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ÚSTAV MATEMATIKY

NUMERICAL METHODS FOR MISSING IMAGE PROCESSING DATA RECONSTRUCTION
NUMERICKÉ METODY PRO REKONSTRUKCI CHYBĚJÍCÍ OBRAZOVÉ INFORMACE

MASTER'S THESIS
DIPLOMOVÁ PRÁCE

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Specification Master's Thesis

Department:	Institute of Mathematics
Student:	**Ebrima M. Bah**
Study programme:	Applied Sciences in Engineering
Study field:	Mathematical Engineering
Supervisor:	**Ing. Milan Hnízdil, Ph.D.**
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Pursuant to Act no. 111/1998 concerning universities and the BUT study and examination rules, you have been assigned the following topic by the institute director Master's Thesis:

Numerical methods for missing image processing data reconstruction

Concise characteristic of the task:

Combination of various climatic conditions causes a fogging of inert automotive headlamp "glass". Automotive headlamp producers try to use micro–ventilation for faster “glass” defogging. Heat transfer and fluid flow laboratory developed experimental stand for simulation of lamps defogging process and data evaluation by image processing. A camera is placed in front of the fogged “glass” and takes pictures in defined intervals during experiments. Inert headlamp reflecting parts causes local light reflection spots on “glass”. It is not possible to decide about defogging in these spots. Student task is to use and compare numerical methods to reconstruct missing image informations.

Goals Master's Thesis:

Chose and describe appropriate numerical methods for reconstruction of missing data in image analysis.
Create numerical tool to calculate missing data.
Create testing data and compare used numerical methods.
Use created numerical tool on real experimental data obtained from laboratory measurements.

Recommended bibliography:

DRUCKMÜLLER, Miloslav, HERIBAN, Pavel. Digital image processing systems for windows version 5.0. Brno, Sofo, 2004.

STARHA, Pavel, DRUCKMÜLLER, Miloslav. Adaptive contrast control, image structure and object analyser manual.

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Deadline for submission Master's Thesis is given by the Schedule of the Academic year 2018/19

In Brno,

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Abstract

The Diploma thesis deals with reconstruction of Missing data of an Image. It is done by the use of appropriate Mathematical theory and numerical algorithm to reconstruct missing information. The result of this implementation is the reconstruction of missing image information. The thesis also compares different numerical methods, and see which one of them perform best in terms of efficiency and accuracy of the given problem, hence it is used for the reconstruction of missing data.

keywords

Reconstruction of Missing data, Image processing, Gaussian function, convolution, interpolations, regression polynomial, pixels, data-set, restoration of image.

Bah, Ebrima. M.: *Numerical Methods for Missing Image Processing Data Reconstruction*, Brno University of Technology, Faculty of Mechanical Engineering, 2019. xx pp. Supervisor: Ing. Milan Hnízdil, Ph.D.

I declare that I have written the diploma thesis *Numerical Methods for Missing Image Processing Data Reconstruction*, on my own according to advice of my diploma thesis supervisor Ing. Milan Hnízdil, Ph.D., and using the sources listed in references.

May 30, 2019 Ebrima M. BAH

Acknowledgement
I would like to express my sincere gratitude to *Ing. Milan Hnízdil, Ph.D* the supervisor of this work, for his great help, availability and patience during the course of this study.

Ebrima M. BAH

Dedication
This copy is dedicated to my two lovely parents, *Alhagie Mamadou Bah* and *Wury Bah*, and to my entire family for the support, love and patience during the time spent away from home. Thank you, *Tumbulu Jallow* for you encouragement and help during this time.

Thanks to *Mike McConnell* and *Andrea Danielson* for the support during these two years. It would have been very hard without which.

Finally, I would like to thank *Almighty Allah* for the energy, strength and consciousness, both within and without, Which made it possible for me to complete this study.

Ebrima M. BAH

Contents

1 Introduction

Natural Sciences is based on Experimental data, but the development of new branches has been greatly stimulated by theories, however, measurement and experimentation has always proven beyond doubt in different scientific disciplines.

As light of an Automotive headlamp is turns on, the air that circulates inside begins to heats up, this causes an expansion of the air, this dry air if further remove and replaced by the surrounding air by a micro-ventilation slit. Then in the reverse process, when the light bulb is turn off, the air of the inside gets cooled slowly, as this process causes a saturation of the 'humid air' from the surrounding environment to 'leaked' into the inside of the head lamp. This is due to the higher humid and the different in temperature between the inside air and the 'leaked' in air, this causes condensation on the inside and the 'glass', that covers the lens, therefore causes 'fogging', which in-turn reduces the brightness or efficiency of the Automotive-headlamp. "Nowadays, anti-fog coating is use to solve this problem, it involves painting of the inside of an Automotive headlamp with anti-fogging polymeric coating" [13]

Figure 1: A picture of a fogged Automotive headlamp

The lost of information has been a common phenomenon in many Scientific and Engineering areas, just in the last years, there has been an increase demand on image data exploration and image miming. Medical imaging has also gained high attention, there has been proposed a "classification scheme" for medical images, but such solution needs image processing.

These problems are caused by various reasons, such as, the failure of the machine in use, failing to maintain routine procedures, unfavourable environmental condition, and even human error, and there fore result to an incomplete data. The incomplete data set, could have a lot of negative effects on the further analysis of the observed dataset. The degree of this effects also depends on the deviation or the systematic difference between the measured data and the unmeasured data. The result of such deviation causes bias in the data processing. Hence, there need to come-up with a suitable mechanism to estimating missing values, this is very important, it ensures that the data that is analysed is representable and of a high quality. There are different methods employed in this study to see which does what, and then decided base on efficiency and precision, we shall then be able to conclude on the one to use for the reconstruction of missing data of images obtain in a laboratory and the performance of each of the methods were compared [5]. It is always good to avoid incomplete data matrices, as it can cause problems, because incomplete dataset can leads to conclusion that is different from those that would have been obtained from a complete dataset. There are three main issues that could arise when handling an incomplete dataset. Firstly, there is always inadequate or loss of the information, and this reduces efficiency. Second, there are many difficulties when it comes to 'computations and analysis'[5], as a result of the irregularities that occurs in the data structure, in-relation to this is that, the "impossibility of using standard software"[5]. Third, which is the most crucial, there may be bias as a result of the "systematic differences between observed and unobserved data"[5]. There exist an approach to solve the problem of incomplete data, which is to use "adoption of imputation techniques", therefore, this study discussed and compared the implementation and performance between different Numerical method for the reconstruction or restoration of missing values of laboratory images[5].

2 Mathematical Background

We selected some mathematical definition use in the field on image processing, particularly for missing data reconstruction, for which this study was based on. This definition were obtained in the following resources.[1], [5], [9], [13].

2.1 Vector spaces

A vector space is a non-empty set $\boldsymbol{V}$, whose objects are called vectors, equipped with two operations, call addition and scalar multiplication: For any two vectors $\boldsymbol{u}$**,** $\boldsymbol{v}$ in $\boldsymbol{V}$ and a scalar $\boldsymbol{c}$, there are unique vectors $\boldsymbol{u} + \boldsymbol{v}$ and $\boldsymbol{cu}$ in $\boldsymbol{V}$ such that the following properties are satisfied.

- $\forall u,v \in \mathbf{V} : u{+}v{=}w \in \mathbf{V}$(other law)
- $\forall u,v \in \mathbf{V} : u{+}v{=}v{+}u$ (commutative law)
- $\forall u,v \in \mathbf{V} : u + (v + w) = (u + v) + w$ (associative law)
- $\exists 0 \in \mathbf{V} : \forall u \in \mathbf{V} : (u + 0) = (0 + u)$ (addition of a zero)
- $\forall u \in \mathbf{V} \exists {-u} \in \mathbf{V} : u + (-u) = (-u) + u = 0$ (additions of an opposite vector)

Multiplications with a real number with the above properties.

- $\forall u \in \mathbf{V}, \forall a \in \mathbf{R} : a.u = w \in \mathbf{V}$
- $\forall a \in \mathbf{R}, \forall u,v \in \mathbf{V} : \alpha.(u{+}v) = \alpha.u + \alpha.v$
- $\forall \alpha, \beta \in \mathbf{R}, \forall u \in \mathbf{V} : (\alpha + \beta).u = \alpha.u + \beta.u$
- $\forall \alpha, \beta \in \mathbf{R}, \forall u \in \mathbf{V} : \alpha.\beta.u = \alpha.\beta.u$
- $\forall u \in \mathbf{V} : 1.u = u$

Let $\mathbf{V}$ = (V,+), is a linear Vector space in **Rn**. Elements V we call the vectors and denote them in small bold letters: **u, v**,..., real numbers. we call scalars.

The vector u - v is called the difference of vectors u and v and is defined by the relation.

$u - v = u + (-v)$

From the definition of vector space it is easy to see that for any vector u and scalar c.

Definition 2.1. Let V and W be vector spaces, and $W \in V$. If the addition and scalar multiplication in W are the same as the addition and scalar multiplication in V, then W is called a subspace of V .

If H is a subspace of V , then H is closed for the addition and scalar multiplication of V , i.e., for any u, v $\in H$ and scalar c $\in$ R, we have.

$u{+}v \in H$, cv $\in H$

Theorem 2.1. *Let H be a non-empty subset of a vector space V . Then H is a subspace of V if and only if H is closed under addition and scalar multiplication, i.e.*

1. For any vectors $u, v \in H$, we have $u + v \in H$.
2. For any scalar c and a vector $v \in H$, we have $cv \in H$.

2.2 Some special subspaces

Null Space: Let A be an mxn matrix. The null space of A, denoted by $NulA$, is the space of solutions of the linear system $Ax = 0$, that is.

$$NulA = X \in R^n : Ax = 0 \tag{2.1}$$

column space: The column space of A, denoted by $ColA$, is the span of the column vectors of A, that is, if:

$$A = [a_1, a_2, ..., a_n] \tag{2.2}$$

then,

$$ColA = Spana_1, a_2, ...a_n \tag{2.3}$$

row space: The row space of the row vector of A is the span of the row vectors of$A$, and is denoted by $RowA$.

