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FUNCTIONAL ANALYSIS ON TIME SCALES AND ITS APPLICATIONS IN THE THEORY OF DYNAMIC EQUATIONS
FUNKCIONÁLNÍ ANALÝZA NA ČASOVÝCH ŠKÁLÁCH A JEJÍ APLIKACE V TEORII DYNAMICKÝCH ROVNIC

DIPLOMA THESIS
DIPLOMOVÁ PRÁCE

AUTHOR — Bc. JINDŘICH KOSÍK
AUTOR PRÁCE

SUPERVISOR — prof. Mgr. PAVEL ŘEHÁK, Ph.D.
VEDOUCÍ PRÁCE

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Institut:	Institute of Mathematics
Student:	**Bc. Jindřich Kosík**
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Recommended bibliography:

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Abstrakt
Cílem práce bylo shrnout základní výsledky kalkulu na časových škálách, zpracovat nástroje z funkcionální analýzy v kontextu časových škál a využít je při studiu kvalitativních vlastností řešení konkrétních nelineárních dynamických rovnic. Práce obsahuje detailně zpracovanou problematiku derivace a integrace na časových škálách s důrazem na integrál Lebesgueova typu. Detailně jsou rozebrány alternativy k řetězovému pravidlu z klasického kalkulu. Podrobně jsou studovány prostory funkcí na časových škálách, zejména pak prostor rd-spojitých funkcí na kompaktním intervalu a prostor ohraničených spojitých funkcí na nekompaktním intervalu. Zvláštní pozornost je kladena na klíčové vlastnosti prostorů jako jsou úplnost a relativní kompaktnost, které jsou doplněny detailními důkazy. Zavedené matematické prostředky jsou později využity při studiu kvalitativních vlastností konkrétních nelineárních dynamických rovnic.

Summary
The aim of the thesis was to summarize the basic results of calculus on time scales and elaborate in detail on the tools from functional analysis in the context of the time scales and to use them in the study of the qualitative properties of the solution of specific nonlinear dynamic equations. The thesis focuses in detail on the problem of derivation and integration on time scales with an emphasis on the Lebesgue-type integral. Alternatives to the chain rule from classical calculus are discussed in detail. Spaces of functions on time scales are analyzed in depth, especially the space of rd-continuous functions on a compact interval and the space of bounded continuous functions on a noncompact interval. Emphasis is placed on key properties of spaces such as completeness and relative compactness, which are complemented by detailed proofs. Introduced mathematical instruments are later used for a study of qualitative properties of concrete nonlinear dynamic equations.

Klíčová slova
dynamické rovnice, časová škála, prostory funkcí na časových škálách, integrace na časových škálách, rd-spojitá funkce, zobecněná exponenciální funkce, věty o pevných bodech

Keywords
dynamic equations, time scale, functional spaces on time scales, rd-continuous function, integration on time scales, generalized exponential function, fixed point theorems,

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Rozšířený abstrakt

V této práci jsou detailně zpracovány základy teorie časových škál s cílem vytvořit funkcionálně analytický aparát pro studium dynamických rovnic a ukázat jeho uplatnění na konkrétních nelineárních dynamických rovnicích.

Na začátku je představen koncept časové škály a kalkulu na časových škálách ($\mathbb{T}$), který je sjednocujícím přístupem (nejen) klasického diferenciálního a diferenčního kalkulu. Jsou zavedeny základní pojmy, jako je dopředný skok (σ), zpětný skok (ϱ) a zrnitost (μ), které slouží jako základ pro klasifikaci bodů na časových škálách.

Důkladně je zkoumána problematika derivace na časových škálách. Je zavedena takzvaná delta derivace (pro funkci f značeno f^Δ). Jednou z klíčových odlišností oproti tradiční derivaci z diferenciálního kalkulu je, že není obecně možné přímo aplikovat tradiční řetězové pravidlo. Z tohoto důvodu jsou prezentovány tři alternativy, jež slouží jako jeho náhrada. Kromě toho je také představena modifikace věty o střední hodnotě, aby byla použitelná v rámci kalkulu na časových škálách. Delta derivace je analyzována na různých časových škálách, zejména platí

$$f^\Delta = \begin{cases} f' & \text{pro } \mathbb{T} = \mathbb{R}, \\ \Delta f & \text{pro } \mathbb{T} = \mathbb{Z}. \end{cases}$$

Velká část práce je pak věnována problematice integrace na časových škálách. Jsou zavedeny tři různé druhy integrálů s důrazem na korektnost a detailní rozbor jejich vzájemné rozdílnosti a také jejich podobnosti s analogiemi z diferenciálního kalkulu. Nejprve je definován integrál Cauchyova typu na základě pojmů rd-spojité funkce a antiderivací. Následuje integrál Riemannova typu, který je v práci zkonstruován obdobně jako v případě tradičního diferenciálního kalkulu pomocí Darbouxových horních a dolních sum. Ani jeden z těchto integrálů nemá dostatečně širokou množinu integrovatelných funkcí a nenabízí analogie vět o limitním přechodu. S využitím Carathéodoryho přístupu je proto představena nejprve delta míra a následně pomocí jednoduchých funkcí a obecné teorie míry integrál Lebesgueova typu na časových škálách. Množina integrovatelných funkcí u tohoto integrálu je dostatečně velká a umožňuje zformulovat Leviho větu o monotónní konvergenci a Lebesgueovu větu o dominantní konvergenci. Rozbor integrace na časových škálách je doplněn o analogie stěžejních výsledků z tradičního diferenciálního kalkulu, jako je základní věta integrálního počtu, věta o substituci nebo integrace per partes. Integrace je studována na různých časových škálách, zejména platí pro $a < b$

$$\int_a^b f(t)\Delta t = \begin{cases} \int_a^b f(t)\mathrm{d}t & \text{pro } \mathbb{T} = \mathbb{R}, \\ \sum_{k=a}^{b-1} f(k) & \text{pro } \mathbb{T} = \mathbb{Z}. \end{cases}$$

Dále se práce věnuje prostorům funkcí na časových škálách. Pro analýzu dynamických rovnic hrají důležitou roli věty o pevných bodech, zejména pak věta Banachova a Schauderova. V předpokladech obou vět figuruje vlastnost úplnosti a pro Schauderovu větu je klíčový pojem relativní kompaktnosti. V kapitole jsou proto uvedené prostory doplněny o studium úplnosti a relativní kompaktnosti. Podrobně je studován prostor rd-spojitých funkcí na kompaktním intervalu se supremovou normou. Důkaz úplnosti je detailně zpracován na základě důkazu úplnosti prostoru spojitých funkcí z klasického kalkulu. Ve stručnosti jsou zmíněny také idey dalších možných přístupů k tomuto důkazu. Dále jsou

v práci zevrubně rozebrány prostor regresivních a pozitivně regresivních funkcí. U prostoru pozitivně regresivních rd-spojitých funkcí na kompaktním intervalu jsou zmíněny různé přístupy k důkazu úplnosti, zejména pak přístup založený na izometrické izomorfii s prostorem rd-spojitých funkcí. Dále se práce věnuje prostoru omezených spojitých funkcí na nekompaktním intervalu a Lebesgueovým delta prostorům. Pro zmíněné prostory jsou formulovány kritéria relativní kompaktnosti s důrazem především na prostor neohraničených spojitých funkcí na nekompaktním intervalu, který je klíčový pro analýzu zvolených rovnic v další části práce.

Stručně je zavedena zobecněná exponenciální funkce s využitím takzvané cylindrické transformace a pojmu regresivní funkce. Zmíněn je také možný přístup definice jako řešení počátečního problému

$$y^{\Delta} = p(t)y, \quad y(t_0) = 1,$$

kde p je regresivní funkce. Jsou uvedeny a rozpracovány příklady pro různé časové škály.

Zavedený aparát funkcionální analýzy je využit při analýze kvalitativních vlastností nelinerání dynamické rovnice druhého řádu

$$y^{\Delta\Delta} = p(t)g(y^{\sigma}). \tag{A}$$

Jsou vyjádřeny předpoklady pro funkce g a p. Uvažuje se, že p je kladná rd-spojitá funkce a g je funkce, která je spojitá a splňuje

$$xg(x) > 0$$

pro $x \neq 0$. Pro takto zformulované předpoklady studujeme existenci řešení, které má kladnou limitu a pro velká t je kladné a neklesající. V práci je s využitím Schauderovy věty o pevném bodě dokázano, že

$$\int_a^{\infty} \int_t^{\infty} p(s)\, \Delta s \Delta t < \infty \tag{B}$$

je nutnou a postačující podmínkou existence řešení s požadovanými vlastnostmi. Jsou rozebrány možné modifikace podmínky (B), které zaručí existenci řešení s požadovanými vlastnostmi nejenom pro velká t, ale na celé uvažované časové škále.

Dále je studován případ, kdy je k předpokladům pro funkci g přidán také předpoklad na lipschitzovskou spojitost. S využitím Banachovy věty o pevném bodě je dokázáno, že podmínka (B) zaručí vedle existence i jednoznačnost řešení s požadovanými vlastnostmi pro velká t.

Nakonec je analyzována obecnější rovnice

$$(r(t)y^{\Delta})^{\Delta} = p(t)g(y^{\sigma})$$

K předpokladům pro p a g z původně rozebraného problému jsou připojeny předpoklady pro r. Opět je studována existence kladného a neklesajícího řešení pro velká t s kladnou limitou ve vztahu ke splnění podmínky

$$\int_a^{\infty} \frac{1}{r(t)} \int_t^{\infty} p(s)\, \Delta s \Delta t < \infty. \tag{C}$$

Důkaz je tentokrát založen na využití věty o substituci. Původní časová škála je transformována na novou a studovaná rovnice na rovnici

$$u^{\widetilde{\Delta}\widetilde{\Delta}} = \widetilde{p}(t)g(y^{\widetilde{\sigma}})$$

a podmínka (C) na

$$\int_{\widetilde{a}}^{\infty} \int_{t}^{\infty} \widetilde{p}(s)\, \widetilde{\Delta} s \widetilde{\Delta} t < \infty. \tag{D}$$

Následně je ukázáno, že takto převedený problém je ekvivalentní problému (A), (B).

I hereby declare that I have written my Master's Thesis Functional Analysis on Time Scales and its Application in the Theory of Dynamic Equations under the supervision of prof. Mgr. Pavel Řehák, Ph.D. using literature listed in the bibliography section.

Bc. Jindřich Kosík

I would like to express my gratitude to my supervisor prof. Mgr. Pavel Řehák, PhD. for his willingness, professional guidance and valuable advice he provided. I would also like to thank my family and friends for their support during my studies.

Bc. Jindřich Kosík

Contents

1 Introduction

The theory of time scales aims to unify differential and difference calculus. It provides elegant means to describe differences between continuous and discrete case. Functional analysis is crucial, when studying dynamic equations. It is therefore important to introduce correctly formulated functional-analysis theory regarding dynamic equations and demonstrate its proper application on concrete dynamic equations.

In Chapter 2, we summarize fundamentals of calculus on time scales. We explore the differentiation on time scales and related theorems. We proceed with several variations of the chain rule substituting the version from traditional calculus. We conclude the chapter with examples of various time scales and differentiation on them.

Chapter 3 focuses on the problem of the integration on time scales. Three different types of integrals are explored: the Cauchy-type integral, the Riemann-type integral, and the Lebesgue-type integral. It is demonstrated that, for our purposes, the Lebesgue-type integral on time scales possesses convenient properties, such as a broad set of integrable functions and analogies to the monotone convergence theorem and the dominated convergence theorem.

In Chapter 4, an examination of functional spaces on time scales is conducted. The focus is on the study of various types of functions, including continuous functions, rd-continuous functions, regressive and positively regressive functions on compact intervals, bounded continuous functions on non-compact intervals, and spaces of Lebesgue-integrable functions. Emphasis is placed on investigating the properties of these spaces, particularly their completeness. The chapter concludes with a comprehensive exploration of relative compactness within these spaces.

Chapter 5 focuses on the concept of a generalized exponential function. Several examples for various time scales are given.

In Chapter 6, we utilize the mathematical tools introduced in the preceding chapters to analyze a second-order equation presented in the form

$$y^{\Delta\Delta} = p(t)g(y^{\sigma}).$$

on a noncompact time scale interval. We set assumptions on p and g functions and formulate a condition necessary and sufficient for the existence of a solution positive and nondecreasing for large t with a positive limit. We subsequently refine the assumptions in order to ensure with the formulated condition not only the existence but also the uniqueness of the solution.

Subsequently, we delve into the examination of a more general equation

$$(r(t)y^{\Delta})^{\Delta} = p(t)g(y^{\sigma}).$$

We establish assumptions on functions r, p and g and again formulate a necessary and sufficient condition for the existence of a solution with properties identical to the previous case.

Chapter 7 is appendix dedicated to selected concepts from functional analysis. Emphasis is placed on completeness of metric spaces, relative compactness and related notions, isometry and homeomorphism of normed spaces, and especially fixed point theorems, which are the key tool for the proofs in Chapter 6.

2 Time scales

Theory of time scales was introduced by Stefan Hilger at the end of the 20th century, and it has gained a lot of popularity. Time scales provide an elegant way to unify discrete and continuous analysis. In this chapter, we cover fundamental principles of time scale theory. We introduce a concept of delta (Hilger) differentiation and several variations of the chain rule. The chapter concludes with a comprehensive exploration of various examples of time scales, serving to provide further clarity and illustration of the discussed concepts. Note that we do not provide detailed proofs, since we explore standard results from time scale theory that are well-established. We use [1], [2], [3] and [4] as the main sources.

2.1 Fundamentals

The aim of this section is to provide an overview of the fundamental definitions and theorems associated with time scales.

Definition 2.1.1 (Time scale)**.** A *time scale* is an arbitrary nonempty closed subset of the real line $\mathbb{R}$. Time scale is commonly denoted by $\mathbb{T}$.

Example 2.1.2. It is possible to list a lot of examples of time scales, since the definition is broad. The natural numbers ($\mathbb{N}$), integers ($\mathbb{Z}$), real numbers ($\mathbb{R}$), union of closed intervals, set of isolated points combined with a union of closed intervals or the Cantor set are just some of many. However, not all sets qualify as time scales, rational numbers ($\mathbb{Q}$) or open intervals are examples of such sets.

For following definitions we put $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$

Definition 2.1.3 (Forward jump operator)**.** Let $t \in \mathbb{T}$, we define a mapping $\sigma : \mathbb{T} \to \mathbb{T}$ denoted as the *forward jump operator* as follows

$$\sigma(t) = \inf \{s \in \mathbb{T}, s > t\}.$$

Definition 2.1.4 (Backward jump operator)**.** Let $t \in \mathbb{T}$, we define a mapping $\varrho : \mathbb{T} \to \mathbb{T}$ denoted as the *backward jump operator* as follows

$$\varrho(t) = \sup \{s \in \mathbb{T}, s < t\}.$$

Definition 2.1.5 (Right-scattered, left-scattered and isolated points)**.** We say that $t \in \mathbb{T}$ is *right-scattered* if

$$\sigma(t) > t,$$

on the other hand if

$$\varrho(t) < t,$$

we say t is *left-scattered.* If t is both right- and left-scattered, we say that it is *isolated.*

Definition 2.1.6 (Right-dense, left-dense and dense points)**.** A point $t \in \mathbb{T}$ satisfying $t < \sup \mathbb{T}$ and

$$\sigma(t) = t$$

is called *right-dense.* Similarly, a point t satisfying $t > \inf \mathbb{T}$ and

$$\varrho(t) = t$$

is called *left-dense.* A point t both right- and left-dense is called *dense.*

Remark 2.1.7. Note that
$$\sigma(\varrho(t)) = t$$
does not hold in general, suppose that $t \in \mathbb{T}$ is left-dense and right-scattered, then $\sigma(\varrho(t)) = \sigma(t) \neq t$. Similarly,
$$\varrho(\sigma(t)) = t$$
does not hold for left-scattered and right-dense points in $\mathbb{T}$.

Figure 2.1: Point classification: t_1 – dense, t_2 – left-dense and right-scattered, t_3 – isolated, t_4 – left-scattered and right-dense

Definition 2.1.8 (Graininess)**.** Let μ be a mapping $\mu : \mathbb{T} \setminus \{\sup \mathbb{T}\} \to [0, \infty)$ defined as
$$\mu(t) = \sigma(t) - t,$$
we call this mapping the *graininess*.

Remark 2.1.9. Note that we can distinguish between the right- and left-graininess. The previously defined graininess is sometimes referred to as the *right-graininess*, and the *left-graininess* is defined as the mapping $\nu : \mathbb{T} \setminus \inf \mathbb{T} \to [0, \infty)$ given by
$$\nu(t) = t - \varrho(t).$$
It is important to note that the domain of ν is $\mathbb{T} \setminus \inf \mathbb{T}$, i.e., the left endpoint of $\mathbb{T}$ is excluded.

Remark 2.1.10. Since every time scale $\mathbb{T}$ is a closed set, the definitions given above imply that both $\sigma(t)$ and $\varrho(t)$ belong to $\mathbb{T}$ when $t \in \mathbb{T}$.

