Palacký University
Faculty of science
Department of optics

BACHELOR THESIS

New identification of nonclassical features of single photon states

Thesis supervisor:
doc. Radim Filip, Ph.D.

Author:
Lukáš Lachman

Olomouc

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Abstract

The nonclassicality of light has been studying for many years. The definition of those states offers no methods to decide from measurement if a state is nonclassical. This thesis searchs for new experimentally verifiable criteria that give sufficient conditions for nonclassicality of single photon states, also tests their power in realistic models.

Keys words:

Nonclassicality of light, single photon sources.

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Bakalářská práce

Nová identifikace neklasického charakteru jednofotonových stavů

Vedoucí práce:
doc. Radim Filip, Ph.D.

Autor:
Lukáš Lachman

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Abstract

Neklasičinost světla je studována po mnoho let. Definice ale nenabízí žádný způsob, jak z měření rozhodnout, zda je stav skutečně neklasický. Tato bakalářská práce se zabývá novými kritérii, která dávají postačující podmínky neklasiňosti a která lze verifikovat. V práci je dále testována jejich síla na reálných modelech jednofotonových zdrojů.

Klíčová slova:

Neklasičnost světla, jednofotonové zdroje.

Declaration

I hereby state that I have written this thesis myself under the supervision of Doc. Radim Filip, Ph.D. All used resource are listed under Bibliography.

Acknowledgement

I'd like to thank my supervision doc. Radim Filip, Ph.D for his patience, willingness and inspiring discussions, that leads to this bachelor thesis. Also I express gratitude to my family for patient and support during my studies.
In Olomouc, May 2012

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Chapter 1

Introduction

Quantum optics is very fast developing discipline. And it gives large possibilities in experiments and applications of optics, many of those experiments are based on nonclassical states of light. Therefore we need to distinguish states of light with very nonclassical features from those that tender weaker quantum effects. It is long term problem, still not solved completely. The first step in this course was done by Glauber who explained theory of coherence in the words of quantum optics. He defined quantum states that are describable in terms of classical optics. His definition is unfortunately limitedly provable from measurement. The definition corresponds to infinite number of conditions for classicality of state. It has yielded to sequence of different methods to decide if measured data correspond to nonclassical states.

Since this time it has been many times confirmed that some sources like single photon sources break rules or criteria that should be valid from the perspective of classical optics. We can example it by first pioneering resonance fluorescence measured by H. J. Kimble, M. Dagenais and L. Mandel in the year 1956 or later, measuring the values of α-parameter quantifying anti-correlation effect of photons at a splitter or recently, measuring negative Wigner's function of single photon state. Main application of single photon states is advanced quantum key distribution with repeaters.

The first two Chapters in this Thesis contain historical introduction to nonclassicality of light. In the third Chapter criterion involving exact photon distribution is discussed. This criterion was introduced by Radim Filip and Ladislav Mišta, but contrary to the topic of this Thesis, it was tested for mixtures of Gaussian states. In this chapter we suggest new interpretation of this criterion. The next Chapter discusses criteria that are less powerful but better testable because they involve the probabilities measured by avalanche detectors. We extend it for basic higher-ordered criteria and in the Chap. 7 we compare all these criteria at realistic models of one photon sources.

Chapter 2

Light and Photons

All features of classical optics can be derived from the solution of Maxwell's equations [1]. These equations connect spatial and temporal evolution of the electric intensity **E** and magnetic induction **B**. It's interesting simplification for optics to decompose these quantities in time independent and space independent components. They are given by electromagnetic wave. A general solution is a superposition of these waves. The detectors are sensitive in $|\mathbf{E}|^2$ mainly, therefore we are interested in behaviour of the vector **E**.

We will suppose the scalar approximation of electric intensity for simplicity. So we can write:

$$E = \sum_k c_k u_k(\mathbf{r}) \exp(-i\omega_k t), \tag{2.1}$$

where we develop electric intensity in Fourier series, i.e. in superposition of harmonic waves. Each plane wave defines one mode, characterized by time independent component $u_k(r)$, angular velocity ω_k and Fourier coefficient c_k. Summation index k labels different classical properties of harmonic waves. It's convenient to express it in two component and use a different constants:

$$E^{(+)} = i \sum_k \left(\frac{1}{2}\hbar\omega_k\right)^{1/2} a_k u_k(\mathbf{r}) \exp(-i\omega_k t), \tag{2.2}$$

$$E^{(-)} = -i \sum_k \left(\frac{1}{2}\hbar\omega_k\right)^{1/2} a_k^* u_k(\mathbf{r}) \exp(i\omega_k t)$$

The resulted electric intensity E is equalled to sum of both expressions. The a_k and a_k^* are dimensionless amplitudes of linear harmonic oscillator (LHO).

The extension from classical to quantum optics is guaranteed by substitution of a_k and a_k^* by operators [2]. The description of light as classical LHO yields to quantum LHO. The Fourier's coefficients a_k and a_k^* become the annihilation and creation operators, which have to fulfil the canonical commutation relations:

$$\left[a_k, a_{k'}^\dagger\right] = \delta_{kk'}, \tag{2.3}$$

$$[a_k, a_{k'}] = \left[a_k^\dagger, a_{k'}^\dagger\right] = 0$$

One can see the operator $E^{(+)}$ component contains only the annihilation operator and the operator $E^{(-)}$ only the creation operator. The total energy of the field is given by Hamiltonian $H = 1/2 \int (\epsilon_0 E^2 + 1/\mu_0 B^2) d\mathbf{r}$. Inserting the annihilation and creation operator leads to[2]:

$$H = 1/2 \sum_k \hbar\omega_k (a_k^\dagger a_k + a_k a_k^\dagger). \tag{2.4}$$

The energy of monochromatic electromagnetic modes is given by energy of set of LHOs. One of the properties of LHO is that its energy is quantized, i. e. eigenvalues of $a^\dagger a$ operator can be only a discrete values. Let's define a new operator $n_k = a_k^\dagger a_k$ and its eigenstate $n_k|n_k\rangle = n'_k|n_k\rangle$. The field with given classical properties we call a mode and one quantum of energy *photon*. The eigenvalue n'_k says how many photons are there in the k-mod. The state $|n_k\rangle$ we call Fock state. Further we are omitting the index k and working only with single mode[3].

If we let act Hamiltonian H on $a^\dagger|n\rangle$ we find the creation operator adds a photon. Similarly the annihilation operator takes off a photon. When we use normalization conditions we can write [2]:

$$\begin{aligned} a^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle, \\ a|n\rangle &= \sqrt{n}|n-1\rangle. \end{aligned} \tag{2.5}$$

The prefactors $\sqrt{n}$ and $\sqrt{n+1}$ in this relations are consequences of quantum physics.

The field from which it's impossible to remove some energy is called vacuum and is defined by equation: $a|0\rangle = 0$. The energy of vacuum is given by (2.4) and is equalled to $E_{vac} = \frac{1}{2}\hbar\omega$ for each mode [3].

If a state can be describe by linear combination of Fock state basis we know maximum information about it. This state is called pure state. But we don't know sometimes in which pure state is light prepared. The density matrix $\rho = \sum_{m,n=0}^{\infty} \rho_{m,n}|m\rangle\langle n|$ is introduced to describe such situations. The density matrix is semi-definite operator. Its trace is equalled to one and the probabilities p_n of n-measured photons are given by:

$$p_n = \rho_{n,n} = \langle n|\rho|n\rangle, \tag{2.6}$$

The m-th moment of operator n is yielded by:

$$\langle n^m \rangle = Tr\left[n^m \rho\right]. \tag{2.7}$$

The variance can be expressed so this: $\langle(\Delta n)^2\rangle = \langle n^2\rangle - \langle n\rangle^2$. It is one from quantities of amount of photon noise presented in the state [2].

Single photon state $|1\rangle$ is important building block in quantum optics. Let us consider beam splitter has transitivity very near to one. The incident state is a single state $|1\rangle$ that interacts with arbitrary state $|\psi\rangle$. The unitary operator of the beam splitter is given by $U_{BS} \approx 1 + k(ab^\dagger - a^\dagger b)$, where a ($a^\dagger$) is annihilation (creation) operator acting on the single photon state and b ($b^\dagger$) is annihilation (creation) operator acting on the state $|\psi\rangle$. If we measure vacuum in one out-coming mode, the state is then given by: $\langle 0|_a U_{BS} |1\rangle_a |\psi\rangle_b = kb^\dagger |\psi\rangle_b$ [4]. We see that single photon state is resource to build operator $a^\dagger$ on any state. Linear combination of operators $a^\dagger$ then can be used to conditionally build any multimode state in quantum optics, also that states are required in advanced QKD [5]. But how one can recognise that single photon states is prepared in experiment?

2.1 First and Second Ordered Correlation function

Historically quantum optics started from a development of quantum coherence theory[3]. Correlation functions are main tools of coherence theory of light. The first ordered correlation function corresponds to measuring interference in Mach-Zehnder interferometer. It is defined thus:

$$g^{(1)}(\tau) = \frac{\langle E^{(-)}(t)E^{(+)}(t+\tau)\rangle}{|\langle E(t)\rangle|^2}. \tag{2.8}$$

Let's quantise it by replacing amplitudes by normally ordered operators introduced in (2.2). Normally ordering corresponds to particle counting experiments. This yields to[3]:

$$g^{(1)}(\tau) = \frac{\langle a^\dagger(t)a(t+\tau)\rangle}{\langle a^\dagger(t)a(t)\rangle}. \tag{2.9}$$

For $\tau = 0$ this function gets uniting always. Therefore it gives us no information about the fluctuations of the light. Those effects are better described by second ordered correlation function.

The pioneering experiment in measuring the fluctuation of light was arranged by R. Hanbury Brown and R. Q. Twiss (1956) [6]. The beam of thermal source was split in semi-transparent mirror in two beams that were detected. Then one of the outputs was time delayed and coincidence signal of the delayed and non-delayed outputs were measured. For classical thermal light sources, there was observed correlation between the two beams. The correlation of intensities for short time delays were greater than the one for long times. Figure (2.1) shows the scheme of the experiment. This effect is called bunching and is caused by tend of photons to bunch and come in couples. For coherent light from laser the correlation of intensities is independent on time delay. This behaviour of light can be explained by both the classical and quantum optics. But the contrary effect - antibunching only by quantum theory.

