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Faculty of Mechanical Engineering

MASTER'S THESIS

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ÚSTAV MATEMATIKY

RIGID BODY MOTION FROM THE GEOMETRIC VIEWPOINT

GEOMETRICKÝ POHLED NA POHYB TUHÉHO TĚLESA

MASTER'S THESIS

DIPLOMOVÁ PRÁCE

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BRNO UNIVERSITY OF TECHNOLOGY
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Assignment Master's Thesis

Institut:	Institute of Mathematics
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As provided for by the Act No. 111/98 Coll. on higher education institutions and the BUT Study and Examination Regulations, the director of the Institute hereby assigns the following topic of Master's Thesis:

Rigid body motion from the geometric viewpoint

Brief Description:

The motion of a rigid body can be solved as an optimal control problem on a Lie group. It is therefore possible to use the procedures of geometric control theory and simplify the resulting equation using the group structure. This procedure enables the direct acquisition of equations for the description of motion in the language of geometric algebras, as well as various modifications and generalizations of the motion.

Master's Thesis goals:

1) study the basics of the theory of Lie groups and algebras
2) study the basics of geometric control theory
3) understand the formulation of the motion of a rigid body as a left–invariant optimal problem on a Lie group with an invariant Hamiltonian
4) obtain the relevant equations in the formalism of geometric algebras
5) study potential variations of this task

Recommended bibliography:

AGRACHEV, A., BARILARI, D., BOSCAIN, U. A comprehensive introduction to sub-Riemannian geometry. From the Hamiltonian viewpoint, Cambridge Studies in Advanced Mathematics, Vol. 181, Cambridge University Press, 2020.

AGRACHEV, A. A., SACHKOV, Y. Control Theory from the Geometric Viewpoint. Encyclopedia of Mathematical Sciences. Berlin: Springer-Verlag, 2004.

Deadline for submission Master's Thesis is given by the Schedule of the Academic year 2022/23

In Brno,

L. S.

doc. Mgr. Petr Vašík, Ph.D. Director of the Institute	doc. Ing. Jiří Hlinka, Ph.D. FME dean

ABSTRACT
The main objective of this thesis is to derive the Hamiltonian equations for left-invariant problems on Lie groups. Our motivation is as follows. The motion of a 3D rigid body can be formulated as an optimal control problem in $\mathbb{R}^3$. The Pontryagin's Maximum Principle (PMP) can be applied to solve such a problem. However, the motion of a rigid body can also be viewed as a problem on the Lie group SE(3). This problem belongs to the class of left-invariant problems. To further simplify the problem, we assume a left-invariant Hamiltonian function. The usual approach in studying such problems involves first defining the Lagrangian function, then obtaining the Hamiltonian function, and finally formulating the Hamiltonian equations. However, we take a different approach. We derive the Hamiltonian equations for a general Lie group and a general left-invariant Hamiltonian, and then explore the types of problems that can be described by choosing specific Lie groups and Hamiltonian functions. The theoretical results obtained are then applied in the development of simulation scripts for both rigid body motion and soft body motion which utilizes CGA as its computational core. We have opted for CGA due to its remarkable computational capabilities in this context. By utilizing CGA, we naturally obtain dimension independence without any additional effort.

KEYWORDS
Lie group, Lie algebra, left-invariant systems, left-invariant Hamiltonian, control theory on Lie groups, rigid body motion, soft body motion, conformal geometric algebra, CGA

ABSTRAKT
Cílem této práce je odvodit rovnice levo-invariantních Hamiltonovských systémů na Lieových grupách. Naše motivace je následující. Pohyb tuhého tělesa v 3D prostoru lze formulovat jako úlohu optimálního řízení na $\mathbb{R}^3$. Pro takto formulovanou úlohu lze využít Pontryaginův princip maxima (PMP). Nicméně pohyb tuhého tělesa lze také chápat jako úlohu na Lieově grupě SE(3). Tato úloha patří do skupiny tzv. *levo-invariantních úloh*. Jako další zjednodušení volíme také *levo-invariantní Hamiltoniány*. Běžný postup při studiu takových úloh je, že formulujeme Lagrangián této úlohy, odvodíme Hamiltonián a následně formulujeme Hamiltonovy rovnice. Náš postup je opačný. Odvodíme Hamiltonovy rovnice pro obecnou Lieovu grupu a obecný levo-invariantní Hamiltonián a následně zkoumáme, jaké typy úloh můžeme popsat volbou konkrétní Lieovy grupy a konkrétního Hamiltoniánu. Teoretické výsledky poté využijeme k vytvoření simulačního skriptu pohybu tuhého a pružného tělesa, který využije konformní geometrickou algebru (CGA) jako své výpočetní jádro. CGA je totiž nesmírně silný nástroj pro popis této problematiky, jelikož využitím CGA lze vyvinout kód, který je nezávislý na dimenzi uvažovaného prostoru bez větší námahy.

KLÍČOVÁ SLOVA
Lieova grupa, Lieova algebra, levo-invariantní systémy, levo-invariantní Hamiltonián, teorie řízení na Lieových grupách, pohyb tuhého tělesa, pohyb pružného tělesa, konformní geometrická algebra, CGA

ROZŠÍŘENÝ ABSTRAKT

Cílem této práce je odvodit rovnice levo-invariantních Hamiltonovských systémů na Lieových grupách. Naše motivace je následující. Pohyb tuhého tělesa v 3D prostoru lze formulovat jako úlohu optimálního řízení na $\mathbb{R}^3$. Pro takto formulovanou úlohu lze využít Pontryaginův princip maxima (PMP). Nicméně pohyb tuhého tělesa lze také chápat jako úlohu na Lieově grupě SE(3). Tato úloha patří do skupiny tzv. *levo-invariantních úloh*. Jako další zjednodušení volíme také *levo-invariantní Hamiltoniány*. Běžný postup při studiu takových úloh je, že formulujeme Lagrangián této úlohy, odvodíme Hamiltonián a následně formulujeme Hamiltonovy rovnice. Náš postup je opačný. Odvodíme Hamiltonovy rovnice pro obecnou Lieovu grupu a obecný levo-invariantní Hamiltonián a následně zkoumáme, jaké typy úloh můžeme popsat volbou konkrétní Lieovy grupy a konkrétního Hamiltoniánu. Teoretické výsledky poté využijeme k vytvoření simulačního skriptu pohybu tuhého a pružného tělesa, který využije konformní geometrickou algebru (CGA) jako své výpočetní jádro. CGA je totiž nesmírně silný nástroj pro popis této problematiky. Tento fakt můžeme velmi jednoduše ilustrovat. Uvažujme grupy SE(2) a SE(3). Jejich prvky jsou pochopitelně velmi rozdílné matice a vývoj kódu, který by byl nezávislý na tom, kterou z nich zvolíme by byl velmi složitý. Ve velmi ostrém kontrastu pak máme 2D CGA a 3D CGA. Báze vektorů v 3D CGA obsahuje oproti 2D CGA jeden prvek navíc. Ostatní blady (včetně bivektorů, které jsou pro náš zvlášť podstatné) jsou generovány touto bází. Ve výsledku je tedy popis využívající CGA naprosto nezávislý na dimenzi a můžeme využít stejné výpočetní jádro pro 2D CGA i 3D CGA.

Jak vyplývá z předchozího odstavce, tato práce využívá teorii z několika různých oblastí matematické teorie, které jsou v práci popsány. V první kapitole se věnujeme zavedení pojmů potřebných algebraických struktur a funkcí na nich. Vycházíme z pojmů diferenciální geometrie. Připomene definici n-dimenzionální hladké variety M, jejího tečného prostoru v bodě T_qM a tečného bandlu TM. Dále definujeme hladká vektorová pole na hladké varietě, což jsou hladká zobrazení $X : q \mapsto T_qM$. Definujeme tento pojem (a relevantní pojmy s ním spojené) také pro hladkou varietu, protože řešení úlohy levo-invariantních Hamiltonovských systémů na Lieových grupách je integrální křivka v Lieově grupě. Zavedené pojmy následně demonstrujeme na 2-dimenzionální sféře S^2. Následuje zavedení Lieovy závorky vektorových polí. V této práci nevycházíme z axiomatického zavedení obecné Lieovy závorky, ale využíváme její vlastnosti na vektorových polích a axiomy poté ukazujeme jako její vlastnosti. Tento alternativní postup je běžnější při zavádění pro účely teorie Lieových grup, proto jsme jej zvolili též. Dalším krokem je definice Poissonovy závorky, to je bilineární a antisymetrický operátor na $C^\infty(T^*M)$ – hladkých funkcích na tzv. kotečném bandlu. Začneme tedy s definicí T^*M a poté uvádíme definici Pois-

sonovy závorky jako takové. Definice Poissonovy závorky je pro nás důležitá, protože nám umožňuje zavést pojem Hamiltonovského vektorového pole přidruženého Hamiltoniánu.

S využitím těchto pojmů můžeme přistoupit k definici Lieovy grupy a Lieovy algebry. Uvádíme jednak přesné definice a jednak příklady maticových Lieových grup (obecnou lineární grupu $\mathrm{GL}(n)$, speciální ortogonální grupu $\mathrm{SO}(n)$ a Euklidovskou grupu $\mathrm{SE}(n)$) a jejich příslušných Lieových algeber ($\mathrm{gl}(n)$, $\mathrm{so}(n)$, $\mathrm{se}(n)$). Tyto příklady ukazujeme, protože na nich budeme zkoumat Hamiltonovské systémy. Dále ukážeme, že tečné prostory (resp. kotečné prostory) Lieovy grupy jsou levo-invariantní. Díky tomu máme globální trivializaci tečného bandlu (resp. kotečného bandlu): $TG \cong G \times L$ (resp. $T^*G \cong G \times L^*$), kde G je Lieova grupa, L její Lieova algebra a L^* duál Lieovy algebry. Tato vlastnost, která pro obecnou hladkou varietu platí jen lokálně, nám umožní zjednodušit Hamiltonovy rovnice.

V druhé kapitole se věnujeme teorii optimálního řízení. Nejprve připomeneme úlohu optimálního řízení a PMP na $\mathbb{R}^n$ a poté ukážeme, jak tuto úlohu a PMP formulujeme pro případ obecné hladké variety. Následně uvádíme, jak vypadají Hamiltonovy rovnice pro zjednodušený případ levo-invariantního systému a levo-invariantního Hamiltoniánu. Tvrzení o tom, jak rovnice můžeme zjednodušit také dokazujeme, jelikož je klíčové pro tuto práci.

V třetí kapitole zavádíme *indefinitní speciální ortogonální grupu* $\mathrm{SO}(\mathrm{p}, \mathrm{q})$, která je Lieovou grupou, její Lieovu algebru a konformní geometrickou algebru (CGA). Jak čtenář jistě tuší, v následujících kapitolách budeme volit $\mathrm{SO}(3)$, $\mathrm{SE}(3)$ a také $\mathrm{SO}(4, 1)$ jako Lieovy grupy, na kterých budeme pozorovat výsledky ze sekce 2.1. Protože bivektory v CGA jsou izomorfní Lieově algebře $\mathrm{so}(4, 1)$, můžeme úlohu také zformulovat na CGA. Toho využijeme v kapitole 5.

Ve čtvrté kapitole začínáme komentářem ohledně koadjungovaného operátoru, který se vyskytuje v rovnici 2.8. Dále se již věnujeme příkladům Hamiltonovského formalismu. Jak jsme již zmínili, volíme postupně grupy $\mathrm{SO}(3)$, $\mathrm{SE}(3)$ a $\mathrm{SO}(4, 1)$. Ve všech třech případech volíme Hamiltoniány ve formě kvadratické formy, lineární formy a kvadratické funkce bez absolutního členu na dané Lieově algebře.

Ne příliš překvapivě jsme zjistili, že kvadratická forma na $\mathrm{SO}(3)$ modeluje rotační pohyb tuhého tělesa, nicméně volbou koeficientů kvadratické formy lze také získat vyjádření rovnic geodetiky, případně sub-Riemannovské geodetiky. Ukazuje se, že lineární Hamiltonián nemá nijak zvlášť hezké aplikace, což plyne z toho, že pro něj nelze zformulovat Lagrangián. Hamiltonián v podobě kvadratické funkce bez absolutního členu lze využít jako model modelu pohybu tuhého tělesa, na které působí síly konstantní v referenční soustavě tělesa. Jelikož Lagrangián této úlohy je také kvadratická forma, lze takto modelovat geodetiky. Volba $\mathrm{SE}(3)$ přidává navíc tři dimenze v Lieově algebře. Pomocí nich lze nyní modelovat nejen rotační, ale i translační pohyb tuhého tělesa. Jelikož zbylé aplikace jsou podobné těm na SO(3), věnujeme se blíže především pohybu tuhého tělesa. Přidáním lineárního členu získáme navíc působení sil, ale opět jen těch, které jsou konstatní v body framu. V opačném případě by Hamiltonián nemohl být levo-invariantní. Zvolením $\mathrm{SO}(4,1)$ získáme navíc k translačnímu a rotačnímu pohybu involuce, inverze, škálování a další operace. Pro nás je především zajímavé škálování. V aplikacích tohoto případu už nemůžeme mluvit o tuhém tělese, jelikož dochází k jeho deformaci. V oblasti výpočetní grafiky a modelů je toto těleso nazýváno „soft body“ (srov. s „rigid body“). Český ekvivalent tohoto pojmu je pružné těleso. Hamiltonián ve formě kvadratické formy modeluje pohyb pružného tělesa bez působení sil, které lze přidat pomocí lineárního členu v Hamiltoniánu.

V poslední kapitole jsou popsány tři skripty, které byly naprogramovány k této práci. V prvním ukazujeme pohyb tuhého tělesa (popsaný v projektivní geometrické algebře), nicméně tento příklad je koncipován tak, že můžeme demonstrovat jednu vlastnost popisu levo-invariantních systémů pomocí Lieovy algebry a to, že tento popis nezávisí na dimenzi prostoru, který uvažujeme. Každý příklad tedy ukazujeme pro 2D i 3D prostor v tom smyslu, že tuto dimenzi volíme jako proměnnou skriptu, veškerý ostatní kód zůstává stejný. Byly vyvinuty dvě verze toho skriptu, jedna generující animace pohybu a druhá, která pohyb vygeneruje na nějakém časovém intervalu a poté zobrazí několik řezů v čase, abychom mohli výsledky prezentovat v této práci. Výsledky simulací se shodovaly s výsledky kapitoly 4.

Následující skript ukazuje výpočty v CGA, modelujeme tedy pohyb pružného tělesa s i bez působení sil. V první řadě demonstrujeme, že CGA (a tedy i $\mathrm{so}(4,1)$) opravdu obsahuje translace a rotace, následně ukazujeme efekty smršťování a roztahování při volbě nenulového koeficientu u bivektoru $e_{0\infty}$, který toto chování způsobuje. Podobně jako v předchozím případě byly vytvořeny dvě verze tohoto skriptu, jeden tvořící animace a jeden tvořící časové řezy pohybu. Výsledky opět reflektovaly ty z kapitoly 4.

Výše popsané skripty byly napsány v jazyce JavaScript, poslední je napsaný v Matlabu, ukazuje řešení úloh na $\mathrm{SO}(4,1)$, tedy v maticové formulaci. K řešení rovnice 2.8 využíváme funkci `ode45`, vývoj na grupě počítáme pomocí analytického řešení. Jelikož nemáme zvlášť vhodnou možnost prezentace výsledné křivky na maticové grupě, vykreslujeme graf rychlostí v čase a kontrolujeme, že matice na křivce opravdu patří do grupy $\mathrm{SO}(4,1)$, tedy jestli ve výpočtu nedochází k velkým numerickým chybám. Výpočty jsme prováděli na datech ekvivalentních datům z předchozího skriptu. Výsledné rychlosti reflektovaly výsledky animací, křivka také zůstala na grupě. Celkově tedy máme pozitivní výsledek.

Author's Declaration

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Topic:	Geometric Numerical Integration Methods In Control Theory

I declare that I have written this paper independently, under the guidance of the advisor and using exclusively the technical references and other sources of information cited in the paper and listed in the comprehensive bibliography at the end of the paper.

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Brno ....

author's signature*

*The author signs only in the printed version.

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To my son, I extend a special thank you for being my guiding light during the darkest nights.

Contents

Introduction

This thesis seeks to unify and generalize the concept of left-invariant Hamiltonian systems on Lie groups which is currently regarded as state of art theory for the area of simulation of rigid body motion in computer science ([1]). The following objectives have been identified:

- Formulate Hamiltonian equations for general Lie groups and arbitrary left-invariant Hamiltonian functions.
- Explore potential applications by selecting specific Lie groups and Hamiltonian functions.
- Develop a computational tool for simulating rigid body motion using the theoretical findings.

While the concept of rigid body motion is widely recognized, its formulation as an optimal control problem on the Lie group SE(3), particularly the simplification of the Hamiltonian equations resulting from the group structure, is not as well-known. Furthermore, the representation of this problem in geometric algebra is a relatively novel research outcome from recent years.

The thesis will begin by defining the concepts of differential geometry, which serve as the building blocks for more complex structures. This will be followed by the definitions of Lie groups, Lie algebras, and left-invariant Hamiltonian systems, which are crucial for the main statement and its proof. The subsequent chapter will recall notions of control theory on $\mathbb{R}^n$ and extend it to smooth manifolds. We will then demonstrate the simplifications that arise when focusing on Lie groups and left-invariant Hamiltonian functions, substantiating this claim with a proof.

To illustrate the practical implications of the research, we will apply the developed framework to real-world problems. In addition to well-established Lie groups such as SO(3) and SE(3), we will introduce the Lie group SO(4,1), which encompasses both SO(3) and SE(3) as subgroups. Furthermore, we will introduce the Conformal Geometric Algebra (CGA). With these concepts established, our investigation will delve into each of the three Lie groups, presenting the Hamiltonian equations resulting from different choices of left-invariant Hamiltonian functions.

Finally, to visualize the outcomes in the context of rigid body motion, we will develop an animation tool that leverages CGA as its computational core. We have opted for CGA due to its remarkable computational capabilities in this context. Let us consider the cases of SE(2) and SE(3) as illustrative examples. Those are fundamentally different matrix groups and it would be extremely challenging to develop a unified code that accommodates both groups. In stark contrast, the utilization of 2D CGA and 3D CGA demonstrates the converse situation. The basis of vectors of 3D CGA has one more element in addition to the basis of vectors of 2D CGA. Since all other blades, including bivectors which are crucial in our approach, are generated by base vectors, we naturally obtain dimension independence as a valuable attribute, without any additional effort.

1 Mathematical Background

In this chapter, our objective is to present a set of concepts that will aid in streamlining the formulations of control theory. We will begin by providing an overview of fundamental notions in differential geometry. Subsequently, we will employ these concepts to construct the Lie group, Lie algebra, and other relevant entities. The primary references utilized for this chapter are [2] and [3].

1.1 Introduction to Differential Geometry

In this section, our focus will be on introducing the fundamental concepts of differential geometry. Specifically, we will cover topics such as smooth manifolds, tangent bundles, vector fields, and the flow of vector fields. It is assumed that the reader has prior knowledge of basic notions in topology and analysis, chiefly the notion of manifold.

Recall that on any manifold M there are *charts* (V, ψ), with $V \subset M$ and homeomorphism $\psi : V \to \mathbb{R}^n$, and the *coordinate functions* $x_j : V \to \mathbb{R}$. Thus, a point p can be identified with an n-tuple:

$$\psi(p) = (x_1(p), \ldots, x_n(p)) .$$

Now, a smooth manifold is, roughly speaking, a manifold endowed with smooth maps between its charts. What exactly we mean by these smooth maps is shown in the precise definition below.

Definition 1.1.1. Let M be an n-dimensional manifold and let there be a collection of charts $\{(V_\alpha, \psi_\alpha)\}_{\alpha \in I}$, where I is a set of indices. Suppose that

$$\cup_{\alpha \in I} V_\alpha = M,$$

and that $\forall \alpha, \beta \in I$, $V_\alpha \cap V_\beta \neq \emptyset$, the map

$$\Psi_{\alpha\beta} = \psi_\alpha \circ \psi_\beta^{-1} \colon \psi_\beta \left(V_\alpha \cap V_\beta \right) \to \psi_\alpha \left(V_\alpha \cup V_\beta \right)$$

is smooth. Then M is called *smooth* (differentiable) *manifold.*

By our construction, the map $\Psi_{\alpha\beta}$ is map from $\mathbb{R}^n$ to itself and thus the smoothness is meant in the usual way.

1.1.1 Tangent Bundle and Vector Field

We now shift our focus to tangent bundles and vector fields. To begin, we explore the concept of curves on smooth manifolds, which allows us to construct tangent vectors at points. Within the domain of differential geometry, smooth curves are defined as smooth maps from an interval to a smooth manifold. However, it's important to note that this parametrization is not unique. We say that two smooth curves, denoted as γ_1 and γ_2, both mapping from the interval I to the manifold M and based at the point $q = \gamma_1(0) = \gamma_2(0) \in M$, are considered *equivalent* if they share the same 1st order Taylor polynomial within some coordinate chart. It becomes evident that this notion of equivalence satisfies the properties of an equivalence relation, namely, reflexivity, symmetry, and transitivity. By utilizing these equivalent curves, we can define the tangent space at a point on a smooth manifold.

Definition 1.1.2. Let M be an n-dimensional smooth manifold and let γ be a smooth curve, s.t. $\gamma(0) = q \in M$. Its *tangent vector* at q, denoted by

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \gamma(t), \qquad \text{or} \qquad \dot{\gamma}(0), \tag{1.1}$$

is the equivalence class in the space of all smooth curves in M such that $\gamma(0) = q$. Moreover, the set of all tangent vectors at point q is called the *tangent space*, and we denote it by T_qM,

$$T_qM = \left\{ \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \gamma(t),\ \gamma\colon I \to M,\ \gamma(0) = q \right\}.$$

The tangent space has a structure of an n-dimensional vector space. Moreover, by $TM = \bigcup_{p\in M} T_pM$ we denote the *tangent bundle*.

Remark 1.1.3. The tangent bungle TM of a smooth manifold M is locally isomorphic to $M \times T_qM$.

Definition 1.1.4. Let M be a smooth manifold, $q \in M$. Then a smooth map $X\colon q \mapsto X(q) \in T_qM$ is called a *smooth vector field*. The set of all smooth vector fields on M we denote by $\mathrm{Vec}\,(M)$.

Now, let us shift our attention towards a few properties of the vector field. Drawing an analogy from physics, we can envision the vector field as guiding us along a path of least resistance. In classical calculus, we achieve this by solving certain types of differential equations, be they ordinary or partial. Building upon this intuitive notion, we aim to formalize these concepts based on the principles of classical calculus.

An *ordinary differential equation* (ODE) on a smooth manifold M given by a vector field $X \in \mathrm{Vec}\,(M)$ is the following equation.

$$\dot{q} = X(q), \qquad q \in M \tag{1.2}$$

A *solution* of 1.2 is every smooth curve $\gamma\colon J \to M$, where $J \subset \mathbb{R}$ is an open interval, s.t.

$$\dot{\gamma}(t) = X(\gamma(t)), \qquad \forall t \in J. \tag{1.3}$$

We also call γ the *integral curve of the vector field* X. The standard theorem on ODEs guarantees the existence of a unique solution to the equation 1.2 for any initial condition within an open interval I. For full statements, proofs and results of classical ODE, we advise to see [4]. Hence, we can formulate the Cauchy problem on a smooth manifold.

