A model of sets with incomplete information

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Dissertation

Faculty of Science
Palacky University Olomouc
2018

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Declaration
Hereby I declare that the thesis is my original work.

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Abstract

The presented mathematical model of sets with incomplete information is based on L-valued sets in universes endowed with symmetric and transitive L-valued relations $\approx$ where L is a complete and atomic Boolean algebra. Values $x \approx x$ express incomplete information about the presence of elements in universes. In addition, incomplete information about the equality of elements and membership relations of sets is modeled. The work introduces a logic for structures with incomplete information and preliminary results on ordered sets and concept lattices with incomplete information.

Acknowledgements

Dedicated to Michal Krupka.

This thesis was supported by grant No. PrF 2016 027 of IGA of Palacky University Olomouc.

Contents

Introduction

This thesis deals with modeling of sets with incomplete information. We distinguish two different use of sets: universes and ordinary sets. Assume we have a universe U with missing information. We might be missing the information whether two elements of U are equal or whether an element is in U. More precisely, the relation of equality of elements and the membership relation are not completely known. For example, we do not know if for an element u it holds $u \in U$ or $u \notin U$ and for elements u_1 and u_2 whether $u_1 = u_2$ or $u_1 \neq u_2$. For the second use of sets, assume we have a set V with missing information where the membership relation of V can be only partially known. In general, we are missing the information whether the partially known set V is in the universe U.

We present a model of the above described sets with missing information. A universe U with missing information is modeled by an ordinary set X with an L-valued binary relation $\approx : X \times X \to L$ where L is a complete and atomic Boolean algebra. We call X a *conditional universe*. Elements of L are called *conditions* and model missing pieces of information. The structure of L models dependencies between conditions.

A set V with missing information is modeled by an L-valued set $A \colon X \to L$ called a *conditional set* in X. Complete homomorphisms $h \colon L \to 2$ (the two element Boolean algebra) model possible completion of missing parts of information. They are called *total realities* and represent possible worlds. Each complete homomorphism $h \colon L \to 2$ transforms X to an ordinary set X^h, the L-relation $\approx$ to the ordinary equality on X^h and the L-set A to an ordinary set A^h. Only some sets A^h are in universes X^h. Sets X^h and A^h (called *realizations*) match all possible forms of U and V, respectively.

We require that the L-relation $\approx$ is symmetric and transitive. Generally, it is not reflexive. The condition $x \approx x$ expresses what we know about the presence of x in X. The condition expressing presence of A in X is derived from the information about the membership relation of A and presence of elements in X.

The presented work is a continuation of [19] where foundations of conditional universes and sets were given. However, definitions of conditional universes in the paper and in this work differ. Namely, L-relations $\approx$ used in the paper are reflexive. The motivation for the generalization comes from the theory of Boolean-valued and Heyting-valued sets [14, 6] developed in the connection with Boolean-valued and Heyting-valued models [27, 2, 26, 1]. Since conditional sets are fuzzy sets, we use results from the fuzzy set theory [34, 10, 21, 12, 3]. However, many results are formulated for fuzzy sets in X with reflexive $\approx$ and can not be used. For fuzzy sets in X with non-reflexive $\approx$ see [16, 17, 33]. Yet, literature covers only conditional sets completely present in X. The way how we treat conditional sets partially present in X is an original result of this work.

The main motivation of this work is to study concept lattices with missing information. A concept lattice [7] is a hierarchy of formal concepts found in a formal context (a binary relation between a universe of objects and a universe of attributes). Formal contexts and concept lattices with missing information are modeled by *conditional contexts* and *conditional concept lattices*, respectively. They were introduced in [20] and can be used in practice. The presented definition of conditional contexts, in addition, model missing information about the presence of objects and attributes in contexts. Consequently, we cover also the empty context as a realization of a conditional context and the reduction of conditional contexts can be performed.

The work is divided into two chapters. The first chapter provides basic definitions and results on conditions, conditional universes and sets. A generalization of extensionality of conditional sets is proposed here. The second chapter focuses on models of structures with missing information called *conditional structures*. A logic of conditional structures is the topic of the first part of the second chapter. A section with an application on concept lattices is presented in the second chapter. An illustrative example can be found in the end of the second chapter. Some ideas for further research are described in conclusions.

Chapter 1

Incomplete information

This chapter presents a model of sets with missing information.

1.1 Conditions

Conditions represents missing pieces of information and dependencies between them. A *Boolean algebra of conditions* L is defined to be a complete and atomic Boolean algebra L. Elements of L are called *conditions*.

Recall that a structure of Boolean algebra [9, 11, 30] consists of elements 1 and 0, and operations $\wedge$ (join), $\vee$ (meet) and $'$ (complement). $(L, \wedge, \vee)$ is a distributive lattice. We use the induced order relation $\leq$ given by $c \leq d$ iff $c \wedge d = c$ iff $c \vee d = d$. The element 0 is the least element and 1 is the greatest element and for each $c \in L$,

$$c \wedge c' = 0, \qquad\qquad c \vee c' = 1.$$

A lattice L is called *complete* if for every set $M \subseteq L$, the infimum $\bigwedge M$ and supremum $\bigvee M$ exist. A non-zero element $a \in L$ is called an *atom* if from $c \leq a$ it follows $c = 0$ or $c = a$. L is called *atomic* if for each non-zero $c \in L$ there is an atom $a \in L$ below c. Complete atomic Boolean algebras are exactly Boolean algebras isomorphic to powersets [30, Theorem 25.1.].

We also use derived operations $\rightarrow$ (residuum) and $\leftrightarrow$ (biresiduum) defined by $c \rightarrow d = c' \vee d$ and $c \leftrightarrow d = (c \rightarrow d) \wedge (d \rightarrow c)$. The residuum and join satisfy the so-called *adjointness* *property* $c_1 \wedge c_2 \leq c_3$ iff $c_1 \leq c_2 \rightarrow c_3$ for each $c_1, c_2, c_3 \in L$. Therefore, L has also the structure of residuated lattice [31].

A surjective complete homomorphism $h\colon L \to K$ where K is another Boolean algebra of conditions is called a *reality*. Realities model filling of missing information. Recall that a homomorphism $h\colon L \to K$ between complete lattices is called *complete* if $h(\bigvee M) = \bigvee h(M)$ and $h(\bigwedge M) = \bigwedge h(M)$ for each $M \subseteq L$. The set of all realities $h\colon L \to K$ is denoted $\mathrm{CHom}(L,K)$. If for a condition $c \in L$ it holds $h(c) = 1$ then we say that c is *satisfied in* h. If h is obvious from the context then we simply say that c *is satisfied*. Clearly, 1 is satisfied in every reality and 0 is not satisfied in any reality.

A reality $h\colon L \to 2$ (the standard two-element Boolean algebra) is called a *total reality*. Each condition is either satisfied or not satisfied in a total reality. Total realities represent possible worlds.

It is easy to see that c is satisfied in h and d is satisfied in h iff $c \wedge d$ is satisfied in h. In short, c and d are satisfied iff $c \wedge d$ is satisfied. Similar statements hold for meet, complement, residuum and biresiduum. Moreover, it holds for each $M \subseteq L$ that every $c \in M$ is satisfied iff $\bigwedge M$ is satisfied.

Atomicity of L implies that $c = 1$ iff c is satisfied in every total reality h. Moreover, we have $c \leq d$ iff for every total reality h it holds that c is satisfied in h implies d is satisfied in h. We can easily see that $c = d$ if and only if for every total reality h it holds that c is satisfied in h iff d is satisfied in h.

We can construct a Boolean algebra of conditions from a set of propositional formulas. The details follow. Suppose we identify missing pieces of information with a finite set of propositional variables P and represent dependencies between missing pieces by a set T (theory) of propositional formulas over P. Now, we define an equivalence relation $\sim$ on the set of all propositional formulas $\mathscr{F}(P)$ over P given by $\varphi \sim \psi$ iff the formula $\varphi \Leftrightarrow \psi$ (the symbol $\Leftrightarrow$ denotes the equivalence connective) is provable from T. Let $L = \mathscr{F}(P)/\sim$ be the quotient set of $\mathscr{F}(P)$ by $\sim$. We consider the structure of Boolean algebra on L given by $[\varphi]_\sim \wedge [\psi]_\sim = [\varphi \wedge \psi]_\sim$ ($[\varphi]_\sim$ denotes the class of φ w.r.t. $\sim$ and the second occurrence of $\wedge$ the conjunction connective) similarly for meet and complement, $0 = [\varphi \wedge \neg\varphi]_\sim$ and $1 = [\varphi \vee \neg\varphi]_\sim$ ($\neg$ and $\vee$ denote negation and disjunction connectives, respectively). The constructed Boolean algebra L is the Lindenbaum algebra [24] of T. From finiteness of P follows that L is complete and atomic, i.e. L is a Boolean algebra of conditions.

Models of T are in a one-to-one correspondence with total realities $L \to 2$ (where an evaluation $e\colon P \to 2$ is called a *model of* T if for each formula $\varphi \in T$ it holds that the truth value of φ under e is 1). Now, we describe the correspondence. If e is a model of T

then $h_e\colon L \to 2$ given by $h_e([\varphi]_\sim) = ||\varphi||_e$ is a total reality and if h is a total reality then the evaluation $e_h\colon P \to 2$ given by $e_h(p) = h([p]_\sim)$ is a model of T.

If $T = \emptyset$ then the above constructed Lindenbaum algebra of T is a freely generated Boolean algebra over the set of generators P. Note that there is no freely generated complete Boolean algebra over infinite set of generators [13]. Therefore, the above construction for infinite set P does not need to produce a Boolean algebra of conditions.

A Boolean algebra of conditions can be also constructed from a given set of evaluations $E \subseteq 2^P$ where P is possibly infinite set of propositional variables. E represents dependencies between missing pieces identified with elements of P. Let $L = 2^E$ and $\iota : P \to L$ given by $(\iota(p))(e) = e(p)$ for $p \in P$. Then we associate with a total reality h an evaluation $e_h\colon P \to 2$ given by $e_h(p) = h(\iota(p))$. Now, the set of all e_h ($h \in \mathrm{CHom}(L,2)$) equals E [19, Theorem 1].

1.2 Conditional universes

Suppose we have a set U about which we do not have complete information. It can be unknown whether some elements belong to the set and whether some elements are equal. We model this situation by a conditional universe X so that standard realizations of X corresponds to all possible forms of U.

Elements of the conditional universe represent possible elements of the set U. For simplicity, we talk directly about elements of X as about elements of U. By the uncertainty of the set U, we have to formulate statements hypothetically with respect to a possible reality. Therefore, we say that an element x does not exist in a reality h (it is not present in the universe) or elements x_1 and x_2 are equal in a reality h.

For example, consider a set U that definitely contains elements u_1, u_2, u_3 and possibly an element u_4. U does not contain any other element. Moreover, we do not know whether u_2 is equal to u_3. We model this situation by a universe $X = \{x_1, x_2, x_3, x_4\}$ (which contains exactly 4 elements, while the set U may contains only two) with certain L-relation. X is designed so that standard realizations of X capture exactly four possible forms of the unknown set U.

The first section explains why we model incomplete information on presence of elements by non-reflexive relations.

1.2.1 Non-existing elements

We open with a quote from Plato [23, 185a]:

> SOCRATES Now in regard to sound and color, you have, in the first place, this thought about both of them, that they both exist?
>
> THEAETETUS Certainly.
>
> SOCRATES And that each is different from the other and the same as itself?
>
> THEAETETUS Of course.

This is the first occurence of the law of identity stated that each (existing) thing is the same with itself and different from another. For the following discussion the first part is more important. More explicitly, it expresses that if x exists then $x = x$. If we define an element x and derive, when reasoning about it, that it is not equal to itself ($x \neq x$) then we conclude that x does not exist. For example take a division by zero: if we assume that $x = 0/0$ is a number then we easily conclude that $x \neq x$ and therefore x does not exist. Indeed, $0/0$ should be the unique number x such that $0 \cdot x = 0$, but x can be (among other numbers) equal to zero or one. From the fact that $0 \neq 1$ we conclude that $x \neq x$. Generally, we can reach any contradiction to conclude non-existence of x. However, the contradiction $x \neq x$ is the simplest one contradiction with a reference to the element x.

In what follows, we materialize non-existing elements and add them to a universe. Existence is then a property of elements. Note that we follow this approach although in philosophy [22][25] there is a long debate whether existence is a property of individuals or a property of concepts.

There is evident need to distinguish non-existing elements from existing. This can be done by considering the set of existing elements. From our point of view, it seems to be more natural to embed existence of an element to a binary transitive and symmetric relation on the universe. More precisely, we assume that an element exists if and only if it is related to itself. Similar approach is studied by Scott in [28] and in free logic with negative semantics [25]. Note that in some programming languages (e.g. JavaScript) there is a special object called `NaN` (not a number) which is not equal to itself. `NaN` is returned for example when `0/0` is evaluated. Therefore, in JavaScript `document.write(0/0==0/0)` displays `false`.

We end this part with a quote from Romeo and Juliet [29, Act 1, Scene 1] illustrating that in fiction not equality with itself implies non-existence:

> ROMEO Tut, I have lost myself; I am not here; This is not Romeo, he's some other where.

1.2.2 Definition of conditional universes

Let L be a Boolean algebra of conditions. An *L-conditional universe* is defined to be a set X together with a mapping $\approx : X \times X \to L$ satisfying

$$x \approx y = y \approx x, \qquad \text{(symmetry)}$$
$$(x \approx y) \wedge (y \approx z) \leq x \approx z. \qquad \text{(transitivity)}$$

The mapping $\approx$ is called the *conditional equality* of X. The L-conditional universe is also shortly denoted by $(X, \approx)$. If $L = 2$ then $\approx$ is an ordinary symmetric and transitive relation on X.

For $x, y \in X$, the value $x \approx y$ is interpreted as the condition under which x is equal to y and the value $x \approx x$ is interpreted as the condition under which x is present in X.

For a reality h, $h(x \approx y) = 1$ means that the condition under which x is equal to y is satisfied in h or shortly that it is satisfied in h that x is equal to y. Similarly, $h(x \approx x) = 1$ is shortly read as it is satisfied in h that x is present.

By ${}^1X$ we denote the set $\{x \in X \mid x \approx x = 1\}$ of *completely present* elements of X.

The theory of sets endowed with symmetric and transitive L-valued relations $\approx$ where values $x \approx x$ are interpreted as degrees of existence is well-developed [14, 33, 16, 17].

Due to the symmetry and the transitivity of $\approx$ we have $x \approx y = (x \approx y) \wedge (x \approx y) = (x \approx y) \wedge (y \approx x) \leq x \approx x$. Similarly can be proved that $x \approx y \leq y \approx y$. Thus, for each $x, y \in X$ it holds

$$x \approx y \leq (x \approx x) \wedge (y \approx y). \qquad (1.1)$$

The preceding inequality is to be read as follows: if it is satisfied that x is equal to y then it is also satisfied that x and y are present.

Let $x \in X$. Clearly, $\bigvee_{y \in X} x \approx y \geq x \approx x$. By (1.1), $x \approx y \leq x \approx x$ for each $y \in X$ which yields $\bigvee_{y \in X} x \approx y \leq x \approx x$. Therefore, for each $x \in X$ it holds

$$x \approx x = \bigvee_{y \in X} y \approx x. \tag{1.2}$$

The conditional equality $\approx$ is called *reflexive* iff $x \approx x = 1$ for all $x \in X$. Elements $x, y \in X$ satisfying $x \approx y = x \approx x = y \approx y$ are called *extensionally equal* and denoted by $x \sim y$. Extensionall equality of x and y is read as follows: if it is satisfied that x or y is present then it is satisfied that the other is present and they are both equal. The conditional equality $\approx$ is called *separated* iff for each $x, y \in X$ it holds that from $x \sim y$ it follows $x = y$. Reflexive and separated conditional equalities are well known in the fuzzy set theory as L-equalities [3]. Conditional universes with reflexive and separated conditional equalities were studied in [19].

If $L = 2$ and $\approx$ is reflexive and separated then $\approx$ is just the ordinary equality on X and X is an ordinary universe.

1.2.3 Realizations of conditional universes

A reality $h\colon L \to K$ transforms an L-conditional universe X to a K-conditional universe Y. The transformation of X to Y is described by a partial mapping from X to Y.

Recall that a *partial mapping* $f\colon X \nrightarrow Y$ from a set X to a set Y is a binary relation between X and Y such that for each $x \in X$ and $y_1, y_2 \in Y$ it holds that if (x, y_1) and (x, y_2) are in f then $y_1 = y_2$. If for $x \in X$ there is an $y \in Y$ such that $(x, y) \in f$ then we write $f(x) = y$ and say that $f(x)$ *is defined*. By $\mathrm{dom} f$ we denote the set of all elements $x \in X$ such that $f(x)$ is defined. We use the following convention in definitions of partial mappings. Assume we define a partial mapping $f\colon X \nrightarrow Y$ by an expression. Then we suppose that $f(x)$ is defined for all $x \in X$ for which the expression makes sense. For example, a partial mapping $f\colon R \nrightarrow R$ from the set of real numbers to the set of real numbers given by $f(x) = 1/x$ is defined for all non-zero numbers. A partial mapping $f\colon X \nrightarrow Y$ is called *surjective* if for each $y \in Y$ there is $x \in \mathrm{dom} f$ such that $f(x) = y$. In the theory of partial mappings the term "total" is used to state that a partial mapping is an ordinary mapping. However, we use the term "total" with the different meaning. Therefore, we never use the term "total" with the first mentioned meaning.

Let X be an L-conditional universe, $h\colon L \to K$ a reality and Y a K-conditional universe with a conditional equality $\approx_Y$. An *h-realization of* X is a surjective partial mapping $f\colon X \nrightarrow Y$

which satisfies the following two conditions. First,

$$f(x_1) \approx_Y f(x_2) = h(x_1 \approx x_2) \tag{1.3}$$

holds for each $x_1, x_2 \in \mathrm{dom} f$. Second,

$$h(x \approx x) \leq \bigvee_{\bar{x} \in \mathrm{dom} f} h(\bar{x} \approx x) \tag{1.4}$$

holds for each $x \in X$. By (1.4), if h is total and $h(x \approx x) = 1$ then there is $\bar{x} \in X$ such that $f(\bar{x})$ is defined and $h(\bar{x} \approx x) = 1$. By the first requirement, a realization preserve equality and presence of elements and by the second the realization is defined for sufficiently many elements.

Note that $f(x)$ does not have to be defined, even if the condition under which an element $x \in X$ is present is satisfied in h. If $f(x)$ is defined whenever $h(x \approx x) > 0$ then the h-realization f of X is called *faithful*. If the conditional equality $\approx_Y$ is separated then f is called *merging*. If $h(x \approx x) = 0$ then the requirement (1.4) never enforces $f(x)$ to be defined. If $h(x \approx x) = 0$ implies that $f(x)$ is not defined for each $x \in X$ then f is called *omitting*.

Sometimes, we also call the K-conditional universe Y itself an h-realization of X. In this case, we suppose we are given a surjective partial mapping $f : X \nrightarrow Y$ satisfying (1.3) and (1.4).

Suppose that h is a total reality. Then any faithful, merging and omitting h-realization $f : X \nrightarrow Y$ of X is called *standard* and in this case Y is an ordinary universe ($\approx_Y$ is the ordinary equality on Y). Let $f : X \nrightarrow Y$ and $g : X \nrightarrow Z$ be two merging and omitting h-realizations of X. Consider a binary relation r between Y and Z consisting of pairs $(f(x_1), g(x_2))$ for $x_1 \in \mathrm{dom} f$ and $x_2 \in \mathrm{dom} g$ such that $h(x_1 \approx x_2) = 1$. For $x_1 \in \mathrm{dom} f$ we have $h(x_1 \approx x_1) = 1$ (f is omitting). By (1.4), there is $x_2 \in \mathrm{dom} g$ such that $h(x_1 \approx x_2) = 1$. If $(f(x_1), g(x_2)), (f(x_1), g(x_3)) \in r$ then by the transitivity of $\approx$, $h(x_2 \approx x_3) = 1$ and by g is merging, $g(x_2) = g(x_3)$. We showed that r is a function $Y \to Z$. As the definition of r is symmetric, injectivity and surjectivity can be shown similarly. Therefore, r is a bijection between Y and Z. We showed that there is a bijection between any two total, merging and omitting h-realizations Y and Z of X. Any total, merging and omitting h-realization Y of X can represent the unknown set modeled by X in the total reality h. Note that standard realizations are special cases of such realizations. We show a more general result on isomorphism of any two h-realizations of X for any reality h in Section 1.3.5.

For $x \in X$, the value $f(x)$ is called the *h-realization of x*. When an h-realization $f : X \nrightarrow Y$ is given, we denote for simplicity f by h^X, Y by X^h, $\mathrm{dom} f$ by X_h, $\approx_Y$ by $\approx^{|h}$ and $f(x)$

by x^h. The reason why $\approx_Y$ is denoted by $\approx^{|h}$ and not by $\approx^h$ will be explained in Section 1.3.1. Now, the equality (1.3) can be rewritten as

$$x_1^h \approx^{|h} x_2^h = h(x_1 \approx x_2) \tag{1.5}$$

for all $x_1^h, x_2^h \in X^h$ (or $x_1, x_2 \in X_h$). If the left-hand side is equal to 1, we say that x_1 *is equal to x_2 in the h-realization f*. The right-hand side being equal to 1 means, as we know, that the condition $x_1 \approx x_2$ is satisfied in the reality h. Thus, the equality is read as x_1 *is equal to x_2 (in the h-realization f) if and only if it is satisfied (in the reality h) that x_1 is equal to x_2.*

We say that x_1 *is equal to x_2 in a reality h* if x_1 and x_2 are equal in any h-realization of X. By (1.5), Let f and g be two h-realizations of X. Then the following holds. If x_1 and x_2 are equal in f and $g(x_1)$ and $g(x_2)$ are defined then x_1 and x_2 are also equal in g.

Moreover,

$$x^h \approx^{|h} x^h = h(x \approx x). \tag{1.6}$$

Similarly as above, if the left-hand side is equal to 1, we say that *x is present in the h-realization f*. The equality is to be read as *x is present (in the h-realization f) if and only if it is satisfied (in the reality h) that x is present*. We say that *x is present in a reality h* if x is present in any h-realization of X.

As shown above, there is a bijection between two standard h-realizations of X. Therefore, we not distinguish between standard h-realizations of X and refer to any of them as *the standard h-realization of X*. For simplicity, we denote the standard h-realization of X by X^h.

Four examples of realizations of a three-element 2-conditional universe are described in Fig. 1.1. In what follows, we give general examples of h-realizations of X. Let $(Y, \approx_Y)$ be the K-conditional universe given by $Y = X$ and $x_1 \approx_Y x_2 = h(x_1 \approx x_2)$ for each $x_1, x_2 \in X$. Then the mapping $X \to Y$, $x \mapsto x$ is an h-realization of X.

The next example is an h-realization of X obtained by means of factorization. Let $X_1 = \{x \in X \mid h(x \approx x) > 0\}$. Denote, for a moment, by ${}^h\!\approx$ the (ordinary) equivalence on X_1 given by $x\,{}^h\!\approx y$ if $h(x \approx y) = h(x \approx x) = h(y \approx y) > 0$. Denote by $X_1/_{^h\approx}$ the quotient space of X by ${}^h\!\approx$.

Theorem 1. *Let $Y = X_1/_{^h\approx}$, $f\colon X \nrightarrow Y$ be defined by $f(x) = [x]_{^h\approx}$ for $x \in X_1$ and $\approx_Y\colon Y \times Y \to K$ be defined by*

$$[x]_{^h\approx} \approx_Y [y]_{^h\approx} = h(x \approx y). \tag{1.7}$$

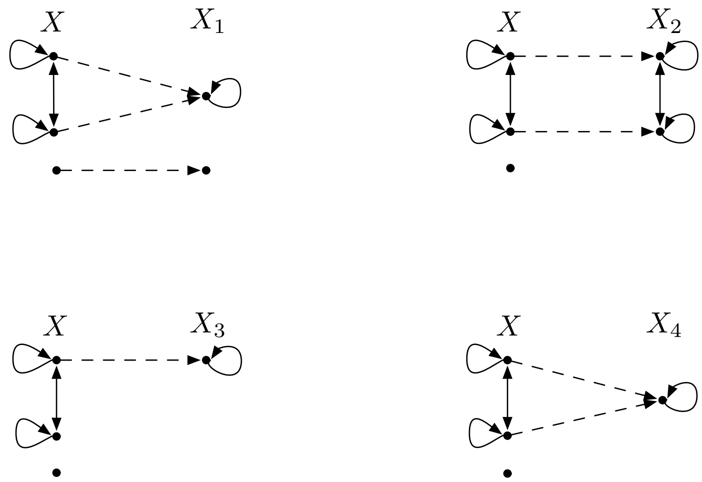

Figure 1.1: Four realizations of a conditional universe X consisting of three elements. Elements of X are represented by circles below X. The underlying Boolean algebra of condition is the two-element Boolean algebra $\mathbb{2}$. The considered reality h is the identity on $\mathbb{2}$. Elements of h-realizations X_i of X are represented by circles below X_i. Solid arrows indicate conditional equalities of X and X_i. Namely, there is a solid arrow from an element x to an element y iff the condition under which x equals y is 1. Dashed arrows indicate realizations $f_i\colon X \nrightarrow X_i$. Namely, there is a dashed arrow from an element x to an element y iff y is the h-realization of x. If there is no arrow leading from an element x of X then the h-realization of x is not defined. The top-left realization of X is merging, faithful but not omitting, the top-right realization is faithful, omitting but not merging, the bottom-left realization is omitting, merging but not faithful and the bottom-right realization is faithful, merging and omitting (it is a standard h-realization of X).

Then $\approx_Y$ is defined correctly, the pair $(Y, \approx_Y)$ is a separated K-conditional universe, f is a faithful, merging and omitting h-realization of X. Moreover, if h is total then f is standard.

Proof. Let $\bar{x}\overset{h}{\approx}x$ and $\bar{y}\overset{h}{\approx}y$. By definition, $h(\bar{x} \approx \bar{x}) = h(x \approx \bar{x})$ and $h(\bar{y} \approx \bar{y}) = h(\bar{y} \approx y)$. By (1.1) and the transitivity of $\approx$,

$$
\begin{aligned}
h(\bar{x} \approx \bar{y}) &= h(\bar{x} \approx \bar{x}) \wedge h(\bar{x} \approx \bar{y}) \wedge h(\bar{y} \approx \bar{y}) \\
&= h(x \approx \bar{x}) \wedge h(\bar{x} \approx \bar{y}) \wedge h(\bar{y} \approx y) \\
&= h\big((x \approx \bar{x}) \wedge (\bar{x} \approx \bar{y}) \wedge (\bar{y} \approx y)\big) \leq h(x \approx y)
\end{aligned}
$$

and, symmetrically, $h(x \approx y) \leq h(\bar{x} \approx \bar{y})$. This proves correctness of (1.7).

Denote the class $[x]_{h\approx}$ of $x \in X$ by x^h. We have for each $x, y, z \in X$

$$
\begin{aligned}
& x^h \approx_Y y^h = h(x \approx y) = h(y \approx x) = y^h \approx_Y x^h, \\
& (x^h \approx_Y y^h) \wedge (y^h \approx_Y z^h) = h(x \approx y) \wedge h(y \approx z) \\
& \quad = h((x \approx y) \wedge (y \approx z)) \leq h(x \approx z) = x^h \approx_Y z^h.
\end{aligned}
$$

Thus, $\approx_Y$ is symmetric and transitive. By definition, $x^h \approx_Y x^h > 0$ and $y^h \approx_Y y^h > 0$ for each $x^h, y^h \in X^h$. Moreover, if $x^h \approx_Y y^h = x^h \approx_Y x^h = y^h \approx_Y y^h$ then $h(x \approx y) = h(x \approx x) = h(y \approx y) > 0$. So, $x\overset{h}{\approx}y$ and, consequently, $\approx_Y$ is separated.

If $x \notin \mathrm{dom} f$ then $h(x \approx x) = 0$. So, (1.4) is trivially satisfied. (1.3) follows directly from (1.7). The surjectivity of f follows from the surjectivity of quotient projections. Therefore, f is an h-realization of X.

Since $f(x)$ is defined iff $h(x \approx x) > 0$, f is faithful and omitting and since $\approx_Y$ is separated, f is merging. The fact that f is standard for h total follows from the definition. □

Note that if h is total then the above construction is presented e.g. in [14] and subsequent papers.

For two partial mappings $f\colon X \nrightarrow Y$ and $g\colon Y \nrightarrow Z$, the *composition of g with f* is the partial mapping $g \circ f\colon X \nrightarrow Z$ defined by $(g \circ f)(x) = g(f(x))$.

Lemma 1. *Let $h_1\colon L \to K_1$ and $h_2\colon K_1 \to K_2$ be realities, $f_1\colon X \nrightarrow Y$ an h_1-realization of X, $f_2\colon Y \nrightarrow Z$ an h_2-realization of Y. Then $f_2 \circ f_1$ is an $h_2 \circ h_1$-realization of X.*

Proof. The surjectivity of $f_2 \circ f_1$ follows from the surjectivity of f_1 and f_2. (1.3) can be checked by a direct verification. We prove (1.4). From the fact that f_1 is an h_1-realization of X it follows

$$h_2(h_1(x \approx x)) \leq \bigvee_{\bar{x} \in \mathrm{dom} f_1} h_2(h_1(\bar{x} \approx x)) \tag{1.8}$$

for each $x \in X$.

For $\bar{x} \in \mathrm{dom} f_1$ we have $h_2(h_1(\bar{x} \approx \bar{x})) = h_2(f_1(\bar{x}) \approx_Y f_1(\bar{x})) \leq \bigvee_{\bar{y} \in \mathrm{dom} f_2} h_2(f_1(\bar{x}) \approx_Y \bar{y}) = \bigvee_{\bar{y} \in \mathrm{dom} f_2} \bigvee_{f_1(\bar{\bar{x}}) = \bar{y}} h_2(f_1(\bar{x}) \approx_Y f_1(\bar{\bar{x}})) = \bigvee_{\bar{\bar{x}} \in \mathrm{dom}(f_2 \circ f_1)} h_2(h_1(\bar{x} \approx \bar{\bar{x}}))$.

Now, by the transitivity of $\approx$ and (1.1),

$$\begin{aligned}
\bigvee_{\bar{x} \in \mathrm{dom} f_1} h_2(h_1(\bar{x} \approx x)) &= \bigvee_{\bar{x} \in \mathrm{dom} f_1} h_2(h_1((\bar{x} \approx x) \wedge (\bar{x} \approx \bar{x}))) \\
&\leq \bigvee_{\bar{x} \in \mathrm{dom} f_1} \bigvee_{\bar{\bar{x}} \in \mathrm{dom}(f_2 \circ f_1)} h_2(h_1((\bar{x} \approx x) \wedge (\bar{x} \approx \bar{\bar{x}}))) \leq \bigvee_{\bar{\bar{x}} \in \mathrm{dom}(f_2 \circ f_1)} h_2(h_1(x \approx \bar{\bar{x}})).
\end{aligned}$$

The preceding with (1.8) prove (1.4). □

Let $f_1 : X \nrightarrow Y$ be an h_1-realization of X and $f_2 : Y \nrightarrow Z$ be an h_2-realization of Y. Then we associate with the reality $h_2 \circ h_1$ the $h_2 \circ h_1$-realization $f_2 \circ f_1 : X \nrightarrow Z$ of X. Remind that Y can be denoted by X^{h_1} and Z by $(X^{h_1})^{h_2}$ or $X^{h_2 \circ h_1}$. We denote $(X^{h_1})^{h_2}$ simply by $X^{h_1 h_2}$ and obviously have $X^{h_1 h_2} = X^{h_2 \circ h_1}$. By the definition of the composition of partial mappings, $f_2(f_1(x))$ is defined if any only if $(f_2 \circ f_1)(x)$ is defined and $g(f(x)) = (g \circ f)(x)$ for each $x \in \mathrm{dom}(g \circ f)$. Which can be reformulated as the follows. $x^{h_1 h_2}$ is defined if and only if $x^{h_2 \circ h_1}$ is defined and $x^{h_1 h_2} = x^{h_2 \circ h_1}$ for each $x \in X_{h_2 \circ h_1}$.

Since the conditional equality of any standard realization of a conditional universe is an ordinary equality, the composition of any realization and standard realization is a merging and omitting realization, but it does not have to be faithful.

Lemma 2. *The composition of two faithful realizations is faithful.*

Proof. Let $f_1 : X \nrightarrow Y$ be a faithful h_1-realization of X and $f_2 : Y \nrightarrow Z$ a faithful h_2-realization of Y.