A classical correlation function which characterizes this experiment is defined thus [6]:

$$g^{(2)}(\tau) = \frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle^2}, \tag{2.10}$$

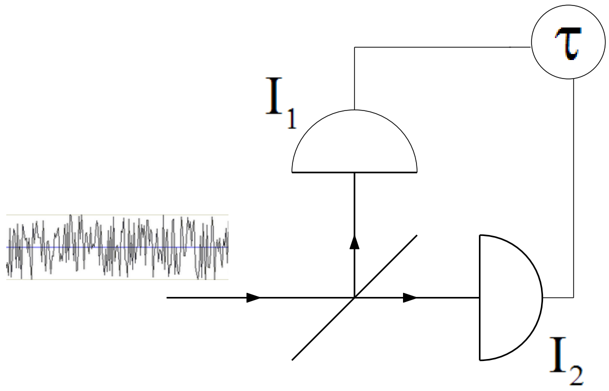

Figure 2.1: *Scheme of HBT experiment. The sourced beam is split to two intensities detectors. The coincidence of intensities I_1 and I_2 time delayed is measured.*

where τ is the time delay. Let's suppose the incident beam has intensity I. If detectors have efficiency η and τ_r is the time of measuring, the measured intensity is given by: $K(\tau) = 1/4\eta\tau_r\langle I(t)I(t+\tau)\rangle$. We express $I(t) = \langle I\rangle + i(t)$, where $i(t)$ stands for fluctuation of intensity. Because $\langle i(t)\rangle = 0$ we can write $K(0) = 1/4\eta\tau_r(\langle I^2\rangle + \langle i^2\rangle)$. It guarantees the condition: $g^{(2)}(0) \geq 1$. For very long times τ we can put $\langle i^2\rangle = 0$, i. e. the intensities are uncorrelated.

On the other hand quantum physics gives different look. Let's replace the intensity in the expression (2.10) with normal ordered annihilation and creation operators. We get:

$$g^{(2)}(\tau) = \frac{\langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t)\rangle}{\langle a^\dagger(t)a(t)\rangle} \tag{2.11}$$

The function $g^{(2)}(0)$ can exhibit smaller values than one. A simple example is the state $p|1\rangle\langle 1| + (1-p)|0\rangle\langle 0|$, for any $p > 0$ we get zero. This effect doesn't correspond with classical optics therefore we call such states non-classical and this effect anti-correlation of photons. In all natural sources the photons are bunched therefore anti-correlation cannot be observed. The $g^{(2)}(0)$ function (2.11) for thermal sources is equalled to 2 [6]. The first experiment which proved anti-correlation was arranged by Kimbel, Dagenais and Mandel in the year 1978. They let interact sodium atoms with laser light in resonance. The atoms behaved like two-level system and anti-correlation was measured. They used the same arrangement and clearly observed $g^{(2)}(0)$ less than one[7]. For very long times τ, $g^{(2)}(\tau)$ is limited to one, since photons become anti-correlated.

But anti-correlation is only sufficient condition for features of light beyond classical optics. The necessary and sufficient condition haven't been found yet.

2.2 Coherent state and Glauber-Sudarshan P-representation

We can search for a state which contains uncorrelated photons and its normally ordered normalized correlation functions get value one. This state is called coherent and is defined as the eigenstate of annihilation operator:

$$a|\alpha\rangle = \alpha|\alpha\rangle. \tag{2.12}$$

Clearly, because of this eigenstate property, a mean number of $\langle \left(a^\dagger\right)^m a^n \rangle$ is equalled to $\left(\alpha^*\right)^m \alpha^n$ for any n and m.

We can derive the coherent states also by different more operational definition. Coherent state is operationally a product of acting the displacement operator $D(\alpha)$ on vacuum state:

$$|\alpha\rangle = D(\alpha)|0\rangle. \tag{2.13}$$

The displacement operator is in fact evolution operator $U = \exp\left(i\frac{H_I}{\hbar}t\right)$ with interaction Hamiltonian H_I between classical driving harmonic oscillator and vacuum quantum oscillator. If such oscillator interacts with the vacuum for very short time and driving is strong, the interaction Hamiltonian approaches:

$$H_I = \hbar(\beta^* a + \beta a^\dagger), \tag{2.14}$$

where β is constant that characterizes the driving. Putting this Hamiltonian into Schrödinger equation and using Baker-Hausdorf theorem yields to:

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{i\alpha a^\dagger}|0\rangle, \tag{2.15}$$

where α is β multiplied by the time of interaction. Both expressions (2.13) and (2.12) leads to coherent state expanded in Fock state basis[3]:

$$|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} e^{-|\alpha|^2/2}|n\rangle. \tag{2.16}$$

The photon distribution is given by Poissonian distribution:

$$P_n = \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}, \tag{2.17}$$

where $|\alpha|^2$ stands for the mean number of photons in this state (further the mean number of photons is signed $\bar{n}$). The figure (2.2) shows such distribution for different cases of mean number.

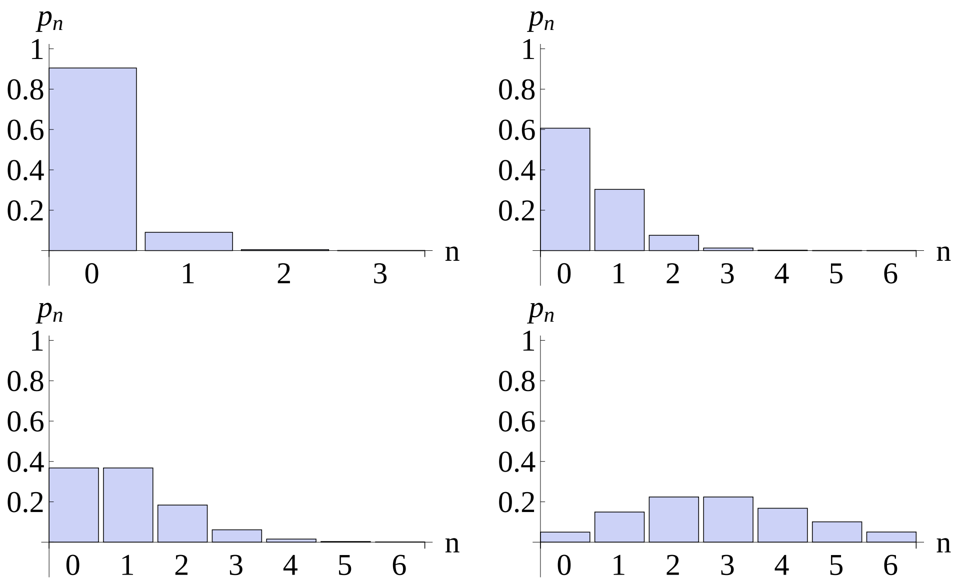

Figure 2.2: *Photon distribution of coherent state for different mean number of photon $\bar{n}$: the top and left figure corresponds to $\bar{n}$=0.1, the top and right $\bar{n}$=0.5, the bottom and left $\bar{n}$=0.5 and the bottom and right $\bar{n}$=3.*

For coherent light $g^{(2)}(0) = 1$, the photons are independent of each other but still they are not generated individually enough. On the other hand, thermal light, given by Bose - Einstein statistic [2]:

$$P_n = \frac{1}{1+\bar{n}} \left(\frac{\bar{n}}{1+\bar{n}} \right)^n , \tag{2.18}$$

where $\bar{n}$ is mean number of photons, shows bunching, $g^{(2)}(0) = 2$. The figure (2.3) shows the distribution for different $\bar{n}$. Differently to the coherent states maximum remains in origin and variance increases for larger $\bar{n}$. They are more noisy than coherent states.

General framework of classical state from point of view of coherent theory was given by Glauber in the year 1963. Let's define the $P(\alpha)$ representation of a state [3]:

$$\rho = \int P(\alpha)|\alpha\rangle\langle\alpha| d^2\alpha. \tag{2.19}$$

A state ρ is called classical (from the point of view of coherent theory) if the $P(\alpha)$ function can be interpreted as density of probability. The other states are called non-classical. If $P(\alpha)$ gets negative values or can't be generally interpreted as probability distribution, the state is non-classical from definition. For pure coherent state the $P(\alpha)$ function is equalled to Dirac's delta function. The coherent states can be obtained from very good laser. All the classical states are therefore obtainable from laser light by adding noise [3].

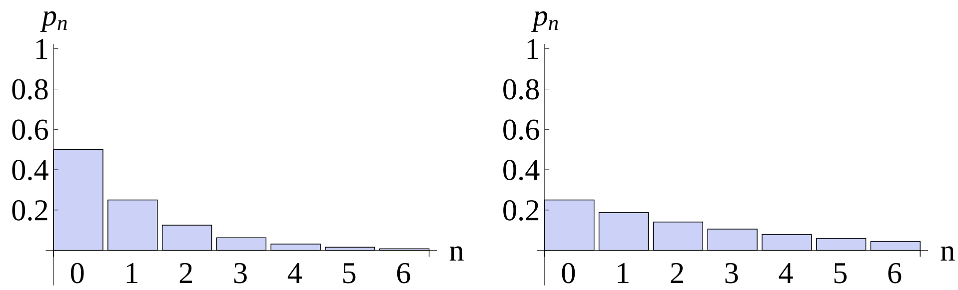

Figure 2.3: *Photon distribution of thermal state for different mean number of photon $\bar{n}$: the left figure corresponds to $\bar{n}$=1, the right $\bar{n}$=3.*

For example the thermal noise is very usual natural source therefore we would expect this state is classical. It can be proven that its $P(\alpha)$ is given by function [2]:

$$P_{th}(\alpha) = \frac{1}{\pi \bar{n}} \exp\left(-\frac{|\alpha|^2}{\bar{n}}\right). \tag{2.20}$$

We see that thermal light has Gaussian distribution $P(\alpha)$, it is one way how to obtain single mode thermal state from laser light.

Unfortunately $P(\alpha)$ isn't directly measurable. Experimental data are badly classified from this definition. Therefore sufficient criteria which would decide directly from measurement if a state is non-classical are necessary [11]. One of such criterion is the correlation function described previously. But such the criterion cannot be applied for very weak photon beams, since exact photon-number detection is still challenging problem.