Theorem 1.1.5. *Let $X \in \mathrm{Vec}\,(M)$ and consider following problem*

$$\begin{cases} \dot{q}(t) = X(q(t)), \\ q(0) = q_0. \end{cases} \tag{1.4}$$

Then $\forall q_0 \in M$, $\exists \delta > 0$ and there is an unique solution $\gamma\colon (-\delta, \delta) \to M$ of 1.4, denoted by $\gamma(t; q_0)$. Moreover, the map $(t, q) \mapsto \gamma(t; q)$ is smooth on some neighborhood of $(0, q_0)$.

The uniqueness of the solution is meant in following sense. Let $\gamma_1\colon I_1 \to M$, $\gamma_2\colon I_2 \to M$ be two solutions of 1.4 on two intervals I_1, I_2 containing zero $\Rightarrow \forall t \in I_1 \cap I_2$, $\gamma_1(t) = \gamma_2(t)$. Thus definition of a *maximal* solution is sensible. The definition is very natural. It is such a solution $\gamma\colon I \to M$ of 1.4 that it is not extendable to any interval $J \supset I$. The vector field $X \in \mathrm{Vec}\,(M)$ from the problem 1.4 is called *complete* if $\forall q_0 \in M$, the maximal solution $\gamma(t; q_0)$ is defined on $I = \mathbb{R}$.

With complete vector fields we are able to study following family of maps called the flow of the vector field.

Definition 1.1.6. Let $X \in \mathrm{Vec}\,(M)$ be a complete vector field and let $\gamma(t; q)$ be the integral curve of X, starting at q for $t = 0$. The family of maps

$$\phi_t\colon M \to M, \qquad \phi_t(q) = \gamma(t; q), \qquad \forall t \in \mathbb{R}, \tag{1.5}$$

is called the *flow generated by* X.

From theorem 1.1.5 it follows, that the map $\phi\colon \mathbb{R} \times M \to M$ is smooth in both variables. The flow satisfies following identities:

$$\begin{aligned} \phi_0 &= \mathrm{Id}, \\ \phi_s \circ \phi_r &= \phi_r \circ \phi_s = \phi_{r+s}, \qquad && \forall r, s \in \mathbb{R}, \\ (\phi_t)^{-1} &= \phi_{-t}, && \forall t \in \mathbb{R}. \end{aligned} \tag{1.6}$$

An essential outcome arising from our construction of the flow is

$$\frac{\partial \phi_t(q)}{\partial t} = X(\phi_t(q)), \qquad \phi_o(q) = q, \qquad \forall q \in M. \tag{1.7}$$

In addition, the exponential notation is frequently employed, which naturally follows as a corollary of equation 1.7

$$\phi_t = \exp\{tX\}, \qquad \forall t \in \mathbb{R}.$$

The identities 1.6 and 1.7 then take the form, $\forall t, r, s \in \mathbb{R}, \forall q \in M$,

$$\begin{aligned}
e^{0X} &= \mathrm{Id}, \\
e^{sX} \circ e^{rX} &= e^{rX} \circ e^{sX} = e^{(s+r)X}, \\
\left(e^{tX}\right)^{-1} &= e^{-tX}, \\
\frac{\mathrm{d}}{\mathrm{d}t} e^{tX}(q) &= X\left(e^{tX} q\right).
\end{aligned}$$

Another essential property of the vector fields is that they differentiate smooth functions on M along the integral curves. Specifically, for any $X \in \operatorname{Vec}(M)$ and $a \in C^\infty(M)$, X *induces action* of a on $C^\infty(M)$ defined as

$$X\colon C^\infty(M) \to C^\infty(M), \qquad a \mapsto Xa,$$

where

$$(Xa)(q) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} a\left(e^{tX}(q)\right), \qquad \forall q \in M. \tag{1.8}$$

It might be beneficial to show what does the function $a_t = a \circ e^{tX}$ look like more precisely. As the map $t \mapsto a_t$ is smooth, we can expand this function as a sum in terms of the parameter t. The first element in this expansion is obviously just a. From 1.8 we immediately get the first-order element, Xa. Thus

$$a_t = a + tXa + O\left(t^2\right),$$

where $O(t^2)$ represents the term of order t^2 or higher-order terms in the expansion. In the next theorem, an expression of form X^n, $X \in \operatorname{Vec}(M)$, $n \in \mathbb{N}$ is used. For function $a \in C^\infty(M)$, the term $X^n a$ signifies repeated action of X on a. We will give a precise depiction of X^2 in the proof of the following theorem.

Theorem 1.1.7. *Let $a \in C^\infty(M)$, $X \in \operatorname{Vec}(M)$. Denote $a_t = a \circ e^{tX}$. Then the formulas*

$$\frac{\mathrm{d}}{\mathrm{d}t} a_t = Xa_t, \tag{1.9}$$

$$a_t = a + tXa + \frac{t^2}{2!}X^2 a + \frac{t^3}{3!}X^3 a + \cdots + \frac{t^k}{k!}X^k a + O\left(t^{k+1}\right) \tag{1.10}$$

hold.

Proof. The second formula can be derived as an extension of 1.8. Consider function $b = Xa$, where $a \in C^\infty(M)$ then $\forall q \in M$ by 1.8

$$
\begin{aligned}
(X^2 a)(q) = (Xb)(q) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} b\left(e^{tX}(q)\right) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} Xa\left(e^{tX}(q)\right) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} a\left(e^{sX}\left(e^{tX}(q)\right)\right) = \left.\frac{\mathrm{d}^2}{\mathrm{d}r^2}\right|_{r=0} a\left(e^{rX}(q)\right)
\end{aligned}
$$

In the last step, we use substitution $r = s + t$. Thus we have obtained the second-order element of the expansion, $\frac{t^2}{2!}X^2 a$. And the form so far

$$a_t = a + tXa + \frac{t^2}{2!}X^2 a + O\left(t^3\right).$$

By induction we would get the full expression. The formula 1.9 immediately arises from 1.10.

$$\frac{\mathrm{d}}{\mathrm{d}t}a_t = Xa + tX^2 a + \frac{t^2}{2!}X^3 a + \cdots + \frac{t^k}{k!}X^{k+1}a + O\left(t^{k+1}\right) = Xa_t$$

□

Thus the vector fields act as derivation of the smooth functions on M. As a corollary, we obtain another notation for the exponential

$$e^{tX} = \mathrm{Id} + tX + \frac{t^2}{2!}X^2 + \frac{t^3}{3!}X^3 + \ldots$$

It could be beneficial to illustrate the key concepts of this section on an example. For this purpose, we have selected the S^2 sphere.

Example 1.1.8. The S^2 sphere is defined as follows:

$$S^2 = \left\{x \in \mathbb{R}^3\colon \|x\| = 1\right\},$$

where $\|\cdot\|$ is the Euclidean metric. To show that this is a manifold, more precisely a 2-manifold, we construct 2 charts. Let us denote

$$U_1 = S^2 \setminus (0,0,1), \qquad U_2 = S^2 \setminus (0,0,-1),$$

spheres without a pole. The exact choice of this excluded point is not relevant. Those open sets endowed with a stereographic projection are indeed charts covering the whole sphere. The map between the charts from definition 1.1.1 is smooth, thus S^2 sphere is a smooth manifold, which isn't that surprising. The tangent plane at point $P = (x_p, y_p, z_p)$ is defined by the equation

$$x_p x + y_p y + z_p z = \|P\| = 1, \qquad \forall x, y, z \in \mathbb{R}.$$

A plane is evidently a two-dimensional vector space. The construction of the tangent plane in this particular example is highly intuitive. By utilizing the definitions provided earlier, we can take, for instance, an equator and the main meridian, and generate two linearly independent vectors at their intersection. Subsequently, we can form the tangent space as the linear combinations of these vectors.

1.1.2 Lie Bracket

The notion of Lie bracket of two vector fields X and Y holds significant importance for us. Essentially, the Lie bracket of two vector field represents the infinitesimal motion of the vector field Y along the flow of X. In another words, Lie bracket measures how much is Y changed by the flow of X and vice versa.

Definition 1.1.9. Let $X, Y \in \mathrm{Vec}\,(M)$. The *Lie bracket of vector fields* X and Y is the vector field $[X, Y] \in \mathrm{Vec}\,(M)$ such that $\forall q \in M$

$$\gamma(t) = \mathrm{Id} + t^2\,[X, Y]\,(q) + O(t^3) \tag{1.11}$$

where the curve γ is defined as

$$\gamma(t) = e^{-tY} \circ e^{-tX} \circ e^{tY} \circ e^{tX}(q).$$

There exists an alternative, more convenient representation for the Lie bracket. As previously mentioned, we interpret the product of two vector fields as a derivation of functions. In the specific context of matrix Lie groups and algebras (which will be introduced in detail later), this interpretation aligns with the matrix commutator, and the exponential notation corresponds to the matrix exponentiation.

Theorem 1.1.10. *Let $X,\ Y \in \mathrm{Vec}\,(M)$. The Lie bracket of vector fields X and Y can be equivalently expressed as follows:*

$$[X, Y] = XY - YX. \tag{1.12}$$

Proof. First, let us expand the exponential notation of the curve $\gamma(t)$.

$$\begin{aligned}
\gamma(t) &= \left(\mathrm{Id} + tX + \frac{t^2}{2!}X^2 + O(t^3)\right)\left(\mathrm{Id} + tY + \frac{t^2}{2!}Y^2 + O(t^3)\right)\\
&\qquad \left(\mathrm{Id} - tX + \frac{t^2}{2!}X^2 + O(t^3)\right)\left(\mathrm{Id} - tY + \frac{t^2}{2!}Y^2 + O(t^3)\right)(q)\\
&= \left(\mathrm{Id} + t(X+Y) + \frac{t^2}{2!}\left(X^2 + 2XY + Y^2\right) + O(t^3)\right)\\
&\qquad \left(\mathrm{Id} - t(X+Y) + \frac{t^2}{2!}\left(X^2 + 2XY + Y^2\right) + O(t^3)\right)(q)\\
&= \left(\mathrm{Id} + t^2(XY - YX) + O(t^3)\right)(q)
\end{aligned}$$

□

Another important properties of Lie bracket of vector fields are

- bilinearity:

$$\begin{aligned}
[aX + bY, Z] &= a\,[X, Z] + b\,[Y, Z]\,,\\
[Z, aX + bY] &= a\,[Z, X] + b\,[Z, Y]\,,
\end{aligned}$$

- skew-symmetry: $[X, Y] = -[Y, X]$,
- satisfies the Jacobi identity:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

In fact, the general Lie bracket is defined by those properties and our definition of Lie bracket of vector fields is subsequently derived as a theorem.

Theorem 1.1.11. *Let $X, Y \in \mathrm{Vec}(M)$, $a \in C^\infty(M)$. The Lie bracket satisfies the Leibniz rule*

$$[X, aY] = a[X, Y] + (Xa)Y. \tag{1.13}$$

Proof. For any function $a \in C^\infty(M)$ and for any point $q \in M$ we use 1.12 to obtain

$$[X, aY](q) = (X(aY))(q) - ((aY)X)(q).$$

Let us recall that X acts as a derivation (1.9), and applying the product rule to the first composition, we obtain

$$(X(aY))(q) = X(aY(q)) = X(a)Y(q) + a(XY(q)).$$

For the second composition, we can derive the following expression:

$$((aY)X)(q) = a(YX(q)).$$

Now it remains to combine these two results.

$$\begin{aligned} [X, aY](q) &= (X(aY))(q) - ((aY)X)(q) \\ &= X(a)Y(g) + a(XY(q)) - a(YX(q)) \\ &= a[X, Y](q) + X(a)Y(q) \end{aligned}$$

Since q was arbitrary, the proposition 1.1.11 holds. $\square$

We will show an explicit example of Lie bracket later after definition of Lie algebra (ex. 1.3.8).

1.2 Hamiltonian Vector Field

Hamiltonian vector field is a vector field associated with a function, more precisely with a derivation of a function. Our objective here is clear: the solution to Hamiltonian equations corresponds to an integral curve, where the derivative at each point is equal to the value of a certain vector field at that point. Unsurprisingly, this vector field is known as the Hamiltonian vector field.

The conventional approach to constructing such vector fields involves defining canonical and symplectic forms, which can be found, for example, in references such as [5] or [1]. However, there exists an alternative definition that is more straightforward in our approach and is often obtained as a corollary of the aforementioned theory: the notion of the Poisson bracket. The Poisson bracket is an operator on functions on cotangent bundle, so we give its definition first.

Let us introduce the covectors, the linear functionals on the tangent space. The space of all covectors at a point $q \in M$, the cotangent space, is just the usual dual space formed by one-forms on the tangent space.

Definition 1.2.1. Let M be a smooth manifold, $q \in M$. By the *cotangent space at* q we understand the set

$$T_q^* M = (T_q M)^* = \{\lambda \colon T_q M \to \mathbb{R},\ \lambda \text{ linear}\}.$$

By $T^*M = \bigcup_{q \in M} T_q^* M$ we denote the *cotangent bundle* of M. Let $\lambda \in T_q^* M$ and let $v \in T_q M$. We denote the evaluation of the covector λ on the vector v by $\langle \lambda, v \rangle = \lambda(v)$.

Remark 1.2.2. The cotangent bundle T^*M is locally isomorphic to $M \times T_q^* M$.

1.2.1 Poisson Bracket

The Lie bracket was defined as an operation on the $\mathrm{Vec}\,(M)$. The Poisson bracket is an analogous operation on the space of smooth functions on T^*M. We start with definition of the operation on the smooth linear functions, $C^\infty_{lin}(T^*M)$, and then we continue with an extension to the whole space $C^\infty(T^*M)$.

We will make use of functions on the cotangent bundle associated with a vector field. Let $X \in \mathrm{Vec}\,(M)$ and let $\varphi \in T^*M$ be arbitrary. Then, by $a_X : T^*M \to \mathbb{R}$ we denote function given by the assignment $\varphi \mapsto \varphi(X)$.

Definition 1.2.3. Let $a_X, a_Y \in C^\infty_{lin}(T^*M)$ be two linear functions associated with vector fields $X, Y \in \mathrm{Vec}\,(M)$. Their *Poisson bracket* is defined by

$$\{a_X, a_Y\} = a_{[X,Y]}, \tag{1.14}$$

where $a_{[X,Y]}$ is the function in $C^\infty_{lin}(T^*M)$ associated with the vector field $[X, Y]$.

Since the Lie bracket is bilinear, skew-symmetric and satisfies the Leibniz rule, as consequence, the Poisson bracket is bilinear, skew-symmetric and satisfies

$$\{a_X, \alpha a_Y\} = \{a_X, a_{\alpha Y}\} = a_{[X,\alpha Y]} = \alpha a_{[X,Y]} + (X\alpha) a_Y, \qquad \forall \alpha \in C^\infty(M). \tag{1.15}$$

Now to the extension.

Theorem 1.2.4. *There exists a unique bilinear and skew-symmetric map*

$$\{\cdot,\cdot\}\colon C^\infty(T^*M)\times C^\infty(T^*M)\to C^\infty(T^*M),$$

that extends 1.2.3 to $C^\infty(T^*M)$ and that is a derivation in each argument. i.e., it satisfies

$$\{a,bc\}=\{a,b\}\,c+\{a,c\}\,b,\qquad \forall a,b,c\in C^\infty(T^*M). \tag{1.16}$$

We call this operation the Poisson bracket *on $C^\infty(T^*M)$.*

Proof. The proof of this proposition consists of two steps. First we extend the Poisson bracket to all smooth affine functions on T^*M, $C^\infty_{aff}(T^*M)$. Using that structure we show the extension to $C^\infty(T^*M)$ in canonical coordinates on T^*M. The proof is rather technical, for the full formulation we recommend [1]. Among other, during the proof we obtain an explicit way to compute the Poisson bracket in canonical coordinates $(x_1,\ldots,x_n,p_1,\ldots,p_n)$. Let $a,b\in C^\infty(T^*M)$, then

$$\{a,b\}=\sum_{i=1}^{n}\frac{\partial a}{\partial p_i}\frac{\partial b}{\partial x_i}-\frac{\partial a}{\partial x_i}\frac{\partial b}{\partial p_i}. \tag{1.17}$$

□

The identity 1.16 provides the expression for the Poisson bracket of a product of smooth functions and another function. Similarly, in the forthcoming statement we will show how the Poisson bracket acts on a composition of functions and another function. This will prove significant later in simplification of the Hamiltonian equations.

Theorem 1.2.5. *Let $h_i\colon T^*M\to\mathbb{R}$, $g\colon T^*M\to\mathbb{R}$ and $\varphi\colon\mathbb{R}\to\mathbb{R}$ be smooth functions. Denote by $\varphi_{h_i}=\varphi\circ h_i$. Then*

$$\{\varphi_{h_i},g\}=\frac{\partial\varphi}{\partial h_i}\{h_i,g\}\,. \tag{1.18}$$

Proof. The proof is very simple. It is sufficient to use 1.17.

$$\{\varphi_{h_i},g\}=\sum_{j=1}^{n}\frac{\partial\varphi_{h_i}}{\partial p_j}\frac{\partial g}{\partial x_j}-\frac{\partial\varphi_{h_i}}{\partial x_j}\frac{\partial g}{\partial p_j}=\sum_{j=1}^{n}\frac{\partial\varphi}{\partial h_i}\frac{\partial h_i}{\partial p_j}\frac{\partial g}{\partial x_j}-\frac{\partial\varphi}{\partial h_i}\frac{\partial h_i}{\partial x_j}\frac{\partial g}{\partial p_j}=\frac{\partial\varphi}{\partial h_i}\{h_i,g\}\,.$$

□

1.2.2 Definition of Hamiltonian Vector Fields

With the aid Poisson bracket, we can finally begin with the construction of Hamiltonian vector field. In general, Hamiltonian vector field $\vec{H}$ is a vector field on the cotangent bundle T^*M associated with a smooth function H on T^*M. Let us introduce the following operator:

$$\vec{H}\colon C^\infty(T^*M) \to C^\infty(T^*M), \qquad \vec{H}(b) = \{H, b\}\,. \tag{1.19}$$

This operator is linear and acts as a derivation on the space of smooth functions $C^\infty(T^*M)$. Therefore $\vec{H}$ can be identified with an element of $\mathrm{Vec}\,(T^*M)$. We continue with the precise definition.

Definition 1.2.6. The vector field $\vec{H}$ defined by 1.19 is called the *Hamiltonian vector field* associated with the smooth function $H \in C^\infty(T^*M)$.

1.3 Lie Group and Algebra

Having introduced the concepts of manifolds, tangent and cotangent bundles, we now possess sufficient theoretical groundwork to delve into control theory. However, one challenge we would encounter in this endeavor is that the tangent bundle TM and $M \times T_qM$ are only locally isomorphic. Similarly, the cotangent bundle T^*M is only locally isomorphic to $M \times T_q^*M$ as well. By narrowing our focus to Lie groups, we can simplify the problem and transform the aforementioned isomorphisms into a global property. This leads to a particularly intriguing outcome: we can solve the problem at the identity element of the Lie group and subsequently, with the aid of the isomorphism, compute the evolution of the system which greatly enhances numerical stability during the computation process.

1.3.1 Lie Group

Definition 1.3.1. A set G is called a *Lie group*, if

1. G is a smooth manifold,
2. G is a group,
3. the group operations in G are smooth.

In our forthcoming examples, we will primarily focus on Lie groups that are matrix groups. Let us denote the linear space of all real $n \times n$ matrices by

$$\mathrm{M}(n, \mathbb{R}) = \{X = (x_{ij}) \mid x_{ij} \in \mathbb{R},\ i, j \in \{1, \ldots, n\}\}\,.$$

Now, we can proceed with the introduction of general linear group, i.e., the most general group.

Example 1.3.2. The *general linear group* consists of all $n \times n$ invertible matrices.

$$\mathrm{GL}(n, \mathbb{R}) = \mathrm{GL}(n) = \{X \in \mathrm{M}(n) \mid \det X \neq 0\}\,.$$

We will show that the general linear group $\mathrm{GL}(n)$ is a Lie group. By the continuity of the determinant $\det\colon \mathrm{M}(n) \to \mathbb{R}$, the set $\mathrm{GL}(n)$ is an open domain. Thus, $\mathrm{GL}(n)$ is a smooth submanifold in the linear space $\mathrm{M}(n)$. From introductory courses of linear algebra it is known that $\mathrm{GL}(n)$ is a group with respect to the matrix product. Since the matrix product at every element are polynomials of elements and inverse are rational functions of elements, the group operations are smooth.

In the following, we will present several significant Lie groups. We highly recommend referring to [5] or [6] for detailed proofs confirming their classification as Lie groups.

Example 1.3.3. The *special orthogonal group* $\mathrm{SO}(n)$ consists of all unimodular orthogonal $n \times n$ matrices, that is

$$\mathrm{SO}(n) = \left\{X \in \mathrm{M}(n) \mid XX^\top = \mathrm{Id},\ \det X = 1\right\}.$$

Example 1.3.4. The *Euclidean group* $\mathrm{SE}(n)$ consists of matrices of the following form:

$$\mathrm{SE}(n) = \left\{X = \begin{pmatrix} Y & b \\ 0 & 1 \end{pmatrix} \in \mathrm{M}(n+1) \mid Y \in \mathrm{SO}(n),\ b \in \mathbb{R}^n\right\}.$$

The extension of our discussion to complex matrices is indeed possible, but for the current context, it is not crucial. Therefore, we will omit delving into that topic. For further details, refer to [5].

1.3.2 Lie Algebra

There are many ways to define the Lie algebra. In some cases the definition introduces the Lie bracket, but we are going to define those notions separately.

Definition 1.3.5. The tangent space L to a Lie group G at the identity element is called the *Lie algebra of the Lie group G*:

$$L = T_{\mathrm{Id}}G.$$

Furthermore, let $A,\ B \in L$. The *Lie bracket* $[A, B]$, given by $[A, B] = AB - BA$, is meant in the sense of Lie bracket of vector fields.

It is evident that the Lie bracket, as we have defined it, is bilinear and skew-symmetric. Moreover, the Jacobi identity also holds, thus Lie bracket on Lie algebra satisfies the axioms of a general Lie bracket.

Now, let us show the Lie algebras of the groups above.

Example 1.3.6. The Lie algebra of the general linear group we denote by $\mathrm{gl}(n)$. In fact, it is possible to show that

$$\mathrm{gl}(n) = \mathrm{M}(n). \tag{1.20}$$

By definition $\mathrm{gl}(n) = \left\{\dot{X}(0) \mid X(t) \in \mathrm{GL}(n), X(0) = \mathrm{Id}\right\}$. Since $\dot{X}(t)$ is an $n \times n$ matrix, $\mathrm{gl}(n) \subset \mathrm{M}(n)$. Now, for small $\varepsilon > 0$ and $|t| < \varepsilon$ and for any $A \in \mathrm{M}(n)$, the curve $X(t) = \mathrm{Id} + tA \in \mathrm{GL}(n)$. Because $\dot{X}(0) = A$ and $X(0) = \mathrm{Id}$, we obtain the equality from above.

Example 1.3.7. The Lie algebra of $\mathrm{SO}(n)$ we denote by $\mathrm{so}(n)$. By definition of the group, $\forall X(t) \in \mathrm{SO}(n)$, $X(t)X^\top(t) = \mathrm{Id}$. Now we apply the definition of the algebra to get

$$0 = \dot{X}(0)X^\top(0) + X(0)\dot{X}^\top(0) = \dot{X}(0) + \dot{X}^\top(0).$$

And we obtain the shape of the algebra

$$\mathrm{so}(n) = \left\{A \in \mathrm{M}(n) \mid A + A^\top = 0\right\}, \tag{1.21}$$

the skew-symmetric matrices.

