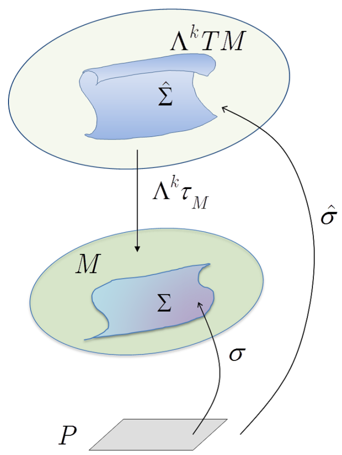

Figure 4.2: lift of parameterisation for Kawaguchi area

manifold, if the subject of discussion is clear.

Let $(V, \psi), \psi = (x^\mu, y^{\mu_1 \cdots \mu_k})$, $\mu, \mu_1, \cdots, \mu_k = 1, \cdots, n$ be a chart on $\Lambda^k TM$, then the local expression of the above condition can be written as

$$K(x^\mu, \lambda y^{\mu_1 \cdots \mu_k}) = \lambda K(x^\mu, y^{\mu_1 \cdots \mu_k}), \quad \lambda > 0. \tag{4.3.2}$$

The condition (4.3.2) implies the following,

$$\frac{1}{k!} \frac{\partial K}{\partial y^{\mu_1 \cdots \mu_k}} y^{\mu_1 \cdots \mu_k} = K, \tag{4.3.3}$$

Which corresponds to the Euler's homogeneous theorem.

4.3.2 Parameterisation invariant k-area of Kawaguchi geometry

In this section we will define the object k-curves, k-patches, their parameterisations and Kawaguchi area. Kawaguchi area is invariant with respect to the reparameterisation we describe in the following, and the notion is naturally extended to k-dimensional immersed submanifolds. Similarly as in the case of Finsler length and Finsler-Kawaguchi length, the reparameterisation invariance of Kawaguchi area is due to the homogeneity of k-areal Kawaguchi function.

Definition 4.11. C^r-k-curves
Let M be a smooth manifold, and $\sigma : P \to M$ a C^r-mapping, where P is an open

rectangle of $\mathbb{R}^k$. We denote the image of P by Σ; $\Sigma := \sigma(P) \subset M$, and call Σ, the C^r-*k-curve* on M.

Definition 4.12. Lift of C^r-k-curves
Consider a k-multivector bundle $(\Lambda^k TM, \Lambda^k \tau_M, M)$ where $\Lambda^k \tau_M$ is the natural projection, and (V, ψ), $\psi = (x^\mu, y^{\mu_1 \cdots \mu_k})$, $\mu, \mu_1, \cdots, \mu_k = 1, \cdots, n$ be the induced chart on $\Lambda^k TM$, and $(t^1, t^2, \cdots, t^k)$ the global chart of $\mathbb{R}^k$. Let $\sigma : P \to M$ be a C^r-mapping where P is an open rectangle of $\mathbb{R}^k$. Consider a map $\hat{\sigma} : P \to \Lambda^k TM$, such that its image is denoted by $\hat{\Sigma} := \hat{\sigma}(P) \subset \Lambda^k TM$, where $\Lambda^k \tau_M(\hat{\Sigma}) = \Sigma$, and its coordinate expressions given by

$$\hat{\sigma}(t) = \left.\frac{\partial(x^{\mu_1} \circ \sigma)}{\partial t^1}\right|_t \cdots \left.\frac{\partial(x^{\mu_k} \circ \sigma)}{\partial t^k}\right|_t \left(\frac{\partial}{\partial x^{\mu_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_k}}\right)_{\sigma(t)}, \tag{4.3.4}$$

for $t \in P$. The map $\hat{\sigma}$ and its image $\hat{\Sigma}$ is called the *lift* or the *multi-tangent lift* of σ (resp. Σ).

In coordinate charts, (4.3.4) is expressed as

$$(x^\mu \circ \hat{\sigma})(t) = (x^\mu \circ \sigma)(t), (y^{\mu_1 \cdots \mu_k} \circ \hat{\sigma})(t) = \left.\frac{\partial(x^{[\mu_1} \circ \sigma)}{\partial t^1}\right|_t \cdots \left.\frac{\partial(x^{\mu_k]} \circ \sigma)}{\partial t^k}\right|_t. \tag{4.3.5}$$

Definition 4.13. Regularity of σ
The C^r-map σ is called *regular*, if its lift $\hat{\sigma}$ is nowhere 0.

Definition 4.14. Parameterisation of an immersed curve
The C^r-map $\sigma : P \to M$is called an *immersion*, if its lift $\hat{\sigma}$ is injective, and the image Σ is called an *immersed k-curve*. The map σ is called a *parameterisation* of immersed k-curve C, and P is called a *parameter space*, when σ is an immersion, and preserves orientation.

Definition 4.15. Parameterisation
Let $\sigma : P \to M$ be a C^r-map, and $\Sigma = \sigma(P)$. The map σ is called a *parameterisation* of Σ, and P is called a *parameter space*, when σ is injective, and preserves orientation.

Definition 4.16. Lift of a parameterisation
We call $\hat{\sigma}$ the *lift of parameterisation* σ, when σ is a parameterisation.

Given a k-curve Σ on M, more than one map and open rectangle may exist, namely for cases such as $\Sigma = \sigma(P) = \rho(Q)$, where $\sigma,\ \rho$ are the maps, and P, Q are the open rectangles in $\mathbb{R}^k$. We can classify the k-curves by considering the properties of these maps.

Definition 4.17. Regular k-curve
Let Σ be a C^r-k-curve on M. Σ is called a *regular k-curve* on M, if there exists a regular C^r map σ and an open rectangle P of $\mathbb{R}^k$ such that $\sigma(P) = \Sigma$.

Definition 4.18. Parameterisable immersed k-curve
Let Σ be a C^r-k-curve on M. Σ is called a *parameterisable immersed k-curve* on M, if there exists an immersion σ and an open rectangle P of $\mathbb{R}^k$ such that $\sigma(P) = \Sigma$.

Definition 4.19. Parameterisable curve
Let Σ be a C^r-k-curve on M. Σ is called a *parameterisable k-curve* on M, if there exists an injective C^r-map σ and an open rectangle P of $\mathbb{R}^k$ such that $\sigma(P) = \Sigma$.

Occasionally, we implicitly refer to the pair (σ, P) by the parameterisation σ. In the following discussion, we will only consider regular, parameterisable k-curves.

Now we will introduce the concept of a k-dimensional area of a k-curve by integration on the parameter space. For simplicity, we will restrict ourselves to curves that are parameterisable, and consider its closed subset, which we define below.

Definition 4.20. k-patch
Let $\tilde{\Sigma}$ be a parameterisable k-curve on M with some parameterisation σ. A subset of $\tilde{\Sigma}$ given by $\Sigma := \sigma([t_i^1, t_f^1] \times [t_i^2, t_f^2] \times \cdots \times [t_i^k, t_f^k]) \subset \tilde{\Sigma}$, where $[t_i^1, t_f^1] \times [t_i^2, t_f^2] \times \cdots \times [t_i^k, t_f^k] \subset P$ is called the *k-patch* on M, σ is called the *parameterisation of the k-patch* and the closed rectangle $[t_i^1, t_f^1] \times [t_i^2, t_f^2] \times \cdots \times [t_i^k, t_f^k]$ is called the *parameter space of the k-patch*.

The k-areal Kawaguchi function defines a geometrical area for a k-patch Σ on M.

Definition 4.21. Kawaguchi k-area
Let (M, K) be the n-dimensional k-areal Kawaguchi manifold, and Σ the k-patch on M such that $\Sigma = \sigma([t_i^1, t_f^1] \times [t_i^2, t_f^2] \times \cdots \times [t_i^k, t_f^k])$. We assign to Σ the following integral

$$l^K[\Sigma] = \int_{t_i^1}^{t_f^1} dt^1 \int_{t_i^2}^{t_f^2} dt^2 \cdots \int_{t_i^k}^{t_f^k} dt^k K\left(\hat{\sigma}(t)\right). \tag{4.3.6}$$

We call this number $l^K(\Sigma)$ the *Kawaguchi area* or *Kawaguchi k-area* of Σ.

Let (V, ψ), $\psi = (x^\mu, y^{\mu_1 \cdots \mu_k})$, $\mu, \mu_1, \cdots, \mu_k = 1, \cdots, n$ be the induced chart on $\Lambda^k TM$. By chart expression, (4.3.6) is,

$$l^K[\Sigma] = \int_{t_i^1}^{t_f^1} dt^1 \int_{t_i^2}^{t_f^2} dt^2 \cdots \int_{t_i^k}^{t_f^k} dt^k K\left(x^\mu(\hat{\sigma}(t)), y^{\mu_1 \cdots \mu_k}(\hat{\sigma}(t))\right)$$

$$= \int_{t_i^1}^{t_f^1} dt^1 \int_{t_i^2}^{t_f^2} dt^2 \cdots \int_{t_i^k}^{t_f^k} dt^k K\left(x^\mu(\sigma(t)), \frac{\partial(x^{[\mu_1}(\sigma(t)))}{\partial t^1} \cdots \frac{\partial(x^{\mu_k]}(\sigma(t)))}{\partial t^k}\right), \quad (4.3.7)$$

where we used the definition of $\hat{\sigma}$, and definition of induced coordinates of $\Lambda^k TM$ (4.3.5).

Let $\rho : Q \to \Sigma$, $Q \subset \mathbb{R}^k$ be another parameterisation of Σ. When there exists a diffeomorphism $\phi : Q \to P$ such that $\rho = \sigma \circ \phi$, this gives an equivalence relation $\sigma \sim \rho$. As in the case of 1-dimensional parameter space, in Kawaguchi geometry of k-dimensional parameter space, the Kawaguchi area defined above has the important property of reparameterisation invariance.

Lemma 4.22. Reparameterisation invariance of Kawaguchi k-area The Kawaguchi k-area does not change by the reparameterisation $\rho = \sigma \circ \phi$, where $\phi : Q \to P$ is a diffeomorphism, and preserves the orientation.

Proof. Dividing the rectangle $\bar{P} := [t_i^1, t_f^1] \times [t_i^2, t_f^2] \times \cdots \times [t_i^k, t_f^k]$ if necessary into smaller closed sub-rectangles, we can suppose without loss of generality that the set $\Sigma = \sigma(\bar{P})$ lies in the coordinate neighbourhood of a chart (U, φ), $\varphi = (x^\mu)$. Then the lift of ρ becomes,

$$\begin{aligned}
\hat{\rho}(s) = & \left.\frac{\partial(x^{\mu_1} \circ \sigma \circ \phi)}{\partial s^1}\right|_s \cdots \left.\frac{\partial(x^{\mu_k} \circ \sigma \circ \phi)}{\partial s^k}\right|_s \left(\frac{\partial}{\partial x^{\mu_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_k}}\right)_{\sigma \circ \phi(s)} \\
= & \left.\frac{\partial(x^{\mu_1} \circ \sigma)}{\partial t^{a_1}}\right|_{\phi(s)} \cdots \left.\frac{\partial(x^{\mu_k} \circ \sigma)}{\partial t^{a_k}}\right|_{\phi(s)} \left.\frac{\partial(t^{a_1} \circ \phi)}{\partial s^1}\right|_s \cdots \left.\frac{\partial(t^{a_k} \circ \phi)}{\partial s^k}\right|_s \left(\frac{\partial}{\partial x^{\mu_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_k}}\right)_{\sigma \circ \phi(s)} \\
= & \, \varepsilon_{a_1 \cdots a_k} \left.\frac{\partial(t^{a_1} \circ \phi)}{\partial s^1}\right|_s \cdots \left.\frac{\partial(t^{a_k} \circ \phi)}{\partial s^k}\right|_s \hat{\sigma}(\phi(s)),
\end{aligned} \quad (4.3.8)$$

for $s \in Q$, $a_1, \cdots, a_k = 1, 2, \cdots, k$, and since ρ is a regular parameterisation that preserves orientation,

$$\varepsilon_{a_1 \cdots a_k} \left.\frac{\partial(t^{a_1} \circ \phi)}{\partial s^1}\right|_s \cdots \left.\frac{\partial(t^{a_k} \circ \phi)}{\partial s^k}\right|_s > 0. \quad (4.3.9)$$

The k-dimensional area of Σ is preserved by

$$\begin{aligned}
l^K[\Sigma] = & \int_{s_i^1}^{s_f^1} ds^1 \int_{s_i^2}^{s_f^2} ds^2 \cdots \int_{s_i^k}^{s_f^k} ds^k K\left(\hat{\rho}(s)\right) \\
= & \int_{s_i^1}^{s_f^1} ds^1 \cdots \int_{s_i^k}^{s_f^k} ds^k K\left(\varepsilon_{a_1 \cdots a_k} \left.\frac{\partial(t^{a_1} \circ \phi)}{\partial s^1}\right|_s \cdots \left.\frac{\partial(t^{a_k} \circ \phi)}{\partial s^k}\right|_s \hat{\sigma}(\phi(s))\right)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\phi^{-1}(s_i^1)}^{\phi^{-1}(s_f^1)} \cdots \int_{\phi^{-1}(s_i^k)}^{\phi^{-1}(s_f^k)} ds^1 \wedge \cdots \wedge ds^k \varepsilon_{a_1 \cdots a_k} \left.\frac{\partial (t^{a_1} \circ \phi)}{\partial s^1}\right|_s \cdots \left.\frac{\partial (t^{a_k} \circ \phi)}{\partial s^k}\right|_s K\left(\hat{\sigma}(\phi(s))\right) \\
&= \int_{t_i^1}^{t_f^1} \cdots \int_{t_i^k}^{t_f^k} dt^1 \wedge dt^2 \wedge \cdots \wedge dt^k K\left(\hat{\sigma}(t)\right) \\
&= \int_{t_i^1}^{t_f^1} dt^1 \cdots \int_{t_i^k}^{t_f^k} dt^k K\left(\hat{\sigma}(t)\right), \qquad (4.3.10)
\end{aligned}
$$

where $s_i^1, s_f^1, s_i^2, s_f^2, \cdots, s_i^k, s_f^k$ are the pre-image of the boundary points $t_i^1, t_f^1, t_i^2, t_f^2, \cdots, t_i^k, t_f^k$ by ϕ. For the third equality of (4.3.10), we have used the pulled back homogeneity condition of K, and the definition of integration of k-form in accord to Section 2.2. □

We conclude that the homogeneity of K and parameterisation invariance of Kawaguchi k-area is an equivalent property.

Remark 4.23. As similarly in the case of a curve, the "Kawaguchi area" does not have the properties of a "standard" area, considered by Euclid or Riemannian geometry, since we require only homogeneity condition of the Kawaguchi function. For instance, when one changes the orientation of the k-curve, in general, it gives different values (not just signatures). Nevertheless, in our following discussion of the calculus of variations, we can still use this concept to obtain extremals and equations of motion. It is especially designed to use for the applications for extensions of field theory, which unifies spacetime and fields. Such situation appears commonly in modern theoretical physics, and Kawaguchi area will be a good geometrical object to consider for constructing viable models.

4.3.3 Kawaguchi k-form

Given a n-dimensional k-areal Kawaguchi manifold (M, K), we can obtain an important geometrical structure, which we will call a *Kawaguchi k-form*. Kawaguchi k-form is constructed in accord with the homogeneity condition, and gives the Lagrangian of a field theory when pulled back to the parameter space, namely the spacetime, by a certain parameterisation.

Definition 4.24. Kawaguchi k-form (first order field theory)
Let (V, ψ), $\psi = (x^\mu, y^{\mu_1 \cdots \mu_k})$, $\mu, \mu_1, \cdots, \mu_k = 1, \cdots, n$ be the induced chart on $\Lambda^k TM$. The *Kawaguchi k-form* $\mathcal{K}$ is a k-form on $\Lambda^k TM$, which in local coordinates are expressed by

$$
\mathcal{K} = \frac{1}{k!} \frac{\partial K}{\partial y^{\mu_1 \cdots \mu_k}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}. \qquad (4.3.11)
$$

This expression corresponds to the homogeneity condition (4.3.3).

Proposition 4.25. The Kawaguchi form is invariant with respect to the coordinate transformations.

Proof. Let $\left(\bar{V},\bar{\psi}\right), \bar{\psi} = (\bar{x}^{\mu}, \bar{y}^{\mu_1\cdots\mu_k})$ be another chart on $\Lambda^k TM$ with intersection $V \cap \bar{V} \neq \emptyset$. Then by the coordinate transformation

$$\begin{aligned} &x^{\mu} \to \bar{x}^{\mu} = \bar{x}^{\mu}(x^{\nu}), \\ &y^{\mu_1\cdots\mu_k} \to \bar{y}^{\mu_1\cdots\mu_k}(x^{\mu}, y^{\mu_1\cdots\mu_k}) = \frac{\partial \bar{x}^{\mu_1}}{\partial x^{\nu_1}}\cdots\frac{\partial \bar{x}^{\mu_k}}{\partial x^{\nu_k}} y^{\nu_1\cdots\nu_k}, \end{aligned} \tag{4.3.12}$$

we have

$$\begin{aligned} \mathcal{K} &= \frac{1}{k!}\frac{\partial K}{\partial y^{\mu_1\cdots\mu_k}} dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_k} = \frac{1}{k!}\frac{\partial K}{\partial \bar{y}^{\nu_1\cdots\nu_k}}\frac{\partial \bar{y}^{\nu_1\cdots\nu_k}}{\partial y^{\mu_1\cdots\mu_k}}\frac{\partial x^{\mu_1}}{\partial \bar{x}^{\nu_1}}\cdots\frac{\partial x^{\mu_k}}{\partial \bar{x}^{\nu_k}} d\bar{x}^{\mu_1}\wedge\cdots\wedge d\bar{x}^{\mu_k} \\ &= \frac{1}{k!}\frac{\partial K}{\partial \bar{y}^{\nu_1\cdots\nu_k}}\frac{\partial \bar{x}^{\nu_1}}{\partial x^{[\mu_1}}\cdots\frac{\partial \bar{x}^{\nu_k}}{\partial x^{\mu_k]}}\frac{\partial x^{\mu_1}}{\partial \bar{x}^{\nu_1}}\cdots\frac{\partial x^{\mu_k}}{\partial \bar{x}^{\nu_k}} d\bar{x}^{\mu_1}\wedge\cdots\wedge d\bar{x}^{\mu_k} \\ &= \frac{1}{k!}\frac{\partial K}{\partial \bar{y}^{\mu_1\cdots\mu_k}} d\bar{x}^{\mu_1}\wedge\cdots\wedge d\bar{x}^{\mu_k}. \end{aligned} \tag{4.3.13}$$

□

Proposition 4.26. Let $\mathcal{K}$ be the Kawaguchi k-form on $\Lambda^k TM$, $\Sigma = \sigma(\bar{P})$ the k-patch on M, with $\bar{P} = [t_i^1, t_f^1] \times [t_i^2, t_f^2] \times \cdots \times [t_i^k, t_f^k]$ a closed rectangle in $\mathbb{R}^k$. Then,

$$\int_{\hat{\Sigma}} \mathcal{K} = l^K(\Sigma). \tag{4.3.14}$$

Proof. The simple calculation leads,

$$\begin{aligned} \int_{\hat{\Sigma}} \mathcal{K} &= \int_{\hat{\sigma}(\bar{P})} \frac{1}{k!}\frac{\partial K}{\partial y^{\mu_1\cdots\mu_k}} dx^{\mu_1\cdots\mu_k} = \int_{t_i^1}^{t_f^1}\cdots\int_{t_i^k}^{t_f^k} \frac{1}{k!}\frac{\partial K}{\partial y^{\mu_1\cdots\mu_k}}\circ\hat{\sigma}\, d(x^{\mu_1}\circ\hat{\sigma})\wedge\cdots\wedge d(x^{\mu_k}\circ\hat{\sigma}) \\ &= \int_{t_i^1}^{t_f^1}\cdots\int_{t_i^k}^{t_f^k} \frac{1}{k!}\frac{\partial K}{\partial y^{\mu_1\cdots\mu_k}}\left(x^{\mu}(\sigma(t)), \left.\frac{\partial(x^{\mu_1}\circ\sigma)}{\partial t^1}\right|_t \cdots \left.\frac{\partial(x^{\mu_k}\circ\sigma)}{\partial t^k}\right|_t\right) \\ &\qquad\qquad\qquad\qquad\times \frac{\partial(x^{\mu_1}\circ\sigma)}{\partial t^1}\cdots\frac{\partial(x^{\mu_k}\circ\sigma)}{\partial t^k} dt^1\wedge\cdots\wedge dt^k \\ &= \int_{t_i^1}^{t_f^1}\cdots\int_{t_i^k}^{t_f^k} \frac{1}{k!}\frac{\partial K}{\partial y^{\mu_1\cdots\mu_k}}\left(x^{\mu}(\sigma(t)), y^{\mu_1\cdots\mu_k}(\hat{\sigma}(t))\right) y^{\mu_1\cdots\mu_k}(\hat{\sigma}(t))\, dt^1\wedge\cdots\wedge dt^k \\ &= \int_{t_i^1}^{t_f^1} dt^1\cdots\int_{t_i^k}^{t_f^k} dt^k K\left(x^{\mu}(\sigma(t)), \frac{\partial(x^{[\mu_1}(\sigma(t))}{\partial t^1}\cdots\frac{\partial(x^{\mu_k]}(\sigma(t))}{\partial t^k}\right) = l^K(\Sigma) \end{aligned}$$

(4.3.15)

where we used the pull-back homogeneity condition

$$\frac{1}{k!}\frac{\partial K}{\partial y^{\mu_1\cdots\mu_k}}\circ\hat{\sigma}\cdot y^{\mu_1\cdots\mu_k}\circ\hat{\sigma} = K\circ\hat{\sigma}. \tag{4.3.16}$$

$\square$

Remark 4.27. This lemma extends the notion of Kawaguchi area given by (4.3.6). Namely, since the Kawaguchi form can be integrated over *any* k-dimensional submanifold of M, the identity (4.3.14) suggests to extend the integration over k-patches to *arbitrary* k-dimensional submanifold of M.

Remark 4.28. Now that we showed that Kawaguchi k-form gives the Kawaguchi k-area (and in a more general situation of a submanifold), we redefine the pair $(M,\mathcal{K})$ as the n-dimensional k-areal Kawaguchi manifold instead of the pair (M,K). This is a more geometrical definition of a Kawaguchi manifold, similar as in the case of Finsler geometry.

4.4 Second order, k-dimensional parameter space

Here in this section, combining the previous two directions of generalisation, we will consider the case of second order, k-dimensional parameter space. However, unlike the previous discussions, we will consider only the case of $M=\mathbb{R}^n$, and leave the global construction for the future work. In this section, $M=\mathbb{R}^n$ is assumed.

4.4.1 Basic definitions of Kawaguchi space (second order k-multivector bundle)

We will first define the geometric structure on the total space of a second order k-multivector bundle $((\Lambda^kT)^2M,\Lambda^k\tau_M^{2,0},M)$ with $\Lambda^k\tau_M^{2.0}=\Lambda^k\tau_M\circ\Lambda^k\tau_M^{2,1}$, $\Lambda^k\tau_M^{2,1}:=\Lambda^k\tau_{\Lambda^kTM}\big|_{(\Lambda^kT)^2M}$. We will call this structure a second order k-areal Kawaguchi function. For visibility, the multi-index notation (see Section 2.1.4) are used.