Chapter 3

Nonclassicality of single photon

3.1 α - parameter

The correlation function $g^{(2)}(0)$ is function of the first and the second momentum of the photon number operator n. P. Grangier et al. introduced in 1985 different method, which directly compare probabilities of detection [8].

The scheme of experiment is inspired by HBT experiment but very weak beams with $\langle n \rangle \ll 1$ are measured. For this measurement we have to use very sensitive avalanche photodetectors which, however, are not able to distinguish the number of photons [8]. The source is two photon radiative cascade. It emits pairs of photons with different frequencies. One of them is a trigger which open a gate of the detectors and the other is split further in beam splitter (BS). Two photomultiplier measure probabilities the photon is reflected or transmitted. P_s stand for probability that one detector clicks independently the other one and P_c both detectors click simultaneously. The Fig. (3.1) visualises the scheme of the experiment.

The mean photocurrent is given by (in normal ordering): $\langle i \rangle = \eta \langle a^\dagger a \rangle$. The coincidence photocurrent: $\langle i^2 \rangle = \eta^2 \langle \left(a^\dagger\right)^2 a^2 \rangle$, where η is efficiency of the detector. The source is very weak, that means we can approximate it by the state: $\rho = \rho_0 |0\rangle\langle 0| + \rho_1 |1\rangle\langle 1| + \rho_2 |2\rangle\langle 2|$, where $\rho_1 \gg \rho_2$. For this approximation we get: $\langle i \rangle = \eta\rho_1$, $\langle i^2 \rangle = 2\eta^2\rho_2$. But the probability P_s is approximated by: $P_s = \eta\rho_1/2$ and $P_c = \eta^2\rho_2/2$. We see that for weak source of photons we have $g^{(2)}(0) = \langle i^2 \rangle / \langle i \rangle^2 \approx P_c/P_s^2$. The Cauchy-Schwartz inequality guarantees: $\langle i^2 \rangle \geq \langle i \rangle^2$. It leads to $P_c \geq P_s^2$. One can arrange α parameter [8]:

$$\alpha = \frac{P_c}{P_s^2} \tag{3.1}$$

obeys the classical condition $\alpha \geq 1$. For very weak photon beams, α-parameter approaches value of $g^{(2)}(0)$ function. Because one photon is very anti-correlated,

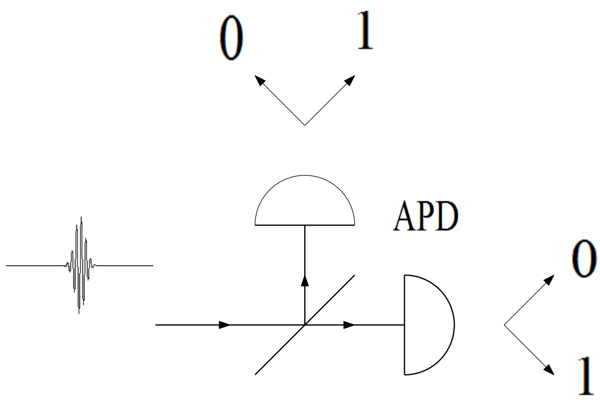

Figure 3.1: *Schema of experiment for measuring α parameter. The incident beam is split in beam splitter. They are two avalanche photomultiplier detectors (APD). They differ only two inputs: no signal (no photon) and signal (at least one photon).*

the described experiment measured by Grangier et. al broke the inequality.

3.2 Klyshko's criterion

In the year 1996 Klyshko introduced another criterion[9]. Differently to α parameter, it involves probabilities P_k of Fock states. Klyshko found a sequence of semi-definite expressions that fulfils the following condition for non-classical states:

$$\frac{(k+1)P_{k-1}P_{k+1}}{P_k^2} > 1, \tag{3.2}$$

that is satisfied for every $k \geq 1$. For classification of non-classicality of one photon sources is the most useful the case with $k = 1$. This criterion needs information about exact P_0, P_1 and P_2 probabilities. Photomultiplier detectors or avalanche photodiods can't precisely measure P_1 or P_2 probabilities. But we can suppose such ideal detector that is capable to measure them, or use homodyne detector and estimate them. The criterion is than very powerful and reveals areas of non-classicality where $g^{(2)}(0)$ function or α - parameter fails [9].

The formula (3.2) can be proven by simple inserting the classical state probabilities given by Glauber-Sadarshan P-representation. We prove it for the case $k = 1$, when the classical state is obeyed to $P_1^2 - 2P_2P_0 \leq 0$. The inserting the probabilities leads to:

$$P_1^2 - 2P_2P_0 = \int P(\alpha)P(\beta)(|\alpha|^2|\beta|^2 - |\alpha|^4)\exp(-|\alpha|^2 - |\beta|^2)d\alpha^2 d\beta^2 \quad (3.3)$$

or written equivalently with interposed α and β:

$$P_1^2 - 2P_2P_0 = \int P(\alpha)P(\beta)(|\alpha|^2|\beta|^2 - |\beta|^4)\exp(-|\alpha|^2 - |\beta|^2)d\alpha^2 d\beta^2, \quad (3.4)$$

We sum both expressions and simplify by relation $-|\alpha|^4 + 2|\alpha|^2|\beta|^2 - |\beta|^4 = -(|\alpha|^2 - |\beta|^2)^2$. Thus we get:

$$2(P_1^2 - 2P_2P_0) = -\int P(\alpha)P(\beta)(|\alpha|^2 - |\beta|^2)^2\exp(-|\alpha|^2 - |\beta|^2)d\alpha^2 d\beta^2 \quad (3.5)$$

If the function $P(\alpha)$ and $P(\beta)$ are nonnegative, the expression on the right side of (3.5) is never positive. All classical states fulfils condition $P_1^2 - 2P_2P_0 \leq 0$. If a state breaks it, it is certainly a nonclassical state.

The prove was derived with assumption of one mode of a classical state. In the Chap. 6 we show the multimode classical states have photon distribution that corresponds to distribution of one mode case. Therefore no classical multimode state can fulfils the inequality (3.2) as well.

3.3 Homodyne Detection and Tomography

We discuss different measurement technique - homodyne detection to complete basic description of nonclassicality criteria. Homodyne detection measures quadratures of generalized coordinate and momentum of the light wave. It is convenient to define the phase space quasiprobability representation of a state. This representation is called Wigner's function and is define so this[8]:

$$W(Q,P) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\langle Q + \frac{1}{2}Q'|\rho|Q - \frac{1}{2}Q'\rangle e^{-iPQ'}dQ', \quad (3.6)$$

where ρ is density matrix and $|Q\rangle$ are eigenstates of quadrature $Q = (a+a^\dagger)/\sqrt{2}$. This function can be considered as another definition of state in phase space with Q and P variables.

The scheme of the homodyne measuring consists of signal input that interferes with laser beam on a balance beam splitter. There are two detectors which measure photon numbers of the strong out-going optical signals from the beam splitter. This guarantees the quadrature amplitude $Q_\theta = (ae^{-i\theta} + a^\dagger e^{i\theta})/\sqrt{2}$ is measured[10]. The amplitude of quadrature is associated with Wigner's function by this expression:

$$\langle Q_\theta, \theta|\rho|Q_\theta, \theta\rangle = \int_{-\infty}^{\infty} W(Q_\theta\cos\theta - P_\theta\sin\theta, Q_\theta\sin\theta + P_\theta\cos\theta)dP_\theta, \quad (3.7)$$

where $|Q_\theta, \theta\rangle$ is eigenstate of the operator Q_θ. The angular θ is parameter which is changeable. If a state is measured in all θ in interval $\theta \in (-\pi/2, \pi/2)$ then we get the full information about the Wigner's function. This procedure is called tomography. The photon distribution is determined from the tomography [10]. It effectively inverts (3.6) and gain ρ_{nm} in Fock state basis from $W(Q, P)$. From reliable estimation of P_n probabilities, nonclassicality of the state can be proven using either $g^{(2)}(0)$ function or Klyshko's criterion.

By integration the Wigner's function over some area in $Q - P$ phase space we get the probability that a state is located in this area. But the Wigner's function can gets even negative values. Therefore it's called quasiprobability density function. All states with negative Wigner's function are non-classical A clear example of negative Wigner function is single photon state. Wigner function in the coordinate origin is given by: $W(0,0) = \frac{1}{\pi} Tr\left[\rho(-1)^n\right]$. We see that in the origin the single photon state gets negative value $W(0,0) = -\frac{1}{\pi}$ [10]. However, for $p|1\rangle\langle 1| + (1-p)|0\rangle\langle 0|$ negative values vanish already for $p \leq 0.5$.

Vogel introduced more sensitive criterion that classified some states with positive Wigner's function as nonclassical. He defined the function $G(k, \theta)$ by Fourier transformation:

$$G(k, \theta) = \frac{1}{2\pi} \int dk \langle Q_\theta, \theta|\rho|Q_\theta, \theta\rangle e^{ikx} \tag{3.8}$$

And he derived the sufficient condition of nonclassicality: there exists such k that [11]:

$$|G(k, \theta)| > e^{-k^2/2}. \tag{3.9}$$

Unfortunately, his criterion mixes non-classicality of single photon and very different squeezed states. Further he attempted to find hierarchy of condition which would guarantees a state is classical. It's yielded by characteristic function of Glaubner-Sudarshan P- function:

$$\Phi(u, \upsilon) = \int_{-\infty}^{\infty} P(\alpha_r, \alpha_i) \exp\left[2i(\upsilon\alpha_r - u\alpha_i)\right] d\alpha_r d\alpha_i \tag{3.10}$$

where index r and i stands for the real and imaginary part of argument α. The equivalent definition of classicality is given by infinite number of condition. For arbitrary real numbers u_k, υ_k, arbitrary complex number ξ_k $(k = 1, ..., n)$, the expression:

$$\sum_{i,j=1}^{n} \Phi(u_i - u_j, \upsilon_i - \upsilon_j)\xi_i \xi_j^* > 0 \tag{3.11}$$

is satisfied. That's means $\Phi(u, \upsilon)$ is positive semi-definite. It guarantees the $P(\alpha_r, \alpha_i)$ is probability density [11]. It is witnessing that to conclusively decide whether state is nonclassical needs to check infinite number of conditions.