Exercise 1.3.8 (Lie bracket on $\mathrm{so}(3)$)**.** Consider $A, B \in \mathrm{so}(3)$. Let us write out the elements explicitly as

$$A = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix}.$$

Let us show that $[A, B] \in \mathrm{so}(3)$. To accomplish this, we are going to use 1.12.

$$\begin{aligned}
[A, B] = AB - BA &= \begin{pmatrix} -a_3b_3 - a_2b_2 & a_2b_1 & a_3b_1 \\ a_1b_2 & -a_3b_3 - a_1b_1 & a_3b_2 \\ a_1b_3 & a_2b_3 & -a_2b_2 - a_1b_1 \end{pmatrix} \\
&- \begin{pmatrix} -a_3b_3 - a_2b_2 & a_1b_2 & a_1b_3 \\ a_2b_1 & -a_3b_3 - a_1b_1 & a_2b_3 \\ a_3b_1 & a_3b_2 & -a_2b_2 - a_1b_1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & a_2b_1 - a_1b_2 & a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 & 0 & a_3b_2 - a_2b_3 \\ a_1b_3 - a_3b_1 & a_3b_2 - a_3b_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix} \in \mathrm{so}(3)
\end{aligned}$$

Example 1.3.9. The Lie algebra of Euclidean group is denoted by $\mathrm{se}(n)$. Since we know the structure of $\mathrm{SE}(n)$ and the Lie algebra of $\mathrm{SO}(n)$, the Lie algebra of Euclidean group is rather easy to find:

$$\mathrm{se}(n) = \left\{\mathcal{A} = \begin{pmatrix} A & b \\ 0 & 0 \end{pmatrix} \in \mathrm{M}(n+1) \mid A \in \mathrm{so}(n), b \in \mathbb{R}^n \right\}. \tag{1.22}$$

1.4 Left-Invariant Hamiltonian Systems on Lie Groups

In our formulation of control theory, the concept of left-invariance plays a central role. As mentioned earlier, the presence of left-invariant tangent spaces enables us to solve problems at the identity element of the group. This is facilitated by the global isomorphism between the cotangent bundle T^*G and $G \times L^*$. Additionally, the utilization of left-invariant Hamiltonians provides further simplification of the problems.

1.4.1 Left-Invariant Vector Fields

Definition 1.4.1. By $L_g : G \to G$ we denote *left translation*,

$$L_g(h) = gh, \qquad g,\ h \in G.$$

Theorem 1.4.2. *Let G be a linear Lie group ($G \subset \mathrm{GL}(n)$), L its Lie algebra and let $g \in G$ be arbitrary. Then*

$$T_gG = gT_{\mathrm{Id}}G = gL = \{gA \mid A \in L\}.$$

Proof. First, let us clearify, what do we mean by the multiplication gA, where $g \in G$, $A \in L$. It is a vector field and $\forall h \in G$, $gA(h) = A(gh)$. In fact, this is a tangent map to the left translation. The tangent space T_gG has the following form.

$$T_gG = \{\dot{g}(0) \mid g(t) \in G, g(0) = g\}$$

If $g(t)$ is a smooth curve, we can easily construct curve from the identity

$$Y(t) = g^{-1}g(t), \qquad\qquad Y(0) = g^{-1}g = \mathrm{Id}.$$

Thus $\dot{Y}(0) = g^{-1}\dot{g}(0) \in L$. In conclusion, $T_gG \subset gL$. Since both linear spaces have the same finite dimension, we obtain

$$T_gG = gL.$$

□

Thus we have translation of the tangent space L from the identity to a tangent space gL at an arbitrary point g of the group G. This translation we denote by $L_{g*} : L \to gL$, where $g \in G$. Moreover, $L_{g^{-1}*}$ (resp. L_g^*) trivializes TG (resp. T^*G) to $G \times L$ (resp. $G \times L^*$).

Remark 1.4.3. We've proved the theorem 1.4.2 for a linear Lie group, it is possible to extend this to an arbitrary Lie group (see [1]).

Definition 1.4.4. Let G be a Lie group and let $X \in \mathrm{Vec}\,(G)$. We say that X is *left-invariant* on a Lie group G, if $\forall g,\ h \in G$

$$L_{g*}X(h) = gX(h) = X(gh).$$

Using the notion of the left-invariant vector fields, we can equivalently define Lie algebra as the algebra of the left-invariant vector fields on G endowed with the Lie bracket of vector fields.

1.4.2 Coordinates on TG and T^*G

We introduce *vertical and horizontal coordinates* on TG and T^*G. Though this terminology is somewhat misleading, it is widely recognized, thus we will comply. Indeed, every element of TG can be represented as a pair

$$(g, v),\ g \in G,\ v \in T_gG. \tag{1.23}$$

Since we have just established the notion of the left-invariant vector field and we know that the vector fields of the Lie algebra are left-invariant, we may simplify the coordinates 1.23. Consider a basis $\{e_1, \ldots, e_n\}$ in L. This basis induces global coordinates on TG as the induced basis in T_gG is $\{L_{g*}e_1, \ldots, L_{g*}e_n\}$ and thus the element $(g, v) \in TG$ can be represented as

$$(g, v) = \left(g, \sum_{i=1}^{n} v_i L_{g*} e_i\right). \tag{1.24}$$

The coordinates $v_1, \ldots, v_n$ are called the *vertical coordinates* in TG. The misleading part about this is, that g is then referred to as the *horizontal coordinate* even though it is not a coordinate at all, it is an element of the group G.

Furthermore, the element $(g, v) \in TG$ given by 1.24 may be identified with an element in $G \times L$:

$$(g, \zeta) = \left(g, \sum_{i=1}^{n} v_i e_i\right) \in G \times L.$$

Thus we have an isomorphism between TG and $G \times L$ given by

$$L_{g*}^{-1} : TG \ni (g, v) \mapsto (g, \zeta) \in G \times L, \tag{1.25}$$

with $\zeta = L_{g*}^{-1} v$. So any point in both TG and $G \times T_{\mathrm{Id}}G$ can be represented by coordinates

$$(g, v_1, \ldots, v_n)\,.$$

As this isomorphism extends to the cotangent bundle, the isomorphism between T^*G and $G \times L^*$ is given by L_g^*,

$$T^*G \ni (g, p) \mapsto (g, \xi) \in G \times L^*,$$

with $\xi = L_g^* p$. Let us denote the basis on L^* dual to the basis $\{e_1, \dots, e_n\}$ on L by $\{e_1^*, \dots, e_n^*\}$. The duality here is meant in the usual sense,

$$e_i^*(e_j) = \delta_{ij}.$$

We introduce the vertical coordinates on the dual of Lie algebra $\xi_1, \dots, \xi_n$. Any point then can be represented by coordinates

$$(g, \xi_1, \dots, \xi_n)$$

both in T^*G and $G \times L^*$. The reasoning of this is identical to the one above.

1.4.3 Left-Invariant Hamiltonians

Consider a smooth function $H\colon T^*G \to \mathbb{R}$. We shall refer to such a function as a *Hamiltonian function.* As discussed in subsection 1.4.2, the isomorphism between T^*G and $G \times L^*$ allows us to interpret this function as a function on $G \times L^*$. Let $\mathcal{H}\colon G \times L^* \to \mathbb{R}$ be defined by

$$\mathcal{H}(g, \xi) = H(g, L_{g^{-1}}^* \xi).$$

If $\mathcal{H}$ is independent on g, then H is said to be left-invariant and $\mathcal{H}$ is then called its trivialized Hamiltonian. But we can define those terms equivalently as follows.

Definition 1.4.5. Let $H\colon T^*G \to \mathbb{R}$ be a Hamiltonian function. H is said to be *left-invariant* if there is a function $\mathcal{H}\colon L^* \to \mathbb{R}$ such that

$$H(g, p) = \mathcal{H}(L_g^* p).$$

The function $\mathcal{H}$ is called its *trivialized Hamiltonian.*

Now, let $p = \sum_{i=1}^n \xi_i L_{g^{-1}}^* e_i^*$, then

$$H(g, p) = H\left(g, \sum \xi_i L_{g^{-1}}^* e_i^*\right) = \mathcal{H}\left(L_g^* \sum \xi_i L_{g^{-1}}^* e_i^*\right) = \mathcal{H}\left(\sum \xi_i e_i^*\right).$$

Thus, for left-invariant Hamiltonian we have

$$H(g, \xi_1, \dots, \xi_n) = \mathcal{H}(\xi_1, \dots, \xi_n). \tag{1.26}$$

2 Control Theory on Lie Groups

2.1 Hamiltonian Formalism

The objective of this thesis is to demonstrate the application of the Hamiltonian formalism to Lie groups. In order to provide the necessary background and context for the extended statements, it is relevant to present the formulation of the original problem. While the theory is widely recognized, we will provide a concise overview of the fundamental concepts and propositions. For comprehensive details, we recommend referring to [1].

2.1.1 Hamiltonian Formalism on $\mathbb{R}^n$

The optimal control problem is an optimization problem of the following form

$$\begin{aligned} \dot{x} &= f(x,u), \qquad u \in U \subset \mathbb{R}^m, \\ x(0) &= x_0, \qquad x(t_f) = x_f, \\ J(u) &= \int_0^{t_f} f_0(x(t),u(t))dt \to \min. \end{aligned} \tag{2.1}$$

By U we denote the control region, the function $u(t)$ is the control and the functions $f(x,u)$ and $f_0(x,u)$ are smooth. Moreover, we define the adjoint system for variables $\lambda(t) = (\lambda_1(t),\ldots,\lambda_n(t))^\top$ and finally the Hamiltonian function

$$H(x(t),u(t),\lambda(t),\lambda_0) = \lambda_0 f_0(x(t),u(t)) + \langle \lambda(t); f(x(t),u(t))\rangle,$$

where $\langle \cdot;\cdot\rangle$ is the Euclidean inner product. Using the Hamiltonian function we can rewrite the differential system from 2.1 and the adjoint system into the form

$$\begin{aligned} \dot{x}_i(t) &= \frac{\partial H}{\partial \lambda_i}(x(t),u(t),\lambda(t),\lambda_0),\ i \in \{1,\ldots,n\}, \\ \dot{\lambda}_i(t) &= -\frac{\partial H}{\partial x_i}(x(t),u(t),\lambda(t),\lambda_0),\ i \in \{1,\ldots,n\}. \end{aligned} \tag{2.2}$$

Necessary conditions for optimality of a solution of this system of ODEs is given by the Pontryagin Maximum Principle (PMP).

Theorem 2.1.1 (Pontryagin Maximum Principle)**.** *Let $\hat{u}(t),\ t \in \langle 0,\hat{t}_f\rangle$, be a solution of the problem 2.1 and let $\hat{x}(t),\ t \in \langle 0,\hat{t}_f\rangle$, be the corresponding optimal trajectory. Then there exists non-positive constant λ_0 and non-zero continuous solution $\lambda(t)$ of the adjoint system*

$$\dot{\lambda}_i(t) = -\frac{\partial H}{\partial x_i}\left(\hat{x}(t),\hat{u}(t),\lambda(t),\lambda_0\right), \qquad \forall i \in \{1,\ldots,n\},\ \forall t \in \langle 0,\hat{t}_f\rangle,$$

such that the Hamiltonian function satisfies the maximum condition

$$\max_{u \in U} H\left(\hat{x}(t), u, \lambda(t), \lambda_0\right) = H\left(\hat{x}(t), \hat{u}(t), \lambda(t), \lambda_0\right), \qquad \forall t \in \langle 0, \hat{t}_f \rangle.$$

Moreover, $H\left(\hat{x}(t), \hat{u}(t), \lambda(t), \lambda_0\right) \equiv 0, \qquad \forall t \in \langle 0, \hat{t}_f \rangle.$

Theorem 2.1.1 is applicable to problems with free time, where the right endpoint of the interval $\langle 0, t_f \rangle$ is considered as part of the solution. Nevertheless, there exists a version of this theorem for problems with fixed time. This version is structurally similar, with the main distinction being that the Hamiltonian is generally constant along the optimal trajectory and the optimal control rather than zero.

2.1.2 Hamiltonian Formalism on Smooth Manifolds

In [1], the reformulation of the original problem 2.1 on general smooth manifold M is presented. For the sake of completeness, we will also state it here. Consider the following optimal control problem

$$\begin{aligned} \dot{q} &= f(q, u), \qquad q \in M,\ u \in U \subset \mathbb{R}^m, \\ q(0) &= q_0, \qquad q(t_f) = q_f, \\ J(u) &= \int_0^{t_f} \varphi(q(t), u(t))dt \to \min. \end{aligned} \tag{2.3}$$

Here $\varphi : M \times U \to \mathbb{R}$, $f : M \times U \to TM$. Furthermore, let $\lambda \in T^*M$ be a covector, $\nu \in \mathbb{R}$ a parameter and $u \in U$ a control parameter. The Hamiltonian $h_u^\nu : T^*M \to \mathbb{R}$ is defined as follows

$$h_u^\nu(\lambda) = \langle \lambda; f(q, u) \rangle + \nu\varphi(q, u). \tag{2.4}$$

Since the Hamiltonian is a smooth function on T^*M, we can associate a vector field $\vec{h}^\nu_{\hat{u}(t)} \in \operatorname{Vec}(T^*M)$ given by the Poisson bracket, as we have shown in 1.19. The PMP for this problem is stated next, its proof can be found in [1].

Theorem 2.1.2 (PMP on smooth manifolds)**.** *Let $\hat{u}(t),\ t \in \langle 0, \hat{t}_f \rangle$, be a solution of the problem 2.3. Then there exists non-positive constant ν and a Lipschitzian curve $\lambda_t \in T^*_{q(t)}M,\ t \in \langle 0, t_f \rangle$ such that $\forall t \in \langle 0, t_f \rangle$*

$$\begin{aligned} \dot{\lambda}_t &= \vec{h}^\nu_{\hat{u}(t)}(\lambda_t), \\ \max_{u \in U} h_u^\nu(\lambda_t) &= h^\nu_{\hat{u}(t)}(\lambda_t), \\ (\lambda_t, \nu) &\neq (0, 0). \end{aligned} \tag{2.5}$$

Moreover, $h^\nu_{\hat{u}(t)}(\lambda_t) \equiv 0, \qquad \forall t \in \langle 0, \hat{t}_f \rangle.$

From there directly follow the Hamiltonian equations on smooth manifold. Since the cotangent bundle T^*M is isomorphic to $M \times T^*_x M$ in some neighborhood of point x, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(x, p) = \vec{H}(x, p), \tag{2.6}$$

where $x \in M$ is position in the manifold, $p = (p_1, \ldots, p_n) \in T^*_x M$ and the vector field $\vec{H} \in \mathrm{Vec}(T^*M)$ is the Hamiltonian vector field associated with some Hamiltonian function H. For the case of canonical coordinates $(x_1, \ldots, x_n, p_1, \ldots, p_n)$ on M, we obtain the Hamiltonian equations described in 2.2:

$$\dot{x}_i = \partial_{p_i} H,$$
$$\dot{p}_i = -\partial_{x_i} H.$$

2.1.3 Hamiltonian Equations on Lie Group

As was said in 1.1, we could solve problems of control theory using 2.6. But the important thing why we want to study Hamiltonian formalism on Lie groups is, that for general smooth manifold M, the isomorphism $T^*M \cong M \times T^*_q M$ holds only locally, but $T^*G \cong G \times L^*$ is a global property in the case of Lie group G and its Lie algebra L. That is very useful simplification. Finally, we present the theorem showing the form of Hamiltonian equations on Lie groups.

Theorem 2.1.3. *Let $H : T^*G \to \mathbb{R}$ be a left-invariant Hamiltonian on a Lie group G, $\mathcal{H} : L^* \to \mathbb{R}$ its triavialized Hamiltonian and $(g, \xi) \in T^*G = G \times L^*$. Moreover, let $\mathrm{d}\mathcal{H}$ be the differential of $\mathcal{H}$ seen as an element of L. Then the Hamiltonian equations 2.6, with $p = L^*_{g^{-1}}\xi$, may be expressed in the following form*

$$\dot{g} = L_{g*}\,\mathrm{d}\mathcal{H}, \tag{2.7}$$
$$\dot{\xi} = (\mathrm{ad}\ \mathrm{d}\mathcal{H})^*\xi. \tag{2.8}$$

As a reminder, L^*_g is the left translation on the cotangent bundle, L_{g*} is the left translation on the tangent bundle. In the following subsection we will prove the equations 2.8 and 2.7 respectively.

2.1.4 Proof of Hamiltonian Equations

We begin with the proof of the vertical part, 2.8. We have $\xi = \sum_{i=1}^n \xi_i e^*_i$. Utilizing the equation 2.6, left-invariance of the cotangent bundle and definition of the Hamiltonian vector field, we obtain

$$\dot{\xi}_i = \{H, \xi_i\}, \qquad i \in \{1, \ldots, n\}, \tag{2.9}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket. However, it is possible to further manipulate this equation. We employ 1.2.5, the Poisson bracket of composition of functions, and the relationship between Poisson and Lie brackets (1.14) for $i \in \{1,\dots,n\}$:

$$\dot{\xi}_i = \sum_{j=1}^{n} \frac{\partial H}{\partial \xi_j} \{\xi_j, \xi_i\} = \sum_{j=1}^{n} \frac{\partial H}{\partial \xi_j} \langle \xi, [e_j, e_i] \rangle = \left\langle \xi, \left[\sum_{j=1}^{n} \frac{\partial H}{\partial \xi_j} e_j, e_i \right] \right\rangle. \tag{2.10}$$

Consider the trivialized Hamiltonian $\mathcal{H}$. Since it is a function on the linear space L^*, then $\mathrm{d}\mathcal{H}(\xi_1,\dots,\xi_n)$ is an element of $(L^*)^* = L$ thanks to the linear structure. For an element $\xi_1 e_1^* + \dots + \xi_n e_n^* \in L^*$, the element of its tangent space at $(\xi_1,\dots,\xi_n)$ is $v_1 \partial_{\xi_1} + \dots + v_n \partial_{\xi_n}$, with $\partial_{\xi_i} = e_i^*$ thanks to the linear structure. The element of its cotangent space $(L^*)^*$ at $(\xi_1,\dots,\xi_n)$ is $\omega_1 \,\mathrm{d}\xi_1 + \dots + \omega_n \,\mathrm{d}\xi_n$, with $\mathrm{d}\xi_i = (e_i^*)^* = e_i$ once more thanks to the linear structure. Then we obtain

$$\mathrm{d}\mathcal{H}(\xi_1,\dots,\xi_n) = \sum_{j=1}^{n} \frac{\partial \mathcal{H}}{\partial \xi_j} \,\mathrm{d}\xi_j = \sum_{j=1}^{n} \frac{\partial \mathcal{H}}{\partial \xi_j} e_j = \sum_{j=1}^{n} \frac{\partial H}{\partial \xi_j} e_j. \tag{2.11}$$

If we apply 2.11 to 2.10, we obtain

$$\begin{aligned} \dot{h}_i &= \langle \xi, [\mathrm{d}\mathcal{H}, e_i] \rangle \\ &= \langle \xi, (\mathrm{ad}\ \mathrm{d}\mathcal{H}) e_i \rangle \\ &= \langle (\mathrm{ad}\ \mathrm{d}\mathcal{H})^* \xi, e_i \rangle \end{aligned} \tag{2.12}$$

or equivalently

$$\dot{\xi} = (\mathrm{ad}\ \mathrm{d}\mathcal{H})^* \xi, \tag{2.13}$$

which is exactly 2.8.

The proof of 2.7 is simpler. Consider function $\beta \in \mathcal{C}^\infty(T^*G)$ that is constant on the vertical fibers. This basically means that $\beta \in \mathcal{C}^\infty(G)$. Now, for every solution of the horizontal part of the system associated with H, represented by curves $g(\cdot)$, we obtain (by 1.2.5)

$$\frac{\mathrm{d}}{\mathrm{d}t} \beta(g(t)) = \{H, \beta\}_{(p(t),g(t))} = \sum_{j=1}^{n} \frac{\partial H}{\partial \xi_j} \{\xi_j, \beta\}_{(p(t),g(t))}.$$

Denote by $X_j = L_{g*} e_j$ the translation of j-th base vector of L. Moreover, utilizing an identity for affine functions on L, $\{a_X + \alpha, a_y + \beta\} = a_{[X,Y]} + X\beta - Y\alpha$, we obtain $\{\langle p, X_j \rangle, \beta\} = X_j \beta$. But we have $\{\xi_j, \beta\} = \{\langle p, X_j \rangle, \beta\} = X_j \beta = (L_{g*} e_j)\beta$. Thus

$$\frac{\mathrm{d}}{\mathrm{d}t} \beta(g(t)) = \sum_{j=1}^{n} \frac{\partial H}{\partial \xi_j} (L_{g*} e_j) \beta \bigg|_{g(t)} = \left(L_{g*} \sum_{j=1}^{n} \frac{\partial H}{\partial \xi_j} e_j \right) \beta \bigg|_{g(t)} = (L_{g*}\, \mathrm{d}\mathcal{H}) \beta|_{g(t)}.$$

Since β is arbitrary function, we get

$$\dot{g} = L_{g*}\, \mathrm{d}\mathcal{H}. \tag{2.14}$$

Thus, the theorem 2.1.3 has been proved.

2.1.5 The Case of Compact Lie Groups

The system 2.7, 2.8 can be further simplified in the case of a compact Lie group G. This is mentioned in [5] and derived for $\mathrm{SO}(n)$ (which is compact), we will paraphrase. In the case of compact Lie group, there is an invariant inner product g on $\mathbb{R}^n$, i.e.,

$$g(Xu, Xv) = g(u, v), \qquad \forall X \in G,\ \forall u, v \in \mathbb{R}^n.$$

For example, the invariant inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathrm{so}(n)$ has form

$$\langle A, B \rangle = -\operatorname{tr}(AB)$$

and the invariance is meant in the following way:

$$\langle e^{t\,\mathrm{ad}\,C} A, e^{t\,\mathrm{ad}\,C} B \rangle = \langle A, B \rangle, \qquad \forall A,\ B,\ C \in \mathrm{so}(n),\ \forall t \in \mathbb{R}. \tag{2.15}$$

That is, the operator $e^{t\,\mathrm{ad}\,C}$ is orthogonal and can be expressed as $e^{t\,\mathrm{ad}\,C} = e^{tC} A e^{-tC}$. The infinitesimal version of 2.15 is obtained by differentiation with respect to t at $t = 0$:

$$\langle \mathrm{ad}\,C(A), B \rangle + \langle A, \mathrm{ad}\,C(B) \rangle = 0, \qquad \forall A, B, C \in \mathrm{so}(n).$$

Thus, the operator $\mathrm{ad}\,C$ is skew-symmetric. As a consequence, the Lie algebra is endowed with invariant scalar product which allows us to construct the canonical map between the algebra L and its dual L^*:

$$A \leftrightarrow \tilde{A} = \langle A, \cdot \rangle, \qquad A \in L, \tilde{A} \in L^*.$$

Thus we have a way of representing the coadjoint operator in equation 2.8 by an element from the Lie algebra itself. By direct computation it can be shown that $(\mathrm{ad}\,A)^* : L^* \to L^*$ is identified with $(-\,\mathrm{ad}\,A) : L \to L$.

$$\begin{aligned}\left((\mathrm{ad}\,A)^* \tilde{B}\right)(C) &= \tilde{B}\left((\mathrm{ad}\,A)\,C\right) = \langle B, (\mathrm{ad}\,A)\,C \rangle = -\langle (\mathrm{ad}\,A)\,B, C \rangle \\ &= -(\widetilde{(\mathrm{ad}\,A)B})(C)\end{aligned}$$

Thus, in case of compact Lie group the system 2.7, 2.8 can be expressed in the following form:

$$\begin{aligned}\dot{g} &= g\,\mathrm{d}\mathcal{H}, \\ \dot{\xi} &= -(ad\ \mathrm{d}\mathcal{H})\xi = [\xi, \mathrm{d}\mathcal{H}]\,.\end{aligned} \tag{2.16}$$

3 Matrix Lie Group SO(4, 1) and CGA

In this chapter, we will utilize two distinct representations of the Lie group $\mathrm{SO}(4,1)$: the matrix representation and the Conformal Geometric Algebra (CGA), as defined in [7]. Our approach involves presenting the matrix representation and subsequently demonstrating the isomorphism between the Lie algebra and bivectors of the geometric algebra.