Suppose $h_2(h_1(x \approx x)) > 0$. Since h_2 is isotone, $h_1(x \approx x) > 0$. By the faithfulness of f_1, the h_1-realization $f_1(x)$ of x is defined and we have $h_2(f_1(x) \approx_Y f_1(x)) > 0$ and again, since f_2 is faithful, the h_2-realization $f_2(f_1(x))$ is defined. The proof is concluded by the fact that $f_2(f_1(x))$ equals $(f_2 \circ f_1)(x)$. □

Lemma 3. *Let X be a set together with a system of surjective partial mappings $f_h\colon X \nrightarrow X_h$ ($h \in \mathrm{CHom}(L,2)$). Then there is a unique mapping $\approx\colon X \times X \to L$ such that for each $h \in \mathrm{CHom}(L,2)$ and $x_1, x_2 \in X$ it is satisfied:*

1. if $f_h(x_1)$ and $f_h(x_2)$ are defined and $f_h(x_1) = f_h(x_2)$ then $h(x_1 \approx x_2) = 1$;

2. otherwise, $h(x_1 \approx x_2) = 0$.

The set X together with $\approx$ is an L-conditional universe and for each total reality h the partial mapping f_h is a standard h-realization of X.

Proof. By basic properties of complete atomic Boolean algebras, for any $x_1, x_2 \in X$, the value $x_1 \approx x_2$ satisfying the requirements of the theorem exists and is unique. This proves the existence and uniqueness of $\approx$. It remains to be shown that $\approx$ is a conditional equality. This is easy, as symmetry and transitivity are obvious. □

There are two canonical ways how to construct conditional universes from a family of sets. They can be used in a situation when we have the form of an unknown set in every possible world. Let I be a non-empty set and X_i ($i \in I$) a family of sets.

We first construct a suitable Boolean algebra of conditions. We set $L = 2^I$. Since L is a power set, L is complete and atomic Boolean algebra, i.e. a Boolean algebra of conditions. For each $i \in I$ there is a total reality $h_i\colon L \to 2$ given by $h_i(c) = 1$ if $i \in c$ and $h_i(c) = 0$ otherwise. Since $\{i\}$ is an atom of L, the construction of total realities h_i is valid and each total reality is equal to h_i for some $i \in I$.

In the first construction, we assume that all X_i ($i \in I$) are non-empty. Let X be the product of X_i and $f_i\colon X \to X_i$ be the family of projections of X. Then by the surjectivity of projections and Lemma 3 there is the unique conditional equality $\approx$ on X such that f_i are standard realizations of $(X, \approx)$. It holds that $\approx$ is reflexive and $x_1 \approx x_2 = \{i \in I \mid f_i(x_1) = f_i(x_2)\}$ for all $x_1, x_2 \in X$.

The second construction follows. Sets X_i ($i \in I$) can be possibly empty. Let $X = \dot{\bigcup}_{i \in I} X_i$ (the disjoint union) and $f_i\colon X \nrightarrow X_i$ be partial mappings given by $f_i(x) = x$ if $x \in X_i$. By definition, f_i are surjective and again, by Lemma 3, there is the unique conditional equality $\approx$ such that f_i are standard realizations of $(X, \approx)$. For each $x_1, x_2 \in X$ it holds that $x_1 \approx x_2 = \{i\}$ if $x_1 = x_2$ and $x_1, x_2 \in X_i$, and $x_1 \approx x_2 = 0$ otherwise.

Realizations merge elements in the first construction, while omit elements in the second. In finite case, the size of the result of the first construction is the product of sizes of X_i, while the size of the result of the second construction is the sum of sizes of X_i.

Any ordinary subset $Y \subseteq X$ with the induced conditional equality $\approx_Y$ (i.e. the conditional equality given by $y_1 \approx_Y y_2 = y_1 \approx y_2$) is a conditional universe called a *subuniverse* of X. We usually treat ordinary subsets of conditional universes as conditional subuniverses without mentioning it explicitly.

The *product* of L-conditional universes $(X, \approx_X)$ and $(Y, \approx_Y)$ is defined as the set $X \times Y$ together with the mapping $\approx_{X\times Y} \colon (X \times Y) \times (X \times Y) \to L$ given by

$$(x_1, y_1) \approx_{X\times Y} (x_2, y_2) = (x_1 \approx_X x_2) \wedge (y_1 \approx_Y y_2). \tag{1.9}$$

The following result can be proved by a direct verification.

Lemma 4. *$X \times Y$ with $\approx_{X\times Y}$ is an L-conditional universe. For any reality h, h-realizations f and g of X and Y, respectively, the pair $(X^h \times Y^h, \approx^h_{X\times Y})$ where $\approx^h_{X\times Y}$ is given by $(x_1^h, y_1^h) \approx^h_{X\times Y} (x_2^h, y_2^h) = h((x_1, y_1) \approx_{X\times Y} (x_2, y_2))$ is a K-conditional universe. The partial mapping $(f \times g) \colon X \times Y \nrightarrow X^h \times Y^h$ defined by $(f \times g)(x, y) = (x^h, y^h)$ is an h-realization of $(X \times Y, \approx_{X\times Y})$.*

Unless stated differently, for h-realizations f and g of X and Y, respectively, we always use as an h-realization of $X \times Y$ the h-realization $f \times g$ from the above lemma.

1.3 Sets with missing information

As we already know, a universe U with missing information is modeled by a conditional universe X. In this section, we introduce models of sets with missing information partially present in U.

Take for an example an unknown universe U which certainly contains an element u_1 and possibly an element u_2. It does not contain any other elements. We are sure that u_1 is not equal to u_2. Moreover, consider a set V which possibly contains u_2 and does not contain any other element. Now, in a possible world where U does not contain u_2 but V contains u_2 holds that the set V is not in U. In all other three possible worlds it holds that V is in U.

We model missing pieces of information in the example by a Boolean algebra of conditions freely generated by two elements c_1 and c_2. The unknown universe U is represented by a conditional universe $X = \{x_1, x_2\}$. The conditional equality $\approx$ on X is given by $x_1 \approx x_1 = 1$, $x_2 \approx x_2 = c_1$ and $x_1 \approx x_2 = x_2 \approx x_1 = 0$. The partially known set V is represented by a mapping $A \colon X \to L$ given by $A(x_1) = 0$ and $A(x_2) = c_2$. The condition c_1 represents the condition under which u_2 is in U and c_2 the condition under which u_2 in in V.

We have four total realities $h_1, h_2, h_3, h_4 : L \to 2$ given by $h_1(c_1) = h_1(c_2) = h_2(c_1) = h_3(c_2) = 1$ and $h_2(c_2) = h_3(c_1) = h_4(c_1) = h_4(c_2) = 0$. Standard realizations of X are $X^{h_1} = X^{h_2} = \{x_1, x_2\}$ and $X^{h_3} = X^{h_4} = \{x_1\}$. Standard h-realizations of A are $A^{h_1} = A^{h_3} = \{x_2\}$ and $A^{h_2} = A^{h_4} = \emptyset$. We have $A^{h_1} \subseteq X^{h_1}$, $A^{h_2} \subseteq X^{h_2}$, $A^{h_4} \subseteq X^{h_4}$, but $A^{h_3} = \{x_2\} \nsubseteq \{x_1\} = X^{h_3}$. Therefore, we say that A is present in h_1, h_2 and h_4, but A is not present in h_3.

1.3.1 Conditional sets and relations

Let X be an L-conditional universe. A mapping $A : X \to L$ is called a *conditional set* in X. When L needs to be emphasized, we call conditional sets as *L-conditional sets*. But this is usually not necessary as L is given by $\approx$. We will use this convention also for other conditional mathematical objects defined later. We use the term "conditional" and called them "L-conditional" if needed. We denote by L^X the set of all conditional sets in X.

For $x \in X$, the value $A(x)$ is interpreted as the condition under which x is an element of A; a *membership condition*.

If for all $x \in X$ distinct from $x_1, x_2, \ldots, x_n$ we have $A(x) = 0$, A is also denoted $\{{}^{A(x_1)}/x_1, {}^{A(x_2)}/x_1, \ldots, {}^{A(x_n)}/x_n\}$. We usually write x instead of ${}^1/x$. We denote by $\emptyset$ the conditional set in X given by $\emptyset(x) = 0$ for all $x \in X$. By ${}^1A$ we denote the set $\{x \in X \mid A(x) = 1\}$ in X of elements known to be in A.

We adopt definitions of subsethood, intersection and union from the theory of L-sets. For two conditional sets A, B in X we say that A is a *subset* of B, writing $A \subseteq B$, if $A(x) \le B(x)$ for all $x \in X$. The *intersection* $A \cap B$ and *union* $A \cup B$ of A and B are defined by $(A \cap B)(x) = A(x) \wedge B(x)$ and $(A \cup B)(x) = A(x) \vee B(x)$, respectively. The intersection and union of a family A_j ($j \in J$) of conditional sets in X are given by

$$\bigcap_{j \in J} A_j(x) = \bigwedge_{j \in J} A_j(x), \qquad \bigcup_{j \in J} A_j(x) = \bigvee_{j \in J} A_j(x) \tag{1.10}$$

for all $x \in X$.

For a condition $c \in L$ and conditional set A in X we define the conditional set $c \to A$ in X by $(c \to A)(x) = c \to A(x)$ for $x \in X$. The conditional set $c \to A$ is called the *shift of A by c*.

A conditional set A in X is called *completely present* if

$$A(x) \le x \approx x \tag{1.11}$$

for all $x \in X$. The inequality (1.11) is read as follows: "if it is satisfied that x is an element of A then it is also satisfied that x is present".

Note that the above requirement is called strictness in literature, e.g. [16]. Only strict L-sets are covered by the literature. We consider also non-strict conditional sets and derive conditions under which conditional sets are present from presence of their elements. The *condition under which A is present* is given by

$$\mathrm{E}A = \bigwedge_{x \in X} A(x) \to (x \approx x). \tag{1.12}$$

The condition $\mathrm{E}A$ under which A is present is satisfied in a reality h if and only if $h(A(x)) \leq h(x \approx x)$ holds for all $x \in X$. Clearly, A is completely present iff $\mathrm{E}A = 1$.

We denote by X_{E} the greatest (with respect to $\subseteq$) completely present conditional set in X and call it the *conditional set of present elements*. We have $X_{\mathrm{E}}(x) = x \approx x$ for each $x \in X$. The conditional set X_{E} is interpreted as the conditional set of elements present in the conditional universe $(X, \approx)$. The condition under which x is in X_{E} is equal to the condition under which x is present.

Let $h\colon L \to K$ be a reality, f an h-realization of X and A a conditional set in X satisfying the following requirement: for each $x \in X$ it holds that if the h-realization x^h of x is not defined then $h(A(x)) = 0$. Then the *h-realization of A* is a K-conditional set A^h in X^h defined by

$$A^h(x^h) = \bigvee_{y^h = x^h} h(A(y)) \tag{1.13}$$

("$y^h = x^h$" means we are taking supremum over all $y \in X$ satisfying $y^h = x^h$). When the h-realization f needs to be emphasized, we denote A^h also by $f(A)$. If there is $x \in X$ such that $h(A(x)) \neq 0$ and the h-realization x^h of x is not defined then the h-realization of A is not defined.

If A^h is completely present then we say that *A is present in the h-realization f*. If it holds $A^h(x^h) = 1$, we say that *x belongs to A in the h-realization f*.

We say that *A is present in a reality h* if A is present in any h-realization of X. We say that *x belongs to A* if x belongs to A in any h-realization of X.

Suppose f is a standard h-realization of X. Then A^h is defined if and only if it is satisfied in h that A is present. If A^h is defined then it is an ordinary subset of X^h. We view A^h as the form that the unknown set represented by A takes in the reality h. For $x_h \in X^h$, we have $x_h \in A^h$ iff there exists $x \in X$ satisfying $x^h = x_h$ and $h(A(x)) = 1$.

Lemma 5. *Let $h_1 : L \to K_1$ and $h_2 : K_1 \to K_2$ be realities, $f_1 : X \twoheadrightarrow Y_1$ an h_1-realization of X and $f_2 : Y_1 \twoheadrightarrow Y_2$ an h_2-realization of Y_1. Then for each conditional set A in X such that $A^{h_1 h_2}$ is defined it holds that $A^{h_2 \circ h_1}$ is defined and $A^{h_1 h_2} = A^{h_2 \circ h_1}$.*

Proof. Let $x \in X$. Suppose $A^{h_1 h_2}$ is defined and $x^{h_2 \circ h_1}$ is not defined. We distinguish two cases. First, if x^{h_1} is not defined then since A^{h_1} is defined, $h_1(A(x)) = 0$ and so $h_2(h_1(A(x))) = 0$. Second, if x^{h_1} is defined but $x^{h_1 h_2}$ is not then as $A^{h_1 h_2}$ is defined $h_2(A^{h_1}(x^{h_1})) = 0$ and by (1.13) we have $h_2(h_1(A(x))) = 0$. In both cases $h_2(h_1(x)) = 0$. Therefore, $A^{h_2 \circ h_1}$ is defined.

For each $x^{h_1 h_2} \in X^{h_1 h_2}$ by Lemma 1 we have $A^{h_1 h_2}(x^{h_1 h_2}) = \bigvee_{y^{h_1 h_2} = x^{h_1 h_2}} h_2(A^{h_1}(y^{h_1})) = \bigvee_{y^{h_1 h_2} = x^{h_1 h_2}} \bigvee_{z^{h_1} = y^{h_1}} h_2(h_1(A(z))) = \bigvee_{y^{h_1 h_2} = x^{h_1 h_2}} h_2(h_1(A(y))) = A^{h_2 \circ h_1}(x^{h_1 h_2})$. □

Generally, the fact that $A^{h_2 \circ h_1}$ is defined does not imply that $A^{h_2 h_1}$ is defined.

A conditional set R in $X \times Y$ is called a *binary conditional relation between X and Y*. If $X = Y$, we also talk about a binary conditional relation *on* X. Membership conditions $R(x,y)$ are interpreted as conditions under which elements of X and Y are related. The conditional equality $\approx$ is an example of a binary conditional relation on X. One can also define ternary and, in general, n-ary conditional relations. On the other hand, conditional sets can be regarded as unary conditional relations and single values from L as nullary conditional relations.

For $x \in X$ we define a conditional set R_x in Y by $R_x(y) = R(x,y)$. Similarly, for $y \in Y$ we set $R_y(x) = R(x,y)$, obtaining a conditional set R_y in X.

By definition, R is completely present iff

$$R(x,y) \leq (x \approx x) \wedge (y \approx y) \tag{1.14}$$

for all $x \in X$ and $y \in Y$. By (1.1), the conditional equality $\approx$ is completely present.

Let f and g are h-realizations of X and Y, respectively. Then the h-realization R^h of a binary conditional relation R between X and Y is defined if and only if for each $x \in X$ and $y \in Y$ it holds that x^h or y^h is not defined implies $h(R(x,y)) = 0$. If the h-realization R^h of R is defined then it is a binary conditional relation between X^h and Y^h and satisfies

$$R^h(x^h, y^h) = \bigvee_{\bar{x}^h = x^h} \bigvee_{\bar{y}^h = y^h} h(R(\bar{x},\bar{y})) = \bigvee_{\bar{x}^h = x^h} (R_{\bar{x}})^h(y^h) = \bigvee_{\bar{y}^h = y^h} (R_{\bar{y}})^h(x^h). \tag{1.15}$$

From the technical point of view, conditional sets and relations are L-fuzzy sets and relations in the sense of [10]. Throughout the thesis, we use known results on L-fuzzy sets and relations for L being a complete residuated lattice [3].

The *h-realization c^h of a condition c* is defined by $c^h = h(c)$.

Restricted realizations. Generally, an h-realization A^h of A does not have to be defined even if $h(\mathrm{E}A) = 1$. For example, let $L = 2$, $X = \{x_1, x_2\}$, $x_1 \approx x_1 = x_1 \approx x_2 = x_2 \approx x_1 = x_2 \approx x_2 = 1$, A be the conditional set in X given by $A(x_1) = A(x_2) = 1$, h be the identity on L, $Y = \{x_1\}$, f be the h-realization $X \twoheadrightarrow Y$ of X given by $f(x_1) = x_1$ and $f(x_2)$ be not defined. Then $\mathrm{E}A = 1$, but the h-realization A^h of A is not defined.

For a conditional set A in X, a reality h and an h-realization $f : X \twoheadrightarrow Y$ we define a conditional set $A|_f$ in X by $A|_f(x) = A(x)$ if x^h is defined, otherwise $A|_f(x) = 0$. The conditional set $A|_f$ is also denoted simply by $A|_h$ and called the *restriction of A to f*. For a nullary conditional relation $c \in L$ we set $c|_f = c$. The h-realization $(A|_h)^h$ of $A|_h$ is always defined and it holds

$$(A|_h)^h(x^h) = \bigvee_{\bar{x}^h = x^h} h(A(\bar{x})) \tag{1.16}$$

for each $x \in X_h$. The realization $(A|_h)^h$ is also denoted by $A^{|h}$ and called the *restricted h-realization of A*.

If the h-realization A^h of A is defined then clearly $A = A|_h$ and thus $A^h = A^{|h}$.

Lemma 6. *Let $h_1 : L \to K_1$ and $h_2 : K_1 \to K_2$ be realities, Y an h_1-realization of X and Z an h_2-realization of Y. Then for each conditional set A in X it holds*

$$A^{|h_1|h_2} = A^{|(h_2 \circ h_1)}.$$

Proof. The proof is analogous to the proof of Lemma 5.

For each $x^{h_1 h_2} \in X^{h_1 h_2}$ by Lemma 1 and (1.16) we have $A^{|h_1|h_2}(x^{h_1 h_2}) = \bigvee_{y^{h_1 h_2} = x^{h_1 h_2}} h_2(A^{|h_1}(y^{h_1})) = \bigvee_{y^{h_1 h_2} = x^{h_1 h_2}} \bigvee_{z^{h_1} = y^{h_1}} h_2(h_1(A(z))) = \bigvee_{y^{h_1 h_2} = x^{h_1 h_2}} h_2(h_1(A(y))) = A^{|(h_2 \circ h_1)}(x^{h_1 h_2})$. □

If f is a standard h-realization of X then easily

$$A^{|h} = (A \cap X_{\mathrm{E}})^h. \tag{1.17}$$

Let h be a total reality, f and g two merging and omitting h-realization of X. Then there may be no bijection between restricted realizations $f(A|_f)$ and $g(A|_g)$. For example, let

$L = 2$, $X = \{x_1, x_2\}$, $x_1 \approx x_1 = x_1 \approx x_2 = x_2 \approx x_1 = x_2 \approx x_2 = 1$, $A = \{x_1\}$, h be the identity on 2, $Y = \{x_1\}$, $f\colon X \to Y$ be the h-realization of X given by $f(x_1) = f(x_2) = x_1$, $Z = \{x_2\}$, $g\colon X \nrightarrow Z$ be the h-realization of X given by $g(x_1)$ be not defined and $g(x_2) = x_2$. Then $f(A|_f) = \{x_1\}$, but $g(A|_g) = \emptyset$.

It can be directly checked that for a family A_j ($j \in J$) of conditional sets in X, reality h and h-realization f of X it holds

$$\left(\bigcup_{j\in J} A_j\right)^{|h} = \bigcup_{j\in J} A_j^{|h}. \tag{1.18}$$

Let X and Y be L-conditional universes, $h\colon L \to K$ a reality, f and g be h-realizations of X and Y, respectively. Then the restricted h-realization of a binary conditional relation R between X and Y is a binary conditional relation between X^h and Y^h and satisfies

$$\begin{aligned} R^{|h}(x^h, y^h) &= \bigvee_{\bar{x}^h = x^h} \bigvee_{\bar{y}^h = y^h} h(R(\bar{x}, \bar{y})) \\ &= \bigvee_{\bar{x}^h = x^h} (R_{\bar{x}})^{|h}(y^h) = \bigvee_{\bar{y}^h = y^h} (R_{\bar{y}})^{|h}(x^h). \end{aligned} \tag{1.19}$$

It can be easily checked that for $R = \approx$, the K-conditional equality $\approx^{|h}$ from Sec. 1.2.3 satisfies (1.19).

Respectability. Let X be an L-conditional universe, A a conditional set in X, $h\colon L \to K$ a reality and f an h-realization of X. Generally, it does not hold that $h(A(x)) = A^h(x^h)$ for all $x \in X_h$. By (1.13), only $h(A(x)) \le A^h(x^h)$ holds. For example, let $L = 2$, $X = \{x_1, x_2\}$, $x_1 \approx x_1 = x_1 \approx x_2 = x_2 \approx x_1 = x_2 \approx x_2 = 1$, $A = \{x_1\}$, h be the identity on L, $Y = \{x_1\}$, $f\colon X \to Y$ be the h-realization of X given by $f(x_1) = f(x_2) = x_1$. Then $h(A(x_2)) = 0$, but $A^h(x_2^h) = 1$.

We say that an h-realization f of X *respects* a conditional set A in X if for each $x_1, x_2 \in X_h$ from $x_1^h = x_2^h$ it follows that $h(A(x_1)) = h(A(x_2))$.

The following theorem captures the relationship between respectability and restricted realizations of conditional sets.

Theorem 2. *Let h be a reality, f an h-realization of X and A a conditional set in X. Then the following two statements are equivalent.*

1. *f respects A.*
2. *$A^{|h}(x^h) = h(A(x))$ for each $x \in X_h$.*

Proof. We first prove the implication from 1. to 2. By (1.16) we have $A^{|h}(x^h) = \bigvee_{\bar{x}^h = x^h} h(A(\bar{x})) = \bigvee_{\bar{x}^h = x^h} h(A(x)) = h(A(x))$ for each $x \in X_h$.

Now, we prove the converse implication. Let $x_1, x_2 \in X_h$ such that $x_1^h = x_2^h$. Then by second statement we have $h(A(x_1)) = A^{|h}(x_1^h) = A^{|h}(x_2^h) = h(A(x_2))$. □

Compatibility. We study relationships between compatibility of conditional sets with some special conditional relations and respectability of realizations.

We call a conditional set A in X *compatible with a binary conditional relation R on X* if

$$A(x) \wedge R(x,y) \leq A(y) \tag{1.20}$$

for all $x, y \in X$.

Let R_X be a binary conditional relation on X, R_Y be a binary conditional relation on Y and $R_X \times R_Y$ be the binary conditional relation on $X \times Y$ defined by $R_X \times R_Y((x_1,y_1),(x_2,y_2)) = R_X(x_1,x_2) \wedge R_Y(y_1,y_2)$ for each $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. If a binary conditional relation R between X and Y (a conditional set in $X \times Y$) is compatible with $R_X \times R_Y$ then we say that R is *compatible with R_X and R_Y*. By definition we have that R is compatible with R_X and R_Y iff

$$R(x_1,y_1) \wedge R_X(x_1,x_2) \wedge R_Y(y_1,y_2) \leq R(x_2,y_2) \tag{1.21}$$

for each $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Note that compatibility of R with R_X and R_Y does not generally imply that $R(x_1,y) \wedge R_X(x_1,x_2) \leq R(x_2,y)$ and $R(x,y_1) \wedge R_Y(y_1,y_2) \leq R(x,y_2)$ for each $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$. If the right hand side of the previous implication is satisfied then we say that R is *compatible with R_X and R_Y from both sides*. If R_X and R_Y are reflexive then the implication holds. If R and R_X are binary conditional relations on X then we say just that R is compatible with R_X (from both sides) instead of R is compatible with R_X and R_X (from both sides). It can be easily checked that R is compatible with R_X and R_Y from both sides if and only if a conditional set R_x in Y is compatible with R_Y and a conditional set R_y in X is compatible with R_X for each $x \in X$ and $y \in Y$.

Let $\mathrm{E}_\approx$ be the binary conditional relation $\mathrm{E}_\approx$ on X defined by

$$\mathrm{E}_{\approx}(x_1,x_2) = ((x_1 \approx x_1) \to (x_1 \approx x_2)) \wedge ((x_2 \approx x_2) \to (x_1 \approx x_2)) \tag{1.22}$$

for $x_1,x_2 \in X$. The conditional relation $\mathrm{E}_{\approx}$ is called the *conditional extensional equality* on X. The L-relation $\mathrm{E}_{\approx}$ is studied in [14]. Let h be a reality and f an h-realization of X. Then for $x_1,x_2 \in \mathrm{dom} f$ we have $h(\mathrm{E}_{\approx}(x_1,x_2)) = 1$ if and only if it hods if x_1 or x_2 is present in f then so is the other and x_1 and x_2 are equal in f.

It can be directly checked that $\mathrm{E}_{\approx}(x,y) = 1$ if and only if x and y are extensionally equal. Clearly, $\mathrm{E}_{\approx}$ is reflexive and symmetric. To prove transitivity we express $\mathrm{E}_{\approx}(x_1,x_2)$ as

$$\mathrm{E}_{\approx}(x_1,x_2) = \mathrm{R}_{\approx}(x_1,x_2) \wedge \mathrm{R}_{\approx}(x_2,x_1) \tag{1.23}$$

where

$$\mathrm{R}_{\approx}(x_1,x_2) = (x_1 \approx x_1) \to (x_1 \approx x_2) \tag{1.24}$$

and prove the transitivity of $\mathrm{R}_{\approx}$. Notice that the L-relation $\mathrm{R}_{\approx}$ also appears in [15]. The inequality $\mathrm{R}_{\approx}(x_1,x_2) \wedge \mathrm{R}_{\approx}(x_2,x_3) \leq \mathrm{R}_{\approx}(x_1,x_3) = (x_1 \approx x_1) \to (x_1 \approx x_3)$ is by adjointness equivalent to $(x_1 \approx x_1) \wedge \mathrm{R}_{\approx}(x_1,x_2) \wedge \mathrm{R}_{\approx}(x_2,x_3) \leq x_1 \approx x_3$. The last inequality holds since $(x_1 \approx x_1) \wedge \mathrm{R}_{\approx}(x_1,x_2) \wedge \mathrm{R}_{\approx}(x_2,x_3) = (x_1 \approx x_1) \wedge ((x_1 \approx x_1) \to (x_1 \approx x_2)) \wedge \mathrm{R}_{\approx}(x_2,x_3) \leq (x_1 \approx x_2) \wedge \mathrm{R}_{\approx}(x_2,x_3) \leq x_2 \approx x_2 \wedge ((x_2 \approx x_2) \to (x_2 \approx x_3)) \leq x_2 \approx x_3$. Therefore, $\mathrm{E}_{\approx}$ is a reflexive conditional equality. If X is an ordinay universe then $\mathrm{E}_{\approx}$ is the ordinary equality on X.

For $x_1,x_2 \in X_h$ we have $h(\mathrm{R}_{\approx}(x_1,x_2)) = h((x_1 \approx x_1) \to (x_1 \approx x_2)) = (x_1^h \approx^{|h} x_1^h) \to (x_1^h \approx^{|h} x_2^h) = \mathrm{R}_{\approx^{|h}}(x_1^h,x_2^h)$ proving

$$h(\mathrm{R}_{\approx}(x_1,x_2)) = \mathrm{R}_{\approx^{|h}}(x_1^h,x_2^h) \tag{1.25}$$

and by (1.23) also

$$h(\mathrm{E}_{\approx}(x_1,x_2)) = \mathrm{E}_{\approx^{|h}}(x_1^h,x_2^h). \tag{1.26}$$

Lemma 7. *Every completely present conditional set in X compatible with $\approx$ is also compatible with $\mathrm{E}_{\approx}$.*

Proof. Let A be a completely present conditional set in X compatible with $\approx$. Then for $x_1,x_2 \in X$ we have $A(x_1) \wedge \mathrm{E}_{\approx}(x_1,x_2) \leq A(x_1) \wedge (x_1 \approx x_1) \wedge ((x_1 \approx x_1) \to (x_1 \approx x_2)) \leq A(x_1) \wedge (x_1 \approx x_2) \leq A(x_2)$. We showed that A is compatible with $\mathrm{E}_{\approx}$. □

Lemma 8. *For each conditional set A in X compatible with* $\mathrm{E}_{\approx}$ *it holds that any realization of X respects A.*

Proof. Let f be an h-realization of X, $x_1, x_2 \in X_h$ such that $x_1^h = x_2^h$. By (1.26) and reflexivity of $\mathrm{E}_{\approx|^h}$, we have $h(\mathrm{E}_{\approx}(x_1,x_2)) = \mathrm{E}_{\approx|^h}(x_1^h, x_2^h) = \mathrm{E}_{\approx|^h}(x_1^h, x_1^h) = 1$. Therefore, $h(A(x_1)) = h(A(x_1) \wedge \mathrm{E}_{\approx}(x_1,x_2)) \leq h(A(x_2))$. The converse inequality can be shown similarly. □

Lemma 9. *Suppose that any merging total realization of X which is a function respects a conditional set A in X. Then A is compatible with* $\mathrm{E}_{\approx}$.

Proof. Let h be a total reality, $f : X \to Y$ a merging h-realization of X such that f is a function, $x_1, x_2 \in X$. We show that $h(A(x_1) \wedge \mathrm{E}_{\approx}(x_1,x_2)) \leq h(A(x_2))$. The fact that f is a function implies that h-realizations $x_1^h, x_2^h, \approx^h$ and A^h of $x_1, x_2, \approx$ and A, respectively, are defined. Suppose $h(A(x_1)) = 1$ and $h(\mathrm{E}_{\approx}(x_1,x_2)) = 1$. By (1.26), $h(\mathrm{E}_{\approx}(x_1,x_2)) = \mathrm{E}_{\approx^h}(x_1^h, x_2^h) = 1$ and thus x_1^h is extensionally equal to x_2^h. As f is merging, X^h is separated which implies $x_1^h = x_2^h$. Now, the assumption yields $1 = h(A(x_1)) = h(A(x_2))$.

We showed that $h(A(x_1) \wedge \mathrm{E}_{\approx}(x_1,x_2)) \leq h(A(x_2))$ holds for each total reality h. As L is complete and atomic, A is compatible with $\mathrm{E}_{\approx}$. □

Theorem 3. *Let A be a conditional set in X. Then the following statements are equivalent.*

1. *A is compatible with* $\mathrm{E}_{\approx}$.
2. *Any total merging realization of X which is a function respects A.*
3. *Any realization of X respects A.*

Proof. The implication from the second to the first statement follows directly from Lemma 9. The implication from 1. to 3. is due to Lemma 8. The implication from 3. to 2. is trivial. □

Lemma 10. *If A is completely present and compatible with $\approx$ then any realization of X respects A.*

Proof. By Lemma 7, A is compatible with $\mathrm{E}_{\approx}$. Therefore, the claim follows from Lemma 8. □

The requirement on compatibility of A with $\mathrm{E}_{\approx}$ is usually not satisfied. We make the requirement weaker.

Lemma 11. *If A is a conditional set in X compatible with $\approx$ then any total and omitting realization of X respects A.*

Proof. Let h be a total reality and f an omitting h-realization of X. If $x_1, x_2 \in X_h$ and $x_1^h = x_2^h$ then as the h-realization f of X is omitting, $1 = x_1^h \approx^h x_1^h = x_1^h \approx^h x_2^h = h(x_1 \approx x_2)$ and compatibility of A with $\approx$ yields $h(A(x_1)) = h(A(x_1)) \wedge h(x_1 \approx x_2) \leq h(A(x_2))$. The converse inequality can be shown similarly. □

Lemma 12. *If each standard realization of X respects A then A is compatible with $\approx$.*

Proof. It suffices to show that $h(A(x_1) \wedge (x_1 \approx x_2)) \leq h(A(x_2))$ for each $x_1, x_2 \in X$ and total reality h. The only interesting case is when $h(A(x_1)) = 1$ and $h(x_1 \approx x_2) = 1$. Then by (1.1), $h(x_1 \approx x_1) = 1$ and $h(x_2 \approx x_2) = 1$. Let f be a standard h-realization of X. Then x_1^h and x_2^h are defined and $x_1^h = x_2^h$. Now by the assumption, $1 = h(A(x_1)) = A^{|h}(x_1^h) = A^{|h}(x_2^h) = h(A(x_2))$, proving that also $h(A(x_2)) = 1$. □

Theorem 4. *Let A be a conditional set in X. Then the following statements are equivalent.*

1. *A is compatible with $\approx$.*
2. *Any standard realization of X respects A.*
3. *Any total and omitting realization of X respects A.*

Proof. The implication from 2. to 1. statement follows directly from Lemma 12. The implication from 1. to 3. is due to Lemma 11. The implication from 3. to 2. is trivial. □

We conclude this part with two useful consequences of compatibility with $\approx$.