Definition 2.1.11 ($\mathbb{T}^\kappa$)**.** Let $\mathbb{T}$ be a time scale. We define $\mathbb{T}^\kappa$ as
$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} - (\varrho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup(\mathbb{T}) < \infty, \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty. \end{cases}$$

Remark 2.1.12. We introduce the following convention. Let $a, b \in \mathbb{R}$, then
$$[a, b]_\mathbb{T} = [a, b] \cap \mathbb{T}.$$
Similarly for $(a, b) \cap \mathbb{T}$, $[a, b) \cap \mathbb{T}$ and $(a, b] \cap \mathbb{T}$.

2.2 Differentiation

In this section, we discuss the concept of differentiation on time scales and provide some useful formulas and demonstrate their practical applications.

Definition 2.2.1 (Delta derivative)**.** We say that a function $f : \mathbb{T} \to \mathbb{R}$ has a *delta derivative* $f^{\Delta}(t)$ at $t \in \mathbb{T}^{\kappa}$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $s \in (t-\delta, t+\delta)_{\mathbb{T}}$

$$\big|(f(\sigma(t)) - f(s)) - f^{\Delta}(t)(\sigma(t) - s)\big| \leq \varepsilon|\sigma(t) - s| \tag{2.2.1}$$

holds.

Remark 2.2.2 (Nabla derivative)**.** We can define another type of derivative called *nabla derivative*, we need to replace (2.2.1) with

$$\big|(f(\varrho(t)) - f(s)) - f^{\nabla}(t)(\varrho(t) - s)\big| \leq \varepsilon|\varrho(t) - s|.$$

Example 2.2.3. If $\mathbb{T} = \mathbb{R}$, then the delta and nabla derivatives coincide with the usual derivatives, and we have $f^{\Delta} = f^{\nabla} = f'$. On the other hand, if $\mathbb{T} = \mathbb{Z}$, then the delta and nabla derivatives coincide with the forward and backward difference operators, respectively. Specifically, we have $f^{\Delta}(t) = f(t+1) - f(t) = \Delta f(t)$ and $f^{\nabla}(t) = f(t) - f(t-1) = \nabla f(t)$.

Remark 2.2.4. There are some other types of derivatives, e.g. diamond, we do not study derivative types other than delta in this thesis any further. In the remainder of the text, we focus only on the delta derivative. If not stated otherwise, by "derivative" we mean the delta derivative.

Theorem 2.2.5. *Let $f : \mathbb{T} \to \mathbb{R}$ be a function and $t \in \mathbb{T}^{\kappa}$. Then the following hold:*

1. *Assume f is delta differentiable at t, then f is continuous at t.*
2. *Assume t is right-scattered and f is continuous at t, then f is delta differentiable at t and*
$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$
3. *Assume t is right-dense, then f is delta differentiable at t if and only if*
$$a = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$
exists, then $f^{\Delta}(t) = a$.
4. *Assume t is delta differentiable t, then*
$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

Theorem 2.2.6. *Let $f, g : \mathbb{T} \to \mathbb{R}$ be delta differentiable at $t \in \mathbb{T}^{\kappa}$. Then*

1. *$f + g$ is delta differentiable at t and*
$$(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t),$$
2. *for any $\alpha \in \mathbb{R}$ $\alpha f : \mathbb{T} \to \mathbb{R}$ is delta differentiable at t and*
$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t),$$

3. *fg is delta differentiable at t and*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)),$$

4. *if $f(t)f(\sigma(t)) \neq 0$, then $1/f$ is delta differentiable at t and*

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))},$$

5. *if $g(t)g(\sigma(t)) \neq 0$, then f/g is delta differentiable at t and*

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

Example 2.2.7. Let $\alpha \in \mathbb{R}$ and $m \in \mathbb{N}$. Then

1. for f defined as $f(t) = (t-\alpha)^m$

$$f^{\Delta}(t) = \sum_{\nu=0}^{m-1}(\sigma(t)-\alpha)^{\nu}(t-\alpha)^{m-1-\nu},$$

2. for g defined as $g(t) = 1/(t-\alpha)^m$

$$g^{\Delta}(t) = -\sum_{\nu=0}^{m-1}\frac{1}{(\sigma(t)-\alpha)^{m-\nu}(t-\alpha)^{\nu+1}}.$$

Remark 2.2.8. From now on we use the following convention

$$f^{\sigma} = (f \circ \sigma).$$

Example 2.2.9. Let f, g and h be delta differentiable at t. We can calculate the delta derivative of fgh. Since Theorem 2.2.6 holds, we can state gh is delta differentiable at t and therefore also $f \cdot (gh) = fgh$ is delta differentiable at t and the delta derivative of fgh is given by following formula

$$(fgh)^{\Delta} = f^{\Delta}gh + f^{\sigma}(gh)^{\Delta} = f^{\Delta}gh + f^{\sigma}g^{\Delta}h + f^{\sigma}g^{\sigma}h^{\Delta}.$$

We can generalize this for n functions, consider a function $p = f_1 f_2 \dots f_n$ and suppose f_i is delta differentiable at t for $i \in \{1, \dots, n\}$, then

$$p^{\Delta} = \sum_{j=1}^{n}\prod_{i=0}^{j-1} f_i^{\sigma} f_j^{\Delta} \prod_{k=n-j}^{n} f_k.$$

We might prove this using mathematical induction.

Example 2.2.10. Again by Theorem 2.2.6

$$(f^2)^{\Delta} = (f \cdot f)^{\Delta} = f^{\Delta}f + f^{\sigma}f^{\Delta} = (f + f^{\sigma})f^{\Delta}.$$

This expression can be again generalized for f^{n+1} as

$$(f^{n+1})^{\Delta} = \left\{\sum_{k=0}^{n} f^k (f^{\sigma})^{n-k}\right\} f^{\Delta}.$$

Definition 2.2.11 (Second delta derivative)**.** Let $f : \mathbb{T} \to \mathbb{R}$ be a function, suppose f^Δ is delta differentiable on $(\mathbb{T}^\kappa)^\kappa$ with delta derivative $f^{\Delta^2} = f^{\Delta\Delta} = (f^\Delta)^\Delta : (\mathbb{T}^\kappa)^\kappa \to \mathbb{R}$, we denote this function as *second delta derivative.*

Remark 2.2.12. Function fg does not need to be twice delta differentiable even if f and g are so. We know that

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta.$$

Now when considering the derivative of $(fg)^\Delta$, it is important to note that the requirement for delta differentiability extends beyond just f and g. In this case, the existence of f^σ is also necessary, which may not always be satisfied. If the requirement is satisfied, then

$$\begin{aligned}(fg)^{\Delta^2} &= (f^\Delta g + f^\sigma g^\Delta)^\Delta \\ &= f^{\Delta\Delta} g + f^{\Delta\sigma} g^\Delta + f^{\sigma\Delta} g^\Delta + f^{\sigma\sigma} g^{\Delta\Delta}.\end{aligned}$$

2.3 Chain rule

It is well-known that if $f, g : \mathbb{R} \to \mathbb{R}$ and g is differentiable at t and f is differentiable at $g(t)$, then

$$(f \circ g)' = f'(g(t))g'(t).$$

We show that this chain rule does not hold in general for the time scale calculus and provide several alternatives.

Example 2.3.1. Assume $\mathbb{T} = \mathbb{Z}$ and consider $f(t) = g(t) = t^2$. Then

$$(f \circ g)^\Delta = (t^4)^\Delta = \frac{(t+1)^4 - t^4}{t+1-t} = 4t^3 + 6t^2 + 4t + 1$$

and

$$f^\Delta(g(t))g^\Delta(t) = \frac{(t^2+1)^2 - (t^2)^2}{t^2+1-t^2}\frac{(t+1)^2 - t^2}{t+1-t} = 4t^3 + 2t^2 + 2t + 1,$$

then apparently for $t \neq 0$

$$(f \circ g)^\Delta \neq f^\Delta(g(t))g^\Delta.$$

We introduce three alternatives to the classical chain rule.

Theorem 2.3.2 (Chain rule)**.** *Let $g : \mathbb{R} \to \mathbb{R}$ be continuous and suppose $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}^\kappa$ and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, then there exists c in the real interval $[t, \sigma(t)]$ such that*

$$(f \circ g)^\Delta(t) = f^\Delta(g(c))g^\Delta(t).$$

Theorem 2.3.3 (Chain rule)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $f \circ g$ is delta differentiable and the formula*

$$(f \circ g)^\Delta(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t)g^\Delta(t)\mathrm{d}h \right\} g^\Delta(t)$$

holds.

Let $\mathbb{T}$ be a time scale and $\nu : \mathbb{T} \to \mathbb{R}$ be a strictly increasing function such that $\widetilde{\mathbb{T}} = \nu(\mathbb{T})$ is also a time scale. We denote the forward jump operator on $\widetilde{\mathbb{T}}$ by $\widetilde{\sigma}$ and the corresponding delta derivative by $\widetilde{\Delta}$. It is true that $\nu \circ \sigma = \widetilde{\sigma} \circ \nu$. This allows us to introduce another chain rule.

Theorem 2.3.4 (Chain rule)**.** *Let $\nu : \mathbb{T} \to \mathbb{R}$ be strictly increasing and suppose $\widetilde{\mathbb{T}} = \nu(\mathbb{T})$ is a time scale and $w : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^{\Delta}(t)$ and $w^{\widetilde{\Delta}}(\nu(t))$ exist for $t \in \mathbb{T}^{\kappa}$, then*

$$(w \circ \nu)^{\Delta} = (w^{\widetilde{\Delta}} \circ \nu)\nu^{\Delta}.$$

Remark 2.3.5. We use Theorem 2.3.4 in order to transform a dynamic equation to a simpler form later in Chapter 6.

2.4 Mean value theorem

In this section, we present the mean value theorem and related remarks.

Theorem 2.4.1 (Mean value theorem)**.** *Let $a, b \in \mathbb{T}$ and consider a continuous function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ that is differentiable on $[a, b)_{\mathbb{T}}$. Then there exist $\xi, \tau \in [a, b]_{\mathbb{T}}$ such that*

$$f^{\Delta}(\tau) \leq \frac{f(b) - f(a)}{b - a} \leq f^{\Delta}(\xi).$$

Remark 2.4.2. Let f be a continuous function on $[a, b]_{\mathbb{T}}$ that is differentiable on $[a, b)_{\mathbb{T}}$. If $f^{\Delta}(t) = 0$ for all $t \in [a, b)_{\mathbb{T}}$, then f is constant function on $[a, b]_{\mathbb{T}}$.

Remark 2.4.3. Let f be a continuous function on $[a, b]_{\mathbb{T}}$ that is differentiable on $[a, b)_{\mathbb{T}}$. Then f is increasing, decreasing and nonincreasing on $[a, b]_{\mathbb{T}}$ if $f^{\Delta}(t) > 0$, $f^{\Delta}(t) < 0$, $f^{\Delta}(t) \geq 0$ and $f^{\Delta}(t) \leq 0$ for all $t \in [a, b)_{\mathbb{T}}$, respectively.

Theorem 2.4.4. *Suppose f and g are continuous functions on $[a, b]_{\mathbb{T}}$ that are differentiable on $[a, b)_{\mathbb{T}}$. Let moreover $g^{\Delta}(t) > 0$ for all $t \in [a, b)_{\mathbb{T}}$. Then there exist $\xi, \tau \in [a, b)_{\mathbb{T}}$ such that*

$$\frac{f^{\Delta}(\tau)}{g^{\Delta}(\tau)} \leq \frac{f(b) - f(a)}{b - a} \leq \frac{f^{\Delta}(\xi)}{g^{\Delta}(\xi)}.$$

2.5 Examples of time scales

Several examples of time scales have already been mentioned, most importantly the real numbers ($\mathbb{R}$). In this section, we explore these time scales in more detail and introduce other cases that have been less frequently studied.

Example 2.5.1 ($\mathbb{R}$)**.** Let us focus on the time scale $\mathbb{T} = \mathbb{R}$.

Forward jump, backward jump and graininess
Consider $\mathbb{T} = \mathbb{R}$, then for all $t \in \mathbb{T}$

$$\sigma(t) = \inf\{s \in \mathbb{T}, s > t\} = t,$$

similarly

$$\varrho(t) = \sup\{s \in \mathbb{T}, s < t\} = t.$$

Thus for all $t \in \mathbb{T}$

$$\mu(t) = \sigma(t) - t = t - t = 0.$$

Derivative

Suppose $\mathbb{T} = h\mathbb{Z}$, then for a function $f : \mathbb{T} \to \mathbb{R}$ we have

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} = f'(t)$$

for all $t \in \mathbb{T}$.

Example 2.5.2 ($h\mathbb{Z}$)**.** Now we focus on the time scale $\mathbb{T} = h\mathbb{Z} = \{hk, k \in \mathbb{Z}\}$, where $h > 0$.

Forward jump, backward jump and graininess

Consider $\mathbb{T} = h\mathbb{Z}$, then for all $t \in \mathbb{T}$ the following holds

$$\sigma(t) = \inf\{s \in \mathbb{T}, s > t\} = \inf\{t + nh, n \in \mathbb{T}\} = t + h$$

and

$$\varrho(t) = t - h.$$

Therefore for all $t \in \mathbb{T}$

$$\mu(t) = \sigma(t) - t = t + h - t = h.$$

That means the graininess is constant.

Derivative

Let $\mathbb{T} = h\mathbb{Z}$, then for a function $f : \mathbb{T} \to \mathbb{R}$ we have

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(t+h) - f(t)}{h}$$

for all $t \in \mathbb{T}$.

Example 2.5.3 ($\mathbb{P}_{a,b}$)**.** In this section, we explore the time scale

$$\mathbb{T} = \mathbb{P}_{a,b} = \bigcup_{k=0}^{\infty} [k(a+b), k(a+b) + a],$$

where $a, b > 0$.

Forward jump and graininess

Consider $\mathbb{T} = \mathbb{P}_{a,b}$, then the following holds

$$\sigma(t) = \begin{cases} t & \text{if } t \in \bigcup_{k=0}^{\infty} [k(a+b), k(a+b) + a], \\ t + b & \text{if } t \in \bigcup_{k=0}^{\infty} \{k(a+b) + a\}. \end{cases}$$

Therefore

$$\mu(t) = \begin{cases} 0 & \text{if } t \in \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a], \\ b & \text{if } t \in \bigcup_{k=0}^{\infty} \{k(a+b)+a\}. \end{cases}$$

That means the graininess is not constant and not continuous. This time scale faithfully models the life span of cicadas or a common mayfly.

Example 2.5.4 ($q^{\mathbb{Z}}$)**.** Let us consider the time scale

$$q^{\mathbb{Z}} = \{q^k, k \in \mathbb{Z}\},$$

where $q > 1$. We now take the time scale $\mathbb{T} = \overline{q^{\mathbb{Z}}}$.

Forward jump, backward jump and graininess
Assume $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, then

$$\sigma(t) = \inf\{q^n, n \in [m+1, \infty)\} = q^{m+1} = qq^m = qt$$

if $t = q^m \in \mathbb{T}$ and $\sigma(0) = 0$. That means we obtain for all $t \in \mathbb{T}$

$$\sigma(t) = qt,$$

$$\varrho(t) = \frac{t}{q}$$

and thus

$$\mu(t) = \sigma(t) - t = (q-1)t.$$

That means $\mathbb{T}$ has one right dense point 0 and every other point is isolated and the graininess is an unbounded function.

Derivative
Let $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, then for a function $f : \mathbb{T} \to \mathbb{R}$ we have

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(qt) - f(t)}{(q-1)t}$$

for all $t \in \mathbb{T} \setminus \{0\}$ and provided the limit exists

$$f^{\Delta}(0) = \lim_{s \to 0} \frac{f(0) - f(s)}{0 - s} = \lim_{s \to 0} \frac{f(s) - f(0)}{s}.$$

Example 2.5.5. ($2^{2^{\mathbb{N}}}$) Let us consider the time scale

$$2^{2^{\mathbb{N}}} = \{2^{2^k}, k \in \mathbb{N}\}.$$

Forward jump, backward jump and graininess
Asssume $\mathbb{T} = 2^{2^{\mathbb{N}}}$, then

$$\sigma(t) = \inf\{2^{2^n}, n \in [m+1, \infty)\} = 2^{2^{m+1}} = 2^{2^m} 2^{2^m} = t^2$$

if $t = 2^{2^m} \in \mathbb{T}$. That means we obtain for all $t \in \mathbb{T}$

$$\sigma(t) = t^2$$

and for $t \in [16, \infty) \cap \mathbb{T}$

$$\varrho(t) = \sqrt{t}$$

and thus

$$\mu(t) = \sigma(t) - t = t(t-1).$$

That means every point in $\mathbb{T} = 2^{2^{\mathbb{N}}}$ is isolated.

Derivative
Let $\mathbb{T} = 2^{2^{\mathbb{N}}}$, then for a function $f : \mathbb{T} \to \mathbb{R}$ we have

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(t^2) - f(t)}{t(t-1)}$$

for all $t \in \mathbb{T}$.