Let $T : R^n \to R^m, T(x) = Ax$ be a linear transformation. Then$NulA$ is the set of inverse images of 0 under T and $ColA$ is the image of T, that is.

$$NulA = T^{-1}(0) \ and \ ColA = T(R^n) \tag{2.4}$$

2.3 Independent sets and bases

Definition 2.2. The Vectors $v_1, v_2, ..., v_p$ of a vector space V are called linearly independent if there are constants $c_1, c_2, ..., c_p$ such that.

$$c_1v_1 + c_2v_2 + ... + c_pv_p = 0 \tag{2.5}$$

we have

$$c_1 = c_2 = ... = c_p = 0 \tag{2.6}$$

The vectors $v_1; v_2, ..., v_p$ are called linearly dependent if there exist constants $c_1, c_2, ..., c_p$, not all zero,such that.

$$c_1v_1 + c_2v_2 + ..., +c_pv_p = 0 \tag{2.7}$$

Theorem 2.2. *Vectors $v_1, v_2, ..., v_k (k \geq 2)$ are linearly dependent if and only if one of the vectors is a linear combination of the others, i.e., there is one i such that.*

$$v_i = a_1v_1 + ... + a_{i-1}v_{i-1} + a_{i+1} + v_{i+1} + ... + a_kv_k \tag{2.8}$$

Theorem 2.3. *Let $S = v_1, v_2, ..., v_k$ be a subset of independent vectors in a vector space V . If a vector v can be written in two linear combinations of$v_1, v_2, ..., v_k$, say,*

$$v = c_1v_1 + c_2v_2 + ... + c_kv_k = d_1v_1 + d_2v_2 + ... + d_kv_k \tag{2.9}$$

then
$c_1 = d_1; c_2 = d_2, c_k = d_k$

Definition 2.3. Let H be a subspace of a vector space V . An ordered set $B = v_1v_2, ..., v_p$ of vectors in V is called a basis for H if

- B is a linearly independent se1; t, and
- B spans H, that is, H = Span $(v_1, v_2, ..., v_p\ )$

2.4 Dimensions of vector spaces

A vector space V is said to be finite dimensional if it can be spanned by a set of finite number of vectors. The dimension of V , denoted by $dimV$, is the number of vectors of a basis of V . The dimension of the zero vector space 0 is zero.If V cannot be spanned by any finite set of vectors, then V is said to be in finite dimensional.

Theorem 2.4. *Let H be a subspace of a finite dimensional vector space V . Then any linearly independent subset of H can be expanded to a basis of H . Moreover, H is finite dimensional and $dimH \leq dimV$.*

Theorem 2.5. *(BasisTheorem): Given a set $S = \{v_1, v_2, ..., v_n\}$ of n vectors of an n - dimensional vector space V.*

- If $v_1, v_2, ..., v_n$ is linearly independent, then $v_1, v_2, ..., v_n$ is a basis of V .
- If Span $\{ v_1, v_2, ..._n \} = V, then v_1, v_2, ...v_n$ is a basis of $v_1, v_2, ..., v_n$.

2.5 Matrix and its inverse

Definition 2.4. **Matrix**: A matrix is a two dimensional array of numbers or expressions arranged in a set of rows and columns. An MxN matrix A has m rows and n-columns and is written

Definition 2.5. **Inverse of a Matrix**: The inverse of a matrix is the multiplicative inverse of a square matrix. If a matrix A has an inverse, then A is said to be **nonsingular** or **invertible**. A singular matrix does not have an **inverse**. To find the inverse of a square matrix A , you need to find a matrix $^{-1}$ such that the product of A and A^{-1} is the **identity matrix**.

This is to say that for any square matrix A, and it is a nonsingular matrix, there exists an inverse matrix, with the property. $AA^{-1} = A^{-1}A = I$

where I is the identity matrix.

2.6 Numerical Methods

This is a Science and Engineering tool use to model a problem that is very difficult to solve mathematically, either it does not have solution or its very complicated to fine, Numerical methods develop an accurate and fast approximation to such a problem.

2.6.1 1D Gaussian function

Gaussian function, often simply referred to as a Gaussian distribution or the normal distribution, is a function of the form.

$$f(x) = ae^{-\frac{(x-b)^2}{2c^2}} \tag{2.10}$$

where a, b,are arbitrary constants and c is a non-zero constant . The graph of Gaussian is a symmetric with a "bell curve" shape. The constant a represent the height of the peak, b represent the position of the centre of the peak and c, is the standard deviation, the width of the bell depends on the standard deviation.

Gaussian function is use to show a probability density function of a normally distributed random variable, it is also use in signal processing to represent Gaussian filters, in image processing where a 2-dimensional Gaussian is use for blurring, with expected value $\mu = b$, and variance $\sigma^2 = c^2$. We can represent it as [13].

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} \tag{2.11}$$

2.6.2 2D Gaussian Blurring

Gaussian Blurs is also called Gaussian smoothing, it is normally used in ”graphics manipulating software”, mainly to decrease image noise. The Gaussian Blurs uses the idea of ”convolution” a convolution kernel is a result of a Gaussian function, with the pixels of the image. The 2D Gaussian function,$g(x, y)$, is represented as.

$$g(x, y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x^2+y^2)}{2\sigma^2}} \tag{2.12}$$

where x and y are the horizontal and vertical space variables, *sigma* is the standard deviation of the Gaussian distribution, the *sigma* determines the degree of the blurring . The Gaussian distribution is approximated by a convolution kernel. Therefore , the, values of the distribution is used construct a convolution matrix which is applied to the original image. After this we obtain a square matrix whose highest value is at the centre, that is the original pixel's value as a result the process goes outwards, the neighbouring pixel values get smaller since their distance to the original pixel increases [13].

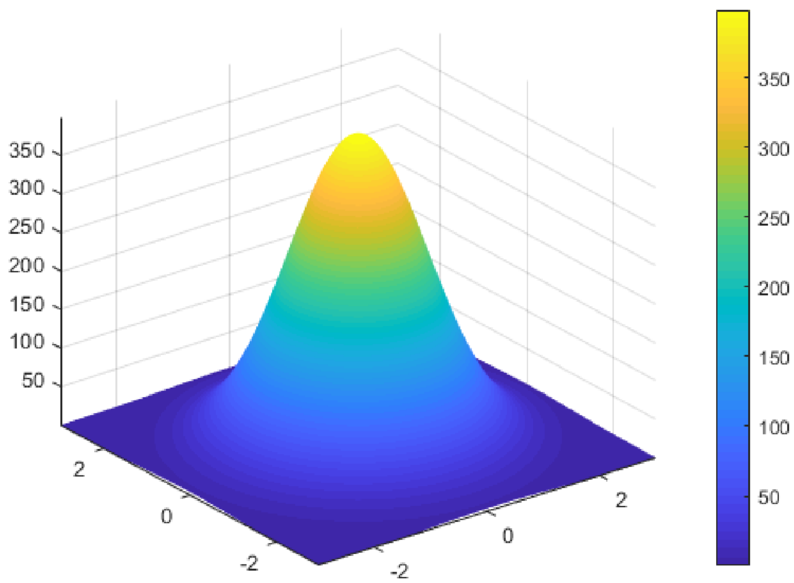


Figure 2: A surface plot of Gaussian convolution matrix

The Gaussian convolution with a particular Gaussian kernel is a low-pass filter, it smooths the edges of an image by reducing its high frequency components. The Gaussian kernel on a square support is a "linearly separable"[13]. Therefore it is computationally effective to decompose the two-dimensional Gaussian kernels into a series of one-dimensional kernel for rows and columns. The 2-Dimensional Gaussian filter produces an image convolution where each pixel value of the image is merely dependent on the surrounding pixel values and based on the on the size of the kernel used for the the implementation.

2.7 Convolution

A Convolution is either discrete or continuous, is considered as a function that forms the "integral(that is continuous) or summation(that is discrete) of two component functions", and that measures the amount of overlap as one function is shifted over the other.

convolution is the product of two functions. When doing a convolution between two functions, one function is changed with respect to the independent variable before shifting, and a transformation of variables say from t to τ is used in the shifting operation. The mathematical definition of one dimensional convolution, in discrete and continuous time are indicated by the "$*$" operator.

If f and g are functions that depends on t, then the convolution of f and g over an infinite interval is an integral given by[13].

$$f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau \tag{2.13}$$

where the minus sign indicates the change with respect to the independent function, t is the displacement that is required to slide one function and past the other, and T is a dummy variable that is integrated out. for simplicity we assume that the range of interval of the function extends from -∞ to ∞ [2]

However if the If the convolution is operated over a finite range [0, t], then we have the following expression.

$$f(t) * h(t) = \int_{-0}^{t} f(\tau)h(t-\tau)d\tau \tag{2.14}$$

To understand these equations, we can make some simple observations.It is clear to see that, the change-of-variables, f and g are functions of τ under the integral, although $f * g$ depends on t. we can see that the symmetric reflection of function *p(x)* is given by *p(-x)* If $g(t-\tau)$ is a symmetric reflection of $g(\tau)$, and which is shifted by a value t on the τ axis. Then the integral with respect to τ is computed, the amount that the function $g(t-\tau)$ changes, rather, the function g "slides" from -∞ to ∞ (or from *0* to t).