Chapter 4

Simplest criterion of nonclassicality

The Klyshko's criterion of nonclassicality is given by formula (3.2). The case with $k = 1$ leads to inequality:

$$\frac{2P_0P_2}{P_1^2} < 1, \tag{4.1}$$

which has to be satisfied for all nonclassical states. We show simpler criterion which only involves exactly probabilities P_0 and P_1.

The probability $P_1 = \langle 1|\rho|1\rangle$ of measuring exactly single photon with ideal detector we call success probability. The probability of measuring more then one photon $P_{2+} = \sum_{n=2}^{\infty}\langle n|\rho|n\rangle$ we call the error probability and finally the probability of measuring vacuum $P_0 = \langle 0|\rho|0\rangle$ we call failure probability. They have to fulfil $P_0 + P_1 + P_{2+} = 1$.

First, we find the maximum value of success probability over all classical states. According to Glauber-Sudarshan P(α)-representation (2.19) the success probability is given by:

$$P_1 = \int P(\alpha)|\alpha_i|^2 \exp(-|\alpha_i|^2) d\alpha^2 \tag{4.2}$$

Because this functional is linear, the maximum of (4.2) has to be in some pure coherent states. The extreme is satisfied for $|\alpha|^2 = 1$, all states with $P_1 > 1/e \doteq 0.368$ [12] are certainly nonclassical. The closest coherent state to $|1\rangle$ has the same mean number of photons. However, for many practical sources with low efficiency $P_1 > 1/e$ is to far to be satisfied.

The criterion becomes more powerful when we involve more information from the density matrix. We make an analogy procedure for the expression [13]:

$$P_1 + a(1 - P_0 - P_1), \tag{4.3}$$

where a is real parameter. The threshold function $F(a)$ is define as maximum of expression (4.3) over all classical states with fixed a. Because expression (4.3) is linear functional of state ρ we optimize it over coherent state $|\alpha\rangle$. We compare this function with measured probabilities Q_1 and Q_{2+}. If there is such a that inequality:

$$Q_1 + aQ_{2+} > F(a) \tag{4.4}$$

is valid, the measured state can't be interpreted as a mixture of coherent states. This gives us a sufficient condition for non-classicality based only on P_0 and P_1 probabilities.

The threshold function $F(a)$ is yielded by zero equalled derivative of (4.3). It is satisfied for $|\alpha|^2 = \frac{1}{1-a}$ and then:

$$F(a) = (1-a)\exp\left(-\frac{1}{1-a}\right) + a, \tag{4.5}$$

where $a \in (-\infty, 1)$. The existence of a in the inequality (4.4) is equivalent to condition $Q_1 \geq Q(a_0)$, where $Q(a_0)$ is minimum of function $Q(a) = F(a) - aQ_{2+}$. The a_0 is satisfied for $Q_{2+} = 1 + \frac{2-a_0}{1-a_0}\exp\left(-\frac{1}{1-a_0}\right)$ and $Q(a_0) = \frac{1}{1-a_0}\exp\left(-\frac{1}{1-a_0}\right)$. The condition of nonclassicality is given by:

$$\begin{aligned} Q_1 &> \frac{1}{1-a_0}\exp\left(-\frac{1}{1-a_0}\right), \\ Q_{2+} &= 1 + \frac{2-a_0}{1-a_0}\exp\left(-\frac{1}{1-a_0}\right). \end{aligned} \tag{4.6}$$

The criterion is unchanged if we multiplied expression (4.3) by any constant or we add one. Therefore the criterion (4.3) is equivalent to this one:

$$P_1 + aP_0. \tag{4.7}$$

The same optimization derivation gives threshold function $F(a) = \exp(a-1)$ for $a \in (-\infty, 1)$. The minimum of function $Q(a) = \exp(a-1) - aQ_0$ is satisfied for $a = 1 + \ln Q_0$. Substituting it in $Q(a)$ function we get the explicit expression of non-classical condition:

$$Q_1 > -Q_0 \ln Q_0. \tag{4.8}$$

The equivalence of both criteria can be seen when we make substitution $Q_0 = \exp\left(-\frac{1}{1-a_0}\right)$ in relations (4.6):

$$\begin{aligned} Q_1 &> -Q_0 \ln Q_0, \\ Q_{2+} &= 1 - Q_0 + Q_0 \ln Q_0 \end{aligned} \tag{4.9}$$

We see, by changing parametrization of the problem, we can solve it more easily and go back to original physical representation.

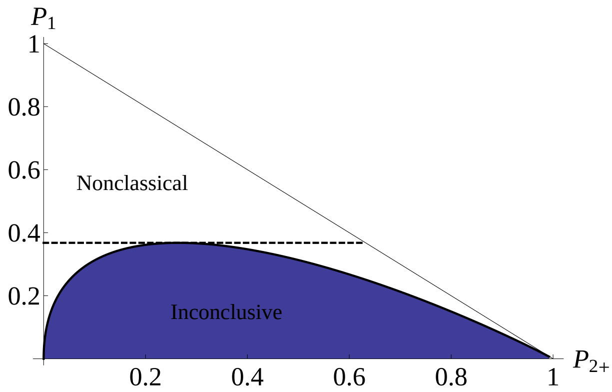

Figure 4.1: *The simplest criterion gives sufficient condition of non-classicality. All states that belongs to the white area are non-classical. The dashed line remarks the maximum of success probability of classical states.*

Our linear criterion (4.3) leads to very non-linear condition of non-classicality (4.8) or (4.10) which is full equivalent to previous method deriving same threshold. The sufficient condition of nonclassicality is visualized in Fig. (4.1). One can see the boundary of non-classicality touches the P_{2+} axes in two point. They correspond to regions with low success but low or high errors.

The criterion (4.3) can be interpreted as maximizing P_1 with condition $P_{2+} = Q_{2+}$ and Lagrangian multiplier a. Because we optimize over pure state we have:

$$\begin{aligned} \frac{d}{d|\alpha|^2}\left[P_1(|\alpha|^2) + a\left[P_{2+}(|\alpha|^2) - Q_{2+}\right]\right] &= 0 \\ Q_{2+} &= P_{2+}(|\alpha|^2). \end{aligned} \tag{4.10}$$

Thus we have two equations for two variables a and $|\alpha|^2$. From the first we can express $|\alpha|^2$ and put it in the second one. The second equation would then leads to relation between a and Q_{2+}. But because it has no analytical solution the a is understood as parameter. The result corresponds to relations (4.6).

Although one can consider $\frac{P_{2+}}{2P_1^2}$ as some function approaching $g^{(2)}(0)$ in limit of $P_1 \gg P_{2+}$, we skip this historical link and rather follow more complete evaluation based on $P_1 - P_{2+}$ diagram.

4.1 Robustness of nonclassicality

The visualization of the criterion in 2D plot of the success and error probabilities is useful to evaluate a quality of the state. We suggest method that enable us to assign to a state a number, that characterise robustness of its nonclassicality. It can be guaranteed by tolerance against an additional losses, which are typically

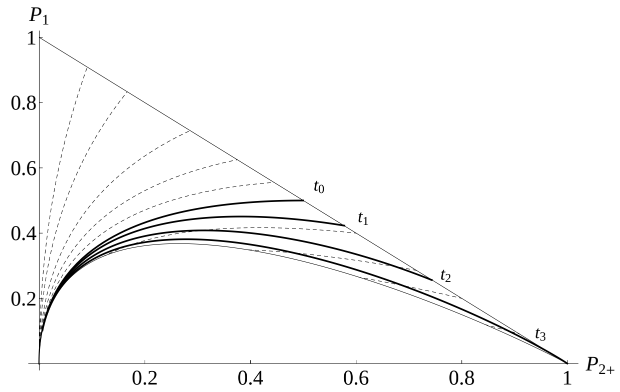

Figure 4.2: *Robustness of one photon mode with distinguishable thermal mode. The state moves along the dashed lines during attenuation and crosses the boundary of nonclassicality if $p < \bar{n}$. Each thick line connect states with the same robustness. The transmittances of individual case are given by: $t_1 = 0.2$, $t_2 = 0.5$ and $t_3 = 0.8$.*

present in any application of single photon state. The non-classical state is more robust when its nonclassicality is more tolerant against the attenuation.

Some states are always detected as nonclassical by that simplest criterion even when they are attenuated. This case is exampled by a distinguishable mode of Poissonian noise added to single photon or N-distinguishable modes of single photon state. The thermal noise added to state $p|1\rangle\langle + (1-p)|0\rangle\langle 0|$ in a distinguishable mode can break the nonclassical border by attenuation if $p < \bar{n}$, where $\bar{n}$ is mean number of thermal photon. The figure (4.2) shows the situation. The dashed lines stands for the states with $p = k\bar{n}$, where k is constant fixed for each line. A state located in a line in this diagram follows the line during attenuation. The thick line t_0 stands for the case $p = \bar{n}$. All states with lower rate $p/\bar{n}$ reach the boundary of nonclassicality. The others thick lines connect states with the same robustness. The necessary transmittances to reach the boundary are given by $t_1 = 0.2$, $t_2 = 0.5$ and $t_3 = 0.8$. With respect to this criterion, nonclassicality is more robust when it survives larger loss.

Consider two states: states A donated by probabilities P_0, P_1, P_{2+} and state B donated by Q_0, Q_1 and Q_{2+}. How can be compare? If we can prepare both states with the same rate $1 - P_0$ and $1 - Q_0$, the greater success probability, the better source. Otherwise we have to externally equalize failure or error probability of both states A and B.

1) If the P_0 probability is not important for us, we can equalize the errors by attenuation. We can generally say that the state A is better if $Q_{2+} > P_{2+}$ and $Q_1 < P_1$. Otherwise we equalize the errors by attenuation. The attenuation make the errors smaller, therefore if $Q_{2+} > P_{2+}$ we can attenuate the state B to yield $Q'_{2+} = P_{2+}$. But the attenuation makes the errors lower and makes the state B better therefore only if $P_1 > Q'_1$ the state A is better. The result is

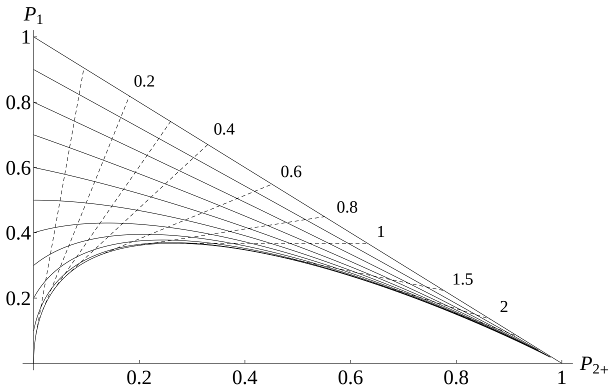

Figure 4.3: *Poissonian noise. Full lines express how is a states changed if the noise is added. The dashed line determine the mean number of photon to get the state to different spot. The numbers above the graph are related to case of one photon noised and express the mean number of photon in the mode of the noise.*

inconclusive in the opposite case.