Definition 4.29. Kawaguchi space (Second order k-dimensional parameter space) Let (M,K) be a pair of n-dimensional Cartesian space $M=\mathbb{R}^n$ and a function $K\in C^\infty((\Lambda^kT)^2M)$, $k\leqslant n$, which for the induced global chart $\varphi^2=(x^\mu,y^{\mu_1\cdots\mu_k},z^{I_1\nu_2\cdots\nu_k},$ $z^{I_1I_2\nu_3\cdots\nu_k},\cdots,z^{I_1I_2\cdots I_k})$, $\mu,\mu_1,\cdots,\mu_k,\nu_2,\cdots,\nu_k=1,\cdots,n$, $I_j:=\mu_j^{i_1}\cdots\mu_j^{i_k}$, on $(\Lambda^kT)^2M$ satisfies the following *second order homogeneity condition*,

$$
\begin{aligned}
&K\left(x^{\mu},\lambda y^{\mu_1\cdots\mu_k},(\lambda)^2 z^{I_1\nu_2\cdots\nu_k}+\lambda^{\nu_2\cdots\nu_k}y^{I_1},(\lambda)^2 z^{I_1I_2\nu_3\cdots\nu_k}+\lambda^{\nu_3\cdots\nu_k}y^{I_1}y^{I_2},\right.\\
&\qquad\qquad\qquad\qquad\left.\dots,(\lambda)^2 z^{I_1I_2\cdots I_k}+\lambda^0 y^{I_1}y^{I_2}\cdots y^{I_k}\right)\\
&\quad=\lambda K(x^{\mu},y^{\mu_1\cdots\mu_k},z^{I_1\nu_2\cdots\nu_k},z^{I_1I_2\nu_2\cdots\nu_k},\cdots,z^{I_1I_2\cdots I_k}),
\end{aligned}
\tag{4.4.1}
$$

for $\lambda > 0$, and $\lambda^{\nu_2\cdots\nu_k},\lambda^{\nu_3\cdots\nu_k},\cdots,\lambda^{\nu_k},\lambda^0$ being arbitrary constants. We will call the function with such properties, a *second order k-areal Kawaguchi function* on $M=\mathbb{R}^n$, and the pair (M,K) a *second order n-dimensional k-areal Kawaguchi space*, or simply *second order Kawaguchi space*, if the subject of discussion is clear.

As in the case of second order Finsler-Kawaguchi geometry, $(\Lambda^k T)^2 M$ is not a vector space. Therefore, we have only second order homogeneity conditions in chart expressions. This condition (4.4.1) implies the following conditions,

$$
\begin{cases}
\dfrac{\partial K}{\partial y^{I_1}}y^{I_1}+\dfrac{2}{(k-1)!}\dfrac{\partial K}{\partial z^{I_1\nu_2\cdots\nu_k}}z^{I_1\nu_2\cdots\nu_k}+\dfrac{2}{2!(k-2)!}\dfrac{\partial K}{\partial z^{I_1I_2\nu_3\cdots\nu_k}}z^{I_1I_2\nu_3\cdots\nu_k}+\\
\quad\cdots+\dfrac{2}{k!}\dfrac{\partial K}{\partial z^{I_1I_2\cdots I_k}}z^{I_1I_2\cdots I_k}=K,\\
\dfrac{\partial K}{\partial z^{I_1\nu_2\cdots\nu_k}}y^{I_1}=0,\\
\dfrac{\partial K}{\partial z^{I_1I_2\nu_3\cdots\nu_k}}y^{I_1}y^{I_2}=\cdots=\dfrac{\partial K}{\partial z^{I_1I_2\cdots I_k}}y^{I_1}y^{I_2}\cdots y^{I_k}=0.
\end{cases}
\tag{4.4.2}
$$

The last row is simply an identity, by the symmetric and anti-symmetric property of indices. However, this expression is not coordinate invariant, and therefore, we cannot deduce the corresponding Kawaguchi k-form of the second order by just these expressions. In this text we will only consider the case of $M=\mathbb{R}^n$.

4.4.2 Parameterisation invariant k-area of second order Kawaguchi geometry

In this section we will define the k-dimensional area, which we will call the Kawaguchi area of second order. This area is invariant with respect to reparameterisation. Similarly as in the previous cases, this is due to the homogeneity of second order k-areal Kawaguchi function. We will begin by introducing the second order lift of parameterisation. In this section, $M=\mathbb{R}^n$ is assumed.

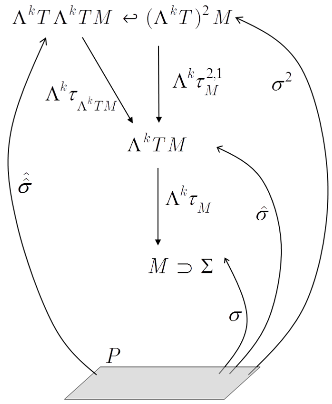

Figure 4.3: lift of parameterisation for Kawaguchi area

Definition 4.30. Second order lift of parameterisation
Consider a second order k-multivector bundle $((\Lambda^kT)^2M, \Lambda^k\tau_M^{2,0}, M)$ defined in Section 2.1.4, where $M = \mathbb{R}^n$ and the induced global chart $\varphi^2 = (x^\mu, y^{\mu_1\cdots\mu_k}, z^{I_1\nu_2\cdots\nu_k}, z^{I_1I_2\nu_3\cdots\nu_k}, \cdots, z^{I_1I_2\cdots I_k})$, $\mu, \mu_1, \cdots, \mu_k, \nu_2, \cdots, \nu_k = 1, \cdots, n$, $I_j := \mu_j^{i_1}\cdots\mu_j^{i_k}$, on $(\Lambda^kT)^2M$. Let σ be a parameterisation of Σ, namely $\Sigma = \sigma(P)$ defined in Section 4.3.2, where P is an open k-rectangle. We call the map $\sigma^2 : P \to \Sigma^2 \subset (\Lambda^kT)^2M$, such that its coordinate expression is given by

$$
\begin{aligned}
\sigma^2(t) = {} & \left.\frac{\partial(x^{\mu_1}\circ\sigma)}{\partial t^1}\right|_t \cdots \left.\frac{\partial(x^{\mu_k}\circ\sigma)}{\partial t^k}\right|_t \left(\frac{\partial}{\partial x^{\mu_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{\mu_k}}\right)_{\hat{\sigma}(t)} \\
& + \varepsilon^{a_1\cdots a_k}\left.\frac{\partial(y^{I_1}\circ\hat{\sigma})}{\partial t^{a_1}}\right|_t \left.\frac{\partial(x^{\mu_2}\circ\sigma)}{\partial t^{a_2}}\right|_t \cdots \left.\frac{\partial(x^{\mu_k}\circ\sigma)}{\partial t^{a_k}}\right|_t \left(\frac{\partial}{\partial y^{I_1}}\wedge\frac{\partial}{\partial x^{\mu_2}}\wedge\cdots\wedge\frac{\partial}{\partial x^{\mu_k}}\right)_{\hat{\sigma}(t)} \\
& + \varepsilon^{a_1\cdots a_k}\left.\frac{\partial(y^{I_1}\circ\hat{\sigma})}{\partial t^{a_1}}\right|_t \left.\frac{\partial(y^{I_2}\circ\hat{\sigma})}{\partial t^{a_2}}\right|_t \left.\frac{\partial(x^{\mu_3}\circ\sigma)}{\partial t^{a_3}}\right|_t \cdots \left.\frac{\partial(x^{\mu_k}\circ\sigma)}{\partial t^{a_k}}\right|_t \left(\frac{\partial}{\partial y^{I_1}}\wedge\frac{\partial}{\partial y^{I_2}}\wedge\frac{\partial}{\partial x^{\mu_3}}\wedge\cdots\wedge\frac{\partial}{\partial x^{\mu_k}}\right)_{\hat{\sigma}(t)} \\
& + \cdots + \varepsilon^{a_1\cdots a_k}\left.\frac{\partial(y^{I_1}\circ\hat{\sigma})}{\partial t^{a_1}}\right|_t \left.\frac{\partial(y^{I_2}\circ\hat{\sigma})}{\partial t^{a_2}}\right|_t \cdots \left.\frac{\partial(y^{I_k}\circ\hat{\sigma})}{\partial t^{a_k}}\right|_t \left(\frac{\partial}{\partial y^{I_1}}\wedge\frac{\partial}{\partial y^{I_2}}\wedge\cdots\wedge\frac{\partial}{\partial y^{I_k}}\right)_{\hat{\sigma}(t)}, \\
\frac{\partial(y^{I_j}\circ\hat{\sigma})}{\partial t^a} = {} & \frac{\partial}{\partial t^a}\left(\frac{\partial(x^{[\mu_1^j}\circ\sigma)}{\partial t^1}\cdots\frac{\partial(x^{\mu_k^j]}\circ\sigma)}{\partial t^k}\right), \quad a, a_1, \cdots, a_k, b_1, \cdots, b_k = 1, \cdots, k,
\end{aligned}
\tag{4.4.3}
$$

the *second order lift of parameterisation* σ to $(\Lambda^kT)^2M$. The image $\Sigma^2 = \sigma^2(P)$ is called the *second order lift of* Σ.

The above second order lift of parameterisation σ is constructed similarly as in the case of 1-dimensional parameter space (Section 4.2.2), by considering the subset of iterated tangent lift. Namely, we first construct the tangent lift $\hat{\hat{\sigma}} : P \to \Lambda^k T \Lambda^k TM$ of the parameterisation $\hat{\sigma} : P \to \Lambda^k TM$, and then take its subset by $\sigma^2 := \{\hat{\hat{\sigma}} | \Lambda^k T_{\hat{\sigma}(t)} \Lambda^k \tau_M(\hat{\hat{\sigma}}(t)) = \Lambda^k \tau_{TM}(\hat{\hat{\sigma}}(t)), t \in P\}$. The iterated tangent lift $\hat{\hat{\sigma}}(t)$ has the coordinate expressions,

$$
\begin{aligned}
\hat{\hat{\sigma}}(t) = {} & \left.\frac{\partial(x^{\mu_1}\circ\hat{\sigma})}{\partial t^1}\right|_t \cdots \left.\frac{\partial(x^{\mu_k}\circ\hat{\sigma})}{\partial t^k}\right|_t \left(\frac{\partial}{\partial x^{\mu_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{\mu_k}}\right)_{\hat{\sigma}(t)} \\
& + \varepsilon^{a_1\cdots a_k} \left.\frac{\partial(y^{I_1}\circ\hat{\sigma})}{\partial t^{a_1}}\right|_t \left.\frac{\partial(x^{\mu_2}\circ\hat{\sigma})}{\partial t^{a_2}}\right|_t \cdots \left.\frac{\partial(x^{\mu_k}\circ\hat{\sigma})}{\partial t^{a_k}}\right|_t \left(\frac{\partial}{\partial y^{I_1}}\wedge\frac{\partial}{\partial x^{\mu_2}}\wedge\cdots\wedge\frac{\partial}{\partial x^{\mu_k}}\right)_{\hat{\sigma}(t)} \\
& + \varepsilon^{a_1\cdots a_k} \left.\frac{\partial(y^{I_1}\circ\hat{\sigma})}{\partial t^{a_1}}\right|_t \left.\frac{\partial(y^{I_2}\circ\hat{\sigma})}{\partial t^{a_2}}\right|_t \left.\frac{\partial(x^{\mu_3}\circ\hat{\sigma})}{\partial t^{a_3}}\right|_t \cdots \left.\frac{\partial(x^{\mu_k}\circ\hat{\sigma})}{\partial t^{a_k}}\right|_t \left(\frac{\partial}{\partial y^{I_1}}\wedge\frac{\partial}{\partial y^{I_2}}\wedge\frac{\partial}{\partial x^{\mu_3}}\wedge\cdots\wedge\frac{\partial}{\partial x^{\mu_k}}\right)_{\hat{\sigma}(t)} \\
& + \cdots + \varepsilon^{a_1\cdots a_k} \left.\frac{\partial(y^{I_1}\circ\hat{\sigma})}{\partial t^{a_1}}\right|_t \left.\frac{\partial(y^{I_2}\circ\hat{\sigma})}{\partial t^{a_2}}\right|_t \cdots \left.\frac{\partial(y^{I_k}\circ\hat{\sigma})}{\partial t^{a_k}}\right|_t \left(\frac{\partial}{\partial y^{I_1}}\wedge\frac{\partial}{\partial y^{I_2}}\wedge\cdots\wedge\frac{\partial}{\partial y^{I_k}}\right)_{\hat{\sigma}(t)},
\end{aligned}
\tag{4.4.4}
$$

and the condition for $\hat{\hat{\sigma}}(t)$ to be in $(\Lambda^k T)^2 M$ will give us the coordinates of $\sigma^2(t)$,

$$
\begin{aligned}
& (x^{\mu}\circ\sigma^2)(t) = (x^{\mu}\circ\hat{\sigma})(t) = (x^{\mu}\circ\sigma)(t), \\
& (y^{\mu_1\cdots\mu_k}\circ\sigma^2)(t) = \left.\frac{\partial(x^{[\mu_1}\circ\hat{\sigma})}{\partial t^1}\right|_t \cdots \left.\frac{\partial(x^{\mu_k]}\circ\hat{\sigma})}{\partial t^k}\right|_t = \left.\frac{\partial(x^{[\mu_1}\circ\sigma)}{\partial t^1}\right|_t \cdots \left.\frac{\partial(x^{\mu_k]}\circ\sigma)}{\partial t^k}\right|_t \\
& \quad = (y^{\mu_1\cdots\mu_k}\circ\hat{\sigma})(t), \\
& (z^{I_1\nu_2\cdots\nu_k}\circ\sigma^2)(t) = \varepsilon^{a_1\cdots a_k} \left.\frac{\partial(y^{I_1}\circ\hat{\sigma})}{\partial t^{a_1}}\right|_t \left.\frac{\partial(x^{[\nu_2}\circ\sigma)}{\partial t^{a_2}}\right|_t \cdots \left.\frac{\partial(x^{\nu_k]}\circ\sigma)}{\partial t^{a_k}}\right|_t, \\
& \quad = \varepsilon^{a_1\cdots a_k} \frac{\partial}{\partial t^{a_1}} \left(\frac{\partial(x^{[\mu_i^1}\circ\sigma)}{\partial t^1}\cdots\frac{\partial(x^{\mu_k^1]}\circ\sigma)}{\partial t^k}\right)_t \left.\frac{\partial(x^{[\nu_2}\circ\sigma)}{\partial t^{a_2}}\right|_t \cdots \left.\frac{\partial(x^{\nu_k]}\circ\sigma)}{\partial t^{a_k}}\right|_t, \\
& (z^{I_1 I_2\nu_3\cdots\nu_k}\circ\sigma^2)(t) \\
& \quad = \varepsilon^{a_1\cdots a_k} \frac{\partial}{\partial t^{a_1}} \left(\frac{\partial(x^{[\mu_i^1}\circ\sigma)}{\partial t^1}\cdots\frac{\partial(x^{\mu_k^1]}\circ\sigma)}{\partial t^k}\right)_t \\
& \qquad\qquad \times \frac{\partial}{\partial t^{a_2}} \left(\frac{\partial(x^{[\mu_i^2}\circ\sigma)}{\partial t^1}\cdots\frac{\partial(x^{\mu_k^2]}\circ\sigma)}{\partial t^k}\right)_t \left.\frac{\partial(x^{[\nu_3}\circ\sigma)}{\partial t^{a_3}}\right|_t \cdots \left.\frac{\partial(x^{\nu_k]}\circ\sigma)}{\partial t^{a_k}}\right|_t, \\
& \vdots \\
& (z^{I_1 I_2\cdots I_k}\circ\sigma^2)(t) = \varepsilon^{a_1\cdots a_k} \frac{\partial}{\partial t^{a_1}} \left(\frac{\partial(x^{[\mu_i^1}\circ\sigma)}{\partial t^1}\cdots\frac{\partial(x^{\mu_k^1]}\circ\sigma)}{\partial t^k}\right)_t \cdots \frac{\partial}{\partial t^{a_k}} \left(\frac{\partial(x^{[\mu_i^k}\circ\sigma)}{\partial t^1}\cdots\frac{\partial(x^{\mu_k^k]}\circ\sigma)}{\partial t^k}\right)_t,
\end{aligned}
\tag{4.4.5}
$$

giving the formula (4.4.3).

The parameterisation σ where its second order lift σ^2 is nowhere 0 is called a *regular parameterisation of order* 2. Apparently, if σ^2 is nowhere 0, also the tangent lift $\hat{\sigma}$ is nowhere 0. In the discussions concerning second order k-dimensional parameter space Kawaguchi geometry, we will only consider the regular parameterisation.

The r-th order parameterisation $\sigma^r : P \to (\Lambda^k T)^r M$ can be obtained by iterative process. Namely, construct the lift $\widehat{(\sigma)^{r-1}} : P \to \Lambda^k T((\Lambda^k T)^{r-1} M)$ of the parameterisation $\sigma^{r-1} : P \to (\Lambda^k T)^{r-1} M$, and then regarding the construction on the higher-order multi tangent bundle (2.1.88), take its subset by

$$\sigma^r := \{\widehat{(\sigma)^{r-1}} | \, \Lambda^k T_{\sigma^{r-1}(t)} \tau^{r-1,r-2}_{\Lambda^k TM}(\widehat{(\sigma)^{r-1}}(t)) = \iota_{r-1} \circ \tau_{(\Lambda^k T)^{r-1}M}(\widehat{(\sigma)^{r-1}}(t)), \, t \in P\}. \tag{4.4.6}$$

Definition 4.31. r-th order parameterisation
Let σ be a parameterisation of the k-curve Σ on M. The map $\sigma^r : P \to (\Lambda^k T)^r M$ given by (4.4.6) is called the *r-th order lift of parameterisation σ* .

The second order k-areal Kawaguchi function defines a geometrical area for a k-patch Σ on M.

Definition 4.32. Kawaguchi k-area (second order)
Let (M, K), $M = \mathbb{R}^n$ be the second order n-dimensional k-areal Kawaguchi space, and Σ the k-patch on M such that $\Sigma = \sigma(\bar{P})$, $\bar{P} = [t_i^1, t_f^1] \times [t_i^2, t_f^2] \times \cdots \times [t_i^k, t_f^k]$. We assign to Σ the following integral

$$l^K(\Sigma) = \int_{t_i^1}^{t_f^1} dt^1 \int_{t_i^2}^{t_f^2} dt^2 \cdots \int_{t_i^k}^{t_f^k} dt^k K\left(\sigma^2(t)\right), \qquad t \in R \tag{4.4.7}$$

We call this number $l^K(\Sigma)$ the (second order) *Kawaguchi area* or *Kawaguchi k-area* of Σ.

Let $\varphi^2 = (x^\mu, y^{\mu_1 \cdots \mu_k}, z^{I_1 \nu_2 \cdots \nu_k}, z^{I_1 I_2 \nu_3 \cdots \nu_k}, \cdots, z^{I_1 I_2 \cdots I_k})$, $\mu, \mu_1, \cdots, \mu_k, \nu_2, \cdots, \nu_k = 1, \cdots, n$, $I_j := \mu_j^{i_1} \cdots \mu_j^{i_k}$, be the induced global chart on $(\Lambda^k T)^2 M$. Then by chart expression, this is,

$$l^K(\Sigma) = \int_{t_i^1}^{t_f^1} dt^1 \cdots \int_{t_i^k}^{t_f^k} dt^k K\big(x^\mu(\sigma^2(t)), y^{\mu_1 \cdots \mu_k}(\sigma^2(t)), z^{I_1 \nu_2 \cdots \nu_k}(\sigma^2(t)), \cdots, z^{I_1 \cdots I_k}(\sigma^2(t))\big) \tag{4.4.8}$$

with the components given by (4.4.5).

Also for the second order, the Kawaguchi k-area defined above is reparameterisation invariant.

Lemma 4.33. Reparameterisation invariance of Kawaguchi k-area
The second order Kawaguchi k-area does not change by the reparameterisation $\rho = \sigma\circ\phi$, where $\phi : Q \to P$ is a diffeomorphism, and preserves the orientation.

Proof. By (4.4.3), the second order lift of $\rho = \sigma\circ\phi$ is,

$$\begin{aligned}
\rho^2(s) = {} & \left.\frac{\partial(x^{\mu_1}\circ\sigma\circ\phi)}{\partial s^1}\right|_s \cdots \left.\frac{\partial(x^{\mu_k}\circ\sigma\circ\phi)}{\partial s^k}\right|_s \left(\frac{\partial}{\partial x^{\mu_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{\mu_k}}\right)_{\widehat{\sigma\circ\phi}(s)} \\
& + \varepsilon^{a_1\cdots a_k}\left.\frac{\partial(y^{I_1}\circ\widehat{\sigma\circ\phi})}{\partial s^{a_1}}\right|_s \left.\frac{\partial(x^{\mu_2}\circ\sigma\circ\phi)}{\partial s^{a_2}}\right|_s \cdots \left.\frac{\partial(x^{\mu_k}\circ\sigma\circ\phi)}{\partial s^{a_k}}\right|_s \left(\frac{\partial}{\partial y^{I_1}}\wedge\frac{\partial}{\partial x^{\mu_2}}\wedge\cdots\wedge\frac{\partial}{\partial x^{\mu_k}}\right)_{\widehat{\sigma\circ\phi}(s)} \\
& + \varepsilon^{a_1\cdots a_k}\left.\frac{\partial(y^{I_1}\circ\widehat{\sigma\circ\phi})}{\partial s^{a_1}}\right|_s \left.\frac{\partial(y^{I_2}\circ\widehat{\sigma\circ\phi})}{\partial s^{a_2}}\right|_s \left.\frac{\partial(x^{\mu_3}\circ\sigma\circ\phi)}{\partial s^{a_3}}\right|_s \cdots \left.\frac{\partial(x^{\mu_k}\circ\sigma\circ\phi)}{\partial s^{a_k}}\right|_s \\
& \qquad \times \left(\frac{\partial}{\partial y^{I_1}}\wedge\frac{\partial}{\partial y^{I_2}}\wedge\frac{\partial}{\partial x^{\mu_3}}\wedge\cdots\wedge\frac{\partial}{\partial x^{\mu_k}}\right)_{\widehat{\sigma\circ\phi}(s)} \\
& + \cdots + \varepsilon^{a_1\cdots a_k}\left.\frac{\partial(y^{I_1}\circ\widehat{\sigma\circ\phi})}{\partial s^{a_1}}\right|_s \cdots \left.\frac{\partial(y^{I_k}\circ\widehat{\sigma\circ\phi})}{\partial s^{a_k}}\right|_s \left(\frac{\partial}{\partial y^{I_1}}\wedge\frac{\partial}{\partial y^{I_2}}\wedge\cdots\wedge\frac{\partial}{\partial y^{I_k}}\right)_{\widehat{\sigma\circ\phi}(s)},
\end{aligned} \tag{4.4.9}$$

with $a_1, \cdots, a_k, b_1, \cdots, b_k = 1, \cdots, k$. By the chain rule, we have relations such as

$$\begin{aligned}
\frac{\partial(x^{\mu_1}\circ\sigma\circ\phi)}{\partial s^{a_1}} &= \frac{\partial(t^c\circ\phi)}{\partial s^{a_1}}\left.\frac{\partial(x^{\mu_1}\circ\sigma)}{\partial t^c}\right|_{\phi(\cdot)}, \\
\frac{\partial(y^{I_j}\circ\widehat{\sigma\circ\phi})}{\partial s^{a_1}} &= \mathcal{T}\frac{\partial(t^c\circ\phi)}{\partial s^{a_1}}\frac{\partial}{\partial t^c}(y^{I_j}\circ\hat{\sigma}) + \left(\frac{\partial}{\partial s^{a_1}}\mathcal{T}\right)(y^{I_j}\circ\hat{\sigma}),
\end{aligned} \tag{4.4.10}$$

where we put

$$\mathcal{T} := \varepsilon_{b_1\cdots b_k}\frac{\partial(t^{b_1}\circ\phi)}{\partial s^1}\cdots\frac{\partial(t^{b_k}\circ\phi)}{\partial s^k} = \varepsilon^{a_1\cdots a_k}\frac{\partial(t^1\circ\phi)}{\partial s^{a_1}}\cdots\frac{\partial(t^k\circ\phi)}{\partial s^{a_k}}. \tag{4.4.11}$$