2.7.1 1D-convolution

The one-dimensional convolution are the most basic representation of a convolution operation. Thus, because images are naturally two-dimensional,the 2-D adjustment of the discrete convolution is needed for the convolution operation on images. The two-dimensional and one-dimensional versions are very similar, as a result the two are identical save for an additional set of indices[2].

If *f[n]* and *g[n]* are functions with respect to a single discrete n such as the digital signals, then convolution takes the following form:

$$A[n_a] * B[n_b] = C[n_c] = \sum_{r=0}^{n_a-1} A[\tau]B[n_c - \tau] \tag{2.15}$$

where $0 \leq n_c < n_a + n_b - 1$

2.7.2 2D-convolution

As we discussed previously, here too we can imagine that the two-dimensional case when one matrix "sliding" over the other at a unit time, with the sum of the element-wise products of the two matrices as a result. The complete convolution is found by repeating the process until the kernel has passed over every possible pixel of the source matrix. where these two matrices are a source image and a filter kernel respectively, "the result of convolution is a filtered version of the source image"[2].

$$A[i_{a,j}\, a] * B[i_b, j_b] = C[i_c, j_c] = \sum_{\tau_1=0}^{i_a-1} \sum_{\tau_2=0}^{ja-1} A[\tau_1, \tau_2]B[i - \tau_1, j - \tau_2] \tag{2.16}$$

2.8 Differentiation and Integration

In this section, we briefly introduce the notion of Numerical Derivatives and integration, as as a tool use in approximation of function.

2.8.1 Differentiation

We use the notion of Taylor's series expansion to derived finite divided difference approximation of derivatives.

Note that, higher-accuracy can be obtain by addition of higher order terms from the Taylor's series expansion.

Theorem 2.6. *Taylor's Theorem: Let f be a function, where n+1 derivatives continuous on an interval containing a and x, then the value of the function at x is given by.*

$$f(x)(a) + f'(a)(x - a) + \frac{f''(a)(x-a)^2}{2!} + ... + \frac{f^n(a)(x-a)^n}{n!} + R_n \tag{2.17}$$

Where R_n is the remainder and it is given as.

$$R_n = \int_a^x \frac{(x-t)^n}{n!} f^{n+1}(t)dt \tag{2.18}$$

Where t is a dummy variable.

The above expression, can be written as.

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2}h^2 + ... \tag{2.19}$$

If we solve for the first derivatives , we have

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{f''(x_i)h}{2} + O(h^2) \tag{2.20}$$

If we truncate the above results, we have.

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + O(h) \tag{2.21}$$

2.8.2 Integration

Integral is either finite(definite) or infinite(indefinite), of a function of one or more real variable, we discuss numerical method for approximating definite integrals, for example *f(x, y)* or *f(x,y,z)*, we can compute integrals over a region of $R^2 or R^3$.

Let f be a continuous function and derivatives exists.

$$I(f) := \int_a^x f(x)dx \tag{2.22}$$

The most common method of this is the interpolation. There are two ways of this.

- Interpolation *f(x)* by a polynomial *h(x)* at some point *(a,b)*
- Approximation *I(f)* by the exact integral of *h(x)*.

$$I(h) \approx \int_a^b h(x)dx \tag{2.23}$$

2.9 Interpolations

Interpolation is a techniques that allows us to approximate the behaviour of true underline function. If you would like to estimate an intermediate value between data points, you had use interpolation techniques.There are different types of interpolation, Liner, polynomial, cubic, etc, depending on the degree of the function. Among these, the polynomial is the mostly use method for approximation. The general expression for an nth-order polynomial is[1],[2].

$$f(x) = a_0 + a_1x + a_2x^2 + ...a_nx^n \tag{2.24}$$

For $n + 1$ data points, there is one and only one polynomial of order n that passes through all the points. For example, there is only one straight line (that is, a first-order polynomial) that connects two points[1]. Similarly, only one parabola connects a set of three points. Polynomial interpolation consists of determining the unique nth-order polynomial that fits *n+1* data points. This polynomial then provides a formula to compute intermediate values[1].

There are many type of expression for polynomial There is one and only one nth-order polynomial that fits *n+1* points, however, there are many formats in which this can be express mathematically.

2.9.1 Linear Interpolation

The simplest form of interpolation is to connect two data points with a straight line. This technique is called linear interpolation.

$$\frac{f_1(x) - f(x_0)}{x - x_0} = \frac{f_1(x) - f(x_0)}{x_1 - x_0} \tag{2.25}$$

the above equation can be rearranged to give this below.

$$f_1(x) = f(x_0) + \frac{f_1(x) - f(x_0)}{x - x_0}(x - x_0) \tag{2.26}$$

The above expression is a linear-interpolation formula. The notation *f1(x)* denotes the first-order interpolating polynomial. It is seen that, besides representing the slope of the line connecting the points, the term $[f_1(x) - f(x_0)/x - x_0]$ is a finite-divided-difference[1].

"where x is the independent variable, x_1 and x_0 are known values of the independent variable and *f(x)* is the value of the dependent variable for a value x of the independent variable"[2]. In general, the smaller the interval between the data points, the better the approximation. This is due to the fact that, as the interval decreases, a continuous function will be better approximated by a straight line.

2.9.2 Quadratic Interpolation

The error associated by the use of linear interpolation, that is approximating a curve with a straight line can be minimize by quadratic interpolation. Consequently, a strategy for improving the estimate is to introduce some curvature into the line connecting the points. If three data points are available, this can be accomplished with a second-order polynomial (also called a quadratic polynomial or a parabola). A particularly convenient form for this purpose is.

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) \tag{2.27}$$

Note that although Eq. (2.27) might seem to differ from the general polynomial [Eq. (2.25)],the two equations are equivalent. This can be shown by multiplying the terms in Eq. (2.27) to yield.

$$f_2(x) = b_0 + b_1x_1 - b_1x_0 + b_2x^2 + b_2x_0x_1 - b_2xx_0 - b_2xx_1 \tag{2.28}$$

or, collecting terms,

$$f_2(x) = a_0 + a_1x + a_2x^2 \tag{2.29}$$

where,

$$a_0 = b_0 - b_1x_0 + b_2x_0x_1 \tag{2.30}$$

$$a_1 = b_1 - b_2x_0 + b_2x_1 \tag{2.31}$$

$$a_2 = b_2 \tag{2.32}$$

Thus, Eqs. (2.25) and (2.27) are alternative, equivalent formulations of the unique second-order polynomial joining the three points. A simple procedure can be used to determine the values of the coefficients. For *b0*, Eq. (2.27) with $x = x_0$ can be used to compute.

$$b_0 = f(x_0) \tag{2.33}$$

Equation (2.32) can be substituted into Eq. (2.27), which can be evaluated at $x = x_0$ for.

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \tag{2.34}$$

Finally, Eqs. (2.32) and (2.33) can be substituted into Eq. (2.27), which can be evaluated at $x = x_0$ and solved (after some algebraic manipulations) of ......

Notice that, as was the case with linear interpolation, b1 still represents the slope of the line connecting points *x0* and *x1*. Thus, the first two terms of Eq. (2.27) are equivalent to linear interpolation from *x0* to *x1*, as specified previously in Eq. (2.26). The last term, $b_2(x - x_0)(x - x_1)$, introduces the second-order curvature into the formula.

2.9.3 Newton divided difference Interpolation

Suppose that $f_n(x)$ is the *nth* order polynomial that is consistent with the function f, at different values, $x_0, x_1, ..., x_n$.

The divided difference of f with respect to $x_0, x_1, ...x_n$is used to represent $f_n(x)$ in the form.

$$f_n(x) = b_0 + b_1(x - x0) + b_2(x - x0)(x - x1) + ... + b_n(x - x0)(x - x1)(x - xn - 1) \tag{2.35}$$

where the constants:

$$b_0 = f(x_0)$$
$$b_1 = f[x_1, x_0]$$
$$b_2 = f[x_2, x_1, x_0]...$$

$$b_n = f[x_n, x_{n-1}...x_1, x_0]$$

The above expression are finite divided difference are computed repeatedly. The following expression represent the *1st,2nd* and *nth* finite divided difference respectively.

$$f[x_i, x_j]\frac{f(x_i) - f(x_j)}{x_i - x_j} \tag{2.36}$$

$$f[x_i, x_j x_k] = \frac{f[x_i, x_j] - f[x_j, x_k]}{x_i - x_k} \tag{2.37}$$

$$f[x_n, x_{n-1}...x_1, x_0] = \frac{f[x_n, x_{n-1}, ..., x_1] - f[x_{n-1}, ...x_1, x_0]}{x_n - x_0} \tag{2.38}$$

The nth order Newton divided difference interpolation polynomial is given as.

$$f_n(x) = f(x_0)+(x-x_0)f[x_1,x_0]+(x-x_0)(x-x_1)f[x_2,x_1,x_0]+...+(x-x_0)(x-x_1)...(x-x_{n-1})f[x_n,x_{n-1},... \tag{2.39}$$

2.10 Polynomial regression approximation

Polynomial regression is a regression technique which is used to model the relationship between a dependent variable (denoted by y) and an independent variable (denoted by x) to a polynomial over variable x in degree n. A polynomial regression equation of degree n can be represented using the following equation.

$y_i \;=\; \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \cdots + \beta_m x_i^m + \varepsilon_i \ (i = 1, 2, \dots, n)$

The goal here is to computes all the coefficients β in this equation, and use it to approximate an intermediate point.

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^m \\ 1 & x_2 & x_2^2 & \dots & x_2^m \\ 1 & x_3 & x_3^2 & \dots & x_3^m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^m \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \vdots \\ \varepsilon_n \end{bmatrix},$$

$\vec{y} = \mathbf{X}\vec{\beta} + \vec{\varepsilon}.$

$\widehat{\vec{\beta}} = (\mathbf{X}^\mathsf{T}\mathbf{X})^{-1}\,\mathbf{X}^\mathsf{T}\vec{y},$

3 Image and Image Matrix

In the field of imaging and image processing systems, it is always suitable and most importantly ideal to categorized the image that is going to be processed mathematically. This categories are of two main point. The two main mathematical categories are of key values, that is, deterministic categories and statistical categories. For the deterministic categories, image representation is done by , a consideration of the mathematical image function and properties of a particular points of the image is considered.