3.1 Indefinite Special Orthogonal Group and its Lie Algebra

In next chapter will follow solved problems on groups $\mathrm{SO}(3)$, $\mathrm{SE}(3)$ and $\mathrm{SO}(4,1)$. We have introduced the groups $\mathrm{SO}(3)$, $\mathrm{SE}(3)$ in 1.3.3 and 1.3.4 respectively. The general group $\mathrm{SO}(p,q)$ is called *indefinite special orthogonal group*. And as a matter of fact $\mathrm{SO}(4,1)$ contains both $\mathrm{SO}(3)$ and $\mathrm{SE}(3)$ as its subgroups.

3.1.1 Definitions of the Structures

Definition 3.1.1. Let $B \in M(p+q)$ with $p, q \in \mathbb{N}$ be a matrix of symmetric non-degenerate bilinear form of signature (p,q). Then the set

$$\mathrm{SO}(p,q) = \{A \in M(p+q) |\ A^\top BA = B,\ \det A = 1\} \tag{3.1}$$

is called *indefinite special orthogonal group.*

Theorem 3.1.2. *The indefinite special ortogonal group (endowed with matrix multiplication and inverse) is a Lie group.*

Proof. The proof is a corollary of the Cartan's closed subgroup theorem ([3]) which states that if H is a closed subgroup of a Lie group G then H is a Lie group as well. Since elements of $\mathrm{SO}(p,q)$ are regular matrices, i.e., $\mathrm{SO}(p,q) \subseteq \mathrm{GL}(p+q)$, it suffices to show, that $\mathrm{SO}(p,q)$ is closed under matrix multiplication and inversion.

$$X, Y \in \mathrm{SO}(p,q) \Rightarrow B = Y^\top X^\top BXY = (XY)^\top BXY \Rightarrow XY \in \mathrm{SO}(p,q),$$
$$X \in \mathrm{SO}(p,q) \Rightarrow X^\top BX = B \Rightarrow B = \left(X^{-1}\right)^\top BX^{-1} \Rightarrow X^{-1} \in \mathrm{SO}(p,q).$$

□

Let us show a property of $\mathrm{SO}(p,q)$ that can serve as an alternative definition of $\mathrm{SO}(p,q)$. Consider the inner product $\mathcal{B} : \mathbb{R}^{(p+q)} \times \mathbb{R}^{(p+q)} \to \mathbb{R}$, given by a bilinear form $\mathcal{B}(x,y) = x^\top By$, then

$$\forall A \in \mathrm{SO}(p,q) : \mathcal{B}(x,y) = \mathcal{B}(Ax, Ay).$$

In other words, the inner product $\mathcal{B}$ remains invariant under the transformation of the vectors x and y by A. This reinforces the concept of an invariant inner product and its preservation under transformations by elements of the indefinite special orthogonal group $\mathrm{SO}(p,q)$. Now, let us turn our attention to the Lie algebra of this group. We can construct the algebra directly using definition 1.3.5,

$$\forall X \in \mathrm{SO}(p,q)\colon 0 = \dot{X}^\top(0)BX(0) + X^\top(0)B\dot{X}(0) = \dot{X}^\top(0)B + B\dot{X}(0).$$

Thus, the Lie algebra of $\mathrm{SO}(p,q)$ is the following matrix algebra.

$$\mathrm{so}(p,q) = \{A \in M(p+q)|\ A^\top B = -BA\} \tag{3.2}$$

3.1.2 Lie Group SO(4, 1)

At the outset of this section, we expressed our intention to investigate the Hamiltonian formalism within the group $\mathrm{SO}(4,1)$. There are a couple of important considerations to be made. Given the definition, the most common and widely used form that naturally arises is

$$\mathrm{SO}(4,1) = \{A \in M(5)|\ A^\top BA = B,\ \det A = 1,\ B = \mathrm{diag}(1,1,1,1,-1)\}. \tag{3.3}$$

Solving the equation 3.2 for this bilinear form, we obtain Lie algebra of the following structure

$$\mathrm{so}(4,1) = \left\{ \begin{pmatrix} 0 & a & b & d & g \\ -a & 0 & c & e & h \\ -b & -c & 0 & f & i \\ -d & -e & -f & 0 & j \\ g & h & i & j & 0 \end{pmatrix}, a,b,c,d,e,f,g,h,i,j \in \mathbb{R} \right\}. \tag{3.4}$$

But for reasons we will discuss later on, we are going to choose the matrix B as follows.

$$B = \begin{pmatrix} 0 & & -1 \\ & \mathbb{1}_3 & \\ -1 & & 0 \end{pmatrix} \tag{3.5}$$

The Lie algebra is then a matrix algebra of the following form:

$$\mathrm{so}(4,1) = \left\{ \sum_{i=1}^{10} a_i E_i = \begin{pmatrix} a_{10} & a_7 & a_8 & a_9 & 0 \\ a_4 & 0 & -a_3 & a_2 & a_7 \\ a_5 & a_3 & 0 & -a_1 & a_8 \\ a_6 & -a_2 & a_1 & 0 & a_9 \\ 0 & a_4 & a_5 & a_6 & -a_{10} \end{pmatrix}, a_i \in \mathbb{R} \right\}. \tag{3.6}$$

The indices in the matrix show the base we will assume in the sequel.

3.2 Introduction to CGA

We wish to express the Hamiltonian equations in the language of CGA. The motivation for that we will explain in due time. As a quick introduction, CGA is graded space of outer products over basis of vectors. There are two usual choices in the terms of the vector basis. The first one would be $\{e_1,\ e_2,\ e_3,\ e_+,\ e_-\}$ which corresponds to the matrix B given by 3.3. The second one, and the one we will assume, corresponds to the matrix B from 3.5. We denote the basis by $\{e_1,\ e_2,\ e_3,\ e_0,\ e_\infty\}$. The outer product of k vectors $e_1 \wedge \cdots \wedge e_k$ is called k-*blade*, linear combinations of k-blades are called k-vectors (we will frequently use terms *bivectors* for $k = 2$ and *trivectors* for $k = 3$). The linear space of k-vectors of CGA we denote by $\bigwedge^k \mathbb{R}^5$. Linear combinations of general k-vectors and l-vectors we call *multivectors*.

The essential operation on CGA is the *geometric product* (usually denoted by juxtaposition). Mathematically, geometric product of two vectors a and b is sum of their inner product and outer product

$$ab = a \cdot b + a \wedge b, \tag{3.7}$$

thus it is a map from $\mathbb{R}^5 \times \mathbb{R}^5$ to $\mathbb{R} \cup \bigwedge^2 \mathbb{R}^5$ – multivectors. Geometric product of the general blades can be defined in similar way, but the role of inner product is played by *contractions*. For complete definitions we advise to consult [7] or [8].

We will also make use of Projective Geometric Algebra. Its basis for 3D space is $\{e_0,\ e_1,\ e_2,\ e_3\}$ and thus PGA can be seen as part of CGA.

3.3 Dual Space of Vectors of CGA

The bivectors of CGA are isomorphic to the Lie algebra $\mathrm{so}(4,1)$. However, before we approach the isomorphism itself, we have to mention dual space of vectors of CGA.

The basis of one forms of CGA are determined by a bilinear form:

$$e_i^*(e_j) = b_{ij},$$

where e_j is a basis vector, e_i^* is a basis one form and b_{ij} are elements of the matrix B from the definition 3.1.1. There comes in play our choice of the form of the matrix B. With that said, we can express the isomorphism between base vectors and their duals.

$$e_0^* \mapsto -e_\infty \qquad e_i^* \mapsto e_i \qquad e_\infty^* \mapsto -e_0, \qquad i \in \{1,2,3\} \tag{3.8}$$

For the case of PGA, recall that we have basis $\{e_0,\ e_1,\ e_2,\ e_3\}$. From the isomorphism we have described above we obtain, that the dual space of vectors of PGA has basis $\{e_1,\ e_2,\ e_3,\ e_\infty\}$. But this space is isomorphic to the trivectors of PGA. The isomorphism of vectors of PGA to trivectors of PGA is called *Poincaré duality* and in this case can be defined as $e_i \wedge e_i^* = e_{0123}$.

3.4 Isomorphism between Bivectors of CGA and so(4, 1)

Since both bivectors of CGA and the Lie algebra $\mathrm{so}(4,1)$ are vector spaces of the same finite dimension ($\dim \mathrm{so}(4,1) = 10$), there has to be an isomorphism of vector spaces. However, it turns out that there is even an isomorphism of algebras between the Lie group $\mathrm{so}(4,1)$ endowed with matrix commutator and bivectors of CGA endowed with a version of the commutator for CGA. We construct this isomorphism in the following way. An element of $\mathrm{so}(4,1)$ is a linear map on $\mathbb{R}^5$, i.e., $\forall A \in \mathrm{so}(4,1), A : \mathbb{R}^5 \to \mathbb{R}^5$. Obviously, $\mathbb{R}^5$ is a vector space of finite dimension and thus we can associate A with an element of the space $(\mathbb{R}^5)^* \otimes \mathbb{R}^5$, where $\otimes$ is a tensor product. Since $\mathbb{R}^5$ is a subspace of CGA, the tensor product coincides with the geometric product of the algebra which can be expressed as the outer product. The isomorphism is defined by the following map for the base matrix e_{ij} of $\mathrm{M}(5)$ that has all elements zero apart from the element at i-th row and j-th column which is equal to one:

$$e_{ij} \mapsto -\frac{1}{2} e_j^* \otimes e_i. \tag{3.9}$$

As a final touch, we will express this map for the basis matrices E_i of $\mathrm{so}(4,1)$, which were introduced in 3.6, using the isomorphism between dual space of vectors and vectors of CGA given by 3.8.

$$\begin{aligned}
E_1 &\mapsto -\frac{1}{2} e_2^* \wedge e_3 \mapsto -\frac{1}{2} e_2 \wedge e_3 \\
E_2 &\mapsto -\frac{1}{2} e_3^* \wedge e_1 \mapsto -\frac{1}{2} e_3 \wedge e_1 \\
E_3 &\mapsto -\frac{1}{2} e_1^* \wedge e_2 \mapsto -\frac{1}{2} e_1 \wedge e_2 \\
E_{i+3} &\mapsto -\frac{1}{2} e_0^* \wedge e_i \mapsto -\frac{1}{2} e_i \wedge e_\infty, \qquad i \in \{1,2,3\} \\
E_{i+6} &\mapsto -\frac{1}{2} e_i^* \wedge e_0 \mapsto \frac{1}{2} e_0 \wedge e_i, \qquad i \in \{1,2,3\} \\
E_{10} &\mapsto -\frac{1}{2} e_0^* \wedge e_0 \mapsto -\frac{1}{2} e_0 \wedge e_\infty
\end{aligned} \tag{3.10}$$

4 Examples of Hamiltonian Formalism Problems

In this chapter, we will explore several problems related to left-invariant systems on Lie groups. We have set out with the following vision: the equations of rigid body motion have been extensively studied and are widely understood. The modern approach is to solve them employing Hamilton's principle of least action which leads to the Lagrangian formulation, formulation of the problem as an optimization problem on $\mathbb{R}^3$ in the context of control theory. The Lagrangian formulation then can be transformed into the Hamiltonian formulation as an optimal control problem on Lie group SE(3) and further analyzed using the Pontryagin Maximum Principle. Indeed, in our study, we have focused on the examination of left-invariant problems with left-invariant Hamiltonian functions on Lie groups. By adopting this perspective, we can simplify the equations of the Hamiltonian formalism on smooth manifolds. This approach allows us to exploit the inherent structure of Lie groups, leading to a more streamlined analysis and understanding of the underlying dynamics. The aim of this subsection is to establish connection between this mathematical formulation and the real world problems studied in physics. Moreover, we will try to answer what applications have the simplified equations for various choices of Lie group and Hamiltonian function.

As mentioned previously, the Hamiltonian formalism and the Lagrangian formalism are closely interconnected. Many texts in this field often begin by defining the Lagrangian function and then deriving the Hamiltonian function from it. This preference might be attributed to the fact that the Lagrangian serves as the integrand of the minimized of the problem 2.3. Consequently, the solution of the Hamiltonian equations minimizes the Lagrangian function. However, in our approach, we start with the concepts of the Hamiltonian formalism. Nevertheless, we recognize the significance of the Lagrangian function and will provide its expression whenever possible. For a Lagrangian function denoted as $L(g, \omega)$ and a Hamiltonian function denoted as $H(g, p)$, the following identities generally hold true:

$$L(g, \omega) = \omega p - H(g, p),$$
$$\omega = \frac{\partial H}{\partial p}.$$

However, in the case of a left-invariant system and left-invariant Hamiltonian function, they take the following form:

$$L(\omega) = \sum \omega_i \xi_i - \mathcal{H}(\xi), \tag{4.1}$$
$$\omega_i = \frac{\partial \mathcal{H}}{\partial \xi_i}. \tag{4.2}$$

This form is precisely derived in [9], let us outline the idea. Essentially, the trivialized Hamiltonian corresponds to the reduced Lagrangian $L(\omega)$. The function ω may be obtained by the means of inverted Legendre transformation. The Lagrangian may be then expressed directly from the equation 2.4.

In the subsequent sections, we will discuss problems on SO(3), SE(3) and SO(4, 1). For each group, we will initially present the general structure of the coadjoint operator, followed by the presentation of three specific problems. We will express the Hamiltonian function in the form of a quadratic form, a linear form, or a combination thereof. The general form of the coadjoint operator will be particularly useful in the second and third subsections, enabling us to represent the equations in a more concise manner.

4.1 The Coadjoint Operator

Let us regard the expression of the coadjoint operator. Now, consider a Lie group G, its Lie algebra L, $\xi \in T^*_{\mathrm{Id}}G$, $A,\ B \in L$. Then,

$$\langle \xi, [A, B] \rangle = \langle \xi, (\mathrm{ad}\, A)\, B \rangle = \langle (\mathrm{ad}\, A)^* \, \xi, B \rangle.$$

Expressing the operators $\mathrm{ad}\, A$ and $(\mathrm{ad}\, A)^*$ as $n \times n$ matrices can be achieved through straightforward manipulation of the elements. This is facilitated by representing ξ as a row vector in basis $\{e_1^* \dots, e_n^*\}$, and representing B as a column vector in basis $\{e_1, \dots, e_n\}$. Thus,

$$\xi^\top (\mathrm{ad}\, A)\, B = \left((\mathrm{ad}\, A)^\top \xi \right)^\top B = ((\mathrm{ad}\, A)^* \, \xi)^\top B \quad \Rightarrow \quad (\mathrm{ad}\, A)^* = (\mathrm{ad}\, A)^\top . \tag{4.3}$$

4.2 Hamiltonian Formalism on SO(3)

The general form of the coadjoint operator on SO(3) is rather easy to find. Let $A = \sum_{i=1}^{3} a_i e_i$, $B = \sum_{i=1}^{3} b_i e_i$. In exercise 1.3.8 we have obtained

$$[A, B] = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

thus the coadjoint operator $(\mathrm{ad}\, A)^*$ has the form

$$(\mathrm{ad}\, A)^* = \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix}. \tag{4.4}$$

4.2.1 Hamiltonian as a Quadratic Form on $\mathrm{so}^*(3)$

Let us begin by presenting the form of the Hamiltonian function under consideration, denoted as $H(R,p)$. Our objective is to utilize the trivialized form of Hamiltonian functions, so it's important to clarify how we represent them. By definition, the trivialized Hamiltonian function is expressed as $\mathcal{H}(\xi) = H(R,p)$, where $\forall R \in \mathrm{SO}(3)$ and $\xi = Rp$. Furthermore, since $\xi = \xi_1 e_1^* + \xi_2 e_2^* + \xi_3 e_3^*$, we can restate the trivialized Hamiltonian function as a function of its coordinates, $\mathcal{H}(\xi) = \mathcal{H}(\xi_1,\xi_2,\xi_3)$. Hence, we obtain the final form of the Hamiltonian function for this subsection:

$$\mathcal{H}(\xi_1,\xi_2,\xi_3) = c_1\xi_1^2 + c_2\xi_2^2 + c_3\xi_3^2, \tag{4.5}$$

where ξ_1,ξ_2,ξ_3 are the coordinate functions. Taking $\xi = (\xi_1,\xi_2,\xi_3)^\top$, we can rewrite the Hamiltonian in the following form.

$$\mathcal{H}(\xi) = \xi^\top C\xi \tag{4.6}$$

The matrix of this quadratic form

$$C = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}$$

is for simplicity's sake considered as a diagonal matrix, obviously, for choice of arbitrary symmetric and regular matrix, we could transform the basis of $\mathrm{so}^*(3)$, so that the matrix of quadratic form would be diagonal.

As for the Lagrangian function $L(\omega)$, it can be derived from the Hamiltonian function using the identity 4.2:

$$\omega = \frac{\partial\mathcal{H}}{\partial\xi} = 2C\xi.$$

Thus using 4.1 and symmetry of the matrix of the quadratic form,

$$L(\omega) = \omega^\top\frac{1}{2}C^{-1}\omega - \frac{1}{2}\omega^\top\frac{1}{2}C^{-1}\omega = \frac{1}{4}\omega^\top C^{-1}\omega.$$

Let us present couple of applications arising from the specific choice of the matrix C.

- For $c_i \neq 0,\ i \in \{1,\ 2,\ 3\}$, the Lagrangian has form $L(\omega) = \frac{1}{4}\sum_{i=1}^3 c_i\omega_i^2$, which is exactly Lagrangian for geodesics on SO(3), thus the solution relates to minimization of length of a curve and we obtain equations for geodesics.
- If some coefficients are zero, e.g., $L(\omega) = c_1\omega_1^2$, the problem relates to sub-Riemannian geodesics. That is, we minimize length of the curve on a substructure of SO(3). Since sub-Riemannian theory is yet another challenging field of mathematical theory, we will not pursue further exploration in this direction.

- For the specific choice $c_i = \frac{1}{2I_i}$, the Lagrangian $L(\omega) = \frac{1}{2}\sum_{i=1}^{3}\frac{\omega_i^2}{I_i}$ is related to free rigid body rotations, i.e., rotation of rigid body without influence of forces. The matrix $J = \frac{1}{2}\,\mathrm{diag}(I_1,\ I_2,\ I_3)$ is the inertia matrix of the rigid body.

And now, let us formulate the equations of Hamiltonian formalism 2.7, 2.8 for the case of rigid body rotations. First, we need to find the differential of the Hamiltonian function $\mathrm{d}\mathcal{H}$.

$$\mathrm{d}\mathcal{H}(\xi_1,\xi_2,\xi_3) = \frac{\xi_1}{I_1}e_1 + \frac{\xi_2}{I_2}e_2 + \frac{\xi_3}{I_3}e_3$$

This implies that the adjoint operator in 2.8 has the form

$$(\mathrm{ad}\,\mathrm{d}\mathcal{H})^* = \begin{pmatrix} 0 & \xi_3/I_3 & -\xi_2/I_2 \\ -\xi_3/I_3 & 0 & \xi_1/I_1 \\ \xi_2/I_2 & -\xi_1/I_1 & 0 \end{pmatrix}. \tag{4.7}$$

The horizontal equation is of form

$$\dot{g} = g\,\mathrm{d}\mathcal{H} = g\left(\frac{\xi_1}{I_1}e_1 + \frac{\xi_2}{I_2}e_2 + \frac{\xi_3}{I_3}e_3\right). \tag{4.8}$$

The vertical equation we will list using coordinates, as it is a bit clearer for a human reader.

$$\dot{\xi}_1 = \xi\left([\mathrm{d}\mathcal{H}, e_1]\right) = \xi\left(-\frac{\xi_2}{I_2}e_3 + \frac{\xi_3}{I_3}e_2\right) = \left(\frac{1}{I_3} - \frac{1}{I_2}\right)\xi_2\xi_3 \tag{4.9}$$

$$\dot{\xi}_2 = \xi\left([\mathrm{d}\mathcal{H}, e_2]\right) = \xi\left(-\frac{\xi_3}{I_3}e_1 + \frac{\xi_1}{I_1}e_3\right) = \left(\frac{1}{I_1} - \frac{1}{I_3}\right)\xi_1\xi_3 \tag{4.10}$$

$$\dot{\xi}_3 = \xi\left([\mathrm{d}\mathcal{H}, e_3]\right) = \xi\left(-\frac{\xi_1}{I_1}e_2 + \frac{\xi_2}{I_2}e_1\right) = \left(\frac{1}{I_2} - \frac{1}{I_1}\right)\xi_1\xi_2 \tag{4.11}$$

As can be observed, the vertical part of the equations are decoupled from the horizontal part. The evolution on the cotangent bundle is independent on the position in the Lie group. Moreover, the horizontal part can be very easily solved. As $\mathrm{d}\mathcal{H} \in \mathrm{so}(3)$, the solution of this equation can be written as follows.

$$g = A \cdot \exp\left\{t\left(\frac{\xi_1}{I_1}e_1 + \frac{\xi_2}{I_2}e_2 + \frac{\xi_3}{I_3}e_3\right)\right\}, \qquad A \in \mathrm{so}(3), \tag{4.12}$$

on some open interval $t \in (0, t_f)$.

4.2.2 Hamiltonian as a Linear Form on $\mathrm{so}^*(3)$

As in the previous subsection we start with definition of the Hamiltonian function. This time however, we consider

$$\mathcal{H}(\xi) = \mathcal{H}(\xi_1,\xi_2,\xi_3) = a_1\xi_1 + a_2\xi_2 + a_3\xi_3, \qquad a_1, a_2, a_3 \in \mathbb{R}. \tag{4.13}$$

The differential of the Hamiltonian function is constant. Intuitively this does not seem very useful. The first complication arises as we try to formulate the Lagrangian. As

$$\omega = \frac{\partial \mathcal{H}}{\partial \xi} = \vec{a}, \qquad \vec{a} = (a_1, a_2, a_3)^\top$$

is also constant, we cannot express ω as a function of ξ and vice versa. Thus we cannot find the Lagrangian.