2) In some situation the failures can be an important error. If $Q_0 > P_0$ and $Q_1 < P_1$, the state A is better. Otherwise we equalize the failures by adding a distinguishable mode of Poissonian noise. The yielded probabilities are given by:

$$\begin{aligned} P_0' &= P_0 e^{-\bar{n}} \\ P_1' &= P_0 \bar{n} e^{-\bar{n}} + P_1 e^{-\bar{n}}, \end{aligned} \quad (4.11)$$

where $\bar{n}$ is mean number of photons. It leads the P_0' is always smaller. We noise the state A until $P_0' = Q_0$. We can then compare the states A and B. Only if $Q_1 > P_1'$ the state B is better. In opposite case, the result is inconclusive. Figure (4.3) show where a state is relocated in $P_1 - P_{2+}$ diagram after adding the Poissonian noise. The full lines are given by (4.11). No full line cross the border of nonclassicality. The dashed lines determine the mean number of photon to get the state to different spot and are tangential to the border of non-classicality.

4.2 Estimation of success probability

To detect weak photon streams we usually use avalanche photo-detectors, as in case of anticorrelaction and measurement described in Sec. 3.1. They differ only two inputs: one or more photons detected or vacuum detected. All imperfections which cannot improve nonclassical character are pessimistically consider in following tests as part of state.

Because the previously described criterion requires the probability of one photon Q_1 we need to reliably estimate it from measured probabilities.

In experiment from Sec. 3.1, a state of light is split by semi-transparent mirror in two beams. Both are detected by two avalanche photo-detectors. We define the single probability Q_s (just one detector clicks), coincidence probability Q_c (both detectors click) and POVM operators that measure those events: $\Pi_s = (1_a - |0\rangle\langle 0|_a) \otimes |0\rangle\langle 0|_b + (1_b - |0\rangle\langle 0|_b) \otimes |0\rangle\langle 0|_a$ corresponds to just one detector clicks and $\Pi_c = (1_a - |0\rangle\langle 0|_a) \otimes (1_b - |0\rangle\langle 0|_b)$ both detectors click. Let's define density matrix for the split beams $\rho_{a,b}$. The indexes a and b differ two modes of transmitted and reflected light. The single and coincidence probabilities are given by:

$$\begin{aligned} Q_s &= Tr(\Pi_s \rho_{a,b}) \\ Q_c &= Tr(\Pi_c \rho_{a,b}). \end{aligned} \tag{4.12}$$

If Q_n is equalled to $\langle n|\rho|n\rangle$, where ρ is state before the beam splitter, the single probability is given by $Q_s = 2\sum_{n=1}^{\infty} (1/2)^n p_n$ and $Q_c = 1 - Q_s - Q_0$, where Q_0 corresponds to measuring the vacuum. We see, two photons can pass the splitter together and be detected by single detector only. We have to eliminate influence of this in data.

Because the criterion (4.3) leads to extremal problem we want the condition $Q_1^e \leq Q_1$ to be satisfied, where Q_1^e is estimated[14]. It ensures that all states classified as non-classical are non-classical really. The estimation $Q_1^e = Q_s - Q_c$ yields to $Q_1^e = Q_1 - \sum_{n=3}^{\infty}(1 - 1/2^{n-2})Q_n$ and meets the requirement.

The estimated Q_1^e has to fulfils the statement (4.4) to be the state interpreted as non-classical. We use an analogy sequence of calculations as previously to get the solution. We consider equation $Q_s - Q_c + a(1 - Q_0 - Q_s + Q_c) = F(a)$, where the threshold function corresponds with function in (4.5). The expression of Q_s probability yields to function $Q_s(a) = (1-a)\exp\left(-\frac{1}{1-a}\right) + a + Q_c - 2aQ_c$. We find the minimum of the $Q_s(a)$. It is satisfied for $Q_c = \frac{a\exp[1/(a-1)]-a+1}{-2a+2}$. Putting the Q_c in expression $Q_s(a)$ we get the parametrization of the boundary of nonclassicality. It yields to this condition of nonclassicality:

$$\begin{aligned} Q_s &> \frac{a\exp\left[1/(a-1)\right] - a + 1}{-2a + 2}, \\ Q_c &= \frac{(a-2)\exp\left[1/(a-1)\right] - a + 1}{2(a-1)}. \end{aligned} \tag{4.13}$$

From these probabilities we can easily get the estimated probabilities $Q_1^e = Q_s - Q_c$ and $Q_{2+}^e = 2Q_c$. After the substitution we get the same condition of non-classicality like in (4.6).

We can also estimate probabilities in criterion $P_1 + aP_0$. The failure probability corresponds to no detector clicks probability Q_0. We need only estimate the success probability: $P_1^e = Q_s - Q_c$. The threshold function is given by $F(a) = e^{a-1}$. We search for minimum value of Q_s probability given by $Q_s - Q_c + aQ_0 = F(a)$. It is satisfied for $Q_c = \frac{1}{2}(1 - 2e^{a-1}(a-2))$ and

$Q_s = 1/2(1 - ae^{a-1})$. We can easily convince ourselves this estimation is equivalent with (4.13) by substituting it by $a = b/(b-1)$. It leads to the same parametrization of the boundary of nonclassical states.

Chapter 5

Criteria based on the measured probabilities

Beside the correct estimation of P_1 there is another way. We can include the anti-correlation type of setup and measured probabilities in the criterion:

$$P_s + aP_c. \tag{5.1}$$

It's beneficial to define POVM Π_s so this: $\Pi_s = (1_a - |0\rangle\langle 0|_a) \otimes 1_b$, i. e. we measure clicks of one detector independently the other. The $\Pi_c = (1_a - |0\rangle\langle 0|_a) \otimes (1_b - |0\rangle\langle 0|_b)$ as in previous section. The pure coherent state is uncorrelated in the beam splitter, thus its density matrix is factorizable:

$$\rho_{a,b} = |\alpha/\sqrt{2}\rangle\langle\alpha/\sqrt{2}|_a \otimes |\alpha/\sqrt{2}\rangle\langle\alpha/\sqrt{2}|_b. \tag{5.2}$$

It yields to $P_s = 1 - e^{-|\alpha|^2/2}$ and $P_c = \left(1 - e^{-|\alpha|^2/2}\right)^2$. The optimization of criterion (5.1) gives function $F(a) = -\frac{1}{4a}$, for $|\alpha|^2 = -2\ln\left(\frac{2a+1}{2a}\right)$, where $a \in (-\infty, 0)$. We compare this function with measured probabilities Q_s and Q_c and find the set of non-classical states. The function $Q_s(a) = -\frac{1}{4a} - aQ_c$ gets minimum in $a = -\frac{1}{2\sqrt{Q_c}}$. By inserting it instead of a we get:

$$Q_s > \sqrt{Q_c}. \tag{5.3}$$

If we define $\alpha = Q_c/Q_s^2$ this result is fully equivalent to α-parameter derived by Cauchy-Schwartz in-equality and photon detection formula introduced by Grangier et al in Ref. [8]. However, this criterion (5.3) derived ab initio without any approximation presents nonclassicality in 2D $P_s - P_c$ plane which contains more information.

We compare this result with estimation. But there are two different definition of P_s probability. Let donate P'_s probability of single detection used in the estimation case and P_s probability of single detection used in this Section. The

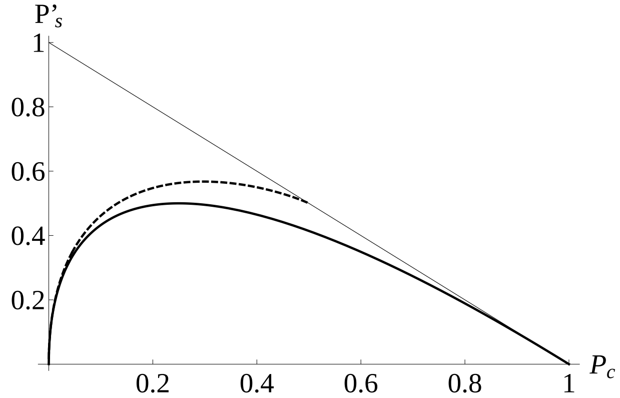

Figure 5.1: *Comparison of α-parameter and estimation of probabilities. The dashed curve stands for the boundary for estimation and the full one stand for α-parameter.*

optimal result is given by coherent state, thus we can write:

$$P_s' = 2p_0(1-p_0) = 2\left[1 - p_0 - (1-p_0)^2\right], \tag{5.4}$$

where p_0 is given by $p_0 = e^{-|\alpha|^2}$. Using $P_c = (1-p_0)^2$ it leads to $P_s' = 2(\sqrt{P_c} - P_c)$ and condition of non-classicality:

$$Q_s' > 2(\sqrt{Q_c} - Q_c). \tag{5.5}$$

This is comparable with the estimation. The figure (5.1) shows this criterion (5.5) against estimation version of criterion (4.13), (5.5) clearly catches more states in region of higher P_c. However, for low P_c both methods well coincide.

5.1 Higher ordered criteria based on measurement probabilities

Our optimization approach allows to extend concept of criterion (5.1) being equivalent to α-parameter. Can we detect more states as nonclassical involving extended scheme with three detectors as shown in Fig. (5.2)? There are two BS: the first in propagation of the beam has transitivity $t_1 = 1/3$ and the second one has transitivity $t_2 = 1/2$. We define probability of single P_s (detector A clicks), probability of coincidence P_c (detector A and B detect simultaneously) and triple probability P_t (all three detectors click). The operators of these events are given by: $\Pi_s = (1 - |0\rangle\langle 0|)_A \otimes 1_B \otimes 1_C$, corresponds to A detector clicks, $\Pi_c = (1 - |0\rangle\langle 0|)_A \otimes (1 - |0\rangle\langle 0|)_B \otimes 1_C$, corresponds to A and B detectors click and finally $(1 - |0\rangle\langle 0|)_A \otimes (1 - |0\rangle\langle 0|)_B \otimes (1 - |0\rangle\langle 0|)_C$ corresponds to all detectors click.