The second relation of (4.4.10) follows from:

$$\frac{\partial(y^{I_j}\circ\widehat{\sigma\circ\phi})}{\partial s^{a_1}} = \frac{\partial}{\partial s^{a_1}}\left(\frac{\partial(x^{[\mu_1^j}\circ\sigma\circ\phi)}{\partial s^1}\cdots\frac{\partial(x^{\mu_k^j]}\circ\sigma\circ\phi)}{\partial s^k}\right)$$

$$
\begin{aligned}
&= \frac{\partial}{\partial s^{a_1}} \left(\frac{\partial(t^{b_1}\circ\phi)}{\partial s^1} \cdots \frac{\partial(t^{b_k}\circ\phi)}{\partial s^k} \left.\frac{\partial(x^{[\mu_1^j}\circ\sigma)}{\partial t^{b_1}}\right|_{\phi(\cdot)} \cdots \left.\frac{\partial(x^{\mu_k^j]}\circ\sigma)}{\partial t^{b_k}}\right|_{\phi(\cdot)} \right) \\
&= \frac{\partial(t^{b_1}\circ\phi)}{\partial s^1} \cdots \frac{\partial(t^{b_k}\circ\phi)}{\partial s^k} \frac{\partial(t^{c}\circ\phi)}{\partial s^{a_1}} \frac{\partial}{\partial t^c} \left(\frac{\partial(x^{[\mu_1^j}\circ\sigma)}{\partial t^{b_1}} \cdots \frac{\partial(x^{\mu_k^j]}\circ\sigma)}{\partial t^{b_k}} \right) \\
&+ \frac{\partial}{\partial s^{a_1}} \left(\frac{\partial(t^{b_1}\circ\phi)}{\partial s^1} \cdots \frac{\partial(t^{b_k}\circ\phi)}{\partial s^k} \right) \left.\frac{\partial(x^{[\mu_1^j}\circ\sigma)}{\partial t^{b_1}}\right|_{\phi(\cdot)} \cdots \left.\frac{\partial(x^{\mu_k^j]}\circ\sigma)}{\partial t^{b_k}}\right|_{\phi(\cdot)} \\
&= \mathcal{T} \frac{\partial(t^{c}\circ\phi)}{\partial s^{a_1}} \frac{\partial}{\partial t^c}(y^{I_j}\circ\hat{\sigma}) + \left(\frac{\partial}{\partial s^{a_1}} \mathcal{T} \right) (y^{I_j}\circ\hat{\sigma}).
\end{aligned}
\tag{4.4.12}
$$

Considering each components of ρ^2, and using the above relation, we can find the relations between the two parameterisations ρ, σ, in its coordinate representation:

$$
\begin{cases}
(x^{\mu}\circ\rho^2)(s) = (x^{\mu}\circ\sigma^2)(\phi(s)) = (x^{\mu}\circ\sigma)(\phi(s)), \\
(y^{\mu_1\cdots\mu_k}\circ\rho^2)(s) = \mathcal{T}(y^{\mu_1\cdots\mu_k}\circ\sigma^2)(\phi(s)), \\
(z^{I_1\mu_2\cdots\mu_k}\circ\rho^2)(s) = (\mathcal{T})^2(z^{I_1\mu_2\cdots\mu_k}\circ\sigma^2)(\phi(s)) + \beta^{\mu_2\cdots\mu_k}(y^{I_1}\circ\sigma^2)(\phi(s)), \\
(z^{I_1 I_2\mu_3\cdots\mu_k}\circ\rho^2)(s) = (\mathcal{T})^2(z^{I_1 I_2\mu_3\cdots\mu_k}\circ\sigma^2)(\phi(s)) + \beta^{\mu_3\cdots\mu_k}(y^{I_1}\circ\sigma^2)\cdot(y^{I_2}\circ\sigma^2)(\phi(s)), \\
\vdots \\
(z^{I_1 I_2\cdots I_k}\circ\rho^2)(s) = (\mathcal{T})^2(z^{I_1 I_2\cdots I_k}\circ\sigma^2)(\phi(s)) + \beta^{0}(y^{I_1}\circ\sigma^2)\cdot(y^{I_2}\circ\sigma^2)\cdot\cdots\cdot(y^{I_k}\circ\sigma^2)(\phi(s)),
\end{cases}
\tag{4.4.13}
$$

where we set

$$
\begin{aligned}
\beta^{\mu_2\cdots\mu_k} &:= \varepsilon^{a_1\cdots a_k} \left(\frac{\partial}{\partial s^{a_1}}\mathcal{T} \right) \frac{\partial(t^{c_2}\circ\phi)}{\partial s^{a_2}} \cdots \frac{\partial(t^{c_k}\circ\phi)}{\partial s^{a_k}} \left.\frac{\partial(x^{\mu_2}\circ\sigma)}{\partial t^{c_2}}\right|_{\phi(\cdot)} \cdots \left.\frac{\partial(x^{\mu_k}\circ\sigma)}{\partial t^{c_k}}\right|_{\phi(\cdot)}, \\
\beta^{\mu_3\cdots\mu_k} &:= \varepsilon^{a_1\cdots a_k} \left(\frac{\partial}{\partial s^{a_1}}\mathcal{T} \right) \left(\frac{\partial}{\partial s^{a_2}}\mathcal{T} \right) \left.\frac{\partial(x^{\mu_3}\circ\sigma)}{\partial t^{c_3}}\right|_{\phi(\cdot)} \cdots \left.\frac{\partial(x^{\mu_k}\circ\sigma)}{\partial t^{c_k}}\right|_{\phi(\cdot)} \frac{\partial(t^{c_3}\circ\phi)}{\partial s^{a_3}} \cdots \frac{\partial(t^{c_k}\circ\phi)}{\partial s^{a_k}}, \\
&\vdots \\
\beta^{0} &:= \varepsilon^{a_1\cdots a_k} \left(\frac{\partial}{\partial s^{a_1}}\mathcal{T} \right) \cdots \left(\frac{\partial}{\partial s^{a_k}}\mathcal{T} \right).
\end{aligned}
\tag{4.4.14}
$$

Below we will show some intermediate calculations. For example, the second formula of (4.4.13) is obtained by

$$
\left(\frac{\partial(x^{[\mu_1}\circ\sigma\circ\phi)}{\partial s^1} \cdots \frac{\partial(x^{\mu_k]}\circ\sigma\circ\phi)}{\partial s^k} \right)(s) = \frac{\partial(t^{c_1}\circ\phi)}{\partial s^1} \cdots \frac{\partial(t^{c_k}\circ\phi)}{\partial s^k} \left.\frac{\partial(x^{[\mu_1}\circ\sigma)}{\partial t^{c_1}}\right|_{\phi(s)} \cdots \left.\frac{\partial(x^{\mu_k]}\circ\sigma)}{\partial t^{c_k}}\right|_{\phi(s)}
$$

$$= \mathcal{T}(y^{\mu_1\cdots\mu_k}\circ\sigma)(\phi(s)). \tag{4.4.15}$$

The third formula uses (4.4.10), namely

$$\varepsilon^{a_1\cdots a_k}\frac{\partial(y^{I_j}\circ\widehat{\sigma\circ\phi})}{\partial s^{a_1}}\frac{\partial(x^{\mu_2}\circ\sigma\circ\phi)}{\partial s^{a_2}}\cdots\frac{\partial(x^{\mu_k}\circ\sigma\circ\phi)}{\partial s^{a_k}} = (\mathcal{T})^2(z^{I_j\mu_2\cdots\mu_k}\circ\sigma^2)+\beta^{\mu_2\cdots\mu_k}(y^{I_j}\circ\sigma^2), \tag{4.4.16}$$

The above follows from

$$\begin{aligned}
&\varepsilon^{a_1\cdots a_k}\frac{\partial(y^{I_j}\circ\widehat{\sigma\circ\phi})}{\partial s^{a_1}}\frac{\partial(x^{\mu_2}\circ\sigma\circ\phi)}{\partial s^{a_2}}\cdots\frac{\partial(x^{\mu_k}\circ\sigma\circ\phi)}{\partial s^{a_k}}\\
&=\varepsilon^{a_1\cdots a_k}\left(\mathcal{T}\frac{\partial(t^{c_1}\circ\phi)}{\partial s^{a_1}}\frac{\partial}{\partial t^{c_1}}(y^{I_j}\circ\hat\sigma)+\left(\frac{\partial}{\partial s^{a_1}}\mathcal{T}\right)(y^{I_j}\circ\hat\sigma)\right)\\
&\qquad\qquad\times\frac{\partial(t^{c_2}\circ\phi)}{\partial s^{a_2}}\left.\frac{\partial(x^{\mu_2}\circ\sigma)}{\partial t^{c_2}}\right|_{\phi(\cdot)}\cdots\frac{\partial(t^{c_k}\circ\phi)}{\partial s^{a_k}}\left.\frac{\partial(x^{\mu_k}\circ\sigma)}{\partial t^{c_k}}\right|_{\phi(\cdot)}\\
&=\mathcal{T}\varepsilon^{a_1\cdots a_k}\frac{\partial(y^{I_j}\circ\hat\sigma)}{\partial t^{c_1}}\frac{\partial(t^{c_1}\circ\phi)}{\partial s^{a_1}}\frac{\partial(t^{c_2}\circ\phi)}{\partial s^{a_2}}\cdots\frac{\partial(t^{c_k}\circ\phi)}{\partial s^{a_k}}\left.\frac{\partial(x^{\mu_2}\circ\sigma)}{\partial t^{c_2}}\right|_{\phi(\cdot)}\cdots\left.\frac{\partial(x^{\mu_k}\circ\sigma)}{\partial t^{c_k}}\right|_{\phi(\cdot)}\\
&\quad+\varepsilon^{a_1\cdots a_k}\left(\frac{\partial}{\partial s^{a_1}}\mathcal{T}\right)(y^{I_j}\circ\hat\sigma)\frac{\partial(t^{c_2}\circ\phi)}{\partial s^{a_2}}\cdots\frac{\partial(t^{c_k}\circ\phi)}{\partial s^{a_k}}\left.\frac{\partial(x^{\mu_2}\circ\sigma)}{\partial t^{c_2}}\right|_{\phi(\cdot)}\cdots\left.\frac{\partial(x^{\mu_k}\circ\sigma)}{\partial t^{c_k}}\right|_{\phi(\cdot)}\\
&=(\mathcal{T})^2\cdot\varepsilon^{c_1\cdots c_k}\frac{\partial(y^{I_j}\circ\hat\sigma)}{\partial t^{c_1}}\left.\frac{\partial(x^{\mu_2}\circ\sigma)}{\partial t^{c_2}}\right|_{\phi(\cdot)}\cdots\left.\frac{\partial(x^{\mu_k}\circ\sigma)}{\partial t^{c_k}}\right|_{\phi(\cdot)}\\
&\quad+\varepsilon^{a_1\cdots a_k}\left(\frac{\partial}{\partial s^{a_1}}\mathcal{T}\right)\frac{\partial(t^{c_2}\circ\phi)}{\partial s^{a_2}}\cdots\frac{\partial(t^{c_k}\circ\phi)}{\partial s^{a_k}}\left.\frac{\partial(x^{\mu_2}\circ\sigma)}{\partial t^{c_2}}\right|_{\phi(\cdot)}\cdots\left.\frac{\partial(x^{\mu_k}\circ\sigma)}{\partial t^{c_k}}\right|_{\phi(\cdot)}(y^{I_j}\circ\hat\sigma)\\
&=(\mathcal{T})^2(z^{I_j\mu_2\cdots\mu_k}\circ\sigma^2)\\
&\quad+\varepsilon^{a_1\cdots a_k}\left(\frac{\partial}{\partial s^{a_1}}\mathcal{T}\right)\frac{\partial(t^{c_2}\circ\phi)}{\partial s^{a_2}}\cdots\frac{\partial(t^{c_k}\circ\phi)}{\partial s^{a_k}}\left.\frac{\partial(x^{\mu_2}\circ\sigma)}{\partial t^{c_2}}\right|_{\phi(\cdot)}\cdots\left.\frac{\partial(x^{\mu_k}\circ\sigma)}{\partial t^{c_k}}\right|_{\phi(\cdot)}(y^{I_j}\circ\hat\sigma).
\end{aligned} \tag{4.4.17}$$

Since we assumed ρ is a regular parameterisation that preserves orientation, $\mathcal{T}>0$, and we see that (4.4.13) are in the form of the homogeneity conditions (4.4.1), the second order k-dimensional area of Σ is preserved by

$$\begin{aligned}
l^K(\Sigma)=&\int_{s_i^1}^{s_f^1}ds^1\cdots\int_{s_i^k}^{s_f^k}ds^kK\left(\rho^2(s)\right)\\
=&\int_{s_i^1}^{s_f^1}ds^1\cdots\int_{s_i^k}^{s_f^k}ds^kK\left((x^\mu\circ\rho^2)(s),(y^I\circ\rho^2)(s),(z^{I_1\mu_2\cdots\mu_k}\circ\rho^2)(s),\cdots,(z^{I_1I_2\cdots I_k}\circ\rho^2)(s)\right)
\end{aligned}$$

$$
\begin{aligned}
=&\int_{s_i^1}^{s_f^1} ds^1 \cdots \int_{s_i^k}^{s_f^k} ds^k K\left((x^\mu \circ \sigma^2)(\phi(s)), \mathcal{T}(y^{\mu_1\cdots\mu_k} \circ \sigma^2)(\phi(s)),\right.\\
&(\mathcal{T})^2(z^{I_1\mu_2\cdots\mu_k} \circ \sigma^2)(\phi(s)) + \beta^{\mu_2\cdots\mu_k}(y^{I_1} \circ \sigma^2)(\phi(s)), \cdots,\\
&\left.(\mathcal{T})^2(z^{I_1 I_2\cdots I_k} \circ \sigma^2)(\phi(s)) + \beta^0(y^{I_1} \circ \sigma^2)\cdot(y^{I_2} \circ \sigma^2)\cdot \cdots \cdot (y^{I_k} \circ \sigma^2)(\phi(s))\right)\\
=&\int_{s_i^1}^{s_f^1} ds^1 \cdots \int_{s_i^k}^{s_f^k} ds^k K\Big(x^\mu(\sigma(\phi(s))), \mathcal{T} y^I(\sigma^2(\phi(s))), \big((\mathcal{T})^2 z^{I_1\mu_2\cdots\mu_k} +!\beta^{\mu_2\cdots\mu_k} y^{I_1}\big)(\sigma^2(\phi(s))),\\
&\cdots, \big((\mathcal{T})^2 z^{I_1 I_2\cdots I_k} + \beta^0 y^{I_1} y^{I_2}\cdots y^{I_k}\big)\,(\sigma^2(\phi(s)))\Big)\\
=&\int_{\phi^{-1}(t_i^1)}^{\phi^{-1}(t_f^1)} \cdots \int_{\phi^{-1}(t_i^k)}^{\phi^{-1}(t_f^k)} ds^1 \wedge ds^2 \wedge \cdots \wedge ds^k \mathcal{T} K\left(\sigma^2(\phi(s))\right)\\
=&\int_{t_i^1}^{t_f^1} \cdots \int_{t_i^k}^{t_f^k} dt^1 \wedge dt^2 \wedge \cdots \wedge dt^k K\left(\sigma^2(t)\right)\\
=&\int_{t_i^1}^{t_f^1} dt^1 \cdots \int_{t_i^k}^{t_f^k} dt^k K\left(\sigma^2(t)\right),
\end{aligned}
\tag{4.4.18}
$$

where $s_i^1, s_f^1, s_i^2, s_f^2, \cdots, s_i^k, s_f^k$ are the pre-image of the boundary points $t_i^1, t_f^1, t_i^2, t_f^2, \cdots, t_i^k, t_f^k$ by ϕ. We have used the homogeneity condition of K, and the definition of integration of k-form in accord to Section 2.2. □

We can conclude that in the second order case, homogeneity of K and parameterisation invariance of Kawaguchi k-area is an equivalent property, provided that we are considering the case of $M = \mathbb{R}^n$.

4.4.3 Second order Kawaguchi k-form

Now we will turn to defining a second order Kawaguchi k-form, which we construct to have the same property as the previous cases, namely, it should be constructed by referring to the conditions given by (4.4.2); and should be equivalent to giving a second order k-dimensional area, when its pull back is integrated over the parameter space. However, since the conditions (4.4.2) depend on coordinates, from these alone we cannot construct a global form for general manifolds. For this reason we also restrict our model for $M = \mathbb{R}^n$ case, and leave the general case for future research. Nevertheless, the obtained form could be used for the consideration of second order field theories with the restriction of $M = \mathbb{R}^n$.

Definition 4.34. Second order Kawaguchi k-form
Let $\varphi^2 = (x^\mu, y^{\mu_1\cdots\mu_k}, z^{I_1;\nu_2\cdots\nu_k}, z^{I_1 I_2;\nu_3\cdots\nu_k}, \cdots, z^{I_1 I_2\cdots I_k})$, $\mu, \mu_1, \cdots, \mu_k, \nu_2, \cdots, \nu_k = 1, \cdots, n,$

$I_j, := \mu_j^{i_1} \cdots \mu_j^{i_k}$, be the induced global chart on $(\Lambda^k T)^2 M$, where $M = \mathbb{R}^n$. The *second order Kawaguchi k-form* $\mathcal{K}$ is a k-form on $(\Lambda^k T)^2 M$, which in coordinates are expressed by

$$\mathcal{K} = \frac{1}{k!} \frac{\partial K}{\partial y^{\mu_1 \cdots \mu_k}} dx^{\mu_1 \cdots \mu_k} + \frac{2}{(k-1)!} \frac{\partial K}{\partial z^{I_1 \nu_2 \cdots \nu_k}} dy^{I_1} \wedge dx^{\nu_2 \cdots \nu_k}$$
$$+ \frac{2}{2!(k-2)!} \frac{\partial K}{\partial z^{I_1 I_2 \nu_3 \cdots \nu_k}} dy^{I_1} \wedge dy^{I_2} \wedge dx^{\nu_3 \cdots \nu_k} + \cdots + \frac{2}{k!} \frac{\partial K}{\partial z^{I_1 I_2 \cdots I_k}} dy^{I_1} \wedge \cdots \wedge dy^{I_k}. \tag{4.4.19}$$

We used the abbreviation such as

$$dx^{\mu_1 \cdots \mu_k} := dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}, \quad dy^{I_1} \wedge dx^{\nu_2 \cdots \nu_k} := dy^{I_1} \wedge dx^{\nu_2} \wedge \cdots \wedge dx^{\nu_k}.$$

This expression (4.4.19) corresponds to the first homogeneity condition in (4.4.2).

As we already mentioned, in general the above form is not invariant with respect to the coordinate transformations given by (2.1.4).

Proposition 4.35. Let $\mathcal{K}$ be the second order Kawaguchi k-form on $(\Lambda^k T)^2 M$, $\Sigma = \sigma(\bar{P})$ the k-patch on M, with $\bar{P} = [t_i^1, t_f^1] \times [t_i^2, t_f^2] \times \cdots \times [t_i^k, t_f^k]$ a closed rectangle in $\mathbb{R}^k$. Then,

$$\int_{\Sigma^2} \mathcal{K} = l^K(\Sigma). \tag{4.4.20}$$

Proof. The simple calculation leads,

$$\begin{aligned}
\int_{\Sigma^2} \mathcal{K} &= \int_{\sigma^2(P)} \frac{1}{k!} \frac{\partial K}{\partial y^{\mu_1 \cdots \mu_k}} dx^{\mu_1 \cdots \mu_k} + \int_{\sigma^2(P)} \frac{2}{(k-1)!} \frac{\partial K}{\partial z^{I_1 \nu_2 \cdots \nu_k}} dy^{I_1} \wedge dx^{\nu_2 \cdots \nu_k} \\
&\quad + \int_{\sigma^2(P)} \frac{2}{2!(k-2)!} \frac{\partial K}{\partial z^{I_1 I_2 \nu_3 \cdots \nu_k}} dy^{I_1} \wedge dy^{I_2} \wedge dx^{\nu_3 \cdots \nu_k} \\
&\quad + \cdots + \int_{\sigma^2(P)} \frac{2}{k!} \frac{\partial K}{\partial z^{I_1 I_2 \cdots I_k}} dy^{I_1} \wedge \cdots \wedge dy^{I_k} \\
&= \int_{t_i^1}^{t_f^1} \cdots \int_{t_i^k}^{t_f^k} \frac{1}{k!} \frac{\partial K}{\partial y^{\mu_1 \cdots \mu_k}} \circ \sigma^2 \, d(x^{\mu_1} \circ \sigma^2) \wedge \cdots \wedge d(x^{\mu_k} \circ \sigma^2) \\
&\quad + \int_{\sigma^2(P)} \frac{2}{2!(k-2)!} \frac{\partial K}{\partial z^{I_1 I_2 \nu_3 \cdots \nu_k}} \circ \sigma^2 \, d(y^{I_1} \circ \sigma^2) \wedge d(y^{I_2} \circ \sigma^2) \wedge d(x^{\nu_3 \cdots \nu_k} \circ \sigma^2) \\
&\quad + \cdots + \int_{\sigma^2(P)} \frac{2}{k!} \frac{\partial K}{\partial z^{I_1 I_2 \cdots I_k}} \circ \sigma^2 \, d(y^{I_1} \circ \sigma^2) \wedge \cdots \wedge d(y^{I_k} \circ \sigma^2) \\
&= \int_{t_i^1}^{t_f^1} \cdots \int_{t_i^k}^{t_f^k} \frac{\partial K}{\partial y^{\mu_1 \cdots \mu_k}} \left(\sigma^2(t)\right) y^{\mu_1 \cdots \mu_k}(\sigma^2(t)) \, dt^1 \wedge \cdots \wedge dt^k
\end{aligned}$$

$$
\begin{aligned}
&+\int_{t_i^1}^{t_f^1}\cdots\int_{t_i^k}^{t_f^k} 2\frac{\partial K}{\partial z^{I_1\nu_2\cdots\nu_k}}\left(\sigma^2(t)\right) z^{I_1\nu_2\cdots\nu_k}(\hat{\sigma}(t))dt^1\wedge\cdots\wedge dt^k \\
&+\cdots+\int_{\sigma^2(P)} 2\frac{\partial K}{\partial z^{I_1I_2\cdots I_k}}\left(\sigma^2(t)\right) z^{I_1I_2\cdots I_k}(\sigma^2(t))\,dt^1\wedge\cdots\wedge dt^k \\
=&\int_{t_i^1}^{t_f^1} dt^1\cdots\int_{t_i^k}^{t_f^k} dt^k K\left(\sigma^2(t)\right) \\
=&\, l^K(\Sigma)
\end{aligned}
\tag{4.4.21}
$$

where we used the pull-back homogeneity condition

$$
\begin{aligned}
&\frac{\partial K}{\partial y^{I_1}}\circ\sigma^2\cdot y^{I_1}\circ\sigma^2+\frac{2}{(k-1)!}\frac{\partial K}{\partial z^{I_1\nu_2\cdots\nu_k}}\circ\sigma^2\cdot z^{I_1\nu_2\cdots\nu_k}\circ\sigma^2 \\
&\quad+\frac{2}{2!(k-2)!}\frac{\partial K}{\partial z^{I_1I_2\nu_3\cdots\nu_k}}\circ\sigma^2\cdot z^{I_1I_2\nu_3\cdots\nu_k}\circ\sigma^2+\cdots+\frac{2}{k!}\frac{\partial K}{\partial z^{I_1I_2\cdots I_k}}\circ\sigma^2\cdot z^{I_1I_2\cdots I_k}\circ\sigma^2 \\
&=K\circ\sigma^2.
\end{aligned}
\tag{4.4.22}
$$

□

Remark 4.36. Similarly as in the case of Finsler manifold and second order Finsler-Kawaguchi manifold, we can redefine the pair $(M,\mathcal{K})$ as the second order n-dimensional k-areal Kawaguchi manifold, instead of the pair (M,K), where $M=\mathbb{R}^n$.

Chapter 5

Lagrangian formulation of Finsler and Kawaguchi geometry

In the preceding chapters, we have prepared the foundations for considering the parameterisation invariant theory of calculus of variation. Here in this chapter, we will interpret the Finsler length as the Lagrangian, and derive the Euler-Lagrange equations and conservation laws by considering the calculus of variation. We interpret the Finsler manifold as a dynamical system, and in this context we will also call the Euler-Lagrange equations the *equations of motion*. We will begin with the standard first order mechanics, which will be based on Finsler geometry (Chapter 3), and then extend it to second order mechanics, based on Finsler-Kawaguchi geometry (Chapter 4, Section 4.2), and finally to the field theory (first and second order), based on Kawaguchi geometry (Chapter 4, Section 4.3, 4.4). In all cases, the dynamics of the object (particle, field) is described as a motion of a k-patch (arc segment) Σ of n-dimensional manifold M.

5.1 First order mechanics

Here we introduce the theory of first order mechanics, in terms of Finsler geometry. By the term *first order*, we mean that the total space we are considering is the tangent bundle, and by *mechanics*, we mean that we are considering the arc segment on M.

The basic structure we consider in this section is introduced in Chapter 2 and 3, namely the n-dimensional Finsler manifold $(M, \mathcal{F})$, the tangent bundle (TM, τ_M, M), and a 1-dimensional curve (arc segment) C on M, parameterised by σ. The curve (arc segment) describes the trajectory of the object on M.