Lemma 13. *If A is compatible with $\approx$ then for any reality h and h-realization f of X it holds that the restricted h-realization $A^{|h}$ of A is compatible with $\approx^{|h}$.*

Proof. Let $x_1^h, x_2^h \in X^h$. Then by (1.5) and (1.16) we have $A^{|h}(x_1^h) \wedge (x_1^h \approx^{|h} x_2^h) = \bigvee_{\bar{x}_1^h = x_1^h} h(A(\bar{x}_1)) \wedge (x_1^h \approx^{|h} x_2^h) = \bigvee_{\bar{x}_1^h = x_1^h} h(A(\bar{x}_1)) \wedge (\bar{x}_1^h \approx^{|h} x_2^h) = \bigvee_{\bar{x}_1^h = x_1^h} h(A(\bar{x}_1) \wedge (\bar{x}_1 \approx x_2)) \leq \bigvee_{\bar{x}_1^h = x_1^h} h(A(x_2)) = h(A(x_2)) \leq A^{|h}(x_2^h)$. □

Lemma 14. *Let A be a conditional set in X compatible with $\approx$, h a total reality, f and g two merging and omitting h-realizations of X. Then there is a bijection between restricted h-realizations $f(A|_f)$ and $g(A|_g)$.*

Proof. Let h be a total reality, $f : X \twoheadrightarrow Y$ and $g : X \twoheadrightarrow Z$ two merging and omitting h-realizations of X. We consider the bijection r between Y and Z defined in Sec. 1.2.3. We show that the restriction of r to $f(A|_f)$ denoted by r_A is a bijection between $f(A|_f)$ and $g(A|_g)$. Suppose that $f(x) \in f(A|_f)$ where $x \in \mathrm{dom} f$. Then there is $\bar{x} \in \mathrm{dom} g$ such that $r(f(x)) = g(\bar{x})$ and by the definition of r, $h(x \approx \bar{x}) = 1$. Now, by the compatibility of A with $\approx$, $h(A(x)) = 1$ and $h(A(x)) \wedge h(x \approx \bar{x}) \leq h(A(\bar{x}))$ and thus $h(A(\bar{x})) = 1$. By (1.16), $g(A|_g)(g(x)) = 1$. We showed that r_A is a mapping, similarly can be shown that r_A is surjective. (The injectivity of r_A follows from the injectivity of r.) Therefore, r_A is a bijection between $f(A|_f)$ and $g(A|_g)$. □

Relational products. We introduce the relational product operators $\circ$ and $\triangleleft$, well-known in the fuzzy set theory [18]. For two conditional sets A_1, A_2 in X we set

$$A_1 \circ A_2 = \bigvee_{x \in X} A_1(x) \wedge A_2(x), \qquad A_1 \triangleleft A_2 = \bigwedge_{x \in X} A_1(x) \to A_2(x). \tag{1.27}$$

Relational products simplify many proofs in the thesis.

We denote the relational product $\circ$ with the same symbol as the composition of partial mappings. From context it is always clear what is the intended meaning of $\circ$.

For conditional sets A in X and B in Y and binary conditional relations R in $X \times Y$ and S in $Y \times Z$ we set

$$\begin{aligned}
(A \circ R)(y) &= A \circ R_y, & (R \circ B)(x) &= R_x \circ B, & (R \circ S)(x,z) &= R_x \circ S_z,\\
(A \triangleleft R)(y) &= A \triangleleft R_y, & (R \triangleleft B)(x) &= R_x \triangleleft B, & (R \triangleleft S)(x,z) &= R_x \triangleleft S_z,
\end{aligned}$$

obtaining conditional sets $A \circ R$ and $A \triangleleft R$ in Y, conditional sets $R \circ B$ and $R \triangleleft B$ in X and binary conditional relations $R \circ S$ and $R \triangleleft S$ between X and Z. For any conditional relations R, S, T of any arity for which the products make sense it holds

$$R \circ (S \circ T) = (R \circ S) \circ T, \qquad R \triangleleft (S \triangleleft T) = (R \circ S) \triangleleft T \tag{1.28}$$

and for any two collections A_i $(i \in I)$ and B_j $(j \in J)$ of conditional sets in the same universe it holds

$$\begin{aligned}
\bigvee_{i \in I} \bigvee_{j \in J} A_i \circ B_j &= \Big(\bigcup_{i \in I} A_i\Big) \circ \Big(\bigcup_{j \in J} B_j\Big),\\
\bigwedge_{i \in I} \bigvee_{j \in J} A_i \triangleleft B_j &= \Big(\bigcup_{i \in I} A_i\Big) \triangleleft \Big(\bigcup_{j \in J} B_j\Big).
\end{aligned} \tag{1.29}$$

The first equality in (1.28) allows us to omit parentheses and write simply $R \circ S \circ T$.

Lemma 15. *The following holds for any binary conditional relation R on X.*

1. If R is reflexive then $R \circ R \supseteq R$.

2. If R is transitive then $R \circ R \subseteq R$.

3. If R is reflexive and transitive then $R \circ R = R$.

Proof. 1. For $x,z \in X$ we have $R \circ R(x,z) = \bigvee_{y\in X} R(x,y) \wedge R(y,z) \geq R(x,x) \wedge R(x,z) = R(x,z)$.

2. For $x,z \in X$ we have $R \circ R(x,z) = \bigvee_{y\in X} R(x,y) \wedge R(y,z) \leq \bigvee_{y\in X} R(x,z) = R(x,z)$.

3. It follows directly from the first and second part. □

Lemma 16. *Let R be a reflexive binary conditional relation on X. Then for each conditional set A in X the following statements are equivalent.*

1. A is compatible with R.

2. $A \circ R = A$.

3. $R \triangleleft A = A$.

Proof. Suppose $A \circ R = A$. Then for $y \in X$ we have $A \circ R(y) = \bigvee_{x\in X} A(x) \wedge R(x,y) = A(y)$. From which it follows $A(x) \wedge R(x,y) \leq A(y)$ for each $x,y \in X$. Therefore, A is compatible with R. Note that this implication holds even if R is not reflexive.

Suppose A is compatible with R. Then for $x \in X$ by reflexivity of R we have $A \circ R(x) = \bigvee_{y\in X} A(y) \wedge R(y,x) \geq A(x) \wedge R(x,x) = A(x)$ and by compatibility of A with R we have $A \circ R(x) = \bigvee_{y\in X} A(y) \wedge R(y,x) \leq \bigvee_{y\in X} A(x) = A(x)$. We showed that $A \circ R \supseteq A$ and $A \circ R \subseteq A$, thus the second statement holds. The equivalence of the first and second statement was proved.

Finally, we have $A \circ R(x) = \bigvee_{y\in X} A(y) \wedge R(y,x) \leq A(x)$ for each $x \in X$ if and only if $A(y) \wedge R(y,x) \leq A(x)$ for each $x,y \in X$ iff $A(y) \leq R(y,x) \to A(x)$ for each $x,y \in X$ iff $A(y) \leq \bigwedge_{x\in X} R(y,x) \to A(x) = R \triangleleft A(y)$ for each $y \in X$. Now, as we know from the second part of the proof, reflexivity of R implies $A \circ R(x) \geq A(x)$ for $x \in X$. By reflexivity of R we also have $R \triangleleft A(y) = \bigwedge_{x\in X} R(y,x) \to A(x) \leq R(y,y) \to A(y) = A(y)$ for $y \in X$. Together we obtain $A \circ R(x) = A(x)$ for each $x \in X$ if and only if $R \triangleleft A(y) = A(y)$ for each $y \in X$. Which proves the equivalence of the second and third statement.

□

From the previous lemma and the definition of compatibility of a binary conditional relation (from both sides) it follows:

Lemma 17. *If R is a binary conditional relation between X and Y, R_X is a reflexive binary conditional relation on X and R_Y is a reflexive binary conditional relation on Y then the following statements are true.*

1. R is compatible with R_X and R_Y if and only if $R_X \circ R \circ R_Y = R$.

2. R is compatible with R_X and R_Y from both sides if and only if $R_X \circ R = R = R \circ R_Y$.

Since $\approx$ is not generally reflexive, we can not use Lemmas 16 and 17 on $\approx$. Therefore, we introduce the following reflexive binary relation. Let $\mathrm{T}_{\approx}$ be a binary conditional relation on X given by

$$\mathrm{T}_{\approx}(x_1,x_2) = \begin{cases} 1 & \text{if } x_1 = x_2, \\ x_1 \approx x_2 & \text{otherwise.} \end{cases} \tag{1.30}$$

We can easily check that $\mathrm{T}_{\approx}$ is reflexive, transitive, symmetric and $x_1 \approx x_2 \le \mathrm{T}_{\approx}(x_1,x_2)$ for each $x_1,x_2 \in X$. $\mathrm{T}_{\approx}$ is also compatible with $\approx$. Indeed, for each $x_1,x_2,x_1',x_2' \in X$ by the transitivity of $\mathrm{T}_{\approx}$, we have $(x_1' \approx x_1) \wedge \mathrm{T}_{\approx}(x_1,x_2) \wedge (x_2 \approx x_2') \le \mathrm{T}_{\approx}(x_1',x_1) \wedge \mathrm{T}_{\approx}(x_1,x_2) \wedge \mathrm{T}_{\approx}(x_2,x_2') \le \mathrm{T}_{\approx}(x_1',x_2')$.

For a reality h, h-realization f of X and $x_1,x_2 \in \mathrm{dom} f$ generally it does not hold that $h(\mathrm{T}_{\approx}(x_1,x_2)) = \mathrm{T}_{\approx^{|h}}(x_1^h,x_2^h)$. For example, let $L = 2$, $X = \{x_1,x_2\}$, $x_1 \approx x_1 = x_1 \approx x_2 = x_2 \approx x_1 = x_2 \approx x_2 = 0$, h be the identity on 2, $Y = \{x_1\}$ and $f : X \to Y$ be the h-realization of X given by $f(x_1) = f(x_2) = x_1$. Then $h(\mathrm{T}_{\approx}(x_1,x_2)) = h(x_1 \approx x_2) = h(0) = 0$, but $\mathrm{T}_{\approx^{|h}}(x_1^h,x_2^h) = \mathrm{T}_{\approx^{|h}}(x_1,x_1) = 1$.

Lemma 18. *For a reality h and h-realization f of X it holds $T_{\approx}{}^{|h} = T_{\approx^{|h}}$.*

Proof. For $x^h \in X^h$ we have $\mathrm{T}_{\approx}{}^{|h}(x^h,x^h) = \bigvee_{y^h=x^h,z^h=x^h} h(\mathrm{T}_{\approx}(y,z)) \ge h(\mathrm{T}_{\approx}(x,x)) = 1 = \mathrm{T}_{\approx^{|h}}(x^h,x^h)$.

Let $x_1^h,x_2^h \in X^h$ such that $x_1^h \ne x_2^h$. First observe that if for $\bar{x}_1,\bar{x}_2 \in X_h$ holds $\bar{x}_1^h = x_1^h$ and $\bar{x}_2^h = x_2^h$ then $\bar{x}_1 \ne \bar{x}_2$. Now $\mathrm{T}_{\approx}{}^{|h}(x_1^h,x_2^h) = \bigvee_{\bar{x}_1^h=x_1^h} \bigvee_{\bar{x}_2^h=x_2^h} h(\mathrm{T}_{\approx}(\bar{x}_1,\bar{x}_2)) = \bigvee_{\bar{x}_1^h=x_1^h} \bigvee_{\bar{x}_2^h=x_2^h} h(\bar{x}_1 \approx \bar{x}_2) = \bigvee_{\bar{x}_1^h=x_1^h} \bigvee_{\bar{x}_2^h=x_2^h} \bar{x}_1^h \approx^{|h} \bar{x}_2^h = \bigvee_{\bar{x}_1^h=x_1^h} \bigvee_{\bar{x}_2^h=x_2^h} x_1^h \approx^{|h} x_2^h = x_1^h \approx^{|h} x_2^h = \mathrm{T}_{\approx^{|h}}(x_1^h,x_2^h)$. □

We can easily check that $A \circ \mathrm{T}_{\approx} = A \cup (A \circ \approx)$.

Lemma 19. *Let A be a conditional set in X. Then A is compatible with $\approx$ if and only if $A \circ T_{\approx} = A$.*

Proof. Assume A is compatible with $\approx$. By $\mathrm{T}_\approx$ is reflexive, we have $A \subseteq A \circ \mathrm{T}_\approx$. Conversely, for $x \in X$ we have $A \circ \mathrm{T}_\approx(x) = \bigvee_{y \in X} A(y) \wedge \mathrm{T}_\approx(y,x) = A(x) \vee \bigvee_{y \in X} A(y) \wedge (y \approx x) \leq A(x) \vee A(x) = A(x)$.

Suppose $A \circ \mathrm{T}_\approx = A$. For $x_1, x_2 \in X$ we have $A(x_1) \wedge (x_1 \approx x_2) \leq A(x_1) \wedge \mathrm{T}_\approx(x_1, x_2) \leq A(x_2)$ showing that A is compatible with $\approx$. □

Lemma 20. *If R is a binary conditional relation between X and Y such that $R \circ T_{\approx_Y} = R$ then for every conditional set A in X it holds that $A \triangleleft R$ is compatible with $\approx_Y$.*

Proof. Observe that for $c_1, c_2, c_3 \in L$ we have $(c_1 \to c_2) \wedge c_3 = (c_1' \vee c_2) \wedge c_3 = (c_1' \wedge c_3) \vee (c_2 \wedge c_3) \leq c_1' \vee (c_2 \wedge c_3) = c_1 \to (c_2 \wedge c_3)$.

By the observation and the compatibility of R_x with $\approx_Y$, for each $x \in X$ we have for $y_1, y_2 \in Y$, $A \triangleleft R(y_1) \wedge (y_1 \approx_Y y_2) = \bigwedge_{x \in X} (A(x) \to R(x, y_1)) \wedge (y_1 \approx_Y y_2) \leq \bigwedge_{x \in X} A(x) \to (R(x, y_1) \wedge (y_1 \approx_Y y_2)) \leq \bigwedge_{x \in X} A(x) \to R(x, y_2) = A \triangleleft R(y_2)$.

□

Lemma 21. *Let h be a reality and f a faithful h-realization of X. Then for a conditional set A in X such that the h-realization A^h of A is defined it holds that the h-realization $(A \circ T_\approx)^h$ of $A \circ T_\approx$ is also defined.*

Proof. We need to show that $h(A \circ \mathrm{T}_\approx(x)) = 0$ for every $x \in X \setminus X_h$. Let $x \in X \setminus X_h$. We have $h(A \circ \mathrm{T}_\approx(x)) = h(\bigvee_{\bar{x} \in X} A(\bar{x}) \wedge \mathrm{T}_\approx(\bar{x}, x)) = 0$ iff $h(A(\bar{x}) \wedge \mathrm{T}_\approx(\bar{x}, x)) = 0$ for all $\bar{x} \in X$. We distinguish two cases. First, if $x = \bar{x}$ then $h(A(\bar{x}) \wedge \mathrm{T}_\approx(\bar{x}, x)) = h(A(x))$ and as A^h is defined, $h(A(x)) = 0$. Second, suppose $x \neq \bar{x}$. Then $h(A(\bar{x}) \wedge \mathrm{T}_\approx(\bar{x}, x)) = h(A(\bar{x})) \wedge h(\bar{x} \approx x)$. By X^h is faithful, we have $h(x \approx x) = 0$ and thus $h(\bar{x} \approx x) = 0$. Therefore, $h(A(\bar{x})) \wedge h(\bar{x} \approx x)$ is also equal to 0. □

Partial respectability. Generally, it does not hold that a realization of X respects a conditional set in X compatible with $\approx$. For example, let $L = 2$, $X = \{x_1, x_2\}$, $x_1 \approx x_1 = x_1 \approx x_2 = x_2 \approx x_1 = x_2 \approx x_2 = 0$, h be the identity on 2, $Y = \{x_1\}$, $f : X \to Y$ be the h-realization of X given by $f(x_1) = f(x_2) = x_1$ and $A = \{x_1\}$. Then $x_1^h = x_2^h$ but $h(A(x_1)) = 1 \neq 0 = h(A(x_2))$.

For a reality h we say that an h-realization f of X *partially respects a conditional set* A if for each $x_1, x_2 \in \mathrm{dom} f$ such that $h(x_1 \approx x_1) < 1$, $h(x_2 \approx x_2) < 1$ and $x_1^h = x_2^h$ it holds $h(A(x_1)) = h(A(x_2))$. By definition, if f respect A then also f partially respect A.

For a binary conditional relation R between X and Y, reality h and h-realizations f and g of X and Y, respectively, we say that *f and g partially respect R* if $f \times g$ partially respects R.

Lemma 22. *Let A be a conditional set in X compatible with $\approx$. If an h-realization f of X partially respects A then f respects A.*

Proof. It remains to show that for $x_1, x_2 \in \operatorname{dom} f$ such that $h(x_1 \approx x_1) = h(x_2 \approx x_2) = 1$ and $x_1^h = x_2^h$ it holds $h(A(x_1)) = h(A(x_2))$. Which is indeed true as we have $h(x_1 \approx x_2) = x_1^h \approx^{|h} x_2^h = x_1^h \approx^{|h} x_1^h = h(x_1 \approx x_1) = 1$ and the compatibility of A with $\approx$ implies $h(A(x_1)) = h(A(x_1) \wedge (x_1 \approx x_2)) \leq h(A(x_2))$. The converse inequality can be proven analogously. □

Theorem 5. *Let A be a conditional set in X compatible with $\approx$, h a reality and f an h-realization of X which partially respects A. Then for each $x \in X_h$ we have*

$$A^{|h}(x^h) = h(A(x)).$$

Proof. The claim directly follows from Lemma 22 and Theorem 2. □

Characterization of respectability. In this part, we introduce a conditional relation such that the compatibility with the conditional relation is equivalent to respectability. For an h-realization $f\colon X \nrightarrow Y$ of X we define a binary conditional relation T_f on X given by

$$\mathrm{T}_f(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 = x_2, \\ a_h & \text{if } f(x_1) \text{ and } f(x_2) \text{ are defined, } x_1 \neq x_2 \text{ and } f(x_1) = f(x_2), \\ 0 & \text{otherwise,} \end{cases} \tag{1.31}$$

where $a_h = \bigwedge h^{-1}(1) = \bigwedge\{c \in L \mid h(c) = 1\}$. The conditional relation T_f is also denoted by T_{h^X}, T_{X^h} or T_h. For $x_1, x_2 \in X_h$ we have $h(\mathrm{T}_h(x_1, x_2)) = 1$ if and only if $x_1^h = x_2^h$.

It is easy to see that T_h is a reflexive conditional equality on X. Clearly, the restricted h-realization $(\mathrm{T}_h)^{|h}$ of T_h is an identity on X^h and if for a conditional set A in X is the h-realization A^h of A defined then also the h-realization $(A \circ \mathrm{T}_h)^h$ of $A \circ \mathrm{T}_h$ is defined.

Theorem 6. *The following statements are equivalent for a conditional set A in X and an h-realization f of X.*

1. *A is compatible with T_h.*

2. *X^h respects A.*

Proof. Suppose that A is compatible with T_h. Let $x_1, x_2 \in X_h$ such that $x_1^h = x_2^h$. If $x_1 = x_2$ then clearly $h(A(x_1)) = h(A(x_2))$. Assume $x_1 \neq x_2$. Then by the definition of $\mathrm{T}_{\approx}$, $h(\mathrm{T}_h(x_1,x_2)) = h(\bigwedge h^{-1}(1)) = 1$ and compatibility of A with T_h yields $h(A(x_1)) = h(A(x_1) \wedge \mathrm{T}_h(x_1,x_2)) \leq h(A(x_2))$. The converse inequality can be proven analogously. We proved that the h-realization f of X respects A.

Suppose that f respects A. Clearly, $A(x_1) \wedge \mathrm{T}_h(x_1,x_2) \leq A(x_2)$ holds for all $x_1, x_2 \in X$ such that $x_1 = x_2$ or x_1^h is not defined or x_2^h is not defined or $x_1^h \neq x_2^h$. Thus, suppose $x_1 \neq x_2$, x_1^h and x_2^h are defined and $x_1^h = x_2^h$. For each total reality h', we show that $h'(A(x_1) \wedge \mathrm{T}_h(x_1,x_2)) \leq h'(A(x_2))$. The only important case here is when $h'(A(x_1)) = 1$ and $h'(\mathrm{T}_h(x_1,x_2)) = 1$. By the definition of T_h we have $1 = h'(\mathrm{T}_h(x_1,x_2)) = h'(\bigwedge h^{-1}(1))$. By the properties of complete and atomic Boolean algebras there is a total reality h'' such that $h' = h'' \circ h$. Since f respects A, it holds $h''(h(A(x_1))) = h''(h(A(x_2)))$ and thus $h'(A(x_2)) = 1$. By completeness and atomicity of L, A is compatible with T_h. □

Conditional representation. It this subsection, we deal with representation of ordinary sets by conditional sets. The main result in this part can be used to find a conditional set A in a conditional universe X in a situation when we have a form of an unknown set in every reality. Existence of X is proved by preceding results.

Lemma 23. *For each reality h, h-realization $f : X \to Y$ of X and conditional set A_h in Y there exists the greatest conditional set A in X satisfying $A^h = A_h$. If f is a standard h-realization of X then A is compatible with $\approx$. If h is total and $\bar{h}$ is a total reality such that $\bar{h} \neq h$ then $\bar{h}(A(x)) = 1$ for each $x \in X$.*

Proof. Set $A(x) = \bigvee h^{-1}(A_h(x^h))$ if $x \in X_h$, otherwise $A(x) = \bigvee h^{-1}(0)$. By the completeness of h, $h(A(x)) = A_h(x^h)$ if $x \in X_h$, otherwise $h(A(x)) = 0$. Therefore, the h-realization A^h of A is defined and evidently, if $\bar{x}^h = x^h$ then $A(\bar{x}) = A(x)$. Now, $A^h(x^h) = h(\bigvee_{\bar{x}^h = x^h} A(\bar{x})) = \bigvee_{\bar{x}^h = x^h} A_h(x^h) = A_h(x^h)$. Maximality of A follows from the fact that if $\bar{A}^h = A_h$ then, by definition, $h(\bigvee_{\bar{x}^h = x^h} \bar{A}(\bar{x})) = A_h(x^h)$, yielding $h(\bar{A}(x)) \leq A_h(x^h) = h(A(x))$ for each $x \in X_h$ and $h(A(x)) = h(\bar{A}(x)) = 0$ for each $x \notin X_h$ and so $\bar{A}(x) \leq A(x)$.

Let h be total and $\bar{h} \neq h$ be also total reality. Then for each $x \in X$, $A(x)$ is either 1 or the coatom in L such that $h(A(x)) = 0$. Thus $\bar{h}(A(x)) = 1$.

Suppose f is standard. Since $\approx^h$ is the ordinary equality on Y, A_h is compatible with $\approx^h$. We show that for each total reality $\bar{h}$, $x, y \in X$ it holds $\bar{h}(A(x_1)) \wedge \bar{h}(x_1 \approx x_2) \leq \bar{h}(A(x_2))$. The only interesting case here is when $\bar{h}(A(x_1)) = \bar{h}(x_1 \approx x_2) = 1$. By the previous part of the proof, if $\bar{h} \neq h$ then $\bar{h}(A(x_2)) = 1$. So suppose $h = \bar{h}$ and by (1.1), $h(x_1 \approx x_1) = h(x_2 \approx x_2) = 1$ and by X^h is standard, h-realizations x_1^h and x_2^h of x_1 and x_2, respectively, are defined. Now, by the compatibility of A_h with $\approx^h$ we have $h(A(x_1)) \wedge h(x_1 \approx x_2) = A_h(x_1^h) \wedge (x_1^h \approx^h x_2^h) \leq A_h(x_2^h) = h(A(x_2))$. Since L is complete and atomic, A is compatible with $\approx$. $\square$

Theorem 7 (conditional representation of sets)**.** *Suppose we have for each total reality h a standard h-realization f_h of X and a subset A_h of X^h. Then there is a unique completely present conditional set A in X compatible with $\approx$ such that for each total reality h it holds $A^h = A_h$.*

Proof. Let for each total reality h, B_h be the greatest conditional set in X such that $(B_h)^h = A_h$ (Lemma 23). For any total reality $\bar{h} \neq h$ we have $h(B_{\bar{h}}(x)) = 1$. Set $A = \bigcap_{\bar{h} \in \mathrm{CHom}(L,2)} B_{\bar{h}}$. Now, $A^h(x^h) = \bigvee_{y^h = x^h} \bigwedge_{\bar{h} \in \mathrm{CHom}(L,2)} h(B_{\bar{h}}(y)) = (B_h)^h(x^h) = A_h(x^h)$.

The compatibility of A with $\approx$ follows from the compatibility of B_h with $\approx$. Since each standard realization of A is defined, A is completely present. $\square$

Inverse conditional relations. Let R be a conditional relation between X and Y. Then the *inverse conditional relation* R^{-1} of R is a conditional relation between Y and X defined by $R^{-1}(y,x) = R(x,y)$.

Lemma 24. *For a reality h and h-realizations f and g of X and Y, respectively, it holds $(R^{-1})^{|h} = (R^{|h})^{-1}$.*

Proof. For $x^h \in X^h$ and $y^h \in Y^h$ we have $R^{-1|h}(y^h, x^h) = \bigvee_{\bar{y}^h = y^h} \bigvee_{\bar{x}^h = x^h} h(R^{-1}(\bar{y}, \bar{x})) = \bigvee_{\bar{x}^h = x^h} \bigvee_{\bar{y}^h = y^h} h(R(\bar{x}, \bar{y})) = R^{|h}(x^h, y^h) = R^{|h-1}(y^h, x^h)$. $\square$

Height of conditional sets. We conclude this subsection with a study of height of conditional sets. The *height* of a conditional set A is the value $\bigvee_{x \in X} A(x)$ (cf. [35]).

Lemma 25. *Let A be a conditional set in X. Then the following holds.*

1. If the height of A is 1 then so is the height of any realization of A.

2. If each standard realization f of X it holds that the h-realization A^h of A is nonempty then the height of A is 1.

Proof. 1. Let h be a reality, f an h-realization of X and A^h the h-realization of A. Since the h-realization A^h of A is defined, $h(A(x)) = 0$ if x^h is not defined. Now, the claim follows by $\bigvee_{x \in X} A^h(x^h) = \bigvee_{x \in X} \bigvee_{y^h = x^h} h(A(y)) = h\left(\bigvee_{x \in X} A(x)\right) = h(1) = 1$.

2. It suffices to show that for any total reality h, $\bigvee_{x\in X} h(A(x)) = 1$. This is indeed true because if A^h is nonempty, there exists $x_h \in A^h$, i.e. $A^h(x_h) = 1$, which implies that there is an element $x \in X$ such that $x^h = x_h$ and so $h(A(x)) = 1$. □

1.3.2 Realizations of relations products

In this subsection, we discuss in which situations it holds $(R \circ S)^{|h} = R^{|h} \circ S^{|h}$ and $(R \triangleleft S)^{|h} = R^{|h} \triangleleft S^{|h}$ for conditional relations R and S.

Let X be an L-conditional universe, h a reality, f an h-realization of X, R and S unary conditional relations on X. Then we say that the product $R \circ S$ is *safe* with respect to f if $h(R \circ S) = h((R \cap X_h) \circ S) = h(R \circ (S \cap X_h))$ and the product $R \triangleleft S$ is *safe* with respect to f if $h(R \triangleleft S) = h((R \cap X_h) \triangleleft S)$. We also say that a product is safe w.r.t. h.

Let X, Y and Z be L-conditional universes, f_X, f_Y and f_Z h-realizations of X, Y and Z, respectively, R a binary conditional relation between X and Y and S a binary conditional relation between Y and Z. Then we say that the *product* $R \circ S$ *is safe* w.r.t. f_X, f_Y and f_Z if $R_x \circ S_z$ is safe w.r.t. f_Y for all $x \in X_h$ and $z \in Z_h$. Similarly, we say that the *product* $R \triangleleft S$ *is safe* w.r.t. f_X, f_Y and f_Z if $R_x \triangleleft S_z$ is safe w.r.t. f_Y for all $x \in X_h$ and $z \in Z_h$. Again, we also say that a product is safe w.r.t. h.

If needed, we identify a unary conditional relation R on Y with a binary conditional relation between a one point set $X = \{x\}$, with a reflexive conditional equality $\approx_X$, and Y. Similarly, we identify an unary conditional relation S on Y with a binary conditional relation between Y and $Z = \{z\}$ also with a reflexive conditional equality $\approx_Z$. Clearly, for any h-realization f of X it holds that X^h is an one-element set $\{x^h\}$ with $x^h \approx_X^h x^h = 1$ and thus $X_h = X$ and similarly for Z.

We give two sufficient conditions for products to be safe:

Lemma 26. *Let R, S be unary conditional relations on X, h a reality and f an h-realization of X. Then it holds:*

1. If the h-realization R^h of R is defined or the h-realization S^h of S is defined then $R \circ S$ is safe w.r.t. h.

2. If the h-realization R^h of R is defined then $R \triangleleft S$ is safe w.r.t. h.

Proof. The result follows easily from the fact that if R^h is defined then by definition $h(R(x)) = 0$ for $x \in X \setminus X_h$ and similarly for S. □

Lemma 27. *Let R and S be unary conditional relations on X compatible with $\approx$. Then the following holds for every reality h and h-realization f of X.*

1. If R or S is completely present then $R \circ S$ is safe w.r.t. f.

2. If R is completely present then $R \triangleleft S$ is safe w.r.t. f.

Proof. 1. By (1.4) we have $h(R \circ S) = h(\bigvee_{x \in X} R(x) \wedge S(x)) = h(\bigvee_{x \in X} R(x) \wedge (x \approx x) \wedge S(x)) \leq h(\bigvee_{x \in X} \bigvee_{y \in X_h} R(x) \wedge (x \approx y) \wedge S(x)) \leq h(\bigvee_{x \in X} \bigvee_{y \in X_h} R(y) \wedge S(y) = h((R \cap X_h) \circ S)$. The converse inequality is obvious.

2. First observe that $R(y) \to S(y) \leq (y \approx x) \to (R(x) \to S(x))$ holds for each $x, y \in X$. Indeed, the inequality is equivalent to $(y \approx x) \wedge R(x) \wedge (R(y) \to S(y)) \leq S(x)$ which is true since we have, by compatibility of R and S with $\approx$, that $(y \approx x) \wedge R(x) \wedge (R(y) \to S(y)) = (y \approx x) \wedge R(y) \wedge (R(y) \to S(y)) \leq (y \approx x) \wedge S(y) \leq S(x)$.

Now, also by (1.4) we have $h(R \triangleleft S) = h(\bigwedge_{x \in X} R(x) \to S(x)) = h(\bigwedge_{x \in X} (R(x) \wedge (x \approx x)) \to S(x)) = h(\bigwedge_{x \in X} (x \approx x) \to (R(x) \to S(x))) \geq h(\bigwedge_{x \in X} (\bigvee_{y \in X_h} y \approx x) \to (R(x) \to S(x))) = h(\bigwedge_{x \in X} \bigwedge_{y \in X_h} (y \approx x) \to (R(x) \to S(x))) \geq h(\bigwedge_{x \in X} \bigwedge_{y \in X_h} R(y) \to S(y)) = h(\bigwedge_{y \in X_h} R(y) \to S(y)) = h((R \cap X_h) \triangleleft S)$. The converse inequality is easy. □

Let h be a reality and f an h-realization of X. Then a *conditional set A in X is safe* w.r.t. f if for every conditional set S in X compatible with $\approx$ the products $A \circ S$ and $A \triangleleft S$ are safe w.r.t. f. By Lemmas 26 and 27, we have that if A is completely present and compatible with $\approx$ or the h-realization A^h of A is defined then A is safe w.r.t. f.

The following technical lemma will be used in the next theorem.

Lemma 28. *Let R and S be unary conditional relations on X, h a reality and f an h-realization of X. Then the following holds.*

1. If $R \circ S$ is safe w.r.t. h and $R \circ T_h \circ S = R \circ S$ then $(R \circ T_h) \circ (T_h \circ S)$ is also safe w.r.t. h.

2. If $R \triangleleft S$ is safe w.r.t. h and $T_h \triangleleft S = S$ then $(R \circ T_h) \triangleleft S$ is also safe w.r.t. h.

Proof. 1. We have $h((R \circ \mathrm{T}_h) \circ (\mathrm{T}_h \circ S)) = h(R \circ \mathrm{T}_h \circ S) = h(R \circ S) = h(\bigvee_{x \in X_h} R(x) \wedge S(x)) \leq h(\bigvee_{x \in X_h} (R \circ \mathrm{T}_h)(x) \wedge (\mathrm{T}_h \circ S)(x))$. The converse inequality is trivially satisfied.