On the other hand, the statistical image categorisation, the image is specified on the basis of average properties of the image inconsideration.The below sections discussed the deterministic and statistical categories of a continuous images. "Although the analysis is presented in the context of visual images, many of the results can be extended to general two-dimensional time-varying signals and fields"[4].

3.1 Image Sampling and Image Quantization

There are many different ways to obtain images, but the purpose remains the same, to create a digital image from an available identified data. The output of many sensors is usually a continuous voltage waveform whose amplitude and spatial characteristic are related to the physical phenomenon that is being identified. In-order to acquire a digital image, it is required to convert the continuous identified data into digital format. There are two procedures, involves in this, that is the "Sampling and Quantization".[2] The main Idea about is, given a continuous image, the idea is, we wish to transform it to a digital form, and this is term "sampling and quantization"[2].

An Image may be continuous with respect to the coordinates x and y, also taking in to consideration of the amplitude, In-order to sampled the function in coordinates and in amplitude, there is need to convert it to a digital form.

"Digitizing the coordinates values is called sampling. Digitizing the amplitude values is called Quantization"[2].

3.2 Image Representation

Let (x, y) be the spatial coordinate that represents the spatial energy distribution of an image source of radiant energy, and at a wavelength λ and at a time t.It is known that, the intensity of light is a real positive quantity, and that is to say, the intensity is directly proportional to the square of the modulus of the electric field, hence the image light function is always real and non-negative.

Moreover, some quantity of light in the background is always present in many practical imaging systems, there are some physical Imaging system that causes reduction on the maximum intensity of the image, e.g file saturation the phosphor heating in Cathode ray tube [4],[2]. Sampling and Quantization techniques are used to convert the image function to a digital image, if we take into consideration, and sample the continuous function image into a 2D array, containing $M - rows$ and $N - columns$, and(x, y) be the discrete coordinates.

For the purpose of convenient, we use integer values for this discrete coordinates: $x = 0, 1, 2, ..., M - 1 andy = 0, 1, 2..., N - 1$. The $f(0, 0)$, $f(0, 1)$ are the values of the digital image at the origin and at the next coordinate value along the first row.

The notations $(0,1)$ specify the second sample along the first row, but this does not ensures that these are actual values of the physical coordinates during in which the image was sampled. In general, the coordinate (x, y) is used to represent the values of an image function, $f(x, y)$ at any coordinate, where x and y are integer numbers. The spatial domain where x and y are known to be the spatial variables and spatial coordinate, of the real plane which is spanned by the coordinate of the image[2]. Therefore we can consider that[4].

$$0 < E(x, y, t, \lambda) \leq \infty \tag{3.1}$$

The maximum image intensity of a physical image is limited with respect to the imaging system and the image recording parameters. For mathematical point of view, all images are considered to be non-zero over a rectangular region, for which we have[4].

$$-J_x \leq x \leq J_x \tag{3.2}$$

$$-J_y \leq y \leq J_y \tag{3.3}$$

The observation of the physical image is only for a small amount of time interval, hence we have.

$$-T \leq t \leq T \tag{3.4}$$

"The image light function *E(x,y,t)* is, therefore, a bounded four-dimensional function with bounded independent variables"[4]. For the final restriction, we considered that the image function is continuous over its domain of definition. The intensity perceived by a standard human observer to the image light function is usually measured in terms of the instantaneous luminance of the light field, as given below [4].

$$Y(x, y, t) = \int_0^\infty E(x, y, t, \lambda)V(\lambda)d\lambda \tag{3.5}$$

where $V(\lambda)$ represents the relative luminous efficiency function, that is, the spectral response of human vision.

Similarly, the tristimulus values is a set of values of a response of a colour of a standard observer that is linearly proportional to the amounts of red, green and blue light required to match a coloured light, it is also use to profile and calibrate output devices. For any arbitrary red–green–blue coordinate system, the instantaneous tristimulus values are given as[4].

$$R(x, y, t) = \int_0^\infty E(x, y, t, \lambda)R_S(\lambda)d\lambda \tag{3.6}$$

$$G(x, y, t) = \int_0^\infty E(x, y, t, \lambda)G_S(\lambda)d\lambda \tag{3.7}$$

$$B(x, y, t) = \int_0^\infty E(x, y, t, \lambda)B_S(\lambda)d\lambda \tag{3.8}$$

Where $R_S\lambda, G_S\lambda, B_S\lambda$ are spectral tristimulus values for the set of red, green and blue primaries. The spectral tristimulus values are, in effect, the tristimulus values required to match a unit amount of narrowband light at wavelength λ In a multispectral imaging system, the image field observed is modelled as a spectrally weighted integral of the image light function. The *ith* spectral image field is then given as[4].

$$F_i(x, y, t) = \int_0^{\infty} E(x, y, t, \lambda) S_i(\lambda) d\lambda \tag{3.9}$$

where $S_i\lambda$ is the spectral response of the *ith* sensor. For a representational simplicity, a single image function *F(x,y,t)* is selected to be an image field in a physical imaging system.

For a monochrome imaging system, the image function $F(x, y, t)$ usually denotes the image luminance, or few converted or corrupted physical representation of the luminance, whereas in a colour imaging system, *F(x,y,t)* indicates one of the tristimulus values, or other function of the tristimulus value. The image function $F(x, y, t)$ is also used to represent a general three-dimensional fields, for example, the time-depending noise of an image scanner. With regards to the standard definition for one-dimensional time signals,the time average of an image function at a given coordinate (x, y) is defined as[4].

$$\langle F(x, y, t)\rangle_T = \lim_{T\to\infty} \left[\frac{1}{2T} \int_{-T}^{T} f(x, y, t) J(t) dt \right] \tag{3.10}$$

where $L(t)$ is a time-weighting function, and the spatial average is use to denote the average image brightness for a given time, t [4].

$$\langle F(x, y, t)\rangle_T = \lim_{J_x\to\infty J_y\to\infty} \left[\frac{1}{4J_xJ_y} \int_{-J_x}^{J_x} \int_{-J_y}^{J_y} f(x, y, t) dx dy \right] \tag{3.11}$$

Image projection devices as applies to imaging systems that does a constant with the increase in time, and the time variable could be dropped from the image function.For other types of systems, such as movie pictures, the image function is time sampled . It is also possible to convert the spatial variation into time variation, as in television, using an image scanning process. In the next discussion, the time variable is dropped from the image field notation unless precisely required[4].

From the above paragraph is is known that an image can be represented in a most useful way. Image displays permits us to view a result of a processed image. Numerical arrays are used for processing and algorithm development. we write the representation of an MxN of a numerical arrays as [2].

$$f(x, y) = \begin{bmatrix} f(0,0) & f(0,1) & ...\ f(0, N-1) \\ f(1,0) & f(1,1) & ...\ f(1, N-1) \\ f(M-1,0) & f(M-1,1) & ...\ f(M-1, N-1) \end{bmatrix} \tag{3.12}$$

Both the left and the right sides of the above equation are Identical way of representing a digital image quantitatively. the right side is a matrix of real elements. Each of this matrix is called an $Imageelement$,*pictureelements*, *pixelsorpel*. the term image and pixel are used throughout this study to denote digital image and its elements [2].

$$A = \begin{bmatrix} a_{0,0} & a_{0,1} & a_{0,N-1} \\ a_{1,0} & a_{1,1} & a_{1,N-1} \\ a_{M-1,0} & a_{M-1,1} & a_{M-1,N-1} \end{bmatrix} \tag{3.13}$$

clearly, $a_{i,j} = f(x = i, y = j) = f(i, j)$

3.3 Relationship Between Pixels

There are many important relationship between pixels in a digital image, as we indicated before, an image function is denoted by $f(x, y)$. When we are referring to a particular pixel, we normally use a lowercase letters for example p and q [2].

3.3.1 Neighbors of a Pixel

A pixel p at a coordinate (x, y) has four horizontal and vertical Neighbors whose coordinate are given as[2].

$$(x+1, y), (x-1, y), (x, y+1), (x, y-1) \tag{3.14}$$

The set of pixels, called the 4-Neighbors of p, denoted by $N_4(p)$. Each pixels is a unit distance from (x, y) and some of the Neighbors location of p lies outside the digital image if (x, y) is on the border of the image[2].

the four diagonal Neighbors of p have coordinates given as.

$$(x+1, y+1), (x+1, y-1), (x-1, y+1), (x-1, y-1) \tag{3.15}$$

These above points and together with the 4-Neighbors are called the 8-neighbors of p, which is denoted by $N_8(p)$. As indicated already, of the Neighbors location in $N_D(p)$ and $N_8(p)$ falls outside the image when (x, y) is on the edge of the image[2].

3.4 Colour Models

The importance of colour model (at times called colour space or colour system) is to enable the specification of colour in a standard, or in general accepted way. in essence, a colour model is a specification of a coordinate system and a subspace within that system where each colour is represented by a single point[2].
Many of the colour models use today are focused either on hardware (for example a colour monitors and printers) or on application where colour manipulation is a of an interest (for example in the creation of colour graphics for animation).

Considering a digital image processing, the hardware oriented models most commonly used in practice are the RGB (red, green, blue) model for colour monitors and a broad class of colour video cameras. The CMY (cyan, magenta, yellow) and CMYK (cyan, magenta, yellow, black) model for colour printing, and the HSI (hue, saturation,intensity) model, which corresponds closely with the way human describe and interprets colour. The HSI model model also has an important features that its decouples the colour and Gray-scale mechanism developed. there are various colour models use today as a result to the fact that colour science is a large field that covers many areas of application it is important to venture on to some of these models here simply because they are interesting and revealing, especially in image processing[2].