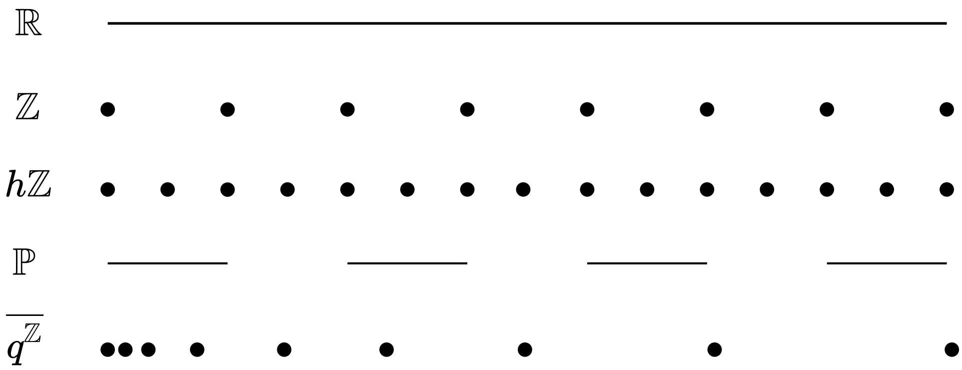

Figure 2.2: Selected time scales considered in this section

3 Integration on time scales

This chapter focuses on the problem of integration on time scales. First, we introduce the notion of the Cauchy-type integral defined by means of antiderivatives. However, similar to classical calculus, this integral is not sufficient for our purposes. Therefore, we briefly explore the Riemann-type integral and then the Lebesgue-type integral. It is worth noting that, similar to derivatives, we can define delta and nabla integrals. However, we focus only on the delta cases. Our references for this chapter are [1], [2], [4] and [6].

3.1 Cauchy-type integral

First, we introduce the Cauchy delta integral defined by means of antiderivatives (or pre-antiderivatives). This is the original integral used on time scales introduced by Hilger.

Remark 3.1.1. Note that we use terminology based on the work of Dieudonne. The notion of the Cauchy-type integral might denote another type of integral in other sources.

3.1.1 Construction of the integral

For the construction of the Cauchy-type integral, we need to introduce notion of regulated and pre-differentiable functions.

Definition 3.1.2 (Regulated function)**.** Let $f : \mathbb{T} \to \mathbb{R}$ be a function and suppose there exists a right-sided limit for every right-dense point in $\mathbb{T}$ and there exists a left-sided limit for every left-dense point in $\mathbb{T}$. Then we call this function *regulated*.

Definition 3.1.3 (Pre-differentiable function)**.** Let $f : \mathbb{T} \to \mathbb{R}$ be a continuous function, suppose $D \subset \mathbb{T}^\kappa$, $\mathbb{T}^\kappa \setminus D$ is countable and contains no right-scattered elements of $\mathbb{T}$ and f is differentiable on D. Then we call f *pre-differentiable* with the region of differentiation D.

Theorem 3.1.4. *Let f be a regulated function. Then there exists a pre-differentiable function with the region of differentiation D such that for all $t \in D$ the following holds*

$$F^\Delta(t) = f(t). \tag{3.1.1}$$

Definition 3.1.5 (Pre-antiderivatives)**.** Let f be a regulated function, then F is called *pre-antiderivative* if it is pre-differentiable with region of differentiation D and (3.1.1) holds for every $t \in D$.

Definition 3.1.6. Assume f is a regulated function, then we define the *indefinite integral* of this fuction by

$$\int f(t)\,\Delta t = F(t) + C,$$

where F is a pre-antiderivative of f and C is an arbitrary constant.

Following this, we can define the *Cauchy delta integral* by

$$\int_r^s f(t)\,\Delta t = F(s) - F(r),$$

for all $s, t \in \mathbb{T}$.

A function $F : \mathbb{T} \to \mathbb{T}$ is called an *antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ if (3.1.1) holds for all $t \in \mathbb{T}^\kappa$.

In the following, we state a theorem regarding the existence of antiderivatives. Before we do so, it is necessary to introduce the notion of an rd-continuous function.

Definition 3.1.7 (Rd-continuous function)**.** Let $f : \mathbb{T} \to \mathbb{R}$ be a function and suppose that f is regulated and continuous at every right-dense point of $\mathbb{T}$, then we call this function *rd-continuous.*

Theorem 3.1.8. *Let f be an rd-continuous function if $t_o \in \mathbb{T}$, then F defined by*

$$F(t) := \int_{t_0}^{t} f(x)\,\Delta x$$

for $t \in \mathbb{T}$ is an antiderivative of f.

Remark 3.1.9. The major advantage of the Cauchy-type integral is the simple way, in which we construct it. Unlike other types of integrals, it does not require a limiting process to be constructed. On the other hand, the main disadvantage is that it has a strict restriction on integrability, as it can only be applied to regulated functions.

3.1.2 Improper integrals

Definition 3.1.10. Let $a \in \mathbb{T}$, $\sup \mathbb{T} = \infty$, and assume f is rd-continuous on $[a, \infty)$, then we define the *improper integral* by

$$\int_{a}^{\infty} f(t)\,\Delta t := \lim_{b\to\infty} \int_{a}^{b} f(t)\,\Delta t.$$

We say the improper integral *converges* provided the limit exists. Otherwise we say it *diverges.*

3.2 Riemann-type integral

In the traditional calculus, the Riemann-type integral is usually defined using either the Riemann sums or the Darboux sums. On time scales, we can also construct the Riemann delta integral using both methods. In this section, we present a brief construction using Darboux sums.

3.2.1 Construction of the integral

Consider points $a, b \in \mathbb{T}$ such that $a < b$ and let $[a, b]_{\mathbb{T}}$ be a closed and thus bounded interval in $\mathbb{T}$. A partition of $[a, b]_{\mathbb{T}}$ is any finite ordered subset

$$P = \{t_0, t_1, \dots, t_n\} \subset [a, b]_{\mathbb{T}},$$

where $a = t_0 < t_1 < \ldots < t_n = b$. Let us consider the set of all partitions of $[a, b]_{\mathbb{T}}$ and denote it by $\mathcal{P}(a, b)$. Let f be a real-valued bounded function on $[a, b]_{\mathbb{T}}$, then

$$M = \sup\{f(t), t \in [a, b)_{\mathbb{T}}\}, \quad m = \inf\{f(t), t \in [a, b)_{\mathbb{T}}\}$$

and for every $i \in \{1, 2, \ldots, n\}$

$$M_i = \sup\{f(t), t \in [t_{i-1}, t_i)_{\mathbb{T}}\}, \quad m_i = \inf\{f(t), t \in [t_{i-1}, t_i)_{\mathbb{T}}\}.$$

Let us now consider a partition $P \in \mathcal{P}$, then

$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}) \quad \text{and} \quad L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}),$$

where by $U(f, P)$ we denote the upper (delta) sum and by $L(f, P)$ lower (delta) sum of f with respect to P. Note that

$$m(b - a) \leq L(f, P) \leq U(f, P) \leq M(b - a). \tag{3.2.1}$$

Now we can define the upper delta $U(f)$ and lower delta $L(f)$ integrals of f from a to b by

$$U(f) = \inf\{U(f, P), P \in \mathcal{P}\}$$

and

$$L(f) = \sup\{U(f, P), P \in \mathcal{P}\}.$$

In view of (3.2.1), we state that $L(f)$ and $U(f)$ are finite.

Definition 3.2.1 (The Riemann delta integral)**.** Let $f : \mathbb{T} \to \mathbb{R}$ be a bounded function. We say f is delta integrable from a to b if $L(f) = U(f)$. We denote this value by

$$\int_a^b f(t)\,\Delta t$$

and call this integral the *Riemann delta integral.*

Remark 3.2.2. The construction we used is called the Darboux construction. The original Riemann construction of the Riemann-type integral is slightly different. We can prove the equivalency of the two constructions analogically to the classical calculus.

Remark 3.2.3. It is possible to prove analogies of many theorems regarding ordinary Riemann-type integral for the calculus on time scales, for more read [2].

Remark 3.2.4. Any bounded function on $[a, b]_{\mathbb{T}}$ with finitely many discontinuity points is integrable. It can be proven, that this set is larger than the set of regulated functions. Moreover, the Riemann-type integral suffers from problems related to a lack of reasonable convergence results, which can restrict its applicability.

3.2.2 Improper integrals

Definition 3.2.5. Suppose $a \in \mathbb{T}$ and and $\sup \mathbb{T} = \infty$. Now assume a real-valued function f is defined on $[a, \infty)_{\mathbb{T}}$ and is integrable on the interval $[a, b]_{\mathbb{T}}$ for any $b \in \mathbb{T}$ with $b \geq a$. Consider the integral

$$\int_a^b f(t)\,\Delta t.$$

We define *improper integral* of f from a to ∞ by

$$\int_a^\infty f(t)\,\Delta t = \lim_{b\to\infty} \int_a^b f(t)\,\Delta t.$$

We say the integral *converges*, provided the limit exists. Otherwise, we say the improper integral *diverges*.

Remark 3.2.6. The improper integral defined above is the improper integral of the first kind. We could define also the improper integral of the second kind. We consider a time scale $\mathbb{T}$ and an interval $[a, b]_{\mathbb{T}}$, where b is left-dense and let f be integrable on any interval $[a, c]_{\mathbb{T}}$ and unbounded on $[a, b)_{\mathbb{T}}$. We call the formal expression

$$\int_a^b f(t)\,\Delta t$$

an improper integral of *second kind*. We say f has a *singularity* at $t = b$. If the left-sided limit

$$\lim_{c\to b^-} \int_a^c f(t)\,\Delta t$$

exists and is finite, then we say the improper integral converges. Otherwise, we say the improper integral diverges.

3.3 Lebesgue-type integral

In this section, we focus on the Lebesgue delta integral. We introduce the Lebesgue delta measure by means of Carathéodory-like approach. We utilize measure theory to construct the integral. Later we present monotone and dominated convergence theorems.

3.3.1 Lebesgue delta measure

Let us first recall some concepts from general measure theory.

Definition 3.3.1 (Measure)**.** Let X be a set. Then the system Σ of subsets of X satisfying

- $X \in \sigma$,
- $X \setminus A \in \Sigma$ for all sets $A \in \Sigma$,
- for every countable system $\{A_k\}_{k\in\mathbb{N}}$ of sets, if $A_k \in \Sigma$ then $\bigcup_{k\in\mathbb{N}} A_k \in \Sigma$.

is called *σ-algebra*. The pair (X, σ) is called a *measurable space*.

Definition 3.3.2. Let (X, Σ) be a measurable space, a function $\mu : \Sigma \to [0, \infty) \cup \{\infty\}$ satisfying

- $\mu(\emptyset) = 0$,
- $\mu(\bigcup_{k\in\mathbb{N}} A_k) = \sum_{k\in\mathbb{N}} \mu(A_k)$ for all countable collections $\{A_k\}_{k=1}^{\infty}$ of pairwise disjoint sets in Σ

is called a *measure*. A triple (X, Σ, μ) is called a *measure space*.

Now we can proceed with the construction of the Lebesgue delta measure. Let $\mathbb{T}$ be a time scale and suppose $\sup \mathbb{T} = \infty$. Let us consider the set of all left closed and right open intervals on $\mathbb{T}$ of the form

$$[a, b)_{\mathbb{T}} = \{t \in \mathbb{T}, a \le t < b\},$$

where $a, b \in \mathbb{T}$ and $a \le b$, we denote this set as $\mathcal{F}_1$.

We define a mapping $m_1 : \mathcal{F}_1 \to [0, \infty) \cup \{\infty\}$, that assigns to every interval $[a, b)_{\mathbb{T}}$ its length, that is

$$m_1([a, b)_{\mathbb{T}}) = b - a.$$

This mapping is a countably additive measure. The interval $[a, a)_{\mathbb{T}}$ is understood as the empty set.

Using m_1, we generate the outer measure m_1^*. Let E be a subset of $\mathbb{T}$. Assume there exists an at most countable system of intervals $V_j \in \mathcal{F}_1$ for $j \in \mathbb{N}$ such that

$$E \subset \bigcup_{j\in\mathbb{N}} V_j,$$

then we set

$$m_1^*(E) = \inf \sum_j m_1(V_j),$$

where the infimum is taken over all coverings of E by the mentioned system of intervals.

We say a set $N \subset \mathbb{T}$ is m_1^*-measurable if

$$m_1^* = m_1^*(E \cap N) + m_1^*(E \cap (\mathbb{T} \setminus N))$$

for any $E \subset \mathbb{T}$. It can be shown, that the set of all m_1^*-measurable subsets of $\mathbb{T}$ is a σ-algebra, we denote it as $\mathcal{M}(m_1^*)$.

Definition 3.3.3 (The Lebesgue delta measure)**.** Let m_1^* be an outer measure on the family of all subsets of bounded time scale $\mathbb{T}$ and $\mathcal{M}(m_1^*)$ be a family of all m_1^* measurable subsets of $\mathbb{T}$, then the restriction μ_Δ of m_1^* to $\mathcal{M}(m_1^*)$ is called the *Lebesgue delta measure*.

We may extend the the Lebesgue delta measure to other types of intervals, one can show that if $a, b \in \mathbb{T}$ and $a \le b$, then

$$\mu_\Delta([a, b)_{\mathbb{T}}) = b - a \quad \text{and} \quad \mu_\Delta((a, b)_{\mathbb{T}}) = b - \sigma(a).$$

and if $a, b \in \mathbb{T} \setminus \{\max \mathbb{T}\}$ and $a \le b$, then

$$\mu_\Delta((a, b]_{\mathbb{T}}) = \sigma(b) - \sigma(a) \quad \text{and} \quad \mu_\Delta([a, b]_{\mathbb{T}}) = \sigma(b) - a.$$

The Lebesgue delta measure satisfies all the axioms of a measure and therefore it is a measure (Definition 3.3.2) in the sense of measure theory. Similarly, $(\mathbb{T}, \mathcal{M}(m_1^*))$ is a measurable space and $(\mathbb{T}, \mathcal{M}(m_1^*), \mu_\Delta)$ is a measure space.

3.3.2 Construction of the integral

We present some of the concepts from general measure and integration applied to the measure space $(\mathbb{T}, \mathcal{M}(m_1^*), \mu_\Delta)$.

Definition 3.3.4 (Delta measurable function)**.** Consider $(\mathbb{T}, \mathcal{M}(m_1^*), \mu_\Delta)$, then we say $f : \mathbb{T} \to [-\infty, \infty] = \mathbb{R} \cup \{-\infty, \infty\}$ is *delta measurable* if

$$f^{-1}([-\infty, \alpha)) = \{t \in \mathbb{T}, f(t) < \alpha\} \in \mathcal{M}(m_1^*)$$

for any $\alpha \in \mathbb{R}$.

Let $E \subseteq \mathbb{T}$ be delta measurable. A function $f : E \to [-\infty, \infty]$ is *delta measurable on E*, if its zero extension on $\mathbb{T}$ is a delta measurable function.

Definition 3.3.5 (Simple function)**.** We say a function $S : \mathbb{T} \to \mathbb{R}$ is *simple* if it is delta measurable and takes only a finite number of different values $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Remark 3.3.6. Every simple function S can be expressed using

$$S = \sum_{i=1}^{n} \alpha_i \chi_{A_i},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$, $A_1, A_2, \ldots, A_n$ are delta measurable sets and χ_{A_i} is the characteristic function of A_i, i.e.,

$$\chi_{A_i}(t) = \begin{cases} \chi_{A_i}(t) = 1 \text{ for } t \in A_i, \\ \chi_{A_i}(t) = 0 \text{ for } t \in \mathbb{T} \setminus A_i. \end{cases}$$

For the following definitions we use the convention that $0 \cdot \infty = 0$.

Definition 3.3.7. Suppose $E \subseteq \mathbb{T}$ is a delta measurable set and let $S : \mathbb{T} \to [0, \infty)$ be a simple delta measurable function with

$$S = \sum_{j=1}^{n} \alpha_j \chi_{A_j}.$$

The Lebesgue delta integral of S is defined by

$$\int_E S(s)\,\Delta s = \sum_{j=1}^{n} \alpha_j \mu_\Delta(A_j \cap E).$$

Definition 3.3.8. Suppose $E \subset \mathbb{T}$ is a delta measurable set and let $f : \mathbb{T} \to [0, \infty]$ be a delta measurable function. The Lebesgue delta integral of f on E is then defined by

$$\int_E f(s)\,\Delta s = \sup \left\{ \int_E S(s)\,\Delta s, S \text{ is simple delta measurable, } 0 \le S(t) \le f(t) \text{ for } t \in \mathbb{T} \right\}.$$

Definition 3.3.9. Suppose $E \subset \mathbb{T}$ is a delta measurable set and let $f : \mathbb{T} \to [-\infty, \infty]$ be a delta measurable function and let

$$f^+ := \max\{f, 0\}, \quad \text{and} \quad f^- := \max\{-f, 0\}.$$

Function f is then Lebesgue delta integrable on E if at least one of the integrals

$$\int_E f^+(s)\,\Delta s \quad \text{and} \quad \int_E f^-(s)\,\Delta s$$

is finite. The Lebesgue delta integral of f on E is then defined by

$$\int_E f(s)\,\Delta s = \int_E f^+(s)\,\Delta s - \int_E f^-(s)\,\Delta s.$$

3.3.3 Convergence theorems

We know that $(\mathbb{T}, \mathcal{M}(m_1^*))$ is a measurable space, thus we might utilize existing results from measure theory. In this section, we formulate the Levi monotone convergence theorem and the Lebesgue dominated convergence theorem adapted to $(\mathbb{T}, \mathcal{M}(m_1^*))$.