The adjoint operator has the form

$$(\operatorname{ad} \mathrm{d}\mathcal{H})^* = \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix}. \tag{4.14}$$

We can construct the Hamiltonian equations, but they don't yield any relevant results.

$$\dot{g} = g \cdot \mathrm{d}\mathcal{H},$$
$$\dot{\xi} = \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix} \xi$$

4.2.3 Hamiltonian as a Quadratic Function on $\mathrm{so}^*(3)$

Lastly, we combine the two previous cases. The trivialized Hamiltonian is of form

$$\mathcal{H}(\xi_1, \xi_2, \xi_3) = \frac{\xi_1^2}{2I_1} + a_1\xi_1 + \frac{\xi_2^2}{2I_2} + a_2\xi_2 + \frac{\xi_3^2}{2I_3} + a_3\xi_3. \tag{4.15}$$

It is worth mentioning that this is not only the most general case so far but it also yields the most natural applications. But now, let us do the same derivations as in the case 4.2.1. Again, let us denote by $J = \operatorname{diag}(I_1,\ I_2,\ I_3)$ the matrix as before and moreover let us denote by $\vec{a} = (a_1, a_2, a_3)^\top$ the vector of the linear form. Using this notation, we are able to rewrite the Hamiltonian as follows.

$$\mathcal{H}(\xi) = \frac{1}{2}\xi^\top J^{-1}\xi + \vec{a}^\top \xi, \tag{4.16}$$

Now, we focus on the Lagrangian of this case. Using 4.2,

$$\omega = \frac{\partial \mathcal{H}}{\partial \xi} = J^{-1}\xi + \vec{a}.$$

Thus, in this case we are able to express ξ in terms of ω:

$$\xi = J(\omega - \vec{a}).$$

Using 4.1:

$$\begin{aligned} L(\omega) &= \omega^\top \xi - \frac{1}{2}\xi^\top J^{-1}\xi - \vec{a}^\top \xi \\ &= \omega^\top J(\omega - \vec{a}) - \frac{1}{2}(\omega - \vec{a})^\top J(\omega - \vec{a}) - \vec{a}^\top J(\omega - \vec{a}) \\ &= (\omega - \vec{a})^\top J(\omega - \vec{a}) - \frac{1}{2}(\omega - \vec{a})^\top J(\omega - \vec{a}) \\ &= \frac{1}{2}(\omega - \vec{a})^\top J(\omega - \vec{a}) \end{aligned} \tag{4.17}$$

This is once again a quadratic Lagrangian corresponding to a minimization of length of some curve. Now, let us turn our attention to the equations of Hamiltonian formalism. First, we need to find the differential of the Hamiltonian and the adjoint operator.

$$\mathrm{d}\mathcal{H}(\xi_1, \xi_2, \xi_3) = \left(\frac{\xi_1}{I_1} + a_1\right) e_1 + \left(\frac{\xi_2}{I_2} + a_2\right) e_2 + \left(\frac{\xi_3}{I_3} + a_3\right) e_3 \tag{4.18}$$

$$(\mathrm{ad}\,\mathrm{d}\mathcal{H})^* = \begin{pmatrix} 0 & \xi_3/I_3 + a_3 & -\xi_2/I_2 - a_2 \\ -\xi_3/I_3 - a_3 & 0 & \xi_1/I_1 + a_1 \\ \xi_2/I_2 + a_2 & -\xi_1/I_1 - a_1 & 0 \end{pmatrix} \tag{4.19}$$

Substituting into the vertical equation 2.8 and expressing the result in coordinates we obtain

$$\begin{aligned} \dot{\xi}_1 &= \xi\left([\mathrm{d}\mathcal{H}, e_1]\right) = \xi\left(-\left(\frac{\xi_2}{I_2} + a_2\right) e_3 + \left(\frac{\xi_3}{I_3} + a_3\right) e_2\right) \\ &= \left(\frac{1}{I_3} - \frac{1}{I_2}\right)\xi_2\xi_3 + a_3\xi_2 + a_2\xi_3 \\ \dot{\xi}_2 &= \xi\left([\mathrm{d}\mathcal{H}, e_2]\right) = \xi\left(-\left(\frac{\xi_3}{I_3} + a_3\right) e_1 + \left(\frac{\xi_1}{I_1} + a_1\right) e_3\right) \\ &= \left(\frac{1}{I_1} - \frac{1}{I_3}\right)\xi_1\xi_3 + a_3\xi_1 + a_1\xi_3 \\ \dot{\xi}_3 &= \xi\left([\mathrm{d}\mathcal{H}, e_3]\right) = \xi\left(-\left(\frac{\xi_1}{I_1} + a_1\right) e_2 + \left(\frac{\xi_2}{I_2} + a_2\right) e_1\right) \\ &= \left(\frac{1}{I_2} - \frac{1}{I_1}\right)\xi_1\xi_2 + a_2\xi_1 + a_1\xi_2 \end{aligned} \tag{4.20}$$

Substituting into the horizontal equation 2.7 we obtain very similar result as in 4.2.1.

$$\dot{g} = g\,\mathrm{d}\mathcal{H} = g\left[\left(\frac{\xi_1}{I_1} + a_1\right) e_1 + \left(\frac{\xi_2}{I_2} + a_2\right) e_2 + \left(\frac{\xi_3}{I_3} + a_3\right) e_3\right] \tag{4.21}$$

Again, the equations on vertical coordinates are decoupled from the horizontal equation. The solution of the horizontal equation is an exponential. As a matter of fact, it can be written in the same form as 4.12.

4.3 Hamiltonian Formalism on SE(3)

Analogously to the previous section we will present three problems of Hamiltonian formalism, this time on the group SE(3). Refer to examples 1.3.4 and 1.3.9 for introduction of the matrix structures. These structures are used to describe the full motion of rigid body, meaning not only rotational motion (which is inherited from the SO(3) group) but also translations. Let us mention the form of the base of the algebra se(3). As said, the algebra is 6-dimensional space. For arbitrary element $A \in \mathrm{se}(3)$:

$$A = \sum_{i=1}^{6} a_i e_i = \begin{pmatrix} 0 & -a_3 & a_2 & a_4 \\ a_3 & 0 & -a_1 & a_5 \\ -a_2 & a_1 & 0 & a_6 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Alternatively, the group $\mathrm{SO}(4,1)$ we have introduced in subsection 3.1.2 contains SE(3) as its subgroup. Thus $\mathrm{se}(3) \subset \mathrm{so}(4,1)$ and we could represent an element $A \in \mathrm{se}(3)$ as

$$A = \sum_{i=1}^{6} a_i E_i = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ a_4 & 0 & -a_3 & a_2 & 0 \\ a_5 & a_3 & 0 & -a_1 & 0 \\ a_6 & -a_2 & a_1 & 0 & 0 \\ 0 & a_4 & a_5 & a_6 & 0 \end{pmatrix}, a_i \in \mathbb{R}. \tag{4.22}$$

Last step before we continue to the individual problems, let us show the form of the coadjoint function on se(3). Since

$$\begin{aligned}(\mathrm{ad}\, A)\, B &= \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \\ a_2b_6 - a_3b_5 + a_5b_3 - a_6b_2 \\ a_3b_4 - a_1b_6 - a_4b_3 + a_6b_1 \\ a_1b_5 - a_2b_4 + a_4b_2 - a_5b_1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -a_3 & a_2 & 0 & 0 & 0 \\ a_3 & 0 & -a_1 & 0 & 0 & 0 \\ -a_2 & a_1 & 0 & 0 & 0 & 0 \\ 0 & -a_6 & a_5 & 0 & -a_3 & a_2 \\ a_6 & 0 & -a_4 & a_3 & 0 & -a_1 \\ -a_5 & a_4 & 0 & -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{pmatrix},\end{aligned} \tag{4.23}$$

the coadjoint operator on se(3) has the following form.

$$(\mathrm{ad}\,A)^* = \begin{pmatrix} 0 & a_3 & -a_2 & 0 & a_6 & -a_5 \\ -a_3 & 0 & a_1 & -a_6 & 0 & a_4 \\ a_2 & -a_1 & 0 & a_5 & -a_4 & 0 \\ 0 & 0 & 0 & 0 & a_3 & -a_2 \\ 0 & 0 & 0 & -a_3 & 0 & a_1 \\ 0 & 0 & 0 & a_2 & -a_1 & 0 \end{pmatrix} \tag{4.24}$$

4.3.1 Hamiltonian as a Quadratic Form on $\mathrm{se}^*(3)$

Again, we begin with the specification of the considered Hamiltonian. Let $g \in \mathrm{SE}(3)$, $\xi \in \mathrm{se}^*(3)$, $p = L^*_{g^{-1}}\xi \in T^*_g\mathrm{SE}(3)$.

$$H(g,p) = \mathcal{H}(\xi), \qquad \mathcal{H}(\xi_1,\ldots,\xi_6) = \frac{1}{2}\sum_{i=1}^{3}\frac{\xi_i^2}{I_i} + \frac{1}{2}\sum_{i=1}^{3}\frac{\xi_{i+3}^2}{m} \tag{4.25}$$

Using the matrix of this quadratic form, we may represent the Hamiltonian as

$$\mathcal{H}(\xi) = \frac{1}{2}\xi^\top J^{-1}\xi, \tag{4.26}$$

where

$$J = \mathrm{diag}\,(I_1,\ I_2,\ I_3,\ m,\ m,\ m)\,,\ I_1,\ I_2,\ I_3 \in \mathbb{R}\backslash\{0\},\ m > 0. \tag{4.27}$$

This form of Hamiltonian was chosen because this example is a model of the free rigid body motion in space, i. e., translations and rotations of a rigid body without influence of forces.

The Lagrangian of this problem has the following form.

$$\begin{aligned} \omega &= \frac{\partial\mathcal{H}}{\partial\xi} = J^{-1}\xi, \\ L(\omega) &= \frac{1}{2}\omega^\top J\omega \end{aligned}$$

Now, to form the equations, we need the differential of the Hamiltonian, which is

$$\mathrm{d}\mathcal{H}(\xi_1,\ldots,\xi_6) = \sum_{i=1}^{3}\frac{\xi_i}{I_i}e_i + \sum_{i=1}^{3}\frac{\xi_{i+3}}{m}e_{i+3}. \tag{4.28}$$

Thus the coadjoint operator associated with the differential has the following form.

$$(\mathrm{ad}\,\mathrm{d}\mathcal{H})^* = \begin{pmatrix} 0 & \xi_3/I_3 & -\xi_2/I_2 & 0 & \xi_6/m & -\xi_5/m \\ -\xi_3/I_3 & 0 & \xi_1/I_1 & -\xi_6/m & 0 & \xi_4/m \\ \xi_2/I_2 & -\xi_1/I_1 & 0 & \xi_5/m & -\xi_4/m & 0 \\ 0 & 0 & 0 & 0 & \xi_3/I_3 & -\xi_2/I_2 \\ 0 & 0 & 0 & -\xi_3/I_3 & 0 & \xi_1/I_1 \\ 0 & 0 & 0 & \xi_2/I_2 & -\xi_1/I_1 & 0 \end{pmatrix} \tag{4.29}$$

From there all that remains is formulate the equations of Hamiltonian formalism.

$$\begin{pmatrix} \dot\xi_1 \\ \dot\xi_2 \\ \dot\xi_3 \\ \dot\xi_4 \\ \dot\xi_5 \\ \dot\xi_6 \end{pmatrix} = \begin{pmatrix} 0 & \xi_3/I_3 & -\xi_2/I_2 & 0 & \xi_6/m & -\xi_5/m \\ -\xi_3/I_3 & 0 & \xi_1/I_1 & -\xi_6/m & 0 & \xi_4/m \\ \xi_2/I_2 & -\xi_1/I_1 & 0 & \xi_5/m & -\xi_4/m & 0 \\ 0 & 0 & 0 & 0 & \xi_3/I_3 & -\xi_2/I_2 \\ 0 & 0 & 0 & -\xi_3/I_3 & 0 & \xi_1/I_1 \\ 0 & 0 & 0 & \xi_2/I_2 & -\xi_1/I_1 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \end{pmatrix} \tag{4.30}$$

$$\dot g = g\,\mathrm{d}\mathcal{H} = g\left(\sum_{i=1}^{3}\frac{\xi_i}{I_i}e_i + \sum_{i=1}^{3}\frac{\xi_{i+3}}{m}e_{i+3}\right) \tag{4.31}$$

The solution itself is very similar to the previous cases. We would solved the vertical part 4.30 numerically, the horizontal part 4.31 has an analytical solution, an exponential. At this point it is rather obvious that this is no coincidence but a property of this class of problems.

4.3.2 Hamiltonian as a Linear Form on $\mathrm{se}^*(3)$

We want to show this example as it is an extension of subsection 4.2.2. Let $g \in \mathrm{SE}(3)$, $\xi \in \mathrm{se}^*(3)$, $p = L^*_{g^{-1}}\xi \in T^*_g\mathrm{SE}(3)$. We take the Hamiltonian

$$H(g,p) = \mathcal{H}(\xi), \qquad \mathcal{H}(\xi_1,\dots,\xi_6) = \sum_{i=1}^{6} a_i\xi_i, \qquad a_1,\dots,a_6 \in \mathbb{R}, \tag{4.32}$$

$$\mathcal{H}(\xi) = \vec{a}^\top\xi, \qquad \vec{a} = (a_1,\dots,a_6)^\top. \tag{4.33}$$

Once again it is not possible to formulate the Lagrangian function of this problem because

$$\omega = \frac{\partial\mathcal{H}}{\partial\xi} = \vec{a}^\top$$

is not a function of ξ.

The differential of the Hamiltonian is $\mathrm{d}\mathcal{H}(\xi_1,\dots,\xi_2) = \sum_{i=1}^{6} a_ie_i = \mathit{const.}$, thus the coadjoint operator has the form

$$(\mathrm{ad}\,\mathrm{d}\mathcal{H})^* = \begin{pmatrix} 0 & a_3 & -a_2 & 0 & a_6 & -a_5 \\ -a_3 & 0 & a_1 & -a_6 & 0 & a_4 \\ a_2 & -a_1 & 0 & a_5 & -a_4 & 0 \\ 0 & 0 & 0 & 0 & a_3 & -a_2 \\ 0 & 0 & 0 & -a_3 & 0 & a_1 \\ 0 & 0 & 0 & a_2 & -a_1 & 0 \end{pmatrix}. \tag{4.34}$$

Then the Hamiltonian equations have the from

$$\dot{g} = g \cdot \mathrm{d}\mathcal{H} = g \cdot \left(\sum_{i=1}^{6} a_i e_i \right),$$

$$\dot{\xi} = \begin{pmatrix} 0 & a_3 & -a_2 & 0 & a_6 & -a_5 \\ -a_3 & 0 & a_1 & -a_6 & 0 & a_4 \\ a_2 & -a_1 & 0 & a_5 & -a_4 & 0 \\ 0 & 0 & 0 & 0 & a_3 & -a_2 \\ 0 & 0 & 0 & -a_3 & 0 & a_1 \\ 0 & 0 & 0 & a_2 & -a_1 & 0 \end{pmatrix} \xi.$$

This example again doesn't seem to resemble any real world problem. The vertical part models the change in momentum, the horizontal part models the change in position. In this particular example the momentum is being rotated but the change in position is constant, which is not what we would expect in any real world application.

4.3.3 Hamiltonian as a Quadratic Function on $\mathrm{se}^*(3)$

In the last example in this section we combine 4.3.1 and 4.3.2. Let $g \in \mathrm{SE}(3)$, $\xi \in \mathrm{se}^*(3)$, $p = L^*_{g^{-1}}\xi \in T^*_g\mathrm{SE}(3)$ and $a_1, \ldots, a_6 \in \mathbb{R}$. The Hamiltonian therefore is of form

$$H(g,p) = \mathcal{H}(\xi), \qquad \mathcal{H}(\xi_1, \ldots, \xi_6) = \frac{1}{2}\sum_{i=1}^{3} \frac{\xi_i^2}{I_i} + \frac{1}{2}\sum_{i=1}^{3} \frac{\xi_{i+3}^2}{m} + \sum_{i=1}^{6} a_i \xi_i, \tag{4.35}$$

$$\mathcal{H}(\xi) = \frac{1}{2}\xi^\top J^{-1}\xi + \vec{a}^\top \xi. \tag{4.36}$$

The matrix of the quadratic form is the same as the one given in 4.27. As the quadratic part models the free rigid body motion and the linear part rotates the vertical part of the cotangent bundle, this can be seen as a example of rigid body that is acted upon by some kind of forces that are constant in the body frame.

The Lagrangian of this problem is, to no surprise, very similar to its counterpart from 4.17.

$$L(\omega) = \frac{1}{2}\left(\omega - \vec{a}\right)^\top J \left(\omega - \vec{a}\right)$$

Now, let us state the Hamiltonian equations associated with this problem. The differential of the Hamiltonian has the form

$$\mathrm{d}\mathcal{H}(\xi_1, \ldots, \xi_6) = \sum_{i=1}^{3} \frac{\xi_i}{I_i} e_i + \sum_{i=1}^{3} \frac{\xi_{i+3}}{m} e_{i+3} + \sum_{i=1}^{6} a_i e_i \tag{4.37}$$

and the coadjoint operator has the form of block matrix

$$(\mathrm{ad}\,\mathrm{d}\mathcal{H})^* = \begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{0} & \mathbb{A} \end{pmatrix}, \tag{4.38}$$

where $\mathbb{0}$ is zero matrix and

$$\mathbb{A} = \begin{pmatrix} 0 & \xi_3/I_3 + a_3 & -\xi_2/I_2 - a_2 \\ -\xi_3/I_3 - a_3 & 0 & \xi_1/I_1 + a_1 \\ \xi_2/I_2 + a_2 & -\xi_1/I_1 - a_1 & 0 \end{pmatrix}, \tag{4.39}$$

$$\mathbb{B} = \begin{pmatrix} 0 & \xi_6/m + a_6 & -\xi_5/m - a_5 \\ -\xi_6/m - a_6 & 0 & \xi_4/m + a_4 \\ \xi_5/m + a_5 & -\xi_4/m - a_4 & 0 \end{pmatrix}. \tag{4.40}$$

Considering the increasing size of the matrices involved, we will now incorporate their block structure whenever feasible. The equations corresponding to this problem are

$$\dot{\xi} = (\operatorname{ad} \mathrm{d}\mathcal{H})^* \xi = \begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{0} & \mathbb{A} \end{pmatrix} \xi, \tag{4.41}$$

$$\dot{g} = g \cdot \mathrm{d}\mathcal{H} = g \cdot \left(\sum_{i=1}^{3} \frac{\xi_i}{I_i} e_i + \sum_{i=1}^{3} \frac{\xi_{i+3}}{m} e_{i+3} + \sum_{i=1}^{6} a_i e_i \right). \tag{4.42}$$

The solution of the horizontal part is again an exponential, the vertical part we would solve numerically.

4.4 Hamiltonian Formalism on SO(4, 1)

For this final section, we will focus on the most general group. As mentioned earlier, both SO(3) and SE(3) are subgroups of SO(4,1). In Chapter 3, we introduced this group and its associated Lie algebra. Namely, recall 3.6, the structure of the Lie algebra we will assume. Before we explore individual problems we will show the structure of general coadjoint operator on this Lie algebra.

$$(\operatorname{ad} A)\, B = \begin{pmatrix} a_2b_3 - a_3b_2 - a_5b_9 + a_6b_8 - a_8b_6 + a_9b_5 \\ a_3b_1 - a_1b_3 + a_4b_9 - a_6b_7 + a_7b_6 - a_9b_4 \\ a_1b_2 - a_2b_1 - a_4b_8 + a_5b_7 - a_7b_5 + a_8b_4 \\ a_2b_6 - a_3b_5 + a_5b_3 - a_6b_2 + a_4b_{10} - a_{10}b_4 \\ a_3b_4 - a_1b_6 - a_4b_3 + a_6b_1 + a_5b_{10} - a_{10}b_5 \\ a_1b_5 - a_2b_4 + a_4b_2 - a_5b_1 + a_6b_{10} - a_{10}b_6 \\ a_2b_9 - a_3b_8 + a_8b_3 - a_9b_2 - a_7b_{10} + a_{10}b_7 \\ a_3b_7 - a_1b_9 - a_7b_3 + a_9b_1 - a_8b_{10} + a_{10}b_8 \\ a_1b_8 - a_2b_7 + a_7b_2 - a_8b_1 - a_9b_{10} + a_{10}b_9 \\ a_7b_4 - a_4b_7 - a_5b_8 + a_8b_5 - a_6b_9 + a_9b_6 \end{pmatrix}$$

$$(\mathrm{ad}\, A)^* = \begin{pmatrix} \mathbb{A} & \mathbb{B} & \mathbb{C} & \mathbb{0} \\ \mathbb{C}^\top & \mathbb{D} & \mathbb{0} & \vec{w} \\ \mathbb{B}^\top & \mathbb{0} & \mathbb{E} & -\vec{x} \\ \mathbb{0} & \vec{x}^\top & -\vec{w}^\top & 0 \end{pmatrix} \tag{4.43}$$

$$\mathbb{A} = \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix} \qquad \mathbb{B} = \begin{pmatrix} 0 & a_6 & -a_5 \\ -a_6 & 0 & a_4 \\ a_5 & -a_4 & 0 \end{pmatrix}$$

$$\mathbb{C} = \begin{pmatrix} 0 & a_9 & -a_8 \\ -a_9 & 0 & a_7 \\ a_8 & -a_7 & 0 \end{pmatrix} \qquad \mathbb{D} = \begin{pmatrix} -a_{10} & a_3 & -a_2 \\ -a_3 & -a_{10} & a_1 \\ a_2 & -a_1 & -a_{10} \end{pmatrix}$$

$$\mathbb{E} = \begin{pmatrix} a_{10} & a_3 & -a_2 \\ -a_3 & a_{10} & a_1 \\ a_2 & -a_1 & a_{10} \end{pmatrix}$$

$$\vec{x} = (a_4, a_5, a_6)^\top \qquad \vec{w} = (a_7, a_8, a_9)^\top$$

The coadjoint operator can be represented by a 10×10 matrix, it would be challenging to incorporate it directly into the text without the block structure. Consequently, we will solely utilize the block structure approach to present the coadjoint operator.

4.4.1 Hamiltonian as a Quadratic Form on $\mathrm{so}^*(4,1)$

Once again, we begin with the definition of the Hamiltonian. Let $g \in \mathrm{SO}(4,1)$, $\xi \in \mathrm{so}^*(4,1)$, $p = L^*_{g^{-1}}\xi \in T^*_g\mathrm{SO}(4,1)$.

$$H(g,p) = \mathcal{H}(\xi), \qquad \mathcal{H}(\xi_1,\dots,\xi_{10}) = \frac{1}{2}\sum_{i=1}^{10}\frac{\xi_i^2}{j_i} \tag{4.44}$$

We can reformulate the Hamiltonian using matrix notation. Take

$$J = \mathrm{diag}\,(j_1,\dots,j_{10})\,, \qquad j_1,\dots,j_{10} \in \mathbb{R}\backslash\{0\},$$

then

$$\mathcal{H}(\xi) = \frac{1}{2}\xi^\top J^{-1}\xi. \tag{4.45}$$

The Lagrangian function associated with this problem can be formulated in the same manner as in the previous examples.

$$L(\omega) = \frac{1}{2}\omega^\top J\omega$$

The form is symbolically the same as in the case of quadratic form on SE(3), the difference is that ω has the dimension of 10 instead of 6 and the matrix J is also different.

Finding a real world example is trickier than in the previous cases. The reason for that is that the tenth dimension of the Lie algebra (in the definition we use) is responsible for scaling. Therefore it is really no longer appropriate to talk about motion of a rigid body, because we expect that the body would deform. But it turns out that the group $\mathrm{SO}(4,1)$ models more complicated physics, which is not that surprising given that it is an extension of SE(3). For instance, in [10], the group $\mathrm{SO}(4,1)$ (in the sense of 3.3) is employed to express the equations of general relativity in a dimensionless formulation. Similarly, all computations behind the animations featured in [11] (scientifically accurate visualizations of Schwarzschild and Reissner–Nordström black hole models) are conducted using the $\mathrm{SO}(4,1)$ Lie group. Considering that this case is a generalization of 4.2.1 and 4.3.1, it can be assumed that a real-world application of this scenario would involve free motion in close proximity to a black hole.