2. By the second associative law (1.28), we have $h((R \circ \mathrm{T}_h) \triangleleft S) = h(R \triangleleft (\mathrm{T}_h \triangleleft S)) = h(R \triangleleft S) = h(\bigwedge_{x \in X_h} R(x) \to S(x)) \geq h(\bigwedge_{x \in X_h} R \circ \mathrm{T}_h(x) \to S(x))$. The converse inequality is trivial.

□

Theorem 8. *Let R and S be (unary or binary) conditional relations such that all compositions below make sense. Then the following holds.*

1. If $R \circ T_h \circ S = R \circ S$ and $R \circ S$ is safe w.r.t. h then

$$(R \circ S)^{|h} = R^{|h} \circ S^{|h}. \tag{1.32}$$

2. If $T_h \triangleleft S = S$ and $R \triangleleft S$ is safe w.r.t. h and, in the case R is binary, $T_h \triangleleft (R \triangleleft S) = R \triangleleft S$ then

$$(R \triangleleft S)^{|h} = R^{|h} \triangleleft S^{|h}. \tag{1.33}$$

Proof. The proof is an analogy of the proof of [19, Lemma 3].

1. First suppose that R and S are conditional sets (i.e. unary conditional relations) in the same universe X. By the definition of $\circ$, $(R \circ S)^{|h} = h(R \circ S) = h(\bigvee_{x \in X} R(x) \wedge S(x)) = \bigvee_{x \in X_h} h(R(x)) \wedge h(S(x)) \leq \bigvee_{x^h \in X^h} R^{|h}(x^h) \wedge S^{|h}(x^h) = R^{|h} \circ S^{|h}$ where $\leq$ holds as $h(R(x)) \leq R^{|h}(x^h)$ and $h(S(x)) \leq S^{|h}(x^h)$. Moreover, if R and S are compatible with T_h, we can by Theorem 2 put $=$ in the place of $\leq$. To prove the converse inequality, we observe that the conditional sets $R \circ \mathrm{T}_h$ and $\mathrm{T}_h \circ S$ are compatible with T_h and satisfy $(R \circ \mathrm{T}_h) \supseteq R$ and $(\mathrm{T}_h \circ S) \supseteq S$. Therefore, by 1. of Lemma 28, $R^{|h} \circ S^{|h} \leq (R \circ \mathrm{T}_h)^{|h} \circ (\mathrm{T}_h \circ S)^{|h} = (R \circ \mathrm{T}_h \circ S)^{|h} = (R \circ S)^{|h}$.

Now we will prove (1.32) for binary R and S. Let R be a binary conditional relation between X and Y and S a binary conditional relation between Y and Z. Thus, by (1.19), the first part of the proof and the first distributivity law (1.29),

$$\begin{aligned}
(R \circ S)^{|h}(x^h, z^h) &= \bigvee_{\bar{x}^h = x^h} \bigvee_{\bar{z}^h = z^h} h((R \circ S)(\bar{x}, \bar{z})) = \bigvee_{\bar{x}^h = x^h} \bigvee_{\bar{z}^h = z^h} h(R_{\bar{x}} \circ S_{\bar{z}}) \\
&= \bigvee_{\bar{x}^h = x^h} \bigvee_{\bar{z}^h = z^h} (R_{\bar{x}})^{|h} \circ (S_{\bar{z}})^{|h} = \left(\bigcup_{\bar{x}^h = x^h} (R_{\bar{x}})^{|h} \right) \circ \left(\bigcup_{\bar{z}^h = z^h} (S_{\bar{z}})^{|h} \right) \\
&= (R^{|h})_{x^h} \circ (S^{|h})_{z^h} = (R^{|h} \circ S^{|h})(x^h, z^h).
\end{aligned}$$

2. The proof goes similarly as the proof of the first part. First, suppose that R and S are unary. We have $(R \triangleleft S)^{|h} = h(R \triangleleft S) = h(\bigwedge_{x \in X} R(x) \to S(x)) = \bigwedge_{x \in X_h} h(R(x)) \to h(S(x)) \geq \bigwedge_{x^h \in X^h} R^{|h}(x^h) \to S^{|h}(x^h) = R^{|h} \triangleleft S^{|h}$. Here, the inequality follows from $h(R(x)) \leq R^{|h}(x^h)$, $h(S(x)) = S^{|h}(x^h)$ and antitony of $\to$ in the first argument. Moreover, if R is compatible with T_h, $=$ can be used instead. Now, since $\triangleleft$ is antitone in the first argument, $R \circ \mathrm{T}_h \supseteq R$ and $R \circ \mathrm{T}_h$ is compatible with T_h, we obtain by 2. of Lemma 28, $R^{|h} \triangleleft S^{|h} \geq (R \circ \mathrm{T}_h)^{|h} \triangleleft S^{|h} =$

$((R\circ \mathrm{T}_h)\triangleleft S)^{|h} = (R\triangleleft(\mathrm{T}_h\triangleleft S))^{|h} = (R\triangleleft S)^{|h}$. This finishes the proof of (1.33) for unary R and S.

To prove the assertion for binary R and S, we first make the following observation. If T is a binary conditional relation between X and Z satisfying $\mathrm{T}_h\triangleleft T = T$ then for any $z\in Z$ the conditional set T_z is compatible with T_h. Thus, (1.19) and Theorem 2 give $T^{|h}(x^h,z^h) = \bigvee_{\bar{z}^h=z^h}(T_{\bar{z}})^{|h}(x^h) = \bigvee_{\bar{z}^h=z^h} h(T_{\bar{z}}(x)) = \bigvee_{\bar{z}^h=z^h} h(T(x,\bar{z}))$. We apply this result to $T = R\triangleleft S$ and use the second distributivity law (1.29):

$$
\begin{aligned}
(R\triangleleft S)^{|h}(x^h,z^h) &= \bigwedge_{\bar{x}^h=x^h}(R\triangleleft S)^{|h}(\bar{x}^h,z^h) = \bigwedge_{\bar{x}^h=x^h}\bigvee_{\bar{z}^h=z^h} h(R_{\bar{x}}\triangleleft S_{\bar{z}})\\
&= \bigwedge_{\bar{x}^h=x^h}\bigvee_{\bar{z}^h=z^h}(R_{\bar{x}}\triangleleft S_{\bar{z}})^{|h} = \bigwedge_{\bar{x}^h=x^h}\bigvee_{\bar{z}^h=z^h}(R_{\bar{x}})^{|h}\triangleleft(S_{\bar{z}})^{|h}\\
&= \left(\bigcup_{\bar{x}^h=x^h}(R_{\bar{x}})^{|h}\right)\triangleleft\left(\bigcup_{\bar{z}^h=z^h}(S_{\bar{z}})^{|h}\right) = (R^{|h})_{x^h}\triangleleft(S^{|h})_{z^h} = (R^{|h}\triangleleft S^{|h})(x^h,z^h). \qquad \square
\end{aligned}
$$

1.3.3 Completely present conditional sets compatible with $\approx$

Completely present L-sets compatible with $\approx$ were also studied in [14] and subsequent papers. We first summarize basic known facts and then present new results.

Let X be an L-conditional universe. The conditional set X_{E} is clearly completely present and compatible with $\approx$. Therefore, we associate with each conditional universe X a completely present conditional set X_{E} compatible with $\approx$.

On the other hand, let A be a completely present conditional set in X compatible with $\approx$. Then we have a binary conditional relation $\approx_A$ on X given by

$$x_1 \approx_A x_2 = A(x_1)\wedge(x_1\approx x_2) = (x_1\approx x_2)\wedge A(x_2) \tag{1.34}$$

for $x_1,x_2\in X$. Clearly, $\approx_A$ is symmetric and transitive. Therefore, we associate with A a conditional universe $(X_A,\approx_A)$ where $X_A = X$.

As we will see below, completely present conditional sets compatible with $\approx$ are exactly fixpoints of the following operator. For a conditional set A in X we define a conditional set $\mathrm{C}_{\approx}A$ in X by

$$\mathrm{C}_{\approx}A(x) = (A\circ\approx)(x) = (\approx\circ A)(x) = \bigvee_{x'\in X} A(x')\wedge(x'\approx x). \tag{1.35}$$

The following lemma gives basic properties of $\mathrm{C}_{\approx}$.

Lemma 29. *For each conditional sets A and B in X it holds*

$$A \triangleleft B \le \mathrm{C}_{\approx}A \triangleleft \mathrm{C}_{\approx}B, \tag{1.36}$$

$$\mathrm{C}_{\approx}(\mathrm{C}_{\approx}A) = \mathrm{C}_{\approx}A. \tag{1.37}$$

Proof. The proof is taken from [3, Lemma 7.57]. The inequality (1.36) is true iff for each $x \in X$ we have $\mathrm{C}_{\approx}A(x) \wedge (A \triangleleft B) \le \mathrm{C}_{\approx}B(x)$ which is true since

$$\begin{aligned}
\mathrm{C}_{\approx}A(x) \wedge (A \triangleleft B) &= \left(\bigvee_{y \in X} A(y) \wedge (x \approx y)\right) \wedge \left(\bigwedge_{z \in X} A(z) \to B(z)\right) \\
&= \bigvee_{y \in X} A(y) \wedge (x \approx y) \wedge \bigwedge_{z \in X} A(z) \to B(z) \\
&\le \bigvee_{y \in X} A(y) \wedge (x \approx y) \wedge (A(y) \to B(y)) \le \bigvee_{y \in X} B(y) \wedge (x \approx y) \\
&= \mathrm{C}_{\approx}B(x).
\end{aligned}$$

To prove (1.37) we check both inequalities. For $x \in X$ we have

$$\begin{aligned}
\mathrm{C}_{\approx}(\mathrm{C}_{\approx}A)(x) &= \bigvee_{y \in X} \mathrm{C}_{\approx}(A)(y) \wedge (x \approx y) = \bigvee_{y \in X} \left(\bigvee_{z \in X} A(z) \wedge (z \approx y)\right) \wedge (x \approx y) \\
&= \bigvee_{y \in X} \bigvee_{z \in X} A(z) \wedge (z \approx y) \wedge (x \approx y) \le \bigvee_{y \in X} \bigvee_{z \in X} A(z) \wedge (z \approx x) \\
&= \bigvee_{z \in X} A(z) \wedge (z \approx x) = \mathrm{C}_{\approx}A(x)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{C}_{\approx}(\mathrm{C}_{\approx}A)(x) &= \bigvee_{y \in X} \left(\bigvee_{z \in X} A(z) \wedge (z \approx y)\right) \wedge (x \approx y) \\
&\ge \bigvee_{y \in X} A(y) \wedge (y \approx y) \wedge (x \approx y) = \bigvee_{y \in X} A(y) \wedge (x \approx y) \\
&= \mathrm{C}_{\approx}A(x).
\end{aligned}$$

$\square$

Generally, $\mathrm{C}_{\approx}$ is not a fuzzy closure operator [3] with respect to $\subseteq$, since in general $A \nsubseteq \mathrm{C}_{\approx}A$. In the end of Subsection 1.4.1, we define a conditional version of a closure operator which covers also $\mathrm{C}_{\approx}$.

In what follows, we introduce new results. Recall that a fixpoint of a mapping $f : X \to X$ is an $x \in X$ such that $f(x) = x$. The following lemma gives a characterization of fixpoints of $\mathrm{C}_{\approx}$.

Theorem 9. *A conditional set is a fixpoint of* $\mathrm{C}_{\approx}$ *if and only if it is completely present and compatible with* $\approx$*.*

Proof. Let A be a conditional set in X such that $A = A \circ \approx$. For $\bar{x} \in X$ we have $(A \circ \approx)(x') = \bigvee_{x \in X} A(x) \wedge (x \approx x') \leq A(x')$ which yields $A(x) \wedge (x \approx x') \leq A(x')$ (for each $x, x' \in X$) proving that A is compatible with $\approx$. By (1.2), for $x \in X$ we have $A(x) = (A \circ \approx)(x) = \bigvee_{x' \in X} A(x') \wedge (x' \approx x) \leq \bigvee_{x' \in X} x \approx x' = x \approx x$ showing that A is completely present.

Conversely, assume that a conditional set A in X is completely present and compatible with $\approx$. Let $x \in X$. Then for each $x' \in X$ we have $A(x') \wedge (x' \approx x) \leq A(x)$ which yields $\mathrm{C}_{\approx}A(x) = \bigvee_{x' \in X} A(x') \wedge (x' \approx x) \leq A(x)$. Since $A(x) \leq x \approx x$, we have $A(x) = A(x) \wedge (x \approx x) \leq \bigvee_{x' \in X} A(x') \wedge (x' \approx x) = \mathrm{C}_{\approx}A(x)$. □

Lemma 30. *For each conditional set A in X compatible with $\approx$ it holds* $\mathrm{C}_{\approx}A = A \cap E_X$*.*

Proof. Clearly, for $x \in X$ we have $A(x) \wedge (x \approx x) \leq \bigvee_{y \in X} A(y) \wedge (y \approx x) = (A \circ \approx)(x)$. The converse inequality holds since the compatibility of A with $\approx$ and (1.1) implies $A(y) \wedge (y \approx x) \leq A(x) \wedge (x \approx x)$ for all $x, y \in X$. □

1.3.4 Power conditional relations

Let X be an L-conditional universe. According to the theory of fuzzy power structures [8], any binary L-relation R on X can be extended to a binary L-relation R^+ on L^X, called a *power relation*. The definition is the following. For conditional sets A, B in X we set

$$R^{\to}(A,B) = \bigwedge_{x \in X} \left(A(x) \to \bigvee_{y \in X} R(x,y) \wedge B(y) \right) = A \triangleleft (R \circ B), \tag{1.38}$$

$$R^{\leftarrow}(A,B) = \bigwedge_{y \in X} \left(B(y) \to \bigvee_{x \in X} R^{-1}(y,x) \wedge A(x) \right) = B \triangleleft (R^{-1} \circ A) \tag{1.39}$$

and define

$$R^{+}(A,B) = R^{\rightarrow}(A,B) \wedge R^{\leftarrow}(A,B). \tag{1.40}$$

Theorem 10. *If R is compatible with $\approx$ then for any reality h, h-realization f of X which partially respects R and safe conditional sets A and B w.r.t. f it holds $R^{|h\rightarrow}(A^{|h},B^{|h}) = h(R^{\rightarrow}(A,B))$, $R^{|h\leftarrow}(A^{|h},B^{|h}) = h(R^{\leftarrow}(A,B))$ and $R^{|h+}(A^{|h},B^{|h}) = h(R^{+}(A,B))$.*

Proof. By Lemma 22, the h-realization f of X respects R. By Lemma 6, it holds $\mathrm{T}_h \circ R \circ \mathrm{T}_h = R$. Now, by safeness of A and B and Theorem 8, we have $R^{|h\rightarrow}(A^h,B^h) = A^{|h} \triangleleft (R^{|h} \circ B^{|h}) = A^{|h} \triangleleft (R \circ B)^{|h} = (A \triangleleft (R \circ B))^{|h} = h(A \triangleleft (R \circ B)) = h(R^{\rightarrow}(A,B))$. The second equality is proved similarly and the third one follows by (1.40). □

1.3.5 Conditional mappings

A conditional relation F between L-conditional universes X and Y is called a *conditional mapping from X to Y*, in short $F\colon X \to Y$, if for each total reality h, standard h-realizations f and g of X and Y, respectively, the restricted h-realization $F^{|h}$ of F is an ordinary mapping.

For a total reality h and standard h-realizations f and g of X and Y, respectively, the ordinary binary relation $F^{|h}$ can be viewed as a 2-conditional relation, ordinary relation, or an ordinary mapping, as needed. The three following expressions mean all the same: $F^{|h}(x_h,y_h) = 1$, $(x_h,y_h) \in F^{|h}$, $F^{|h}(x_h) = y_h$.

Generally, a restricted realization of a conditional mapping is not a conditional mapping. For example, let $L = 2$, $X = \{x_1,x_2\}$, $x_1 \approx_X x_1 = x_1 \approx_X x_2 = x_2 \approx_X x_1 = x_2 \approx_X x_2 = 1$, $Y = \{y\}$, $y \approx_Y y = 1$, $F = \{(x_1,y)\}$, h be the identity on 2, $X' = \{x_2\}$, $f\colon X \to X'$ be the h-realization of X given by $f(x_1)$ is not defined, $f(x_2) = x_2$, $Y' = Y$ and $g\colon Y \to Y'$ be the h-realization of Y given by $g(y) = y$. Then $F\colon X \to Y$ is a conditional mapping, but $F^{|h} = \emptyset$ is not a conditional mapping.

The following two lemmas study situations in which a restricted realization of a conditional mapping is a conditional mapping.

Lemma 31. *Let h be a reality, f and g faithful h-realizations of X and Y. Then the restricted realization of any conditional mapping from X to Y is a conditional mapping.*

Proof. Let h be a reality $L \to K$, F be a conditional mapping from X to Y. Then for each total reality $\bar{h}\colon K \to 2$, $\bar{f}$ and $\bar{g}$ standard $\bar{h}$-realizations of X^h and Y^h, respectively, we have by Lemma 6, $F^{|h|\bar{h}} = F^{|(\bar{h}\circ h)}$. Since f and g are faithful, $\bar{f}\circ f$ and $\bar{g}\circ g$ are standard realizations of X and Y, respectively (Lemma 2). Since F is a conditional mapping, $F^{|(\bar{h}\circ h)}$ is an ordinary mapping. We showed that each standard restricted realization of $F^{|h}$ is an ordinary mapping, i.e. $F^{|h}$ is a conditional mapping. $\square$

Lemma 32. *Any restricted realization of a conditional mapping $X \to Y$ compatible with $\approx_X$ and $\approx_Y$ is a conditional mapping.*

Proof. The result is proved similarly as the preceding lemma using Lemma 14. $\square$

A conditional mapping F is called a *conditional bijection* if F^{-1} is also a conditional mapping. By Lemma 24, a conditional relation $F\colon X \to Y$ is a conditional bijection if and only if for each standard realizations f and g of X and Y, respectively, it holds that the restricted realization of F is an ordinary bijection.

We call two L-conditional universes X_1 and X_2 *conditionally isomorphic* if there is a conditional bijection between X_1 and X_2. The conditional equality $\approx$ of a conditional universe X is a binary conditional relation between X and X. For each total reality h and standard h-realization f of X it holds that the h-realization $\approx^h$ of $\approx$ is an ordinary equality on X^h and thus a bijection. Therefore, $\approx$ is a conditional bijection between X and X.

Theorem 11. *Let h be a reality, $f\colon X \nrightarrow Y$ and $g\colon X \nrightarrow Z$ be two h-realizations of an L-conditional universe X. Then Y and Z are conditionally isomorphic.*

Proof. Let $(f\times g)\colon X\times X \nrightarrow Y\times Z$ be an h-realization of $X\times X$ defined by $(f\times g)(x,\bar{x}) = (f(x), g(\bar{x}))$ (see Lemma 4). Then since $\approx$ is a compatible conditional bijection between X and X, the restricted h-realization $(f\times g)(\approx|_{f\times g})$ of $\approx$ is (Lemma 32) also a conditional bijection between Y and Z. Therefore, Y and Z are conditionally isomorphic. $\square$

For any conditional relation F between conditional universes X and Y and a conditional set A in X we denote $F(A) = A\circ \mathrm{T}_{\approx}\circ F$. For single elements $x\in X$ we usually write $F(x)$ instead of $F(\{x\})$. By Lemma 21 and 1. of Theorem 8, for each reality h, faithful h-realization f of X such that the h-realization A^h of A is defined and $A\circ \mathrm{T}_{\approx}\circ \mathrm{T}_h\circ F = A\circ \mathrm{T}_{\approx}\circ F$, and any h-realization g of Y it holds

$$F(A)^{|h} = F^{|h}(A^h). \tag{1.41}$$

Particularly, the equality (1.41) holds if f is a standard h-realization of X.

Note that if it is satisfied in a total reality h that A is present then it does not need to be satisfied in h that $F(A)$ is present. This seems unnatural since if h-realizations f and g of

X and Y, respectively, are standard and A^h is defined then $F^{|h}(A^h)$ is also defined. We call F *presence preserving* if for each conditional set A in X it holds

$$A \approx_X^+ A \leq F(A) \approx_Y^+ F(A). \tag{1.42}$$

Lemma 33. *Let F be a conditional relation between X and Y. Then F is presence preserving if and only if $F(x,y) \wedge (x \approx_X x) \leq y \approx_Y y$ holds for each $x \in X$ and $y \in Y$.*

Proof. Suppose that F is presence preserving. Then since $\{x\} \subseteq \{x\} \circ \mathrm{T}_{\approx}$, it holds $F(x,y) \leq F(x)(y)$, further $\mathrm{E}\{x\} \leq \mathrm{E}F(x)$ yields $F(x)(y) \wedge (x \approx_X x) \leq y \approx_Y y$. Therefore, $F(x,y) \wedge (x \approx_X x) \leq F(x)(y) \wedge (x \approx_X x) \leq y \approx_Y y$.

If the right hand side of the equivalence is true then for each conditional set A in X, $x,x' \in X$, $y \in Y$ it holds $\mathrm{E}A \wedge A(x) \wedge \mathrm{T}_{\approx_X}(x,x') \wedge F(x',y) \leq (x \approx_X x) \wedge \mathrm{T}_{\approx_X}(x,x') \wedge F(x',y) = (x \approx_X x') \wedge F(x',y) \leq y \approx_Y y$. Now $\mathrm{E}A \wedge F(A)(y) = \mathrm{E}A \wedge (A \circ \mathrm{T}_{\approx} \circ F)(y) \leq y \approx_Y y$ which is equivalent to (1.42). □

1.4 More on conditional universes

1.4.1 Conditional universes of conditional sets

Let X be an L-conditional universe, h a reality and f an h-realization of X. We denote $\approx^{\rightarrow}$ by $\mathrm{S}_{\approx}$. For conditional sets A and B in X, the condition $\mathrm{S}_{\approx}(A,B)$ is called the *condition under which A is a subset of B*. If $\mathrm{S}_{\approx^h}(A^h,B^h) = 1$, we say that A *is a subset of B in the h-realization f*. We say that A *is a subset of B in h* if A is a subset of B in any h-realization of X. We have $\approx^{\leftarrow}(A,B) = \mathrm{S}_{\approx}(B,A)$ and, consequently,

$$A \approx^+ B = \mathrm{S}_{\approx}(A,B) \wedge \mathrm{S}_{\approx}(B,A). \tag{1.43}$$

The value $A \approx^+ B$ is called the *condition under which A is equal to B*. If $A^h \approx^{h+} B^h = 1$, we say that A *equals B in the h-realization f*. We say that A *is equal to* B in h if A is equal to B in any h-realization of X. If h is total and f standard, $\mathrm{S}_{\approx^h}$ is the ordinary subset relation and $\approx^{h+}$ is the ordinary set equality relation on X^h.

Lemma 34. *If A and B are compatible with $\approx$ then*

$$\mathrm{S}_{\approx}(A,B) = \mathrm{E}A \wedge (A \triangleleft B), \tag{1.44}$$

$$A \approx^+ B = \mathrm{E}A \wedge \mathrm{E}B \wedge (A \triangleleft B) \wedge (\mathrm{B} \triangleleft A). \tag{1.45}$$

Proof. We first prove (1.44). By Lemma 30, $\mathrm{S}_{\approx}(A,B) = A \triangleleft (\approx \circ B) = A \triangleleft (B \cap E_X) = \bigwedge_{x\in X} A(x) \to (B(x) \wedge (x \approx x)) = \bigwedge_{x\in X} (A(x) \to B(x)) \wedge (A(x) \to (x \approx x)) = (A \triangleleft B) \wedge \mathrm{E}A$. (1.45) new follows from (1.44) and (1.43). □

From the preceding Lemma it follows that for completely present conditional sets A and B in X compatible with $\approx$ it holds

$$\mathrm{S}_{\approx}(A,B) = A \triangleleft B, \tag{1.46}$$

$$A \approx^{+} B = (A \triangleleft B) \wedge (B \triangleleft A). \tag{1.47}$$

Theorem 12. *The condition under which a conditional set A in X is present is equal to the condition under which A is equal to itself. In symbols:* $\mathrm{E}A = A \approx^{+} A$.

Proof. It suffices to show that $\mathrm{S}_{\approx}(A,A) = \mathrm{E}A$. On one hand we have

$$\begin{aligned} \mathrm{S}_{\approx}(A,A) &= \bigwedge_{x_1\in X} A(x_1) \to \bigvee_{x_2\in X} (x_1 \approx x_2) \wedge A(x_2) \geq \bigwedge_{x_1\in X} A(x_1) \to ((x_1 \approx x_1) \wedge A(x_1)) \\ &= \bigwedge_{x_1\in X} (A(x_1) \to (x_1 \approx x_1)) \wedge (A(x_1) \to A(x_1)) = \bigwedge_{x_1\in X} A(x_1) \to (x_1 \approx x_1) = \mathrm{E}A \end{aligned}$$

and on the other hand by (1.2) we have

$$\begin{aligned} \mathrm{S}_{\approx}(A,A) &= \bigwedge_{x_1\in X} A(x_1) \to \bigvee_{x_2\in X} (x_1 \approx x_2) \wedge A(x_2) \leq \bigwedge_{x_1\in X} A(x_1) \to \bigvee_{x_2\in X} x_1 \approx x_2 \\ &= \bigwedge_{x_1\in X} A(x_1) \to (x_1 \approx x_1) = \mathrm{E}A. \end{aligned}$$

□

We prefer to express the condition under which A is present by $A \approx^{+} A$ instead of $\mathrm{E}A$ since it corresponds with expressing the presence of an element $x \in X$ by $x \approx x$.

We have $A \approx^{+} A \leq \mathrm{E}A = \bigwedge_{x\in X} A(x) \to (x \approx x)$ and therefore for each $x \in X$ it holds $A \approx^{+} A \leq A(x) \to (x \approx x)$ which is, by adjointness, equivalent to

$$A(x) \wedge (A \approx^{+} A) \leq x \approx x. \tag{1.48}$$

The inequality (1.48) is read as follows: "If it is satisfied that x is an element of A and A is present then it is also satisfied that x is present."

We show the transitivity of $\mathrm{S}_{\approx}$, i.e. that $\mathrm{S}_{\approx}(A,B) \wedge \mathrm{S}_{\approx}(B,C) \leq \mathrm{S}_{\approx}(A,C)$ holds for all conditional sets A,B and C in X. For $x \in X$ we have $A(x) \wedge \mathrm{S}_{\approx}(A,B) = A(x) \wedge (A \triangleleft \mathrm{C}_{\approx}B) \leq \mathrm{C}_{\approx}B(x)$ proving

$$A(x) \wedge \mathrm{S}_{\approx}(A,B) \leq \mathrm{C}_{\approx}B(x) \tag{1.49}$$

and $A(x) \wedge (A \approx^{+} A) = A(x) \wedge \mathrm{S}_{\approx}(A,A) \leq \mathrm{C}_{\approx}A(x)$ proving

$$A(x) \wedge (A \approx^{+} A) \leq \mathrm{C}_{\approx}A(x). \tag{1.50}$$

For each $x \in X$ it holds

$$\begin{aligned}
\mathrm{C}_{\approx}A(x) \wedge \mathrm{S}_{\approx}(A,B) &= \left(\bigvee_{y \in X} A(y) \wedge (y \approx x) \right) \wedge \mathrm{S}_{\approx}(A,B) \\
&= \bigvee_{y \in X} A(y) \wedge (y \approx x) \wedge \mathrm{S}_{\approx}(A,B) = \bigvee_{y \in X} A(y) \wedge (y \approx x) \wedge \bigwedge_{z \in X} A(z) \to \mathrm{C}_{\approx}B(z) \\
&\leq \bigvee_{y \in X} A(y) \wedge (y \approx x) \wedge (A(y) \to \mathrm{C}_{\approx}B(y)) \leq \bigvee_{y \in X} (x \approx y) \wedge \mathrm{C}_{\approx}B(y) \\
&= \bigvee_{y \in X} (x \approx y) \wedge \bigvee_{z \in X} B(z) \wedge (z \approx y) = \bigvee_{y,z \in X} B(z) \wedge (x \approx y) \wedge (y \approx z) \\
&\leq \bigvee_{y,z \in X} B(z) \wedge (x \approx z) = \bigvee_{z \in X} B(z) \wedge (x \approx z) = \mathrm{C}_{\approx}B(x)
\end{aligned}$$

proving

$$\mathrm{C}_{\approx}A(x) \wedge \mathrm{S}_{\approx}(A,B) \leq \mathrm{C}_{\approx}B(x). \tag{1.51}$$

Now, it is easy to show that

$$\mathrm{S}_{\approx}(A,B) \wedge \mathrm{S}_{\approx}(B,C) \leq \mathrm{S}_{\approx}(A,C). \tag{1.52}$$

Indeed, $\mathrm{S}_{\approx}(A,B) \wedge \mathrm{S}_{\approx}(B,C) \leq \mathrm{S}_{\approx}(A,C) = \bigwedge_{x \in X} A(x) \to \mathrm{C}_{\approx}C(x)$ holds iff for each $x \in X$ we have $\mathrm{S}_{\approx}(A,B) \wedge \mathrm{S}_{\approx}(B,C) \leq A(x) \to \mathrm{C}_{\approx}C(x)$ which is, by adjointness, equivalent to $A(x) \wedge \mathrm{S}_{\approx}(A,B) \wedge \mathrm{S}_{\approx}(B,C) \leq \mathrm{C}_{\approx}C(x)$. The last is true since by (1.49) and (1.51) we have

$$A(x) \wedge \mathrm{S}_{\approx}(A,B) \wedge \mathrm{S}_{\approx}(B,C) \leq \mathrm{C}_{\approx}B(x) \wedge \mathrm{S}_{\approx}(B,C) \leq \mathrm{C}_{\approx}C(x).$$

Theorem 13. *$(L^X, \approx^+)$ is an L-conditional universe.*

Proof. The symmetry of $\approx^+$ is obvious. The transitivity of $\approx^+$ follows from (1.52). The proof is analogous to the proof of [3, Theorem 4.41]. $\square$

We usually consider the set L^X of all conditional sets in X as a conditional universe with the conditional equality $\approx^+$. The following technical lemma is used in the next theorem.

Lemma 35. *Let A be a completely present conditional set in X compatible with $\approx$, h a reality and f an h-realization of X. Then it holds $h(A|_h \circ \approx (x)) = h(A(x))$ for all $x \in X$.*

Proof. By (1.4), we have $h(A(x)) = h(A(x) \wedge (x \approx x)) \leq h(A(x) \wedge \bigvee_{y \in X_h} y \approx x) = h(\bigvee_{y \in X_h} A(y) \wedge (y \approx x)) = h(\bigvee_{y \in X} A|_h(y) \wedge (y \approx x)) = h(A|_h \circ \approx (x))$. $\square$

Theorem 14. *Let $h \colon L \to K$ be a reality and $f \colon X \twoheadrightarrow Y$ an h-realization of X. Then the partial mapping $g \colon L^X \twoheadrightarrow K^Y$ given by $g(A) = A^h$ is an h-realization of $(L^X, \approx^+)$ where K^Y is considered with the conditional equality $\approx^{|h+}$. Moreover, if f is standard then g is also standard.*

Proof. The surjectivity of g follows from Lemma 23. By Theorem 10, g satisfies $g(A_1) \approx^{|h+} g(A_2) = A_1^h \approx^{|h+} A_2^h = h(A_1 \approx^+ A_2)$. We prove (1.4). Let $\bar{A}$ be a conditional set in X given by $\bar{A} = A \circ \approx |_h$. By Lemma 35, the fact $A \circ \approx |_h \subseteq A \circ \approx$ yields $h(A \approx^+ \bar{A}) = h(\mathrm{S}_\approx(A, \bar{A}) \wedge \mathrm{S}_\approx(\bar{A}, A)) = h((A \triangleleft ((A \circ \approx |_h) \circ \approx)) \wedge ((A \circ \approx |_h) \triangleleft (A \circ \approx))) = h(A \triangleleft (A \circ \approx)) = h(\mathrm{S}_\approx(A, A)) = h(A \approx^+ A)$. Now, $h(A \approx^+ A) = h(A \approx^+ \bar{A}) \leq \bigvee_{B^h} h(A \approx B)$, where '$B^h$' means that we take supremum over all $B \in L^X$ such that B^h is defined.

Suppose f is standard. Then $\approx^{h+}$ is an ordinary equality of sets. If $h(A \approx^+ A) = 1$ then by Theorem 12, A^h is defined. We showed that g is also standard. $\square$

The h-realization g of L^X from the preceding theorem is called the *h-realization induced by f*.