3.4.1 The RGB Colour Model

In the RGB model, each colour appears in its primary spectral component of red, green and blue. This model is based on the Cartesian coordinate system. The colour subspace of consideration, In which RGB Primary values are at three corners , the secondary colours cyan, magenta and yellow are on three other corners,and the black is at the origin, and

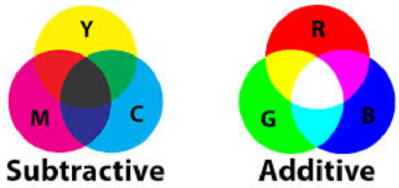


Figure 3: A picture of an subtraction and additive colour space respectively [13].

white is at the corner further away from the origin. In his model, the gray-scale(points of equal RGB values) extends from black to white on the line joining these two points. The different colours in this model are point on or inside the cube and are defined by vectors extending from the origin. For convenience , the assumption is that all colours values have been normalized so that the cube is a unit cube. That is all values of RGB are considered to be on the range [0,1] [2].

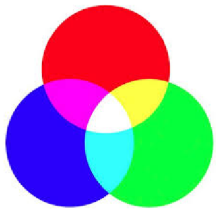
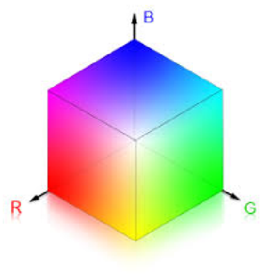


Figure 4: A a pictures of RGB colour space [13]

The pixel depth is the number of bit used to represent each pixel in an RGB space. If we take into consideration an RGB image in which each of red, green, and blue image is an 8-bit image, for these condition each of the RGB colour pixel(that is a triplet of values RGB) is known to have a depth of 24-bit (that is 3 image planes multiply with the number of bits per each plane). The term full-colour image is normally use to represent a 24-bit RGB colour image. The maximum number of colours in a 24-bit RGB is $((2^8)^3)$ which is equals to, 16777216 [2].



.

Figure 5: A a pictures of the combination [13]

3.4.2 The CMY and the CMYK colour model

As we discussed earlier, the cyan, magenta, and yellow are the secondary colour of light or on the other hand, the primary colours of pigment for example, when a surface is coated with cyan pigment it is usually illuminated with white light, and there is no red light that is reflected from that surface[2]. These is to say that, cyan subtracts red light from the reflected white light, which it self is a collection of equal portion of the red, the green and the blue light[2].

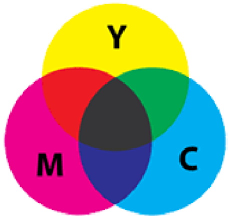


Figure 6: A picture of CYM colour model [13]

Most devices that deposit coloured pigments on paper, for example, a colour "printer and copiers'[2]', this enables the CMY data input to, or to help perform an RGB to CYM conversion on the inside. This conversion is done with the help of a simple operation[2].

$$\begin{bmatrix} C \\ M \\ Y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} R \\ G \\ B \end{bmatrix}$$

We are still with the notion that all colour values have been normalized to the range [0,1].This conversion shows that the reflected light that is from the surface coated with pure cyan does not contain red (which is given by the equation C= 1-R). In the same way, the pure magenta does not reflects green, and the pure yellow, does not reflects blue. It does proves that the RGB value can be obtain easily from a set of CMY values by subtracting the individual CMY values by one (1). As we mentioned earlier, in the field of image processing, this colour model is used in-relation to and obtaining a hand copy output, therefore the inverse operation, that conversion from CMY to RGB is normally of very small practical interest [2].

An Equal portion of the primary pigment, cyan, magenta and yellow when combined should produce a Black colour. In practice, combining those three colours for printing produces "muddy-looking black"[2], a techniques has been use in-order to obtain a true black (which is the predominant colour in printing), there has been need to a fourth colour 'black' that is always added, and this give rise to a next colour component k, and the model becomes, the CMYK colour model[2].

3.4.3 The HSI Color Model

We have learned, making colour in the RGB and the CMY model and converting them from one model to the other is a straightforward process. As we also indicated earlier, these colours system are normally suitable for hardware implementation. Additionally, the RGB system matches appropriately considering the fact that, the human eye is strongly preceptor of red, green and blue primaries.

It is unfortunate that, the RGB, the CMY and other identical colour models are not suitable for explaining colour in a way that are more practical for human interpretation.

Considering the example, we does not refer to the colour of an auto-mobile because of the percentage of each of the primaries constituting its colour. Moreover, we do not think of a colour image as being composed of three primaries images that mixed to form the single image, but rather, when we Human looks at a coloured object, we always described

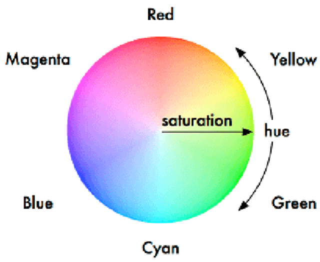


Figure 7: A picture of HSI colour model 13].

it by its brightness, hue, saturation. The hue is a colour attributes that describes a pure colour (pure violet, orange, or blue) on the other hand, Saturation , gives the measure of the degree to which pure colour is diluted or concentrated by white light. Brightness is subjective descriptor that is not physically or experimentally impossible to quantify. It symbolises the achromatic idea of intensity and is one of the main factors in describing colour perception or sensation.

It is know that the brightness or intensity (gra-level) is a most useful descriptor of monochromatic image (one coloured-image) . it is evident that, this quantity is measurable and easily interpretable. The next model we shall present and discuss is called the HSI model (hue, saturation, intensity) this type of a colour model, decouples the intensity component from the colour carrying information (hue, saturation, intensity) in a colour image. As a result, the HSI model is an important tool for the development of image intuitive to humans, who are the developers and end users of this algorithms (when an images captured by a colour camera or when an image is displayed on a monitor screen), its use for colour description but its much limited[2].

3.5 Image compression formats

Images compression and image format are very important tools in image processing, at times there need for an Image to be compressed, and this is more or less depending on a particular purpose or ideally for another application. There are many different ways to perform this image compression algorithms, some are done in a lossless and keep the same information as the original image, and others in loss information during the compressing the image.

most of this these compression methods are set-up for a particular type of images, and so they does not do well for other types of images. Many of these algorithms could even allow you to change the parameters that are used to adjust the compression appropriately to the image in-hand[10].

BMP format: One of the most basic image format is the, Bitmap graphics format (BMP). Windows BMP is an image format operating system "Microsoft windows graphics" (GDI). it is compatible with images of 1,4 8,16, 32 bits per pixels. However, 16 and 32 bit per pixel is very uncommon, as it always uses basic graphics file format and uncompressed format. BMP is normally used with "large blockers" of the same colours, that is why it it is less applicable in compresses format. In-terms of storage data, the BMP format uses complete bytes, hence the ordering of the bit string is not a problem [10],[11].

PNG format:Portable Network Graphics (PNG) is one of the newest graphics format use for image compression, hence it is just beginning to receive popularity on the internet, it also compatible with all image viewing application frequently used [11]. It is a bitmap image format that is build on lossless data compression format[10]. It came in to existence in-order to replace the GIF format additionally with a file format that do not need adjustment during replacement.

There are some applications that does not requires JPEG, PNG is use as an alternative. Also, since it is a lossless data compression for 24-bit images, for an intermediate format for images that are repeated modified, PNG is better use than JPEG.[11] It is also very good for storing images with big field of a unique colour or with small difference of the colour [10]

The storage order of the PNG formats stores multi-byte integers starting with the most important byte first, and the bit string are read from the least, then followed by most important bit. When a bit crosses a byte boundary, the bits that is in the second byte are more important[11]. PNG is of an advantage than the JPEG for storing images that contain text, line art, or other images with sharp transitions that do not transform well into the frequency domain [10].

JPEG format:The Joint Photographic Experts Group (JPEG) has been the most actively used format for storing photographic images, but with widely use, the working principle of JPEG for compression remains un-clear, because instead of defining the image file format, it defines the number of related image compression techniques[11]. It has a lossy compression algorithm, and this algorithm used to compressed images with 24 bits depth or a greyscale images. The algorithm is very flexible, this feature made it very useful for image compression, since the compression rate could be adjusted if needed [10]. However, if we compress a lot on the image, information would likely be lost, in the same way, the resulting image size could also be smaller. therefore, With a smaller compression rate we obtain a better quality of the resulting image, and the size would be bigger. This compression techniques entails in making the coefficients of the quantization matrix bigger if we required to do more compression, and smaller if we want to less compression [10]. The advantages of the JPEG format is that for photographic images, it gives a better compression than of any Bitmap format of the same use. An image that needs 1 MB to be stored in a Windows BMP file can normally be compressed to 50KB by JPEG format[11]. JPEG is computational costly, despite of that, it is still used because of its outstanding ability of compression[11], making it the most used image compression format for storing and transmitting images on the Internet[10].

TIFF format:The Tagged Image File Format is a file format that is also use for storing images, this involves photographs and line art. "It is one of the most popular and flexible of the current public domain raster file formats"[10].

It was initially made by the company "Aldus, jointly with Microsoft, for use with PostScript printing", TIFF is a very famous file format use for images with high colour of depth, besides JPEG and PNG. TIFF format has been widely applicable with "image-manipulation applications, and by scanning, faxing, word processing, optical character recognition, and other applications"[10].

3.6 Image calibration

The defect caused by the non-uniform illumination of the observed sample,this causes change in the intensity of the optical system, and this term is said to be vignetting of the optical system this could also cause impulse noise.