Theorem 3.3.10 (Levi monotone convergence theorem)**.** *Let $E \subset \mathbb{T}$ be a delta measurable set and suppose $(f_n)_{n\in\mathbb{N}}$ is a nondecreasing sequence of nonnegative delta measurable functions $f_n : E \to [0, \infty]$, i.e., for every $t \in E$ and all $n \in \mathbb{N}$*

$$0 \le f_n(t) \le f_{n+1}(t) \le \infty.$$

Further let $t \in E$

$$f(t) = \lim_{n\to\infty} f_n(t).$$

Then f is delta measurable and

$$\lim_{n\to\infty} \int_E f_n(s)\,\Delta s = \int_E f(s)\,\Delta s.$$

Theorem 3.3.11 (Lebesgue dominated convergence theorem)**.** *Let $E \subset \mathbb{T}$ be a delta measurable set and suppose $(f_n)_{n=1}^{\infty}$ is a sequence of delta measurable functions $f_n : E \to [-\infty, \infty]$ such that for $t \in E$*

$$f(t) = \lim_{n\to\infty} f_n(t).$$

Suppose $g : E \to [0, \infty]$ is a delta integrable function such that for all $t \in E$

$$|f_n(t)| \le g(t).$$

Then f is delta measurable and

$$\lim_{n\to\infty} \int_E f_n(s)\,\Delta s = \int_E f(s)\,\Delta s.$$

Remark 3.3.12. The Lebesgue-type integral is far superior to all other notions of the integral on time scales. It provides the largest set of integrable functions and its derivation is based on measure theory and many details can be avoided by quoting standard results from measure theory.

3.3.4 Examples of integration on time scales

In this section we discuss integrals for some special settings.

Example 3.3.13. Let $a, b \in \mathbb{T}$ and suppose f is Lebesgue delta integrable, then

- If $\mathbb{T} = \mathbb{R}$, then
$$\int_a^b f(t)\,\Delta t = \int_a^b f(t)\mathrm{d}t.$$
- If $[a, b]_{\mathbb{T}}$ contains only isolated points, then
$$\int_a^b f(t)\,\Delta t = \begin{cases} \sum_{t\in[a,b)_{\mathbb{T}}} \mu(t)f(t) & \text{if } a < b, \\ 0 & \text{if } a = b, \\ -\sum_{t\in[a,b)_{\mathbb{T}}} \mu(t)f(t) & \text{if } a > b. \end{cases}$$
- If $\mathbb{T} = h\mathbb{Z} = \{hk, k \in \mathbb{Z}\}$, where $h > 0$, then
$$\int_a^b f(t)\,\Delta t = \begin{cases} \sum_{k=a/h}^{b/h-1} f(kh)h & \text{if } a < b, \\ 0 & \text{if } a = b, \\ -\sum_{k=b/h}^{a/h-1} f(kh)h & \text{if } a > b. \end{cases}$$
- If $\mathbb{T} = \mathbb{Z} = \{k, k \in \mathbb{Z}\}$, then
$$\int_a^b f(t)\,\Delta t = \begin{cases} \sum_{k=a}^{b-1} f(k) & \text{if } a < b, \\ 0 & \text{if } a = b, \\ -\sum_{k=b}^{a-1} f(k) & \text{if } a > b. \end{cases}$$
- If $\mathbb{T} = q_0^{\mathbb{N}} = \{q^k, k \in \mathbb{N}\} \bigcup \{0\}$, then
$$\int_a^b f(t)\,\Delta t = \begin{cases} \sum_{k\in\{a,aq,\ldots,b/q\}} (q-1)kf(k) & \text{if } a < b, \\ 0 & \text{if } a = b, \\ -\sum_{k\in\{b,bq,\ldots,a/q\}} (q-1)kf(k) & \text{if } a > b. \end{cases}$$

Remark 3.3.14. The fourth case from Example 3.3.13 is the special case of the third one, where we set $h = 1$.

Example 3.3.15. Consider the time scale $\mathbb{T} = 2^{2^{\mathbb{N}}} = \{t_k, k \in \mathbb{N}\}$. We know that
$$\sigma(t) = t^2$$
and
$$\mu(t) = t(t-1).$$

Then

$$\begin{aligned}\int_{t_1}^{\infty} \frac{1}{t^2}\,\Delta t &= \sum_{k=1}^{\infty} \frac{\mu(t_k)}{t_k^2} \\ &= \sum_{k=1}^{\infty} \frac{t_k(t_k-1)}{t_k^2} \\ &= \sum_{k=1}^{\infty} \left(1-\frac{1}{t_k}\right) \\ &= \infty,\end{aligned}$$

because the general term of the last series tends to 1 as k approaches infinity.

Remark 3.3.16. Example 3.3.15 yields an interesting result,

$$\int_{t_1}^{\infty} \frac{1}{t^2}\,\Delta t < \infty$$

in the time scale calculus, in contrast to the classical differential calculus, does not hold in general.

Remark 3.3.17. Unless stated otherwise, from now on by delta integrability we mean Lebesgue delta integrability.

3.3.5 Properties of the integral

Theorem 3.3.18 (Linearity)**.** *Let $\mathbb{T}$ be a time scale and $c_1, c_2 \in \mathbb{T}$ and suppose u and v are delta integrable functions on $[a,b]_{\mathbb{T}}$, then*

$$\int_a^b (c_1u + c_2v)(t)\,\Delta t = c_1 \int_a^b u(t)\,\Delta t + c_2 \int_a^b v(t)\,\Delta t$$

holds.

Theorem 3.3.19 (Additivity with respect to range of integration)**.** *Let $a \le b \le c \in \mathbb{T}$ and suppose u is integrable on $[a,b]_{\mathbb{T}}$, then*

$$\int_a^b u(t)\,\Delta t = \int_a^c u(t)\,\Delta t + \int_c^b u(t)\,\Delta t$$

holds.

We introduce the fundamental theorem of calculus.

Theorem 3.3.20 (Fundamental theorem of calculus)**.** *Suppose g is a continuous function on $[a,b]_{\mathbb{T}}$ such that g is delta differentiable on $[a,b)_{\mathbb{T}}$. If g^{Δ} is delta differentiable from a to b, then*

$$\int_a^b g^{\Delta}(t)\,\Delta t = g(b) - g(a).$$

The previous statement combined with

$$(uv)^{\Delta} = u^{\Delta}v + u^{\sigma}v^{\Delta}$$

leads to an integration by parts formula.

Theorem 3.3.21 (Integration by parts)**.** *Suppose u and v are continuous functions on $[a,b]_{\mathbb{T}}$ that are delta differentiable on $[a,b)_{\mathbb{T}}$. If u^{Δ} and v^{Δ} are integrable on interval $[a,b]_{\mathbb{T}}$, then*

$$\int_a^b u^{\Delta}(t)v(t)\,\Delta t = u(b)v(b) - u(a)v(a) - \int_a^b u^{\sigma}(t)v^{\Delta}(t)\,\Delta t. \tag{3.3.1}$$

The proof of the next theorem is is based on chain rule (Theorem 2.3.4).

Theorem 3.3.22. *(Substitution) Suppose $\nu : \mathbb{T}_1 \to \mathbb{R}$ is a strictly increasing function such that $\mathbb{T}_2 = \nu(\mathbb{T}_1)$ is a time scale. Denote by Δ_1 the delta derivative on $\mathbb{T}_1$ and Δ_2 the delta derivative on $\mathbb{T}_2$. Let $f : \mathbb{T}_1 \to \mathbb{R}$ be a locally Δ_1-integrable (on each finite interval) function and let ν be a Δ_1-differentiable with locally Δ_1-integrable Δ_1-derivative. Then, if $f\nu^{\Delta}$ has a Δ_1-antiderivative and if $a, b \in \mathbb{T}$,*

$$\int_a^b f(t)\nu^{\Delta_1}(t)\,\Delta_1 t = \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s)\,\Delta_2 s.$$

Theorem 3.3.23. *Let f be a rd-continuous function and $t \in \mathbb{T}^{\kappa}$, then*

$$\int_t^{\sigma(t)} f(\tau)\,\Delta\tau = \mu(t)f(t).$$

4 Function spaces on time scales

In this chapter, we discuss various function spaces endowed with a norm that are commonly used in the study of time scales. These spaces include generalizations of classical function spaces such as the space of continuous functions, the space of bounded functions and L^p spaces. We also introduce function spaces that are specific to time scales. Our primary references for this section are [2], [9], and [10].

4.1 Continuous and rd-continuous functions

Continuity on time scales is defined in the same way as in the classical calculus. We previously defined rd-continuous functions in Definition 3.1.7. In this section, we will discuss the spaces of functions that possess these properties.

Definition 4.1.1. Let $a, b \in \mathbb{T}$, such that $a < b$. Consider the set of rd-continuous functions on interval $[a, b]_{\mathbb{T}}$, we denote this set by $C_{\mathrm{rd}}[a, b]_{\mathbb{T}}$. On this set, we define the so called supremum metric ϱ_∞ as follows

$$\varrho_\infty(f, g) := \sup_{t \in [a,b]_{\mathbb{T}}} |f(t) - g(t)|, \tag{4.1.1}$$

where $f, g \in C_{\mathrm{rd}}[a, b]_{\mathbb{T}}$. The pair $(C_{\mathrm{rd}}[a, b]_{\mathbb{T}}, \varrho_\infty)$ forms the space of *rd-continuous functions with supremum metric.*

Remark 4.1.2. The mapping defined by (4.1.1) satisfies all three properties required for a metric, thus $(C_{\mathrm{rd}}[a, b]_{\mathbb{T}}, \rho_\infty)$ is a metric space. Indeed, suppose $f, g, h \in C_{\mathrm{rd}}[a, b]_{\mathbb{T}}$, then the following properties are satisfied

- identity of elements with zero distance: it is clear that

$$\varrho_\infty(f, g) = \sup_{t \in [a,b]_{\mathbb{T}}} |f(t) - g(t)| = 0,$$

  if and only if $f(t) = g(t)$ for $t \in [a, b]_{\mathbb{T}}$,

- symmetry:

$$\varrho_\infty(f, g) = \sup_{t \in [a,b]_{\mathbb{T}}} |f(t) - g(t)| = \sup_{t \in [a,b]_{\mathbb{T}}} |g(t) - f(t)| = \varrho_\infty(g, f),$$

- triangle inequality: because $|f(t) - g(t)| \le |f(t) - h(t)| + |h(t) - g(t)|$, we have for all $t \in [a, b]_{\mathbb{T}}$

$$\begin{aligned} \varrho_\infty(f, g) &= \sup_{t \in [a,b]_{\mathbb{T}}} |f(t) - g(t)| \\ &\le \sup_{t \in [a,b]_{\mathbb{T}}} |f(t) - h(t)| + \sup_{t \in [a,b]_{\mathbb{T}}} |h(t) - g(t)| \\ &= \varrho_\infty(f, h) + \varrho_\infty(h, g). \end{aligned}$$

Definition 4.1.3. Let $f \in C_{\mathrm{rd}}[a, b]_{\mathbb{T}}$, then we define the *supremum norm* of f by

$$\|f\|_\infty = \sup_{t \in [a,b]_{\mathbb{T}}} |f(t)|. \tag{4.1.2}$$

Remark 4.1.4. It is easy to show that (4.1.2) is indeed a norm, therefore $C_{\mathrm{rd}}[a,b]_{\mathbb{T}}$ and $\|\cdot\|_\infty$ form a normed space.

Theorem 4.1.5. *The space $(C_{\mathrm{rd}}[a,b]_{\mathbb{T}}, \|\cdot\|_\infty)$ is a Banach space.*

Proof. We aim to prove the theorem by showing any Cauchy sequence in $(C_{\mathrm{rd}}[a,b]_{\mathbb{T}}, \|\cdot\|_\infty)$ converges in the sense of the supremum norm to an element of this space. Let $(f_n)_{n=1}^\infty \subseteq C_{\mathrm{rd}}[a,b]_{\mathbb{T}}$ be a Cauchy sequence. For any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that for all $m, n \geq n_\varepsilon$,

$$\|f_n - f_m\|_\infty < \varepsilon.$$

Let us fix arbitrary $t \in [a,b]_{\mathbb{T}}$, then

$$|f_n(t) - f_m(t)| \leq \sup_{\tau \in [a,b]_{\mathbb{T}}} |f_n(\tau) - f_m(\tau)| = \|f_n - f_m\|_\infty < \varepsilon.$$

Hence $(f_n(t))_{n=1}^\infty$ is a Cauchy sequence in $(\mathbb{R}, |\cdot|)$. Since $(\mathbb{R}, |\cdot|)$ is complete, $(f_n(t))_{n=1}^\infty$ is convergent. We denote the limit of $(f_n(t))_{n=1}^\infty$ by

$$\lim_{n\to\infty} f_n(t) = f(t).$$

By taking the limit for all $t \in [a,b]_{\mathbb{T}}$, we construct function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$.

We need to show f is rd-continuous on $[a,b]_{\mathbb{T}}$, that means f is continuous at all right-dense points and has a finite left-sided limit at all ld-dense points. Let us fix $\varepsilon > 0$. We choose $N \in \mathbb{N}$ such that for all $n, m \geq N$

$$\|f_n - f_m\|_\infty < \frac{\varepsilon}{3}.$$

Let $\tau \in [a,b]_{\mathbb{T}}$, then for all $n > N$ and $m = N+1 > N$

$$|f_n(\tau) - f_{N+1}(\tau)| \leq \|f_n - f_{N+1}\|_\infty < \frac{\varepsilon}{3}.$$

We take the limit as $n \to \infty$ in $|f_n(\tau) - f_{N+1}(\tau)|$. Since $(f_n(t))_{n=1}^\infty$ converges in $(\mathbb{R}, |\cdot|)$, we get

$$|f(\tau) - f_{N+1}(\tau)| < \frac{\varepsilon}{3}.$$

By the assumption, we have $f_{N+1} \in C_{\mathrm{rd}}[a,b]_{\mathbb{T}}$. Suppose $t \in [a,b)_{\mathbb{T}}$ is a right-dense point, then f_{N+1} is continuous at t, i.e., there exists $\delta > 0$, such that if

$$|t - \tau| < \delta,$$

then

$$|f_{N+1}(t) - f_{N+1}(\tau)| < \frac{\varepsilon}{3}.$$

Thus

$$\begin{aligned} |f(\tau) - f(t)| &= |f(\tau) - f_{N+1}(\tau) + f_{N+1}(\tau) - f_{N+1}(t) + f_{N+1}(t) - f(t)| \\ &\leq |f(\tau) - f_{N+1}(\tau)| + |f_{N+1}(\tau) - f_{N+1}(t)| + |f_{N+1}(t) - f(t)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

This shows that f is continuous at all right-dense points.

Suppose $t \in (a,b]_{\mathbb{T}}$ is a left-dense point. Then there exists a left-sided limit $L \in \mathbb{R}$ of $(f_n)_{n=1}^{\infty}$ at t. Consequently, there exists $\delta > 0$ such that if

$$\tau \in (t-\delta, t]_{\mathbb{T}},$$

then

$$|f_{N+1}(\tau) - L| < \frac{2\varepsilon}{3}$$

holds. Thus

$$\begin{aligned} |f(\tau) - L| &= |f(\tau) - f_{N+1}(\tau) + f_{N+1}(\tau) - L| \\ &\leq |f(\tau) - f_{N+1}(\tau)| + |f_{N+1}(\tau) - L| \\ &< \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} \\ &= \varepsilon, \end{aligned}$$

which shows that f has left-sided limit at all left-dense points. Hence f is rd-continuous.

As the final step, we prove $(f_n)_{n=1}^{\infty}$ converges to f in the sense of supremum norm. Let $\varepsilon > 0$ be arbitrary and choose $N \in \mathbb{N}$ such that for all $n, m \geq N$

$$\|f_n - f_m\|_{\infty} < \frac{\varepsilon}{2}.$$

Let us fix $n \geq N$ and let $t \in [a,b]_{\mathbb{T}}$, then for all $m \geq N$

$$|f_n(t) - f_m(t)| \leq \|f_n - f_m\|_{\infty}.$$

Taking the limit as $m \to \infty$ in $\|f_n - f_m\|_{\infty}$, we get

$$\sup_{t \in [a,b]_{\mathbb{T}}} |f_n(t) - f(t)| \leq \frac{\varepsilon}{2} < \varepsilon. \tag{4.1.3}$$

We know that (4.1.3) holds for every $n > N$. We chose $\varepsilon > 0$ arbitrarily, therefore for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$

$$\|f_n - f\|_{\infty} < \varepsilon.$$

Thus, f_n converges to f in $(C_{\mathrm{rd}}[a,b]_{\mathbb{T}}, \|\cdot\|_{\infty})$.

□

Remark 4.1.6. The space of continuous functions on a closed interval with the supremum norm from Definition 4.3.2 $(C[a,b]_{\mathbb{T}}, \|\cdot\|_{\infty})$ is a Banach space in the sense of a metric generated by $\|\cdot\|_{\infty}$. The approach we would take to prove this statement is comparable to the demonstrated proof of Theorem 4.1.5.

Remark 4.1.7. It is also important to note that $C_{\mathrm{rd}}[a,b]_{\mathbb{T}}$ is closed in $C[a,b]_{\mathbb{T}}$. Therefore we could approach the proof of completeness of $(C_{\mathrm{rd}}[a,b]_{\mathbb{T}}, \|\cdot\|_{\infty})$ differently. We could first prove the completeness of $(C[a,b]_{\mathbb{T}}, \|\cdot\|_{\infty})$ and then utilize the closedness of $C_{\mathrm{rd}}[a,b]_{\mathbb{T}}$ in $C[a,b]_{\mathbb{T}}$ and employ Theorem 7.1.2.