Generally, g is not faithful even if f is faithful. For example, let L be the four element Boolean algebra $\{1, c, c', 0\}$, $X = \{x_1, x_2\}$, $x_1 \approx x_1 = 1$, $x_1 \approx x_2 = x_2 \approx x_2 = x_2 \approx x_1 = 0$, h be the identity on L, $Y = \{x_1\}$, $f \colon X \twoheadrightarrow Y$ be the h-realization of X given by $f(x_1) = x_1$, $f(x_2)$ be not defined and $A = \{{}^c/x_2\}$. Then f is a faithful h-realization of X, $h(A \approx^+ A) = c'$, but the h-realization $g(A) = A^h$ of A is not defined and thus g is not faithful.

By (1.43) and the transitivity of $\mathrm{S}_\approx$, we have $\mathrm{S}_\approx(A, B_1) \wedge (B_1 \approx^+ B_2) \leq \mathrm{S}_\approx(A, B_1) \wedge \mathrm{S}_\approx(B_1, B_2) \leq \mathrm{S}_\approx(A, B_2)$ for all conditional sets A, B_1, B_2 in X and similarly it can be shown that $(A_2 \approx^+ A_1) \wedge \mathrm{S}_\approx(A_1, B) \leq \mathrm{S}(A_2, B)$ for all conditional sets A_1, A_2, B in X. We proved that $\mathrm{S}_\approx$ is compatible with $\approx^+$ from both sides. By Lemma 11, any standard h-realization of L^X respects $\mathrm{S}_\approx$.

Theorem 15. *Let h be a reality, f an h-realization of X, A and B conditional sets in X such that h-realizations A^h and B^h of A and B, respectively, are defined. Then it holds $\mathrm{S}_{\approx|h}(A^h,B^h) = h(\mathrm{S}_\approx(A,B))$ and $A^h \approx^{|h+} B^h = h(A \approx^+ B)$.*

Proof. By Lemma 10, f respects $\approx$. Now, the claim directly follows from Theorem 10. □

Lemma 36. *Let R be a binary conditional relation on X compatible with $\approx$, h a reality and f any h-realization of X which partially respects R. Then the h-realization g of L^X induced by f respects $R^\rightarrow, R^\leftarrow$ and R^+.*

Proof. Let A_1, A_2, B_1, B_2 be conditional sets in X such that h-realizations $A_1^h, A_2^h, B_1^h, B_2^h$ are defined and $A_1^h = A_2^h$ and $B_1^h = B_2^h$. We show only that g respects $R^\rightarrow$. Proofs for $R^\leftarrow$ and R^+ are similar. By Theorem 10, we have $h(R^\rightarrow(A_1,B_1)) = R^{|h\rightarrow}(A_1^h,B_1^h) = R^{|h\rightarrow}(A_2^h,B_2^h) = h(R^\rightarrow(A_2,B_2))$. □

Theorem 16. *Let h be a reality and f an h-realization of X. Then the h-realization g of L^X induced by f respects $\mathrm{S}_\approx$ and $\approx^+$.*

Proof. By Lemma 10, f respects $\approx$. Now, the claim follows from Lemma 36. □

Now, we study properties of $\mathrm{S}_\approx$ and $\approx^+$.

Lemma 37. *We have*

$$(A_2 \triangleleft A_1) \wedge \mathrm{S}_\approx(A_1,B) \le \mathrm{S}_\approx(A_2,B), \tag{1.53}$$

$$(B_1 \triangleleft B_2) \wedge \mathrm{S}_\approx(A,B_1) \le \mathrm{S}_\approx(A,B_2), \tag{1.54}$$

$$(A \triangleleft B) \wedge (B \approx^+ B) \le A \approx^+ A \tag{1.55}$$

for all conditional sets A, A_1, A_2, B, B_1 and B_2 in X.

Proof. We have $(A_2 \triangleleft A_1) \wedge \mathrm{S}_\approx(A_1,B) = (A_2 \triangleleft A_1) \wedge (A_1 \triangleleft \mathrm{C}_\approx B) \le A_2 \triangleleft \mathrm{C}_\approx B = \mathrm{S}_\approx(A_2,B)$ proving (1.53).

The inequality (1.54) is true since by (1.36) we have $(B_1 \triangleleft B_2) \wedge \mathrm{S}_\approx(A,B_1) = (B_1 \triangleleft B_2) \wedge (A \triangleleft \mathrm{C}_\approx B_1) \le (\mathrm{C}_\approx B_1 \triangleleft \mathrm{C}_\approx B_2) \wedge (A \triangleleft \mathrm{C}_\approx B_1) \le (A \triangleleft \mathrm{C}_\approx B_2) = \mathrm{S}_\approx(A,B_2)$.

Finally, to prove (1.55) we need to show that $(A \triangleleft B) \wedge (B \approx^+ B) \leq A \approx^+ A = \bigwedge_{x \in X} A(x) \to (x \approx x)$. The inequality holds iff for each $x \in X$ it holds $A(x) \wedge (A \triangleleft B) \wedge (B \approx^+ B) \leq x \approx x$, which is true since by (1.48) we have

$$A(x) \wedge (A \triangleleft B) \wedge (B \approx^+ B) \leq B(x) \wedge (B \approx^+ B) \leq x \approx x.$$

□

Lemma 38. *Let R be a completely present binary relation on X. Then for any two conditional sets A and B in X it holds $R^{\to}(A,B) \leq A \approx^+ A$, $R^{\leftarrow}(A,B) \leq B \approx^+ B$ and $R^+(A,B) \leq (A \approx^+ A) \wedge (B \approx^+ B)$.*

Proof. We prove the first inequality. Since R is completely present, $R \circ B$ is also completely present. By (1.55), we have $R^{\to}(A,B) = A \triangleleft (R \circ B) = (A \triangleleft (R \circ B)) \wedge ((R \circ B) \approx^+ (R \circ B)) \leq A \approx^+ A$. The second inequality is similar and the third one follows from (1.40). □

A direct consequence of the preceding lemma is that for conditional sets A_1 and A_2 in X it holds

$$\mathrm{S}_{\approx}(A_1, A_2) \leq A_1 \approx^+ A_1. \tag{1.56}$$

The inequality can be read as follows: "If it is satisfied in a reality h that A_1 is a subset of A_2 then it is also satisfied in h that A_1 is present".

Lemma 39. *If A is a conditional set in X such that $A \approx^+ A \leq \bigvee_{x \in X} A(x)$ then the height of A is* 1.

Proof. We have $\bigvee_{x \in X} A(x) = \big(\bigvee_{x \in X} A(x)\big) \vee (A \approx^+ A) = \big(\bigvee_{x \in X} A(x)\big) \vee \bigwedge_{y \in X} A(y) \to (y \approx y) = \bigwedge_{y \in X} \big(\bigvee_{x \in X} A(x)\big) \vee A(y)' \vee (y \approx y) \geq \bigwedge_{y \in X} A(y) \vee A(y)' = 1$. □

For a conditional set M in L^X (a conditional system of conditional sets) we define the *union* $\bigcup M$ of M as usual by

$$\left(\bigcup M\right)(x) = \bigvee_{A \in L^X} M(A) \wedge A(x) \tag{1.57}$$

and if M is completely present then the *intersection* $\bigcap M$ of M is given by

$$\left(\bigcap M\right)(x) = (x \approx x) \wedge \bigwedge_{A \in L^X} M(A) \to A(x), \tag{1.58}$$

for $x \in X$. Note that $\bigcap \emptyset = X_{\mathrm{E}}$ and thus the condition under which x is in $\bigcap \emptyset$ is equal to the condition under which x is present. The intersection of the empty set is the reason why we consider only completely present conditional sets in the definition of the intersection.

Theorem 17. *Let h be a reality and f an h-realization of X. Denote by M the set of all completely present conditional sets in X compatible with $\approx$ and by M_h the set of all completely present conditional sets in X^h compatible with $\approx^{|h}$. We consider M and M_h with conditional equalities $\approx^h$ and $\approx^{|h+}$, respectively. Then the mapping $r\colon M \to M_h$ defined by $r(A) = A^{|h}$ is an h-realization of M.*

Proof. We first show that the restricted realization $A^{|h}$ of a completely present conditional set A compatible with $\approx$ is a completely present conditional set compatible with $\approx^{|h}$. By Lemma 27, $A\circ \approx$ is safe w.r.t. f. Now, we have $A^{|h} = (\mathrm{C}_{\approx}A)^{|h} = (A\circ \approx)^{|h} = A^{|h}\circ \approx^{|h} = \mathrm{C}_{\approx^{|h}}A^{|h}$ (1. of Theorem 8). By Lemma 9, $A^{|h}$ is completely present and compatible with $\approx^{|h}$.

Next, we show the surjectivity of r. Let $A_h \in M_h$. Then there is the greatest conditional set B in X such that $B^h = A_h$ (Lemma 23). We set $A = \mathrm{C}_{\approx}B$ and show $A^{|h} = A_h$. As B^h is defined, $B\circ \approx$ is safe w.r.t. f. Now, we have $A^{|h} = (\mathrm{C}_{\approx}B)^{|h} = (B\circ \approx)^{|h} = B^h\circ \approx^{|h} = \mathrm{C}_{\approx^{|h}}A_h = A_h$.

The equality (1.3) holds by Theorem 10. The fact that r is a function easily follows from (1.4).

□

Lemma 40. *Let M be a conditional set in the conditional universe of all completely present conditional sets in L^X compatible with $\approx$. Then for each reality h and h-realization f of X it holds $(\bigcap M)^{|h} = \bigcap M^{|h}$.*

Proof. By Theorem 17, for $x \in X_h$ we have

$$\begin{aligned}
&h(\bigcap M(x)) = h((x \approx x) \wedge \bigwedge_{A \in L^X} M(A) \to A(x)) \\
&= (x^h \approx^{|h} x^h) \wedge \bigwedge_{\mathrm{C}_{\approx} A = A} h(M(A)) \to h(A(x)) \\
&= (x^h \approx^{|h} x^h) \wedge \bigwedge_{\mathrm{C}_{\approx} A = A} \bigwedge_{\bar{A}^{|h} = A^{|h}, \mathrm{C}_{\approx} \bar{A} = \bar{A}} h(M(\bar{A})) \to \bar{A}^{|h}(x^h) \\
&= (x^h \approx^{|h} x^h) \wedge \bigwedge_{\mathrm{C}_{\approx} A = A} \left(\bigvee_{\bar{A}^{|h} = A^{|h}, \mathrm{C}_{\approx} \bar{A} = \bar{A}} h(M(\bar{A})) \right) \to A^{|h}(x^h) \\
&= (x^h \approx^{|h} x^h) \wedge \bigwedge_{\mathrm{C}_{\approx} A = A} M^{|h}(A^{|h}) \to A^{|h}(x^h) \\
&= (x^h \approx^{|h} x^h) \wedge \bigwedge_{\mathrm{C}_{\approx^{|h}} A^h = A^h} M^{|h}(A^h) \to A^h(x^h) \\
&= \bigcap M^{|h}(x^h).
\end{aligned}$$

Clearly, $\bigcap M$ is completely present. It can be directly checked that it is also compatible with $\approx$. By Lemma 11 and the first part of the proof, for $x^h \in X^h$ we have $(\bigcap M)^{|h}(x^h) = h(\bigcap M(x)) = \bigcap M^{|h}(x^h)$. $\square$

A conditional mapping $C \colon L^X \to L^X$ is called a *conditional closure operator* on the conditional universe X if for each total reality h and standard h-realization f of X it holds that the restricted h-realization $C^{|h} \colon 2^{X^h} \to 2^{X^h}$ of C is an ordinary closure operator on X^h. Recall that an ordinary mapping $C \colon 2^X \to 2^X$ is called a *closure operator* on a set X if the following statements hold for each sets A and B in X:

$$A \subseteq B \text{ implies } CA \subseteq CB, \qquad \text{(monotony)}$$
$$A \subseteq CA, \qquad \text{(extensivity)}$$
$$C(CA) = CA. \qquad \text{(idempotency)}$$

Theorem 18. *Let $C \colon L^X \to L^X$ be an ordinary mapping which is a presence preserving conditional mapping satisfying $C^{|h} A^h = (CA)^h$ for each total reality h, standard h-realization f of X and conditional set A in X such that the h-realization A^h of A is defined. Then $C \colon L^X \to L^X$ is a conditional closure operator on X if and only if the following* *statements hold for each conditional sets A and B in X:*

$$(B \approx^+ B) \wedge \mathrm{S}_{\approx}(A,B) \leq \mathrm{S}_{\approx}(CA,CB), \qquad \text{(monotony)}$$
$$A \approx^+ A \leq \mathrm{S}_{\approx}(A,CA), \qquad \text{(extensivity)}$$
$$A \approx^+ A \leq C(CA) \approx^+ CA. \qquad \text{(idempotency)}$$

Proof. Suppose first that the right hand side of the equivalence holds. Let h be a total reality and f a standard h-realization of X. We show that the restricted h-realization $C^{|h}$ of C is a closure operator on X^h. Monotony: For sets A_h and B_h in X^h there are conditional sets A and B in X such that $A^h = A_h$ and $B^h = B_h$. Since the h-realization B^h of B is defined, $h(B \approx^+ B) = 1$ and by C is presence preserving, the h-realizations $(CA)^h$ and $(CB)^h$ of CA and CB, respectively, are defined. Now, we have $\mathrm{S}_{\approx^h}(A^h,B^h) = h(\mathrm{S}_{\approx}(A,B)) = h((B{\approx^+}B) \wedge \mathrm{S}_{\approx}(A,B)) \leq h(\mathrm{S}_{\approx}(CA,CB)) = \mathrm{S}_{\approx^h}((CA)^h,(CB)^h) = \mathrm{S}_{\approx^h}(C^{|h}A^h, C^{|h}B^h)$. We showed that if $A_h \subseteq B_h$ then $C^{|h}A_h \subseteq C^{|h}B_h$.

Extensivity: By the h-realization A^h of A is defined, $h(A \approx^+ A) = 1$. We have $1 = h(\mathrm{S}_{\approx}(A,CA)) = \mathrm{S}_{\approx^h}(A^h,(CA)^h) = \mathrm{S}_{\approx^h}(A^h, C^{|h}A^h)$ proving that $A_h \subseteq C^{|h}A_h$.

Idempotency: We have $1 = h(C(CA) \approx^+ CA) = (C(CA))^h \approx^{h+} (CA)^h = C^{|h}(CA)^h {\approx^{h+}} C^{|h}A^h = C^{|h}(C^{|h}A^h) \approx^{h+} C^{|h}A^h$ proving that $C^{|h}C^{|h}A_h = C^{|h}A_h$. We proved that C is a conditional closure operator.

Conversely, suppose that the left hand side of the equivalence holds. Let A and B be conditional sets in X. We prove monotony of C by showing that $h((B \approx^+ B) \wedge \mathrm{S}_{\approx}(A,B)) \leq h(\mathrm{S}_{\approx}(CA,CB))$ holds for each total reality h. Let h be a total reality and f a standard h-realization of X. The interesting case here is when $h((B \approx^+ B) \wedge \mathrm{S}_{\approx}(A,B)) = 1$. The h-realization B^h of B is defined, by (1.56), also the h-realization A^h of A is defined. Since C is presence preserving, the h-realizations $(CA)^h$ and $(CB)^h$ of CA and CB, respectively, are defined. Since C is a conditional closure operator, it holds $C^{|h}A^h \subseteq C^{|h}B^h$ and thus $\mathrm{S}_{\approx^h}(C^{|h}A^h, C^{|h}B^h) = 1$. Now, we have $h(\mathrm{S}_{\approx}(CA,CB)) = \mathrm{S}_{\approx^h}((CA)^h,(CB)^h) = \mathrm{S}_{\approx^h}(C^{|h}A^h, C^{|h}B^h) = 1$. We showed monotony of C. Extensivity and idempotency of C can be shown similarly. □

The extensivity requirement from the preceding theorem is read as follows: "if it is satisfied that A is present then it is also satisfied that A is a subset of CA". Other requirements can be read similarly.

We show that the mapping $\mathrm{C}_{\approx} : L^X \to L^X$ defined by (1.35) is a conditional closure operator. It can be directly checked that $\mathrm{C}_{\approx}$ is a conditional mapping. Clearly, $\mathrm{C}_{\approx}$ is presence preserving. It can be also easily verified that $\mathrm{C}^{|h}_{\approx}(A^h) = (\mathrm{C}_{\approx}A)^h$ holds for each total reality

h, standard h-realization f of X and conditional set A in X such that the h-realization A^h of A is defined.

By (1.36), we have $\mathrm{S}_{\approx}(A,B) = A \triangleleft \mathrm{C}_{\approx}B \leq \mathrm{C}_{\approx}A \triangleleft \mathrm{C}_{\approx}(\mathrm{C}_{\approx}B) = \mathrm{S}_{\approx}(\mathrm{C}_{\approx}A, \mathrm{C}_{\approx}B)$ proving monotonicity. By (1.37), we have $A \approx^{+} A = \mathrm{S}_{\approx}(A,A) = A \triangleleft \mathrm{C}_{\approx}A = A \triangleleft \mathrm{C}_{\approx}(\mathrm{C}_{\approx}A) = \mathrm{S}_{\approx}(A, \mathrm{C}_{\approx}A)$ for each conditional set A in X. Thus, $\mathrm{C}_{\approx}$ is extensive. Directly by (1.37), $\mathrm{C}_{\approx}$ is idempotent.

By Theorem 18, $\mathrm{C}_{\approx}$ is a conditional closure operator on X.

1.4.2 Reflexive conditional equalities

In this section, we show that a conditional universe can be represented by a conditional universe with reflexive conditional equality and a conditional set of present elements. On the other hand, we show that a conditional universe with a reflexive conditional equality can be naturally embedded in a conditional universe with a conditional equality which is not reflexive.

Recall that $\mathrm{E}_{\approx}$ is the reflexive conditional equality on X defined by (1.22). From $x \approx y \leq \mathrm{E}_{\approx}(x,y)$ and (1.1) we get $x \approx y \leq (x \approx x) \wedge \mathrm{E}_{\approx}(x,y)$. The converse inequality also holds as $(x \approx x) \wedge \mathrm{E}_{\approx}(x,y) \leq (x \approx x) \wedge ((x \approx x) \rightarrow (x \approx y)) \leq x \approx y$. Similarly, can be shown that $x \approx y = (y \approx y) \wedge \mathrm{E}_{\approx}(x,y)$. Therefore,

$$x \approx y = (x \approx x) \wedge \mathrm{E}_{\approx}(x,y) = (y \approx y) \wedge \mathrm{E}_{\approx}(x,y). \tag{1.59}$$

for all $x, y \in X$. Recall that X_{E} is the conditional set in X given by $X_{\mathrm{E}}(x) = x \approx x$. The conditional universe $(X, \approx)$ can be equivalently represented as $(X, \mathrm{E}_{\approx}, X_{\mathrm{E}})$. This construction is also described in [14].

It the rest of this section, we suppose that X is an L-conditional universe such that the conditional equality $\approx$ of X is reflexive, i.e. $x \approx x = 1$ for all $x \in X$. By the reflexivity of $\approx$, we have $\mathrm{E}_{\approx} = \mathrm{R}_{\approx} = \approx$. By faithfulness of standard realizations, standard realizations of X are ordinary mappings. Moreover, we suppose that all realizations of X in this section are mappings.

A *conditional point* [19] in X is a conditional set A in X such that $A(x_1) \wedge A(x_2) \leq x_1 \approx x_2$ for all $x_1, x_2 \in X$. When X is an ordinary universe, A is a conditional point if and only if it is empty or a singleton.

The following lemma is a reformulation of [19, Lemma 6].

Lemma 41. *Let A be a conditional set in X.*

1. If A is a conditional point then each realization of A is also a conditional point.

2. If for each standard h-realization f of X it holds that the h-realization A^h of A has at most one element then A is a conditional point.

Conditional points represent single elements that do or do not exist. If for a reality h and any h-realization f of X it holds $A^h = \emptyset$ we say that A *does not exists in h*.

Lemma 42. *Let A be a conditional set in X, h a reality and f an h-realization of X. Then $h(hgtA) = hgtA^h$.*

Proof. We have $\mathrm{hgt}A^h = \bigvee_{x^h \in X^h} A^h(x^h) = \bigvee_{x \in X} \bigvee_{y^h = x^h} h(A(y)) = h\left(\bigvee_{x \in X} A(x)\right) = h(\mathrm{hgt}A)$. □

By the preceding lemma A does not exist in a reality h if and only if $h(\mathrm{hgt}A) = 0$.

Denote by $\mathscr{C}X$ the set of all conditional points in X. We define a binary conditional relation $\approx_{\mathscr{C}}$ on $\mathscr{C}X$ by

$$A_1 \approx_{\mathscr{C}} A_2 = \mathrm{hgt}A_1 \wedge (A_1 \approx^+ A_2) \wedge \mathrm{hgt}A_2 \tag{1.60}$$

for $A_1, A_2 \in \mathscr{C}X$.

Clearly, $\approx_{\mathscr{C}}$ is symmetric and transitive. So, $\approx_{\mathscr{C}}$ is a conditional equality and we have the conditional universe $(\mathscr{C}X, \approx_{\mathscr{C}})$ of conditional points. Since $\emptyset$ is always a conditional point and $\emptyset \approx_{\mathscr{C}} \emptyset = 0$, $\approx_{\mathscr{C}}$ is not reflexive.

The value $A \approx_{\mathscr{C}} A = \mathrm{hgt}A$ is interpreted as the condition under which A exists. Note that in [16, 17] the height of a conditional point compatible with $\approx$ is interpreted as the extent of existence. We have that A does not exist in a reality h if and only if $h(A \approx_{\mathscr{C}} A) = h(\mathrm{hgt}(A)) = 0$.

An *embedding* of an L-conditional universe X to an L-conditional universe Y is an injective mapping $f : X \to Y$ such that $f(x_1) \approx_Y f(x_2) = x_1 \approx_X x_2$ for every $x_1, x_2 \in X$. For a set X we denote by id_X the identity on X.

Theorem 19. *Let $f_{\mathscr{C}} : X \to \mathscr{C}X$ be a mapping given by $x \mapsto \{x\}$. Then $f_{\mathscr{C}}$ is an embedding of X to $\mathscr{C}X$, $f_{\mathscr{C}}^{-1}$ is an id_L-realization of $\mathscr{C}X$. Furthermore, X and $\mathscr{C}X$ are conditionally isomorphic.*

Proof. Since singletons $\{x\}$ for $x \in X$ are conditional points, the mapping $f_{\mathscr{C}}$ is defined correctly and it is obviously injective. Observe that $\mathrm{S}_{\approx}(\{x_1\}, \{x_2\}) = \{x_1\} \triangleleft (\{x_2\} \circ \approx) =$

$\{x_2\}\circ \approx (x_1) = x_1 \approx x_2$ and also $\mathrm{S}_{\approx}(\{x_2\},\{x_1\}) = x_1 \approx x_2$. Thus, $\{x_1\} \approx^+ \{x_2\} = x_1 \approx x_2$. By definition, we have that $f_{\mathscr{C}}(x_1) \approx_{\mathscr{C}} f_{\mathscr{C}}(x_2) = \mathrm{hgt}\{x_1\} \wedge (\{x_1\} \approx^+ \{x_2\}) \wedge \mathrm{hgt}\{x_2\} = 1 \wedge (x_1 \approx x_2) \wedge 1 = x_1 \approx x_2$. We showed that $f_{\mathscr{C}}$ is an embedding of X to $\mathscr{C}X$.

We prove that $f_{\mathscr{C}}^{-1}$ is an id_L-realization of $\mathscr{C}X$. Clearly, $f_{\mathscr{C}}^{-1}$ is surjective and from the first part of the proof it follows that $\{x_1\} \approx_{\mathscr{C}} \{x_2\} = f_{\mathscr{C}}^{-1}(\{x_1\}) \approx f_{\mathscr{C}}^{-1}(\{x_2\})$. It reminds to show that $h(A \approx_{\mathscr{C}} A) \le \bigvee_{x\in X} h(\{x\} \approx^+ A)$ holds for each conditional point A in X.

We first prove that $h(A(x)) \le h(\{x\} \approx^+ A) = h(\mathrm{S}_{\approx}(\{x\},A) \wedge \mathrm{S}_{\approx}(A,\{x\}))$. Clearly, $h(A(x)) \le h(\mathrm{S}_{\approx}(\{x\},A)) = h(\{x\} \triangleleft (A\circ \approx)) = h(A\circ \approx (x))$. Now, $h(A(x)) \le h(\mathrm{S}_{\approx}(A,\{x\})) = h(A \triangleleft (\{x\}\circ \approx))$ holds iff $h(A(x)) \le h(A(\bar{x}) \rightarrow (\{x\}\circ \approx (\bar{x})))$ holds for all $\bar{x} \in X$. The last is true since it is, by adjointness, equivalent with $h(A(x) \wedge A(\bar{x})) \le h(x \approx \bar{x})$ which is true as A is a conditional point.

Since $h(A(x)) \le h(\{x\} \approx^+ A) \le \bigvee_{\bar{x}\in X} h(\{\bar{x}\} \approx^+ A)$ for each $x \in X$, we have that $h(A \approx_{\mathscr{C}} A) = h(\bigvee_{x\in X} A(x)) \le \bigvee_{x\in X} h(\{x\} \approx^+ A)$. We showed that $f_{\mathscr{C}}^{-1}$ is an id_L-realization of $\mathscr{C}X$.

The identity on $\mathscr{C}X$ is an id_L-realization of $\mathscr{C}X$. By Theorem 11, X and $\mathscr{C}X$ are conditionally isomorphic . □

1.5 Extensionality of conditional sets

Let X be an L-conditional universe. Conditional sets A and B in X are called *extensionally equal*, in symbols $A \sim B$, if they are extensionally equal as elements in the conditional universe of all conditional sets $(L^X, \approx^+)$, i.e. if they satisfy $A \approx^+ B = A \approx^+ A = B \approx^+ B$. By Theorem 15 and Theorem 12, conditional sets A and B in X are extensionally equal if and only if it holds that if A or B is present in a total reality h then so is the other and they are equal in h.

We give a necessary and sufficient condition for two conditional sets to be extensionally equal.

Lemma 43. *Let A and B be conditional sets in X. Then it holds that $B \approx^+ B \le \mathrm{S}_{\approx}(A,B)$ and $A \approx^+ A \le \mathrm{S}_{\approx}(B,A)$ if and only if A and B are extensionally equal.*

Proof. By Lemma 38, we have $B \approx^+ B \le \mathrm{S}_{\approx}(A,B) \le A \approx^+ A$ and $A \approx^+ A \le \mathrm{S}_{\approx}(B,A) \le B \approx^+ B$. Therefore, $A \approx^+ A = B \approx^+ B = \mathrm{S}_{\approx}(A,B) = \mathrm{S}_{\approx}(B,A) = A \approx^+ B$. Conversely, suppose $A \approx^+ B = A \approx^+ A = B \approx^+ B$. Then we have $B \approx^+ B = A \approx^+ B \le \mathrm{S}_{\approx}(A,B)$ and $A \approx^+ A = B \approx^+ A \le \mathrm{S}_{\approx}(B,A)$. □

The left-hand side of the equivalence from the preceding lemma is interpreted as follows: "if B is present then A is a subset of B and, conversely, if A is present then B is a subset of A".

We slightly modify the preceding lemma.

Lemma 44. *Let A and B be conditional sets in X such that $A \subseteq B$. Then it holds that $A \approx^+ A \leq \mathrm{S}_\approx(B,A)$ if and only if A and B are extensionally equal.*

Proof. We consider Lemma 43 and prove that $B \approx^+ B \leq \mathrm{S}_\approx(A,B)$. By (1.53), the last inequality is true since $B \approx^+ B = \mathrm{S}_\approx(B,B) \leq \mathrm{S}_\approx(A,B)$. $\square$

In the following, we search for a conditional universe consisting of conditional sets compatible with $\approx$ which is separated and for each conditional set there is an extensionally equal conditional set in the conditional universe. The idea is to choose from each class of extensionally equal conditional sets the greatest element with respect to $\subseteq$. For a conditional set A in X we define a conditional set $\mathrm{C}_\sim A = \bigcup\{B \in L^X \mid A \sim B\}$.

Before we show that $\mathrm{C}_\sim A$ is extensionally equal to A we make the following observation.

By (1.49), for each $x \in X$ we have

$$\begin{aligned}
\mathrm{C}_\sim A(x) \wedge (A \approx^+ A) &= \left(\bigvee_{B\sim A} B(x)\right) \wedge (A \approx^+ A) = \bigvee_{B\sim A} B(x) \wedge (A \approx^+ A) \\
&= \bigvee_{B\sim A} B(x) \wedge (B \approx^+ A) \leq \bigvee_{B\sim A} B(x) \wedge \mathrm{S}_\approx(B,A) \leq \bigvee_{B\sim A} \mathrm{C}_\approx A(x) \\
&= \mathrm{C}_\approx A(x)
\end{aligned}$$

proving

$$\mathrm{C}_\sim A(x) \wedge (A \approx^+ A) \leq \mathrm{C}_\approx A(x). \tag{1.61}$$

Now, it is easy to prove the following.

Lemma 45. *Each conditional set A in X is extensionally equal to $\mathrm{C}_\sim A$.*

Proof. Clearly $A \subseteq \mathrm{C}_\sim A$. By Lemma 44, it remains to show $A \approx^+ A \leq \mathrm{S}_\approx(\mathrm{C}_\sim A, A)$.

The inequality $A \approx^+ A \leq \mathrm{S}_\approx(\mathrm{C}_\sim A, A) = \bigwedge_{x\in X} \mathrm{C}_\sim A(x) \to \mathrm{C}_\approx A(x)$ holds iff for each $x \in X$ it holds $A \approx^+ A \leq \mathrm{C}_\sim A(x) \to \mathrm{C}_\approx A(x)$ which is, by adjointness, equivalent to (1.61). $\square$

Fixpoints of $\mathrm{C}_\sim$ are called *extensional conditional sets*. We denote by $\mathscr{E}X$ the set of all extensional conditional sets in X.

Theorem 20. *The set of all extensional conditional sets $\mathscr{E}X$ in a conditional universe X with the restriction of $\approx^+$ to $\mathscr{E}X$ is a separated conditional universe.*

Proof. It remains to show that the universe of extensional conditional sets is separated. Assume that A and B are extensional conditional sets such that $A \approx^+ B = A \approx^+ A = B \approx^+ B$. Then they are extensionally equal and thus $A = \mathrm{C}_\sim A = \mathrm{C}_\sim B = B$. $\square$

We prove that extensional conditional sets are compatible with $\approx$. First, we introduce the following operator. For a conditional set A in X we define the conditional set $\mathrm{C_t}A = (A{\approx^+}A) \to \mathrm{C}_\approx A$ in X.

Lemma 46. $\mathrm{C_t}A$ *is compatible with $\approx$ for every conditional set A in X.*

Proof. Since $\mathrm{C}_\approx A$ is compatible with $\approx$, for every $x, y \in X$ we have $\mathrm{C_t}A(x) \wedge (x \approx y) = ((A \approx^+ A) \to \mathrm{C}_\approx A(x)) \wedge (x \approx y) \leq (A \approx^+ A) \to (\mathrm{C}_\approx A(x) \wedge (x \approx y)) \leq (A \approx^+ A) \to \mathrm{C}_\approx A(y) = \mathrm{C_t}A(y)$. $\square$

Now, we are going to show that operators $\mathrm{C}_\sim$ and $\mathrm{C_t}$ are identical.

Lemma 47. *For each conditional set A in X it holds $A \subseteq \mathrm{C_t}(A)$.*

Proof. We need to show that for each $x \in X$ it holds $A(x) \leq (A \approx^+ A) \to \mathrm{C}_\approx(A)(x)$ which is, by adjointess, equivalent to $A(x) \wedge (A \approx^+ A) \leq \mathrm{C}_\approx(A)(x)$. The last inequality is (1.50). $\square$

Lemma 48. *Each conditional set A in X is extensionally equal to* $\mathrm{C_t}(A)$.

Proof. By Lemma 44, it reminds to show $A \approx^+ A \leq \mathrm{S}_\approx(\mathrm{C_t}A, A)$. The inequality $A \approx^+ A \leq \mathrm{S}_\approx(\mathrm{C_t}A, A) = \bigwedge_{x \in X}(((A \approx^+ A) \to \mathrm{C}_\approx A(x)) \to \mathrm{C}_\approx A(x)$ holds iff for each $x \in X$ we have

$$A \approx^+ A \leq (((A \approx^+ A) \to \mathrm{C}_\approx A(x)) \to \mathrm{C}_\approx A(x)$$

which is, by adjointness, equivalent to

$$(A \approx^+ A) \wedge (A \approx^+ A \to \mathrm{C}_\approx A(x)) \leq \mathrm{C}_\approx A(x).$$

The last is obviously true.