In our setting, the Hilbert 1-form is the Lagrangian, and the action will be defined by considering the integration over the lift of the parameterisable curve (arc segment) C. The Euler-Lagrange equations are derived by taking the variation of the action with respect to the flow on M that deforms the arc segment C, and fixed on the boundary. We can show that the action and consequently the Euler-Lagrange equations are independent with respect to the parameterisation belonging to the same equivalent class.

5.1.1 Action

Suppose we have a dynamical system (differential equations expressing motions) where the trajectory of the point particle (or any object which dynamics could be considered as a point) is expressed by an arc segment C of a parameterisable curve, such that $C = \sigma(I) \subset M$, where I is a closed interval $I = [t_i, t_f] \subset \mathbb{R}$.

When we can express this system by Finsler geometry, namely the pair $(M, \mathcal{F})$ where $\mathcal{F}$ is a Finsler-Hilbert 1-form, we refer to this dynamical system as *first order mechanics*, and conversely call the pair $(M, \mathcal{F})$ a *dynamical system*.

The action of first order mechanics is defined as follows.

Definition 5.1. Action of first order mechanics
Let $(M, \mathcal{F})$ be a n-dimensional Finsler manifold, (U, φ), $\varphi = (x^\mu)$ be a chart on M, and (V, ψ), $V = {\tau_M}^{-1}(U)$, $\psi = (x^\mu, y^\mu)$ the induced chart on TM. The local coordinate expression of the Finsler-Hilbert form $\mathcal{F} \in \Omega^1(TM)$ is given by $\mathcal{F} = \dfrac{\partial F}{\partial y^\mu} dx^\mu$, where F is the Finsler function. Let C be an arc segment on M, σ its parameterisation, $\sigma(I) = C \subset M$ with $I = [t_i, t_f] \subset \mathbb{R}$, and $\hat{\sigma}$ the tangent lift of σ, defined in Chapter 3 (Definition). We call the functional $S^{\mathcal{F}}(C)$ defined by

$$S^{\mathcal{F}}(C) := l^F(C) = \int_{\hat{C}} \mathcal{F} = \int_{\hat{\sigma}(I)} \frac{\partial F}{\partial y^\mu} dx^\mu, \tag{5.1.1}$$

the *action of first order mechanics associated with* $\mathcal{F}$.

As we have seen in Section 3.3, Lemma 3.19,

Finsler length is invariant with respect to the reparameterisation, therefore the action is also invariant.

5.1.2 Extremal and equations of motion

Having defined the action, we are able to derive the equations of motion by considering the extremal of the action. To make the discussion simple, we only consider global flows in

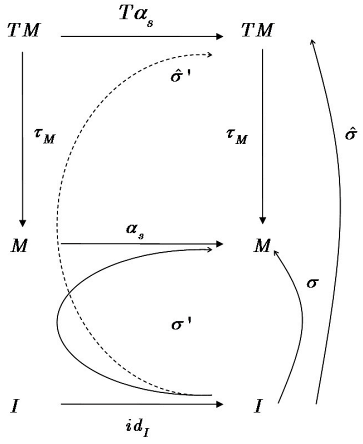

Figure 5.1: First order mechanics

this thesis. Nevertheless, with some details added, the formulation can be set up similarly with local flows.

Consider a C^∞-flow, $\alpha : \mathbb{R} \times M \to M$, and its associated 1-parameter group of transformations $\{\alpha_s\}_{s\in\mathbb{R}}$. The 1-parameter group $\alpha_s : M \to M$ induces a tangent 1-parameter group $T\alpha_s : TM \to TM$ on TM. This will also deform the curve (arc segment) C to $C' = \alpha_s(C)$, and since this is a smooth deformation, it again becomes a parameterisable curve. By the reparameterisation independence, we can always choose the parameterisation of this deformed C' by a new $\sigma' : I \to M$, $\sigma'(I) = C'$, so that it has the same parameter space as C. The variation of the action will be expressed by the small deformations made to the action by α_s.

Definition 5.2. Variation of the action
Let ξ be a vector field on M which generates the 1-parameter group α_s, i.e., $\xi = \left.\dfrac{d\alpha_s}{ds}\right|_{s=0}$.
We call the functional

$$\begin{aligned} \delta_\xi S^{\mathcal{F}}(C) &:= \lim_{s\to 0} \frac{1}{s}\left\{S^{\mathcal{F}}(\alpha_s(C)) - S^{\mathcal{F}}(C)\right\} \\ &= \lim_{s\to 0} \frac{1}{s}\left\{\int_{\widehat{\alpha_s\circ\sigma}(I)} \mathcal{F} - \int_{\hat{\sigma}(I)} \mathcal{F}\right\}, \end{aligned} \tag{5.1.2}$$

the *variation of the action $S^{\mathcal{F}}(C)$ with respect to the flow α, associated to $\mathcal{F}$* .

It is easy to see that the lift of this modified parameterisation σ' is given by, $\hat{\sigma}' = \widehat{\alpha_s\circ\sigma} = T\alpha_s\circ\hat{\sigma} = T\alpha_s\circ\hat{\sigma}\circ {id_I}^{-1}$, which we show in Figure 5.1.

We will get,

$$\begin{aligned}\delta_\xi S^{\mathcal{F}}(C) &:= \lim_{s\to 0}\frac{1}{s}\left\{\int_{T\alpha_s\circ\hat\sigma(I)}\mathcal{F}-\int_{\hat\sigma(I)}\mathcal{F}\right\}=\lim_{s\to 0}\frac{1}{s}\left\{\int_{\hat\sigma(I)}(T\alpha_s)^*\mathcal{F}-\int_{\hat\sigma(I)}\mathcal{F}\right\}\\ &= \int_{\hat\sigma(I)}L_X\mathcal{F}=\int_{\hat C}L_X\mathcal{F}. \qquad (5.1.3)\end{aligned}$$

where X is a vector field on TM that generates the tangent 1-parameter group $T\alpha_s$, i.e., $X=\left.\dfrac{d(T\alpha_s)}{ds}\right|_{s=0}$, and L_X is a Lie derivative with respect to X.

Let us calculate the vector field X in local coordinates. As usual, let (U,φ), $\varphi=(x^\mu)$ be a chart on M, and the induced chart of TM; (V,ψ), $V={\tau_M}^{-1}(U)$, $\psi=(x^\mu,y^\mu)$. Let ξ be be a generator of the 1-parameter group α_s and its local expression $\xi=\xi^\mu\dfrac{\partial}{\partial x^\mu}$, where $\xi^\mu\in C^\infty(M)$. The tangent map $T\alpha_s$ at $p\in M$ sends the vector $v\in T_pM$ to $T_{\alpha_s(p)}M$ by

$$T_p\alpha_s(v)=\left.\frac{\partial(x^\mu\circ\alpha_s\circ\varphi^{-1})}{\partial x^\nu}\right|_{\varphi(p)}v^\nu\left(\frac{\partial}{\partial x^\mu}\right)_{\alpha_s(p)}, \qquad (5.1.4)$$

and since $(T\alpha_s,\alpha_s)$ is a bundle morphism and from the definition of induced coordinates of a tangent bundle, we have for its coordinate expressions,

$$\begin{aligned}x^\mu\circ T_p\alpha_s(v)&=x^\mu\circ\alpha_s\circ\tau_M(v),\\ y^\mu\circ T_p\alpha_s(v)&=\left.\frac{\partial(x^\mu\circ\alpha_s\circ\varphi^{-1})}{\partial x^\nu}\right|_{\varphi(\tau_M(v))}y^\nu(v). \qquad (5.1.5)\end{aligned}$$

By these observations, the vector field X at a point $q\in TM$ has a local expression,

$$\begin{aligned}X_q&=\left.\frac{d(x^\mu\circ T\alpha_s)}{ds}\right|_{s=0}\left(\frac{\partial}{\partial x^\mu}\right)_q+\left.\frac{d(y^\mu\circ T\alpha_s)}{ds}\right|_{s=0}\left(\frac{\partial}{\partial y^\mu}\right)_q\\ &=\left.\frac{d}{ds}(x^\mu\circ\alpha_s\circ\tau_M)\right|_{s=0}\left(\frac{\partial}{\partial x^\mu}\right)_q+y^\nu(q)\frac{d}{ds}\left(\left.\frac{\partial(x^\mu\circ\alpha_s\circ\varphi^{-1})}{\partial x^\nu}\right|_{\varphi(\tau_M(q))}\right)_{s=0}\left(\frac{\partial}{\partial y^\mu}\right)_q\\ &=(\xi^\mu\circ\tau_M)(q)\left(\frac{\partial}{\partial x^\mu}\right)_q+\left(\frac{\partial\xi^\mu}{\partial x^\nu}\circ\tau_M\cdot y^\nu\right)(q)\left(\frac{\partial}{\partial y^\mu}\right)_q, \qquad (5.1.6)\end{aligned}$$

therefore,

$$X=\xi^\mu\circ\tau_M\left(\frac{\partial}{\partial x^\mu}\right)+\frac{\partial\xi^\mu}{\partial x^\nu}\circ\tau_M\cdot y^\nu\left(\frac{\partial}{\partial y^\mu}\right). \qquad (5.1.7)$$

We will call X, the *induced vector field* by ξ, on TM.

The Lie derivative $L_X\mathcal{F}$ in coordinate expression is

$$\begin{aligned}
L_X\mathcal{F} &= L_X\left(\frac{\partial F}{\partial y^\rho}dx^\rho\right) = X\left(\frac{\partial F}{\partial y^\rho}\right)dx^\rho + \frac{\partial F}{\partial y^\rho}dL_X x^\rho \\
&= \left\{\xi^\mu\circ\tau_M\left(\frac{\partial^2 F}{\partial x^\mu\partial y^\rho}\right) + \frac{\partial\xi^\mu}{\partial x^\nu}\circ\tau_M\cdot y^\nu\left(\frac{\partial^2 F}{\partial y^\mu\partial y^\rho}\right)\right\}dx^\rho + \frac{\partial F}{\partial y^\rho}d\left(\xi^\rho\circ\tau_M\right) \\
&= \xi^\mu\circ\tau_M\left\{\frac{\partial^2 F}{\partial x^\mu\partial y^\rho}dx^\rho - d\left(\frac{\partial F}{\partial y^\mu}\right)\right\} + \frac{\partial\xi^\mu}{\partial x^\nu}\circ\tau_M\cdot y^\nu\left(\frac{\partial^2 F}{\partial y^\mu\partial y^\rho}\right)dx^\rho + d\left(\frac{\partial F}{\partial y^\rho}\cdot\xi^\rho\circ\tau_M\right).
\end{aligned} \tag{5.1.8}$$

The result of (5.1.8) is called the *infinitesimal first variation formula* for the Hilbert form $\mathcal{F}$.

The variation of action becomes

$$\begin{aligned}
\delta_\xi S^{\mathcal{F}}(C) &= \int_{\hat\sigma(I)} L_X\mathcal{F} = \int_I \hat\sigma^* L_X\mathcal{F} \\
&= \int_I \hat\sigma^*\left(\xi^\mu\circ\tau_M\left\{\frac{\partial^2 F}{\partial x^\mu\partial y^\rho}dx^\rho - d\left(\frac{\partial F}{\partial y^\mu}\right)\right\} + d\left(\frac{\partial F}{\partial y^\rho}\cdot\xi^\rho\circ\tau_M\right)\right) \\
&= \int_{\hat C}\xi^\mu\circ\tau_M\left\{\frac{\partial^2 F}{\partial x^\mu\partial y^\rho}dx^\rho - d\left(\frac{\partial F}{\partial y^\mu}\right)\right\} + d\left(\frac{\partial F}{\partial y^\rho}\cdot\xi^\rho\circ\tau_M\right),
\end{aligned} \tag{5.1.9}$$

which is called the *integral first variation formula*. We have used the homogeneity condition:

$$\left(\frac{\partial^2 F}{\partial y^\mu\partial y^\rho}\cdot y^\rho\right)\circ\hat\sigma = 0. \tag{5.1.10}$$

(5.3.11) is obtained by taking the derivative of (3.2.3) with respect to y^μ, and then taking the pull back.

Now we can proceed to find the equations of motion to this system. We will first give the definition of an extremal.

Definition 5.3. Extremal of an action

1. We say that an arc segment C is *stable* with respect to a flow α, when it satisfies

$$\delta_\xi S^{\mathcal{F}}(C) = 0, \tag{5.1.11}$$

where ξ is the generator of α.

2. We say that an arc segment C is an *extremal* of the action $S^{\mathcal{F}}$, when it satisfies (5.1.11) for any α such that its associated 1-parameter group α_s satisfies $\alpha_s(\partial C) = \partial C$, $\forall s \in \mathbb{R}$, where ∂C is the boundary of C.

With this concept of an extremal, we can obtain the following theorem.

Theorem 5.4. Extremals
Let C be an arc segment. The following statements are equivalent.

1. C is an extremal.

2. The equation

$$\mathcal{EL}^F{}_{\mu} \circ \hat{\sigma} = 0,$$
$$\mathcal{EL}^F{}_{\mu} := \frac{\partial^2 F}{\partial x^\mu \partial y^\rho} dx^\rho - d\left(\frac{\partial F}{\partial y^\mu}\right), \tag{5.1.12}$$

holds for arbitrary parameterisation σ.

Proof. Suppose C is an extremal. Then, by definition, for all $\alpha_s : M \to M$, such that does not change the boundary of C, we have $\delta_\xi S(C) = 0$. On the other hand, the last term in (5.1.9) becomes 0, since it is the boundary term. Therefore, we have,

$$\int_{\hat{C}} \left(\xi^\mu \circ \tau_M \left\{ \frac{\partial^2 F}{\partial x^\mu \partial y^\rho} dx^\rho - d\left(\frac{\partial F}{\partial y^\mu}\right) \right\} \right) = 0. \tag{5.1.13}$$

Since this relation must be true for all ξ, which is the generator of α_s, we have

$$\left(\frac{\partial^2 F}{\partial x^\mu \partial y^\rho} dx^\rho - d\left(\frac{\partial F}{\partial y^\mu}\right) \right) \circ \hat{\sigma} = 0, \tag{5.1.14}$$

for any parameterisation σ. To prove the converse, it is sufficient to take the similar steps backwards. □

Definition 5.5. Symmetry of the dynamical system
Let u be a vector field over M, and Y an induced vector field by u over TM. We say that $\mathcal{F}$ *is invariant with respect to* u, if

$$L_Y \mathcal{F} = 0, \tag{5.1.15}$$

and u called a *symmetry* of the dynamical system $(M, \mathcal{F})$. We also say that u generates the invariant transformations on the Finsler manifold $(M, \mathcal{F})$.

Now we will have the following important relation between the symmetry and a conserved quantity.

Theorem 5.6. Noether
Suppose we are given a symmetry of $(M, \mathcal{F})$. Then there exists a function f on TM, which along the extremal γ of $S^{\mathcal{F}}$ satisfies,

$$\int_{\hat{\gamma}} df = 0, \tag{5.1.16}$$

for any parameterisation σ which parameterise γ.

Proof. Let the symmetry be u, with its local coordinate expression $u = u^\mu \dfrac{\partial}{\partial x^\mu}$, and the induced vector field Y. Then from (5.1.9), we have

$$\begin{aligned} 0 &= \int_{\hat{\gamma}} L_Y \mathcal{F} \\ &= \int_{\hat{\gamma}} \left(u^\mu \circ \tau_M \left\{ \frac{\partial^2 F}{\partial x^\mu \partial y^\rho} dx^\rho - d\left(\frac{\partial F}{\partial y^\mu} \right) \right\} + d\left(\frac{\partial F}{\partial y^\rho} \cdot u^\rho \circ \tau_M \right) \right) \\ &= \int_{\hat{\gamma}} d\left(\frac{\partial F}{\partial y^\rho} \cdot u^\rho \circ \tau_M \right), \end{aligned} \tag{5.1.17}$$

The second equality comes from the fact we consider along the extremal γ. Therefore we have a function on TM,

$$f = \frac{\partial F}{\partial y^\rho} \cdot u^\rho \circ \tau_M, \tag{5.1.18}$$

such that satisfies the condition. □

We call the relation (5.1.16), the *conservation law*.

We can express the conservation law (5.1.16) by taking arbitrary parameterisation for this γ. For instance, by $\sigma : [t_i, t_f] \to M$

$$0 = \int_{\partial\hat{\gamma}} \frac{\partial F}{\partial y^\rho} \cdot u^\rho \circ \tau_M = \frac{\partial F}{\partial y^\rho} \cdot u^\rho \circ \tau_M(\hat{\sigma}(t_i)) - \frac{\partial F}{\partial y^\rho} \cdot u^\rho \circ \tau_M(\hat{\sigma}(t_f)). \tag{5.1.19}$$

Definition 5.7. Noether current
The quantity f is called the *Noether current associated with* u.

By the coordinate transformation

$$x^\mu \to \tilde{x}^\mu = \tilde{x}^\mu(x^\nu),$$

$$y^\mu \to \tilde{y}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} y^\nu, \tag{5.1.20}$$

the differential 1-form ${\mathcal{EL}^F}_\mu$ in (5.1.12) transforms as

$$\frac{\partial^2 F}{\partial \tilde{x}^\mu \partial \tilde{y}^\rho} d\tilde{x}^\rho - d\left(\frac{\partial F}{\partial \tilde{y}^\mu}\right) = \left(\frac{\partial x^\nu}{\partial \tilde{x}^\mu}\right)\left(\frac{\partial^2 F}{\partial x^\nu \partial y^\rho} dx^\rho - d\left(\frac{\partial F}{\partial y^\nu}\right)\right). \tag{5.1.21}$$

This observation leads us to define a new coordinate invariant form.

Lemma 5.8. Euler-Lagrange form
There exist a global two form on TM, which in local coordinates are expressed by

$$\mathcal{EL}^F := dx^\mu \wedge {\mathcal{EL}^F}_\mu = \left(\frac{\partial^2 F}{\partial x^\mu \partial y^\rho} dx^\mu + d\left(\frac{\partial F}{\partial y^\rho}\right)\right) \wedge dx^\rho. \tag{5.1.22}$$

From the previous coordinate transformations, this form is obviously coordinate independent.

There is a direct relation between the exterior derivative of $\mathcal{F}$ and $\mathcal{EL}$,

$$d\mathcal{F} = \mathcal{EL}^F - \frac{\partial^2 F}{\partial x^\mu \partial y^\nu} dx^\mu \wedge dx^\nu. \tag{5.1.23}$$

It can be also checked easily that this is also a coordinate invariant relation.

Remark 5.9. In Chapter 3, Remark 3.25, we showed that when given a Hilbert form, we can obtain the Cartan form by taking an inclusion map from J^1Y to TY. Here we will show that given a "conventional" Lagrange function on J^1Y, we can also construct its homogeneous counterpart. However, unlike in the case of the former, in general, this cannot be done globally. (It is possible only when $Y = \mathbb{R} \times Q$.) As in the Remark 3.25, let (U, ψ), $\psi = (t, q^i)$, $i = 1, \cdots, n$ be the adapted chart on Y, and the induced chart on $\mathbb{R}$ be $(\pi(U), t)$. We denote the induced chart on J^1Y by $((\pi^{1,0})^{-1}(U), \psi^1), \psi^1 = (t, q^i, \dot{q}^i)$ $((\pi^{1,0})^{-1}(U), \psi^1), \psi^1 = (t, q^i, \dot{q}^i)$. Take the induced chart on TY as $(V, \tilde{\psi}^1)$, $V = (\tau_Y)^{-1}(U)$, $\tilde{\psi}^1 = (x^0, x^i, y^0, y^i)$, $i = 1, \cdots, n$, such that $y^0 \neq 0$. It is always possible to choose such coordinates for a single chart. (In order to avoid confusion we use different symbols, but clearly $x^0 = t \circ \tau_Y, x^i = q^i \circ \tau_Y$.) Now consider a map $\rho : V \hookrightarrow J^1Y$, $\rho(V) = (\pi^{1,0})^{-1}(U)$, which in coordinates are defined by

$$t \circ \rho = x^0, \; q^i \circ \rho = x^i, \; \dot{q}^i \circ \rho = \frac{y^i}{y^0}. \tag{5.1.24}$$

Let F be a function on V, defined by,

$$F/y^0 = \rho^* \mathcal{L}. \tag{5.1.25}$$

In coordinates,

$$F(x^\mu, y^\mu) = \mathcal{L}(t\circ\rho, q^i\circ\rho, \dot{q}^i\circ\rho)y^0 = \mathcal{L}(x^0, x^i, \frac{y^i}{y^0})y^0, \tag{5.1.26}$$

for $\mu = 0, \cdots, n$, $i = 1, \cdots, n$. Then on V, F satisfies the homogeneity function. We now have $\mathcal{F} = \dfrac{\partial F}{\partial y^\mu} dx^\mu$ on V. In this way, for a local coordinate chart, (or for the case of $Y = \mathbb{R} \times Q$, also globally) we can construct a local Hilbert 1-form from a Lagrangian, which also can be used as an reparameterisation invariant action provided that the arc segment where the integration is carried out is covered by this single chart.

Remark 5.10. Locally (on a single chart), we can also construct a different Finsler function and local Hilbert form from $\mathcal{L}$ by choosing an appropriate map for ρ. Though these constructions are local, if the arc segment where the integration is carried out is covered by this single chart, we can take it as a reparameterisation invariant action.