There is a term in which this defects can be reduced, and that is called Calibration, thus improves the quality of the image[14].

3.6.1 Basic type of Image

There are two parameters involves in describing an image, that is. **Temperature**: that is the temperature at which an image is obtained.
Time: also called exposure time, this is the time duration at which the image was captured.
Light Frame: This is the image that is suppose to be calibrated, it is denoted as $L(T, t)$, it depends on temperature and time.
Dark Frame: This is the image with temperature and time parameters, but taken in a complete darkness, with no light on the image. it is denoted as$D(T, t)$
Bias-Frame: This is a form of a Dark-Frame image that was captured with zero-exposure time, $D(T, 0)$. This is however not possible in real life situation. The Bias-Frame is denoted as$B(T)$

$$B(T) = \lim_{t \to 0} D(T, t)$$

Flat-Field: This is the image with absolute homogeneous gray surface, that is there is no saturation of Pixels. It is denoted as$F(T, t)$[14].

3.6.2 Calibration method

Here we discuss how best do we reduce the above mention problems.

Dark-frame subtraction Dark-frame is needed to be form immediately after the light-image, so that the same temperature is retain.
The calibration of this involves:

$$A = L(T, t) - B(T)$$

A calibration fora longer exposure time involves:

$$A = L(T, t) - D(T, t)$$

Flat-field correction. This method involve the correction of the irregular image-light, the different sensitivity of photodiodes, it also reduces vignetting, that is the light lost on the optical system due to increase in distance of the optical axis.

$$A = \frac{L(T_1, t_1) - D(T_1, t_1)}{F(T_2, t_2) - D_F(T_2, t_2)} c$$

where $D_F(T, t)$ is the Dark-frame of the flat-field, and c is a set of scaler values depending on the Dynamic range.

3.7 Digital Image Noise

There is no precise mathematical definition of Noise, it is a random signal or component in a digital image, it can come from various sources. Some are physical, linked to the nature of light and some to other optical properties,oe even during the processing of acquiring the image, the conversion between electrical signal to digital data by the AD-converter. Image Noise reduces the quality of an image. [12],[14]
image distortion is due to many different problems, there are many different types of noise, but we shall discuss few of them here[12],[14].

3.8 Multiplicative noise

This can be cause by different in the properties of the photodiode and the AD-converter in CCD, CMOS chips, since some photodiodes are more effective than the other. It could be remove by image calibration.

3.8.1 Mathematical model

Let A denotes an absolutely perfect and S is an image with pixels and a realization of some random variable X. The final image B, is the product of A and S, given by

$$B = A \cdot S$$

and we can say that, image B has a multiplicative noise, where the 'dot' is an operation between two function[14].

3.9 Additive noise

Additive noise is the most difficult in-terms of filtration. This noise is mainly due to dark-current, adding electrons to pixels that were not generated b photodiode. This can be reduced by cooling, however this cooling destroys the image[14].

3.9.1 Mathematical model

Let A denotes an absolutely perfect and S is an image with pixels and a realization of some random variable X. Where the property of this noise is the property of the random variable eg. mean, median,mode standard deviation, etc. If B is the sum of A and S, denoted as $B = A + S$, we can say image B contain an additive noise. This noise is cause mainly due to Dark-current, adding electrons to pixel which were not generated by photodiode. This can be reduce by cooling the chip, also by reducing the intensity of the light source[12],[14].

3.9.2 Additive noise filtration

The additive noise has two parameters N(μ, σ^2), Gaussian μ is the mean value, and σ is the standard deviation. The μ value is the values of the Dark current and σ is the amount of additive noise[14].

Low-pass filter: This filter that passes signal with a frequency lower than a selected cut-off frequency and attenuates signal with frequencies higher than the cut-off frequency.

Let B be an image which low-pass filter was applied. The most common low-pass filter is the Gaussian function.

$$B = \mathcal{F}^{-1}(\mathcal{F}(A) \cdot \omega)$$

where ω is the Gaussian kernel, $\mathcal{F}$ is the Fourier transformation[14]

3.10 Impulse noise

This is a type of noise that is also call "data drop noise"because its drop the original data statistically, however the image is not fully corrupted instead there are pixel values that are affected in the image. it could be cause dust particle, cosmic rays, during the transmission of the image, or even change in the certain property of the optical axis[12],[14].

3.10.1 Mathematical model

Let A denote an absolutely perfect image, and X is a random variable with alternating distribution(0,1). $X\ A(P)$ where P is the probability of random variable is 1. and Y is another random variable with a distribution, but often a uniform distribution, and x, y denotes a coordinate of pixel. image B is the final image, with impulse noise given by.

3.10.2 Impulse noise filtering

The filtering of impulse noise involves two steps: **defective**: The aim is to fine which pixel is defective. **correction**: This involve a process where defective pixel are replace by fixed value obtained from th neighbourhood.

4 Image Restoration

The principal goal of image restoration techniques is to improve an image in some predefined sense[2]. Image restoration is an objective process, it attempts to recover an image that has been degraded or defective.

4.1 Laboratory Image

This is one of the images that were obtained in an experiment in a laboratory, the Automotive headlamp was made to fogged while a camera was projected taking pictures at defined intervals, while the headlamps in undergoing defogging. After the process, we obtained pictures of the whole set-up. As a result there were some defective pixels on the images. There could be different reasons that would accounts for this, reflection of the surface of the Automotive headlamp, different in temperature or pressure gradient on the inside of the Automotive headlamp, the nature of the Camera used, light source, or even human error. It is important to know the the relations between each of these mentioned and the defective pixels in the entire image, but for this study we would rather consider how to reconstruct the defective pixel values and to fine an appropriate numerical tool to reconstruct missing (defective pixels) information.

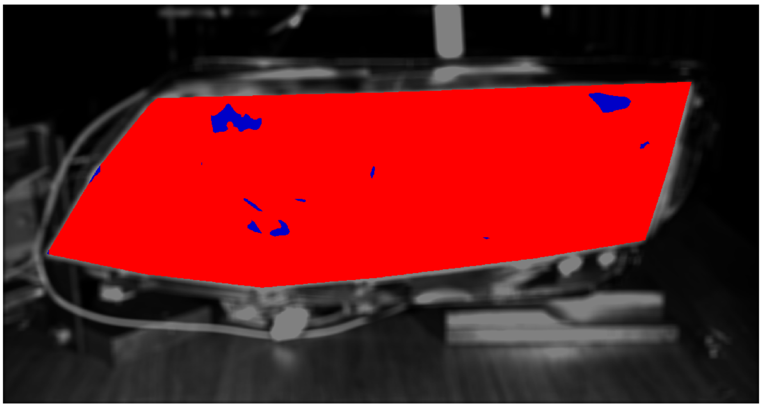


Figure 8: A picture of a laboratory image with defective pixels

4.2 Goal of the study

Part of the goal of this study, is to follow some underlaid guidelines in-order to determine the most appropriate method for image date reconstruction. We highlighted and described different numerical tools used everyday in image processing to calculate missing data.

Firstly, We created test-data that we applied the numerical tools on, some information was erased on the test-data, and texted the already identified numerical tools on the, upon knowing the best working tool, and with respect to the type of problem we have in-hand, we would know which tools to use finally for the image restoration. We used the numerical tools on the laboratory images to reconstruct missing image information.

4.3 Created testing data

We created two different data-set on MATLAB-R2018a, on which we employed and tested numerical methods. These data-set were 2-dimensional of both linear and quadratic equations, that is $z1 = x + y$ and $z2 = x^2 + y^2$, for each of the data-set, we had a dimension corresponding to $[x, y]$, of 201x201 data-points. This was chosen, as part of the procedures t be carried out in this study. We knew that we would have to load images at some point in MATLAB to obtain image-matrix that we would finally do analysis on. In that light, we had to do the first procedures as part of the goals of this study.

On this data set, we applied all studied numerical tools, that is Polynomial interpolation, Gaussian convolution, newton interpolation(divided difference) , Lagrange interpolation, polynomial regression.

We erased some known data point, and we used each of the this and approximated the erased data point, and recorded for each of the separated data-set, that the linear and quadratic data-points. It was shown that, for the linear data-set, linear interpolation gives better results in computing the missing data-point. which we would expect normally. it was seen that, the linear interpolation is the same as taking the average between 2 points, or simply fitting a straight line between two point, and guess that mid-point between them.

Gaussian convolution also gave very good approximation for a smaller kernel, say 3x3, and for this kernel, we obtained the same results for Gaussian and linear interpolation. On the other hand, this is not the same for a Gaussian of larger convolution kernel. We also investigated the effect by changing the kernel size and by varying the standard deviation (sigma). we read and recorded the effect.

For the quadratic data-set, both Gaussian and linear interpolation were compared, but it has a small had deviation, however the Gaussian convolution was better in the approximation of data points, also which goes well facts, when applying a linear interpolation on a quadratic, there is always less accuracy, as the line of best-fit would not pass through all points, as a result, it would have a deviation from the actual points points, as a result, we only used the Gaussian convolution on the quadratic data-set, we also compared with varying the kernel size with and the standard deviation.

4.4 Reconstruction on the image matrix data-set

At this point, we are able to proceed to the next procedures, after testing all the numerical methods on the created data set. we loaded all 113 images obtained from fluid flow laboratory, on MATLAB-R2018a, and first we split them in to RGB-channels, with the dimension of 2064x1082, where R-channel indicated totally fogged region, G-channel indicated totally defogged region, and B-channel indicated defective(missing pixel values) region. The R-channel and G-channel profile maps, has the same area, and the B-channel map remains the same.