4.2 Regressive and positively regressive functions

In this section, we focus on the spaces of regressive functions and positively regressive functions.

Definition 4.2.1 (Regressive function)**.** Let $g : \mathbb{T} \to \mathbb{R}$ be a function on the time scale $\mathbb{T}$. We say g is *regressive* if for all $t \in \mathbb{T}$

$$1 + \mu(t)g(t) \neq 0.$$

Definition 4.2.2 (Positively regressive function)**.** Let $g : \mathbb{T} \to \mathbb{R}$ be a function on the time scale $\mathbb{T}$. We say g is *positively regressive* if for all $t \in \mathbb{T}$

$$1 + \mu(t)g(t) > 0.$$

We denote the set of all regressive (positively regressive) functions on $\mathbb{T}$ by $\Gamma(\mathbb{T})$ $(\Gamma^+(\mathbb{T}))$. The set of regressive (positively regressive) functions that are in addition rd-continuous is denoted by $\mathcal{R}(\mathbb{T})$ $(\mathcal{R}^+(\mathbb{T}))$.

In order to show relation between the newly defined spaces and previously introduced spaces, we define an important mapping known as the cylinder transformation.

Definition 4.2.3 (Cylinder transformation)**.** Suppose $p : \mathbb{T} \to \mathbb{R}$ is a regressive and rd-continuous function. By *cylinder transformation* of p we understand the following function

$$\widetilde{p}(t) = \xi_\mu(p)(t) = \begin{cases} \dfrac{\operatorname{Log}(1 + \mu(t)p(t))}{\mu(t)} & \text{if } \mu(t) > 0, \\ p(t) & \text{if } \mu(x) = 0, \end{cases}$$

where $\operatorname{Log}(z)$ denotes the principal value of the complex logarithm, where $z \neq 0$.

Remark 4.2.4. Note that cylinder transformation preserves rd-continuity and if $p : \mathbb{T} \to \mathbb{R}$ is a positively regressive function, then $\widetilde{p}$ is real-valued, since complex logarithm in this case reduces to the real valued logarithm, thus $\xi_\mu(\mathcal{R}^+(\mathbb{T})) \subset C_{\mathrm{rd}}(\mathbb{T})$.

4.2.1 Circle operations

We defined (positively) regressive functions. In this section, we focus on preserving the property of (positive) regressivity while performing operations on these functions. In order to achieve this goal, we introduce circle operations, which yield yet another (positively) regressive function as their result.

Definition 4.2.5 (Circle addition)**.** Suppose $f : \mathbb{T} \to \mathbb{R}$ and $g : \mathbb{T} \to \mathbb{R}$ are regressive functions. We define *circle addition* "$\oplus$" by

$$(f \oplus g)(x) = f(x) + g(x) + \mu(x)f(x)g(x)$$

for all $x \in \mathbb{T}$.

Definition 4.2.6 (Circle subtraction)**.** Suppose $f : \mathbb{T} \to \mathbb{R}$ and $g : \mathbb{T} \to \mathbb{R}$ are regressive functions. We define a function $\ominus g : \mathbb{T} \to \mathbb{R}$ by

$$\ominus g(x) = \frac{-g(x)}{1 + \mu(x)g(x)}. \tag{4.2.1}$$

for all $x \in \mathbb{T}$. We may now define *circle subtraction* "$\ominus$" by

$$f \ominus g = f \oplus (\ominus g).$$

for $g, f \in \Gamma(\mathbb{T})$.

Remark 4.2.7. Note that addition defined this way preserves not only regressivity, but also rd-continuity. It can be checked that the circle addition is associative and commutative. Zero constant function serves as neutral element for $\Gamma(\mathbb{T})$. For any $g \in \Gamma(\mathbb{T})$, the function $\ominus g$ also preserves regressivity. Moreover, it is the inverse of g under $\oplus$, i.e., $g \oplus (\ominus g) = (\ominus g) \oplus g = 0$. This means $(\Gamma(\mathbb{T}), \oplus)$ constitutes a commutative group and each of the sets $\Gamma^+(\mathbb{T})$, $\mathcal{R}(\mathbb{T})$, $\mathcal{R}^+(\mathbb{T})$ provides subgroup of $(\Gamma(\mathbb{T}), \oplus)$.

Definition 4.2.8 (Circle scalar multiplication)**.** Suppose $\gamma \in \mathbb{R}$ and $g : \mathbb{T} \to \mathbb{R}$ is a positively regressive function. We define the *circle scalar multiplication* "$\odot$" by

$$\gamma \odot g(x) = \begin{cases} \dfrac{(1 + \mu(x)g(x))^\gamma - 1}{\mu(x)} & \text{if } \mu(x) > 0, \\ \gamma g(x) & \text{if } \mu(x) = 0. \end{cases}$$

Remark 4.2.9. The multiplication defined this way preserves not only positive regressivity but also rd-continuity. That means positively regressive functions on a time scale $\mathbb{T}$ provide a real vector space $(\Gamma^+(\mathbb{T}), \oplus, \odot)$ and the set $\mathcal{R}^+(\mathbb{T})$ is a subspace of $\Gamma^+(\mathbb{T})$.

4.2.2 Other properties

Theorem 4.2.10. *Suppose $N \subset \Gamma(\mathbb{T})$. Then the cylinder transformation $\xi_\mu : N \to \xi_\mu(N)$ is a bijection.*

Theorem 4.2.11. *Assume $(\widetilde{V}, +, \cdot)$ is a vector space of functions on $\mathbb{T}$. Let V be the set of positively regressive functions such that*

$$g \in V$$

if and only if

$$\xi_\mu(g) \in \widetilde{V}.$$

Then $(V, \oplus, \odot)$ is another vector space and $\xi_\mu : (V, \oplus, \odot) \to (\widetilde{V}, +, \cdot)$ is an isomorphism between these spaces.

Theorem 4.2.12. *Consider a normed space $((\widetilde{V}, +, \cdot), \|\cdot\|)$ of functions on a time scale $\mathbb{T}$. Then the space $(V, \oplus, \odot)$ constructed in such a way that*

$$g \in V$$

if and only if

$$\xi_\mu(g) \in \widetilde{V}$$

is also a normed space $(V, \|\cdot\|_\mu)$ *with a norm given by*

$$\|g\|_\mu = \|\xi_\mu(g)\|$$

for all $g \in V$.

Remark 4.2.13. This means $\xi_\mu : (V, \|\cdot\|_\mu) \to (\widetilde{V}, \|\cdot\|)$ is an isometry.

Next we state a theorem regarding completness of $((\mathcal{R}^+[a,b]_\mathbb{T}, \oplus, \odot), \|\cdot\|_\mu)$ space, we briefly explore two approaches to the proof.

Theorem 4.2.14. *Let* $a, b \in \mathbb{T}$*, then the normed linear space* $((\mathcal{R}^+[a,b]_\mathbb{T}, \oplus, \odot), \|\cdot\|_\mu)$ *is complete.*

First proof of Theorem 4.2.14. Suppose $(f_n)_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{R}^+[a,b]_\mathbb{T}$ with respect to $\|\cdot\|_\infty$. To prove this theorem directly, we need to show that $(f_n(t))_{n=1}^\infty$ converges to some $f(t) \in \mathbb{R}$ for all $t \in \mathbb{T}$. Then we would show that a function $f : \mathbb{T} \to \mathbb{R}$ constructed this way is positively regressive. We proceed by proving that $(f_n)_{n=1}^\infty$ converges to f in the sense of $\|\cdot\|_\mu$, i.e.,

$$\lim_{n\to\infty} \|f_n \ominus f\|_\mu = 0.$$

As a final step, we would prove that p is rd-continuous. □

The proof, whose idea we have just presented, is quite complicated. There is a much simpler way to prove Theorem 4.2.14 thanks to the following theorem.

Theorem 4.2.15. *The space* $(\mathcal{R}^+[a,b]_\mathbb{T}, \|\cdot\|_\mu)$ *is isometrically isomorphic to the space* $(C_{\mathrm{rd}}[a,b]_\mathbb{T}, \|\cdot\|_\infty)$.

Proof. To prove this theorem, we have to find a bijective linear mapping $F : \mathcal{R}^+[a,b]_\mathbb{T} \to C_{\mathrm{rd}}[a,b]_\mathbb{T}$, which preserves the norm, i.e., for every $f \in \mathcal{R}^+[a,b]_\mathbb{T}$

$$\|F(f)\|_\mu = \|f\|_\infty .$$

Theorem 4.2.12 shows that the cylinder transform ξ_μ satisfies these requirements and therefore the spaces are isometrically isomorphic. □

Second proof of Theorem 4.2.14. Since $(C_{\mathrm{rd}}[a,b]_\mathbb{T}, \|\cdot\|_\infty)$ is isometrically isomorphic to $(\mathcal{R}^+[a,b]_\mathbb{T}, \|\cdot\|_\mu)$, we can utilize Theorem 7.3.4 and prove $(\mathcal{R}^+[a,b]_\mathbb{T}, \|\cdot\|_\mu)$ is a Banach space by proving Theorem 4.1.5. □

4.3 Bounded continuous functions on noncompact interval

Definition 4.3.1 (Bounded continuous functions on noncompact interval)**.** Let $a \in \mathbb{T}$ and $[a,\infty)_\mathbb{T}$. Consider a set of continuous functions $f : [a,\infty)_\mathbb{T} \to \mathbb{R}$, such that $\sup_{t\in[a,\infty)_\mathbb{T}} |f(t)| < \infty$ and let us denote it by $BC[a,\infty)_\mathbb{T}$. We define a metric ϱ_∞ on $BC[a,\infty)_\mathbb{T}$ by

$$\varrho_\infty(f,g) = \sup_{t\in[a,\infty)_\mathbb{T}} |f(t) - g(t)|, \tag{4.3.1}$$

where $f,g \in BC[a,\infty)_\mathbb{T}$. The pair $(BC[a,\infty)_\mathbb{T}, \rho_\infty)$ forms the space of *bounded continuous functions on noncompact interval with supremum metric.*

Definition 4.3.2. Let $f \in BC[a,\infty)_\mathbb{T}$, then we define the norm of f by

$$\|f\|_\infty = \sup_{t\in[a,\infty)_\mathbb{T}} |f(t)|. \tag{4.3.2}$$

Theorem 4.3.3. *The space* $(BC[a,\infty)_\mathbb{T}, \|\cdot\|_\infty)$ *is a Banach space.*

Proof. Our goal is to prove the theorem by showing that any Cauchy sequence $(f_n)_{n=1}^\infty$ in $(BC[a,\infty)_\mathbb{T}, \|\cdot\|_\infty)$ converges in the sense of the norm defined by (4.3.2) to an element of the space. Let $(f_n)_{n=1}^\infty \subseteq (BC[a,\infty)_\mathbb{T}$ be a Cauchy sequence. For any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that for all $m,n \geq n_\varepsilon$,

$$\|f_n - f_m\|_\infty < \varepsilon.$$

Utilizing the ideas of the proof of Theorem 4.1.5, we apply

$$f(t) = \lim_{n\to\infty} f_n(t).$$

for all $t \in [a,\infty)_\mathbb{T}$ and obtain $f : [a,\infty)_\mathbb{T} \to \mathbb{R}$. Analogically to the proof of Theorem 4.1.5, we show f is continuous and $(f_n)_{n=1}^\infty$ converges to a bounded function f in the sense of norm defined by (4.3.2).

□

Definition 4.3.4 (Continuous convergent functions on noncompact interval)**.** Let $a \in \mathbb{R}$, suppose $\mathbb{T}$ is a time scale and consider $[a,\infty)_\mathbb{T}$. Consider a set of functions $f \in BC[a,\infty)_\mathbb{T}$ such that there exists $L \in \mathbb{R}$ and

$$\lim_{t\to\infty} f(t) = L$$

and denote it by $C_L[a,\infty)_\mathbb{T}$. We define a metric ρ_∞ on $C_L[a,\infty)_\mathbb{T}$ by (4.3.1), where $f,g \in C_L[a,\infty)_\mathbb{T}$. The pair $(C_L[a,\infty)_\mathbb{T}, \rho_\infty)$ forms the space of *continuous convergent functions on noncompact interval with supremum metric.*

Remark 4.3.5. Analogically to $BC[a,\infty)_\mathbb{T}$ space, $C_L[a,\infty)_\mathbb{T}$ with the norm defined by (4.3.2) is a Banach space.

Remark 4.3.6. It is important to realise that analogically to $BC[a,\infty)_\mathbb{T}$ and $C_L[a,\infty)_\mathbb{T}$, it is possible to study *bounded rd-continuous functions on noncompact interval* denoted by $BC_{\mathrm{rd}}[a,\infty)_\mathbb{T}$ and *rd-continuous convergent functions on noncompact interval* denoted by $C_L^{\mathrm{rd}}[a,\infty)_\mathbb{T}$.

4.4 Lebesgue delta spaces

Since the delta measure and the Lebesgue delta integral defined in Chapter 3 can be included in general measure theory, we may define the Lebesgue delta spaces on time scales analogically to general measure theory.

Definition 4.4.1 ($\mathcal{L}^p_{\mathbb{T}}(E)$)**.** Suppose $p \in [1, \infty)$ and let $E \subset \mathbb{T}$ be a delta measurable set and $f : \mathbb{T} \to [-\infty, \infty]$ be a delta measurable function. Then if

$$\int_E |f(s)|^p \, \Delta s < \infty,$$

we say f belongs to $\mathcal{L}^p_{\mathbb{T}}(E)$.

Definition 4.4.2. Let $f, g \in \mathcal{L}^p_{\mathbb{T}}(E)$, we define a metric ϱ_p on $\mathcal{L}^p_{\mathbb{T}}(E)$ by

$$\varrho_p(f, g) = \left(\int_E |f(s) - g(s)|^p \, \Delta s \right)^{\frac{1}{p}}.$$

Remark 4.4.3. Similarly to the classical calculus, the problem arises, since for $f, g \in \mathcal{L}^p_{\mathbb{T}}(E)$

$$\rho_p(f, g) = 0$$

does not imply

$$f = g$$

and ρ_p is in fact not a metric on $\mathcal{L}^p_{\mathbb{T}}(E)$. We therefore utilize "almost everywhere equal" relation $\sim$. For $f, g \in \mathcal{L}^p_{\mathbb{T}}(E)$, we set

$$f \sim g$$

if $f(x) = g(x)$ for $x \in E \setminus M$, where $\mu_\Delta(M) = 0$. We consider the Lebesgue delta space as $L^p_{\mathbb{T}}(E) = \mathcal{L}^p_{\mathbb{T}}(E)/\sim$, i.e., as classes of equivalence $\sim$.

Definition 4.4.4. Let $f \in L^p_{\mathbb{T}}(E)$, we define the norm of f by

$$\|f\|_p = \left(\int_E |f(s)|^p \, \Delta s \right)^{\frac{1}{p}}.$$

Theorem 4.4.5. *The space $L^p_{\mathbb{T}}(E)$ is a Banach space.*

Proof. The proof follows from the completeness of the general space with measure. □

4.5 Relative compactness

In this section, we focus on relative compactness in some of the previously introduced spaces.

Theorem 4.5.1 (Relative compactness in $BC[a, \infty)_{\mathbb{T}}$)**.** *Let $N \subset BC[a, \infty)_{\mathbb{T}}$ be bounded, and assume that for every $\varepsilon > 0$, there exists a partition of $[a, \infty)_{\mathbb{T}}$ in a finite numbers of time scale intervals $I_1, \dots, I_n$, such that $\sup_{t,s \in I_i} |f(t) - f(s)| < \varepsilon$ for every $i = \{1, \dots, n\}$ and every $f \in N$. Then N is relatively compact.*

Proof. The space $(BC[a,\infty)\mathbb{T}, \|\cdot\| \infty)$ is Banach. That means, according to Theorem 7.2.5, it is sufficient to prove that N is totally bounded. In order to accomplish that, we need to construct a finite ε-net for arbitrarily chosen ε>0. Let us denote $t_0 = a$ and consider the partition

$$I_1 = [t_0, t_1]_{\mathbb{T}},\ I_2 = [t_1, t_2]_{\mathbb{T}}, \dots, I_{n-1} = [t_{n-2}, t_{n-1}]_{\mathbb{T}},\ I_n = [t_{n-1}, \infty)_{\mathbb{T}},$$

where $t_i \in \mathbb{T}$ for $i \in \{0, 1, \dots, n-1\}$, $t_i \leq t_{i+1}$ for $i \in \{0, 1, \dots, n-2\}$ and $|f(t) - f(s)| < \varepsilon/5$ for all $s, t \in I_j$, where $j \in \{0, 1, \dots, n\}$ and $f \in X$ and $\varepsilon > 0$ is given.