Remark 5.11. We can also check that the equation of motion given by (5.1.12) reduces to the conventional form of Euler-Lagrange equation by considering the same inclusion map ι given in Chapter 3, Remark 3.25, which in coordinate expression were given by

$$x^0\circ\iota = t,\ x^i\circ\iota = q^i,\ y^0\circ\iota = 1,\ y^i\circ\iota = \dot{q}^i, \tag{5.1.27}$$

$i = 1, \cdots, n$. We will use Greek indices such as $\mu, \rho = 0, 1, 2, \cdots, n$, and Latin indices such as $i, j, k = 1, 2, \cdots, n$. Rewrite the 1-form $\mathcal{EL}^F{}_\mu$,

$$\begin{aligned} \mathcal{EL}^F{}_\mu &= \frac{\partial^2 F}{\partial x^\mu \partial y^\rho} dx^\rho - d\left(\frac{\partial F}{\partial y^\mu}\right) = \frac{\partial^2 F}{\partial x^\mu \partial y^0} dx^0 + \frac{\partial^2 F}{\partial x^\mu \partial y^i} dx^i - d\left(\frac{\partial F}{\partial y^\mu}\right) \\ &= \frac{1}{y^0}\left(\frac{\partial F}{\partial x^\mu} - \frac{\partial^2 F}{\partial x^\mu \partial y^j} y^j\right) dx^0 + \frac{\partial^2 F}{\partial x^\mu \partial y^i} dx^i - d\left(\frac{\partial F}{\partial y^\mu}\right), \end{aligned} \tag{5.1.28}$$

the components of $\mu = 0$, and $\mu = 1, 2, \cdots, n$ are

$$\begin{aligned} \mathcal{EL}^F{}_0 &= \frac{\partial^2 F}{\partial x^0 \partial y^\rho} dx^\rho - d\left(\frac{\partial F}{\partial y^0}\right) \\ &= \frac{1}{y^0}\left(\frac{\partial F}{\partial x^0} - \frac{\partial^2 F}{\partial x^0 \partial y^j} y^j\right) dx^0 + \frac{\partial^2 F}{\partial x^0 \partial y^i} dx^i - \frac{1}{y^0} d\left(F - \frac{\partial F}{\partial y^j} y^j\right) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{(y^0)^2}\left(F-\frac{\partial F}{\partial y^j}y^j\right)dy^0,\\
\mathcal{EL}^F{}_i &= \frac{\partial^2 F}{\partial x^i \partial y^\rho}dx^\rho - d\left(\frac{\partial F}{\partial y^i}\right)\\
&= \frac{1}{y^0}\left(\frac{\partial F}{\partial x^i}-\frac{\partial^2 F}{\partial x^i \partial y^j}y^j\right)dx^0+\frac{\partial^2 F}{\partial x^i \partial y^i}dx^i - d\left(\frac{\partial F}{\partial y^i}\right).
\end{aligned}
\tag{5.1.29}
$$

The pull back to J^1Y becomes

$$
\begin{aligned}
\iota^*\mathcal{EL}^F{}_0 &= \left(\frac{1}{y^0}\left(\frac{\partial F}{\partial x^0}-\frac{\partial^2 F}{\partial x^0 \partial y^j}y^j\right)dx^0+\frac{\partial^2 F}{\partial x^0 \partial y^j}dx^j-\frac{1}{y^0}d\left(F-\frac{\partial F}{\partial y^j}y^j\right)\right)\circ\iota\\
&\quad+\left(\frac{1}{(y^0)^2}\left(F-\frac{\partial F}{\partial y^j}y^j\right)dy^0\right)\circ\iota\\
&= \left(\frac{\partial F}{\partial x^0}-\frac{\partial^2 F}{\partial x^0 \partial y^j}y^j\right)\circ\iota\, dt+\frac{\partial^2 F}{\partial x^0 \partial y^j}\circ\iota\, dq^j - d\left(\left(F-\frac{\partial F}{\partial y^j}y^j\right)\circ\iota\right)\\
&= \left(\frac{\partial \mathcal{L}}{\partial t}-\frac{\partial^2 \mathcal{L}}{\partial t \partial \dot{q}^j}\dot{q}^j\right)dt+\frac{\partial^2 \mathcal{L}}{\partial t \partial \dot{q}^j}dq^j - d\left(\mathcal{L}-\frac{\partial \mathcal{L}}{\partial \dot{q}^j}\dot{q}^j\right),\\
\iota^*\mathcal{EL}^F{}_i &= \left(\frac{1}{y^0}\left(\frac{\partial F}{\partial x^i}-\frac{\partial^2 F}{\partial x^i \partial y^j}y^j\right)dx^0+\frac{\partial^2 F}{\partial x^i \partial y^j}dx^j - d\left(\frac{\partial F}{\partial y^i}\right)\right)\circ\iota\\
&= \left(\left(\frac{\partial F}{\partial x^i}-\frac{\partial^2 F}{\partial x^i \partial y^j}y^j\right)\circ\iota\, dt+\frac{\partial^2 F}{\partial x^i \partial y^j}\circ\iota\, dq^j - d\left(\frac{\partial F}{\partial y^i}\circ\iota\right)\right)\\
&= \left(\left(\frac{\partial \mathcal{L}}{\partial q^i}-\frac{\partial^2 \mathcal{L}}{\partial q^i \partial \dot{q}^j}\dot{q}^j\right)dt+\frac{\partial^2 \mathcal{L}}{\partial q^i \partial \dot{q}^j}dq^j - d\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\right)\right).
\end{aligned}
\tag{5.1.30}
$$

Now suppose we have a map $\gamma^1 : \mathbb{R} \to J^1Y$ such that satisfies

$$\iota\circ\gamma^1 = \hat{\sigma}. \tag{5.1.31}$$

Then,

$$\left(\mathcal{EL}^F{}_\mu\circ\iota\right)\circ\gamma^1 = \mathcal{EL}^F{}_\mu\circ\left(\iota\circ\gamma^1\right) = \mathcal{EL}^F{}_\mu\circ\hat{\sigma}, \tag{5.1.32}$$

therefore, the equation of motion $\mathcal{EL}^F{}_\mu\circ\hat{\sigma} = 0$ where $\mathcal{EL}^F{}_\mu$ is a form on TM, can be interpreted as a equation of motion $\left(\mathcal{EL}^F{}_\mu\circ\iota\right)\circ\gamma^1 = 0$, where $\mathcal{EL}^F{}_\mu\circ\iota$ is a form on J^1Y.

In the special case where $Y = \mathbb{R}\times Q$, where Q is the n-dimensional configuration space, and J^1Y is the prolongation of the bundle $(Y, pr_1, \mathbb{R})$, we can consider a section γ of $(Y, pr_1, \mathbb{R})$, and take its prolongation $J^1\gamma$ as γ^1. In such case, the pull back equation

$\left(\mathcal{EL}^F{}_{\mu}\circ\iota\right)\circ\gamma^1=0$ becomes,

$$\begin{aligned}
&\mathcal{EL}^F{}_0\circ\iota\circ J^1\gamma=\left(\left(\frac{\partial\mathcal{L}}{\partial t}-\frac{\partial^2\mathcal{L}}{\partial t\partial\dot q^j}\dot q^j\right)dt+\frac{\partial^2\mathcal{L}}{\partial t\partial\dot q^j}dq^j-d\left(\mathcal{L}-\frac{\partial\mathcal{L}}{\partial\dot q^j}\dot q^j\right)\right)\circ J^1\gamma\\
&=\left(\left(\frac{\partial\mathcal{L}}{\partial t}-\frac{\partial^2\mathcal{L}}{\partial t\partial\dot q^j}\dot q^j\right)\circ J^1\gamma+\left(\frac{\partial^2\mathcal{L}}{\partial t\partial\dot q^j}\dot q^j\right)\circ J^1\gamma-\frac{d}{dt}\left(\left(\mathcal{L}-\frac{\partial\mathcal{L}}{\partial\dot q^j}\dot q^j\right)\circ J^1\gamma\right)\right)dt\\
&=\left(\frac{\partial\mathcal{L}}{\partial t}\right)\circ J^1\gamma\ dt\\
&\quad-\left(\frac{\partial\mathcal{L}}{\partial t}+\frac{\partial\mathcal{L}}{\partial q^j}\dot q^j+\frac{\partial\mathcal{L}}{\partial\dot q^j}\ddot q^j-\frac{\partial^2\mathcal{L}}{\partial t\partial\dot q^j}\dot q^j-\frac{\partial^2\mathcal{L}}{\partial q^k\partial\dot q^j}\dot q^k\dot q^j-\frac{\partial^2\mathcal{L}}{\partial\dot q^k\partial\dot q^j}\ddot q^k\dot q^j-\frac{\partial\mathcal{L}}{\partial\dot q^j}\ddot q^j\right)\circ J^1\gamma\,dt\\
&=-\left(\frac{\partial\mathcal{L}}{\partial q^j}\dot q^j-\frac{\partial^2\mathcal{L}}{\partial t\partial\dot q^j}\dot q^j-\frac{\partial^2\mathcal{L}}{\partial q^k\partial\dot q^j}\dot q^k\dot q^j-\frac{\partial^2\mathcal{L}}{\partial\dot q^k\partial\dot q^j}\ddot q^k\dot q^j\right)\circ J^1\gamma\,dt\\
&=-\left(\frac{\partial\mathcal{L}}{\partial q^j}\circ J^1\gamma-\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot q^j}\circ J^1\gamma\right)\right)\left(\dot q^j\circ J^1\gamma\right)\,dt=0,\\
&\mathcal{EL}^F{}_i\circ\iota\circ J^1\gamma=\left(\left(\frac{\partial\mathcal{L}}{\partial q^i}-\frac{\partial^2\mathcal{L}}{\partial q^i\partial\dot q^j}\dot q^j\right)dt+\frac{\partial^2\mathcal{L}}{\partial q^i\partial\dot q^j}\,dq^j-d\left(\frac{\partial\mathcal{L}}{\partial\dot q^i}\right)\right)\circ J^1\gamma\\
&=\left(\left(\frac{\partial\mathcal{L}}{\partial q^i}-\frac{\partial^2\mathcal{L}}{\partial q^i\partial\dot q^j}\dot q^j\right)\circ J^1\gamma+\left(\frac{\partial^2\mathcal{L}}{\partial q^i\partial\dot q^j}\,\dot q^j\right)\circ J^1\gamma\ -\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot q^i}\circ J^1\gamma\right)\right)dt\\
&=\left(\left(\frac{\partial\mathcal{L}}{\partial q^i}\right)\circ J^1\gamma-\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot q^i}\circ J^1\gamma\right)\right)dt=0,
\end{aligned}\tag{5.1.33}$$

therefore, giving us the well-known form of Euler-Lagrange equations. By fixing the bundle $(Y,pr_1,\mathbb{R})$, the parameterisation independent equation of motion reduces to the standard notion, and the number of equations degenerates to n.

Example 5.12. Every Newtonian mechanics in the form $\mathcal{L}=K-V$ with K the kinematical energy and V a potential term could be expressed in the setting of Finsler manifold.

Let $(M,\mathcal{F})$ be a Finsler manifold with $\dim M=n+1$, and the induced chart $(U,\psi),\psi=(x^\mu,y^\mu)$ on TM. The conventional Lagrangian function of the particle moving in n dimension space with mass m is given on the space J^1M, with $M=\mathbb{R}\times\mathbb{R}^n$. Let the induced chart on J^1M be $(\tilde U,\tilde\psi)$, $\tilde\psi=(t,q^i,\dot q^i)$, then the local expression of the Lagrangian function is,

$$\mathcal{L}=\frac{m}{2}\left((\dot q^1)^2+(\dot q^2)^2+\cdots+(\dot q^n)^2\right)-V(q^1,\ldots,q^n).\tag{5.1.34}$$

Then the local expression of the Finsler function obtained by (5.1.26) is ,

$$F=\frac{m}{2}\frac{(y^1)^2+(y^2)^2+\cdots+(y^n)^2}{y^0}-V(x^1,\ldots,x^n)y^0.\tag{5.1.35}$$

It is apparent that F satisfies the homogeneity condition. In the case of $M = \mathbb{R} \times \mathbb{R}^n$, this construction could be done globally, and we can create a Hilbert form by $\mathcal{F} = \dfrac{\partial F}{\partial y^i} dx^i$, which we take as our Lagrangian. The parameterisation independent equation of motion is obtained by the equation $\left(\mathcal{E}\mathcal{L}^F{}_\mu\right) \circ \hat{\sigma} = 0$, for any parameterisation σ, and the explicit coordinate expression can be calculated as,

$$
\begin{aligned}
\mathcal{E}\mathcal{L}^F{}_0 \circ \hat{\sigma} &= \left(\frac{\partial^2 F}{\partial x^0 \partial y^\rho} dx^\rho - d\left(\frac{\partial F}{\partial y^0}\right)\right) \circ \hat{\sigma} \\
&= \left(\frac{\partial V}{\partial x^k} dx^k - m \frac{(y^1)^2 + \cdots + (y^n)^2}{(y^0)^3} dy^0 + m \frac{y^k}{(y^0)^2} dy^k \right) \circ \hat{\sigma} = 0, \\
\mathcal{E}\mathcal{L}^F{}_i \circ \hat{\sigma} &= \left(\frac{\partial^2 F}{\partial x^i \partial y^\rho} dx^\rho - d\left(\frac{\partial F}{\partial y^i}\right)\right) \circ \hat{\sigma} \\
&= \left(-\frac{\partial V}{\partial x^i} dx^0 + m \frac{y^i}{(y^0)^2} dy^0 - m \frac{1}{y^0} dy^i \right) \circ \hat{\sigma} = 0,
\end{aligned} \tag{5.1.36}
$$

for $i = 1, \cdots, n$. We have used the relation such as,

$$
\begin{aligned}
&\frac{\partial F}{\partial y^0} = -\frac{1}{2} m \frac{(y^1)^2 + \cdots + (y^n)^2}{(y^0)^2}, \quad \frac{\partial F}{\partial y^i} = m \frac{y^i}{y^0}, \quad \frac{\partial^2 F}{\partial x^i \partial y^0} = -\frac{\partial V}{\partial x^i}, \\
&\frac{\partial^2 F}{\partial y^0 \partial y^0} = m \frac{(y^1)^2 + \cdots + (y^n)^2}{(y^0)^3}, \quad \frac{\partial^2 F}{\partial y^0 \partial y^i} = -m \frac{y^i}{(y^0)^2}, \quad \frac{\partial^2 F}{\partial y^i \partial y^j} = m \frac{1}{y^0} \delta^{ij},
\end{aligned} \tag{5.1.37}
$$

during the calculation. There are $n + 1$ equations; however, once a parameterisation is chosen and the equations are pulled back, it reduces to n equations by homogeneity condition. To see this, let us choose some convenient parameterisation $\sigma : I \to M$, such that $\sigma : t \to (t = x^0, x^1, \ldots, x^n)$. x^0 corresponds to the time variable, and now the y^0 coordinate of the tangent lift $\hat{\sigma}$ becomes 1. Consider the pull back of these equations, and we get

$$
\begin{aligned}
\mathcal{E}\mathcal{L}^F{}_0 \circ \hat{\sigma} &= \left(\frac{\partial V}{\partial x^i} \circ \hat{\sigma} + m^i \frac{d(y^i \circ \hat{\sigma})}{dt} \right) (y^i \circ \hat{\sigma}) dt = 0, \\
\mathcal{E}\mathcal{L}^F{}_i \circ \hat{\sigma} &= -\left(\frac{\partial V}{\partial x^i} \circ \hat{\sigma} + m \frac{d(y^i \circ \hat{\sigma})}{dt} \right) dt = 0,
\end{aligned} \tag{5.1.38}
$$

which is the usual Newton's equation of motion. Apparently, the first equation can be derived from the second by multiplying by y^i and summing up, therefore dependent, and we only have n equations.

The choice of σ is arbitrary, and it is not at all necessary to choose the one in Example 5.12. This freedom of choosing the parameterisation could be useful when one tries to find a good variable for solving equations.

5.2 Second order mechanics

Here we will present the Lagrange formulations for the higher order case of mechanics, in terms of Finsler-Kawaguchi geometry introduced in Section 4.2. We discuss especially for the second order. Higher order should follow in the similar extension. By the term *second order*, we mean that the total space we are considering is the second order tangent bundle, and by *mechanics*, we mean that we are considering the arc segment on M.

The basic structure we consider in this section is introduced in Chapter 2 and 4 (Section 4.2), namely the n-dimensional second order Finsler-Kawaguchi manifold $(M, \mathcal{K})$, the second order tangent bundle $(T^2M, \tau_M^{2,0}, M)$, and a 1-dimensional curve (arc segment) C on M, which is parameterised by σ. The curve (arc segment) describes the trajectory of the object on M.

We take the second-order Finsler-Kawaguchi form $\mathcal{K}$ as the Lagrangian, and the action will be defined by considering the integration over the second order lift of the parameterisable curve (arc segment) C. The Euler-Lagrange equations are derived by taking the variation of the action with respect to the flow on M that deforms the arc segment C, and fixed on the boundary. Similarly as in the previous first order case, we can show that the action and consequently the Euler-Lagrange equations are independent with respect to the parameterisation belonging to the same equivalent class.

5.2.1 Action

Suppose we have a dynamical system (differential equations expressing motions) where the trajectory of the point particle (or any object which dynamics could be considered as a point) is expressed by an arc segment C of a parameterisable curve, such that $C = \sigma(I) \subset M$, where I is a closed interval $I = [t_i, t_f] \subset \mathbb{R}$.

When we can express this system by second order Finsler-Kawaguchi geometry, namely the pair $(M, \mathcal{K})$ where $\mathcal{K}$ is a second order Finsler-Kawaguchi 1-form, we refer to this dynamical system as *second order mechanics*.

The action of second order mechanics is defined as follows.

Definition 5.13. Action of second order mechanics
Let $(M, \mathcal{K})$ be a n-dimensional second order Finsler-Kawaguchi manifold, (U, φ), $\varphi =$

(x^μ) be a chart on M, and (V^2, ψ^2), $V^2 = \left(\tau_M^{2,0}\right)^{-1}(U)$, $\psi = (x^\mu, y^\mu, z^\mu)$ the induced chart on T^2M. The local coordinate expression of the Finsler-Hilbert form $\mathcal{K} \in \Omega^1(T^2M)$ is given by

$$\mathcal{K} = \frac{\partial K}{\partial y^\mu} dx^\mu + 2\frac{\partial K}{\partial z^\mu} dy^\mu, \tag{5.2.1}$$

where K is the Finsler-Kawaguchi function. Let C be an arc segment on M, and σ its parameterisation, $\sigma(I) = C \subset M$ with $I = [t_i, t_f] \subset \mathbb{R}$. We call the functional $S^{\mathcal{K}}(C)$ defined by

$$S^{\mathcal{K}}(C) := l^K(C) = \int_{C^2} \mathcal{K} = \int_{\sigma^2(I)} \frac{\partial K}{\partial y^\mu} dx^\mu + 2\frac{\partial K}{\partial z^\mu} dy^\mu, \tag{5.2.2}$$

the *action of second order mechanics associated with* $\mathcal{K}$.

As we have seen in Section 4.2.2, Lemma 4.5, Finsler-Kawaguchi length is invariant with respect to the reparameterisation, therefore the action is also invariant.

5.2.2 Total derivative

Here we will introduce an operator called the *total derivative* that is an identity map on the total space of the bundle we consider. It becomes the derivative with respect to the parameter on the parameter space, namely $\frac{d}{dt}$ in this section, but later we will generalise this concept to the case of k-dimensional parameter space.

Consider the bundle morphism (Tf, f) from (TE, τ_E, E) to (TM, τ_M, M), introduced in (Example 2.12). For the case $E = TM$, there is an identity map called the *total derivative*.

Proposition 5.14. Consider the bundle morphism $(T\tau_M, \tau_M)$ from (TTM, τ_{TM}, TM) to (TM, τ_M, M). Let γ be a section of the sub-bundle $\tau_{TM}|_{T^2M}$ of τ_{TM}, and define a map $D : TM \to TM$ by $D = T\tau_M \circ \gamma$. Then D is an identity map which its local coordinate expression is given by

$$D = y^\mu \cdot \left(\frac{\partial}{\partial x^\mu}\right)_{\tau_M(\cdot)}. \tag{5.2.3}$$

Proof. By the definition of T^2M, we have $T\tau_M(\gamma(p)) = \tau_{TM}(\gamma(p))$ for $\forall p \in TM$. Then since $\tau_{TM} \circ \gamma = \tau_{TM}|_{T^2M} \circ \gamma = id_{TM}$, we have $T\tau_M \circ \gamma = id_{TM}$. This is an identity map of

$$
\begin{tikzcd}
TM \arrow[d, "\tau_M"] & TTM \arrow[l, "T\tau_M"'] \arrow[d, "\tau_{TM}"] \\
M & TM \arrow[l, "\tau_M"] \arrow[ul, bend right, "D"'] \arrow[u, bend right, "\gamma"'] \\
\mathbb{R} \arrow[u, bend left, "\sigma"] \arrow[ur, bend right, "\hat{\sigma}"']
\end{tikzcd}
$$

Figure 5.2: Total derivative

TM, and the coordinate expression becomes, $D(p) = p = y^\mu(p)\left(\frac{\partial}{\partial x^\mu}\right)_{\tau_M(p)}$, therefore, $D = y^\mu \cdot \left(\frac{\partial}{\partial x^\mu}\right)_{\tau_M(\cdot)}$. $\square$

Definition 5.15. Total derivative
We call the map D, the *total derivative* of first order.

Remark 5.16. The pull-back bundle of (TM, τ_M, M) by $\bar{\tau}_M : TM \to M$ is $((\bar{\tau}_M)^*TM, (\bar{\tau}_M)^*\tau_M, TM)$, and the total derivative D defines a unique section δ of $(\bar{\tau}_M)^*\tau_M$, which is called a *vector field along* $\bar{\tau}_M$, by $D = (\tau_M)^*\bar{\tau}_M \circ \delta$. By definition of the pull-back bundle, $(\bar{\tau}_M)^*\tau_M(p,q) = q$, $(\tau_M)^*\bar{\tau}_M(p,q) = p$ where p is a point of the total space of the bundle τ_M, and q is the point of the base space of the bundle $(\bar{\tau}_M)^*\tau_M$. Suppose we have $\delta(q) = (p,q) \in (\bar{\tau}_M)^*TM = TM \times_M TM$, then

$$D(q) = (\tau_M)^*\bar{\tau}_M \circ \delta(q) = (\tau_M)^*\bar{\tau}_M(p,q) = p, \tag{5.2.4}$$

but since D is an identity, we must have $p = q$, for all q, which means δ must be unique.

Indeed, consider a map $\sigma : \mathbb{R} \to M$, and then its tangent map $T\sigma$ will send the total derivative vector field $\dfrac{d}{dt}$ at $s \in \mathbb{R}$ to TM by

$$T_s\sigma(\frac{d}{dt}) = \left.\frac{d(x^\mu \circ \sigma)}{dt}\right|_s \left(\frac{\partial}{\partial x^\mu}\right)_{\sigma(s)}, \tag{5.2.5}$$

where in induced coordinates the components are the y^μ coordinates. To see this in the converse way, consider a smooth function $f \in C^\infty(M)$. $D(p)$ has the coordinate expression

$$D(p) = p = y^\mu(p)\left(\frac{\partial}{\partial x^\mu}\right)_{\tau_M(p)}, \tag{5.2.6}$$

so we can define a new smooth function Df on TM by

$$Df(p) := D(p)f = y^\mu(p)\left(\frac{\partial f}{\partial x^\mu}\right)_{\tau_M(p)}, \tag{5.2.7}$$

for $\forall p \in TM$, and therefore,

$$Df = y^\mu \cdot \left(\left(\frac{\partial f}{\partial x^\mu}\right) \circ \tau_M\right). \tag{5.2.8}$$

Consider a parameterisable curve C on M, and let the parameterisation be $\sigma : \mathbb{R} \to M$, and its lift $\hat{\sigma} : \mathbb{R} \to TM$. Then, along this curve,

$$\hat{\sigma}^* Df = Df \circ \hat{\sigma} = (y^\mu \circ \hat{\sigma}) \cdot \left(\frac{\partial f}{\partial x^\mu} \circ \tau_M\right) \circ \hat{\sigma} = \frac{d(x^\mu \circ \sigma)}{dt} \cdot \left(\frac{\partial f}{\partial x^\mu} \circ \tau_M\right) \circ \hat{\sigma} = \frac{d}{dt}(f \circ \sigma). \tag{5.2.9}$$

The total derivative can be also introduced for the higher order cases (Figure 5.3). Let (V^r, ψ^r), $\psi^r = (x^\mu_1, x^\mu_2, \cdots, x^\mu_{r+1})$ the induced chart on T^rM. Similar to the Proposition 5.14, we have the following.

Proposition 5.17. Consider the bundle morphism $(T\tau_M^{r,r-1}, \tau_M^{r,r-1})$ from $(TT^rM, \tau_{T^rM}, T^rM)$ to $(TT^{r-1}M, \tau_{T^{r-1}M}, T^{r-1}M)$. Let γ be the section of the sub-bundle $\tau_{T^rM}|_{T^{r+1}M}$, and define a map $D_r : T^rM \to TT^rM$ by $D_r = T\tau_M^{r,r-1} \circ \gamma$. Then D_r is an inclusion map $D_r = \iota_r$, $\iota_r : T^rM \to TT^rM$, such that its local coordinate expression is given by

$$D_r = x^\mu_2 \frac{\partial}{\partial x^\mu_1} + x^\mu_3 \frac{\partial}{\partial x^\mu_2} + \cdots + x^\mu_{r+1} \frac{\partial}{\partial x^\mu_r}. \tag{5.2.10}$$

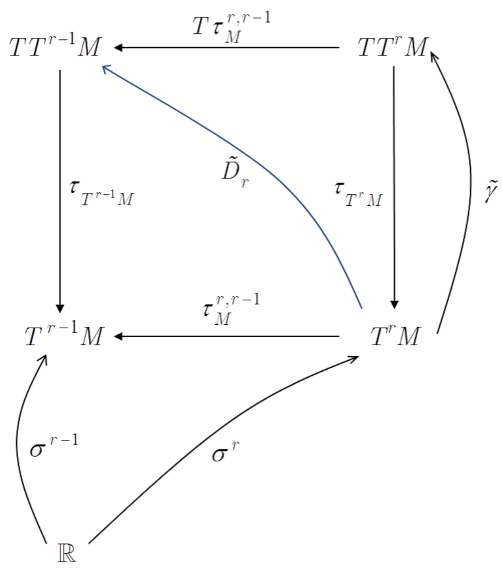

Figure 5.3: r-th order total derivative

Proof. By the definition of $T^{r+1}M$, we have $T\tau_M^{r,r-1}(\gamma(p)) = \iota_{r}\circ\tau_{T^rM}(\gamma(p))$ for $\forall p \in T^rM$. Then since $\tau_{T^rM}\circ\gamma = \tau_{T^rM}|_{T^{r+1}M}\circ\gamma = id_{T^rM}$, we have $T\tau_M^{r,r-1}\circ\gamma = \iota\circ id_{T^rM}$. This becomes an identity map on T^rM, and the coordinate expression becomes,

$$D_r(p) = x_2^\mu(p)\left(\frac{\partial}{\partial x_1^\mu}\right)_{\tau_M^{r,r-1}(p)} + x_3^\mu(p)\left(\frac{\partial}{\partial x_2^\mu}\right)_{\tau_M^{r,r-1}(p)} + \cdots + x_{r+1}^\mu(p)\left(\frac{\partial}{\partial x_r^\mu}\right)_{\tau_M^{r,r-1}(p)},$$

for $\forall p \in T^rM$, therefore,

$$D_r = x_2^\mu \cdot \left(\frac{\partial}{\partial x_1^\mu}\right)_{\tau_M^{r,r-1}(\cdot)} + x_3^\mu \cdot \left(\frac{\partial}{\partial x_2^\mu}\right)_{\tau_M^{r,r-1}(\cdot)} + \cdots + x_{r+1}^\mu \cdot \left(\frac{\partial}{\partial x_r^\mu}\right)_{\tau_M^{r,r-1}(\cdot)}.$$

□

Definition 5.18. r-th order total derivative
The identity map D_r is called the *r-th order total derivative*.