We combined all these with a single Image-matrix, where all the three features appears, that is RGB, this matrix has different pixel values, depending on when and where something changed from red to green, that is from fogged to defogged, and the missing pixel valued. A large portion of this matrix is know, the "defogged" areas, hence the missing areas are smaller compare to whole of the image, this is crucial, as it will help in the comparing and conclusion of the best method for the final restoration of the image.

The R-channel and G-channel indicated known data, we applied this numerical tools, and here we added and extra tool, that is the polynomial regression, to substitute, the linear interpolation, since this data-set was generally not linear on some region, so a better too was to use Gaussian convolution and polynomial regression.

4.4.1 Gaussian convolution approximation

The Gaussian convolution kernel of 3x3, and with Matrix p of 3-Missing pixel Values is tabled below. where h is the convolution kernel and p is the matrix with missing pixel values.

$$h = \begin{bmatrix} 0.0944 & 0.1556 & 0.0944 \\ 0.1556 & 0 & 0.1556 \\ 0.0944 & 0.1556 & 0.0944 \end{bmatrix}$$

$$P = \begin{bmatrix} 12 & 11 & 11 \\ 0 & 0 & 12 \\ 12 & 0 & 0 \end{bmatrix}$$

The table shows the approximation of using Gaussian convolution kernel of 3x3 with 3-missing points with varying standard-deviation(sigma) for the reconstruction of known but erased to compare with the actual values.

Table section 4.7.2 Actual and approximated values using Gaussian convolution.

Actual Value	sigma	Approx. Value
12	1	6.8826
12	2	7.1564
12	4	7.2266

4.4.2 Gaussian convolution approximation

Approximation of Gaussian convolution kernel of 7x7 and 9x9 respectively, in this case we erases 8-points from the matrix, and that simulates 8-Missing Pixel values, with varying the standard-deviation (sigma), we were able to reconstruct missing values, the results is tabled below.

Table section 4.7.2 Actual and approximated values of 7x7 convolution kernel.

Actual Value	sigma	Approx. Value
12	1	9.9585
12	2	10.1768
12	4	10.3334

Table section 4.7.2 Actual and approximated values of 9x9 convolution kernel.

Actual Value	sigma	Approx. Value
12	1	10.243
12	2	11.0662
12	4	11.7954

4.5 Polynomial Regression

The below figure shows the same approximation using regression polynomial of 7th order, knowing that the same missing data-point was used in the Gaussian convolution, so we need to compare the two. This was to fit a curve through each of the date-points, in doing so, the appropriated point is visible on the figure.

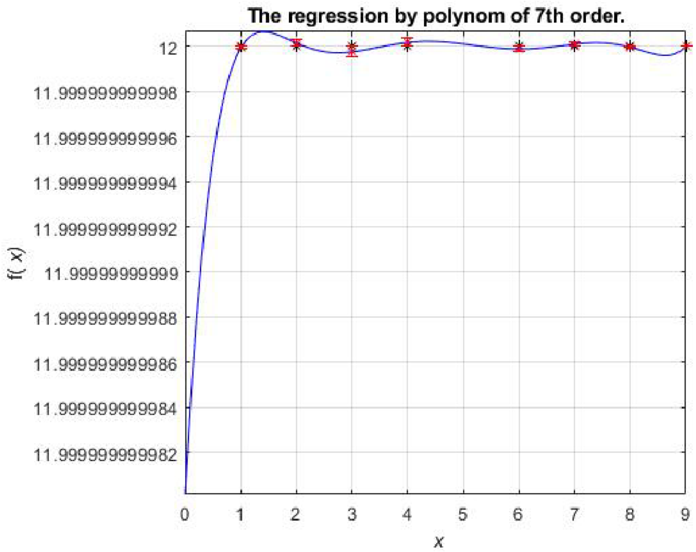


Figure 9: A figure of a regression polynomial fit

4.6 Some Discussion

We have used a Gaussian convolutions kernel of different size on known data, to estimate missing pixel value. It was evident that…

- For a particular Gaussian convolution kernel, the more the number of missing pixel values the higher the deviation from the actual pixel value.
- For a particular Gaussian convolution Kernel, the higher the Standard Deviation, the better the approximation.
- For a particular Standard deviation, the bigger the Gaussian convolution kernel the better the approximation.

The polynomial Regression approximated the missing pixel value, and it was recorded to be 12.00, which was exactly the same as the actual pixel value. However, this was done in only one(1) direction, to obtain the optimum value, we need to generalized to four(4) directions, then if we take the average of these four(4) values, we obtained 11.95, which is the approximated value obtained from regression polynomial.

4.7 Image Restoration process

The restoration of the image was carried out by the programs that has already been texted on the test-data. As we have shown before, we have applied both the Gaussian convolution and polynomial regression to reconstruct missing(erased) points during the testing of the numerical methods for a single missing point where the data was linearly distributed, that means defogging was in one direction. The data points were increasing in magnitude along a particular direction. We also tested on this methods on an area that, linearity does not happen, and that the data points were randomly displayed, and it was independent of direction. for a better choice of the decision of the reconstruction of the image, taking into consideration the type of data set and its distributing, and also the size of the missing area in-comparison with the known area. We came to the conclusion with the use of the Gaussian convolution in the reconstruction and restoration of the missing pixel values.

The image matrix is the matrix of dimension *2046x1048*, that combined all the information of all the 113 laboratory pictures, that is the known area and the unknown information(that is missing pixel values). We characterised earlier that the blue pixels represents the missing information, The red and Green pixels denotes the known information (area). It was seen that the missing information was much less that the know data points, that is to say that there were less defective pixels when compared to the size of the image as a whole. As result of that, we use the Gaussian for the final restoration of the image.

The figure below shows an image with both known and missing information.

Figure 10: A picture of an image with defective pixels

The program was executed on MATLAB R2018a, we scanned on the whole image, pixel-by-pixel to restore missing pixels, with specified parameters, a convolution kernel of 3x3 and 5x5,, a percentage criteria between 19 to 99 percent in-order to obtain higher precision of the restoration. This allows us to apply Gaussian, only when there are enough data point in the sub-matrix that corresponds to the convolution kernel matrix. If there are no enough points to do the convolution, we move to the next, and finally with iterative Gaussian, all pixels were refilled. With the above mentioned parameters, the whole image was finally restored after 123rd iteration, for the 3x3 kernel size, and 155th iteration for the 5x5 kernel size.

The below 3x3 matrix was the kernel matrix that was used to filter the image matrix.

$$h = \begin{bmatrix} 0.3679 & 0.6065 & 0.3679 \\ 0.6065 & 0 & 0.6065 \\ 0.3679 & 0.6065 & 0.3679 \end{bmatrix}$$

$$h = \begin{bmatrix} 0.0183 & 0.0821 & 0.1353 & 0.0821 & 0.0183 \\ 0.0821 & 0.3679 & 0.6065 & 0.3679 & 0.0821 \\ 0.1353 & 0.6065 & 0 & 0.6065 & 0.1353 \\ 0.0821 & 0.3679 & 0.6065 & 0.3679 & 0.0821 \\ 0.0183 & 0.0821 & 0.1353 & 0.0821 & 0.0183 \end{bmatrix}$$

The figure below displays the restored image of.

Figure 11: A figure of the restored image

From the picture above, it is seen that all the missing pixels of the laboratory has been successfully restored with the iterative scheme of a Gaussian convolution. We made a visualization of the original and the restored image, we computed the difference OF the two.

As it is mentioned above, the original image matrix with dimension of 2046x1082, that contains missing pixel values. The contour plot depicts the nature of defogging of the Automotive headlamp. The figures below shows the contour plot of the original image, the restored image and the difference between the two respectively.

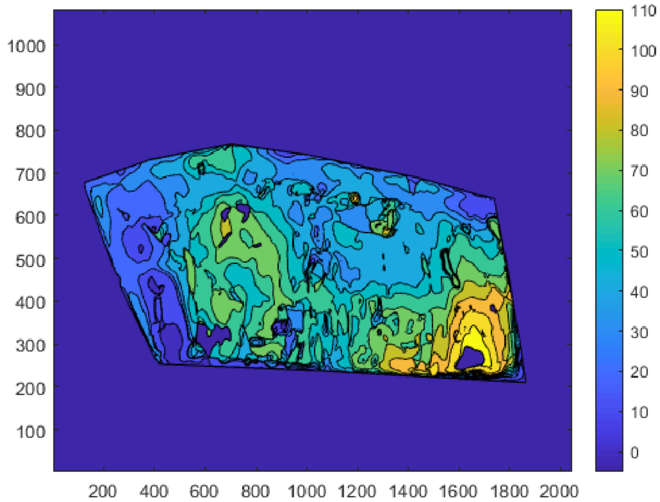


Figure 12: A figure of Original Image

We tried to compare the visualization of the restored image with the same dimension as the original, it is seen that the areas that had blue pixels on the original image, has been modified to yellow, following the restoration of the full image.

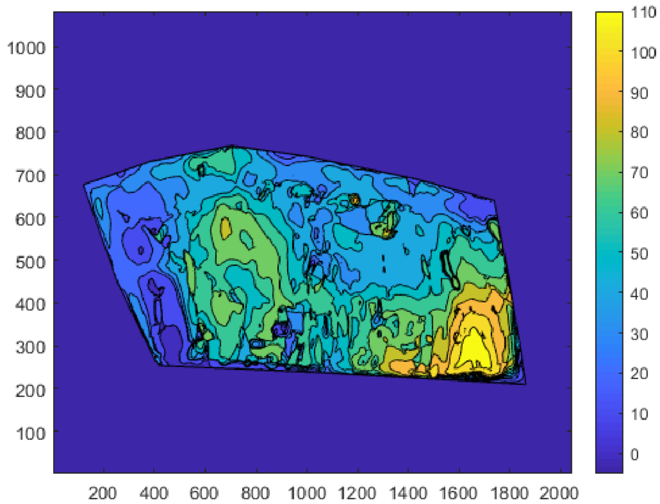


Figure 13: A figure of the restored image

In the next plot, we highlighted the difference between the the Restored image and the original image, of a 5x5 Gaussian convolution kernel.