$\square$

Lemma 49. *Operators* $\mathrm{C_t}$ *and* $\mathrm{C_\sim}$ *are equal.*

Proof. We show that for each conditional set A it holds $\mathrm{C_t}A = \mathrm{C_\sim}A$. Since A and $\mathrm{C_t}A$ are extensionally equal, $\mathrm{C_t}A \subseteq \mathrm{C_\sim}A$. Conversely, $\mathrm{C_\sim}A \subseteq \mathrm{C_t}A$ iff for each $x \in X$ it holds $\mathrm{C_\sim}A(x) \leq (A \approx^+ A) \to \mathrm{C_\approx}A(x)$ which is, by adjointness, equivalent to $\mathrm{C_\sim}A(x) \wedge (A \approx^+ A) \leq \mathrm{C_\approx}A(x)$. The last inequality is (1.61). $\square$

The following summarizes properties of extensional conditional sets.

1. The universe of all extensional conditional sets is separated.
2. Extensional conditional sets are compatible with $\approx$.
3. For each conditional set there is an extensionally equal extensional conditional set.

In the following part, we prove that each shift of a completely present conditional set compatible with $\approx$ is extensional. First, we study shifts of conditional sets. For each condition c and conditional set A in X it holds

$$(c \to A) \approx^+ (c \to A) = (c \vee (X \approx^+ X)) \wedge (A \approx^+ A). \tag{1.62}$$

Indeed, we have

$$\begin{aligned}
(c \to A) \approx^+ (c \to A) &= \bigwedge_{x \in X} (c \to A(x)) \to (x \approx x) \\
&= \bigwedge_{x \in X} (c \vee (x \approx x)) \wedge (A(x) \to (x \approx x)) \\
&= \left(\bigwedge_{x \in X} c \vee (x \approx x) \right) \wedge \bigwedge_{x \in X} A(x) \to (x \approx x) \\
&= \left(c \vee \bigwedge_{x \in X} x \approx x \right) \wedge (A \approx^+ A) = (c \vee (X \approx^+ X)) \wedge (A \approx^+ A).
\end{aligned}$$

In the preceding calculation, we used that $(a \to b) \to c = (a \vee c) \wedge (b \to c)$ holds for each conditions $a, b, c \in L$.

For conditional sets A and B in X it holds

$$\mathrm{E}((B \approx B) \to A) = (B \approx B) \wedge (A \approx A). \tag{1.63}$$

The equality holds since by (1.62) we have $\mathrm{E}((B \approx B) \to A) = ((B \approx B) \vee (X \approx X)) \wedge (A \approx A) = (B \approx B) \wedge (A \approx A)$.

If A is completely present and compatible with $\approx$ then for $x \in X$ it holds

$$\mathrm{C}_{\approx}(c \to A)(x) = (c \to A)(x) \wedge (x \approx x). \tag{1.64}$$

The preceding equality holds since by (1.2) we have

$$\begin{aligned}
\mathrm{C}_{\approx}(c \to A)(x) &= \bigvee_{y \in X} (c \to A(y)) \wedge (y \approx x) = \bigvee_{y \in X} (c' \wedge (y \approx x)) \vee (A(y) \wedge (y \approx x)) \\
&= \left(c' \wedge \bigvee_{y \in X} y \approx x \right) \vee \bigvee_{y \in X} A(y) \wedge (y \approx x) = (c' \wedge (x \approx x)) \vee \mathrm{C}_{\approx} A(x) \\
&= (c' \wedge (x \approx x)) \vee A(x) = (c \to A(x)) \wedge ((x \approx x) \vee A(x)) = (c \to A)(x) \wedge (x \approx x).
\end{aligned}$$

Theorem 21. *Extensional conditional sets in X are exactly shifts of completely present conditional sets in X compatible with $\approx$.*

Proof. By the definition of extensionality of conditional sets, each extensional conditional set in X is a shift of a completely present conditional set in X compatible with $\approx$. Let A be a completely present conditional set in X compatible with $\approx$ and c be a condition. We show that $\mathrm{C}_{\mathrm{t}}(c \to A)(x) = (c \to A)(x)$ holds for each $x \in X$ in order to prove the extensionality of $c \to A$. By (1.63) and (1.64), it holds that

$$\begin{aligned}
\mathrm{C}_{\mathrm{t}}(c \to A)(x) &= ((c \to A) \approx^{+} (c \to A)) \to \mathrm{C}_{\approx}(c \to A)(x) \\
&= (c \vee (X \approx^{+} X)) \to ((c \to A)(x) \wedge (x \approx x)) \\
&= (c \to (c \to A(x))) \wedge (c \to (x \approx x)) \wedge ((X \approx^{+} X) \to (c \to A(x))) \\
&\quad \wedge ((X \approx^{+} X) \to (x \approx x)) \\
&= (c \to A(x)) \wedge (c \to (x \approx x)) \wedge ((X \approx^{+} X) \to (c \to A(x))) \\
&= (c \to A)(x).
\end{aligned}$$

In the preceding calculation, we used that $(X \approx^{+} X) \to (x \approx x) = 1$ and $c \to A(x) \leq c \to (x \approx x)$ holds. $\square$

There is another characterization of extensional conditional sets which uses the conditional relation $\mathrm{R}_{\approx}$ on X. Recall that $\mathrm{R}_{\approx}$ is defined by $\mathrm{R}_{\approx}(x,y) = (x \approx x) \to (x \approx y)$ for $x, y \in X$.

Lemma 50. *For each conditional set A in X it holds $A \circ \mathrm{R}_{\approx} = \mathrm{C}_{\mathrm{t}} A$.*

Proof. For each $x \in X$ we have

$$
\begin{aligned}
A \circ \mathrm{R}_{\approx}(x) &= \bigvee_{y \in X} A(y) \wedge ((y \approx y) \to (y \approx x)) \\
&= \bigvee_{y \in X} (A(y) \wedge (y \approx y)') \vee (A(y) \wedge (y \approx x)) \\
&= \left(\bigwedge_{y \in X} A(y) \to (y \approx y) \right)' \vee \bigvee_{y \in X} A(y) \wedge (y \approx x) \\
&= \mathrm{E}A \to \mathrm{C}_{\approx}A(x) = \mathrm{C}_{\mathrm{t}}A(x).
\end{aligned}
$$

□

From the preceding Lemma easily follows:

Theorem 22. *A conditional set A in X is extensional if and only if it satisfies*

$$A(x) \wedge (x \approx y) \leq A(y), \tag{1.65}$$

$$A(x) \wedge (x \approx x)' \leq A(y) \tag{1.66}$$

for each $x, y \in X$.

Proof. By Lemma 50, extensionality of A is equivalent to $A \circ \mathrm{R}_{\approx} = A$ and, by Lemma 16, equivalent to compatibility of A with $\mathrm{R}_{\approx}$. For $x, y \in X$ we have $A(x) \wedge \mathrm{R}_{\approx}(x, y) = A(x) \wedge ((x \approx x) \to (x \approx y)) = (A(x) \wedge (x \approx x)') \vee (A(x) \wedge (x \approx y)) \leq A(y)$ iff $A(x) \wedge (x \approx x)' \leq A(y)$ and $A(x) \wedge (x \approx y) \leq A(y)$. Which proves the claim. □

If A is compatible with $\mathrm{R}_{\approx}$ then it is also compatible with $\mathrm{E}_{\approx}$. Indeed, for each $x, y \in X$ we have $A(x) \wedge \mathrm{E}_{\approx}(x, y) = A(x) \wedge \mathrm{R}_{\approx}(x, y) \wedge \mathrm{R}_{\approx}(y, x) \leq A(x) \wedge \mathrm{R}_{\approx}(x, y) \leq A(y)$. Therefore, extensional conditional sets are compatible with $\mathrm{E}_{\approx}$.

Lemma 51. *If a conditional set A in X is extensional and* $x_1, x_2 \in X$ *are extensionally equal then* $A(x_1) = A(x_2)$.

Proof. Since L is complete and atomic, it suffices to show that $h(A(x_1)) = h(A(x_2))$ for each total reality h. Let f be a standard h-realization of X. If A, x_1 and x_2 are present in f then by the assumption $x_1^h = x_2^h$. Lemma 8 gives $h(A(x_1)) = A^h(x_1^h) = A^h(x_2^h) = h(A(x_2))$. If A is present in f and both x_1, x_2 are not present in f then (1.12) yields $h(A(x_1)) = h(A(x_2)) = 0$. Finally, if A is not present in f then by extensionality of A we have $1 = h(A(x_1)) = h(A(x_2))$. □

Chapter 2

Structures with incomplete information

A set U with a binary operation r^U is a a simple example of an ordinary mathematical structure. Assume that there is a missing information in the example. Since binary operations are relations, we can represent the structure with missing information by a conditional universe M and a ternary conditional relation r^M. If the amount of the missing information in U is not large then we can still study the structure of U. For example, we can say that U is a group, i.e. it is a group in every possible world.

2.1 Conditional structures

In this section, we present a logic similar to the fuzzy logic discussed in [3, Section 3.2]. Interpretation of quantifiers is adopted from [6, Section 5]. A similar logic is also proposed in [14, Section 4]. As usual, we first present the syntax and then the semantics of our logic.

2.1.1 Sorted languages

Suppose we have a nonempty set S whose elements s will be called *sorts*. An *S-sorted language* $\mathscr{J}$ (*with equality*) is given by the following: a set R of relation symbols, each $r \in R$ with a (possibly empty) string $s_1 s_2 \dots s_n$ of sorts $s_i \in S$ called the *arity* of r, R has to contain a relation symbol $\approx_s$ of the sort ss for each $s \in S$; object variables, each variable x given with its sort $s \in S$, for each sort s there is a denumerable number of variables of sort s; logical connectives $\wedge, \vee, \Rightarrow$ and $\neg$, truth constants 0 and 1, quantifiers $\forall$ and $\exists$, and paranthesis. We denote the arity of $r \in R$ by $\delta(r)$ and the sort of a variable x by $\delta(x)$. Each S-sorted language $\mathscr{J}$ is fully specified by the pair (R, δ), called the *type* of $\mathscr{J}$. Note that there is no function symbol in our logic.

Atomic formulas are of the form $r(x_1, \ldots, x_n)$ where $r \in R$ has arity $s_1 \ldots s_n$ and x_i are variables of sorts s_i, and truth constants; composed formulas are defined us usual from atomic ones using logical connectives and quantifiers.

2.1.2 Conditional structures

An *L-conditional structure* for an S-sorted language of type (R, δ) is a pair (M, R^M) where: $M = \{M_s \mid s \in S\}$ is a system of sets M_s (M_s is called the *universe of the sort* s) and a set $R^M = \{r^M \in L^{M_{s_1} \times \cdots \times M_{s_n}} \mid r \in R, \delta(r) = s_1 \cdots s_n\}$ of conditional relations such that each $\approx_s^M$ is a conditional equality on M_s and each $r^M \in R^M$ with $\delta(r) = s_1 \cdots s_n$ is compatible with $\approx_{s_1}^M, \ldots, \approx_{s_n}^M$. We usually omit the superscripts in r^M and $\approx_s^M$, and write only r and $\approx_s$.

An *M-valuation* v is a mapping assigning to each variable x an element $v(x) \in M_{\delta(x)}$. The *condition* $||\varphi||_{M,v}$ *under which a formula* φ *is true under an M-valuation* v is defined as follows:

(i) for atomic formulas:

$$\begin{aligned} ||r(x_1, \ldots, x_n)||_{M,v} &= r^M(v(x_1), \ldots, v(x_n)), \\ ||0||_{M,v} &= 0, \\ ||1||_{M,v} &= 1; \end{aligned}$$

(ii) if φ and ψ are formulas then

$$\begin{aligned} ||\varphi \wedge \psi||_{M,v} &= ||\varphi||_{M,v} \wedge ||\psi||_{M,v}, \\ ||\varphi \vee \psi||_{M,v} &= ||\varphi||_{M,v} \vee ||\psi||_{M,v}, \\ ||\varphi \Rightarrow \psi||_{M,v} &= ||\varphi||_{M,v} \rightarrow ||\psi||_{M,v}, \\ ||\neg\varphi||_{M,v} &= ||\varphi||'_{M,v}; \end{aligned}$$

(iii) if φ is a formula and x is a variable of the sort s then

$$||(\forall x)\varphi||_{M,v} = \bigwedge_{m\in M_s} (m \approx_s m) \to ||\varphi||_{M,v(x/m)},$$
$$||(\exists x)\varphi||_{M,v} = \bigvee_{m\in M_s} (m \approx_s m) \wedge ||\varphi||_{M,v(x/m)}$$

where $v(x/m) = (v \setminus \{(x, v(x))\}) \cup \{(x,m)\}$.

The value $\bigwedge_v ||\varphi||_{M,v}$ is interpreted as the *condition under which φ is valid in M*. ("v" in the subscript of $\bigwedge$ means we are taking the infimum over all M-valuations v.)

Let $h\colon L \to K$ be a reality. For each $s \in S$ let f_s be an h-realization of M_s such that for each relation symbol r of arity $s_1 \ldots s_n$ it holds that h-realizations $f_{s_1}, \ldots, f_{s_n}$ partially respect r^M. By Lemma 13, for each relation symbol r of arity $s_1 \cdots s_n$ it holds that the restricted h-realization $r^{M|h}$ of r^M is compatible with $\approx_{s_1}^{M|h}, \ldots, \approx_{s_n}^{M|h}$. Then the *$h$-realization of M* is defined to be the K-conditional structure (N, R^N) where $N = \{M_s^h \mid s \in S\}$ and $R^N = \{r^N \mid r \in R\}$ and $r^N = r^{M|h}$ for each $r \in R$. If h-realizations f_s are given then we denote N by M^h. We also call N itself an h-realization of M. In this situation, we assume that h-realizations f_s of M_s are given and satisfy above requirements.

An h-realization N of M is called *total*, *merging*, *omitting*, *faithful* and *standard* if M_s^h is total, merging, omitting, faithful and standard h-realization of M_s, respectively, for each $s \in S$.

Let v be an M-valuation such that $v(x)^h$ is defined for every object variable x. Then the M^h-valuation v^h defined by $v^h(x) = v(x)^h$ for each object variable x is called the *h-realization of v*.

Construction of conditional structures. We fix an S-sorted language. Suppose we have an ordinary structure M_i for each $i \in I$. Let $L = 2^I$ and, for each $i \in I$, $h_i\colon L \to 2$ be the total reality given by $h_i(c) = 1$ iff $i \in c$. Then we can by means of Lemma 3 find conditional universes M_s ($s \in S$) with the following property: $(M_i)_s$ is a standard h-realization of M_s for each total reality h. Two canonical ways how to find M_s are presented in the end of Subsection 1.2.3. We set $M = \{M_s \mid s \in S\}$. Now, for each relation symbol r of arity $s_1 \cdots s_n$ we set r^M to be the unique completely present conditional relation between $M_{s_1}, \ldots, M_{s_n}$ compatible with $\approx_{s_1}, \ldots, \approx_{s_n}$ such that the h_i-realization $(r^M)^{h_i}$ of r^M is r^{M_i} for each $i \in I$ (Theorem 7).

The preceding construction yields a conditional structure M such that M_i is the h_i-realization of M for each $i \in I$.

Isomorphism of conditional structures. Let M and N be L-conditional structures for the same S-sorted language. We call a family of conditional bijections $F_s : M_s \to N_s$ ($s \in S$) a *conditional isomorphism* between M and N if for each total reality h, standard h-realizations f_s and g_s of M_s and N_s ($s \in S$), respectively, and relation symbol r of arity $s_1 \cdots s_n$ it holds $r^{M^h}(x_1^h, \ldots, x_n^h) = r^{N^h}(F_{s_1}^{|h}(x_1^h), \ldots, F_{s_n}^{|h}(x_n^h))$ where $x_1^h \in M_{s_1}^h, \ldots, x_n^h \in M_{s_n}^h$ (*ordinary isomorphism* of M^h and N^h).

Conditional structures M and N are called *conditionally isomorphic* if there is a conditional isomorphism between M and N.

Earlier defined (Sec. 1.3.5) conditionally isomorphic conditional universes are a special case of conditionally isomorphic conditional structures.

2.1.3 Truth in realizations

We will study a connection between the truth of a formula in a conditional structure M and the truth of the formula in realizations of M.

Let M be an L-conditional structure.

Lemma 52. *For each formula φ, reality h, faithful h-realization N of M and M-valuation v such that the h-realization v^h of v is defined it holds $h(||\varphi||_{M,v}) = ||\varphi||_{N,v^h}$.*

Proof. The proof goes by induction on complexity of φ. If $\varphi = r(x_1, \ldots, x_n)$ then by Theorem 5 we have $h(||r(x_1, \ldots)||_{M,v}) = h(r^M(v(x_1), \ldots)) = (r^M)^{|h}(v(x_1)^h, \ldots) = r^N(v(x_1)^h, \ldots) = r^N(v^h(x_1), \ldots) = ||r(x_1, \ldots)||_{N,v^h}$.

If $\varphi = \psi_1 \wedge \psi_2$ then we have $h(||\psi_1 \wedge \psi_2||_{M,v}) = h(||\psi_1||_{M,v} \wedge ||\psi_2||_{M,v}) = h(||\psi_1||_{M,v}) \wedge h(||\psi_2||_{M,v}) = ||\psi_1||_{N,v^h} \wedge ||\psi_2||_{N,v^h} = ||\psi_1 \wedge \psi_2||_{N,v^h}$. The proof is similar for other connectives.

If $\varphi = (\forall x)\psi$ with $s = \sigma(x)$ then since the h-realization N_s of M_s is faithful we have

$$\begin{aligned}
h(||(\forall x)\psi||_{M,v}) &= h\left(\bigwedge_{m\in M_s}(m \approx_s^M m) \to ||\psi||_{M,v(x/m)}\right)\\
&= \bigwedge_{m\in M_s} h(m \approx_s^M m) \to h(||\psi||_{M,v(x/m)})\\
&= \bigwedge_{m\in (M_s)_h} (m^h \approx_s^N m^h) \to ||\psi||_{N,v^h(x/m^h)}\\
&= \bigwedge_{m^h\in N_s} (m^h \approx_s^N m^h) \to ||\psi||_{N,v^h(x/m^h)} = ||(\forall x)\psi||_{N,v^h}.
\end{aligned}$$

The proof is similar for the existential quantifier. □

Theorem 23. *Let φ be a formula, M a conditional structure and v an M-valuation such that $v(x) \approx_{\delta(x)} v(x) = 1$ for each object variable x. Then the following statements are equivalent.*

1. $||\varphi||_{M,v} = 1$.
2. $||\varphi||_{N,v^h} = 1$ *for every reality h and faithful h-realization N of M.*
3. $||\varphi||_{N,v^h} = 1$ *for every total reality h and standard h-realization N of M.*

Proof. Trivially, 2. implies 3. By Lemma 52, 1. implies 2. We prove that 3. implies 1. Suppose $||\varphi||_{M,v} \neq 1$. Then there is a total reality h such that $h(||\varphi||_{M,v}) = 0$. Let N be any standard h-realization of M. The h-realization v^h of v is defined and we have by Lemma 52, $||\varphi||_{N,v^h} = 0$. □

Conditional mappings compatible with conditional equalities. We show an application of Theorem 23. Let F be a conditional relation between conditional universes X and Y compatible with $\approx_X$ and $\approx_Y$. Then we can regard F as a conditional structure for a two sorted language with sorts X and Y and a relation symbol f of the arity XY with the obvious interpretation $f^F = F$. By Theorem 23, we have that F is a conditional mapping if and only if formulas $(\forall x)(\exists y)f(x,y)$ and $(\forall x,y_1,y_2)((f(x,y_1)\wedge f(x,y_2)) \Rightarrow (y_1 \approx_Y y_2))$, where x is of the sort X and y,y_1,y_2 are of the sort Y, are valid in F. Semantic rules of our logic yield that F is a conditional mapping if and only if

$$x \approx_X x \leq \bigvee_{y\in Y}(y \approx_Y y)\wedge F(x,y), \tag{2.1}$$

$$(x \approx_X x)\wedge(y_1 \approx_Y y_1)\wedge(y_2 \approx_Y y_2)\wedge F(x,y_1)\wedge F(x,y_2) \leq y_1 \approx_Y y_2 \tag{2.2}$$

hold for all $x \in X$ and $y, y_1, y_2 \in Y$. Moreover, if F is completely present then the above two inequalities simplify:

$$x \approx_X x \leq \bigvee_{y \in Y} F(x,y), \tag{2.3}$$

$$F(x,y_1) \wedge F(x,y_2) \leq y_1 \approx_Y y_2. \tag{2.4}$$

Note that (2.3) and (2.4) appear also in [33] in the definition of fuzzy functions.

2.2 Conditional ordered sets

We introduce conditional ordered sets as examples of conditional structures.

Let L be a Boolean algebra of conditions and $(V, \approx)$ an L-conditional universe. A *conditional order* on V is a binary conditional relation $\preceq$ on V compatible with $\approx$ from both sides such that for each total reality h and standard h-realization f of V it holds that the restricted h-realization $\preceq^{|h}$ of $\preceq$ is an ordinary order.

V together with $\preceq$ is called a *conditional ordered set* and denoted by $((V, \approx), \preceq)$. If V is an ordinary universe, we treat V as an ordinary ordered set. For elements $v, w \in V$, the condition $v \preceq w \in L$ is interpreted as the condition under which v is less than or equal to w.

We treat conditional ordered sets as conditional structures. Namely, as conditional structures of a one sorted language with binary relations symbols $\preceq$ and $\approx$.

Lemma 53. *A conditional structure $((V, \approx), \preceq)$ such that $\preceq$ is compatible with $\approx$ from both sides is a conditional ordered set if and only if formulas*

$$(\forall u)(u \preceq u), \tag{2.5}$$

$$(\forall u)(\forall v)(((u \preceq v) \wedge (v \preceq u)) \Rightarrow (u \approx v)), \tag{2.6}$$

$$(\forall u)(\forall v)(\forall w)(((u \preceq v) \wedge (v \preceq w)) \Rightarrow (u \preceq w)) \tag{2.7}$$

are valid in V.

Proof. Observe that formulas in the right hand side of the equivalence are axioms of ordered sets. Now, the equivalence is a consequence of the equivalence of the first and third statement of Theorem 23. □

Lemma 54. *Each faithful realization of a conditional ordered set is a conditional ordered set.*

Proof. Let h be a reality and f a faithful h-realization of V.

We first show that $\preceq^{|h}$ is compatible with $\approx^{|h}$ from both sides. Let $v_1, v_2 \in V_h$. As usual denote by $\preceq_{v_1}$ and $\preceq_{v_2}$ conditional sets in V given by $\preceq_{v_1}(v) = v_1 \preceq v$ and $\preceq_{v_2}(v) = v \preceq v_2$. By Lemma 22, f respects $\preceq$. By the compatibility of $\preceq$ with $\approx$ from both sides, conditional sets $\preceq_{v_1}$ and $\preceq_{v_2}$ are compatible with $\approx$ and clearly f partially respect $\preceq_{v_1}$ and $\preceq_{v_2}$. By Lemma 5, $v_1^h \preceq^{|h} v_2^h = h(v_1 \preceq v_2) = \preceq_{v_1}^{|h}(v_2^h) = \preceq_{v_2}^{|h}(v_1^h)$. By Lemma 13, the restricted realization $\preceq^{|h}$ of $\preceq$ is compatible with $\approx^{|h}$ from both sides.

Now, the claim follows from Lemma 53 and the implication from the first to the second statement of Theorem 23. $\square$

Theorem 24. *A conditional universe V with a binary conditional relation $\preceq$ compatible with $\approx$ from both sides is a conditional ordered set if and only if*

$$u \approx u \leq u \preceq u, \tag{2.8}$$

$$(u \approx u) \wedge (v \approx v) \wedge (u \preceq v) \wedge (v \preceq u) \leq u \approx v, \tag{2.9}$$

$$(u \approx u) \wedge (v \approx v) \wedge (w \approx w) \wedge (u \preceq v) \wedge (v \preceq w) \leq u \preceq w \tag{2.10}$$

hold for all $u, v, w \in V$.

Proof. Semantics rules presented in Sec. 2.1.2 and Lemma 53 yield the equivalence. $\square$

If $\approx$ is separated then we denote by $\leq$ the restriction of the ordinary binary relation ${}^1\preceq$ to ${}^1V$. We have that $\leq$ is an ordinary partial order on ${}^1V$. We sometimes use the symbols $\wedge$, $\bigwedge$ and $\vee$, $\bigvee$ for denoting infima and suprema w.r.t. $\leq$, respectively.

Theorem 25. *Let X be an L-conditional universe with the conditional equality $\approx$. Then $((L^X, \approx^+), \mathrm{S}_\approx)$ is a conditional ordered set.*

Proof. In Sec. 1.4.1 we proved that $\mathrm{S}_\approx$ is compatible with $\approx^+$ from both sides. By (1.43), inequalities (2.8) and (2.9) are satisfied. By (1.52), the inequality (2.10) holds. $\square$

The general definition of the isomorphism of conditional structures yields that L-conditional ordered sets V and W are conditionally isomorphic if there is a conditional isomorphism $F : V \to W$. That is a conditional bijection F between V and W such that for each total reality h and standard h-realizations f and g of V and W, respectively, the restricted h-realization $F^{|h}$ of F is an ordinary isomorphism of ordered sets, i.e. an isotone mapping whose inverse is also isotone mapping.

2.2.1 Conditional complete lattices

We define notions of cones and suprema and infima in conditional ordered sets. For any conditional set W in a conditional ordered set V and element $w \in V$ we set

$$\mathscr{L}W(w) = (w \approx w) \wedge \bigwedge_{v \in V} W(v) \to (w \preceq v), \tag{2.11}$$

$$\mathscr{U}W(w) = (w \approx w) \wedge \bigwedge_{v \in V} W(v) \to (v \preceq w). \tag{2.12}$$

Definitions of cones are inspired by [14, Section 4].

The conditions $\mathscr{L}W(w)$ and $\mathscr{U}W(w)$ are interpreted as conditions under which w is a *lower* and *upper bound* of W, respectively. The conditional sets $\mathscr{L}W$ and $\mathscr{U}W$ are called the *lower cone* and the *upper cone* of W, respectively.

We set

$$\operatorname{Sup} W = \mathscr{U}W \cap \mathscr{L}\mathscr{U}W, \qquad \operatorname{Inf} W = \mathscr{L}W \cap \mathscr{U}\mathscr{L}W, \tag{2.13}$$

obtaining conditional sets $\operatorname{Sup} W$ and $\operatorname{Inf} W$ called the *supremum* and *infimum* of W, respectively. $\operatorname{Sup} W(w)$ or $\operatorname{Inf} W(w)$ is interpreted as the condition under which w is the supremum or infimum of W, respectively.

Lemma 55. *For each conditional set W in a conditional ordered set V it holds that conditional sets $\mathscr{L}W, \mathscr{U}W, \operatorname{Sup} W$ and $\operatorname{Inf} W$ are completely present and compatible with $\approx$.*

Proof. It is obvious that the conditional sets are completely present.

We show that $\mathscr{U}W$ is compatible with $\approx$. We have $\mathscr{U}W(v) = (v \approx v) \wedge (W \triangleleft \preceq (v))$ for $v \in V$. By the compatibility of $\preceq$ from both sides with $\approx$ and Lemma 20, $W \triangleleft \preceq$ is compatible with $\approx$. By (1.1), for $v_1, v_2 \in V$ we have $\mathscr{U}W(v_1) \wedge (v_1 \approx v_2) = (v_1 \approx v_1) \wedge (W \triangleleft \preceq (v_1)) \wedge (v_1 \approx v_2) \leq (v_2 \approx v_2) \wedge (W \triangleleft \preceq (v_2)) = \mathscr{U}W(v_2)$. The compatibility of $\mathscr{L}W$ is proved similarly.

We prove only the compatibility of $\operatorname{Sup} W$ with $\approx$. The compatibility of $\operatorname{Inf} W$ with $\approx$ is proved similarly. For $v_1, v_2 \in V$ by compatibility of $\mathscr{L}W$ and $\mathscr{U}W$ with $\approx$ we have $\operatorname{Sup} W(v_1) \wedge (v_1 \approx v_2) = (\mathscr{U} W \cap \mathscr{L}\mathscr{U} W)(v_1) \wedge (v_1 \approx v_2) = \mathscr{U} W(v_1) \wedge \mathscr{L}\mathscr{U} W(v_1) \wedge (v_1 \approx v_2) \leq \mathscr{U} W(v_2) \wedge \mathscr{L}\mathscr{U} W(v_2) = (\mathscr{U} W \cap \mathscr{L}\mathscr{U} W)(v_2) = \operatorname{Sup} W(v_2)$. □

Theorem 26. *For conditional ordered set V, reality h, h-realization f of V and safe conditional set W in V w.r.t. f it holds*

$$(\mathscr{L} W)^{|h} = \mathscr{L} W^{|h}, \qquad (\mathscr{U} W)^{|h} = \mathscr{U} W^{|h}, \tag{2.14}$$

$$(\operatorname{Inf} W)^{|h} = \operatorname{Inf} W^{|h}, \qquad (\operatorname{Sup} W)^{|h} = \operatorname{Sup} W^{|h}. \tag{2.15}$$

Proof. $\mathscr{L} W$ is completely present and compatible with $\approx$ (Lemma 55). The h-realization f of V respects $\mathscr{L} W$ (Lemma 10). By Theorem 2 and Theorem 10, for each $w^h \in V^h$ we have $(\mathscr{L} W)^{|h}(w^h) = h(\mathscr{L} W(w)) = h((w \approx w) \wedge (\{w\} \preceq^{\leftarrow} W)) = (w^h \approx^{|h} w^h) \wedge (\{w\}^h \preceq^{|h\leftarrow} W^{|h}) = \mathscr{L} W^{|h}(w^h)$. The second equality can be proved analogously.

Now, since $\mathscr{L} W$ is safe w.r.t. f and by Theorem 2, $(\operatorname{Inf} W)^{|h}(w^h) = h(\operatorname{Inf} W(w)) = h(\mathscr{L} W(w) \wedge \mathscr{U}\mathscr{L} W(w)) = (\mathscr{L} W)^{|h}(w^h) \wedge (\mathscr{U}\mathscr{L} W)^{|h}(w^h) = \mathscr{L} W^{|h}(w^h) \wedge \mathscr{U}(\mathscr{L} W)^{|h}(w^h) = \mathscr{L} W^{|h}(w^h) \wedge \mathscr{U}\mathscr{L} W^{|h}(w^h) = \operatorname{Inf} W^{|h}(w^h)$ and similarly for Sup. □

Let W be a conditional set in V. The *supremal closure* of W, denoted $\mathrm{C}_{\mathrm{Sup}} W$, is the union of conditional sets $\operatorname{Sup} A$ for all conditional sets $A \subseteq W$. W is said to be *supremally dense* in V if $\mathrm{C}_{\mathrm{Sup}} W = V_{\mathrm{E}}$. Similarly, the *infimal closure* of W, denoted $\mathrm{C}_{\mathrm{Inf}} W$, is the union of conditional sets $\operatorname{Inf} A$ for all conditional sets $A \subseteq W$, and W is said to be *infimally dense* in V if $\mathrm{C}_{\mathrm{Inf}} W = V_{\mathrm{E}}$.

Lemma 56. *For each reality h, h-realization f of V and conditional set W in V such that the h-realization W^h of W is defined we have:*

$$(\mathrm{C}_{\mathrm{Sup}} W)^{|h} = \mathrm{C}_{\mathrm{Sup}} W^h, \qquad (\mathrm{C}_{\mathrm{Inf}} W)^{|h} = \mathrm{C}_{\mathrm{Inf}} W^h.$$

Completely present W is supremally dense in V iff W^h is supremally dense in V^h for each total reality h and standard h-realization f of V and similarly for infimal density.

Proof. First notice that since the h-realization W^h of W is defined, for each conditional set $A \subseteq W$ it holds that the h-realization A^h of A is also defined. By (1.18), Theorem 26 and Lemma 23, $(\mathrm{C}_{\mathrm{Sup}} W)^{|h} = (\bigcup_{A \subseteq W} \operatorname{Sup} A)^{|h} = \bigcup_{A \subseteq W} (\operatorname{Sup} A)^{|h} = \bigcup_{A \subseteq W} \operatorname{Sup} A^h = \bigcup_{A^h \subseteq W^h} \operatorname{Sup} A^h = \mathrm{C}_{\mathrm{Sup}} W^h$ and similarly for Inf.