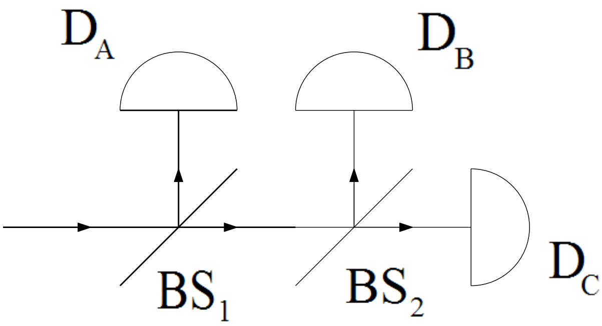

Figure 5.2: *Balanced measurement with three detectors. There are two beam splitters (BS). The first in propagation (BS_1) has transitivity 1/3 and the second (BS_2) has transitivity 1/2. All three out coming modes are detected by avalanche detectors A, B and C.*

We will derive criteria with all combination of these probabilities. We optimize over coherent state, which has no correlation among out coming modes. Therefore the probabilities are given by:

$$P_s = 1 - e^{-|\alpha|^2/3}, P_c = \left(1 - e^{-|\alpha|^2/3}\right)^2, P_t = \left(1 - e^{-|\alpha|^2/3}\right)^3 \tag{5.6}$$

First, criterion has a form $P_c + aP_t$. The nonclassical condition can derived by the same methods as used previously. The threshold function is equalled: $F(a) = -\frac{8}{27a^3}$ and the condition of nonclassicality $P_c > P_t^{2/3}$. Secondly the criterion $P_s + aP_c$, which yields to threshold function $F(a) = -\frac{1}{4a}$ and condition $P_s > \sqrt{P_c}$. Note, although form $P_s > \sqrt{P_c}$ is the same as for previous two detector criterion, but now the detection setup is different. And the last case $P_s + aP_t$ with threshold function $F(a) = \frac{2}{3\sqrt{-3a}}$ and condition $P_s > P_t^{1/3}$. All three threshold functions are define only for negative parameters a.

Because $P_s > P_c^{1/2}$ and $P_c^{1/2} > P_t^{1/3}$ implies $P_s > P_t^{1/3}$ we have in fact only two criteria of nonclassicality and the last one is consequence of the first two ones.

Chapter 6

Multi-mode classical states

We have supposed one mode of a classical state yet. But the realistic states consist of many mode. First, we discuss two modes of a classical mixtures as an example[3]:

$$\rho = \iint P(\alpha, \beta)|\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta| d\alpha^2 d\beta^2 \tag{6.1}$$

where the measured probabilities $P_n = \langle n|\rho|n\rangle$ are given by:

$$P_n = \sum_{k=0}^{n} q_{a,n-k} q_{b,k}, \tag{6.2}$$

The quantities $q_{a,n-k}$ and $q_{b,k}$ stands for the measured probabilities of $n-k$ and k photons in the modes a and b. Inserting in (6.2) mixtures of coherent states we gets:

$$P_n = \iint \sum_{k=0}^{n} P(\alpha, \beta) \exp(-|\alpha|^2 - |\beta|^2) \frac{|\alpha|^{2(n-k)}}{(n-k)!} \frac{|\beta|^{2k}}{k!} d\alpha^2 d\beta^2. \tag{6.3}$$

We modify this expression with binomial theorem and obtain:

$$P_n = \iint P(\alpha, \beta) \exp(-|\alpha|^2 - |\beta|^2)(|\alpha|^2 + |\beta|^2)^n \frac{1}{n!} d\alpha^2 d\beta^2. \tag{6.4}$$

This is an integral over Poissonian distributed probabilities with the effective mean number of photons $\bar{n} = |\alpha|^2 + |\beta|^2$. If we suppose N independent modes, the effective mean number of photons is:

$$\bar{n} = \sum_{i=1}^{N} |\alpha|_i^2. \tag{6.5}$$

The optimization over all these N-modes mixtures yield to optimization over the simple Poissonian distribution with effective parameter $\bar{n}$. It means the previously introduced criterion is valid even for multimodes mixture of coherent states.

Similarly we can prove that optimization of criterion $P_s + aP_c$ yields to optimization over one mode. We can express probability of single and coincidence detection by this way:

$$\begin{aligned} Q_s &= 2(p_0 - p_{00}) \\ Q_c &= 1 - 2p_0 + p_{00}, \end{aligned} \tag{6.6}$$

where p_0 means probability a detector doesn't click and p_{00} both detector don't click.

First, we search for the explicit form of those probabilities in case of two modes. The probability p_{00} corresponds to detection vacuum in state (6.1):

$$p_{00} = \iint P(\alpha, \beta) \exp\left(-|\alpha|^2 - |\beta|^2\right) d\alpha^2 d\beta^2 \tag{6.7}$$

Because coherent state has no correlation at beam splitter, the state (6.1) is split in this state:

$$\rho = \iint P(\alpha, \beta) |\alpha/\sqrt{2}\rangle\langle\alpha/\sqrt{2}|^{\otimes 2} \otimes |\beta/\sqrt{2}\rangle\langle\beta/\sqrt{2}|^{\otimes 2} d\alpha^2 d\beta^2. \tag{6.8}$$

The probability p_0 is then given by:

$$p_0 = \iint P(\alpha, \beta) \exp\left[-\frac{1}{2}(|\alpha|^2 + |\beta|^2)\right] d\alpha^2 d\beta^2 \tag{6.9}$$

One can see that optimizing (5.1) yields to optimizing it over case where $P(\alpha, \beta) = \delta(\alpha, \beta)$, It means we have effective mean number $\bar{n} = |\alpha|^2 + |\beta|^2$ in the case of two modes. It can be easily generalised for N - modes. The mean number is then given by $\bar{n} = \sum_{i=1}^{N} |\alpha|_i^2$ in the case of N - modes.

Maximising any previous criteria over any multimode mixture of coherent states yields to maximising it over single Poissonian distribution, therefore results derived for single mode states remain valid also for multimode case.

Chapter 7

Physical models of single photon sources

It's very hard to generate exactly the Fock state $|1\rangle$. The real single photon sources in solid state physics have effectiveness less then one and gives some incoherent noise to the signal. A physical model of such generated state can be given by the following scheme. A photon state $|1\rangle$ is divided in beam splitter with transmittance p and a distinguishable mode of noise is added to the photon. The outcome is given by multimode structure:

$$\rho = (p|1\rangle\langle 1| + (1-p)|0\rangle\langle 0|) \otimes \rho_{noise}, \tag{7.1}$$

where ρ_{noise} stand for the states of the noise and p is effectiveness of the source. This simple model covers well typical multimode generation of photons to distinguishable modes appearing in modern solid state single photon sources. It's interesting to study influence of different types of noise and its amount.

7.1 Thermal noise

If we add distinguishable mode of thermal state ρ_{th} with mean number of photons $\bar{n}$ to one photon state $p|1\rangle\langle 1|+(1-p)|0\rangle\langle 0|$ we get very realistic one photon source:

$$[p|1\rangle\langle 1| + (1-p)|0\rangle\langle 0|] \otimes \rho_{th}. \tag{7.2}$$

The relevant probabilities are given by:

$$\begin{aligned} P_0 &= (1-p)\frac{1}{1+\bar{n}} \\ P_1 &= p\frac{1}{1+\bar{n}} + (1-p)\frac{\bar{n}}{(1+\bar{n})^2} \end{aligned} \tag{7.3}$$

Further we show in this state the power of previously derived criteria. Firstly the criterion of form $P_1 + aP_{2+}$ which leads to condition of nonclassicality (4.6).

Because we search for boundary we can replace the inequality by equality:

$$\begin{aligned} P_1 &= \frac{1}{1-a}\exp\left(-\frac{1}{1-a}\right), \\ 1-P_0-P_1 &= 1+\frac{2-a}{1-a}\exp\left(-\frac{1}{1-a}\right). \end{aligned} \tag{7.4}$$

Thus we have two equation for two variables p and $\bar{n}$. The solution is satisfied for:

$$\begin{aligned} p &= a-(a-1)\exp\left(\frac{1}{a-1}\right), \\ \bar{n} &= (1-a)\exp\left(\frac{1}{a-1}\right)+a-2 \end{aligned} \tag{7.5}$$

This equations give us the boundary of nonclassicality. All states with p greater than given by these equation are certainly nonclassical.

We can compare this method to more feasible criterion for avalanche detectors based on measurement described in Chap. 5. The probabilities of measuring the state split in balanced BS are Q_s (just one detector clicks) and Q_c (both detecotrs click) and are given by (see Appendix):

$$\begin{aligned} Q_s &= \frac{2(p+\bar{n})}{2+3\bar{n}+\bar{n}^2}, \\ Q_c &= \frac{\bar{n}(p+\bar{n})}{\bar{n}^2+3\bar{n}+2}. \end{aligned} \tag{7.6}$$

We estimate the probabilities $P_1^e = Q_s - Q_c$ and $P_{2+} = 2Q_c$ as shown in previous section:

$$\begin{aligned} P_1^e &= \frac{(2-\bar{n})(p+\bar{n})}{2+3\bar{n}+\bar{n}^2} \\ P_{2+}^e &= \frac{2\bar{n}(p+\bar{n})}{2+3\bar{n}+\bar{n}^2} \end{aligned} \tag{7.7}$$

One can see the estimated P_1^e reaches even negative values for $\bar{n} > 2$. But it is a consequence of our estimation, therefore we can't use it for $\bar{n} > 2$. We put these expressions in (7.4) and find the explicit form of p and $\bar{n}$:

$$\begin{aligned} p &= \frac{1-a+(-3+4a)\exp\left(\frac{1}{a-1}\right)+(4-3a)\exp\left(\frac{2}{a-1}\right)}{1+a\left[\exp\left(\frac{1}{a-1}\right)-1\right]}, \\ \bar{n} &= \frac{2\left[1-a+(-2+a)\exp\left(\frac{1}{a-1}\right)\right]}{1+a\left[\exp\left(\frac{1}{a-1}\right)-1\right]}. \end{aligned} \tag{7.8}$$

These equation parametrised the boundary of nonclassicality with parameter a for criterion with estimated success probability.