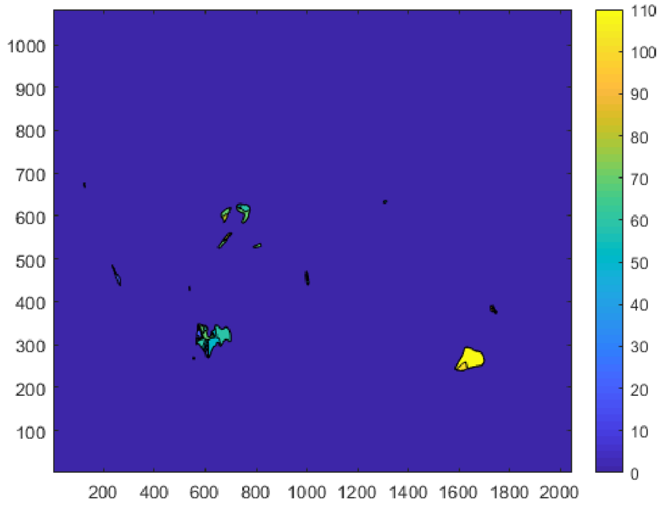


Figure 14: A figure of the error(missing pixel values)

4.7.1 Gaussian on a known data

Gaussian was used to reconstruct information and compared with a known data, that is we compare the original information and restored information.

The Graph below is the contour plot of the difference between the Actual value and restored information of an erased and restored area.

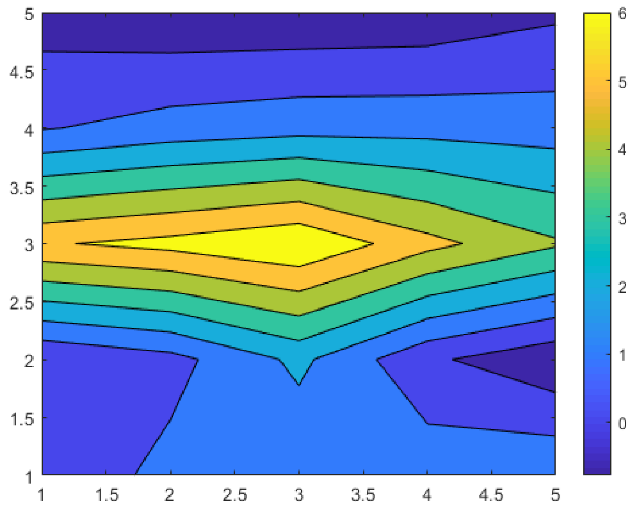


Figure 15: A figure of the restored data of a known region

4.7.2 Regression Polynomial

Aw we have describe in chapter 2, regression polynomial regression of 3rd degree was also use to reconstruct missing information. We erased a region of 5x5 on the Original matrix, and we reconstructed all erased point, we observed that, values around the surrounding were small than the the inside of the 5x5 reconstructed matrix, this was the same thing as for the Gaussian reconstruction. As we did for the Gaussian, we also computed the difference between the original values and the reconstructed values.

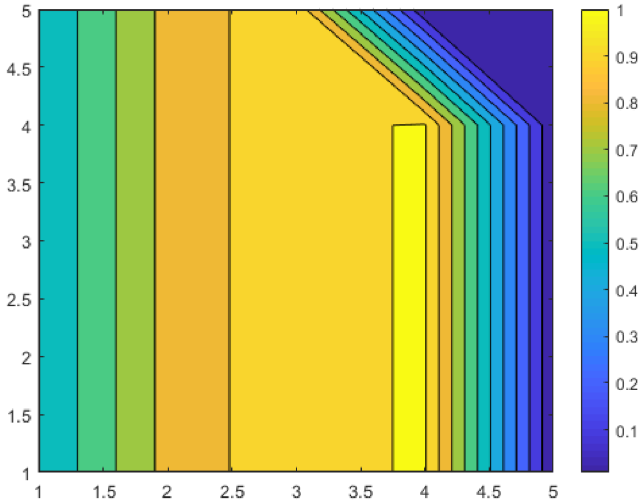


Figure 16: A figure of the difference between the approximated and original values of Polynomial regression

We shown below the difference between the original known value and the approximated using 3rd degree polynomial regression of a 5x5 matrix size. This data points were erased, and a reconstructed.

section 4.7.2 The Error associated with the actual value and the approximated, using Polynomial regression.

0.500	0.833	0.972	1.009	0.010
0.500	0.833	0.977	1.009	0.010
0.500	0.833	0.977	1.009	0.010
0.500	0.833	0.977	1.009	0.010
0.500	0.833	0.977	0.009	0.010

5 Conclusion

We have described some of the mathematical tools that was used for the reconstruction of missing image information, registration, representation of image, methods of image processing and visualization of the reconstruction.

The application was implemented on MATLAB R2018a environment for restoration of the missing information of a digital image. There were 113 PNG-file-format images obtained from fluid flow laboratory experiment that simulates defogging process of an Automotive headlamp. This defogging was independent of direction due to the due to the the behaviour inside, temperature, air pressure, and due to the presence of local light reflection, we could not decide defogging on this areas, and hence there were Missing information on some areas of the obtained digital image.

This PNG-file format of RGB colour scale images were all loaded into MATLAB R2018a environment and separated into RGB-colour channels of pixel values. Furthermore, we iterated on the all images, to depict the nature of defogging , that is from Red colour to Green colour, while the Blue was the region of missing pixels. The obtained profile map has the same area of the missing pixels, This is the area that we are suppose to restore from the surrounding known values.

This needed numerical algorithm , the iterative Gaussian convolution of 5x5 kernel size, was applied for the restoration, of missing pixel values. We set a high percentage criteria, such that there was enough data points on each step-size before the Gaussian is implemented.

Finally, we were able to compare different numerical tools for image restoration, the iterative Gaussian convolution was use for the image restoration of all the missing pixel values of the 113 images. However, there is a need for further studies to improve restoration of missing image information.

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Abbreviations and List of Symbols.

V,W	Vector Spaces
H	subspace
u,v	vectors
m,n	Matrix
(x,y)	Spatial coordinate
f(x,y)	continuous image function
I	Maximum image intensity
$E(x,y,t,\lambda)$	image function
Y(x,y,t)	instantaneous luminance
$V(\lambda)$	luminance efficiency
L(T,t)	Light-Frame
D(T,t)	Dark-Frame
B(T,t)	Bias-Frame
$N(\mu, \sigma)$	Distribution function
ω	Gaussian Kernel
p or q	pixel
RGB	Red-Blue-Green
HSI	Hue-Saturation-Intensity
CMYk	cyan-Magenta-Yellow-Black
BMP	Bitmap
PNG	Portable Network Graphics
JPEG	Joint Photographic Experts Group
TIFF	Tagged Image File Format
GDI	Graphics Device Interface
AD-converter	Analogue-to-Digital converter
CCD	Charged Coupled Device

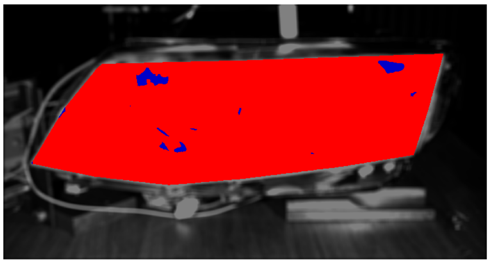
Appendix A

Enclosed CD

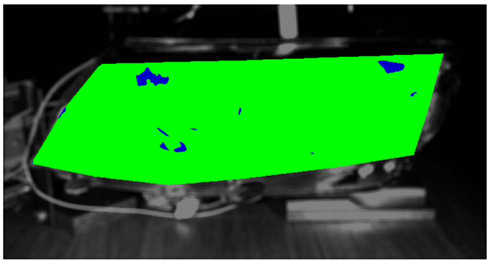
The CD contain the following files:

- Source code of loading of pictures into MATLAB to obtain image matrix with RGB components.
- Source code of restoration using Gaussian Convolution kernel
- Testing images obtained from fluid flow laboratory.
- Program of regression Polynomial.

Appendix B These are among the 113 images used for the study, there are missing information (the blue pixels) on the same area for all the images.

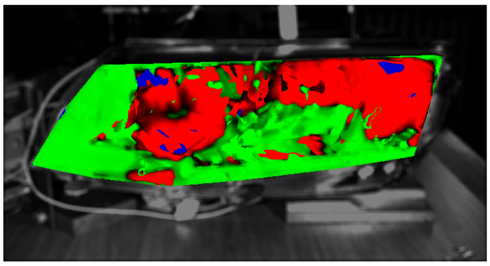


(a) Defogging of first image. .

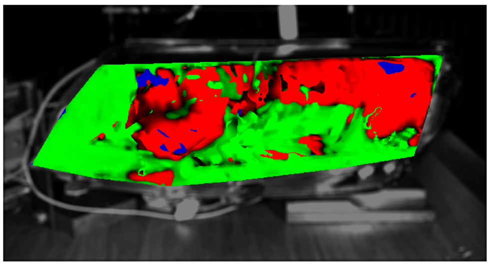


(b) Defogging of second image. .

Figure 17: Images of defogging (from red to green) of Automotive headlamp, all images were obtained from fluid flow laboratory, Faculty of Mechanical Engineering, BUT.



(a) Defogging of Fifty-sixth image. .



(b) Defogging of Fifty-seventh image. .

Figure 18: Images of defogging (from red to green) of Automotive headlamp, all images were obtained from fluid flow laboratory, Faculty of Mechanical Engineering, BUT.

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