Let us finish up this example by formulation of the Hamiltonian equations. The differential of the Hamiltonian in this case has the following form.

$$\mathrm{d}\mathcal{H}\left(\xi_1,\ldots,\xi_{10}\right) = \sum_{i=1}^{10} \frac{\xi_i}{j_i} e_i \tag{4.46}$$

Thus, the coadjoint operator has the form

$$(\mathrm{ad}\, A)^* = \begin{pmatrix} \mathbb{A} & \mathbb{B} & \mathbb{C} & \mathbb{0} \\ \mathbb{C}^\top & \mathbb{D} & \mathbb{0} & \vec{w} \\ \mathbb{B}^\top & \mathbb{0} & \mathbb{E} & -\vec{x} \\ \mathbb{0} & \vec{x}^\top & -\vec{w}^\top & 0 \end{pmatrix} \tag{4.47}$$

$$\mathbb{A} = \begin{pmatrix} 0 & \xi_3/j_3 & -\xi_2/j_2 \\ -\xi_3/j_3 & 0 & \xi_1/j_1 \\ \xi_2/j_2 & -\xi_1/j_1 & 0 \end{pmatrix} \qquad \mathbb{B} = \begin{pmatrix} 0 & \xi_6/j_6 & -\xi_5/j_5 \\ -\xi_6/j_6 & 0 & \xi_4/j_4 \\ \xi_5/j_5 & -\xi_4/j_4 & 0 \end{pmatrix}$$

$$\mathbb{C} = \begin{pmatrix} 0 & \xi_9/j_9 & -\xi_8/j_8 \\ -\xi_9/j_9 & 0 & \xi_7/j_7 \\ \xi_8/j_8 & -\xi_7/j_7 & 0 \end{pmatrix} \qquad \mathbb{D} = \begin{pmatrix} -\xi_{10}/j_{10} & \xi_3/j_3 & -\xi_2/j_2 \\ -\xi_3/j_3 & -\xi_{10}/j_{10} & \xi_1/j_1 \\ \xi_2/j_2 & -\xi_1/j_1 & -\xi_{10}/j_{10} \end{pmatrix}$$

$$\mathbb{E} = \begin{pmatrix} \xi_{10}/j_{10} & \xi_3/j_3 & -\xi_2/j_2 \\ -\xi_3/j_3 & \xi_{10}/j_{10} & \xi_1/j_1 \\ \xi_2/j_2 & -\xi_1/j_1 & \xi_{10}/j_{10} \end{pmatrix}$$

$$\vec{x} = \left(\xi_4/j_4, \xi_5/j_5, \xi_6/j_6\right)^\top \qquad \vec{w} = \left(\xi_7/j_7, \xi_8/j_8, \xi_9/j_9\right)^\top$$

The equations then are

$$
\begin{aligned}
\dot{g} &= g \cdot \left(\sum_{i=1}^{10} \frac{\xi_i}{j_i} e_i \right), \\
\dot{\xi} &= \begin{pmatrix} \mathbb{A} & \mathbb{B} & \mathbb{C} & \mathbb{0} \\ \mathbb{C}^\top & \mathbb{D} & \mathbb{0} & \vec{w} \\ \mathbb{B}^\top & \mathbb{0} & \mathbb{E} & -\vec{x} \\ \mathbb{0} & \vec{x}^\top & -\vec{w}^\top & 0 \end{pmatrix} \xi.
\end{aligned}
\tag{4.48}
$$

4.4.2 Hamiltonian as a Linear Form on $\mathrm{so}^*(4,1)$

Take the Hamiltonian

$$
H(g,p) = \mathcal{H}(\xi), \qquad \mathcal{H}\left(\xi_1, \ldots, \xi_{10}\right) = \sum_{i=1}^{10} a_i \xi_i, \tag{4.49}
$$

where $g \in \mathrm{SO}(4,1)$, $\xi \in \mathrm{so}^*(4,1)$, $p = L^*_{g^{-1}}\xi \in T^*_g\mathrm{SO}(4,1)$. We can reformulate the Hamiltonian using vector notation. Take

$$
\vec{a} = \left(a_1, \ldots, a_{10}\right)^\top, \qquad a_1, \ldots, a_{10} \in \mathbb{R},
$$

then

$$
\mathcal{H}\left(\xi\right) = \vec{a}^\top \xi. \tag{4.50}
$$

The Lagrangian once again cannot be obtained, since $\frac{\partial \mathcal{H}}{\partial \xi}$ is a constant vector.

The differential of Hamiltonian is

$$
\mathrm{d}\mathcal{H}(\xi_1, \ldots, \xi_{10}) = \sum_{i=1}^{10} a_i e_i, \tag{4.51}
$$

thus,

$$
(\mathrm{ad}\, A)^* = \begin{pmatrix} \mathbb{A} & \mathbb{B} & \mathbb{C} & \mathbb{0} \\ \mathbb{C}^\top & \mathbb{D} & \mathbb{0} & \vec{w} \\ \mathbb{B}^\top & \mathbb{0} & \mathbb{E} & -\vec{x} \\ \mathbb{0} & \vec{x}^\top & -\vec{w}^\top & 0 \end{pmatrix} \tag{4.52}
$$

$$
\mathbb{A} = \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix} \qquad \mathbb{B} = \begin{pmatrix} 0 & a_6 & -a_5 \\ -a_6 & 0 & a_4 \\ a_5 & -a_4 & 0 \end{pmatrix}
$$

$$
\mathbb{C} = \begin{pmatrix} 0 & a_9 & -a_8 \\ -a_9 & 0 & a_7 \\ a_8 & -a_7 & 0 \end{pmatrix} \qquad \mathbb{D} = \begin{pmatrix} -a_{10} & a_3 & -a_2 \\ -a_3 & -a_{10} & a_1 \\ a_2 & -a_1 & -a_{10} \end{pmatrix}
$$

$$
\mathbb{E} = \begin{pmatrix} a_{10} & a_3 & -a_2 \\ -a_3 & a_{10} & a_1 \\ a_2 & -a_1 & a_{10} \end{pmatrix}
$$

$$
\vec{x} = \left(a_4, a_5, a_6\right)^\top \qquad \vec{w} = \left(a_7, a_8, a_9\right)^\top
$$

This implies the equations

$$\dot{g} = g \cdot \left(\sum_{i=1}^{10} a_i e_i \right),$$
$$\dot{\xi} = \begin{pmatrix} \mathbb{A} & \mathbb{B} & \mathbb{C} & \mathbb{0} \\ \mathbb{C}^\top & \mathbb{D} & \mathbb{0} & \vec{w} \\ \mathbb{B}^\top & \mathbb{0} & \mathbb{E} & -\vec{x} \\ \mathbb{0} & \vec{x}^\top & -\vec{w}^\top & 0 \end{pmatrix} \xi. \tag{4.53}$$

But, once again, there is no real application to be found that could be modeled by these equations.

4.4.3 Hamiltonian as a Quadratic Function on $\mathrm{so}^*(4,1)$

This last example combines 4.4.1 and 4.4.2 and it is the most general example we will show. Let $g \in \mathrm{SO}(4,1)$, $\xi \in \mathrm{so}^*(4,1)$, $p = L^*_{g^{-1}}\xi \in T^*_g\mathrm{SO}(4,1)$. The Hamiltonian we assume is

$$H(g,p) = \mathcal{H}(\xi), \qquad \mathcal{H}\left(\xi_1, \ldots, \xi_{10}\right) = \frac{1}{2}\sum_{i=1}^{10} \frac{\xi_i^2}{j_i} + \sum_{i=1}^{10} a_i \xi_i. \tag{4.54}$$

We can reformulate the Hamiltonian using matrix notation. Take

$$J = \mathrm{diag}\left(j_1, \ldots, j_{10}\right), \qquad j_1, \ldots, j_{10} \in \mathbb{R}\backslash\{0\},$$
$$\vec{a} = \left(a_1, \ldots, a_{10}\right), \qquad a_1, \ldots, a_{10} \in \mathbb{R}$$

then

$$\mathcal{H}\left(\xi\right) = \frac{1}{2}\xi^\top J^{-1}\xi + \vec{a}^\top \xi. \tag{4.55}$$

The Lagrangian corresponding to this problem can be derived as

$$L(\omega) = \frac{1}{2}\left(\omega - \vec{a}\right)^\top J \left(\omega - \vec{a}\right).$$

The differential of this Hamiltonian is

$$\mathrm{d}\mathcal{H}(\xi_1, \ldots, \xi_{10}) = \sum_{i=1}^{10} \left(\frac{\xi_i}{j_i} + a_i \right) e_i, \tag{4.56}$$

thus the coadjoint operator has the form

$$\left(\mathrm{ad}\, A\right)^* = \begin{pmatrix} \mathbb{A} & \mathbb{B} & \mathbb{C} & \mathbb{0} \\ \mathbb{C}^\top & \mathbb{D} & \mathbb{0} & \vec{w} \\ \mathbb{B}^\top & \mathbb{0} & \mathbb{E} & -\vec{x} \\ \mathbb{0} & \vec{x}^\top & -\vec{w}^\top & 0 \end{pmatrix}, \tag{4.57}$$

where

$$
\begin{aligned}
\mathbb{A} &= \begin{pmatrix} 0 & \xi_3/j_3 + a_3 & -\xi_2/j_2 - a_2 \\ -\xi_3/j_3 - a_3 & 0 & \xi_1/j_1 + a_1 \\ \xi_2/j_2 + a_2 & -\xi_1/j_1 - a_1 & 0 \end{pmatrix}, \\
\mathbb{B} &= \begin{pmatrix} 0 & \xi_6/j_6 + a_6 & -\xi_5/j_5 - a_5 \\ -\xi_6/j_6 - a_6 & 0 & \xi_4/j_4 + a_4 \\ \xi_5/j_5 + a_5 & -\xi_4/j_4 - a_4 & 0 \end{pmatrix}, \\
\mathbb{C} &= \begin{pmatrix} 0 & \xi_9/j_9 + a_9 & -\xi_8/j_8 - a_8 \\ -\xi_9/j_9 - a_9 & 0 & \xi_7/j_7 + a_7 \\ \xi_8/j_8 + a_8 & -\xi_7/j_7 - a_7 & 0 \end{pmatrix}, \\
\mathbb{D} &= \begin{pmatrix} -\xi_{10}/j_{10} - a_{10} & \xi_3/j_3 + a_3 & -\xi_2/j_2 - a_2 \\ -\xi_3/j_3 - a_3 & -\xi_{10}/j_{10} - a_{10} & \xi_1/j_1 + a_1 \\ \xi_2/j_2 + a_2 & -\xi_1/j_1 - a_1 & -\xi_{10}/j_{10} - a_{10} \end{pmatrix}, \\
\mathbb{E} &= \begin{pmatrix} \xi_{10}/j_{10} + a_{10} & \xi_3/j_3 + a_3 & -\xi_2/j_2 - a_2 \\ -\xi_3/j_3 - a_3 & \xi_{10}/j_{10} + a_{10} & \xi_1/j_1 + a_1 \\ \xi_2/j_2 + a_2 & -\xi_1/j_1 - a_1 & \xi_{10}/j_{10} + a_{10} \end{pmatrix}, \\
\vec{x} &= \left(\xi_4/j_4 + a_4, \xi_5/j_5 + a_5, \xi_6/j_6 + a_6\right)^\top, \\
\vec{w} &= \left(\xi_7/j_7 + a_7, \xi_8/j_8 + a_8, \xi_9/j_9 + a_9\right)^\top.
\end{aligned}
$$

The equations have the following form.

$$
\begin{aligned}
\dot{g} &= g \cdot \left(\sum_{i=1}^{10} \left(\frac{\xi_i}{j_i} + a_i \right) e_i \right), \\
\dot{\xi} &= \begin{pmatrix} \mathbb{A} & \mathbb{B} & \mathbb{C} & \mathbb{0} \\ \mathbb{C}^\top & \mathbb{D} & \mathbb{0} & \vec{w} \\ \mathbb{B}^\top & \mathbb{0} & \mathbb{E} & -\vec{x} \\ \mathbb{0} & \vec{x}^\top & -\vec{w}^\top & 0 \end{pmatrix} \xi.
\end{aligned} \tag{4.58}
$$

The application of this particular case are even more complicated than in the case 4.4.1. But seeing this as a generalization of 4.4.1, one might try to model motion of a soft body with forces acting upon it using these equations.

5 Programmed Solutions

We have chosen two main approaches in programmed examples for this thesis. The first are scripts coded in JavaScript for a visualization tool [12] written in JavaScript. The JavaScript scripts can be run locally if one installs the package, but they can be run online at [13]. It was created as a database of free examples for [12] and as an easy way for users to experiment with their own ideas. The second approach is a script in Matlab solving the general problem 4.4.3. In following sections we will visit each of them, describe them, present the results and explain, why is the approach of Lie groups and algebras useful.

5.1 Dimensionless Rigid Body Motion

The first example was chosen as it very clearly demonstrates property of problems described using Lie algebra, that seems almost absurd from the point of view of vector–matrix notation which currently the usual way of computing motion. That is, the form of equations does not depend on the dimension of space the body is in. As we will demonstrate, the same code works for motion of a rectangle in 2D space and a block in 3D space. In some sense, this is similar solution to the Garticle Engine which we have discussed in [14]. However, since we assume left-invariant Hamiltonian functions, this is somewhat simpler case.

5.1.1 Code Description

The dimension is chosen beforehand and defined in the variable called `dimension`. The computations take place in either 2D or 3D Projective Geometric Algebra (PGA). Thorough introduction to PGA can be found in [8]. The reason, why we use PGA and instead of CGA is that it is easier, from programming standpoint, to handle the different number of dimensions. It could be obviously done in CGA as well but it would make the code unnecessarily complicated and that is why we have chosen PGA. Needless to say, that this is caused by default base in [12], which is the base associated with 3.3. If there were by default base vectors e_0 and e_∞ we could write the example as efficiently in CGA.

Following definition of the algebra we define couple of functions. Namely, function

```
var point = x => !(e0 + x * [e1, e2, e3]);
```

which takes an array of coordinates of point in 2D or 3D and returns the representation of this point in PGA (which is a trivector). Keywords `e0`, `e1`, `e2`, `e3` represent the base vectors and the exclamation mark is the duality operator. The fourth-order Runge–Kutta algorithm (RK4, [15]) follows.

```
var RK4 = (f,y,h) => {
  var k1=f(y),
      k2=f(y+0.5*h*k1),
      k3=f(y+0.5*h*k2),
      k4=f(y+h*k3);
  return y+(h/3)*(k2+k3+(k1+k4)*0.5);
}
```

As we mentioned very often, the horizontal part of Hamiltonian equations can be solved analytically, so we will solve only the vertical part using the RK4 and use the exponentiation to find the solution of the horizontal part. But more on that later. Next, we define the commutator for PGA:

```
var commutator = (a, b) => 0.5 * (a * b - b * a);
```

The definition of the body (a rectangle or a block) follows. We choose its physical properties – mass and length of edges – and use this information to calculate the inertia matrix. Then we compute its vertices and edges, we have used an algorithm that can be found in [16] which utilizes the fact you can represent coordinates of vertices of a cube with edge of length one as a binary number and vice versa, then we use binary operations to calculate whether or not there should be an edge between two vertices given their indices. An edge is always between to points whose coordinates differ only in one dimension. Thus we check whether two points satisfy this condition and if it is so, connect them with an edge.

Next step is definition of initial position and momentum of the body. The position is given by a PGA motor (multivector on even blades), momentum is represented by a bivector. As mentioned above, the `initialState`, as we have called the variable, is an array with two elements. For example

```
var initialState = [1-1e01, -0.1e12+0.1e01+0.5e02+0.1e23];
```

is a valid option. Furthermore, variables `momentum` and `position` will be used to store current values.

At last we define the differential of Hamiltonian function.

```
var linearPart = 0,
    dH = h => h.Dual.map((x, i) => x / (inertia[i]||1)) +
        linearPart.Dual;
```

The constant part of $\mathrm{d}\mathcal{H}$ is always taken as a bivector in 3D PGA, that ensures that correct bivectors are used in algebras with fewer dimensions, since for example e_{03} is evaluated as zero in 2D PGA. The vertical equation is not exactly the same as it was shown in previous chapter. Recall 2.1.5. For a compact Lie group we may use the commutator in the vertical equation. But the velocity is dual to momentum, thus we need to first take its dual (this is done directly in dH) and after we compute the commutator we have, so to say, "undual" the result.

```
var undual = h => h.map((x, i) => x * inertia[i]).Dual -
    linearPart,
     verticalEquation = h => commutator(h, dH(h));
```

Then we just run the animation – in each time step we update the state by

```
t += dt; // we usually choose dt = 1/60
velocity = RK4(verticalEquation, velocity, dt).Grade(2);
position = (initialState[0]*((t*velocity).Exp())).Normalized;
```

and draw edges of the body.

5.1.2 Results

In this subsection we will choose some initial data and present the reader with obtained results. However, the code described above is meant as a visualization tool, i.e., it creates animation of the motion, which isn't very useful for displaying on paper. But with slight modification of the script we have created a script that computes the motion beforehand and user can then choose which positions to display. Specifically, the algorithm computing edges leaves zeros in the array if there shouldn't be an edge between points. This doesn't matter in the animation script but it introduces some problems here so we have to clean the `edges` array. The motion is computed in a `for` loop and the motors are stored in an array.

```
for (var i = 0; i < timeSteps; i++) {
  t += dt;
  velocity = RK4(verticalEquation, velocity, dt).Grade(2);
  positions[i] = (initialState[0]*((t*(undual(dH(velocity)))).Exp()
      )).Normalized;
}
```

Then we compute the transformation of cube

```
var movedEdges = positions.map((x,i) => trueEdges.map((y,j) => x
    >>> y));
```

and draw a couple of cubes on screen (the cubes at zero, one and two seconds).

```
document.body.appendChild(this.graph([
  0x5f3110, ...movedEdges[0], "t=0s",
  0x9D86CF, ...movedEdges[59], "t=1s",
  0xCC8ADC, ...movedEdges[119], "t=2s"
]));
```

With that out of our hands, let us turn attention to the examples themselves. We begin with simple examples mainly to show, that the system works the way we would expect it to. By simple we mean that we take cube with length of an edge equal to one and mass 12. Furthermore, we take Hamiltonian function that is a quadratic form. We will present the results in figures of the cube at three moments.

Rotation around one axis

First example is rotation around the z axis. Keep in mind that we start in premise that the same code should work in all dimensions, but we will show only 2D and 3D versions of the same problem, because 1D isn't particularly exciting and anything above 3D isn't very useful anyway. The initial velocity was chosen $\xi = -1e_{12}$, thus we would expect counter-clockwise rotation around z axis, as we already mentioned. As you can see in figures 5.1 and 5.2, this is exactly what we have obtained.

Rotation around all axes

Since there is no axis of rotation in 1D and only one in 2D, this example is interesting for the case of 3D space. We have chosen the velocity as $\xi = -0.3e_{23}+0.1e_{31}-0.2e_{12}$. The result is in figure 5.3. Obviously we could run this in 2D (or even 1D) but it would reduce itself to a rotation with velocity $\xi = -0.2e_{12}$ since the other two bivectors don't exist in 2D PGA (because e_3 does not exist there), which we have demonstrated above.

Translation in one direction

In the two following examples we take look at translations. First, let us take simple translation in one direction. We take the velocity $\xi = e_{01}$, thus we expect the cube to reverse along the x axis. The results for 2D and 3D space can be found in figures 5.4 and 5.5. The results confirm our expectations.

Translation in all directions

In figures 5.6, 5.7 can the reader find translational motion in all directions. We have chosen the bivector of velocity as $\xi = e_{01} - 0.7e_{02} - 0.2e_{03}$. Thus the motion should be in the direction of y and z axes (in the case of 3D space), and against the direction of x axis. This is the case, our assumptions were correct.

Free Rigid Body Motion

Last of the simple motions is combination of the above. This time we take the velocity $\xi = 0.1e_{02} - 0.7e_{03} - 0.1e_{12}$. The results can be found in figures 5.8 and 5.9.

You may notice, that the cube drifts in the direction of the x axis. Since the velocity is in the body frame that makes perfect sense, the rotational motion causes the translational movement to rotate with the cube and thus the movement in x axis.

Non-zero Linear Part in Hamiltonian Function

Now we will experiment with the linear part of the Hamiltonian function. The free motion is described by the quadratic form, thus this term might correspond to a force acting on the body. By definition, left-invariant Hamiltonian cannot depend on the position of the body in space, thus this force would have to be constant in the body frame. With this concept in mind, let us denote by $f = \sum a_i e_i$ the constant part of the differential of Hamiltonian function.

Let us begin from simple translational motion given by $\xi = -0.1e_{02}$. It turns out, that our assumptions were correct, or at least to some extent. Taking $f = -0.25e_{02}$ we obtain results that are in figures 5.10a and 5.10b. As you can clearly see, that is just the effect we would expect a force would have. Taking $f = 0.1e_{12}$ we obtain 5.11a and 5.11b, which again do align with the idea of f playing the role of torque in this case. When we combine torque, forces and non-zero initial velocity we obtain what we would expect, but keep in mind that the forces are in body frame and if the body rotates, the forces rotate with it and the resulting motion copies that. In 5.12a and 5.12b is result obtained for $f = 0.1e_{12} + 0.2e_{01}$.

5.2 Soft Body Motion

Our next script is very similar to the previous one. However, this time we will use CGA and try to demonstrate what is there in addition to the previous case. We have chosen a sphere as the body that we will use in this example rather than a block, because it is for one, very nicely embedded in CGA and for two, due to the changes in relative position of points it might be difficult to see exactly what is happening.

5.2.1 Code Description

This time the algebra is set, we mentioned that we wish to use CGA, thus the $(4, 1)$ metric is chosen. Since the default base vectors correspond to 3.3, we define the base vectors $e_0 = \frac{1}{2}(e_- - e_+)$ and $e_\infty = e_+ + e_-$. Using this new basis we can define the usual point representation in CGA:

```
var point = x => eo + x + 0.5*x*x*einf;
```

For our convenience we define function `sphere` which takes an array of four points and takes their outer product to create a sphere.

```
var sphere = x => x.reduce((a, b) => a ^ b);
```

The commutator follows and it is exactly the same as in previous example. We will again employ fourth order Runge–Kutta algorithm for the vertical part and exponentiation for the horizontal part. The initial positions of the four sphere's points were chosen as follows:

$$P_1 = [0, 0, 1/2], \qquad P_2 = [-1/2, 0, 0],$$
$$P_3 = [1/2, 0, 0], \qquad P_4 = [0, 1/2, 0].$$

We continue with definition of initial motor, velocity and the linear part of the Hamiltonian function. Since we assume the body to be a sphere, we take the matrix of the quadratic part of the Hamiltonian function as an identity matrix. The reasoning behind this is following. Since sphere is obviously symmetric in every basis, it is clear that its inertia is the same in every direction. Then we choose such mass, that there are ones on the diagonal. The last two elements are defined as bivectors of CGA, since it allows very simple formulation of the differential of the Hamiltonian function, which follows next.

```
var dH = h => (h + linearPart).Dual,
    undual = h => h.Dual - linearPart;
```

The final step is to define the Hamiltonian equations, since the matrix of quadratic form is an identity matrix, if the linear part would be zero, we could just take dual of $\mathrm{d}\mathcal{H}$, compute the commutator and then take dual of the result. But this is not the general case, thus we formulate the vertical equation as follows.

```
var verticalEquation = h => -2*undual(commutator(h, dH(h)));
```

In the end we just run the simulation. At each step we update the velocity and position, and then graph the points and sphere.

```
t += dt;
velocity = RK4(verticalEquation, velocity, 1/60).Grade(2);
position = (initialState[0]*((t*(velocity)).Exp())).Normalized;
var newPoints = points.map(p => position >>> p);
```

5.2.2 Results

Similarly to the previous case, the animation script was not very appropriate for presenting results here. Thus we made very much the same modifications to create script that better suits needs of this subsection. Positions are computed beforehand and then we pick three of them (at zero, one and two seconds) and display them.