The map D_r for $r \geq 2$ cannot be expressed by a section of a vector bundle, therefore it is not possible to understand it as a vector field as in the case of D. As in the previous case, we consider the operation of the inclusion map $D_r : T^rM \to TT^{r-1}M$ to a smooth function g on $T^{r-1}M$, and define a new smooth function D_rg on T^rM by,

$$D_rg(w) := D_r(w)g = x_2^\mu(w)\left(\frac{\partial g}{\partial x_1^\mu}\right)_{\tau_M^{r,r-1}(w)} + x_3^\mu(w)\left(\frac{\partial g}{\partial x_2^\mu}\right)_{\tau_M^{r,r-1}(w)}$$

$$+\cdots+x^{\mu}_{r+1}(w)\left(\frac{\partial g}{\partial x^{\mu}_{r}}\right)_{\tau_M^{r,r-1}(w)}, \tag{5.2.11}$$

for $\forall w \in T^rM$, where $\tau_M^{r,r-1}$ is a projection: $\tau_M^{r,r-1} : T^rM \to T^{r-1}M$, and defined iteratively by $\tau_M^{r,r-1} := \tau_{T^{r-1}M}|_{T^rM}$. Now we get,

$$D_r g = x^{\mu}_2\left(\frac{\partial g}{\partial x^{\mu}_1}\circ\tau_M^{r,r-1}\right)+x^{\mu}_3\left(\frac{\partial g}{\partial x^{\mu}_2}\circ\tau_M^{r,r-1}\right)+\cdots+x^{\mu}_{r+1}\left(\frac{\partial g}{\partial x^{\mu}_r}\circ\tau_M^{r,r-1}\right). \tag{5.2.12}$$

Let us see more details, for the case of $r=2$ for simplicity. We will take the induced chart (V^2,ψ^2), $\psi^2=(x^\mu,y^\mu,z^\mu)$ on T^2M for some readability, and denote,

$$D_2 = y^\mu\frac{\partial}{\partial x^\mu}+z^\mu\frac{\partial}{\partial y^\mu}. \tag{5.2.13}$$

$$D_2 g = y^\mu\cdot\left(\frac{\partial g}{\partial x^\mu}\circ\tau_M^{2,1}\right)+z^\mu\cdot\left(\frac{\partial g}{\partial y^\mu}\circ\tau_M^{2,1}\right). \tag{5.2.14}$$

For the case where $g=Df$, we will have

$$\begin{aligned}
D_2(Df) &= y^\mu\cdot\left(\frac{\partial}{\partial x^\mu}Df\circ\tau_M^{2,1}\right)+z^\mu\cdot\left(\frac{\partial}{\partial y^\mu}Df\circ\tau_M^{2,1}\right)\\
&= y^\mu\cdot\left(\frac{\partial}{\partial x^\mu}\left(y^\rho\frac{\partial f}{\partial x^\rho}\circ\tau_M\right)\circ\tau_M^{2,1}\right)+z^\mu\cdot\left(\frac{\partial}{\partial y^\mu}\left(y^\rho\frac{\partial f}{\partial x^\rho}\circ\tau_M\right)\circ\tau_M^{2,1}\right)\\
&= y^\mu y^\rho\cdot\left(\frac{\partial^2 f}{\partial x^\mu\partial x^\rho}\circ\tau_M^{2,0}\right)+z^\mu\cdot\left(\frac{\partial f}{\partial x^\mu}\circ\tau_M^{2,0}\right).
\end{aligned} \tag{5.2.15}$$

To see this becomes the total derivative with respect to the parameterisation space, consider a parameterisable curve C on M, and let the parameterisation be $\sigma:\mathbb{R}\to M$, its lift $\hat\sigma:\mathbb{R}\to TM$, and its second order lift $\sigma^2:\mathbb{R}\to T^2M$. Then, along this curve,

$$\begin{aligned}
(\sigma^2)^*D_2g &= D_2g\circ\sigma^2 = (y^\mu\circ\sigma^2)\cdot\left(\frac{\partial g}{\partial x^\mu}\circ\tau_M^{2,1}\right)\circ\sigma^2+(z^\mu\circ\sigma^2)\cdot\left(\frac{\partial g}{\partial y^\mu}\circ\tau_M^{2,1}\right)\circ\sigma^2\\
&= (y^\mu\circ\hat\sigma)\cdot\left(\frac{\partial g}{\partial x^\mu}\right)\circ\hat\sigma+(z^\mu\circ\sigma^2)\cdot\left(\frac{\partial g}{\partial y^\mu}\right)\circ\hat\sigma\\
&= \frac{d}{dt}(g\circ\hat\sigma).
\end{aligned} \tag{5.2.16}$$

In the case of $g = Df$, we can further use the relation (5.2.9), and get

$$(\sigma^2)^* D_2(Df) = \frac{d}{dt}(Df \circ \hat{\sigma}) = \frac{d^2}{dt^2}(f \circ \sigma). \tag{5.2.17}$$

In order to see the connection to the derivatives with respect to the parameter space, we used the pull-back. Nevertheless, the total derivative on the total space itself can be defined by the bundle morphism only, and no consideration of curves on M or its parameterisation is required.

For general r, similar discussions could be made. For instance, we will have

$$\begin{aligned}(\sigma^r)^* D_r(D_{r-1}(\cdots Df)\cdots) &= \frac{d}{dt}((D_{r-1}(\cdots Df)\cdots)\circ \sigma^{r-1}) \\ = \frac{d^2}{dt^2}((D_{r-2}(\cdots Df)\cdots)\circ \sigma^{r-2}) &= \cdots = \frac{d^{r-1}}{dt^{r-1}}(Df \circ \hat{\sigma}) = \frac{d^r}{dt^r}(f \circ \sigma),\end{aligned} \tag{5.2.18}$$

by iteration.

5.2.3 Extremal and equations of motion

Having defined the action, we are able to derive the equations of motion by considering the extremal of the action. As in the previous section, we only consider global flows. Nevertheless with some details added; the formulation can be made similarly with local flows. Consider a C^∞-flow, $\alpha : \mathbb{R} \times M \to M$, and its associated 1-parameter group of transformations $\{\alpha_s\}_{s\in\mathbb{R}}$. The 1-parameter group $\alpha_s : M \to M$ induces a 1-parameter group $TT\alpha_s : T^2M \to T^2M$ generated by the induced tangent mapping. This will also modify the curve (arc segment) C to $C' = \alpha_s(C)$, which will be now parameterised by σ'. As in the first order case, the variation will be expressed by the small deformations made to the action by α_s.

Before proceeding, we will first check that the induced 1-parameter group $TT\alpha_s : TTM \to TTM$ is also a 1-parameter group on T^2M.

Let (U, φ), $\varphi = (x^\mu)$ be a chart on M, (V, ψ), $V = {\tau_M}^{-1}(U)$, $\psi = (x^\mu, y^\mu)$ the induced chart on TM, $(\tilde{V}, \tilde{\psi})$, $\tilde{V} = {\tau_{TM}}^{-1}(V)$, $\tilde{\psi} = (x^\mu, y^\mu, \dot{x}^\mu, \dot{y}^\mu)$ the induced chart on TTM, and (V^2, ψ^2) $V^2 = \tilde{V}\Big|_{T^2M}$, $\psi^2 = (x^\mu, y^\mu, z^\mu)$ the induced chart on T^2M. The projection maps are denoted by $\tau_M : TM \to M$, $\tau_{TM} : TTM \to TM$, $\tau_M^2 := \tau_M \circ \tau_{TM}$, $\tau_M^{2,1} : T^2M \to TM$, $\tau_M^{2,0} := \tau_M \circ \tau_M^{2,1}$. The local expression of $w_q \in T_qTM$, $q \in TM$ is

$$w_q = w^\mu \left(\frac{\partial}{\partial x^\mu}\right)_q + \tilde{w}^\mu \left(\frac{\partial}{\partial y^\mu}\right)_q . \tag{5.2.19}$$

Then, $TT\alpha_s$ maps $w_q$by

$$TT\alpha_s(w_q) = \left.\frac{\partial(x^\nu\circ T\alpha_s\circ\psi^{-1})}{\partial x^\mu}\right|_{\psi(q)} w^\mu \left(\frac{\partial}{\partial x^\nu}\right)_{T\alpha_s(q)} + \left.\frac{\partial(y^\nu\circ T\alpha_s\circ\psi^{-1})}{\partial x^\mu}\right|_{\psi(q)} w^\mu \left(\frac{\partial}{\partial y^\nu}\right)_{T\alpha_s(q)} + \left.\frac{\partial(y^\nu\circ T\alpha_s\circ\psi^{-1})}{\partial y^\mu}\right|_{\psi(q)} \tilde{w}^\mu \left(\frac{\partial}{\partial y^\nu}\right)_{T\alpha_s(q)}, \tag{5.2.20}$$

Where $T\alpha_s$ is the induced 1-parameter group on TM by α_s.

In components of coordinates of TTM, this is

$$\begin{aligned}
x^\mu\circ TT\alpha_s(w_q) &= x^\mu\circ T\alpha_s(q) = x^\mu\circ\alpha_s\circ\tau_M(q) = x^\mu\circ\alpha_s\circ\tau_M\circ\tau_{TM}(w_q) = x^\mu\circ\alpha_s\circ\tau_M^2(w_q),\\
y^\mu\circ TT\alpha_s(w_q) &= y^\mu\circ T\alpha_s(q) = \left(\left.\frac{\partial(x^\mu\circ\alpha_s\circ\varphi^{-1})}{\partial x^\nu}\right|_{\varphi(\tau_M^2(\cdot))} y^\nu\right)(w_q),\\
\dot{x}^\mu\circ TT\alpha_s(w_q) &= \left.\frac{\partial(x^\mu\circ T\alpha_s\circ\psi^{-1})}{\partial x^\nu}\right|_{\psi(\tau_{TM}(w_q))} w^\nu = \left(\left.\frac{\partial(x^\mu\circ\alpha_s\circ\varphi^{-1})}{\partial x^\nu}\right|_{\varphi(\tau_M^2(\cdot))} \dot{x}^\nu\right)(w_q),\\
\dot{y}^\mu\circ TT\alpha_s(w_q) &= \left.\frac{\partial(y^\mu\circ T\alpha_s\circ\psi^{-1})}{\partial x^\nu}\right|_{\psi(\tau_{TM}(w_q))} w^\nu + \left.\frac{\partial(y^\mu\circ T\alpha_s\circ\psi^{-1})}{\partial y^\nu}\right|_{\psi(\tau_{TM}(w_q))} \tilde{w}^\nu\\
&= \frac{\partial}{\partial x^\nu}\left.\left(\left(\left.\frac{\partial(x^\mu\circ\alpha_s\circ\varphi^{-1})}{\partial x^\rho}\right|_{\varphi(\tau_M(\cdot))} y^\rho\right)\circ\psi^{-1}\right)\right|_{\psi(\tau_{TM}(w_q))} \dot{x}^\nu(w_q)\\
&\quad+ \frac{\partial}{\partial y^\nu}\left.\left(\left(\left.\frac{\partial(x^\mu\circ\alpha_s\circ\varphi^{-1})}{\partial x^\rho}\right|_{\varphi(\tau_M(\cdot))} y^\rho\right)\circ\psi^{-1}\right)\right|_{\psi(\tau_{TM}(w_q))} \dot{y}^\nu(w_q)\\
&= \left(\left.\frac{\partial^2(x^\mu\circ\alpha_s\circ\varphi^{-1})}{\partial x^\nu\partial x^\rho}\right|_{\varphi(\tau_M^2(\cdot))} y^\rho\dot{x}^\nu + \left.\frac{\partial(x^\mu\circ\alpha_s\circ\varphi^{-1})}{\partial x^\nu}\right|_{\varphi(\tau_M^2(\cdot))} \dot{y}^\nu\right)(w_q).
\end{aligned} \tag{5.2.21}$$

In the case when $w_q \in T^2M$, we have $w^\mu = y^\mu(w_q) = \dot{x}^\mu(w_q)$, and the expressions become,

$$\begin{aligned}
x^\mu\circ TT\alpha_s(w_q) &= x^\mu\circ\alpha_s\circ\tau_M^{2,0}(w_q),\\
y^\mu\circ TT\alpha_s(w_q) &= \left(\left.\frac{\partial(x^\mu\circ\alpha_s\circ\varphi^{-1})}{\partial x^\nu}\right|_{\varphi(\tau_M^{2,0}(\cdot))} y^\nu\right)(w_q),\\
\dot{x}^\mu\circ TT\alpha_s(w_q) &= \left(\left.\frac{\partial(x^\mu\circ\alpha_s\circ\varphi^{-1})}{\partial x^\nu}\right|_{\varphi(\tau_M^{2,0}(\cdot))} y^\nu\right)(w_q),
\end{aligned}$$

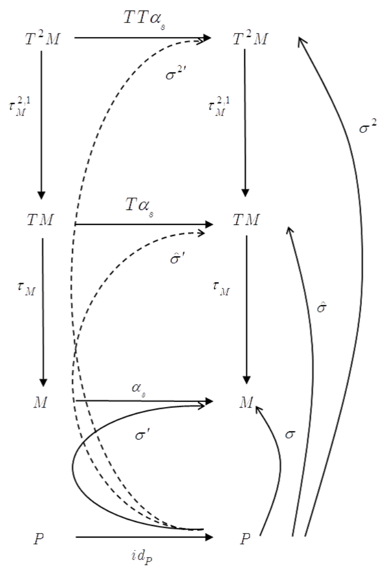

Figure 5.4: Second order mechanics

$$\dot{y}^\mu \circ TT\alpha_s(w_q) = \left(\frac{\partial^2 (x^\mu \circ \alpha_s \circ \varphi^{-1})}{\partial x^\nu \partial x^\rho} \bigg|_{\varphi(\tau_M^{2,0}(\cdot))} y^\rho y^\nu + \frac{\partial (x^\mu \circ \alpha_s \circ \varphi^{-1})}{\partial x^\nu} \bigg|_{\varphi(\tau_M^{2,0}(\cdot))} \dot{y}^\nu \right) (w_q). \tag{5.2.22}$$

Therefore, $y^\mu \circ TT\alpha_s(w_p) = \dot{x}^\mu \circ TT\alpha_s(w_p)$, and the 1-parameter group $TT\alpha_s$will take the elements of T^2M to T^2M.

With the above considerations, we define the following.

Definition 5.19. Variation of the action
Let ξ be a vector field on M which generates the 1-parameter group α_s, i.e., $\xi = \left.\frac{d\alpha_s}{ds}\right|_{s=0}$.
We call the functional

$$\begin{aligned} \delta_\xi S^{\mathcal{K}}(C) &:= \lim_{s\to 0} \frac{1}{s} \left\{ S^{\mathcal{K}}(\alpha_s(C)) - S^{\mathcal{K}}(C) \right\} \\ &= \lim_{s\to 0} \frac{1}{s} \left\{ \int_{(\alpha_s(\sigma))^2(I)} \mathcal{K} - \int_{\sigma^2(I)} \mathcal{K} \right\} \end{aligned} \tag{5.2.23}$$

the *variation of the action $S^{\mathcal{K}}(C)$ with respect to the flow α, associated to $\mathcal{K}$.*

The second order lift of this modified parameterisation σ' is given by,

$$(\sigma')^2 = (\alpha_s \circ \sigma \circ id_I{}^{-1})^2 = TT\alpha_s \circ \sigma^2 \circ id_I{}^{-1}, \tag{5.2.24}$$

which we show in figure 5.4. We can also check this easily. We will get,

$$\begin{aligned}\delta_\xi S^{\mathcal{K}}(C) &= \lim_{s\to 0}\frac{1}{s}\left\{\int_{TT\alpha_s\circ\sigma^2\circ {id_I}^{-1}(I)}\mathcal{K} - \int_{\sigma^2(I)}\mathcal{K}\right\}\\ &= \lim_{s\to 0}\frac{1}{s}\left\{\int_{\sigma^2(I)} TT{\alpha_s}^*\mathcal{K} - \int_{\sigma^2(I)}\mathcal{K}\right\} = \int_{\sigma^2(I)} L_X\mathcal{K}\\ &= \int_{C^2} L_X\mathcal{K}\end{aligned} \tag{5.2.25}$$

where X is a vector field on T^2M that generates the tangent 1-parameter group $TT\alpha_s$, i.e., $X = \left.\dfrac{d(TT\alpha_s)}{ds}\right|_{s=0}$, and L_X is a Lie derivative with respect to X.

We use the same definition of extremal given by Definition 5.3. We will calculate the vector field X and the equation of motion in local coordinates. Let ξ be a vector field related to the 1-parameter group α_s, $\xi = \dfrac{d\alpha_s}{ds}$, and its local expression $\xi = \xi^\mu\dfrac{\partial}{\partial x^\mu}$, where $\xi^\mu \in C^\infty(M)$. We have already calculated the components of the map $TT\alpha_s$ in (5.2.22), which in the chart of T^2M, becomes

$$\begin{aligned}x^\mu\circ TT\alpha_s(w_q) &= x^\mu\circ\alpha_s\circ\tau_M^{2,0}(w_q),\\ y^\mu\circ TT\alpha_s(w_q) &= \left(\left.\frac{\partial(x^\mu\circ\alpha_s\circ\varphi^{-1})}{\partial x^\nu}\right|_{\varphi(\tau_M^{2,0}(\cdot))} y^\nu\right)(w_q),\\ z^\mu\circ TT\alpha_s(w_q) &= \left(\left.\frac{\partial^2(x^\mu\circ\alpha_s\circ\varphi^{-1})}{\partial x^\nu\partial x^\rho}\right|_{\varphi(\tau_M^{2,0}(\cdot))} y^\nu y^\rho + \left.\frac{\partial(x^\mu\circ\alpha_s\circ\varphi^{-1})}{\partial x^\nu}\right|_{\varphi(\tau_M^{2,0}(\cdot))} z^\nu\right)(w_q).\end{aligned} \tag{5.2.26}$$

By these observations, the vector field X on T^2M at a point $w \in T^2M$ has a local expression,

$$\begin{aligned}X_w = &\left.\frac{d(x^\mu\circ TT\alpha_s)}{ds}\right|_{s=0}\left(\frac{\partial}{\partial x^\mu}\right)_w + \left.\frac{d(y^\mu\circ TT\alpha_s)}{ds}\right|_{s=0}\left(\frac{\partial}{\partial y^\mu}\right)_w + \left.\frac{d(z^\mu\circ TT\alpha_s)}{ds}\right|_{s=0}\left(\frac{\partial}{\partial z^\mu}\right)_w\\ = &\left.\frac{d}{ds}(x^\mu\circ\alpha_s\circ\tau_M^{2,0})\right|_{s=0}\left(\frac{\partial}{\partial x^\mu}\right)_w + y^\nu(w)\frac{d}{ds}\left(\left.\frac{\partial(x^\mu\circ\alpha_s\circ\varphi^{-1})}{\partial x^\nu}\right|_{\varphi(\tau_M^{2,0}(w))}\right)_{s=0}\left(\frac{\partial}{\partial y^\mu}\right)_w\\ &+\frac{d}{ds}\left(\left.\frac{\partial^2(x^\mu\circ\alpha_s\circ\varphi^{-1})}{\partial x^\nu\partial x^\rho}\right|_{\varphi(\tau_M^{2,0}(w))} y^\nu y^\rho + \left.\frac{\partial(x^\mu\circ\alpha_s\circ\varphi^{-1})}{\partial x^\nu}\right|_{\varphi(\tau_M^{2,0}(w))} z^\nu\right)\left(\frac{\partial}{\partial z^\mu}\right)_w\\ = &\left(\xi^\mu\circ\tau_M^{2,0}\right)(w)\left(\frac{\partial}{\partial x^\mu}\right)_w + \left(\frac{\partial\xi^\mu}{\partial x^\nu}\circ\tau_M^{2,0}\cdot y^\nu\right)(w)\left(\frac{\partial}{\partial y^\mu}\right)_w\end{aligned}$$

$$+\left(\frac{\partial^2 \xi^\mu}{\partial x^\nu \partial x^\rho} \circ \tau_M^{2,0} \cdot y^\nu y^\rho + \frac{\partial \xi^\mu}{\partial x^\nu} \circ \tau_M^{2,0} \cdot z^\nu\right)(w)\left(\frac{\partial}{\partial z^\mu}\right)_w . \tag{5.2.27}$$

therefore,

$$X = (\xi^\mu \circ \tau_M^{2,0})\frac{\partial}{\partial x^\mu} + \left(\frac{\partial \xi^\mu}{\partial x^\nu} \circ \tau_M^{2,0} \cdot y^\nu\right)\frac{\partial}{\partial y^\mu} + \left(\frac{\partial^2 \xi^\mu}{\partial x^\nu \partial x^\rho} \circ \tau_M^{2,0} \cdot y^\nu y^\rho + \frac{\partial \xi^\mu}{\partial x^\nu} \circ \tau_M^{2,0} \cdot z^\nu\right)\frac{\partial}{\partial z^\mu}. \tag{5.2.28}$$

We can make this expression shorter by using the total derivatives we defined in Definition 5.15, 5.2.11 (or (5.2.13)),

$$X = \xi^\mu \circ \tau_M^{2,0}\frac{\partial}{\partial x^\mu} + D\xi^\mu \circ \tau_M^{2,1}\frac{\partial}{\partial y^\mu} + D_2(D\xi^\mu)\frac{\partial}{\partial z^\mu}. \tag{5.2.29}$$

The Lie derivative $L_X\mathcal{K}$ in coordinate expression becomes,

$$\begin{aligned}
L_X\mathcal{K} &= L_X\left(\frac{\partial K}{\partial y^\rho}dx^\rho + 2\frac{\partial K}{\partial z^\rho}dy^\rho\right)\\
&= X\left(\frac{\partial K}{\partial y^\rho}\right)dx^\rho + \frac{\partial K}{\partial y^\rho}dL_X x^\rho + 2X\left(\frac{\partial K}{\partial z^\rho}\right)dy^\rho + 2\frac{\partial K}{\partial z^\rho}dL_X y^\rho\\
&= \left\{\xi^\mu \circ \tau_M^{2,0}\cdot\left(\frac{\partial^2 K}{\partial x^\mu \partial y^\rho}\right) + D\xi^\mu \circ \tau_M^{2,1}\cdot\left(\frac{\partial^2 K}{\partial y^\mu \partial y^\rho}\right) + D_2(D\xi^\mu)\frac{\partial^2 K}{\partial z^\mu \partial y^\rho}\right\}dx^\rho + \frac{\partial K}{\partial y^\rho}d\xi^\rho\\
&\quad + 2\left\{\xi^\mu \circ \tau_M^{2,0}\cdot\left(\frac{\partial^2 K}{\partial x^\mu \partial z^\rho}\right) + D\xi^\mu \circ \tau_M^{2,1}\cdot\left(\frac{\partial^2 K}{\partial y^\mu \partial z^\rho}\right) + D_2(D\xi^\mu)\frac{\partial^2 K}{\partial z^\mu \partial z^\rho}\right\}dy^\rho\\
&\quad + 2\frac{\partial K}{\partial z^\rho}d\left(D\xi^\mu\right)\\
&= \xi^\mu \circ \tau_M^{2,0}\cdot\left\{\frac{\partial^2 K}{\partial x^\mu \partial y^\rho}dx^\rho - d\left(\frac{\partial K}{\partial y^\mu}\right) + 2\frac{\partial^2 K}{\partial x^\mu \partial z^\rho}dy^\rho\right\}\\
&\quad + D\xi^\mu \circ \tau_M^{2,1}\cdot\left(\frac{\partial^2 K}{\partial y^\mu \partial y^\rho}dx^\rho + 2\frac{\partial^2 K}{\partial y^\mu \partial z^\rho}dy^\rho - 2d\left(\frac{\partial K}{\partial z^\mu}\right)\right)\\
&\quad + D_2(D\xi^\mu)\left(\frac{\partial^2 K}{\partial z^\mu \partial y^\rho}dx^\rho + 2\frac{\partial^2 K}{\partial z^\mu \partial z^\rho}dy^\rho\right)\\
&\quad + d\left(\xi^\mu \circ \tau_M^{2,0}\cdot\frac{\partial K}{\partial y^\mu} + 2D\xi^\mu \circ \tau_M^{2,1}\cdot\frac{\partial K}{\partial z^\mu}\right).
\end{aligned} \tag{5.2.30}$$

The result of (5.2.30) is called the *infinitesimal first variation formula* for the Finsler-Kawaguchi form $\mathcal{K}$.