Suppose f is standard and W completely present. If W is supremally dense then W^h being supremally dense follows from what we just have proved. Conversely, if $(\mathrm{C}_{\mathrm{Sup}} W)^h = \mathrm{C}_{\mathrm{Sup}} W^h = V^h$ for each standard h-realization f then $\mathrm{C}_{\mathrm{Sup}} W = V_E$ by Theorem 7. Similarly for infimal density. □

A conditional ordered set V is called a *conditional complete lattice* if for each total reality h and standard h-realization f of V, the restricted h-realization $\preceq^{|h}$ of $\preceq$ is a complete lattice order on V^h. Since any ordinary complete lattice is nonempty, the height of V_{E} is 1 for every conditional complete lattice V.

Theorem 27. *The following statements are equivalent for each conditional ordered set* $((V,\approx),\preceq)$*:*

1. *V is a conditional complete lattice.*
2. *For each completely present conditional set W in V it holds that the height of* $\operatorname{Inf} W$ *is* 1.
3. *For each completely present conditional set W in V it holds that the height of* $\operatorname{Sup} W$ *is* 1.

Proof. First suppose that 2. is not satisfied. We will prove that then neither 1. is true. By the assumption, there is a completely present conditional set W in V such that the height of $\operatorname{Inf} W$ is not 1, i.e. there exists a total reality h and standard h-realization f of V such that $(\operatorname{Inf} W)^h = \emptyset$. By (2.15), $(\operatorname{Inf} W)^h = \operatorname{Inf} W^h$. So, $\bigwedge W^h$ does not exist, and $\preceq^{|h}$ is not a complete lattice order.

Next, we prove that 2. implies 1. Let h be a total reality, f a standard h-realization of V and $W_h \subseteq V^h$. By Theorem 7, there is a completely present conditional set W in V such that $W^h = W_h$. By the definition of hight of conditional sets, properties of ordinary infimum and (2.15), the set $(\operatorname{Inf} W)^h$ consists of exactly one point, say w^h. But $(\operatorname{Inf} W)^h = \operatorname{Inf} W^h = \operatorname{Inf} W_h$ (Theorem 26) and so $w^h = \bigwedge W_h$. Thus, each subset of V^h has infimum which proves that V^h along with $\preceq^{|h}$ is a complete lattice.

The equivalence of 1. and 3. is dual. □

2.3 Conditional concept lattices

Concept lattices were introduced in [32]. A concept lattice consists of formal concepts extracted from a formal context. A formal context is a binary relation between two sets. Formal contexts with missing information can be represented by binary conditional relations. We extract concepts from a binary conditional relation. An appropriate subset of such concepts can be chosen as a representation of the concept lattice of the unknown context.

2.3.1 Concept lattices

Our basic reference for concept lattices is [7]. A (*formal*) *context* is a triple (X,Y,I) where X is a set of objects, Y a set of attributes and $I \subseteq X \times Y$ a binary relation between X and Y. For $(x,y) \in I$ we say that the object x has the attribute y.

For subsets $A \subseteq X$ and $B \subseteq Y$ we set

$$A^{\uparrow_I} = \{y \in Y \mid \text{for each } x \in A \text{ it holds } (x,y) \in I\}, \tag{2.16}$$

$$B^{\downarrow_I} = \{x \in X \mid \text{for each } y \in B \text{ it holds } (x,y) \in I\}. \tag{2.17}$$

Mappings ${}^{\uparrow_I} \colon 2^X \to 2^Y$ and ${}^{\downarrow_I} \colon 2^Y \to 2^X$ are called the *concept forming operators*. If $A^{\uparrow_I} = B$ and $B^{\downarrow_I} = A$ then the pair (A,B) is called a (*formal*) *concept* of (X,Y,I). Sets A and B are called the *extent* and the *intent* of the concept (A,B), respectively.

A partial order $\leq$ on the set $\mathscr{B}(X,Y,I)$ of all concepts of (X,Y,I) is defined by

$$(A_1,B_1) \leq (A_2,B_2) \quad \text{iff} \quad A_1 \subseteq A_2 \quad \text{iff } B_2 \subseteq B_1 \tag{2.18}$$

and called the *hierarchical order* of the concepts. $\mathscr{B}(X,Y,I)$ together with $\leq$ is called the *concept lattice* of (X,Y,I).

Theorem 28 (basic theorem of concept lattices [32]). *1. $\mathscr{B}(X,Y,I)$ is a complete lattice in which infima and suprema are be described as follows:*

$$\bigwedge_{j\in J}(A_j,B_j) = \left(\bigcap_{j\in J} A_j, \left(\bigcup_{j\in J} B_j\right)^{\downarrow\uparrow}\right), \tag{2.19}$$

$$\bigvee_{j\in J}(A_j,B_j) = \left(\left(\bigcup_{j\in J} A_j\right)^{\uparrow\downarrow}, \bigcap_{j\in J} B_j\right). \tag{2.20}$$

2. A complete lattice V is isomorphic to $\mathscr{B}(X,Y,I)$ iff there are mappings $\gamma \colon X \to V$, $\mu \colon Y \to V$ such that $\gamma(X)$ is supremally dense in V, $\mu(Y)$ is infimally dense in V and $\gamma(x) \leq \mu(y)$ iff $(x,y) \in I$. In particular, V is isomorphic to $\mathscr{B}(V,V,\leq)$.

2.3.2 Conditional contexts

Let L be a Boolean algebra of conditions. A *conditional* (*formal*) *context* is a triple (X,Y,I) where X and Y are conditional universes with associated conditional equalities $\approx_X$ and

$\approx_Y$, respectively, and I is a completely present binary conditional relation between X and Y compatible with $\approx_X$ and $\approx_Y$. Since I is completely present and compatible with $\approx_X$ and $\approx_Y$, I is also compatible with $\approx_X$ and $\approx_Y$ from both sides. Conditional contexts model ordinary formal contexts with missing information. The value $I(x,y)$ is interpreted as the condition under which the object x has the attribute y.

We regard a conditional context as a two sorted conditional structure. Namely, a conditional context (X,Y,I) is viewed as a conditional structure I for a language with sorts X and Y and a binary relation symbol I of arity XY. We apply the definition of a realization of a conditional structure on a conditional context and obtain the following. For a reality h and h-realizations f_X and f_Y of X and Y, respectively, the h-realization of (X,Y,I) is $(X^h,Y^h,I^{|h})$. Realizations f_X and f_Y partially respect I. If f_X and f_Y are standard then the h-realization of (X,Y,I) is a form of the unknown context modeled by (X,Y,I) in the total reality h.

2.3.3 Conditional concept forming operators

Let (X,Y,I) be an L-conditional context. Then for a conditional set A in X we define a conditional set $A^{\uparrow_I}$ in Y by

$$A^{\uparrow_I}(y) = (y \approx_Y y) \wedge \bigwedge_{x\in X} A(x) \to I(x,y) \tag{2.21}$$

for $y \in Y$ and, similarly, for a conditional set B in Y we define a conditional set $B^{\downarrow_I}$ in X by

$$B^{\downarrow_I}(x) = (x \approx_X x) \wedge \bigwedge_{y\in Y} B(y) \to I(x,y) \tag{2.22}$$

for $x \in X$. We denote ${}^{\uparrow_I}$ and ${}^{\downarrow_I}$ also simply by ${}^{\uparrow}$ and ${}^{\downarrow}$, respectively. Generally, $A \nsubseteq A^{\uparrow\downarrow}$ and $B \nsubseteq B^{\downarrow\uparrow}$. We have $A^{\uparrow}(y) = (y \approx_Y y) \wedge (A \triangleleft I(y))$ and $B^{\downarrow}(x) = (x \approx_X x) \wedge (B \triangleleft I^{-1}(x))$.

Lemma 57. *Conditional sets $A^{\uparrow}$ and $B^{\downarrow}$ are completely present and compatible with $\approx_Y$ and $\approx_X$, respectively.*

Proof. It is clear from the definition that $A^{\uparrow}$ and $B^{\downarrow}$ are completely present. We show that $A^{\uparrow}$ is compatible with $\approx_Y$. Similarly can be shown that $B^{\downarrow}$ is compatible with $\approx_X$.

By Lemma 20, $A \triangleleft I$ is compatible with $\approx_Y$. Now, by (1.1), $A^{\uparrow}(y_1) \wedge (y_1 \approx_Y y_2) = (y_1 \approx_Y y_1) \wedge (A \triangleleft I)(y_1) \wedge (y_1 \approx_Y y_2) \leq (y_2 \approx_Y y_2) \wedge (A \triangleleft I)(y_2) = A^{\uparrow}(y_2)$.

□

Let A and B be conditional sets in X and Y, respectively, h a reality, f_X and f_Y h-realizations of X and Y, respectively.

Lemma 58. *If A and B are safe w.r.t. f_X and f_Y, respectively, then $A^{|h\uparrow} = A^{\uparrow|h}$ and $B^{|h\downarrow} = B^{\downarrow|h}$.*

Proof. We show the first equality. The second can be proven similarly.

First, we show that f_Y respects $A \triangleleft I$. Let $y_1, y_2 \in Y_h$ such that $y_1^h = y_2^h$. Since f_X and f_Y respect I, it holds for each $x \in X_h$ that $h(I(x,y_1)) = h(I(x,y_2))$. Now, by the safeness of $A \triangleleft I$ w.r.t. f_Y, $h(A \triangleleft I(y_1)) = h(\bigwedge_{x\in X_h} A(x) \to I(x,y_1)) = h(\bigwedge_{x\in X_h} A(x) \to I(x,y_2)) = h(A \triangleleft I(y_2))$.

We can use Theorem 2 and obtain $(A \triangleleft I)^{|h}(y^h) = h(A \triangleleft I(y))$ for all $y \in Y_h$. Thus, by know facts, for $y^h \in Y^h$ we have $A^{|h\uparrow}(y^h) = (y^h \approx_Y^{|h} y^h) \wedge (A^{|h} \triangleleft I^{|h})(y^h) = (y^h \approx_Y^{|h} y^h) \wedge (A \triangleleft I)^{|h}(y^h) = h((y \approx_Y y) \wedge (A \triangleleft I)(y)) = h(A^{\uparrow}(y)) = A^{\uparrow|h}(y^h)$. □

Lemma 59. *Suppose h is total and h-realizations f_X and f_Y standard. If A^h is not defined then $A^{\uparrow h} = \emptyset$ and similarly if B^h is not defined then $B^{\downarrow h} = \emptyset$.*

Proof. We prove only the first implication. The second can be proven analogously. Since I is completely present, it holds for $y^h \in Y^h$ that $A^{\uparrow h}(y^h) = h(A^{\uparrow}(y)) = h((y \approx_Y y) \wedge \bigwedge_{x\in X} A(x) \to I(x,y)) \leq h(\bigwedge_{x\in X} A(x) \to (x \approx_X x)) = h(\mathrm{E}A) = 0$. □

Lemma 60. *If A and B are completely present then $A^{\uparrow\downarrow\uparrow} = A^{\uparrow}$ and $B^{\downarrow\uparrow\downarrow} = B^{\downarrow}$.*

Proof. Since L is complete and atomic, to prove the first equality it suffices to show that $h(A^{\uparrow\downarrow\uparrow}(y)) = h(A^{\uparrow}(y))$ for each total reality h and $y \in Y$. Let h be a total reality and f_X and f_Y standard h-realizations of X and Y, respectively. Then by Lemma 58 and properties of ordinary concept forming operators, it holds that $h(A^{\uparrow\downarrow\uparrow}(y)) = A^{\uparrow\downarrow\uparrow h}(y^h) = A^{h\uparrow\downarrow\uparrow}(y^h) = A^{h\uparrow}(y^h) = A^{\uparrow h}(y^h) = h(A^{\uparrow}(y))$. The second equality is proved similarly. □

Lemma 61. *For each conditional sets A_1, A_2 in X and B_1, B_2 in Y it holds*

$$\mathrm{S}_{\approx}(A_1,A_2) \leq \mathrm{S}_{\approx}(A_2^{\uparrow},A_1^{\uparrow}), \tag{2.23}$$

$$\mathrm{S}_{\approx}(B_1,B_2) \leq \mathrm{S}_{\approx}(B_2^{\downarrow},B_1^{\downarrow}). \tag{2.24}$$

Proof. We show that $h(\mathrm{S}_{\approx}(A_1,A_2)) \leq h(\mathrm{S}_{\approx}(A_2^{\uparrow},A_1^{\uparrow}))$ for each total reality h. Let h be a total reality and f_X and f_Y standard h-realizations of X and Y, respectively. Then the interesting case is if $h(\mathrm{S}_{\approx}(A_1,A_2)) = 1$. By (1.56), the h-realization A_1^h of A_1 is defined. By $A_1^{\uparrow}$ and $A_2^{\uparrow}$ are completely present, the h-realizations $A_1^{\uparrow h}$ and $A_2^{\uparrow h}$ of $A_1^{\uparrow}$ and $A_2^{\uparrow}$, respectively, are defined. Suppose first that the h-realizations A_1^h and A_2^h of A_1 and A_2, respectively, are defined. Then by Theorem 15 and properties of ordinary concept forming operators, it holds that $h(\mathrm{S}_{\approx}(A_1,A_2)) = \mathrm{S}_{\approx^h}(A_1^h,A_2^h) \leq \mathrm{S}_{\approx^h}(A_2^{h\uparrow},A_1^{h\uparrow}) = \mathrm{S}_{\approx^h}(A_2^{\uparrow h},A_1^{\uparrow h}) = h(\mathrm{S}_{\approx}(A_2^{\uparrow},A_1^{\uparrow}))$.

Now, suppose that the h-realization A_2^h of A_2 is not defined. Then $A_2^{\uparrow h} = \emptyset$ and we have $h(\mathrm{S}_{\approx}(A_2^{\uparrow},A_1^{\uparrow})) = \mathrm{S}_{\approx^h}(A_2^{\uparrow h},A_1^{\uparrow h}) = 1$.

Since L is complete and atomic, the first inequality holds. The second is proved similarly. □

Theorem 29. *Let (X,Y,I) be a conditional context. Then the mappings $A \mapsto A^{\uparrow\downarrow}$ and $B \mapsto B^{\downarrow\uparrow}$ are conditional closure operators on X and Y, respectively. Moreover, they satisfy:*

$$\mathrm{S}_{\approx}(A_1,A_2) \leq \mathrm{S}_{\approx}(A_1^{\uparrow\downarrow},A_2^{\uparrow\downarrow}), \tag{2.25}$$

$$\mathrm{S}_{\approx}(B_1,B_2) \leq \mathrm{S}_{\approx}(B_1^{\downarrow\uparrow},B_2^{\downarrow\uparrow}), \tag{2.26}$$

$$A^{\uparrow\downarrow\uparrow\downarrow} = A^{\uparrow\downarrow}, \tag{2.27}$$

$$B^{\downarrow\uparrow\downarrow\uparrow} = B^{\downarrow\uparrow} \tag{2.28}$$

for each conditional sets A,A_1,A_2 in X and B,B_1,B_2 in Y.

Proof. We prove only that $A \mapsto A^{\uparrow\downarrow}$ is a conditional closure operator on X. The fact that $B \mapsto B^{\downarrow\uparrow}$ is a conditional closure operator on Y can be shown similarly. It can be directly checked that the mapping $A \mapsto A^{\uparrow\downarrow}$, denoted for a moment by r, is a presence preserving conditional mapping and $r^{|h}(A^h) = f(A)^h$ holds for each total reality h, standard h-realizations f_X and f_Y of X and Y, respectively, and conditional set A in X such that the h-realization A^h of A is defined. Therefore, we can use Theorem 18.

Let A_1 and A_2 be conditional sets in X. Then by Lemma 61 we have $\mathrm{S}_{\approx}(A_1,A_2) \leq \mathrm{S}_{\approx}(A_2^{\uparrow},A_1^{\uparrow}) \leq \mathrm{S}_{\approx}(A_1^{\uparrow\downarrow},A_2^{\uparrow\downarrow})$. We proved monotony of the mapping $A \mapsto A^{\uparrow\downarrow}$, (2.25) and (2.25).

Now, we show extensivity. Let A be a conditional set in X. We need to show that $A \approx^+ A \leq \mathrm{S}_{\approx}(A,A^{\uparrow\downarrow})$. The inequality is equivalent to $A(x) \wedge (A \approx^+ A) \leq \mathrm{C}_{\approx}A^{\uparrow\downarrow}(x)$ for each $x \in X$. Which holds if $A(x) \wedge (A \approx^+ A) \leq (x \approx x) \wedge A^{\uparrow\downarrow}(x)$ for each $x \in X$. By (1.48), $A(x) \wedge (A \approx^+ A) \leq x \approx x$. In order to show extensivity, we need to prove that $A(x) \wedge (A \approx^+ A) \leq A^{\uparrow\downarrow}(x)$.

We will show that $h(A(x) \wedge (A \approx^+ A)) \leq h(A^{\uparrow\downarrow}(x))$ holds for each total reality h and $x \in X$. Let h be a total reality, f_X and f_Y standard h-realizations of X and Y, respectively. The only interesting case here is when $h(A(x) \wedge (A \approx^+ A)) = 1$. Since $h(A \approx^+ A) = 1$, the

h-realization A^h of A and the h-realization x^h of x are defined. By Lemma 58 and properties of ordinary concept forming operators, we have $h(A^{\uparrow\downarrow}(x)) = A^{\uparrow\downarrow h}(x^h) = A^{h\uparrow\downarrow}(x^h) \geq A^h(x^h) \geq h(A(x)) = 1$. By L is complete and atomic, $A(x) \wedge (A \approx^+ A) \leq A^{\uparrow\downarrow}(x)$ and extensivity is proved.

Idempotency: (2.27) and (2.28) follow directly from Lemma 60. $\square$

2.3.4 Conditional concepts

For conditional sets A and B in X and Y, respectively, we call the pair (A,B) a *conditional (formal) concept* of (X,Y,I) if $A^{\uparrow_I} = B$ and $B^{\downarrow_I} = A$. A is called the *extent* and B the *intent* of the conditional concept (A,B). By Lemma 57, A and B are completely present and compatible with $\approx_X$ and $\approx_Y$, respectively.

The set of all conditional concepts of (X,Y,I) is denoted by $\mathscr{B}(X,Y,I)$.

For a reality h, h-realizations f_X and f_Y of X and Y, respectively, the pair $(A^{|h}, B^{|h})$ is called the *h-realization of* (A,B). From Lemma 58 it follows that the h-realization $(A^{|h}, B^{|h})$ of (A,B) is a conditional concept of $(X^h, Y^h, I^{|h})$.

Lemma 62. *For conditional sets A and B in X and Y, respectively, we have (A,B) is a conditional concept of (X,Y,I) if and only if A and B are compatible with $\approx_X$ and $\approx_Y$, respectively, and for each total reality h and standard h-realizations f_X and f_Y of X and Y, respectively, the h-realization (A^h, B^h) of (A,B) is defined and is an ordinary concept of (X^h, Y^h, I^h).*

Proof. Above we proved the implication from left to right. So, suppose the right hand side holds. From the fact that for each total reality h and standard h-realizations f_X and f_Y of X and Y, respectively, it holds that the h-realization (A^h, B^h) of (A,B) is defined follows that both A and B are completely present.

Let h be a total reality, f_X and f_Y standard h-realizations of X and Y, respectively, from the claim. For $y \in Y_h$ we have $h(A^{\uparrow}(y)) = A^{\uparrow h}(y^h) = A^{h\uparrow}(y^h) = B^h(y^h) = h(B(y))$. If $y \in Y \setminus Y_h$ then by complete presence of $A^{\uparrow}$ and B, $h(A^{\uparrow}(y)) = 0 = h(B(y))$.

We showed that $h(A^{\uparrow}(y)) = h(B(y))$ for each total reality h and $y \in Y$. Therefore, since L is complete and atomic, $A^{\uparrow} = B$. Similarly can be shown that $B^{\downarrow} = A$. We proved that (A,B) is a conditional concept of (X,Y,I).

$\square$

2.3.5 Conditional concept lattices

We fix for each total reality h standard h-realizations f_h and g_h of X and Y, respectively. For a set V of conditional concepts of (X,Y,I) and total reality h we denote by h^V the mapping $V \to \mathscr{B}(X^h,Y^h,I^h)$ defined by $h^V(A,B) = (A^h,B^h)$.

Let V be a set of conditional concepts of (X,Y,I), $\approx_V$ a conditional equality and $\preceq_V$ a conditional order on $(V,\approx_V)$. The conditional ordered set $((V,\approx_V),\preceq_V)$ is called a *conditional concept lattice* of (X,Y,I) if for each total reality h the mapping h^V is an h-realization of V.

As we can see from the definition, a conditional concept lattice of a conditional context is not defined uniquely. A challenge is to construct conditional concept lattices which meet additional user requirements. Each conditional concept lattice represents the unknown concept lattice of the context with missing information.

Lemma 63. *A conditional ordered set V consisting of conditional concepts of (X,Y,I) is a conditional concept lattice of (X,Y,I) if and only if for each total reality h the mapping h^V is surjective and the conditional relations $\approx_V$ and $\preceq_V$ satisfy*

$$(A_1,B_1) \approx_V (A_2,B_2) = (A_1 \triangleleft A_2) \wedge (A_2 \triangleleft A_1) = (B_1 \triangleleft B_2) \wedge (B_2 \triangleleft B_1), \tag{2.29}$$

$$(A_1,B_1) \preceq_V (A_2,B_2) = A_1 \triangleleft A_2 = B_2 \triangleleft B_1. \tag{2.30}$$

Proof. By (1.47), $(A_1 \triangleleft A_2) \wedge (A_2 \triangleleft A_1) = A_1 \approx_X^+ A_2$ and $(B_1 \triangleleft B_2) \wedge (B_2 \triangleleft B_1) = B_1 \approx_Y^+ B_2$. We use Lemma 3 for $X = V$ and $f_h = h^V$. As we can easily check, $h(A_1 \approx_X^+ A_2) = A_1^h \approx_X^{h+} A_2^h = 1$ iff $A_1^h = A_2^h$ iff $h^V(A_1,B_1) = (A_1^h,B_1^h) = (A_2^h,B_2^h) = h^V(A_2,B_2)$ iff $B_1^h = B_2^h$ iff $1 = B_1^h \approx_Y^{h+} B_2^h = h(B_1 \approx_Y^+ B_2)$. Therefore, $A_1 \approx_X^+ A_2 = B_1 \approx_Y^+ B_2$, $h^V(A_1,B_1) = h^V(A_2,B_2)$ iff $h((A_1,B_1) \approx_V (A_2,B_2)) = 1$ and the L-relation $\approx_V$ is the unique conditional equality on V such that the mappings h^V (h is a total reality) are h-realizations.

Let (A_1,B_1) and (A_2,B_2) be two conditional concepts in V. By (1.46), $A_1 \triangleleft A_2 = \mathrm{S}_{\approx_X}(A_1,A_2)$ and $B_2 \triangleleft B_1 = \mathrm{S}_{\approx_Y}(B_2,B_1)$. We have for each total reality h, $h(\mathrm{S}_{\approx_X}(A_1,A_2)) = \mathrm{S}_{\approx_X^h}(A_1^h,A_2^h) = 1$ iff $A_1^h \subseteq A_2^h$ iff $B_2^h \subseteq B_1^h$ iff $1 = \mathrm{S}_{\approx_Y^h}(B_2^h,B_1^h) = h(\mathrm{S}_{\approx_Y}(B_2,B_1))$ (Theorem 15).

Since $\approx_V$ is reflexive, $\preceq_V$ is completely present. It can be easily checked that $\preceq_V$ is compatible with $\approx_V$ and for each total reality h, the h-realization $\preceq_V^h$ of $\preceq_V$ is equal to the standard partial order (2.18) on $\mathscr{B}(X^h,Y^h,I^h)$. Thus, by Lemma 53, $((V,\approx_V),\preceq_V)$ is a conditional concept lattice of (X,Y,I). Uniqueness of $\preceq_V$ follows by Theorem 7.

□

Denote by $\mathscr{B}_c(X,Y,I)$ the set of all conditional concepts $((B_0 \cap Y_{\mathrm{E}})^{\downarrow}, (B_0 \cap Y_{\mathrm{E}})^{\downarrow\uparrow})$ where B_0 is an ordinary set in Y. Note that concepts in $\mathscr{B}_c(X,Y,I)$ are an analogy of crisply generated concepts [5].

Lemma 64. *$\mathscr{B}_c(X,Y,I)$ is a conditional concept lattice of (X,Y,I).*

Proof. Let h be a total reality, f_X and f_Y standard h-realizations of X and Y, respectively, and (A_h, B_h) a concept of (X^h, Y^h, I^h). Denote by B_0 an ordinary set in Y defined by $y \in B_0$ iff y^h is defined and $y^h \in B_h$. We treat B_0 as a conditional set as usual and show that $(B_0 \cap Y_{\mathrm{E}})^{\downarrow h} = A_h$. It can be easily checked that $(B_0 \cap Y_{\mathrm{E}})^h = B_0^h = B_h$. Now by Lemma 58, $(B_0 \cap Y_{\mathrm{E}})^{\downarrow h} = (B_0 \cap Y_{\mathrm{E}})^{h\downarrow} = B_h^{\downarrow} = A_h$. We showed that (A_h, B_h) is the h-realization of a conditional concept $((B_0 \cap Y_{\mathrm{E}})^{\downarrow}, (B_0 \cap Y_{\mathrm{E}})^{\downarrow\uparrow})$ in $\mathscr{B}_c(X,Y,I)$.

The first part of the proof shows the surjectivity of the mapping $h^{\mathscr{B}_c(X,Y,I)} : \mathscr{B}_c(X,Y,I) \to \mathscr{B}(X^h, Y^h, I^h)$ for each total reality h and by definition $\mathscr{B}_c(X,Y,I)$ is a conditional concept lattice of (X,Y,I). □

As a consequence of the preceding lemma, we obtain that $\mathscr{B}(X,Y,I)$ is a conditional concept lattice of (X,Y,I).

2.3.6 Basic theorem of conditional concept lattices

For a conditional set M of conditional concepts of (X,Y,I), we denote by M_X and M_Y the corresponding conditional set of extents and intents, respectively. These conditional sets satisfy for any $(A,B) \in \mathscr{B}(X,Y,I)$,

$$M_X(A) = M_Y(B) = M(A,B).$$

Theorem 30 (basic theorem of conditional concept lattices)**.** *1. Any conditional concept lattice V of (X,Y,I) is a conditional complete lattice. Suprema and infima in V are given by*

$$\operatorname{Sup} M(A,B) = B \approx_Y^+ \bigcap M_Y, \tag{2.31}$$

$$\operatorname{Inf} M(A,B) = A \approx_X^+ \bigcap M_X, \tag{2.32}$$

for any conditional set M in V.

2. A conditional complete lattice V is conditionally isomorphic to a conditional concept lattice of (X,Y,I) if and only if there exist presence preserving conditional mappings

$\gamma\colon X \to V$ and $\mu : Y \to V$ such that $\gamma(X_{\mathrm{E}})$ is supremally dense in V, $\mu(Y_{\mathrm{E}})$ is infimally dense in V, and for each $x \in X$ and $y \in Y$ it holds

$$I(x,y) = (x \approx_X x) \wedge (y \approx_Y y) \wedge (\gamma(x) \preceq^+ \mu(y)). \tag{2.33}$$

Proof. For each total reality h we fix standard h-realizations f_h^X, f_h^Y and f_h^V of X,Y and V, respectively.

1. By definition, V^h equals the complete lattice $\mathscr{B}(X^h,Y^h,I^h)$ for each total reality h. V is therefore a conditional complete lattice. To prove (2.31), it suffices to show that $h(\operatorname{Sup} M(A,B)) = h(B \approx_Y^+ \bigcap M_Y)$ holds for each total reality h. By the first part of the basic theorem of ordinary concept lattices, Lemma 40 and Theorem 26, $h(\operatorname{Sup} M(A,B)) = (\operatorname{Sup} M)^h(A^h,B^h) = \operatorname{Sup} M^h(A^h,B^h) = 1$ iff $(A^h,B^h) = \bigvee M^h$ iff $B^h = \bigcap M_Y^h$ iff $1 = B^h {\approx_Y^{h+}} \bigcap M_Y^h = B^h \approx_Y^{h+} (\bigcap M_Y)^h = h(B \approx_Y^+ \bigcap M_Y)$. (2.32) can be proven analogously.

2. Let W be a conditional concept lattice of (X,Y,I). By definition, W is conditionally isomorphic to V iff for each total reality h the ordered sets W^h and V^h are isomorphic and $W^h = \mathscr{B}(X^h,Y^h,I^h)$. Therefore, the left-hand side of the equivalence we need to prove means that V^h is isomorphic to $\mathscr{B}(X^h,Y^h,I^h)$ for each total reality h. By 2. of Theorem 28, this is equivalent to the existence of ordinary mappings $\gamma_h\colon X^h \to V^h$ and $\mu_h\colon Y^h \to V^h$ such that $\gamma_h(X^h)$ is supremally dense in V^h, $\mu_h(Y^h)$ is infimally dense in V^h, and $\gamma_h(x^h) \le \mu_h(y^h)$ iff $(x^h,y^h) \in I^h$.

Let $\gamma\colon X \to V$ and $\mu : Y \to V$ be presence preserving conditional mappings. By (1.41), it holds for each total reality h that $\gamma(X_{\mathrm{E}})^h = \gamma^{|h}((X_{\mathrm{E}})^h)$ and $\mu(Y_{\mathrm{E}})^h = \mu^{|h}((Y_{\mathrm{E}})^h)$. Therefore (Lemma 56), $\gamma(X_{\mathrm{E}})$ is supremally dense in V iff $\gamma^{|h}((X_{\mathrm{E}})^h)$ is supremally dense in V^h for each total reality h. Similarly, $\mu(Y_{\mathrm{E}})$ is infimally dense in V iff $\mu^{|h}((Y_{\mathrm{E}})^h)$ is infimally dense in V^h for each total reality h.

In the next step, we show that for total reality h it holds $h(\gamma(x) \preceq^+ \mu(y)) = 1$ iff $\gamma^{|h}(x^h) \le \mu^{|h}(y^h)$ for each $x \in X_h$ and $y \in Y_h$. By Theorem 10, $h(\gamma(x) \preceq^+ \mu(y)) = \gamma(x)^h \preceq^{|h+} \mu(y)^h$. Since h-realizations f_{Xh} and f_{Yh} are standard, $\gamma(x)^h$ and $\mu(y)^h$ are singletons with elements $\gamma^{|h}(x^h)$ and $\mu^{|h}(y^h)$, respectively. The ordinary relation $\preceq^{|h}$ coincides with the partial order $\le$ on V^h. Therefore, $\gamma(x)^h \preceq^{|h+} \mu(y)^h = 1$ iff $\{\gamma^{|h}(x^h)\} \le^+ \{\mu^{|h}(y^h)\}$, which is the same as $\gamma^{|h}(x^h) \le \mu^{|h}(y^h)$.

If $x \in X \setminus X_h$ or $y \in Y \setminus Y_h$ then the complete presence of I implies $0 = h(I(x,y)) = h((x \approx_X x) \wedge (y \approx_Y y) \wedge (\gamma(x) \preceq^+ \mu(y)))$.

Now, we are ready to prove the assertion. Suppose first that V is conditionally isomorphic to a conditional concept lattice of (X,Y,I) and let γ_h and μ_h be the mappings whose existence we have proved above. By Theorem 7, there exist presence preserving conditional mappings $\gamma\colon X \to V$ and $\mu : Y \to V$ such that $\gamma^h = \gamma_h$ and $\mu^h = \mu_h$ for each total reality h. As we have proved, from known properties of γ_h and μ_h we can now derive the right-hand side of the equivalence. The converse implication follows directly from the above considerations by putting $\gamma_h = \gamma^{|h}$ and $\mu_h = \mu^{|h}$. □

2.3.7 Clarified and reduced conditional contexts

We begin with ordinary clarified and reduced contexts as defined in [7]. An ordinary context (X,Y,I) is *clarified* if for any $x_1, x_2 \in X$ from $\{x_1\}^{\uparrow} = \{x_2\}^{\uparrow}$ it follows that $x_1 = x_2$ and, correspondingly, for any $y_1, y_2 \in Y$ from $\{y_1\}^{\downarrow} = \{y_2\}^{\downarrow}$ if follows that $y_1 = y_2$. A clarified context is *reduced* if all objects and attributes are not reducible. Recall that an object x is called *reducible* if there are objects $x_i \neq x$ $(i \in I)$ such that $\{x\}^{\uparrow} = \bigcap_{i \in I}\{x_i\}^{\uparrow}$ and, dually, an attribute y is called *reducible* if there are attributes $y_i \neq y$ $(i \in I)$ such that $\{y\}^{\downarrow} = \bigcap_{i \in I}\{y_i\}^{\downarrow}$ where $y_i \in Y$ and $y_i \neq y$.