Let $\varepsilon > 0$ be arbitrarily given. Let L be such that $\|f\| \leq L$ for all $f \in N$. We know that such L exists thanks to boundedness of N. We now take $y_1, y_2, \dots, y_m \in \mathbb{R}$, such that $-L = y_1 < y_2 < \cdots < y_m = L$ and $y_{i+1} - y_i < \varepsilon/5$ for $i \in \{1, \dots, m-1\}$, as the vertical values of the grid. The horizontal values of the grid are the numbers $x_1, \dots, x_k$, where $x_1 < x_2 < \cdots < x_k$ and

$$\{x_1, x_2, \dots, x_k\} = \{t_0, t_1, \dots t_{n-1}\}.$$

For any $f \in N$, we might now construct a polygon g defined on $[t_0, \infty)$ and linear on (x_j, x_{j+1}) for all $j \in \{0, 1, \dots, k-1\}$ passing through the lattice points closest to the graph of f. Moreover, let g be constant on $[x_k, \infty)$ with the value $v \in \{y_1, y_2, \dots, y_m\}$ such that

$$|f(x_k) - v| = \min_{i \in \{1,2,\dots,m\}} |f(x_k) - y_i|.$$

Now we restrict g to $[t_0, \infty)_{\mathbb{T}}$ and take it as an approximation of f. Suppose x_j is the closest member of $\{x_1, \dots, x_k\}$ to t. For $t = x_j$ the situation is trivial and

$$|f(t) - g(t)| = |f(x_j) - g(x_j)| < \frac{\varepsilon}{5}$$

holds.

On the other hand consider $t \neq x_j$, then

$$\begin{aligned} |f(t) - g(t)| &\leq |f(t) - f(x_j)| + |f(x_j) - g(x_j)| + |g(x_j) - g(t)| \\ &< \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + |g(x_j) - g(t)|. \end{aligned} \tag{4.5.1}$$

The polygon g is monotone between adjacent x_j', that means for $t > x_j$

$$|g(x_j) - g(t)| \leq |g(x_j) - g(x_{j+1})|,$$

where $j \in \{1, \dots, m-1\}$. Thus

$$\begin{aligned} |g(x_j) - g(t)| &\leq |g(x_j) - f(x_j)| + |f(x_j) - f(x_{j+1})| + |f(x_{j+1}) - g(x_{j+1})| \\ &< \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} \end{aligned} \tag{4.5.2}$$

for $t > x_j$. For $t < x_j$ we would make the estimation analogically, instead of x_{j+1} we would consider x_{j-1}.

Now (4.5.1) and (4.5.2) yield

$$\|f - g\|_{\infty} < \varepsilon.$$

Because the number of all paths through the grid is finite (equal to m^n), the set of functions g constructed above forms a finite ε-net and N is therefore totally bounded and since $(BC_{\mathrm{rd}}[a,\infty)_{\mathbb{T}}, \|\cdot\|_{\infty})$ is a Banach space also relatively compact. □

Next, we focus on the spaces $C_L[a,\infty)_{\mathbb{T}}$ and $C[a,b]_{\mathbb{T}}$. We need to introduce modified notions of equicontinuity and equiboundness on time scales.

Definition 4.5.2. Let $a, b \in \mathbb{T}$ such that $a < b$ and consider a set N of functions $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$.

- We say functions in N are *equibounded* on $[a,b]_{\mathbb{T}}$ if there exists a positive real number L satisfying
$$|f(t)| \leq L$$
for all $t \in [a,b]_{\mathbb{T}}$ and every $f \in N$.

- We say functions in N are *equicontinuous* on $[a,b]_{\mathbb{T}}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t, s \in [a,b]_{\mathbb{T}}$ satisfying $|t-s| < \delta$ and for all $f \in N$
$$|f(t) - f(s)| < \varepsilon$$
holds.

Theorem 4.5.3 (Relative compactness in $C_L[a,\infty)_{\mathbb{T}}$)**.** *Let $N \subset C_L[a,\infty)_{\mathbb{T}}$ consist of equicontinuous and equibounded functions in every compact subinterval of $[a,\infty)_{\mathbb{T}}$ and suppose that for any $\varepsilon > 0$ there exist $t_0 > 0$ such that for all $t > t_0$*
$$|f(t) - \lim_{s\to\infty} f(s)| < \varepsilon$$
holds for all $f \in N$. Then N is relatively compact.

Theorem 4.5.4 (Relative compactness in the space of continuous functions)**.** *Let $N \subset C[a,b]_{\mathbb{T}}$ and suppose every sequence $(f_n)_{n=1}^{\infty} \subseteq N$ is made of equicontinuous and equibounded functions in $[a,b]_{\mathbb{T}}$. Then N is relatively compact.*

Remark 4.5.5. The two theorems stated above can be proven using ideas similar to those of the proof of Theorem 4.5.1.

Remark 4.5.6. As far as we know, there does not exist a time scale analogy for the Fréchet–Kolmogorov theorem, which gives a necessary and sufficient condition for relative compactness in L^p spaces. The criterion for relative compactness in the Lebesgue delta spaces has not been derived yet.

5 Generalized exponential function

In this chapter, we introduce and explore the generalized exponential function on time scales. We use [3], [10] as our main sources.

5.1 Construction

The cylinder transformation, defined by (4.2.3), plays an essential role in the construction of generalized exponential function. We use this notion in the following definition.

Definition 5.1.1 (Generalized exponential function)**.** Let $t, t_0 \in \mathbb{T}$ assume $p : \mathbb{T} \to \mathbb{R}$ is a regressive function, then we define the *generalized exponential function* by

$$e_p(t, t_0) = \exp\left\{ \int_{t_0}^{t} \xi_\mu(p)(\tau)\Delta\tau \right\}.$$

Theorem 5.1.2. *Suppose $p \in \mathcal{R}(\mathbb{T})$ and fix $t_0 \in \mathbb{T}$. Then the generalized exponential function $e_p(\cdot, t_0)$ is the unique solution $y : \mathbb{T} \to \mathbb{R}$ of the dynamic initial value problem*

$$y^\Delta = p(t)y, \quad y(t_0) = 1.$$

for all $t \in \mathbb{T}$.

Remark 5.1.3. We defined the generalized exponential function using the cylinder transform. It should be noted that alternative approach consists of defining generalized exponential function as the solution of initial value problem, which is possible thanks to Theorem 5.1.2.

5.2 Examples

We show examples of the generalized exponential function for several time scales derived as the solution of initial value problem

$$y^\Delta = p(t)y, \quad y(t_0) = 1.$$

for all $t \in \mathbb{T}$. We do not provide details on the case when $\mathbb{T} = \mathbb{R}$, since the generalized exponential function in this context is evidently the same as the well/known exponential function from the classical calculus.

Example 5.2.1 ($h\mathbb{Z}$)**.** Consider $\mathbb{T} = h\mathbb{Z}$ for $h > 0$ and let $\alpha \in \mathcal{R}(\mathbb{T})$ be a constant function, i.e., $\alpha \in \mathbb{R}$, then

$$e_\alpha(t, 0) = (1 + \alpha h)^{\frac{t}{h}} \tag{5.2.1}$$

for all $t \in \mathbb{T}$.

Indeed, y defined by (5.2.1) satisfies

$$y(0) = (1 + \alpha h)^0 = 1$$

and

$$\begin{aligned} y^{\Delta}(t) &= \frac{y(t+h)-y(t)}{h} \\ &= \frac{(1+\alpha h)^{\frac{t+h}{h}} - (1+\alpha h)^{\frac{t}{h}}}{h} \\ &= \frac{(1+\alpha h)^{\frac{t}{h}}(1+\alpha h-1)}{h} \\ &= \alpha(1+\alpha h)^{\frac{t}{h}} \\ &= \alpha y(t). \end{aligned}$$

Example 5.2.2 ($q^{\mathbb{N}_0}$)**.** Let $\mathbb{T} = q^{\mathbb{N}_0}$ for $h > 0$ and let $p \in \mathcal{R}(\mathbb{T})$, then the problem

$$y^{\Delta} = p(t)y, \quad y(1) = 1$$

can be rewritten as

$$y^{\sigma} = (1 + (q-1)tp(t))y, \quad y(1) = 1$$

for all $t \in \mathbb{T}$. The solution of this problem is then

$$e_p(t,1) = \prod_{s \in \mathbb{T} \cap (0,t)} (1 + (q-1)sp(s)). \tag{5.2.2}$$

If $\alpha \in \mathcal{R}(\mathbb{T})$ is constant, then we have

$$e_{\alpha}(t,1) = \prod_{s \in \mathbb{T} \cap (0,t)} (1 + (q-1)\alpha s).$$

Indeed, y defined by (5.2.2) clearly satisfies

$$y(1) = \prod_{s \in \mathbb{T} \cap (0,1)} (1 + (q-1)p(s)s) = 1$$

and

$$\begin{aligned} y^{\Delta} &= \frac{y^{\sigma}(t) - y(t)}{(\mu(t)} \\ &= \frac{y(qt) - y(t)}{(q-1)t} \\ &= \frac{(1+(q-1)tp(t))\prod_{s\in\mathbb{T}\cap(0,t)}(1+(q-1)sp(s)) - \prod_{s\in\mathbb{T}\cap(0,t)}(1+(q-1)sp(s))}{(q-1)t} \\ &= \frac{(q-1)tp(t)\prod_{s\in\mathbb{T}\cap(0,t)}(1+(q-1)sp(s))}{(q-1)t} \\ &= p(t)\prod_{s\in\mathbb{T}\cap(0,t)}(1+(q-1)sp(s)) \\ &= p(t)y(t). \end{aligned}$$

6 Analysis of dynamic equations on time scales

In this chapter, we employ the tools from the previous chapters to analyze the qualitative properties of selected dynamic equations.

6.1 Second order nonlinear dynamic equation

In this section, we explore the equation

$$y^{\Delta\Delta} = p(t)g(y^{\sigma}), \tag{6.1.1}$$

which is considered on the interval of the form $[a,\infty)_{\mathbb{T}}$. We suppose $p : [a,\infty)_{\mathbb{T}} \to \mathbb{R}$ is an rd-continuous function such that $p(t) > 0$ for all $t \in [a,\infty)_{\mathbb{T}}$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

$$xg(x) > 0$$

for all $x \neq 0$. For this equation we study a solution, by that we mean a function $y \in C^2_{\mathrm{rd}}[a,\infty)_{\mathbb{T}}$ (an rd-continuous function with rd-continuous first and second delta derivatives) satisfying (6.1.1) for all $t \in [a,\infty)_{\mathbb{T}}$. We aim to show that there exists a solution of (6.1.1) such that

$$\left.\begin{aligned} &y^{\Delta}(t) < 0, \\ &y(t) > 0 \quad \text{for large } t, \\ &\lim_{t\to\infty} y(t) = c, \end{aligned}\right\} \tag{6.1.2}$$

where c is a given positive real number. We also discuss the existence of solutions having a positive limit, which are positive and decreasing on the entire interval $[a,\infty)_{\mathbb{T}}$.

6.1.1 Conditions for existence of solution

In this section, we derive a condition necessary and sufficient for the existence of a solution to (6.1.1) with the properties (6.1.9) for a given $c > 0$.

Theorem 6.1.1. *The coeffient p in the equation (6.1.1) satisfies*

$$\int_a^{\infty}\int_t^{\infty} p(s)\,\Delta s\Delta t < \infty \tag{6.1.3}$$

if and only if there exists a solution of (6.1.1), (6.1.2) for arbitrarily chosen positive real number c.

Remark 6.1.2. The condition (6.1.3) with an additional (but very non-restrictive) assumption

$$\lim_{t\to\infty} t\int_t^{\infty} p(s)\,\Delta s = 0 \tag{6.1.4}$$

is in our case equivalent to

$$\int_a^{\infty} \sigma(t)p(t)\,\Delta t < \infty.$$

Indeed, we may show this using integration by parts as follows

$$\int_a^\infty \int_t^\infty p(s)\,\Delta s \Delta t = \int_a^\infty \left(1 \cdot \int_t^\infty p(s)\,\Delta s\right) \Delta t.$$

Now using additional assumption (6.1.4)

$$\int_a^\infty \left(1 \cdot \int_t^\infty p(s)\,\Delta s\right) \Delta t = \lim_{x\to\infty} \left[(t-a)\int_t^\infty p(s)\,\Delta s\right]_a^x + \int_a^\infty (\sigma(t)-a)p(t)\,\Delta t$$
$$= \int_a^\infty (\sigma(t)-a)p(t)\,\Delta t.$$

Therefore

$$\int_a^\infty \sigma(t)p(t)\,\Delta t < \infty$$

if and only if (6.1.3) holds.

Proof of Theorem 7.1.1. First, we prove the implication from right to left. Suppose y is a solution of (6.1.1) with properties (6.1.2) for fixed $c > 0$. Therefore $y(t) > 0$ and $y^\Delta(t) < 0$ for large t, say $t \geq t_0$ for some $t_0 \in [a,\infty)_\mathbb{T}$. We utilize the relation (3.1.6), i.e.,

$$\int_t^s y^\Delta(x)\Delta x = y(s) - y(t) \tag{6.1.5}$$

for arbitrary $t, s \in \mathbb{T}$ such that $t, s > 0$ and $t \leq s$. By integrating the equation (6.1.1) from a to b, we get

$$y^\Delta(s) - y^\Delta(t) = \int_t^s p(x)g(y^\sigma(x))\Delta x. \tag{6.1.6}$$

Since $p(t) > 0$ for all $t \in [a,\infty)_\mathbb{T}$ and $y(t) > 0$ for $t \geq t_0$, i.e., $g(y^\sigma(t)) > 0$ for $t \geq t_0$, we get for $t \geq t_0$

$$y^{\Delta\Delta}(t) > 0, \tag{6.1.7}$$

thus y^Δ is $t \geq t_0$ increasing. For $t \geq t_0$

$$y^\Delta(t) < 0, \tag{6.1.8}$$

also holds. From (6.1.7) and (6.1.8) we obtain

$$\lim_{t\to\infty} y^\Delta(t) = L,$$

where $L \leq 0$. Suppose $L < 0$, then $y(t) \sim Lt$, i.e.,

$$\lim_{t\to\infty} \frac{Lt}{y(t)} = 1,$$

which is in contradiction with

$$\lim_{t\to\infty} y(t) = c.$$

Therefore

$$\lim_{s\to\infty} y^\Delta(s) = L = 0.$$

By taking the limit in (6.1.6) as $s \to \infty$, we obtain

$$0 - y^{\Delta}(t) = \int_t^{\infty} p(x) g(y^{\sigma}(x))) \, \Delta x.$$

We apply (6.1.5) again and get

$$y(s) = y(u) + \int_s^u \int_t^{\infty} p(x) g(y^{\sigma}(x)) \, \Delta x \Delta t.$$

Now we take the limit as $u \to \infty$ and since

$$\lim_{u \to \infty} y(u) = c,$$

we get

$$y(s) = c + \int_s^{\infty} \int_t^{\infty} p(x) g(y^{\sigma}(x)) \, \Delta x \Delta t.$$

This means

$$\int_{t_0}^{\infty} \int_t^{\infty} p(x) g(y^{\sigma}(x)) \, \Delta x \Delta t < \infty.$$

Thanks to $\lim_{t \to \infty} y(t) = c$, there exists $K > 0$ such that $g(y^{\sigma}(t)) \geq K$ for all $t > a$, therefore

$$K \int_{t_0}^{\infty} \int_t^{\infty} p(x) \, \Delta x \Delta t \leq \int_{t_0}^{\infty} \int_t^{\infty} p(x) g(y^{\sigma}(x)) \, \Delta x \Delta t < \infty$$

and thus

$$\int_{t_0}^{\infty} \int_t^{\infty} p(x) \, \Delta x \Delta t < \infty,$$

which clearly implies (6.1.3). The implication from right to left is proven.

Now we prove the implication from left to right. Using the Schauder fixed point theorem (Theorem 7.4.6), we prove the existence of a solution to (6.1.1) satisfying

$$\left. \begin{array}{l} y^{\Delta}(t) < 0, \quad t \geq t_0 \\ y(t) > 0, \quad t \geq t_0, \\ \lim_{t \to \infty} y(t) = c, \end{array} \right\} \tag{6.1.9}$$

where $t_0 \in \mathbb{T}$ is specified later. Let us consider a real positive number c and denote

$$M = \max_{t \in [c, 2c]} g(t).$$

The integral $\int_a^{\infty} \int_t^{\infty} p(t) \, \Delta t$ converges. This means that for any $\varepsilon > 0$, there exists $t_{\varepsilon} \geq a$ such that

$$\int_{t_{\varepsilon}}^{\infty} \int_t^{\infty} p(x) \, \Delta x \Delta t < \varepsilon.$$

Set $\varepsilon = c/M$ and denote corresponding t_{ε} by t_0. We have previously established that $(BC[t_0, \infty)_{\mathbb{T}}, \|\cdot\|_{\infty})$ is a Banach space for any $t_0 \in \mathbb{T}$ (Theorem 4.3.3). Our current objective is to identify a suitable set $\Omega \subseteq BC[t_0, \infty)_{\mathbb{T}}$ and an operator T such that there exists a fixed point of T in Ω, which corresponds to a solution of (6.1.1) with the desired properties (6.1.9).

The set Ω is considered in the form

$$\Omega = \{f \in BC[t_0, \infty), c \leq f(t) \leq 2c \text{ for } t \geq t_0\}.$$

This set is clearly nonempty and bounded. We need to show that it is closed. Consider a sequence $(f_n)_{n=1}^{\infty} \subset \Omega$ converging to some f in the sense of $\|\cdot\|_{\infty}$. We must show that f belongs to Ω. Since the convergence of a sequence of continuous functions $(f_n)_{n=1}^{\infty}$ in the sense of $\|\cdot\|_{\infty}$ norm is in fact uniform (see proof of Theorem 4.1.5), f is continuous. we can conclude that f is continuous. Next, we need to establish that f satisfies the defining inequality of Ω. It stands that for any $\varepsilon > 0$, there exists $n_{\varepsilon} \in \mathbb{N}$ such that for $n \geq n_{\varepsilon}$

$$\sup_{t \in [t_0, \infty)_{\mathbb{T}}} |f_n(t) - f(t)| < \varepsilon.$$

Let us fix an arbitrary $t \geq t_0$, for such t

$$\lim_{n \to \infty} f_n(t) = f(t).$$

Then for all $n \in \mathbb{N}$ and fixed t

$$c \leq f_n(t) \leq 2c$$

holds, therefore

$$c \leq f(t) \leq 2c.$$

Since t is chosen arbitrarily, $f \in \Omega$ and therefore Ω is closed.