The variation of action becomes,

$$
\begin{aligned}
\delta_\xi S^{\mathcal{K}}(C) &= \int_{\sigma^2(I)} L_X \mathcal{K} = \int_I \sigma^{2*} L_X \mathcal{K} \\
&= \int_{C^3} \xi^\mu \circ \tau_M^{3,0} \cdot \left[\left\{ \frac{\partial^2 K}{\partial x^\mu \partial y^\rho} dx^\rho + 2 \frac{\partial^2 K}{\partial x^\mu \partial z^\rho} dy^\rho - d \left(\frac{\partial K}{\partial y^\mu} \right) \right\} \circ \tau_M^{3,2} + d \left(D_3 \frac{\partial K}{\partial z^\mu} \right) \right] \\
&\quad + \int_{C^3} d \left(\xi^\mu \circ \tau_M^{3,0} \cdot \frac{\partial K}{\partial y^\mu} \circ \tau_M^{3,2} + 2 D \xi^\mu \circ \tau_M^{3,1} \cdot \frac{\partial K}{\partial z^\mu} \circ \tau_M^{3,2} - D_3 \left(\xi^\mu \circ \tau_M^{2,0} \cdot \frac{\partial K}{\partial z^\mu} \right) \right),
\end{aligned}
\tag{5.2.31}
$$

which is called the *integral first variation formula*. D_3 is the total derivative defined by Definition 5.2.11. We used the homogeneity condition:

$$
\begin{aligned}
& K = \frac{\partial K}{\partial y^\mu} y^\mu + 2 \frac{\partial K}{\partial z^\mu} z^\mu, \quad \frac{\partial K}{\partial z^\mu} y^\mu = 0 \\
& \frac{\partial^2 K}{\partial x^\rho \partial y^\mu} y^\mu + 2 \frac{\partial^2 K}{\partial x^\rho \partial z^\mu} z^\mu = \frac{\partial K}{\partial x^\rho}, \\
& \frac{\partial^2 K}{\partial y^\rho \partial y^\mu} y^\mu + 2 \frac{\partial^2 K}{\partial y^\rho \partial z^\mu} z^\mu = 0, \\
& \frac{\partial^2 K}{\partial y^\rho \partial z^\mu} y^\mu + \frac{\partial K}{\partial z^\rho} = 0, \quad \frac{\partial^2 K}{\partial y^\rho z^\mu} y^\mu + \frac{\partial K}{\partial y^\rho} = 0,
\end{aligned}
\tag{5.2.32}
$$

and its pull-back relations on the parameter space.

The detailed calculations are shown in the Appendix.

Now we can obtain the following theorem.

Theorem 5.20. Extremals
Let C be an arc segment. The following statements are equivalent.

1. C is an extremal.

2. The equation

$$
\begin{aligned}
& \mathcal{EL}^K{}_\mu \circ \sigma^3 = 0, \\
& \mathcal{EL}^K{}_\mu := \left\{ \frac{\partial^2 K}{\partial x^\mu \partial y^\rho} dx^\rho + 2 \frac{\partial^2 K}{\partial x^\mu \partial z^\rho} dy^\rho - d \left(\frac{\partial K}{\partial y^\mu} \right) \right\} \circ \tau_M^{3,2} + d \left(D_3 \frac{\partial K}{\partial z^\mu} \right),
\end{aligned}
\tag{5.2.33}
$$

holds for arbitrary parameterisation σ.

Proof. Suppose C is an extremal. Then, by definition, for all $\alpha_s : M \to M$, such that does not change the boundary of C, we have $\delta_\xi S(C) = 0$. On the other hand, the last term in (5.2.31) becomes 0, since it is the boundary term. Therefore, we have,

$$\int_{C^3} \xi^\mu \circ \tau_M^{3,0} \left[\left\{ \frac{\partial^2 K}{\partial x^\mu \partial y^\rho} dx^\rho + 2 \frac{\partial^2 K}{\partial x^\mu \partial z^\rho} dy^\rho - d\left(\frac{\partial K}{\partial y^\mu} \right) \right\} \circ \tau_M^{3,2} + d\left(D_3 \frac{\partial K}{\partial z^\mu} \right) \right] = 0. \tag{5.2.34}$$

Since this relation must be true for all ξ, which is the generator of α, we have

$$\left[\left\{ \frac{\partial^2 K}{\partial x^\mu \partial y^\rho} dx^\rho + 2 \frac{\partial^2 K}{\partial x^\mu \partial z^\rho} dy^\rho - d\left(\frac{\partial K}{\partial y^\mu} \right) \right\} \circ \tau_M^{3,2} + d\left(D_3 \frac{\partial K}{\partial z^\mu} \right) \right] \circ \sigma^3 = 0, \tag{5.2.35}$$

for any parameterisation σ. To prove the converse, take the similar steps backwards. $\square$

Definition 5.21. Symmetry of the dynamical system
Let u be a vector field over M, and Y an induced vector field by u over T^2M. We say that $\mathcal{K}$ *is invariant with respect to* u, if

$$L_Y \mathcal{K} = 0, \tag{5.2.36}$$

and u called a *symmetry* of the dynamical system $(M, \mathcal{K})$. We also say that u generates the invariant transformations on the Finsler-Kawaguchi manifold $(M, \mathcal{K})$.

Now we will have the following conservation law.

Theorem 5.22. Noether (second order)
Suppose we are given a symmetry of second order Finsler-Kawaguchi manifold $(M, \mathcal{K})$. Then there exists a function f on T^3M, which along the extremal γ of $S^{\mathcal{K}}$ satisfies,

$$\int_{\gamma^3} df = 0, \tag{5.2.37}$$

for any parameterisation σ which parameterise γ.

Proof. Let the symmetry be u, with its local coordinate expression $u = u^\mu \dfrac{\partial}{\partial x^\mu}$, and the induced vector field Y. Then from (5.2.31), we have

$$\begin{aligned}
0 &= \int_{\gamma^3} L_Y \mathcal{K} \\
&= \int_{\gamma^3} u^\mu \circ \tau_M^{3,0} \cdot \left[\left\{ \frac{\partial^2 K}{\partial x^\mu \partial y^\rho} dx^\rho + 2 \frac{\partial^2 K}{\partial x^\mu \partial z^\rho} dy^\rho - d\left(\frac{\partial K}{\partial y^\mu} \right) \right\} \circ \tau_M^{3,2} + d\left(D_3 \frac{\partial K}{\partial z^\mu} \right) \right] \\
&\quad + \int_{\gamma^3} d\left(u^\mu \circ \tau_M^{3,0} \cdot \frac{\partial K}{\partial y^\mu} \circ \tau_M^{3,2} + 2 D u^\mu \circ \tau_M^{3,1} \cdot \frac{\partial K}{\partial z^\mu} \circ \tau_M^{3,2} - D_3 \left(u^\mu \circ \tau_M^{2,0} \cdot \frac{\partial K}{\partial z^\mu} \right) \right)
\end{aligned}$$

$$= \int_{\gamma^3} d\left(u^\mu \circ \tau_M^{3,0} \cdot \frac{\partial K}{\partial y^\mu} \circ \tau_M^{3,2} + 2Du^\mu \circ \tau_M^{3,1} \cdot \frac{\partial K}{\partial z^\mu} \circ \tau_M^{3,2} - D_3 \left(u^\mu \circ \tau_M^{2,0} \cdot \frac{\partial K}{\partial z^\mu} \right) \right). \tag{5.2.38}$$

The second equality comes from the fact we consider along the extremal γ. Therefore we have a function on T^3M,

$$f = u^\mu \circ \tau_M^{3,0} \cdot \frac{\partial K}{\partial y^\mu} \circ \tau_M^{3,2} + 2Du^\mu \circ \tau_M^{3,1} \cdot \frac{\partial K}{\partial z^\mu} \circ \tau_M^{3,2} - D_3 \left(u^\mu \circ \tau_M^{2,0} \cdot \frac{\partial K}{\partial z^\mu} \right) \tag{5.2.39}$$

such that satisfies the condition. □

We call the relation (5.2.37), the *conservation law*. We can express the conservation law (5.2.37) by taking arbitrary parameterisation for this γ.

Definition 5.23. Noether current
The quantity f is called the *Noether current associated with* u.

5.3 First order field theory

Here we present the Lagrange formulations for the first order field theory, in terms of Kawaguchi geometry.

By the term *first order field theory*, we mean that the total space we are considering is the first order k-multivector bundle $\Lambda^k TM$ with k-patch in its base space. The total space is sometimes also called the *ambient space*, and its coordinate functions represents the physical fields as well as the spacetime coordinates. In this sense, the coordinate functions of M are regarded as unified variables, and the k-dimensional submanifold of M represents the actual spacetime.

The basic structure we consider in this section is introduced in Chapter 2 and 4 (Section 4.3), namely the n-dimensional k-areal Kawaguchi manifold $(M, \mathcal{K})$, the k-multivector bundle $(\Lambda^k TM, \Lambda^k \tau_M, M)$, and a k-curve (k-patch) Σ on M, parameterised by σ.

We take the Kawaguchi form $\mathcal{K}$ as the Lagrangian, and the action will be defined by considering the integration over the lift of the parameterisable k-curve (k-patch). The Euler-Lagrange equations are derived by taking the variation of the action with respect to the flow on M that deforms the k-patch Σ, and fixed on the boundary. We can show that the action and consequently the Euler-Lagrange equations are independent with respect to the parameterisation belonging to the same equivalent class.

5.3.1 Action

Suppose we have a dynamical system (differential equations expressing motions) where the configurations of the k-dimensional spacetime (or any extended object of dimension k) is expressed as a smooth k-patch Σ of a parameterisable k-area, such that $\Sigma = \sigma(P) \subset M$, where P is a closed rectangle $P = [t_i^1, t_f^1] \times [t_i^2, t_f^2] \times \cdots \times [t_i^k, t_f^k] \subset \mathbb{R}^k$.

When we can express this system by first order k-areal Kawaguchi geometry, namely the pair $(M, \mathcal{K})$ where $\mathcal{K}$ is a first order Kawaguchi k-form, we refer to this dynamical system as *first order field theory*.

The action of first order field theory is defined as follows.

Definition 5.24. Action of first order fields
Let $(M, \mathcal{K})$ be a n-dimensional k-areal Kawaguchi manifold. Consider a k-multivector bundle $(\Lambda^k TM, \Lambda^k \tau_M, M)$ and let (U, φ), $\varphi = (x^\mu)$ be a chart on M, and (V, ψ), $\psi = (x^\mu, y^{\mu_1 \cdots \mu_k})$ the induced chart on $\Lambda^k TM$.

The local expression of the Kawaguchi form $\mathcal{K} \in \Omega^k(\Lambda^k TM)$ is given by

$$\mathcal{K} = \frac{1}{k!} \frac{\partial K}{\partial y^{\mu_1 \cdots \mu_k}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}, \tag{5.3.1}$$

where K is the first order k-areal Kawaguchi function.

Let Σ be a k-patch on M, and σ its parameterisation, $\sigma(P) = \Sigma \subset M$ with $P = [t_i^1, t_f^1] \times [t_i^2, t_f^2] \times \cdots \times [t_i^k, t_f^k] \subset \mathbb{R}^k$, and $\hat{\sigma}$ the lift of σ, defined in chapter 4 (Definition 4.16). We call the functional $S^{\mathcal{K}}(\Sigma)$ defined by

$$S_P{}^{\mathcal{K}}(\sigma) = \int_{\hat{\sigma}(P)} \mathcal{K} = \int_{\hat{\sigma}(P)} \frac{1}{k!} \frac{\partial K}{\partial y^{\mu_1 \cdots \mu_k}} dx^{\mu_1 \cdots \mu_k} \tag{5.3.2}$$

the *action of first order field theory associated with* $\mathcal{K}$.

As we have seen in Section 4.3.2, Lemma 4.22, Kawaguchi area is invariant with respect to the reparameterisation, therefore the action is also invariant.

5.3.2 Extremal and equations of motion

Now we will derive the Euler-Lagrange equations by considering the extremal of the action. Again, we only consider global flows in this text. Nevertheless, with some details added, the formulation can be set up similarly with local flows. Consider a C^∞-flow, $\alpha : \mathbb{R} \times M \to M$, and its associated 1-parameter group of transformations $\{\alpha_s\}_{s \in \mathbb{R}}$. The 1-parameter group $\alpha_s : M \to M$ induces a multi-tangent 1-parameter group $\Lambda^k T\alpha_s$:

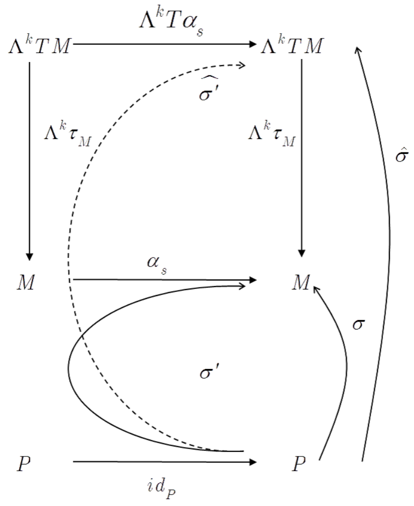

Figure 5.5: First order fields

$\Lambda^k TM \to \Lambda^k TM$ generated by the tangent mapping of multivectors. This will also deform the k-area (k-patch) Σ to $\Sigma' = \alpha_s(\Sigma)$, and since this is a smooth deformation, it again becomes a parameterisable area. By the reparameterisation independence, we can always choose the parameterisation of this deformed Σ' by a new $\sigma' : P \to M$, $\sigma'(P) = \Sigma'$, so that it has the same parameter space as Σ. The variation of the action will be expressed by the small deformations made to the action by α_s.

Definition 5.25. Variation of the action Let ξ be a vector field on M which generates the 1-parameter group α_s, i.e., $\xi = \left.\dfrac{d\alpha_s}{ds}\right|_{s=0}$. We call the functional

$$\begin{aligned} \delta_\xi S^{\mathcal{K}}(\Sigma) &:= \lim_{s\to 0} \frac{1}{s} \left\{ S^{\mathcal{K}}(\alpha_s(\Sigma)) - S^{\mathcal{K}}(\Sigma) \right\} \\ &= \lim_{s\to 0} \frac{1}{s} \left\{ \int_{\widehat{\alpha_s(\sigma)}(P)} \mathcal{K} - \int_{\hat{\sigma}(P)} \mathcal{K} \right\} \end{aligned} \tag{5.3.3}$$

the *variation of the action $S^{\mathcal{K}}(\Sigma)$ with respect to the flow α, associated to $\mathcal{K}$* .

Since the lift of this modified parameterisation is given by, $\widehat{\sigma'} = \widehat{\alpha_s \circ \sigma} = \Lambda^k T\alpha_s \circ \hat{\sigma} = \Lambda^k T\alpha_s \circ \hat{\sigma} \circ {id_P}^{-1}$, We get,

$$\delta_\xi S^{\mathcal{K}}(\Sigma) = \lim_{s\to 0} \frac{1}{s} \left[\int_{\Lambda^k T\alpha_s \circ \hat{\sigma}(P)} \mathcal{K} - \int_{\hat{\sigma}(P)} \mathcal{K} \right] = \lim_{s\to 0} \frac{1}{s} \left[\int_{\hat{\sigma}(P)} (T\alpha_s)^* \mathcal{K} - \int_{\hat{\sigma}(P)} \mathcal{K} \right]$$

$$
\begin{aligned}
&= \int_{\hat{\sigma}(P)} L_X \mathcal{K} \\
&= \int_{\hat{\Sigma}} L_X \mathcal{K}
\end{aligned}
\tag{5.3.4}
$$

where X is a vector field on $\Lambda^k TM$ that generates the multi-tangent 1-parameter group $\Lambda^k T\alpha_s$, i.e., $X = \left.\dfrac{d(\Lambda^k T\alpha_s)}{ds}\right|_{s=0}$, and L_X is a Lie derivative with respect to X.

Now we will calculate the vector field X and the equation of motion in local coordinates. As usual, let (U, φ), $\varphi = (x^\mu)$ be a chart on M, and the induced chart of $\Lambda^k TM$ (V, ψ), $V = {\tau_M}^{-1}(U)$, $\psi = (x^\mu, y^{\mu_1 \cdots \mu_k})$. Let ξ be generator of the 1-parameter group α_s, $\xi = \dfrac{d\alpha_s}{ds}$, and its local coordinate expression $\xi = \xi^\mu \dfrac{\partial}{\partial x^\mu}$, where $\xi^\mu \in C^\infty(M)$. The multi-tangent map $\Lambda^k T_p \alpha_s$ at $p \in M$ sends the vector $v := v_p \in \Lambda^k T_p M$ to $\Lambda^k T_{\alpha_t(p)} M$ by

$$
\Lambda^k T_p \alpha_s(v) = \frac{1}{k!} \left.\frac{\partial x^{\mu_1} \alpha_s \varphi^{-1}}{\partial x^{\nu_1}}\right|_{\varphi(p)} \cdots \left.\frac{\partial x^{\mu_k} \alpha_s \varphi^{-1}}{\partial x^{\nu_k}}\right|_{\varphi(p)} v^{\nu_1 \cdots \nu_k} \left(\frac{\partial}{\partial x^{\mu_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_k}} \right)_{\alpha_s(p)},
\tag{5.3.5}
$$

and since $(\Lambda^k T\alpha_s, \alpha_s)$ is a bundle morphism and from the definition of canonical coordinates of a tangent vector, we have

$$
\begin{aligned}
&x^\mu \circ \Lambda^k T_p \alpha_s(v) = x^\mu \circ \alpha_s \circ \Lambda^k \tau_M(v), \\
&y^{\mu_1 \cdots \mu_k} \circ \Lambda^k T_p \alpha_s(v) = \left.\frac{\partial x^{\mu_1} \alpha_s \varphi^{-1}}{\partial x^{\nu_1}}\right|_{\varphi(\Lambda^k \tau_M(v))} \cdots \left.\frac{\partial x^{\mu_k} \alpha_s \varphi^{-1}}{\partial x^{\nu_k}}\right|_{\varphi(\Lambda^k \tau_M(v))} y^{\nu_1 \cdots \nu_k}(v).
\end{aligned}
\tag{5.3.6}
$$

The induced vector field X by the 1-parameter group $\Lambda^k T\alpha_s$ at a point $q \in \Lambda^k TM$ has a local expression,

$$
\begin{aligned}
X_q &= \left.\frac{d(x^\mu \circ \Lambda^k T\alpha_s)}{ds}\right|_{s=0} \left(\frac{\partial}{\partial x^\mu}\right)_q + \frac{1}{k!} \left.\frac{d(y^{\mu_1 \cdots \mu_k} \circ \Lambda^k T\alpha_s)}{ds}\right|_{s=0} \left(\frac{\partial}{\partial y^{\mu_1 \cdots \mu_k}}\right)_q \\
&= \left.\frac{d}{ds}(x^\mu \circ \alpha_s \circ \Lambda^k \tau_M)\right|_{s=0} \left(\frac{\partial}{\partial x^\mu}\right)_q \\
&\quad + \frac{1}{k!} y^{\nu_1 \cdots \nu_k}(q) \frac{d}{ds} \left(\left.\frac{\partial x^{\mu_1} \alpha_s \varphi^{-1}}{\partial x^{\nu_1}}\right|_{\varphi(\Lambda^k \tau_M(q))} \cdots \left.\frac{\partial x^{\mu_k} \alpha_s \varphi^{-1}}{\partial x^{\nu_k}}\right|_{\varphi(\Lambda^k \tau_M(q))} \right)_{s=0} \left(\frac{\partial}{\partial y^{\mu_1 \cdots \mu_k}}\right)_q
\end{aligned}
$$

$$= \left(\xi^\mu \circ \Lambda^k \tau_M\right)(q)\left(\frac{\partial}{\partial x^\mu}\right)_q + \frac{1}{(k-1)!}\left(\frac{\partial \xi^{\mu_1}}{\partial x^\nu}\circ \Lambda^k \tau_M \cdot y^{\nu\mu_2\cdots\mu_k}\right)(q)\left(\frac{\partial}{\partial y^{\mu_1\cdots\mu_k}}\right)_q, \tag{5.3.7}$$

Therefore,

$$X = \xi^\mu \circ \Lambda^k \tau_M \left(\frac{\partial}{\partial x^\mu}\right) + \frac{1}{(k-1)!}\frac{\partial \xi^{\mu_1}}{\partial x^\nu}\circ \Lambda^k \tau_M \cdot y^{\nu\mu_2\cdots\mu_k}\left(\frac{\partial}{\partial y^{\mu_1\cdots\mu_k}}\right). \tag{5.3.8}$$

The Lie derivative $L_X\mathcal{K}$ in coordinate expression becomes,

$$\begin{aligned}
L_X\mathcal{K} &= L_X\left(\frac{1}{k!}\frac{\partial K}{\partial y^{\rho_1\cdots\rho_k}}dx^{\rho_1\cdots\rho_k}\right)\\
&= X\left(\frac{1}{k!}\frac{\partial K}{\partial y^{\rho_1\cdots\rho_k}}\right)dx^{\rho_1\cdots\rho_k} + \frac{1}{(k-1)!}\frac{\partial K}{\partial y^{\rho_1\cdots\rho_k}}dL_X x^{\rho_1}\wedge dx^{\rho_2\cdots\rho_k}\\
&= \frac{1}{k!}\left\{\xi^\mu \circ \Lambda^k\tau_M\left(\frac{\partial^2 K}{\partial x^\mu \partial y^{\rho_1\cdots\rho_k}}\right) + \frac{\partial \xi^{\mu_1}}{\partial x^\nu}\circ\Lambda^k\tau_M\cdot y^{\nu\mu_2\cdots\mu_k}\left(\frac{\partial^2 K}{\partial y^{\mu_1\cdots\mu_k}\partial y^{\rho_1\cdots\rho_k}}\right)\right\}dx^{\rho_1\cdots\rho_k}\\
&\quad + \frac{1}{(k-1)!}\frac{\partial K}{\partial y^{\rho_1\cdots\rho_k}}d\left(\xi^{\rho_1}\circ\Lambda^k\tau_M\right)\wedge dx^{\rho_2\cdots\rho_k}\\
&= \frac{1}{k!}\xi^\mu\circ\Lambda^k\tau_M\left\{\frac{\partial^2 K}{\partial x^\mu\partial y^{\rho_1\cdots\rho_k}}dx^{\rho_1} - kd\left(\frac{\partial K}{\partial y^{\mu\rho_2\cdots\rho_k}}\right)\right\}\wedge dx^{\rho_2\cdots\rho_k}\\
&\quad + \frac{\partial\xi^{\mu_1}}{\partial x^\nu}\circ\Lambda^k\tau_M\cdot y^{\nu\mu_2\cdots\mu_k}\left(\frac{\partial^2 K}{\partial y^{\mu_1\cdots\mu_k}\partial y^{\rho_1\cdots\rho_k}}\right)dx^{\rho_1\cdots\rho_k}\\
&\quad + \frac{1}{(k-1)!}d\left(\frac{\partial K}{\partial y^{\rho_1\cdots\rho_k}}\cdot\xi^{\rho_1}\circ\Lambda^k\tau_M\right)\wedge dx^{\rho_2\cdots\rho_k}.
\end{aligned} \tag{5.3.9}$$

The result of (5.3.9) is called the *infinitesimal first variation formula* for the Kawaguchi k-form $\mathcal{K}$.