Each finite context can be turned to a clarified and reduced context with the concept lattice isomorphic to the concept lattice of the original context. Note that the finiteness requirement can be weakened, for details see [7]. In order to obtain the clarified and reduced context, we first merge objects with the same extent and attributes with the same intent. Then we delete all reducible objects and attributes. The first step is called the *clarification* and second *reduction*.

For a finite context reducibility of objects and attributes can be tested by the following relations, called *arrow relations*, between X and Y:

$$x \swarrow_I y \quad \text{if} \begin{cases} (x,y) \notin I, \text{ and} \\ \{x\}^{\uparrow} \subseteq \{\bar{x}\}^{\uparrow} \text{ and } \{x\}^{\uparrow} \neq \{\bar{x}\}^{\uparrow} \text{ imply } (\bar{x},y) \in I, \end{cases} \tag{2.34}$$

$$x \nearrow_I y \quad \text{if} \begin{cases} (x,y) \notin I, \text{ and} \\ \{y\}^{\downarrow} \subseteq \{\bar{y}\}^{\downarrow} \text{ and } \{y\}^{\downarrow} \neq \{\bar{y}\}^{\downarrow} \text{ imply } (x,\bar{y}) \in I, \end{cases} \tag{2.35}$$

$$x \swarrow\!\!\!\nearrow_I y \quad \text{if } x \swarrow_I y \text{ and } x \nearrow_I y, \tag{2.36}$$

where $x \in X$ and $y \in Y$.

We usually write $\swarrow$, $\nearrow$ and $\swarrow\!\!\!\nearrow$ instead of $\swarrow_I$, $\nearrow_I$ and $\swarrow\!\!\!\nearrow_I$, respectively. An object x is reducible iff there is $y \in Y$ with $x \swarrow\!\!\!\nearrow y$. An attribute y is reducible iff there is $x \in X$ with $x \swarrow\!\!\!\nearrow y$.

We can compute the clarification and reduction of (X,Y,I) in one step as a standard realization of a suitable 2-conditional context. The details will follow. Consider sets $X' = X$ and $Y' = Y$ equipped with 2-conditional equalities (partial equivalences) R_X and R_Y, respectively, given by $(x_1,x_2) \in \mathrm{R}_X$ iff $\{x_1\}^\uparrow = \{x_2\}^\uparrow$ and there are $y_1, y_2 \in Y$ such that $x_1 \swarrow\!\!\!\nearrow y_1$ and $x_2 \swarrow\!\!\!\nearrow y_2$; and, similarly, $(y_1,y_2) \in \mathrm{R}_Y$ iff $\{y_1\}^\downarrow = \{y_2\}^\downarrow$ and there are $x_1, x_2 \in X$ such that $x_1 \swarrow\!\!\!\nearrow y_1$ and $x_2 \swarrow\!\!\!\nearrow y_2$. Further let I' be a 2-conditional (ordinary) binary relation between X' and Y' defined by $(x,y) \in I'$ iff $(x,y) \in I$, $(x,x) \in \mathrm{R}_X$ and $(y,y) \in \mathrm{R}_Y$.

Lemma 65. *Let (X,Y,I) be a finite ordinary context and h the identity on 2. Then (X',Y',I') is a 2-conditional context and for any standard h-realizations f and g of X' and Y', respectively, it holds that the h-realization (X'^h, Y'^h, I'^h) is isomorphic to the clarification and reduction of (X,Y,I).*

Proof. Clearly, I' is completely present. The compatibility of I' follows from the fact that if $(x_1,y) \in I'$ and $(x_1,x_2) \in \mathrm{R}_X$ then $\{x_1\}^{\uparrow_I}(y) = \{x_2\}^{\uparrow_I}(y)$ and thus $(x_2,y) \in I$, and, by (1.1), $(x_2,x_2) \in \mathrm{R}_X$, proving that $(x_2,y) \in I'$. Similarly can be shown that $(x,y_1) \in I'$ and $(y_1,y_2) \in \mathrm{R}_Y$ imply $(x,y_2) \in I'$. Therefore, (X',Y',I') is a 2-conditional context. The rest follows from the above considerations. $\square$

We generalize the clarification and reduction to the conditional case. We call a conditional context (X,Y,I) *clarified* or *reduced* if for each total realty h, standard h-realizations f and g of X and Y, respectively, it holds that the h-realization (X^h, Y^h, I^h) of (X,Y,I) is clarified or reduced, respectively. We fix a reality h and faithful h-realizations f and g of X and Y, respectively. When no misunderstanding could be caused, we denote $\approx_X$ and $\approx_Y$ by $\approx$. The following conditional relations $\nearrow_I$, $\swarrow_I$ and $\swarrow\!\!\!\nearrow_I$ between X and Y are generalizations of arrow relations to the conditional case:

$$x \swarrow_I y = I(x,y)' \wedge \bigwedge_{\bar{x}\in X} ((\bar{x} \approx \bar{x}) \wedge \mathrm{S}_\approx(\{x\}^\uparrow, \{\bar{x}\}^\uparrow) \wedge (\{x\}^\uparrow \approx^+ \{\bar{x}\}^\uparrow)') \to I(\bar{x},y), \tag{2.37}$$

$$x \nearrow_I y = I(x,y)' \wedge \bigwedge_{\bar{y}\in Y} ((\bar{y} \approx \bar{y}) \wedge \mathrm{S}_\approx(\{y\}^\downarrow, \{\bar{y}\}^\downarrow) \wedge (\{y\}^\downarrow \approx^+ \{\bar{y}\}^\downarrow)') \to I(x,\bar{y}), \tag{2.38}$$

$$x \swarrow\!\!\!\nearrow_I y = (x \swarrow_I y) \wedge (x \nearrow_I y). \tag{2.39}$$

Again we also write $\swarrow$, $\nearrow$ and $\swarrow\!\!\!\nearrow$ instead of $\swarrow_I$, $\nearrow_I$ and $\swarrow\!\!\!\nearrow_I$, respectively. When (X,Y,I) is an ordinary context then $\swarrow_I$, $\nearrow_I$ *and* $\swarrow\!\!\!\nearrow_I$ coincides with (2.34), (2.35) and (2.36), respectively.

Lemma 66. *For each $x \in X_h$ and $y \in Y_h$ we have*

$$x^h \swarrow_{I^h} y^h = h(x \swarrow_I y), \tag{2.40}$$
$$x^h \nearrow_{I^h} y^h = h(x \nearrow_I y), \tag{2.41}$$
$$x^h \swarrow\!\!\!\nearrow_{I^h} y^h = h(x \swarrow\!\!\!\nearrow_I y). \tag{2.42}$$

Proof. We prove only (2.40). The proof of (2.41) is similar and (2.42) follows from (2.41), (2.40) and (2.39). The equality (2.40) can be directly checked using the fact that h-realizations f and g are faithful, Lemmas 5 and 58 and Theorem 15.

□

For an L-conditional context (X,Y,I), we define binary L-conditional relations R_X and R_Y on X and Y, respectively, by

$$\begin{aligned}\mathrm{R}_X(x_1,x_2) = {} & (x_1 \approx x_1) \wedge (x_2 \approx x_2) \wedge (\{x_1\}^{\uparrow} \approx^{+} \{x_2\}^{\uparrow}) \\ & \wedge \bigvee_{y_1,y_2 \in Y} (y_1 \approx y_1) \wedge (y_2 \approx y_2) \wedge (x_1 \swarrow\!\!\!\nearrow y_1) \wedge (x_2 \swarrow\!\!\!\nearrow y_2),\end{aligned} \tag{2.43}$$

$$\begin{aligned}\mathrm{R}_Y(y_1,y_2) = {} & (y_1 \approx y_1) \wedge (y_2 \approx y_2) \wedge (\{y_1\}^{\downarrow} \approx^{+} \{y_2\}^{\downarrow}) \\ & \wedge \bigvee_{x_1,x_2 \in X} (x_1 \approx x_1) \wedge (x_2 \approx x_2) \wedge (x_1 \swarrow\!\!\!\nearrow y_1) \wedge (x_2 \swarrow\!\!\!\nearrow y_2).\end{aligned} \tag{2.44}$$

If (X,Y,I) is an ordinary context then R_X and R_Y coincide with the earlier defined ordinary relations denoted by the same symbols.

Lemma 67. *For each $x_1,x_2 \in X_h$ and $y_1,y_2 \in Y_h$ it holds*

$$\mathrm{R}_{X^h}(x_1^h,x_2^h) = h(\mathrm{R}_X(x_1,x_2)), \tag{2.45}$$
$$\mathrm{R}_{Y^h}(y_1^h,y_2^h) = h(\mathrm{R}_Y(y_1,y_2)). \tag{2.46}$$

Proof. The proof goes similarly as the proof of Lemma 66. We show only (2.45) since the proof of (2.46) is analogous. The equality (2.45) follows from Lemmas 66 and (1.5), Theorem 15 and the fact that h-realizations f and g are faithful. □

Theorem 31. *Let (X,Y,I) be a conditional context, $X' = X$ and $Y' = Y$ be conditional universes with conditional equalities R_X and R_Y, respectively, and I' a conditional relation between X' and Y' given by $I'(x,y) = I(x,y) \wedge \mathrm{R}_X(x,x) \wedge \mathrm{R}_Y(y,y)$. Then (X',Y',I') is a clarified and reduced conditional context and for each total reality h and standard h-realizations f,f',g,g' of X,X',Y,Y', respectively, it holds that the h-realization (X'^h,Y'^h,I'^h)*

of (X', Y', I') is isomorphic to the clarification and reduction of the h-realization (X^h, Y^h, I^h) of (X, Y, I).

Proof. We show that (X', Y', I') is a conditional context. Clearly, I' is completely present. The compatibility of I' with R_X and R_Y: For $x_1, x_2 \in X$ and $y \in Y$ it holds $\{x_1\}^{\uparrow} \approx_Y^{+} \{x_2\}^{\uparrow} \leq \{x_1\}^{\uparrow} \triangleleft \{x_2\}^{\uparrow} \leq \{x_1\}^{\uparrow}(y) \to \{x_2\}^{\uparrow}(y) = I(x_1, y) \to I(x_2, y)$. Now, $I(x_1, y) \wedge \mathrm{R}_X(x_1, x_2) \leq I(x_1, y) \wedge (I(x_1, y) \to I(x_2, y)) \leq I(x_2, y)$. Similarly can be shown $I'(x, y_1) \wedge \mathrm{R}_Y(y_1, y_2) \leq I'(x, y_2)$.

The rest follows from Lemma 67 and Lemma 65. □

2.3.8 Illustrative example

We demonstrate the theory of conditional concept lattices on a small dataset with incomplete information. We use data from a study of belemnites (extinct cephalopod group) [4]. (Species with incomplete information were excluded from the original research and can not be found in the paper.) Figure 2.2 (left) shows a table of four species of belemnites and four rostrum characteristics. Characteristics of the last two species is incomplete. The right side of the figure shows a conditional context representing the data in the table. The underlying Boolean algebra of conditions is a Boolean algebra L freely generated by elements c and d. The Boolean algebra of conditions L is shown in Fig. 2.1. It admits four total realities.

The conditional context (X', Y', I') from Figure 2.3 is the result of clarification and reduction of (X, Y, I) (Theorem 31). Standard realizations of (X', Y', I') are depicted in Fig. 2.4.

The conditional concept lattice $\mathscr{B}_c(X', Y', I')$ consists of the following eight conditional concepts:

$$
\begin{aligned}
&v_1 = (\{{}^{c}/x_1, x_2, x_3, x_4\}, \emptyset), \quad v_2 = (\{{}^{c'\wedge d}/x_2, {}^{c'\wedge d}/x_3, {}^{d}/x_4\}, \{{}^{c\vee d'}/y_1, {}^{c\vee d'}/y_2, {}^{d'}/y_3, {}^{c\vee d'}/y_4\}),\\
&v_3 = (\{x_3, x_4\}, \{y_2, {}^{c}/y_4\}), \quad v_4 = (\emptyset, \{{}^{c\vee d'}/y_1, y_2, y_3, y_4\}),\\
&v_5 = (\{x_2, {}^{c}/x_3, x_4\}, \{y_4\}), \quad v_6 = (\{{}^{c'\wedge d}/x_3, {}^{d}/x_4\}, \{{}^{c\vee d'}/y_1, y_2, {}^{d'}/y_3, {}^{c\vee d'}/y_4\}),\\
&v_7 = (\{{}^{c}/x_3, x_4\}, \{y_2, y_4\}), \quad v_8 = (\{{}^{c'\wedge d}/x_2, {}^{d}/x_4\}, \{{}^{c\vee d'}/y_1, {}^{c\vee d'}/y_2, {}^{d'}/y_3, y_4\}).
\end{aligned}
$$

Let h be a total reality, f and g standard h-realizations of X' and Y', respectively. Then for any conditional concept v in $\mathscr{B}_c(X', Y', I')$ it holds that the h-realization v^h of v is a concept of (X'^h, Y'^h, I'^h). For example, let h_3 be the total reality given by $h_3(c) = 0$ and $h_3(d) = 1$,

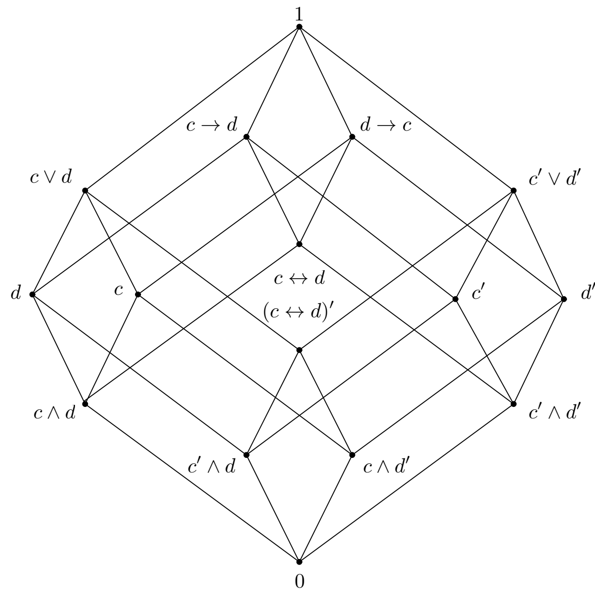

Figure 2.1: Boolean algebra freely generated by elements c and d.

	h	v	γ	λ
Actinocamax verus antefragilis (1)				
Praeactinocamax planus (9)				$\times$
Praeactinocamax aff. plenus		$\times$		?
Praeactinocamax paderbornensis	?	$\times$		$\times$

I	y_1	y_2	y_3	y_4
x_1				
x_2				$\times$
x_3		$\times$		c
x_4	d	$\times$		$\times$

Figure 2.2: Left: a small part of a table of belemnite (extinct cephalopod group) species and their rostrum characteristics from a research published in [4]. The last two species are not present in the paper, since they were excluded due to incomplete information. Symbol "?" means that it is unknown if a specie has the corresponding characteristic. Right: a conditional context (X,Y,I) representing the table on the left. The underlying Boolean algebra of conditions L is a Boolean algebra freely generated by elements c and d. The conditions c and d are interpreted as the conditions under which "P. aff. plenus has the characteristic λ" and "P. paderbornensis has the characteristic h", respectively. Values 0 and 1 in L are depicted by empty space and $\times$ in the table, respectively. The conditional equalities $\approx_X$ and $\approx_Y$ of X and Y, respectively, are both ordinary equalities.

R_X	x_1	x_2	x_3	x_4
x_1	c	0	0	0
x_2	0	1	0	0
x_3	0	0	1	$c \wedge d'$
x_4	0	0	$c \wedge d'$	1

R_Y	y_1	y_2	y_3	y_4
y_1	$c \vee d'$	0	d'	0
y_2	0	1	0	0
y_3	d'	0	1	0
y_4	0	0	0	1

I'	y_1	y_2	y_3	y_4
x_1				
x_2				$\times$
x_3		$\times$		c
x_4	$c \wedge d$	$\times$		$\times$

Figure 2.3: The conditional context (X',Y',I') (bottom middle) is the result of the clarification and reduction of the conditional context (X,Y,I) from Fig. 2.2. The clarification and reduction is described in Sec. 2.3.7. The conditional relation R_X (top left) is the conditional equality of X' and R_Y (top right) the conditional equality of Y'. Crosses represent 1 and empty places 0.

I'^{h_1}	$y_{1,3}$	y_2	y_4
x_2			$\times$
x_3		$\times$	
x_4		$\times$	$\times$

I'^{h_2}	$y_{1,3}$	y_2	y_4
x_1			
x_2			$\times$
$x_{3,4}$		$\times$	$\times$

I'^{h_3}	y_2	y_3	y_4
x_2			$\times$
x_3	$\times$		
x_4	$\times$		$\times$

I'^{h_4}	y_1	y_2	y_3	y_4
x_1				
x_2				$\times$
x_3		$\times$		$\times$
x_4	$\times$	$\times$		$\times$

Figure 2.4: Four standard realizations of the conditional context (X',Y',I') depicted in Fig. 2.3. The realizations correspond to total realities h_1,h_2,h_3,h_4 given by $h_1(c) = h_1(d) = h_2(d) = h_3(c) = 0$ and $h_2(c) = h_3(d) = h_4(c) = h_4(d) = 1$. Standard realizations $f_i : X' \to X'_i$ and $g_i : Y' \to Y'_i$ of X' and Y', respectively, are given by $X'_1 = X'_3 = \{x_2,x_3,x_4\}$, $X'_2 = \{x_1,x_2,x_{3,4}\}$, $X'_4 = X$ and $Y'_1 = Y'_2 = \{y_{1,3},y_2,y_4\}$, $Y'_3 = \{y_2,y_3,y_4\}$, $Y'_4 = Y$, and $f_i(x) = x$ and $g_i(y) = y$ for any $i \in \{1,2,3,4\}$, $x \in X$ and $y \in Y$ with the following exceptions: $f_2(x_3) = f_2(x_4) = x_{3,4}$ and $g_1(y_1) = g_1(y_3) = g_2(y_1) = g_2(y_3) = y_{1,3}$.

$(X'^{h_3}, Y'^{h_3}, I'^{h_3})$, the h_3-realization of (X', Y', I') given in Fig. 2.4 (bottom left) and v_6 the conditional concept $(\{{}^{c'\wedge d}/x_3, {}^{d}/x_4\}, \{{}^{c\vee d'}/y_1, y_2, {}^{d'}/y_3, {}^{c\vee d'}/y_4\})$. Then

$$
\begin{aligned}
v_6^{h_3} &= (\{{}^{c'\wedge d}/x_3, {}^{d}/x_4\}^{h_3}, \{{}^{c\vee d'}/y_1, y_2, {}^{d'}/y_3, {}^{c\vee d'}/y_4\}^{h_3}) \\
&= (\{{}^{h_3(c'\wedge d)}/x_3, {}^{h_3(d)}/x_4\}, \{{}^{h_3(c\vee d')}/y_1, y_2, {}^{h_3(d')}/y_3, {}^{h_3(c\vee d')}/y_4\}) \\
&= (\{{}^{1}/x_3, {}^{1}/x_4\}, \{{}^{0}/y_1, y_2, {}^{0}/y_3, {}^{0}/y_4\}) \\
&= (\{x_3, x_4\}, \{y_2\}).
\end{aligned}
$$

We can easily check that $(\{x_3, x_4\}, \{y_2\})$ is a concept of the context $(X'^{h_3}, Y'^{h_3}, I'^{h_3})$.

Suppose we fix a standard realization of (X', Y', I'). Then we can use conditional relations $\approx_V$ and $\preceq_V$ to reason about equality and order, respectively, of realizations of conditional concepts in $\mathscr{B}_c(X', Y', I')$. For example, let v_3 and v_5 be conditional concepts $(\{x_3, x_4\}, \{y_2, {}^{c}/y_4\})$ and $(\{x_2, {}^{c}/x_3, x_4\}, \{y_4\})$ of (X', Y', I'), respectively. Then by Lemma 63 we have $v_3 \preceq v_5 = \{x_3, x_4\} \triangleleft \{x_2, {}^{c}/x_3, x_4\} = \{y_4\} \triangleleft \{y_2, {}^{c}/y_4\} = c$ and $v_5 \preceq v_3 = \{x_2, {}^{c}/x_3, x_4\} \triangleleft \{x_3, x_4\} = \{y_2, {}^{c}/y_4\} \triangleleft \{y_4\} = 0$. Thus, for each total reality h and standard h-realizations f and g of X' and Y', respectively, it holds that $v_3^h \neq v_5^h$, $v_5^h \nleq v_3^h$ and $v_3^h \leq v_5^h$ iff the condition c is satisfied in h.

Equalities (2.31) and (2.32) describe suprema and infima of conditional sets in $\mathscr{B}_c(X', Y', I')$, respectively. For example, we take again the conditional concepts v_3, v_5 and let v_7 be the conditional concept $(\{{}^{c}/x_3, x_4\}, \{y_2, y_4\})$. Then $\mathrm{Inf}\{v_3, v_5\} = \{{}^{c}/v_3, v_7\}$ and thus for each total reality h and standard h-realizations f and g of X' and Y', respectively, it holds that $v_3^h \wedge v_5^h = v_7^h$ and $v_3^h \wedge v_5^h = v_3^h = v_7^h$ iff the condition c is satisfied in h.

Conclusions

The definition of a realization of a conditional universe admits that there is no realization of an element for which it is satisfied in the respective reality that it is present. The definition of a realization of a conditional universe can be possibly narrowed in such a way that it covers only faithfuls realizations. This would simplify proofs in the theory.

There is another possibility how to simplify the theory. The use of partial mappings as realizations of conditional universes causes many complications in proofs. One can try to leave non-present elements in standard realizations of conditional universes and use ordinary functions as realizations.

Many results on conditional sets can be easily extended for L-sets where L is a complete Heyting algebra. We presented original results which are not covered by the literature. Namely, we study also non-strict L-sets which are not compatible with $\approx$. We also proposed the definition of extensionality of L-sets.

Only conditional relations compatible with conditional equalities can be part of conditional structures. One can try to drop the requirement on compatibility. It should be then possible to define a set of formulas for which the truthfulness is transferred to realizations, i.e. the set of formulas for which Lemma 52 holds.

The conditional concept lattice $\mathscr{B}_c(X,Y,I)$ is a candidate for practical model of the unknown concept lattice of a context with missing information. The conditional concept lattice $\mathscr{B}_c(X',Y',I')$ of the conditional context (X',Y',I') from the illustrative example (Fig. 2.3) can be represented by a labeled Hasse diagram shown in Fig. 2.5. A general method for representing conditional concept lattices by labeled Hasse diagrams is a topic for further research. We could generalize results on crisply generated fuzzy concepts presented in [5] and results on $\bigvee$- and $\bigwedge$-compatible conditional concept lattices presented in [19]. Also results similar to [20] on the size of $\mathscr{B}_c(X,Y,I)$ would be welcomed.

The conditional equality of any conditional concept lattice is reflexive. We can drop the

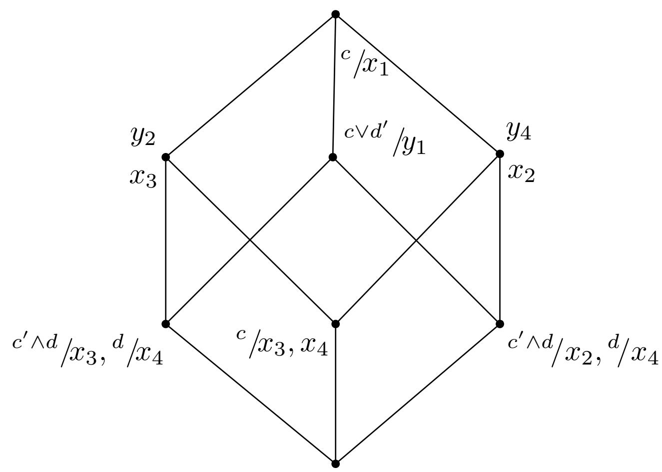

Figure 2.5: Conditional concept lattice $\mathscr{B}_c(X',Y',I')$ of the conditional context (X',Y',I') in Fig. 2.2 (right). The ordinary order $\leq$ on $\mathscr{B}_c(X',Y',I')$ is given by the Hasse diagram. The diagram also specifies conditional mappings $\gamma\colon X \to \mathscr{B}_c(X',Y',I')$ and $\mu : Y \to \mathscr{B}_c(X',Y',I')$: if for $x \in X$ and $v \in \mathscr{B}_c(X',Y',I')$ it holds $\gamma(x,v) = 0$ then the vertex for v is not labeled by x. If $\gamma(x,v) = a > 0$ then the vertex is labeled by $^{a}/x$ if $a < 1$ and by x if $a = 1$. Similarly for μ. We have $I(x,y) = \bigvee_{v_1,v_2 \in V,\ v_1 \leq v_2} \gamma(x,v_1) \wedge \mu(y,v_2)$ for each $x \in X$ and $y \in Y$. For the extent A of a conditional concept v in $\mathscr{B}_c(X',Y',I')$ it holds $A(x) = \bigvee_{\bar{v} \leq v} \gamma(x,\bar{v})$ for each $x \in X$.

requirement on the complete presence of I in the definition of conditional contexts (X,Y,I) and try to describe also partially present concepts in (X,Y,I). An example of such concept is shown in Fig. 2.6.

$$
\begin{array}{c|cc}
I & y_1 & y_2 \\ \hline
x_1 & & \\
x_2 & & \times
\end{array}
\qquad
\begin{array}{c|cc}
\approx_X & x_1 & x_2 \\ \hline
x_1 & 1 & 0 \\
x_2 & 0 & c
\end{array}
\qquad
\begin{array}{c|cc}
\approx_X & y_1 & y_2 \\ \hline
y_1 & 1 & 0 \\
y_2 & 0 & c
\end{array}
$$

$$
\begin{array}{c|cc}
I^{|h_1} & y_1 & y_2 \\ \hline
x_1 & & \\
x_2 & & \times
\end{array}
\qquad
\begin{array}{c|c}
I^{|h_2} & y_1 \\ \hline
x_1 &
\end{array}
$$

Figure 2.6: Top: Non-completely present conditional relation I between X and Y (top left), conditional equalities $\approx_X$ and $\approx_Y$ of X (top middle) and Y (top right), respectively. Empty places represent 0 and crosses 1. The underlying Boolean algebra of conditions is the four-element Boolean algebra $\{0,c,c',1\}$. Bottom: Restricted h_1-realization (bottom left) and h_2-realization (bottom right) of I where h_1 and h_2 are total realities given by $h_1(c) = h_2(c') = 1$. Realizations of X and Y are given by the following. The h_1-realization of X is the identity on X and the h_2-realization of Y is the identity on Y. $f_2 : X \nrightarrow \{x_1\}$ and $g_2 : Y \nrightarrow \{y_1\}$ are h_2-realizations of X and Y, respectively, given by $f_2(x_1) = x_1$, $g_2(y_1) = y_1$, $f_2(x_2)$ and $g_2(y_2)$ are not defined. The pair $(\{x_2\},\{y_2\})$ is present in $X \times Y$ under the condition c and it represents the only non-trivial formal concept of $(X^{h_1},Y^{h_1},I^{|h_1})$.

The theory of conditional structures can be used to model another incompletely defined structures. Especially, conditional structures where partial presence of elements naturally appear. For example, removing unreachable states in finite automatons with missing information could produce conditional structures with non-reflexive conditional equalities.

Bibliography

[1] S. S. Kutateladze A. G. Kusraev. *Boolean Valued Analysis*. Mathematics and Its Applications. Springer, October 2012.

[2] John L. Bell. *Set Theory: Boolean-Valued Models and Independence Proofs*. Oxford Logic Guides. Oxford University Press, 2nd edition, December 2011.

[3] Radim Belohlavek. *Fuzzy Relational Systems: Foundations and Principles*. Kluwer Academic Publishers, Norwell, MA, USA, 2002.

[4] Radim Belohlavek, Martin Kostak, and Petr Osicka. Formal concept analysis with background knowledge: a case study in paleobiological taxonomy of belemnites. *International Journal of General Systems*, 42(4):426–440, 2013.

[5] Radim Belohlavek, Vladimir Sklenar, and Jiri Zacpal. Crisply generated fuzzy concepts. In Bernhard Ganter and Robert Godin, editors, *Formal Concept Analysis*, volume 3403 of *Lecture Notes in Computer Science*, pages 269–284. Springer Berlin / Heidelberg, 2005.

[6] M. P. Fourman and D. S. Scott. *Applications of Sheaves: Proceedings of the Research Symposium on Applications of Sheaf Theory to Logic, Algebra, and Analysis, Durham, July 9–21, 1977*, chapter Sheaves and logic, pages 302–401. Springer Berlin Heidelberg, Berlin, Heidelberg, 1979.

[7] Bernard Ganter and Rudolf Wille. *Formal Concept Analysis – Mathematical Foundations*. Springer, 1999.

[8] George Georgescu. Fuzzy power structures. *Archive for Mathematical Logic*, 47(3):233–261, 2008.

[9] Steven Givant and Paul Halmos. *Introduction to Boolean Algebras*. Springer-Verlag New York, 2009.

[10] J.A. Goguen. L-fuzzy sets. *J. Math. Anal. Appl.*, 18:145–174, 1967.

[11] George A. Gratzer. *General lattice theory*. Academic Press, New York :, 1978.

[12] Petr Hájek. *Metamathematics of Fuzzy Logic*. Kluwer, Dordrecht, 1998.

[13] A. W. Hales. On the non-existence of free complete boolean algebras. *Fund. Math.*, 54:45–66., 1964).

[14] Denis Higgs. A category approach to boolean-valued set theory. *Unpublished typescript, University of Waterloo*, 1973.

[15] Ulrich Höhle. On the fundamentals of fuzzy set theory. *Journal of Mathematical Analysis and Applications*, 201(3):786 – 826, 1996.

[16] Ulrich Höhle. Fuzzy sets and sheaves. part i. *Fuzzy Sets Syst.*, 158(11):1143–1174, June 2007.

[17] Ulrich Höhle. Fuzzy sets and sheaves. part ii. *Fuzzy Sets Syst.*, 158(11):1175–1212, June 2007.

[18] L. J. Kohout and W. Bandler. Relational-product architectures for information processing. *Information Sciences*, 37(1-3):25–37, 1985.

[19] Michal Krupka and Jan Lastovicka. Incomplete information and concept lattices.

[20] Michal Krupka and Jan Lastovicka. Concept lattices of incomplete data. In Florent Domenach, DmitryI. Ignatov, and Jonas Poelmans, editors, *Formal Concept Analysis*, volume 7278 of *Lecture Notes in Computer Science*, pages 180–194. Springer Berlin Heidelberg, 2012.

[21] Rudolf Kruse, Joan E. Gebhardt, and Frank Klawonn. *Foundations of Fuzzy Systems*. John Wiley & Sons, Inc., New York, NY, USA, 1st edition, 1994.

[22] Michael Nelson. Existence. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Winter 2012 edition, 2012.

[23] Plato. *Plato in Twelve Volumes translated by Harold N. Fowler*, volume 12. Cambridge, MA, Harvard University Press, London, William Heinemann Ltd., 1921.

[24] H. Rasiowa and R. Sikorski. *The mathematics of metamathematics*. Panstwowe Wydawnictwo Naukowe, Warszawa, 1963.

[25] Maria Reicher. Nonexistent objects. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Winter 2015 edition, 2015.

[26] J. Barkley Rosser. *Simplified Independence Proofs, Boolean valued models of set theory*. Acedemic Press, 1969.

[27] Dana Scott. A proof of the independence of the continuum hypothesis. *Mathematical systems theory*, 1(2):pp 89 – 111, 6 1967.

[28] Dana Scott. *Applications of Sheaves: Proceedings of the Research Symposium on Applications of Sheaf Theory to Logic, Algebra, and Analysis, Durham, July 9–21, 1977*, chapter Identity and existence in intuitionistic logic, pages 660–696. Springer Berlin Heidelberg, Berlin, Heidelberg, 1979.

[29] William Shakespeare. *Romeo and Juliet*. Arden Shakespeare, 2nd edition, 1980.

[30] Roman Sikorski. *Boolean Algebras*. Academic Press, 2nd edition, 1964.

[31] Dilworth R. P. Ward M. Residuated lattices. *Trans. Amer. Math. Soc. 45*, pages 335–354, 1939.

[32] R. Wille. Restructuring lattice theory: an approach based on hierarchies of concepts. In I. Rival, editor, *Ordered Sets*, pages 445–470. Boston, 1982.

[33] Oswald Wyler. Fuzzy logic and categories of fuzzy sets. pages pp 235–268, 1995.

[34] L. A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965.

[35] Lotfi A. Zadeh. Similarity relations and fuzzy orderings. *Information Sciences*, 3:177–200, 1971.