To prove that Ω is a convex set, we must demonstrate that for any two functions f_1 and f_2 in Ω, their convex combination is also in Ω. Let λ be an arbitrary scalar between 0 and 1. We want to show that $\lambda f_1 + (1 - \lambda) f_2$ belongs to Ω. For $t \geq t_0$

$$\lambda c \leq \lambda f_1(t) \leq \lambda 2c \tag{6.1.10}$$

and

$$(1 - \lambda)c \leq (1 - \lambda) f_2(t) \leq (1 - \lambda)2c. \tag{6.1.11}$$

Now by adding (6.1.10) and (6.1.11) we obtain for $t \geq t_0$

$$c \leq \lambda f(t) + (1 - \lambda) g(t) \leq 2c. \tag{6.1.12}$$

It is evident that $\lambda f_1 + (1-\lambda) f_2 \in BC[t_0, \infty)_{\mathbb{T}}$, therefore thanks to (6.1.12) $\lambda f_1 + (1-\lambda) f_2 \in \Omega$ and the set Ω is convex.

We define the mapping T on Ω for $t \geq t_0$ as follows

$$(Tf)(t) = c + \int_t^{\infty} \int_s^{\infty} p(x) g(f^{\sigma}(x)) \, \Delta x \Delta s,$$

where $f \in \Omega$. We need to prove that for $t \geq t_0$

$$c \leq (Tf)(t) \leq 2c. \tag{6.1.13}$$

Since $p(t) > 0$ for $t \geq a$, $xg(x) > 0$ for $x \neq 0$ and $f^{\sigma}(t) \geq c > 0$ for $t \geq t_0$, it is true that for $t \geq t_0$

$$\int_t^{\infty} \int_s^{\infty} p(x) g(f^{\sigma}(x)) \, \Delta x \Delta s \geq 0.$$

Therefore the first inequality of (6.1.13) holds. To prove the second inequality, we need to show that for $t \geq t_0$

$$\int_t^\infty \int_s^\infty p(x)g(f^\sigma(x))\,\Delta x \Delta s \leq c.$$

We make upper estimates of

$$\int_t^\infty \int_s^\infty p(x)g(f^\sigma(x))\,\Delta x \Delta s$$

as follows

$$\int_t^\infty \int_s^\infty p(x)g(f^\sigma(x))\,\Delta x \Delta s \leq M \int_t^\infty \int_s^\infty p(x)\,\Delta x \Delta s \leq M \int_{t_0}^\infty \int_s^\infty p(x)\,\Delta x \Delta s.$$

Since

$$\int_{t_0}^\infty \int_s^\infty p(x)\,\Delta x \Delta s \leq \frac{c}{M},$$

we may continue with

$$M \int_{t_0}^\infty \int_s^\infty p(x)\,\Delta x \Delta s \leq M \frac{c}{M} = c.$$

Therefore $Tf \in \Omega$.

Next, we need to prove that T is continuous. Let $(f_n)_{n=1}^\infty \subseteq \Omega$ be such that

$$\lim_{n\to\infty} \|f_n - f\|_\infty = 0.$$

We need to prove that

$$\lim_{n\to\infty} \|Tf_n - Tf\|_\infty = 0.$$

Let us denote $A_n = Tf_n - Tf$ and focus on the expression $\|A_n\|_\infty$ and rewrite it as follows

$$\begin{aligned}
\sup_{t\geq t_0} |A_n(t)| &= \sup_{t\geq t_0} \left| \int_t^\infty \int_s^\infty p(x)g(f_n^\sigma(x))\,\Delta x \Delta s - \int_t^\infty \int_s^\infty p(x)g(f^\sigma(x))\,\Delta x \Delta s \right| \\
&= \sup_{t\geq t_0} \left| \int_t^\infty \int_s^\infty p(x)(g(f_n^\sigma(x)) - g(f^\sigma(x)))\,\Delta x \Delta s \right| \\
&\leq \int_{t_0}^\infty \int_s^\infty p(x)|(g(f_n^\sigma(x)) - g(f^\sigma(x)))|\,\Delta x \Delta s.
\end{aligned}$$

Let us now denote the function $g(f_n^\sigma) - g(f^\sigma)$ by G_n. The function $p|G_n|$ satisfies for all $n \in \mathbb{N}$ and all $t \geq t_0$

$$p(t)|G_n(t)| \geq 0.$$

Since f_n converges uniformly to f and g is continuous,

$$\lim_{n\to\infty} p(t)|G_n(t)| = 0$$

holds for all $t \geq t_0$ (pointwise convergence). The function g is continuous and f and f_n for all $n \in \mathbb{N}$ are bounded functions, therefore there exists $L \geq 0$ such that $|G_n(t)| \leq L$ for all $t \geq t_0$ and $n \in \mathbb{N}$ and thus

$$p(t)|G_n(t)| \leq Lp(t).$$

We know that $\int_a^\infty \int_s^\infty p(x)\,\Delta x\Delta s < \infty$, then $\int_{t_0}^\infty p(x)\,\Delta x < \infty$. Therefore the function $p|G_n|$ satisfies the assumptions of the Lebesgue dominated convergence theorem (Theorem 3.3.11). Moreover the function

$$\int_t^\infty p(x)|G_n(x)|\,\Delta x \tag{6.1.14}$$

is also nonnegative for $t \geq t_0$ and for $n \in \mathbb{N}$ satisfies

$$\int_t^\infty p(x)|G_n(x)|\,\Delta x \leq L\int_t^\infty p(x)\,\Delta x.$$

We know that

$$\int_a^\infty \int_s^\infty p(x)|G_n(x)|\,\Delta x\Delta s < \infty$$

and therefore

$$\int_{t_0}^\infty \int_s^\infty p(x)|G_n(x)|\,\Delta x\Delta s < \infty.$$

This means the function (6.1.14) also satisfies assumptions of Theorem 3.3.11. We may now apply this theorem twice as follows

$$\begin{aligned}\lim_{t\to\infty}\int_t^\infty \int_s^\infty p(x)|G_n(x)|\,\Delta x\Delta s &= \int_t^\infty \int_s^\infty p(x)\lim_{x\to\infty}|G_n(x)|\,\Delta x\Delta s\\ &= \int_t^\infty \int_s^\infty p(x)0\,\Delta x\Delta s\\ &= 0.\end{aligned}$$

Consequently,

$$\lim_{n\to\infty}\|A_n\|_\infty = 0$$

and the mapping T is therefore continuous.

It remains to prove $T\Omega$ is relatively compact. We operate on the space $BC[t_0,\infty)_\mathbb{T}$. Therefore to prove relative compactness, we need to demonstrate validity of the assumptions of Theorem 4.5.1. Evidently, $T\Omega$ is bounded, since $T\Omega \subseteq \Omega$ and Ω is bounded. We need to show that we might, for arbitrary $\varepsilon > 0$, divide $[t_0,\infty)_\mathbb{T}$ into subintervals $I_1, I_2, \ldots, I_k$ such that

$$\sup_{t_1,t_2\in I_i}|Tf(t_1) - Tf(t_2)| < \varepsilon \tag{6.1.15}$$

for $i \in \{1, 2, \ldots k\}$ for $f \in \Omega$.

Fix $\varepsilon > 0$ and suppose $f \in \Omega$. Since

$$\lim_{t\to\infty}\int_t^\infty \int_s^\infty p(x)g(f^\sigma(x))\,\Delta x\Delta s = 0,$$

there exists t^* such that for all $t_1, t_2 \geq t^*$

$$\begin{aligned}|Tf(t_1) - Tf(t_2)| = \Bigg|&\int_{t_1}^\infty \int_s^\infty p(x)g(f^\sigma(x))\,\Delta x\Delta s\\ &- \int_{t_2}^\infty \int_s^\infty p(x)g(f^\sigma(x))\,\Delta x\Delta s\Bigg| < \varepsilon.\end{aligned} \tag{6.1.16}$$

On the other hand Tf is delta-differentiable on $[t_0, t^*)_{\mathbb{T}}$ and there exists $K > 0$ such that for $t \geq t_0$

$$|(Tf)^{\Delta}(t)| = \left| \int_t^{\infty} p(x)g(f^{\sigma}(x))\, \Delta x \right| \leq K.$$

We utilize this fact and employ the mean value theorem (Theorem 2.4.1). Therefore for any $t_1, t_2 \in [t_0, t^*)_{\mathbb{T}}$ such that $t_1 < t_2$, there exists $\xi \in [t_1, t_2]_{\mathbb{T}}$

$$|Tf(t_2) - Tf(t_1)| \leq |T^{\Delta} f(\xi)||t_2 - t_1|.$$

If we choose $t_1, t_2 \in [t_0, t^*)_{\mathbb{T}}$ to satisfy

$$|t_2 - t_1| < \frac{\varepsilon}{K},$$

then

$$|Tf(t_2) - Tf(t_1)| \leq |T^{\Delta} f(\xi)||t_2 - t_1| < K\frac{\varepsilon}{K} = \varepsilon. \tag{6.1.17}$$

Using (6.1.16) and (6.1.17), we may produce the desired division of $[t_0, \infty)_{\mathbb{T}}$ and thus $T\Omega$ is relatively compact.

We have proven validity of all assumptions of the Schauder fixed point theorem, therefore there exist (at least one) fixed point y of the mapping T, i.e.,

$$Ty = y. \tag{6.1.18}$$

Now taking the second delta derivative of (6.1.18), we get clearly (6.1.1). Thanks to the form of the set Ω

$$y(t) = (Ty)(t) \geq c > 0,$$

moreover

$$y^{\Delta}(t) = (Ty)^{\Delta}(t) = -\int_t^{\infty} p(x)g(y^{\sigma}(x))\, \Delta x < 0. \tag{6.1.19}$$

Finally

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} (Ty)(t) = \lim_{t\to\infty} \left(c + \int_t^{\infty} \int_s^{\infty} p(x)g(f^{\sigma}(x))\, \Delta x \Delta s \right) = c.$$

The fixed point of T is therefore a solution of (6.1.1) with properties (6.1.2). □

Remark 6.1.3. Under somewhat stronger assumptions, we can guarantee the existence of the solution on the entire interval, i.e., Theorem 6.1.1 holds for the solution with properties

$$\left. \begin{aligned} &y^{\Delta}(t) < 0, \\ &y(t) > 0 \quad \text{for } t \geq a, \\ &\lim_{t\to\infty} y(t) = c. \end{aligned} \right\} \tag{6.1.20}$$

We consider two different situations.

1. Suppose $\mathbb{T}$ is discrete, i.e., consists of isolated points. Then we can prove the existence of the solution of (6.1.1), (6.1.2) for $t \geq t_0$ as in the proof of Theorem 6.1.1 and further extend it using the ρ operator to the interval $[a, t_0)_{\mathbb{T}}$ as follows

$$y(\rho(t_0)) = c + \int_{\rho(t_0)}^{\infty} \int_s^{\infty} p(x)g^{\sigma}(x)\, \Delta x \Delta s.$$

Obviously, $y(\rho(t_0)) > 0$ and since

$$\int_{\rho(t_0)}^{\infty}\int_{s}^{\infty} p(x)g(\sigma(x))\,\Delta x\Delta s > \int_{t_0}^{\infty}\int_{s}^{\infty} p(x)g^{\sigma}(x)\,\Delta x\Delta s,$$

$y^{\Delta}(\rho(t_0)) < 0$ also holds. Repeating this process, thanks to the discrete nature of $\mathbb{T}$, after a final number of steps we reach the point a and thus obtain the solution of (6.1.1) with properties (6.1.20).

2. If we consider an arbitrary time scale $\mathbb{T}$, then we can guarantee existence of the solution of (6.1.1) with properties (6.1.20) if we choose c in Theorem 6.1.1 such that there exists $K > 0$ that satisfies

$$\max_{u\in[c,(1+K)c]} g(u)\int_{a}^{\infty}\int_{s}^{\infty} p(x)\Delta x \le Kc.$$

Now we can consider the set Ω in the form

$$\Omega = \{f \in BC[t_0,\infty), c \le f(t) \le (1+K)c \text{ for } t \ge a\}.$$

Then for $t \ge a$

$$\begin{aligned}(Tf)(t) &= c + \int_{t}^{\infty}\int_{s}^{\infty} p(x)g(f^{\sigma}(x))\Delta x\Delta s\\ &\le c + \max_{u\in[c,(1+K)c]_{\mathbb{T}}} g(u)\int_{t}^{\infty}\int_{s}^{\infty} p(x)\,\Delta x\Delta s\\ &\le c + \max_{u\in[c,(1+K)c]_{\mathbb{T}}} g(u)\int_{a}^{\infty}\int_{s}^{\infty} p(x)\,\Delta x\Delta s\\ &\le (1+K)c,\end{aligned}$$

therefore for $t \ge a$

$$c \le Tf(t) \le (1+K)c$$

and $T\Omega \subseteq \Omega$. Other assumptions would be proven almost without a change as in the proof of Theorem 6.1.1.

6.1.2 Conditions for existence and uniqueness of solution

Let us suppose g is Lipschitz continuous (Definition 7.4.3) on $\mathbb{R}^+$. In this section, we aim to show that (6.1.3) guarantees the existence and uniqueness of the solution of (6.1.1) with properties (6.1.2) for arbitrarily chosen positive real number c. To achieve this we employ the Banach fixed point theorem (Theorem 7.4.5).

Theorem 6.1.4. *Suppose the function g satisfies Lipschitz condition on $\mathbb{R}^+$. Then (6.1.3) holds if and only if there exists a unique solution of (6.1.1), (6.1.2) for arbitrarily chosen positive real number c.*

Proof. Let us start with the implication from right to left. As we have shown in the proof of Theorem 6.1.1, the validity of the condition

$$\int_{a}^{\infty}\int_{t}^{\infty} p(s)\,\Delta s\Delta t < \infty$$

follows already from the existence of the solution with required properties.

Now we focus on the implication from left to right. Suppose g is Lipschitz continuous on $\mathbb{R}^+$ with a constant L. Consider t_0 such that

$$\int_{t_0}^{\infty}\int_{t}^{\infty} p(s)\,\Delta s\Delta t < \frac{1}{2L}. \tag{6.1.21}$$

and let $\Omega = BC[t_0,\infty)_{\mathbb{T}}$. We know $(BC[t_0,\infty)_{\mathbb{T}}, \|\cdot\|_\infty)$ is a Banach space (Theorem 4.3.3). We need to define an operator $T:\Omega\to\Omega$ that is a contraction and its unique fixed point is the solution of (6.1.1), (6.1.2). We may again consider T for $t\geq t_0$ in the following form satisfies

$$(Tf)(t) = c + \int_{t}^{\infty}\int_{s}^{\infty} p(x)g(f^\sigma(x))\Delta x\Delta s.$$

We have to show that there exists $K\in(0,1)$ such that for any $f_1, f_2\in\Omega$

$$\|Tf_1 - Tf_2\|_\infty \leq K\,\|f_1 - f_2\|_\infty\,.$$

We know that

$$\|Tf_1 - Tf_2\|_\infty = \sup_{t\geq t_0}|(Tf_1)(t) - (Tf_2)(t)|$$

and for all $t\in[a,\infty)_{\mathbb{T}}$

$$p(t) > 0.$$

Therefore for all $f\in\Omega$ and $t\geq t_0$

$$\begin{aligned}|(Tf_1)(t) - (Tf_2)(t)| &= \left|\int_{t}^{\infty}\int_{s}^{\infty} p(x)g(f_1^\sigma(x))\,\Delta x\Delta s - \int_{t}^{\infty}\int_{s}^{\infty} p(x)g(f_2^\sigma(x))\,\Delta x\Delta s\right| \\ &\leq \int_{t_0}^{\infty}\int_{s}^{\infty} p(x)\left|g(f_1^\sigma(x)) - g(f_2^\sigma(x))\right|\,\Delta x\Delta s.\end{aligned}$$

That means

$$\sup_{t\geq t_0}|(Tf_1)(t) - (Tf_2)(t)| \leq \int_{t_0}^{\infty}\int_{s}^{\infty} p(x)\left|g(f_1^\sigma(x)) - g(f_2^\sigma(x))\right|\,\Delta x\Delta s. \tag{6.1.22}$$

Next we employ an additional condition of Lipschitz continuity for g and make an upper estimate of

$$\int_{t_0}^{\infty}\int_{s}^{\infty} p(x)\left|g(f_1^\sigma(x)) - g(f_2^\sigma(x))\right|\,\Delta x\Delta s$$

as follows

$$\begin{aligned}\int_{t_0}^{\infty}\int_{s}^{\infty} p(x)\left|g(f_1^\sigma(x)) - g(f_2^\sigma(x))\right|\,\Delta x\Delta s &\leq \int_{t_0}^{\infty}\int_{s}^{\infty} p(x)L\left|f_1^\sigma(x) - f_2^\sigma(x)\right|\,\Delta x\Delta s \\ &\leq L\sup_{t\geq t_0}|f_1(t) - f_2(t)|\int_{t_0}^{\infty}\int_{s}^{\infty} p(x)\,\Delta x\Delta s.\end{aligned}$$

Thanks to t_0 being chosen such that (6.1.21) holds, we can proceed as follows

$$\begin{aligned}L\sup_{t\geq t_0}|f_1(t) - f_2(t)|\int_{t_0}^{\infty}\int_{s}^{\infty} p(x)\,\Delta x\Delta s &< L\sup_{t\geq t_0}|f_1(t) - f_2(t)|\frac{1}{2L} \\ &= \frac{1}{2}\sup_{t\geq t_0}|f_1(t) - f_2(t)|.\end{aligned}$$

Therefore it is true that

$$\sup_{t\geq t_0} |(Tf_1)(t) - (Tf_2)(t)| \leq \frac{1}{2} \sup_{t\geq t_0} |f_1(t) - f_2(t)|$$

and thus

$$\|Tf_1 - Tf_2\|_\infty \leq \frac{1}{2} \|f_1 - f_2\|_\infty .$$

We have proven that T is a contraction. All assumptions of the Banach fixed point theorem are satisfied, therefore there exists a solution of (6.1.1), (6.1.2) and it is unique. □

Remark 6.1.5. The equation (6.1.1) for the discrete case $\mathbb{T} = \mathbb{Z}$ is in detail studied in [5].