Free Rigid Body Motion

Since SE(3) is a subgroup of SO(4, 1), we could go through all the translational and rotational motions separately, but we will show directly the free rigid body motion. Actually, we will take exactly the same velocity bivector $\xi = 0.1e_{2\infty} - 0.7e_{3\infty} - 0.1e_{12}$, only this time expressed in CGA. The results we have obtained are in figure 5.13. To no surprise it is the same motion as in previous section.

Free Soft Body Motion

Bivectors e_{ij} and $e_{i\infty}$, where $i, j \in \{1,\ 2,\ 3\}$ describe, in our application, the rigid body motion, bivectors e_{0i} correspond to transversions, inversion followed by translation followed by inversion. Those are not very interesting for us, since we wish to study real bodies and inversion often doesn't bode too well for them. That leaves the bivector $e_{0\infty}$. As mentioned earlier, this bivector handles scaling. For our next example we have chosen velocity $\xi = -0.6e_{1\infty} - 0.2e_{0\infty}$. The translation is there only that the effect of dialation is visible in the result, which can be found in 5.14. We see that the sphere is shrinking. If we take positive multiple of $e_{0\infty}$, it will expand, see 5.15 ($\xi = -0.6e_{1\infty} + 0.1e_{0\infty}$).

Forces and Torques

Analogously to the previous section, non-zero coefficients in the linear part of Hamiltonian function play role of forces and torques. It is very much the same as in the the previous case apart from the dialation effect of $e_{0\infty}$ which wasn't present in the case of PGA. Thus we will give an example of it. We have chosen the bivector of velocity $v = -0.1e_{1\infty}$ and bivector of forces $f = -0.1e_{1\infty} - 0.15e_{0\infty}$, the result can be found in figure 5.16.

5.3 Matlab Solution of 4.4.3

In this section we take one more look at the example from previous section. However, we will solve it in its matrix formulation in Matlab. The result this time will be a curve $(g, \xi) : \langle 0, t_f \rangle \to \mathrm{SO}(4,1) \times \mathrm{so}^*(4,1)$, that is curve in 5×5 matrices, which is not particularly easy to visualize. Thus we will plot velocities which is rather simple and check whether the properties

- $A^\top B A = B$, where $A \in \mathrm{SO}(4,1)$ and B is given by 3.5,
- $\det A = 1$, $A \in \mathrm{SO}(4,1)$

of the group are satisfied for all matrices on the integral curve.

5.3.1 Code Description

For user's convenience and simplification of the script itself, two functions were written:

- `lieAlgebra` takes an array of length 10 and converts it into corresponding element of $\mathrm{so}(4,1)$
- `coadjointOperator` takes an array `a` of length 10 and converts it into $(\mathrm{ad}\,\mathtt{A})^*$, where `A:=lieAlgebra(a)`.

Both are very simple, they basically programmed form of 3.6 and 4.43 respectively.

The script begins with initialization of data for this problem. We keep the notation introduced in 4.4.3. We choose the diagonal of matrix J, the vector of the linear part a, initial values of ξ and $g(0)$. We will always start from the identity, $g(0) = \mathbb{I}_{5\times 5}$. Next we define the coadjoint operator of $\mathrm{d}\mathcal{H}$ and the equation on $\mathrm{so}^*(4,1)$.

```
addHStar = @(xi) coadjointOperator(xi./J) + coadjointOperator(a);
xiDot = @(t, xi) addHStar(xi) * xi;
```

Since the vertical equation is decoupled from the horizontal, we can solve it on the whole interval $\langle 0, t_f \rangle$. To do so, we use the Matlab function `ode45`.

```
[t, h] = ode45(xiDot, tSpan, xiInitial);
```

Now, we use the solution $\xi(t)$ to obtain the curve on the group $g(t)$.

```
g = zeros([5, 5, length(t)]);
for i=1:length(t)
  g(:, :, i) = gInitial*expm(t(i) * lieAlgebra(h(i, :)./J+a));
end
```

Next we check the properties of $\mathrm{SO}(4,1)$ as we have mentioned at the beginning of this section. To check the first property, we compute the difference

```
g(:, :, i)'*B*g(:, :, i)-B
```

and compare it to zero with chosen tolerance. This creates 5×5 matrix of boolean values and if its sum is 25, the property holds. If it doesn't sum to 25, there has been significantly different value and the property is not satisfied. The second condition is very straight-forward as well. We compute determinant of every matrix on the curve and compare it to one with the chosen tolerance. This is done in one `for` loop that is shown below.

```
for i=1:length(t)
  leftSide = g(:, :, i)'*B*g(:, :, i);
  checkOrthogonality(i) = sum(abs(leftSide - B) < tol, 'all')/25;
  checkDeterminant(i) = abs(det(g(:, :, i)) - 1) < tol;
end
```

The final step is plotting `xi`.

5.3.2 Results

We will compute examples with same initial data as in 5.2.2.

Free Rigid Body Motion

We have chosen bivector $\xi = 0.1e_{2\infty} - 0.7e_{3\infty} - 0.1e_{12}$, this corresponds to initial vector `xiInitial = [0 0 0.2 0 -0.2 1.4 0 0 0 0]` thanks to the isomorphism we have established in section 3.4. Both properties were maintained along the curve in $\mathrm{SO}(4,1)$ and the velocities can be found in 5.17. As you can see, the rotational velocity is constant and the translational velocities are being rotated, which is what we have obtained earlier.

Free Soft Body Motion

For case of shrinking motion was chosen velocity $\xi = -0.6e_{1\infty} - 0.2e_{0\infty}$. This correspond to initial data `xiInitial = [0 0 0 1.2 0 0 0 0 0 0.4]`. Group properties were again satisfied and resulting plot is in 5.18. This again confirms our earlier results. The case of expansion can be found in 5.19 and it is affirmative as well.

Forces and Torques

Finally, the case of non-zero linear part of Hamiltonian function results in 5.20 under the same initial conditions as in the case that we have studied in $\mathrm{SO}(4,1)$. The initial velocity was taken $v = -0.1e_{1\infty}$ and bivector of forces $f = -0.1e_{1\infty} - 0.15e_{0\infty}$. We have used the isomorphism from the section 3.4 to convert them to basis of $\mathrm{so}(4,1)$. Again, the properties were satisfied along the curve and the plot corresponds to the animation obtained earlier.

Conclusion

The aim of this thesis was to unite and generalize the notions of left-invariant Hamiltonian systems on Lie groups with left-invariant Hamiltonian functions and explore their applications. Throughout this research, we have formulated Hamiltonian equations for general Lie groups and investigated the potential uses based on specific choices of Lie groups and Hamiltonian functions. Additionally, we have developed a computational tool for simulating rigid and soft body motion using the concepts of CGA. In this concluding chapter, we will reflect on the key findings and contributions of our work, discuss their implications, and propose potential avenues for future research.

As previously discussed, we have successfully derived the simplified form of the left-invariant Hamiltonian system. However, since Lie groups already encompass a wide range of underlying sets, it is natural to consider extending this theory to incorporate non-left-invariant Hamiltonian functions. Those arise immediately even in the theory of rigid body motion as they represent potential energy and forces associated with them such as gravity or Hooke's Law. Thus we would have different type of forces than those that are constant in the body frame.

Another highly intuitive approach to further this research would be to explore different Hamiltonian functions, including those that are left-invariant. Given that a Hamiltonian function is any smooth function on the cotangent bundle T^*G, there are numerous options to consider, and our examples merely scratched the surface of the possibilities. By investigating a broader range of Hamiltonian functions, we can gain deeper insights into the behavior and dynamics of left-invariant systems on Lie groups.

While the JavaScript scripts serve their purpose effectively, in hindsight, opting for the Python library Clifford ([17]) might have been a more favorable choice. Python is widely regarded as a more intuitive programming language, and the Clifford package itself is extensively documented, providing comprehensive resources for even beginners in code development and science researches.