The variation of action becomes,

$$\begin{aligned}
\delta_\xi S^{\mathcal{K}}(\Sigma) &= \int_{\hat\sigma(P)} L_X\mathcal{K} = \int_P \hat\sigma^* L_X\mathcal{K}\\
&= \int_P \hat\sigma^*\left[\frac{1}{k!}\xi^\mu\circ\Lambda^k\tau_M\left(\frac{\partial^2 K}{\partial x^\mu\partial y^{\rho_1\cdots\rho_k}}dx^{\rho_1} - kd\left(\frac{\partial K}{\partial y^{\mu\rho_2\cdots\rho_k}}\right)\right)\wedge dx^{\rho_2\cdots\rho_k}\right.\\
&\quad + \frac{\partial\xi^{\mu_1}}{\partial x^\nu}\circ\Lambda^k\tau_M\cdot y^{\nu\mu_2\cdots\mu_k}\left(\frac{\partial^2 K}{\partial y^{\mu_1\cdots\mu_k}\partial y^{\rho_1\cdots\rho_k}}\right)dx^{\rho_1\cdots\rho_k}\\
&\quad \left. + \frac{1}{(k-1)!}d\left(\frac{\partial K}{\partial y^{\rho_1\cdots\rho_k}}\cdot\xi^{\rho_1}\circ\Lambda^k\tau_M\right)\wedge dx^{\rho_2\cdots\rho_k}\right]\\
&= \int_{\hat\Sigma}\left[\frac{1}{k!}\xi^\mu\circ\Lambda^k\tau_M\left(\frac{\partial^2 K}{\partial x^\mu\partial y^{\rho_1\cdots\rho_k}}dx^{\rho_1} - kd\left(\frac{\partial K}{\partial y^{\mu\rho_2\cdots\rho_k}}\right)\right)\right.
\end{aligned}$$

$$+\frac{1}{(k-1)!}d\left(\frac{\partial K}{\partial y^{\rho_1\cdots\rho_k}}\cdot\xi^{\rho_1}\circ\Lambda^k\tau_M\right)\Bigg]\wedge dx^{\rho_2\cdots\rho_k}, \tag{5.3.10}$$

which is called the *integral first variation formula*. We used the homogeneity condition:

$$\left(\left(\frac{\partial^2 K}{\partial y^{\mu_1\cdots\mu_k}\partial y^{\rho_1\cdots\rho_k}}\right)\cdot y^{\rho_1\cdots\rho_k}\right)\circ\hat{\sigma}=0. \tag{5.3.11}$$

(5.3.11) is obtained by taking the derivative of (4.3.3) with respect to $y^{\mu_1\cdots\mu_k}$, and then taking the pull back.

Now we can proceed to find the equations of motion to this system. We will begin with the definition of an extremal.

Definition 5.26. Extremal of an action

1. We say that a k-area Σ is *stable* with respect to the flow α, when it satisfies

$$\delta_\xi S^{\mathcal{K}}=0, \tag{5.3.12}$$

   where ξ is the generator of α.

2. We say that a k-area Σ is an *extremal* of the action $S^{\mathcal{K}}$, when it satisfies (5.3.12) for any α such that its associated 1-parameter group α_s satisfies $\alpha_s(\partial\Sigma)=\partial\Sigma$, $\forall s\in\mathbb{R}$, where $\partial\Sigma$ is the boundary of Σ.

Now we can obtain the following theorem.

Theorem 5.27. Extremals
Let Σ be an k-patch. The following statements are equivalent.

1. Σ is an extremal.

2. The equation

$$\mathcal{EL}^K{}_\mu\circ\hat{\sigma}=0,$$
$$\mathcal{EL}^K{}_\mu:=\frac{1}{k!}\left(\frac{\partial^2 K}{\partial x^\mu\partial y^{\rho_1\cdots\rho_k}}dx^{\rho_1}-kd\left(\frac{\partial K}{\partial y^{\mu\rho_2\cdots\rho_k}}\right)\right)\wedge dx^{\rho_2\cdots\rho_k}, \tag{5.3.13}$$

   holds for arbitrary parameterisation σ.

The proof is given similarly as in the case of mechanics. (see Section 5.1.2)

Definition 5.28. Symmetry of the dynamical system
Let u be a vector field over M, and Y an induced vector field by u over $\Lambda^k TM$. We say that $\mathcal{K}$ *is invariant with respect to* u, if

$$L_Y \mathcal{K} = 0, \tag{5.3.14}$$

and u called a *symmetry* of the dynamical system $(M, \mathcal{K})$. We also say that u generates the invariant transformations on the Kawaguchi manifold $(M, \mathcal{K})$.

Now we will have the following conservation law.

Theorem 5.29. Noether (first order field)
Suppose we are given a symmetry of $(M, \mathcal{K})$. Then there exists a $(k-1)$-form f on $\Lambda^k TM$ which along the extremal γ of $S^{\mathcal{K}}$ satisfies,

$$\int_{\hat{\gamma}} df = 0, \tag{5.3.15}$$

for any parameterisation σ which parameterise γ.

Proof. Let the symmetry be u, with its local coordinate expression $u = u^\mu \dfrac{\partial}{\partial x^\mu}$, and the induced vector field Y. Then from (5.3.10), we have

$$\begin{aligned} 0 &= \int_{\hat{\gamma}} L_Y \mathcal{K} \\ &= \int_{\hat{\gamma}} \left[\frac{1}{k!} u^\mu \circ \Lambda^k \tau_M \left(\frac{\partial^2 K}{\partial x^\mu \partial y^{\rho_1 \cdots \rho_k}} dx^{\rho_1} - k d \left(\frac{\partial K}{\partial y^{\mu \rho_2 \cdots \rho_k}} \right) \right) \right. \\ &\quad \left. + \frac{1}{(k-1)!} d \left(\frac{\partial K}{\partial y^{\rho_1 \cdots \rho_k}} \cdot u^{\rho_1} \circ \Lambda^k \tau_M \right) \right] \wedge dx^{\rho_2 \cdots \rho_k} \\ &= \int_{\hat{\gamma}} d \left(\frac{1}{(k-1)!} \frac{\partial K}{\partial y^{\rho_1 \cdots \rho_k}} \cdot u^{\rho_1} \circ \Lambda^k \tau_M \, dx^{\rho_2 \cdots \rho_k} \right) \end{aligned} \tag{5.3.16}$$

The second equality comes from the fact we consider along the extremal γ. Therefore we have a $(k-1)$-form on $\Lambda^k TM$,

$$f = \frac{1}{(k-1)!} \frac{\partial K}{\partial y^{\rho_1 \cdots \rho_k}} \cdot u^{\rho_1} \circ \Lambda^k \tau_M \, dx^{\rho_2 \cdots \rho_k} \tag{5.3.17}$$

such that satisfies the condition. $\square$

We call the relation (5.3.15), the *conservation law*.

Definition 5.30. Noether current
The quantity f is called the *Noether current of first order field theory, associated with* u.

By the coordinate transformation

$$x^\mu \to \tilde{x}^\mu = \tilde{x}^\mu(x^\nu),\ y^{\mu_1\cdots\mu_k} \to \tilde{y}^{\mu_1\cdots\mu_k} = \frac{\partial \tilde{x}^{\mu_1}}{\partial x^{\nu_1}}\cdots\frac{\partial \tilde{x}^{\mu_k}}{\partial x^{\nu_k}} y^{\nu_1\cdots\nu_k}, \tag{5.3.18}$$

the k-form ${\mathcal{E}\mathcal{L}^K}_\mu$ in (5.3.13) transforms as

$$\begin{aligned}&\frac{1}{k!}\left(\frac{\partial^2 K}{\partial \tilde{x}^\mu \partial \tilde{y}^{\rho_1\cdots\rho_k}} d\tilde{x}^\rho - kd\left(\frac{\partial K}{\partial \tilde{y}^{\mu\rho_2\cdots\rho_k}}\right)\right)\wedge d\tilde{x}^{\rho_2\cdots\rho_k}\\ &\qquad = \frac{1}{k!}\left(\frac{\partial x^\nu}{\partial \tilde{x}^\mu}\right)\left(\frac{\partial^2 K}{\partial x^\nu \partial y^{\rho_1\cdots\rho_k}} dx^{\rho_1} - kd\left(\frac{\partial K}{\partial y^{\nu\rho_2\cdots\rho_k}}\right)\right)\wedge dx^{\rho_2\cdots\rho_k}.\end{aligned} \tag{5.3.19}$$

This observation leads us to define a new coordinate invariant form.

Lemma 5.31. Euler Lagrange form
There exist a global $(k+1)$ form on $\Lambda^k TM$, which in local coordinates are expressed by

$$\mathcal{E}\mathcal{L}^K = dx^\mu \wedge {\mathcal{E}\mathcal{L}^K}_\mu = \frac{1}{k!}\left\{\frac{\partial^2 K}{\partial x^\mu \partial y^{\rho_1\cdots\rho_k}} dx^\mu + d\left(\frac{\partial K}{\partial y^{\rho_1\cdots\rho_k}}\right)\right\}\wedge dx^{\rho_1\cdots\rho_k}. \tag{5.3.20}$$

From the previous coordinate transformations, this is obviously coordinate independent.

There is a direct relation between the exterior derivative of $\mathcal{K}$ and $\mathcal{E}\mathcal{L}$,

$$d\mathcal{K} = \mathcal{E}\mathcal{L}^K - \frac{1}{k!}\frac{\partial^2 K}{\partial x^\mu \partial y^{\rho_1\cdots\rho_k}} dx^\mu \wedge dx^{\rho_1\cdots\rho_k}. \tag{5.3.21}$$

It can be also checked easily that this is also a coordinate invariant relation.

Example 5.32. De Broglie field (Schrodinger field)
Here we will give a rather trivial, but instructive example for the case of $M = \mathbb{R}^4$, with Kawaguchi structure corresponding to the De Broglie field on 2-dimensional spacetime $\mathbb{R}^2$.

The conventional Lagrangian function of the De Broglie field is given by

$$\mathcal{L} = \frac{i}{2}\left(\bar{\psi}\cdot\partial_t\psi - \partial_t\bar{\psi}\cdot\psi\right) - \frac{1}{2m}\partial_x\bar{\psi}\cdot\partial_x\psi + e\bar{\psi}\cdot\varphi\cdot\psi, \tag{5.3.22}$$

on $J^1Y = J^1\mathbb{R}^4$, where J^1Y is the prolongation of the total space regarding the bundle $(Y, pr_1, \mathbb{R}^2)$, $Y = \mathbb{R}^2\times\mathbb{R}^2$. We take for the global fibre bundle coordinates; (t, x) for $\mathbb{R}^2$,

$(t, x, \psi, \bar{\psi})$ for Y, and $(t, x, \psi, \bar{\psi}, \partial_t\psi, \partial_t\bar{\psi}, \partial_x\psi, \partial_x\bar{\psi})$ for J^1Y. $t,\ x$ denotes the space-time, $\psi,\ \bar{\psi}$ the fields, $\varphi = \varphi(x)$ the external field, and $m,\ e$ are constants. In an orthodox physics notation, the pull-back of $\mathcal{L}$ to $\mathbb{R}^2$ is also called the Lagrangian, namely

$$L := \mathcal{L}\circ J^1\gamma = \frac{i}{2}\left(\bar{\psi}\frac{\partial\psi}{\partial t} - \frac{\partial\bar{\psi}}{\partial t}\psi\right) - \frac{1}{2m}\frac{\partial\bar{\psi}}{\partial x}\frac{\partial\psi}{\partial x} + e\bar{\psi}\,\varphi\,\psi, \tag{5.3.23}$$

$\psi := \psi\circ\gamma = \psi(t, x)$, $\bar{\psi} = \bar{\psi}\circ\gamma = \bar{\psi}(t, x)$, where γ is a section of the bundle $(Y, pr_1, \mathbb{R}^2)$, and $J^1\gamma$ its prolongation.

We will try to construct the Kawaguchi manifold that corresponds to such model. Let $M = \mathbb{R}^4$, and the parameter space $P = \mathbb{R}^2$. In this case, we have global charts on M and P. Let the canonical coordinates on P and M be (t^0, t^1) and (x^0, x^1, x^2, x^3) respectively, $\sigma : P \to M$ be a parameterisation, and $\hat{\sigma} : P \to \Lambda^2TM$ its lift. The Kawaguchi form is a 2-form on Λ^2TM. Let $(x^\mu, y^{\nu_1\nu_2})$ be the induced global chart on $\Lambda^2TM = \Lambda^2T\mathbb{R}^4$ with $\mu, \nu_1, \nu_2 = 0, 1, 2, 3$. We will consider a pair of maps,

$$f : \Lambda^kTM \to J^1\mathbb{R}^4 \tag{5.3.24}$$

and $id_{\mathbb{R}^2} : P \to \mathbb{R}^2$ such that, $f\circ\hat{\sigma} := J^1\gamma\circ id_{\mathbb{R}^2}$, where γ is a section of the bundle $(Y, pr_1, \mathbb{R}^2)$, and $J^1\gamma$ its prolongation. The identity map on the base space will imply,

$$t^0 = t\circ id_{\mathbb{R}^2},\ t^1 = x\circ id_{\mathbb{R}^2}. \tag{5.3.25}$$

With such f, we will construct the Kawaguchi function K by $K = f^*\mathcal{L}$, then K gives the same Lagrangian for the De Broglie field, when pulled back to the parameter space by $\hat{\sigma}$, by

$$\hat{\sigma}^*K = \hat{\sigma}^*f^*\mathcal{L} = \mathcal{L}\circ f\circ\hat{\sigma} = \mathcal{L}\circ J^1\gamma\circ id_{\mathbb{R}^2}. \tag{5.3.26}$$

From this Kawaguchi function, we can construct the Kawaguchi 2-form $\mathcal{K}$.

The choice of the map f is not unique, nevertheless, we can consider a map that is convenient for calculation. One such choice is the following. For notational convenience, now we will write the coordinates on J^1Y as $(\bar{t}^0, \bar{t}^1, z^0, z^1, z^0_0, z^1_0, z^0_1, z^1_1)$, so that $t = \bar{t}^0,\ x = \bar{t}^1,\ \psi = z^0,\ \bar{\psi} = z^1,\ \partial_t\psi = z^0_0,\ \partial_t\bar{\psi} = z^1_0,\ \partial_x\psi = z^0_1,\ \partial_x\bar{\psi} = z^1_1$. We define the map f such that its coordinate expressions are given by,

$$\begin{aligned} x^\mu &= f^*\bar{t}^\mu, \\ y^{\nu_1\nu_2} &= f^*(z_0^{[\nu_1}z_1^{\nu_2]}) \end{aligned} \tag{5.3.27}$$

then, at point $\hat{\sigma}(p) \in \Lambda^2 TM$, $p \in P$, we will have,

$$\begin{aligned} x^\mu(\hat{\sigma}(p)) &= \bar{t}^\mu(f\circ\hat{\sigma}(p)) = \bar{t}^\mu(J^1\gamma(p)), \\ y^{\nu_1\nu_2}(\hat{\sigma}(p)) &= z_0^{[\nu_1}(f\circ\hat{\sigma}(p))\, z_1^{\nu_2]}(f\circ\hat{\sigma}(p)) = z_0^{[\nu_1}(J^1\gamma(p))\, z_1^{\nu_2]}(J^1\gamma(p)) \\ &= \left.\frac{\partial(\bar{t}^{[\nu_1}\circ\gamma)}{\partial\bar{t}^0}\right|_p \left.\frac{\partial(\bar{t}^{\nu_2]}\circ\gamma)}{\partial\bar{t}^1}\right|_p . \end{aligned} \tag{5.3.28}$$

calculating (5.3.28) explicitly, and rewriting in the original notation gives

$$\begin{aligned} y^{01}\circ\hat{\sigma} &= \frac{\partial(\bar{x}^{[0}\circ\gamma)}{\partial\bar{t}^0}\frac{\partial(\bar{x}^{1]}\circ\gamma)}{\partial\bar{t}^1} = \frac{1}{2}, \quad y^{02}\circ\hat{\sigma} = \frac{\partial(\bar{x}^{[0}\circ\gamma)}{\partial\bar{t}^0}\frac{\partial(\bar{x}^{2]}\circ\gamma)}{\partial\bar{t}^1} = \frac{1}{2}\frac{\partial(\psi\circ\gamma)}{\partial x}, \\ y^{03}\circ\hat{\sigma} &= \frac{\partial(\bar{x}^{[0}\circ\gamma)}{\partial\bar{t}^0}\frac{\partial(\bar{x}^{3]}\circ\gamma)}{\partial\bar{t}^1} = \frac{1}{2}\frac{\partial(\bar{\psi}\circ\gamma)}{\partial x}, \quad y^{12}\circ\hat{\sigma} = \frac{\partial(\bar{x}^{[1}\circ\gamma)}{\partial\bar{t}^0}\frac{\partial(\bar{x}^{2]}\circ\gamma)}{\partial\bar{t}^1} = -\frac{1}{2}\frac{\partial(\psi\circ\gamma)}{\partial t}, \\ y^{31}\circ\hat{\sigma} &= \frac{\partial(\bar{x}^{[3}\circ\gamma)}{\partial\bar{t}^0}\frac{\partial(\bar{x}^{1]}\circ\gamma)}{\partial\bar{t}^1} = \frac{1}{2}\frac{\partial(\bar{\psi}\circ\gamma)}{\partial t}, \\ y^{23}\circ\hat{\sigma} &= \frac{\partial(\bar{x}^{[2}\circ\gamma)}{\partial\bar{t}^0}\frac{\partial(\bar{x}^{3]}\circ\gamma)}{\partial\bar{t}^1} = \frac{1}{2}\left(\frac{\partial(\psi\circ\gamma)}{\partial t}\frac{\partial(\bar{\psi}\circ\gamma)}{\partial x} - \frac{\partial(\psi\circ\gamma)}{\partial x}\frac{\partial(\bar{\psi}\circ\gamma)}{\partial t}\right). \end{aligned} \tag{5.3.29}$$

From these relations, and $\hat{\sigma}^* K = \mathcal{L}\circ J^1\gamma\circ id_{\mathbb{R}^2}$, we obtain our Kawaguchi function K,

$$K = \frac{i}{2}\left(-x^3 y^{12} - x^2 y^{31}\right) - \frac{1}{2m}\frac{y^{03}y^{02}}{y^{01}} + e\varphi(x^1)x^2x^3y^{01} \tag{5.3.30}$$

and the Kawaguchi structure becomes,

$$\mathcal{K} = \left(\frac{1}{2m}\frac{y^{03}y^{02}}{(y^{01})^2} + e\varphi(x^1)x^2x^3\right)dx^{01} - \frac{1}{2m}\left(\frac{y^{03}}{y^{01}}dx^{02} + \frac{y^{02}}{y^{01}}dx^{03}\right) - \frac{i}{2}\left(x^3dx^{12} + x^2dx^{31}\right). \tag{5.3.31}$$

We will take this as the reparameterisation invariant Lagrangian of the De Broglie field theory.

The Euler-Lagrange equations obtained from $\mathcal{K}$ can be calculated by the formula (5.3.13), as

$$\begin{aligned} \mathcal{EL}_0\circ\hat{\sigma} &= d\left(\frac{\partial K}{\partial y^{01}}dx^1 + \frac{\partial K}{\partial y^{02}}dx^2 + \frac{\partial K}{\partial y^{03}}dx^3\right)\circ\hat{\sigma} = 0, \\ \mathcal{EL}_1\circ\hat{\sigma} &= d\left(\frac{\partial K}{\partial y^{10}}dx^1 + \frac{\partial K}{\partial y^{12}}dx^2 + \frac{\partial K}{\partial y^{13}}dx^3\right)\circ\hat{\sigma} = 0, \\ \mathcal{EL}_2\circ\hat{\sigma} &= \left(e\varphi x^3dx^{01} - idx^{31} - \frac{1}{2m}\left(-\frac{y^{03}}{(y^{01})^2}dy^{01} + \frac{1}{y^{01}}dy^{03}\right)\wedge dx^0\right)\circ\hat{\sigma} = 0, \end{aligned}$$

$$\mathcal{E}\mathcal{L}_3 \circ \hat{\sigma} = \left(e\varphi x^2 dx^{01} - i dx^{12} - \frac{1}{2m} \left(-\frac{y^{02}}{(y^{01})^2} dy^{01} + \frac{1}{y^{01}} dy^{02} \right) \wedge dx^0 \right) \circ \hat{\sigma} = 0, \tag{5.3.32}$$

which is true for any parameterisation . To compare these equations with the conventional De Broglie field equations; in other name the Schrodinger equations, choose the parameterisation, which in coordinates are given by

$$(x^0 \circ \sigma, x^1 \circ \sigma, x^2 \circ \sigma, x^3 \circ \sigma) = (t^0, t^1, \psi, \bar{\psi}). \tag{5.3.33}$$

We get,

$$\begin{aligned}
\hat{\sigma}^* \mathcal{E}\mathcal{L}_0 &\equiv 0, \\
\hat{\sigma}^* \mathcal{E}\mathcal{L}_1 &\equiv 0, \\
\hat{\sigma}^* \mathcal{E}\mathcal{L}_2 &= -\left(i\partial_t \bar{\psi} - \frac{1}{2m} \partial_i \partial_i \bar{\psi} - e\varphi \bar{\psi} \right) dt^{01} = 0, \\
\hat{\sigma}^* \mathcal{E}\mathcal{L}_3 &= \left(i\partial_t \psi + \frac{1}{2m} \partial_i \partial_i \psi + e\varphi \psi \right) dt^{01} = 0,
\end{aligned} \tag{5.3.34}$$

which are indeed the well-known Schrodinger equations. The first two equations becomes identity, when the latter two are taken into account, meaning these equations are indeed dependent.

Remark 5.33. The expressions such as (5.3.31), (5.3.32) are reparameterisation invariant, and there are other possibilities to choose different parameterisations such that their pulled back expressions on the parameter space would not look like the conventional expressions. To consider their meaning and applications would be an interesting theme for future research.

Chapter 6

Discussion

In this thesis, we have introduced the foundations needed for the calculus of variation in Finsler and Kawaguchi geometry. For Kawaguchi geometry, we especially constructed the second order 1-dimensional parameter case, and first order k-dimensional case. For the second order k-dimensional case, only local version was presented. We have used a less restricted definition for both Finsler manifold and Kawaguchi manifold compared to the standard definition, which is considered more applicable to the problems of physics. We constructed a global form for the Kawaguchi geometry, which has a similar property as the Finsler-Hilbert form in the Finsler geometry case, in the sense that they define a reparameterisation invariant k-dimensional area on the subset of the base manifold M. Lagrange formulation was introduced on these structures in a natural way, and we obtained the reparameterisation invariant Euler-Lagrange expression. We had compared the results with examples such as Newtonian mechanics and De Broglie field, and confirmed that with a special choice of parameterisation, the results will reduce to the conventional expression of these theories. Throughout the discussion, we only used basic methods in differential geometry, and took the most straightforward path to considering Lagrangian formulation.

There are many issues in the thesis that remains for further discussions and research. The main reason of the difficulty in the case of higher order k-dimensional case comes from the fact that our main pillar; the homogeneity conditions in the simplest expression is not coordinate independent. However, we believe this difficulty can be solved soon, and global Kawaguchi form could be constructed for this case too. Nevertheless, for many concrete problems for physics, our local formalism should also be applicable. In this thesis, we only considered the case where the subset Σ of M is diffeomorphic to the closed k-rectangle in $\mathbb{R}^k$. For a more general case, the action of Σ associated to the Kawaguchi

(Finsler) form is well-defined provided that there exists an inclusion map $\iota : P \to M$, where $\Sigma = \iota(P)$, such that P is an oriented compact manifold. In the case where P has no boundary, we should have to consider an extension of variational principle, such as Cartan's principle, which is also an interesting problem. Since Finsler and Kawaguchi geometry is less restrictive than Riemannian geometry, we expect it should embrace wider area of physics where it cannot be expressed by Riemannian geometry. For instance, system that is irreversible with time, or shows hysteresis, may be a good non-trivial example to be modelled by our approach. However, these problems are for the moment left for the future, and would be presented another time.

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