6.2 More general equation

In this section, we study an equation in a more general form

$$(r(t)y^\Delta)^\Delta = p(t)g(y^\sigma). \tag{6.2.1}$$

We consider the equation on the interval $[a,\infty)_\mathbb{T}$ and assume $p : [a,\infty)_\mathbb{T} \to \mathbb{R}$ is an rd-continuous function that fulfills $p(t) > 0$ for all $t \in \mathbb{T}$. We consider $g : \mathbb{R} \to \mathbb{R}$ as a continuous function satisfying

$$xg(x) > 0$$

for all $x \neq 0$. Additionally, the function $r : [a,\infty)_\mathbb{T} \to \infty$ satisfies $r(t) > 0$ for $t \in [a,\infty)_\mathbb{T}$, $1/r$ is rd-continuous and

$$\int_a^\infty \frac{1}{r(x)} \Delta x = \infty. \tag{6.2.2}$$

Let us denote

$$R(t) = \int_a^t \frac{1}{r(x)} \Delta x.$$

We study a solution, by that we mean a function $y \in C^1_{\mathrm{rd}}[a,\infty)_\mathbb{T}$ such that $ry^\Delta \in C^1_{\mathrm{rd}}[a,\infty)_\mathbb{T}$ satisfying (6.2.1) for all $t \in [a,\infty)_\mathbb{T}$. We aim to show that there exists a solution of (6.2.1) such that

$$\left.\begin{array}{l} y^\Delta(t) < 0, \\ y(t) > 0 \quad \text{for large } t, \\ \lim_{t\to\infty} y(t) = c, \end{array}\right\} \tag{6.2.3}$$

where c is a given positive real number.

6.2.1 Conditions for existence of solution

In this section, we formulate a condition necessary and sufficient for the existence of a solution to (6.2.1) with the properties (6.2.3) for a given $c > 0$. Instead of adopting an approach, where we study the equation in its original general form, we opt to transform it into a known problem described by (6.1.1) with corresponding properties (6.1.2). This approach was introduced in [8].

Theorem 6.2.1. *The functions p and r in the equation (6.2.1) satisfy*

$$\int_a^\infty \frac{1}{r(t)} \int_t^\infty p(s)\,\Delta s\Delta t < \infty \tag{6.2.4}$$

if and only if there exists a solution of (6.2.1), (6.2.3) for arbitrarily chosen positive real number c.

Proof. Let y be a solution of (6.2.1), (6.2.3). We utilize the Theorem 3.3.22. Consider a positive strictly increasing function $\nu \in C^1[a,\infty)_\infty$ on $\mathbb{T}$. Let us set $u(s) = y(t)$, where $s = \nu(t)$ and denote $\widetilde{\mathbb{T}} = \{\nu(t), t \in \mathbb{T}\}$. In the view of chain rule (Theorem 2.3.4), we transform the equation (6.2.1) using

$$y^\Delta = (u^{\widetilde{\Delta}} \circ \nu)\nu^\Delta. \tag{6.2.5}$$

Utilizing the chain rule again, we get

$$\begin{aligned}(ry^\Delta)^\Delta &= (r\nu(u^{\widetilde{\Delta}} \circ \nu))^\Delta \\ &= \Big[[(r\nu^\Delta) \circ \nu^{-1} \circ \nu](u^{\widetilde{\Delta}} \circ \nu)\Big]^\Delta \\ &= \Big[[(r\nu^\Delta) \circ \nu^{-1}]u^{\widetilde{\Delta}}\Big]^{\widetilde{\Delta}} \circ \nu\nu^\Delta.\end{aligned} \tag{6.2.6}$$

Thanks to properties of ν, we have $\nu \circ \sigma = \widetilde{\sigma} \circ \nu$ and therefore $(u \circ \nu)^\sigma = u^{\widetilde{\sigma}} \circ \nu$. Now using (6.2.6), we get on $\widetilde{\mathbb{T}}$

$$\Big(\widetilde{r}(s)u^{\widetilde{\Delta}}\Big)^{\widetilde{\Delta}} = \widetilde{p}(s)g(u^{\widetilde{\sigma}}),$$

where

$$\widetilde{r} = (r\nu^\Delta) \circ \nu^{-1}$$

and

$$\widetilde{p} = \frac{p}{\nu^\Delta} \circ \nu^{-1}. \tag{6.2.7}$$

We set $\nu = R$. In the view of condition (6.2.2), we then get an unbounded time scale $\widetilde{\mathbb{T}} = \nu(\mathbb{T})$. More precisely, the interval $[a,\infty)_\mathbb{T}$ is transformed into $[\widetilde{a},\infty)_{\widetilde{\mathbb{T}}}$, where $\widetilde{a} = \nu(a)$. Further $\nu^\Delta = 1/r$, thus

$$\widetilde{r} = (r/r) \circ \nu^{-1} = 1.$$

This way we transformed (6.2.1) into

$$u^{\widetilde{\Delta}\widetilde{\Delta}} = \widetilde{p}(s)g(u^{\widetilde{\sigma}}) \tag{6.2.8}$$

on $[\widetilde{a},\infty)_{\widetilde{\mathbb{T}}}$.

We intend to utilize the Theorem 6.1.1. We know that

$$\int_{\widetilde{a}}^\infty \int_t^\infty \widetilde{p}(s)\,\widetilde{\Delta} s\widetilde{\Delta} t < \infty. \tag{6.2.9}$$

holds if and only if (6.2.8) has a solution u having properties (6.2.3). We need to show the condition (6.2.10) is equivalent to (6.2.4). Using (6.2.7), we get

$$A = \int_{\widetilde{a}}^{\infty} \int_{t}^{\infty} \widetilde{p}(s)\, \widetilde{\Delta} s \widetilde{\Delta} t = \int_{\widetilde{a}}^{\infty} \int_{t}^{\infty} (pr) \circ R^{-1}(s)\, \widetilde{\Delta} s \widetilde{\Delta} t. \tag{6.2.10}$$

We transform (6.2.10) using the substitution theorem (Theorem 3.3.22). Set $u = R^{-1}(s)$, then

$$A = \int_{\widetilde{a}}^{\infty} \int_{R^{-1}(t)}^{\infty} p(u) r(u) \frac{1}{r(u)}\, \Delta u \widetilde{\Delta} t = \int_{\widetilde{a}}^{\infty} \int_{R^{-1}(t)}^{\infty} p(u)\, \Delta u \widetilde{\Delta} t.$$

We apply the theorem again by setting $\tau = R^{-1}(t)$. We get

$$A = \int_{a}^{\infty} \frac{1}{r(\tau)} \int_{\tau}^{\infty} p(u) \Delta u \Delta \tau.$$

Now it is clear that (6.2.4) holds if and only if (6.2.10).

Assume now (6.2.4) is satisfied. Then (6.2.10) holds as well and applying Theorem 6.1.1, we have guaranteed the existence of the solution u of (6.2.8) with properties (6.2.3). Since $u(s) = y(t)$, where $s = \nu(t)$ and

$$\lim_{s \to \infty} u(s) = c,$$

also

$$\lim_{t \to \infty} y(t) = c.$$

Since ν is increasing, $u(s) > 0$ for large s implies $y(t) > 0$ for large t. Because $r(t) > 0$ for $t \in [a, \infty)_{\mathbb{T}}$, $1/r(t) > 0$ for $t \in [a, \infty)_{\mathbb{T}}$ and since $\nu = R$ is increasing, $\nu^{\Delta}(t) > 0$ for $t \in \mathbb{T}$. Therefore (6.2.5) and $u^{\widetilde{\Delta}}(s) < 0$ for large s imply $y^{\Delta}(t) < 0$ for large t. Thus y satisfies (6.2.3).

As for the opposite direction, if y satisfies (6.2.3), then u satisfies (6.2.3), and hence (6.2.10) holds by Theorem 6.1.1. Consequently, (6.2.4) holds.

□

7 Appendix: selected concepts from functional analysis

In this section, we recall some concepts from functional analysis that are needed for our purposes. We use [4], [5] and [7] as our main sources.

7.1 Completness of metric spaces

In this section, we start with the notion of completness of a metric space.

Definition 7.1.1 (Complete space)**.** A metric space (M, ρ) is called *complete* if every Cauchy sequence of points in M converges in M (has a limit also in M).

We say $N \subseteq M$ is a *complete set* in space (M, ρ) if N with induced metric ρ is complete.

Theorem 7.1.2. *Suppose (M, ρ) is a complete metric space. Then N is a closed subset of M if and only if N is complete.*

7.2 Relative compactness

In this section, we recall the notion of relative compactness and some related notions and facts.

Definition 7.2.1 (Compact set)**.** Let (M, ϱ) be a metric space. We say $N \subseteq M$ is *compact* if every sequence $(x_n)_{n=1}^{\infty} \subseteq N$ contains a converging subsequence $(x_{n_k})_{k=1}^{\infty}$, whose limit is in N.

Definition 7.2.2 (Relatively compact set)**.** Let (M, ϱ) be a metric space. We say $N \subseteq M$ is *relatively compact* if $\overline{N} \subseteq M$ is a compact set in (M, ϱ), where $\overline{N}$ denotes the closure of the set N.

Definition 7.2.3 (ε-net)**.** Let (M, ϱ) be a metric space, ε a positive real number and $N \subseteq M$. A set $A \subseteq M$ is called *ε-net* of N if for every $u \in N$, there exists $v \in A$ such that $\varrho(u, v) \leq \varepsilon$.

Definition 7.2.4 (Totally bounded set)**.** Let (M, ϱ) be a metric space, then $N \subseteq M$ is called *totally bounded* if there exists a finite ε-net for every $\varepsilon > 0$.

Theorem 7.2.5 (Relation between totally bounded and relatively compact)**.** *Let (M, ϱ) be a complete metric space, then a set $N \subseteq M$ is relatively compact if and only if it is totally bounded.*

7.3 Isometry and homeomorphism of normed spaces

Definition 7.3.1 (Isometric isomorphism of normed spaces)**.** Let $(M, \|\cdot\|_M)$, $(N, \|\cdot\|_N)$ be normed vector spaces. We say that these spaces are *isometrically isomorphic* if there exists a bijective linear mapping $T : M \to N$ preserving the norms, i.e., for all $x \in M$

$$\|Tx\|_N = \|x\|_M .$$

Definition 7.3.2 (Homeomorphism of normed spaces)**.** Let $(M, \|\cdot\|_M)$, $(N, \|\cdot\|_N)$ be normed vector spaces. We say that these spaces are *homeomorphic* if there exists a bijective linear mapping $T : M \to N$ and positive real constants a, b satisfying

$$a \|x\|_M \leq \|Tx\|_N \leq b \|x\|_M$$

for all $x \in M$.

Remark 7.3.3. Note that isometric isomorphism evidently implies homeomorphism of normed spaces.

Theorem 7.3.4. *Let $(M, \|\cdot\|_M)$, $(N, \|\cdot\|_N)$ be homeomorphic normed vector spaces. Then $(M, \|\cdot\|_M)$ is a Banach space if and only if $(N, \|\cdot\|_N)$ is a Banach space.*

7.4 Fixed point theorems

In this section, we focus on fixed point theorems and related notions. We give particular attention to Banach and Schauder fixed point theorems. Detailed proofs of these theorems can be found in [5].

Definition 7.4.1 (Fixed point)**.** Let M be a set and let $F : M \to M$. We say $u^* \in M$ is the *fixed point* of the mapping F if $F(u^*) = u^*$ holds.

Definition 7.4.2 (Convex set)**.** Let (M, ϱ) be a metric space and let $N \subseteq M$. We say N is *convex* if for all $x, y \in N$ and $t \in [0, 1]$ an affine combination $(1 - t)x + ty \in N$.

Definition 7.4.3 (Lipschitz continuity)**.** Let (M, ϱ), (N, σ) be metric spaces, a function $f : M \to N$ is called *Liptchitz continuous* if there exists a positive real L such that for all $x, y \in M$

$$\sigma(f(x), f(y)) \leq L\varrho(x, y).$$

Definition 7.4.4 (Contraction)**.** Let (M, ϱ), (N, σ) be metric spaces. We say a function $f : M \to N$ is a *contraction* if there exists $0 \leq L < 1$ such that for all $x, y \in M$,

$$\sigma(f(x), f(y)) \leq L\varrho(x, y).$$

Theorem 7.4.5 (Banach fixed point theorem)**.** *Let (M, ϱ) be a complete metric space and suppose $F : M \to M$ is a contraction. Then there exists a unique fixed point of the mapping F, this point is in addition the limit of the sequence $\{u_n\}_{n=1}^{\infty}$, where $u_1 \in M$ is arbitrary and $u_{n+1} = F(u_n)$ for $n = 2, 3, 4, \ldots$*

Theorem 7.4.6 (Schauder fixed point theorem)**.** *Let M be a Banach space and $N \subseteq M$ a nonempty, convex, bounded and closed set. Moreover, suppose $F : N \to M$ is a continuous mapping, such that $F(N) \subseteq N$ is a relatively compact subset of N. Then the mapping F has a fixed point $u^* \in N$.*

8 Conclusion

The objective of the thesis was to provide an overview of the calculus on time scales, establish a framework of functional analysis on time scales, and utilize this framework to investigate the qualitative properties of specific dynamic equations.

In Chapter 2, we presented a summary of the fundamental concepts in time scales theory. We introduced a concept of the delta derivative as a means of differentiation on time scales. Furthermore, we explored various alternatives of the chain rule adapted for time scales and established mean value theorem.

Chapter 3 of the thesis is dedicated to the integration on time scales. We studied three distinct types of integral. Firstly, we introduced the notion of the Cauchy-type integral defined through the use of antiderivatives. Next, we constructed the Riemann-type integral using the Darboux sums. Additionally, we employed measure theory to define the Lebesgue-type integral, utilizing a Carathéodory-like approach. It became apparent that measure theory was a valuable tool for the development of the integral, as it provided efficient means for its formulation. Moreover, we established key theorems such as the monotone convergence theorem and the dominated convergence theorem, which are essential for the analysis of selected dynamic equations. Furthermore, essential properties of the integrals on time scales were outlined, including the possibility of integration by parts and the substitution theorem.

In Chapter 4 we focused on function spaces on time scales. We discussed continuous and rd-continuous functions on closed interval $[a,b]_{\mathbb{T}}$. We stated detailed proof of completeness of $(C_{rd}[a,b]_{\mathbb{T}}, \|\cdot\|_{\infty})$ space and mentioned briefly other approaches to the proof. We followed with spaces of regressive and positively regressive functions. We introduced an arithmetic on these spaces using circle operation and proved completeness of $(\mathcal{R}^+[a,b]_{\mathbb{T}}, \|\cdot\|_{\mu})$. We followed with bounded continuous functions on noncompact interval and stated and proved that $(BC[a,\infty)_{\mathbb{T}}, \|\cdot\|_{\infty})$ is a Banach space. The second part of the chapter is dedicated to relative compactness of introduced spaces. Special emphasis was placed on the $(BC[a,\infty)_{\mathbb{T}}, \|\cdot\|_{\infty})$ space, with a thorough proof provided for the criterion of relative compactness. Furthermore, relative compactness criteria for other function spaces, which were introduced earlier, were formulated.

In Chapter 7, we applied theoretical tools to analyze the nonlinear dynamic equation (6.1.1). We derived the necessary and sufficient condition for the existence and uniqueness of a solution with the specified properties. By utilizing fixed point theorems, we then provided proofs for the formulated statements concerning the existence (Theorem 6.1.1) and uniqueness (Theorem 6.1.4) of the solution. Additionally, we examined the more general equation (6.2.1) and formulated the necessary and sufficient condition for the existence of the solution (Theorem 6.2.1) with the desired properties. The proof of this theorem was accomplished through a transformation based on the substitution theorem (Theorem 3.3.22).

The main contribution of this thesis is correctly introduced functional analysis apparatus on time scales and demonstration of its proper implementation on the selected problems. The work can be further extended by studying other dynamic equations and exploring the distinctions and similarities across various time scales.

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