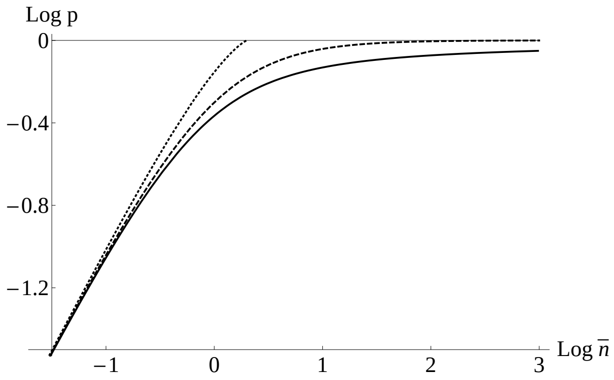

Figure 7.1: *Nonclassicality of single photon with thermal noise. The dotted curve stands for the estimation of probabilities, the dashed one for α-parameter and the full curve corresponds to criterion $P_1 + aP_{2+}$. All states above one of these lines are certainly nonclassical from the view of the relevant criterion.*

In the next we show power of α-parameter. The boundary can be explicitly express by substituting Q_s and Q_c (7.7) in condition (7.6): $Q_s > 2(\sqrt{Q_c} - Q_c)$:

$$p > \frac{\bar{n}}{1+\bar{n}}. \tag{7.9}$$

Thus we get explicit condition of nonclassicality. Figure (7.1) compare the methods. One can see the most powerful criterion is $P_1 + aP_{2+}$ than α-parameter and the weakest one is the estimation. But for very low noise are all approximately just the same. The estimation doesn't catch no states if $\bar{n} \geq 2$. The others converges to $p = 1$ for very high $\log \bar{n}$ of noise.

7.2 Poissonian noise

Poissonian noise is another realistic noise. Comparing to thermal light it corresponds to a stream of independent photons. It can be represent by mixture of coherent states all with the same mean number of photons $\bar{n}$, but random phase of α. The photon distribution is given by Poissonian statistics:

$$p_n = \frac{\bar{n}^n}{n!} e^{-\bar{n}}, \tag{7.10}$$

where $\bar{n}$ is mean number of photons. The probabilities relevant to our criterion are

$$\begin{aligned} P_1 &= pe^{-\bar{n}} + (1-p)\bar{n}e^{-\bar{n}} \\ P_0 &= (1-p)e^{-\bar{n}} \end{aligned} \tag{7.11}$$

Putting the probabilities in condition $P_1 > -P_0 \ln P_0$ we get:

$$pe^{-\bar{n}} + (1-p)\bar{n}e^{-\bar{n}} > -(1-p)e^{-\bar{n}}\left[\ln(1-p) - \bar{n}\right]. \tag{7.12}$$

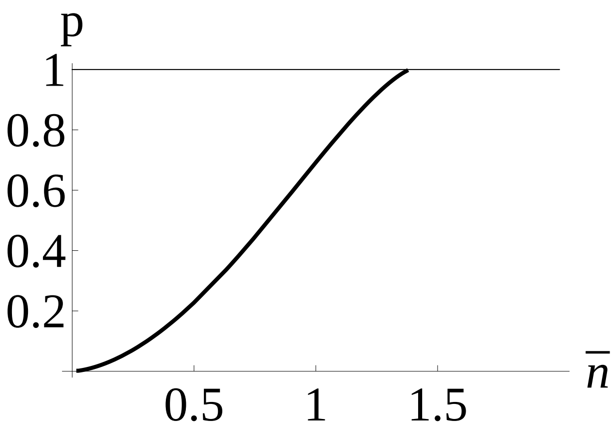

Figure 7.2: *Nonclassicality based on criterion with estimated probabilities P_1^e of Poissonian noise. All state above the curve are nonclasssical.*

It is correct for all $\bar{n} > 0$ and $p > 0$, because it leads to inequality $p > -(1-p)\ln(1-p)$, that is valid for all $p > 0$. Any of those states are therefore nonclassical.

The probability P_s of one detector clicks and P_c of both detectors click are given by (see Appendix):

$$Q_s = 2(-1+p)e^{-\bar{n}} - (p-2)e^{-\bar{n}/2} \tag{7.13}$$
$$Q_c = e^{-\bar{n}}\left(e^{\bar{n}/2}-1\right)\left(-1+p+e^{\bar{n}/2}\right).$$

The α-parameter condition (5.5) can be modified in: $(Q_s/2+Q_c)^2 - Q_c > 0$. We insert the probabilities given by (7.13) and we get: $(Q_s/2+Q_c)^2 - Q_c = \frac{1}{4}p^2 e^{-\bar{n}}$. We see it is always greater than zero for $p > 0$, e.i. the state is nonclassical for such p.

Contrary it, the estimation doesn't catch all states. We estimate $P_1^e = Q_s - Q_c$ and $P_{2+}^e = 2Q_c$. We put these probabilities in equations (4.6) and find the solution. It yields to very extensive expressions, therefore we only plot the result. The figure (7.2) displays the boundary of nonclassicality. We see that for $\bar{n} > 1.386$ the estimation catches no states.

7.3 N-modes of single photon sources

To investigate opposite case typical for solid state sources, we discuss multiple distinguishable single photon sources with comparable rates and focus on sensitivity to number of modes. If N sources independently irradiate one photon with probability p, the state is given by:

$$\rho = \Pi_{i=1}^{N} \otimes \left[p|1\rangle\langle 1| + (1-p)|0\rangle\langle 0|\right]_i \tag{7.14}$$

The success and failure probability are given by:

$$\begin{aligned} P_1 &= Np(1-p)^{N-1} \\ P_0 &= (1-p)^N \end{aligned} \tag{7.15}$$

Inserting them in condition (4.8) we get $P_1 + P_0 \ln P_0 = Np(1-p)^{N-1} + N(1-p)^N \ln(1-p)$. Because $p + (1-p)\ln(1-p)$ is greater then zero, the state is nonclassical for $p > 0$.

The single and coincidence probabilities are guaranteed by expressions (see Appendix):

$$\begin{aligned} Q_s &= 1 - (1-p/2)^N \\ Q_c &= -(1-p)^N + 2(1-p/2)^N - 1 \end{aligned} \tag{7.16}$$

We put them in $(Q_s/2 + Q_c)^2 - Qc$ and get:

$$1 + \frac{1}{4}\left[1 + 2(1-p)^N - 3(1-p/2)^N\right]^2 + (1-p)^N - 2(1-p/2)^N. \tag{7.17}$$

Numerical calculation shows this expression is greater than zero for $p \in (0,1)$ and N belongs to set of natural numbers. This type of source is nonclassical for $p > 0$, irrespective to number of sources N.

The estimation doesn't catch all states similarly as in the case of Poissonian noise. This problem can be solved only numerically from the equation (4.8). The Figure (7.3) shows the result. We see that only single and two modes cases are nonclassical for all p. For very large number of modes the boundary probability p for nonclassical state converges to zero.

7.4 Two-photon state with a distinguishable thermal noise

How efficient is criterion in detection of higher Fock state when they are subjected to loss and noise? Consider two photon state $|2\rangle$. The simplest linear criterion $P_1 + aP_{2+}$ doesn't catch this state, but in the following we show the $P_s + aP_c$ criterion with two detectors classified it as nonclassical state. Let the state $|2\rangle$ incidents in beam splitter with transitivity η. The out-coming state is: $\eta^2|2\rangle\langle 2| + 2\eta(1-\eta)|1\rangle\langle 1| + (1-\eta)^2|0\rangle\langle 0|$. And the distinguishable thermal noise ρ_{th} with mean number of photons $\bar{n}$ is added.

Firstly we search how is the state split in balanced splitter. Q_s (one detector clicks independently) is given by (see Appendix):

$$\begin{aligned} Q_s &= \frac{4\eta - \eta^2 + 2\bar{n}}{4 + 2\bar{n}} \\ Q_c &= \frac{(\eta + \bar{n})^2}{2 + 3\bar{n} + \bar{n}^2} \end{aligned} \tag{7.18}$$

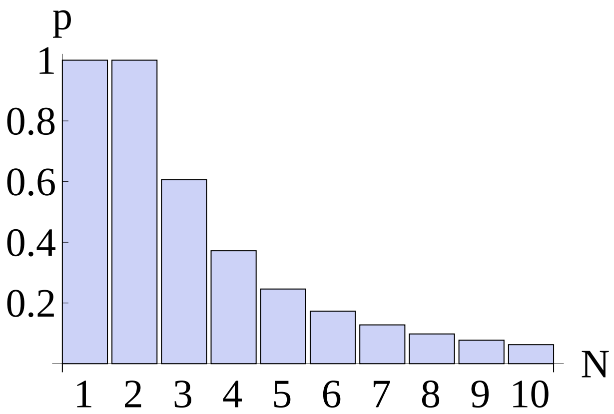

Figure 7.3: *Nonclassicality of estimated probabilities of N distinguishable modes of one photon sources. The bars reach the boundary of nonclassicality. All states with higher p are certainly nonclassical.*

We get the boundary by satisfying the equation $Q_s^2 = Q_c$, where variable is η and $\bar{n}$ is parameter. It leads to four different roots, but only one corresponds to boundary:

$$\eta = 2 + \sqrt{2+\bar{n}} - \sqrt{\frac{(2+\bar{n})(3+\bar{n}+2\sqrt{2+\bar{n}})}{1+\bar{n}}} \tag{7.19}$$

To include the higher ordered criteria we let incident the state in sequence of two BS: the first with transitivity $t_1 = 1/3$ and the second with transitivity $t_2 = 1/2$. It guarantees the state is split balancedly. The probabilities Q_s means the first detector clicks, Q_c the first two click, Q_t all three click. The state yields to (see Appendix):

$$Q_s = \frac{6\eta - \eta^2 + 3\bar{n}}{9+3\bar{n}} \tag{7.20}$$

$$Q_c = \frac{2(\eta+\bar{n})^2}{9+9\bar{n}+2\bar{n}^2}$$

$$Q_t = \frac{2\bar{n}(\eta+\bar{n})^2}{(1+\bar{n})(3+\bar{n})(3+2\bar{n})}$$

The boundary given by: $Q_s = \sqrt{Q_c}$ yields to:

$$\eta = 3 + \sqrt{\frac{3}{2}(3+\bar{n})} - \frac{1}{2}\sqrt{\frac{6(3+\bar{n})(9+2\bar{n}+2\sqrt{6(3+\bar{n})}}{3+2\bar{n}}}. \tag{7.21}$$

It is one the four solution of the equation $Q_s = \sqrt{Q_c}$ again.

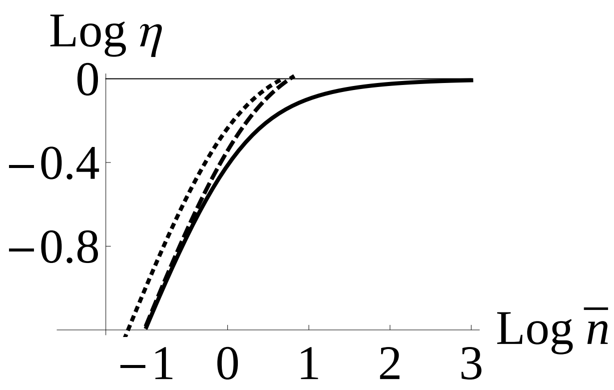

Figure 7.4: *Comparison three different criteria for attenuated two photon state with thermal noise. The dotted curve corresponds to criterion $P_c + aP_t$, the dashed one to $P_s + aP_c$ and the full one to α-parameter.*

The last case is criterion $P_c + aP_t$. The boundary is $Q_c^3 = Q_t^2$. It is satisfied for:

$$\eta = \frac{-2\bar{n}(1+\bar{n}) + \bar{n}\sqrt{2(3+\bar{n})(3+2\bar{n})}}{2(1+\bar{n})}. \tag{7.22}$$

All states with η parameter greater than given be equations (7.19), (7.21) or (7.22) are nonclassical because of the relevant criterion.

We compare all three criteria in one figure (7.4). We can see the $P_s + aP_c$ criterion using two detectors is surprisingly in this case the most powerful criterion, but for low energy of noise it is approximately as power as the criterion $P_s + aP_c$ using three detectors. The case with $\eta = 1$ and $\bar{n} = 0$ corresponds to two photon state $|2\rangle$. We see all three criteria classify it as nonclassical state. The nonclassicality here comes from two-photon states $|2\rangle$, however by loss and detection technique, it is transferred to effect which looks similar as for single photon state. This comparison is stimulating for further investigation of different criteria.

Chapter 8

Summary

In this bachelor Thesis I have tried to show the linear criteria are good tool to detect nonclassicality. They can involve both the probabilities measured by avalanche detectors or probabilities of exact number of photons like in Klyshko's criterion. We have discussed the simplest linear criterion, which is expression only of vacuum probability and one photon probability. Although this criterion is weaker then Klyshko's criterion, the relevant probabilities are more easily estimated then in the Klyshko case, where it would be necessary to estimate the two photons probability from measurement.

We use this criterion to define the robustness of nonclassical states. All states preserves or losses their nonclassical feature after the attenuation. Even the one photon with thermal noise (where the mean number of photons in the thermal noise mode is greater then the mean number of photons the second mode) can loose it completely and such the loss can mark the robustness of nonclassicality. We define comparison of two different sources as well. In future we would like extend this methods in measurable probabilities and compare it with robustness given by α parameter.

The simple criterion $P_s + aP_c$, presented in Sec. 3.3 is equivalent to α-parameter criterion. Our result was derived directly from definition given by Glauber. On the other hand α-parameter was yielded by approximation, that can be used only for weak sources.

We suggest higher ordered criteria that include a setup with more detectors. Probabilities given by events of three avalanche detectors yield to new criteria. It surprisingly appears in the concrete model of source all are weaker then criterion $P_s + aP_c$ with two detectors, but we believe that some linear combination of those criteria can detect new areas of nonclassicality.

All linear criteria were derived for single mode states. But we proved the optimization over Poissonian statistic include optimization over all N-modes classical states. Thus these criteria become more powerful for multimode solid state single photon sources.

We have demonstrated each criterion on realistic physical models. It appears that some states like single photon with Poissonian noise or N-modes of one photon sources are nonclassical always from the point of view criteria $P_s + aP_c$ (the two detectors case) and $P_1 + aP_{2+}$. Contrary the one photon with thermal noise is nonclassical only for some cases. We showed the most powerful criterion is $P_1 + aP_{2+}$. But the success probability P_1 is very badly measured. After the estimation we can see the second type of criteria based on the measurable probabilities is more powerful.

In future we would like to extend the analysis to higher-order criteria and also include other sources of noise in an indistinguishable mode.

Appendix A

Beam splitter

Beams splitter is basic optical device, which is used extensively in the thesis. It is use in such experiments as measuring $g^{(2)}(0)$ function [6], α- parameter [8]. Two incident modes a and b interact in the beam splitter and two output modes c, d come out of the splitter. In all our calculation we are interested in the case when the incident mode interacts only with vacuum. The creation operators of the four modes are bind in these equations:

$$
\begin{aligned}
a_a^\dagger &= t a_c^\dagger + r a_d^\dagger, \\
a_b^\dagger &= -r a_c^\dagger + t a_d^\dagger.
\end{aligned}
\tag{A.1}
$$

The coefficients r and t are obeyed by condition: $t^2 + r^2 = 1$. The same relations are valid for annihilation operators of corresponding modes. We can express all states $|\psi\rangle$ by this way:

$$|\psi\rangle = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \left(a^\dagger\right)^n |0\rangle. \tag{A.2}$$

A state after interaction with vacuum is yielded in form[2]:

$$|\phi\rangle = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} (t a_c^\dagger + r a_a^\dagger)^n |00\rangle_{ac}. \tag{A.3}$$

The equation (A.3) guarantees an out coming state $\rho_{c,d}$ is given by:

$$\rho_{c,d} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} t^k r^{n-k} \rho'_n |k\rangle_c \langle k| \otimes |n-k\rangle_d \langle n-k|, \tag{A.4}$$

where ρ'_n stands for probability of n photons in the incident state: $\rho'_n = \langle n|\rho_a|n\rangle$. We see that photons are split as particles, i. e. there is no interference between them. The probability $\rho_{k,n-k}$ (in one mode is found k photons, in the other $n-k$ photons) is given by binomial distribution:

$$\rho_{k,n-k} = \binom{n}{k} t^k r^{n-k} \rho'_n. \tag{A.5}$$

The problem of getting probability of single click P_s and coincidence probability P_c is then given by these equations:

$$P_s = 2(p_0 - p_{00}) \\ P_c = 1 - P_s - p_{00}, \tag{A.6}$$

where p_0 stands for probability that one of the detector doesn't click (independently the other) and p_{00} no detectors click. Probability p_{00} corresponds to probability of vacuum ρ_0 of the incident state, p_0 is given by summation of (A.5) over all cases, where $k = 0$:

$$p_0 = \sum_{n=0}^{\infty} \frac{1}{2^n} \rho_n. \tag{A.7}$$

For state with Poissonian statistics we get: $p_0 = e^{-\bar{n}/2}$. For thermal state with Bose-Einstein statistics we get $p_0 = \frac{2}{2+\bar{n}}$. If we have two distinguishable modes a and b, the probabilities are independent, therefore : $p_0 = p_{0,a}p_{0,b}$, $p_{00} = p_{00,a}p_{00,b}$, where indexes stand for the different modes a and b. Combined this expressions together we get:

$$p_0 = (1 - p/2)e^{-\bar{n}/2} \\ p_{00} = (1 - p)e^{-\bar{n}} \tag{A.8}$$

for one photon added to Poissonian noise,

$$p_0 = (1 - p/2)\frac{2}{2 + \bar{n}} \\ p_{00} = (1 - p)\frac{1}{1 + \bar{n}} \tag{A.9}$$

for one photon added to thermal noise and

$$p_0 = (1 - p/2)^n \\ p_{00} = (1 - p)^n \tag{A.10}$$

for N-distinguishable N-modes one photon sources as was mentioned previously.

Criteria of form $P_c + aP_t$ etc. involve probability measured in an extended setup with three detectors. The scheme consists of two beam splitter, the first in propagation of the beam is unbalanced with transitivity 2/3. The second one is balanced. It guarantees the beam is consequently split balanced in three modes.

To solve the splitting problem we use the formula (A.5) twice:

$$\rho_{m,n,k} = \frac{(m + n + k)!}{m!n!k!} \left(\frac{1}{3}\right)^{m+n+k} \rho'_{m+n+k}. \tag{A.11}$$

Because any permutation of indexes m, n, k doesn't change the right side of the expression, the setup is really balanced.

Let's define probabilities p_0 (detector A doesn't clicks), p_{00} (detector A and B doesn't click) and p_{000} (no detector click). The probability the detector A clicks is $1 - p_0$, the probability the detectors A and B click simultaneously is $1 - 2p_0 + p_{00}$, the probability all three detectors click is $1 - 3p_0 + 3p_{00} - p_{000}$.

For coherent state we have: $p_0 = e^{-\bar{n}/3}$, $p_{00} = e^{-2\bar{n}/3}$ and $p_{000} = e^{-\bar{n}}$. Thermal noise yields to: $p_0 = \frac{3}{3+\bar{n}}$, $p_{00} = \frac{3}{3+2\bar{n}}$ and $p_{000} = \frac{1}{1+\bar{n}}$. The two-photon state $\rho_0|0\rangle\langle 0| + \rho_1|1\rangle\langle 1| + \rho_2|2\rangle\langle 2|$ yields to: $p_0 = \rho_0 + \frac{2}{3}\rho_1 + \frac{4}{9}\rho_2$, $p_{00} = \rho_0 + \frac{1}{3}\rho_1 + \frac{1}{9}\rho_2$ and $p_{000} = \rho_0$. If we have two distinguishable modes, these probabilities are factorizable in the two modes as well.

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