Figures

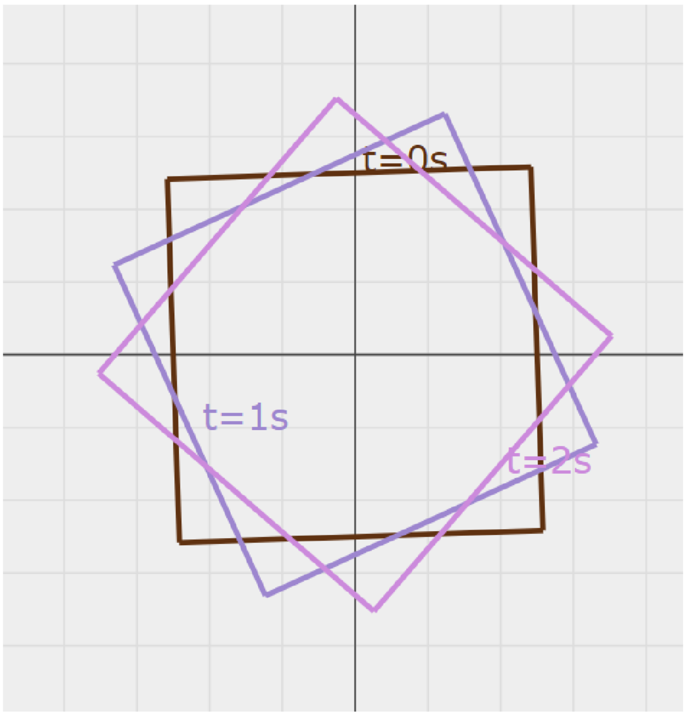


Fig. 5.1: Rotation around one axis in 2D

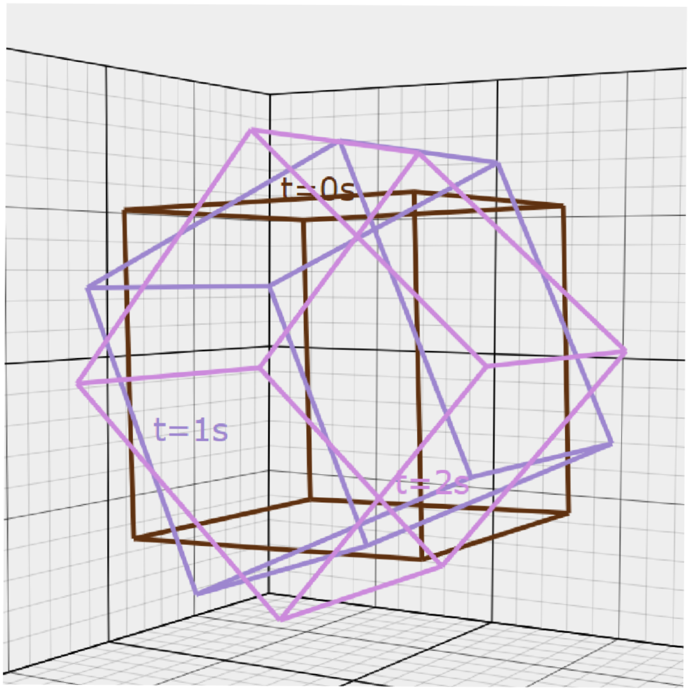


Fig. 5.2: Rotation around one axis in 3D

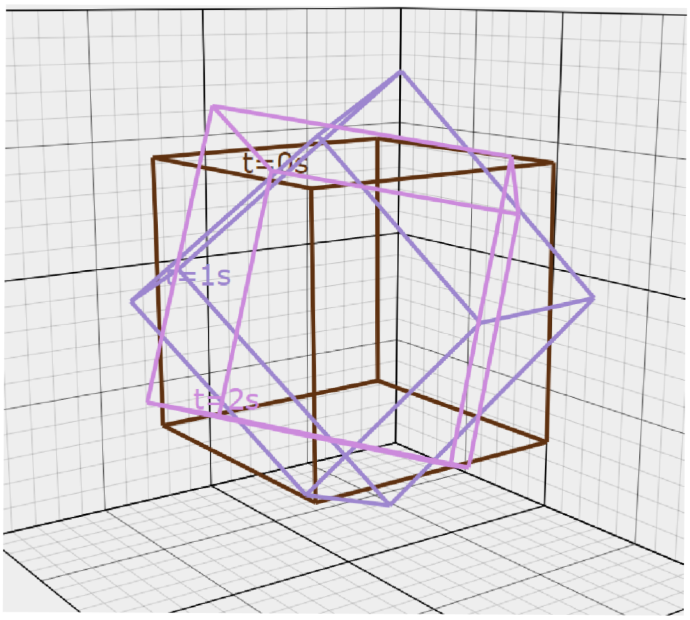


Fig. 5.3: Rotation around all axes

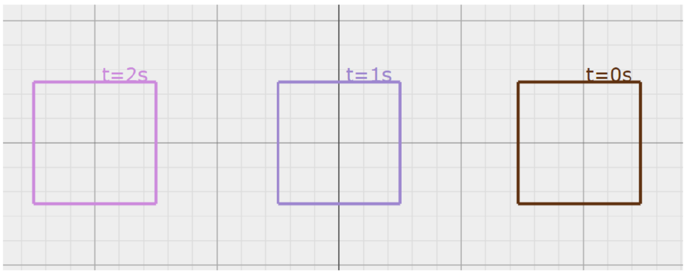


Fig. 5.4: Translation in one direction in 2D

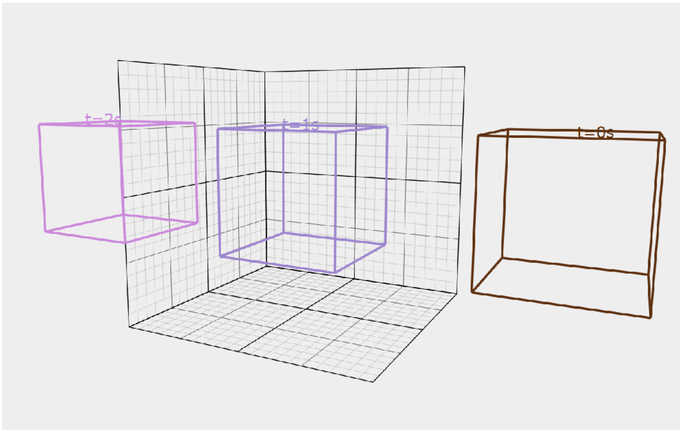


Fig. 5.5: Translation in one direction in 3D

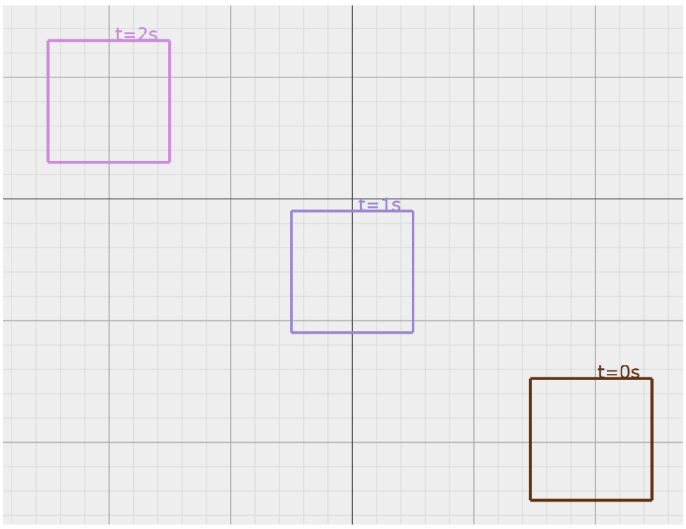


Fig. 5.6: Translation in all directions in 2D

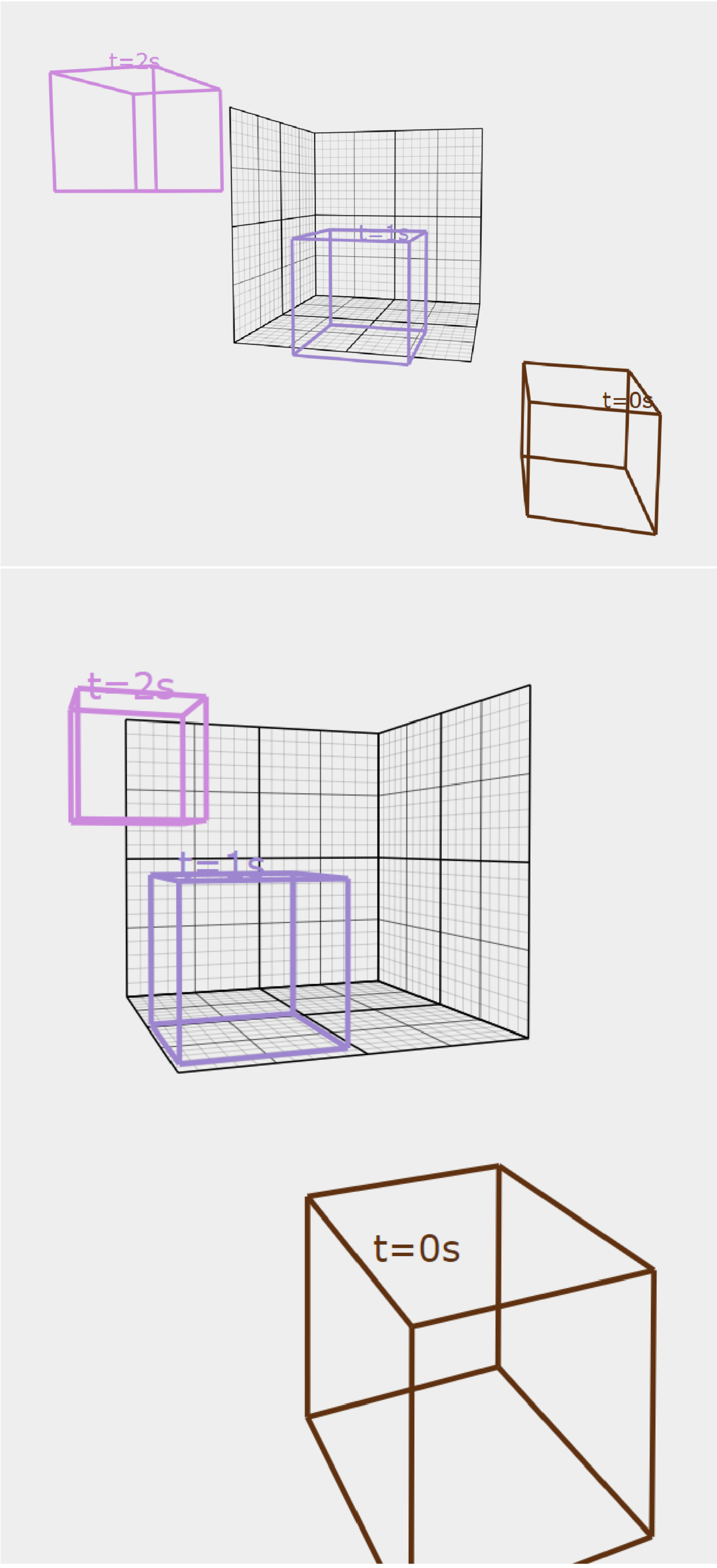


Fig. 5.7: Translation in all directions in 3D

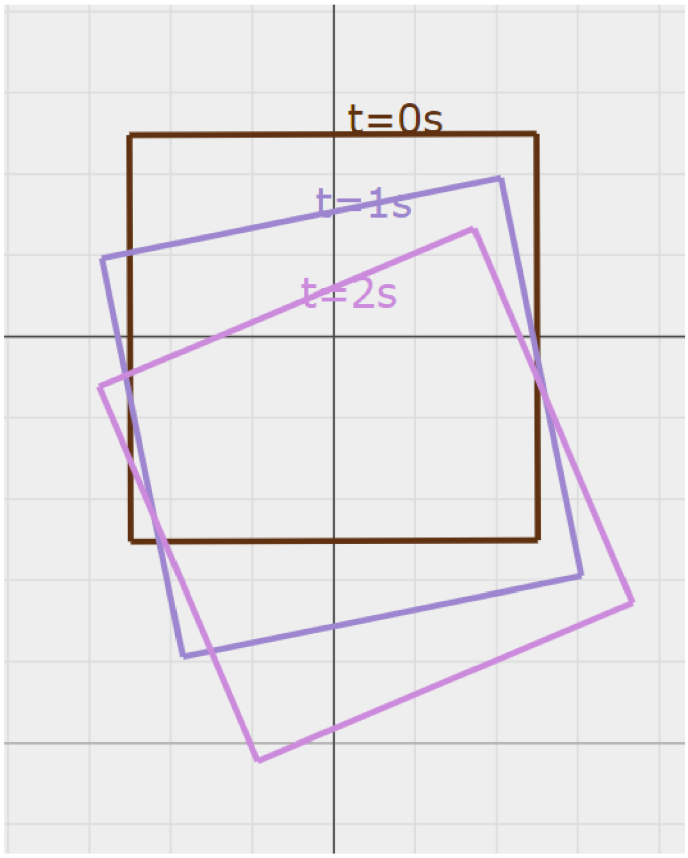


Fig. 5.8: Full motion in 2D

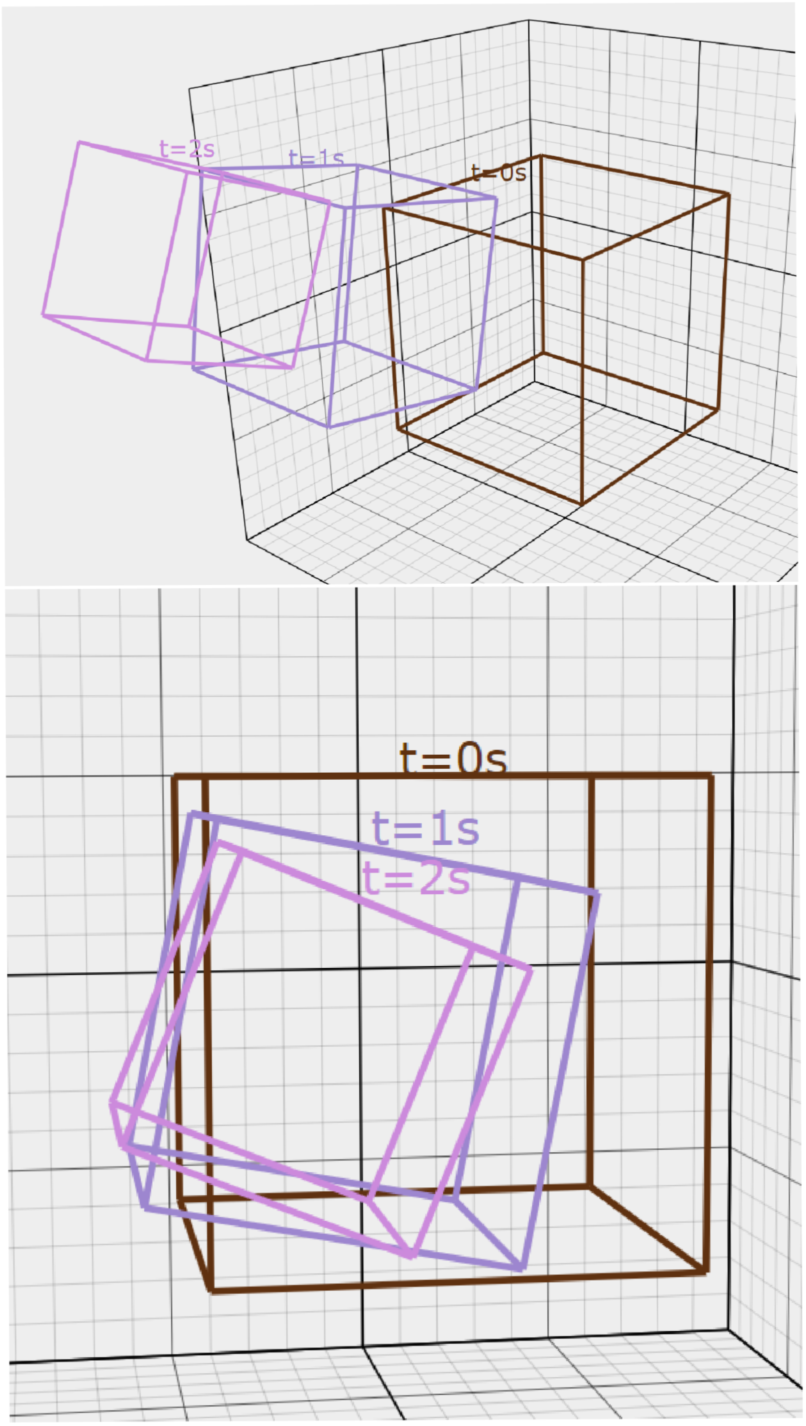
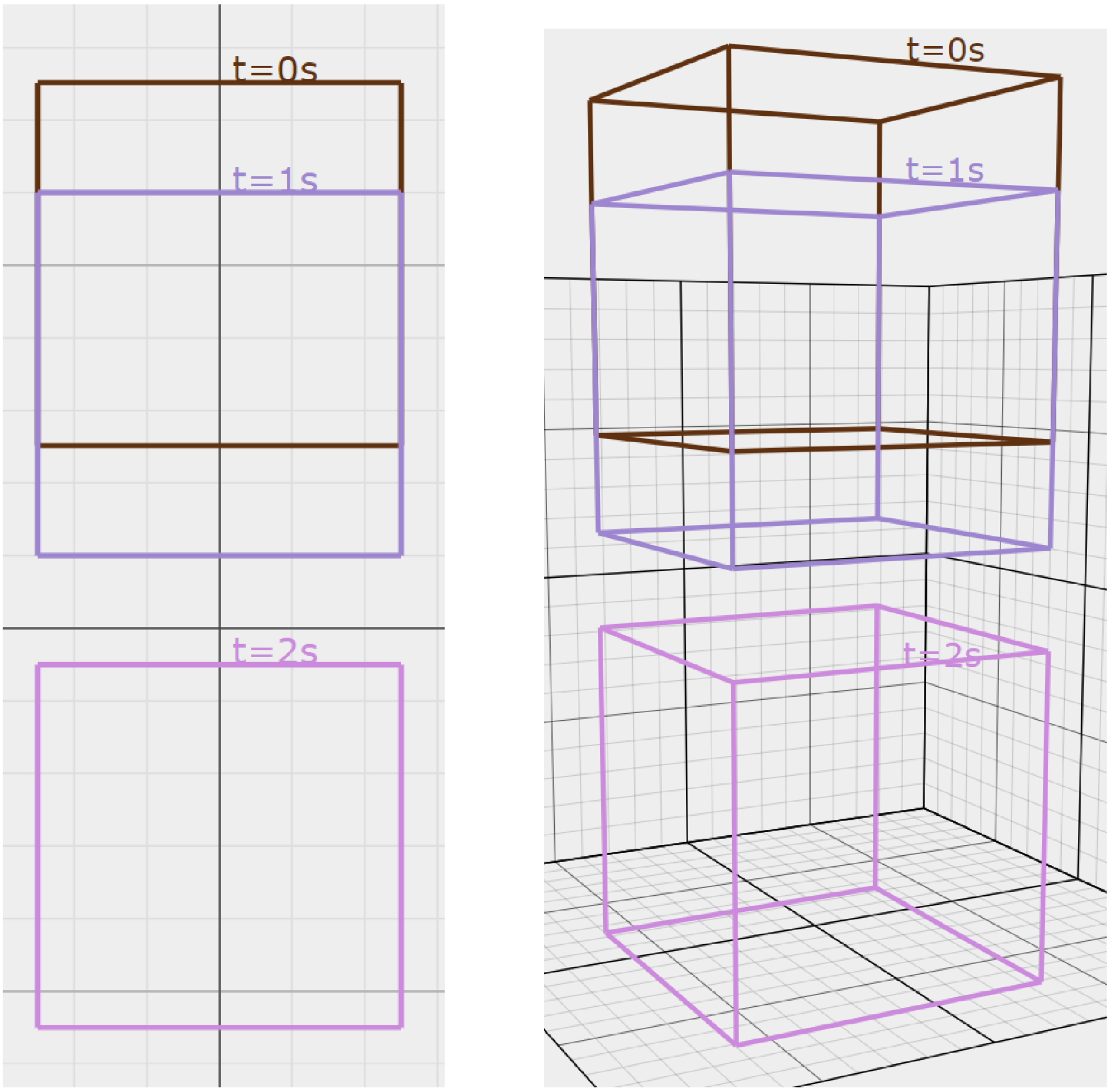


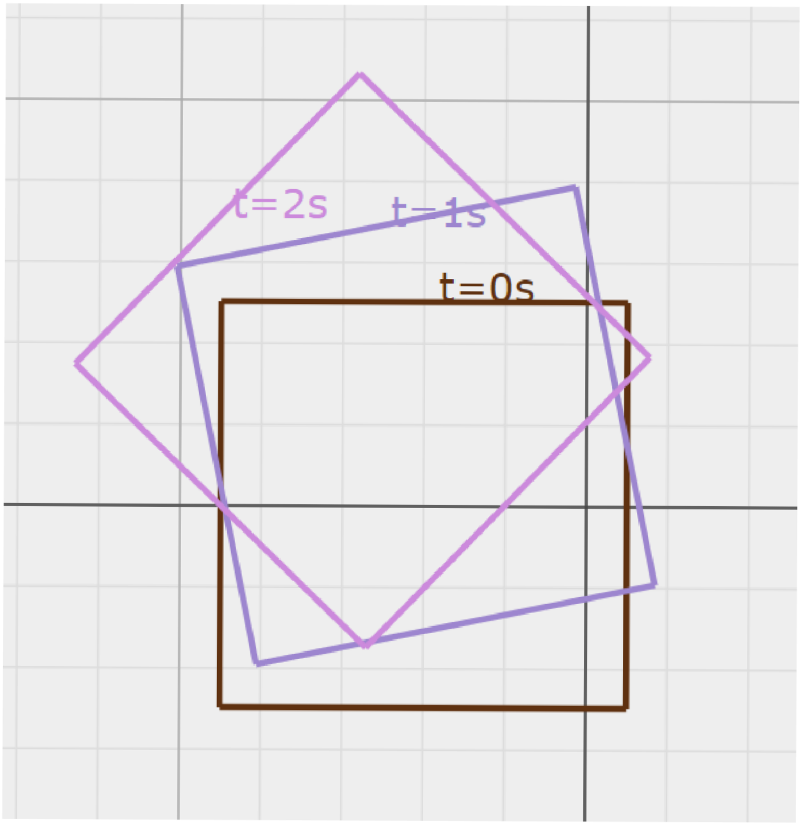
Fig. 5.9: Full motion in 3D



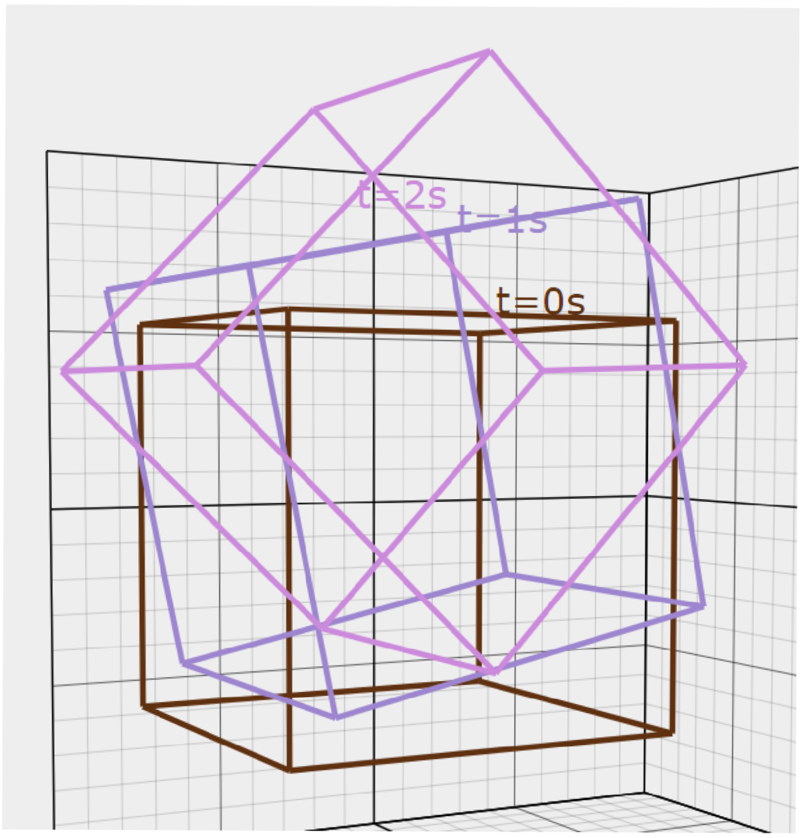
(a) Force in 2D space

(b) Force in 3D space

Fig. 5.10: Body frame forces

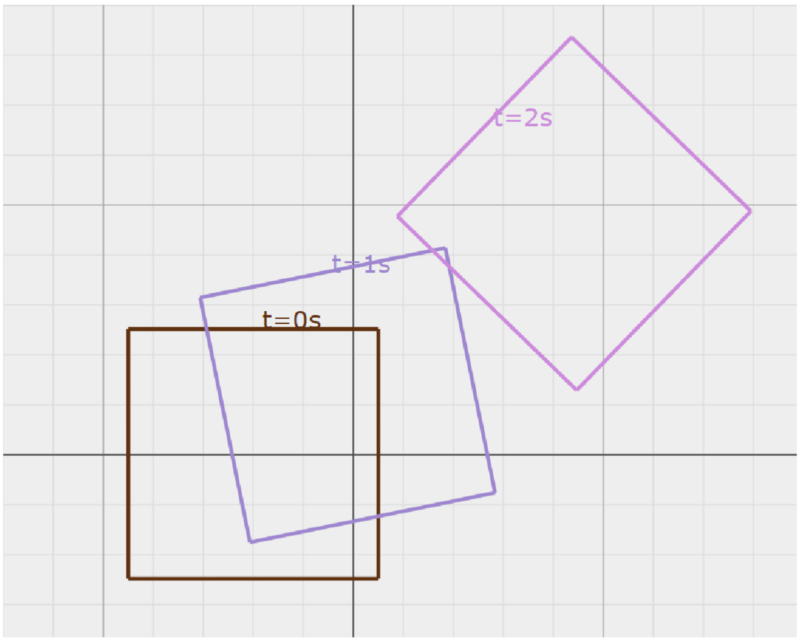


(a) Torque in 2D space

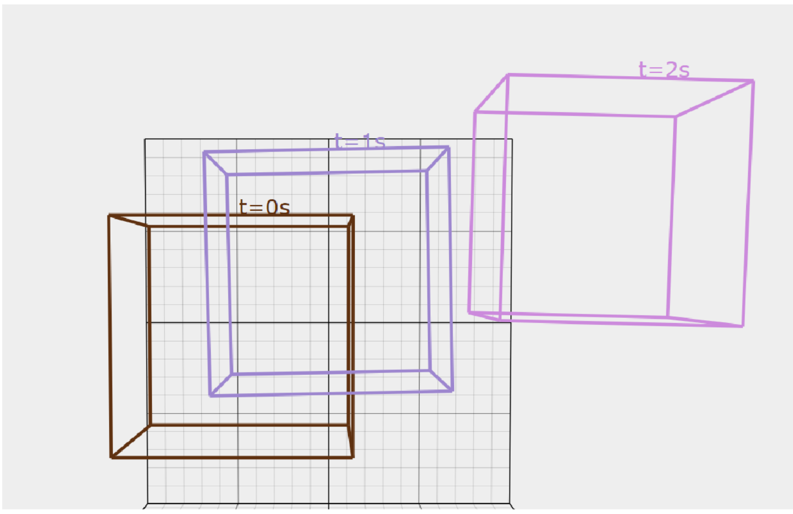


(b) Torque in 3D space

Fig. 5.11: Body frame torques



(a) Force and torque in 2D space



(b) Force and torque in 3D space

Fig. 5.12: Body frame forces and torques

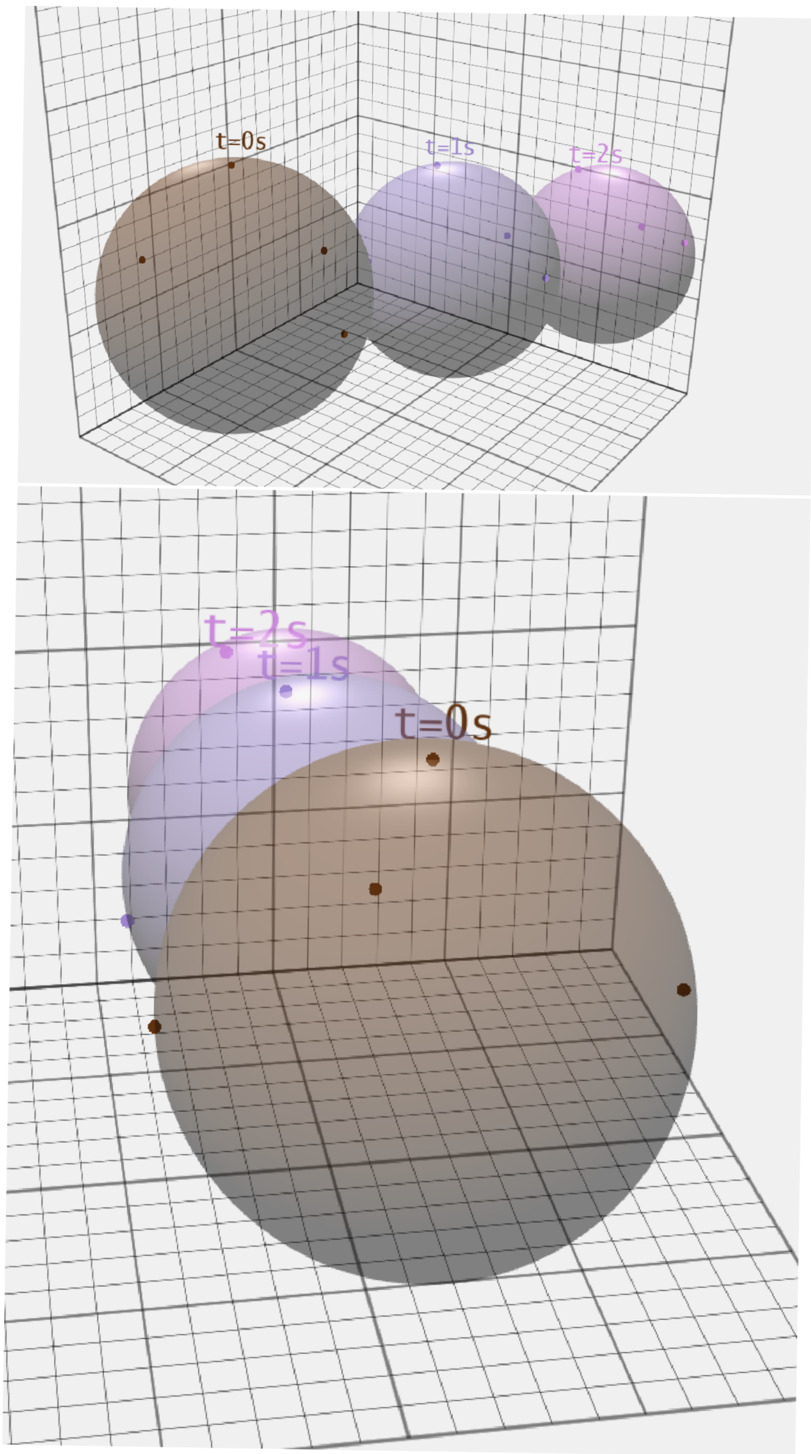


Fig. 5.13: Free rigid body motion in SO(4, 1)

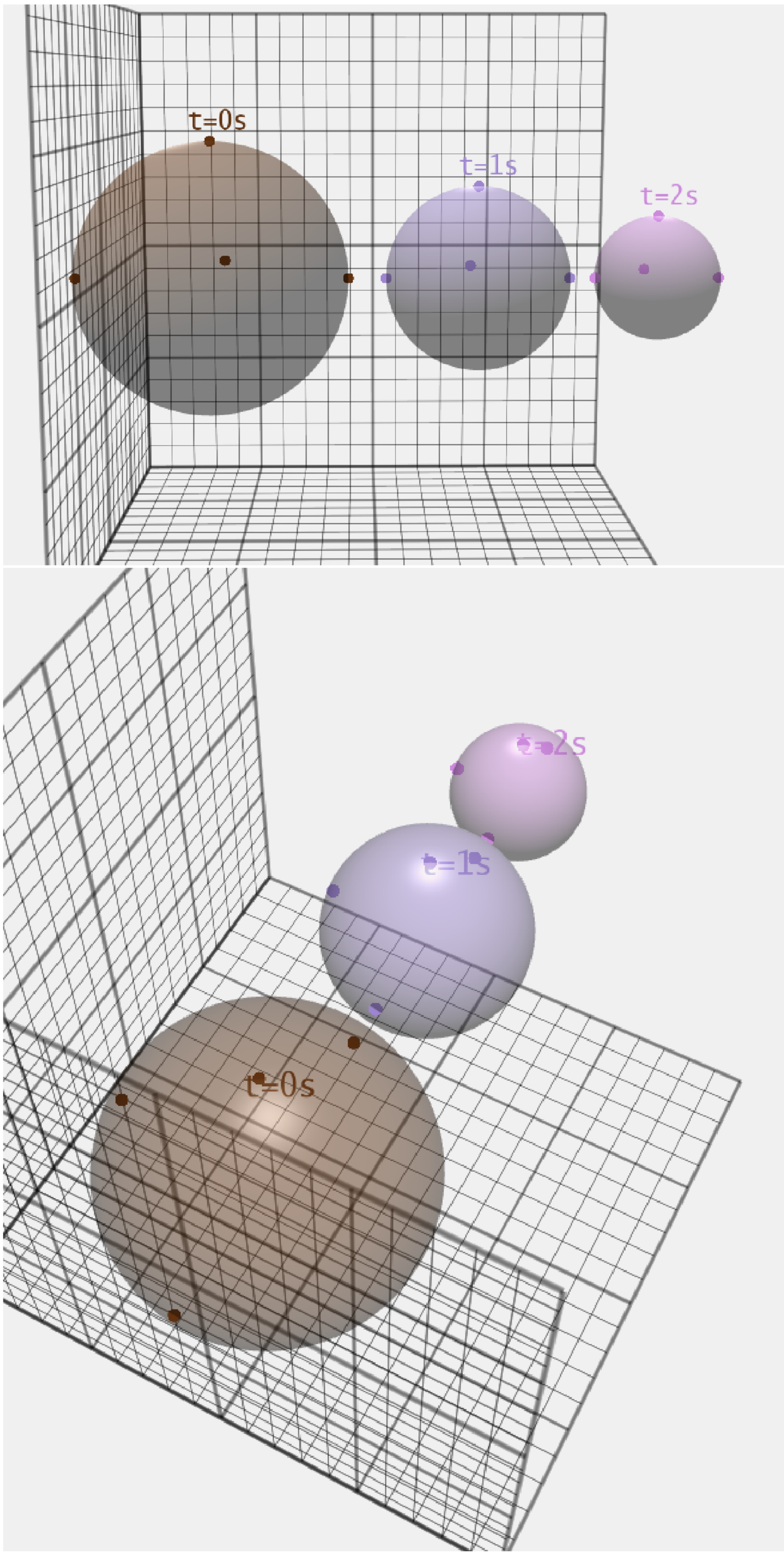


Fig. 5.14: Shrinking of soft body

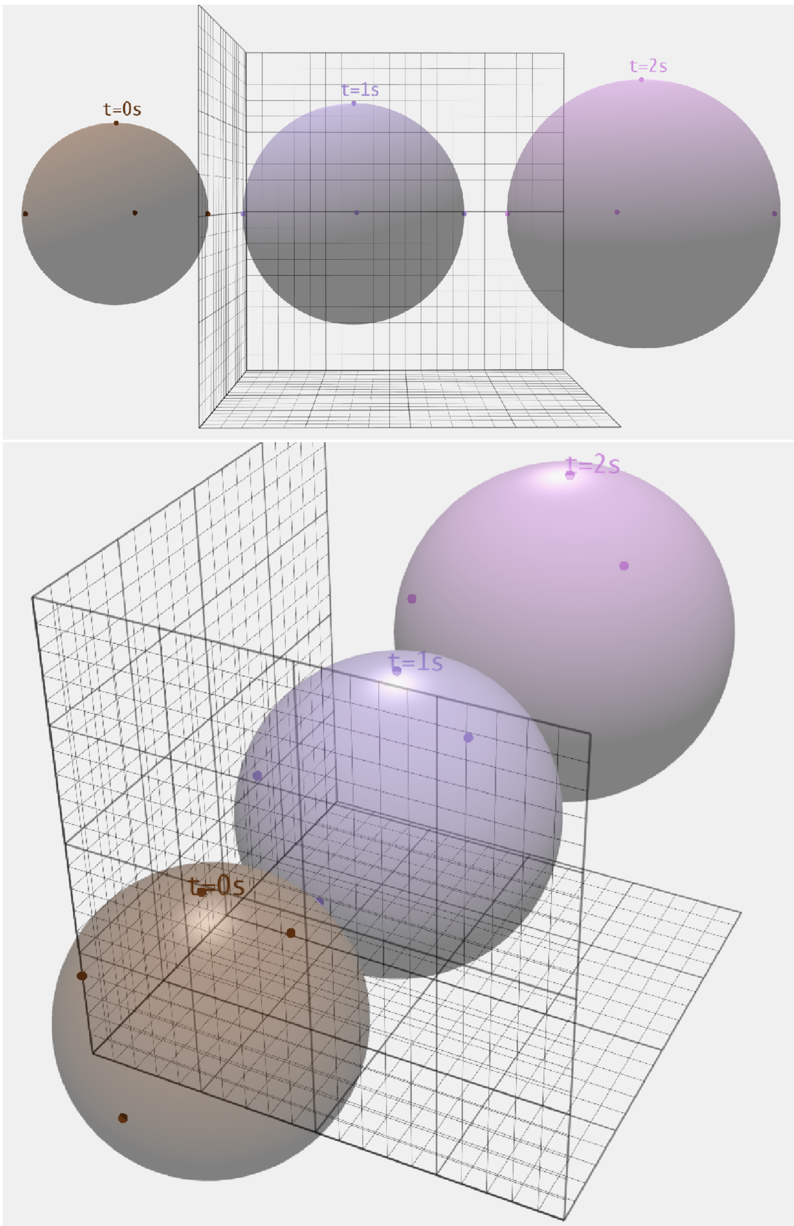


Fig. 5.15: Expansion of soft body

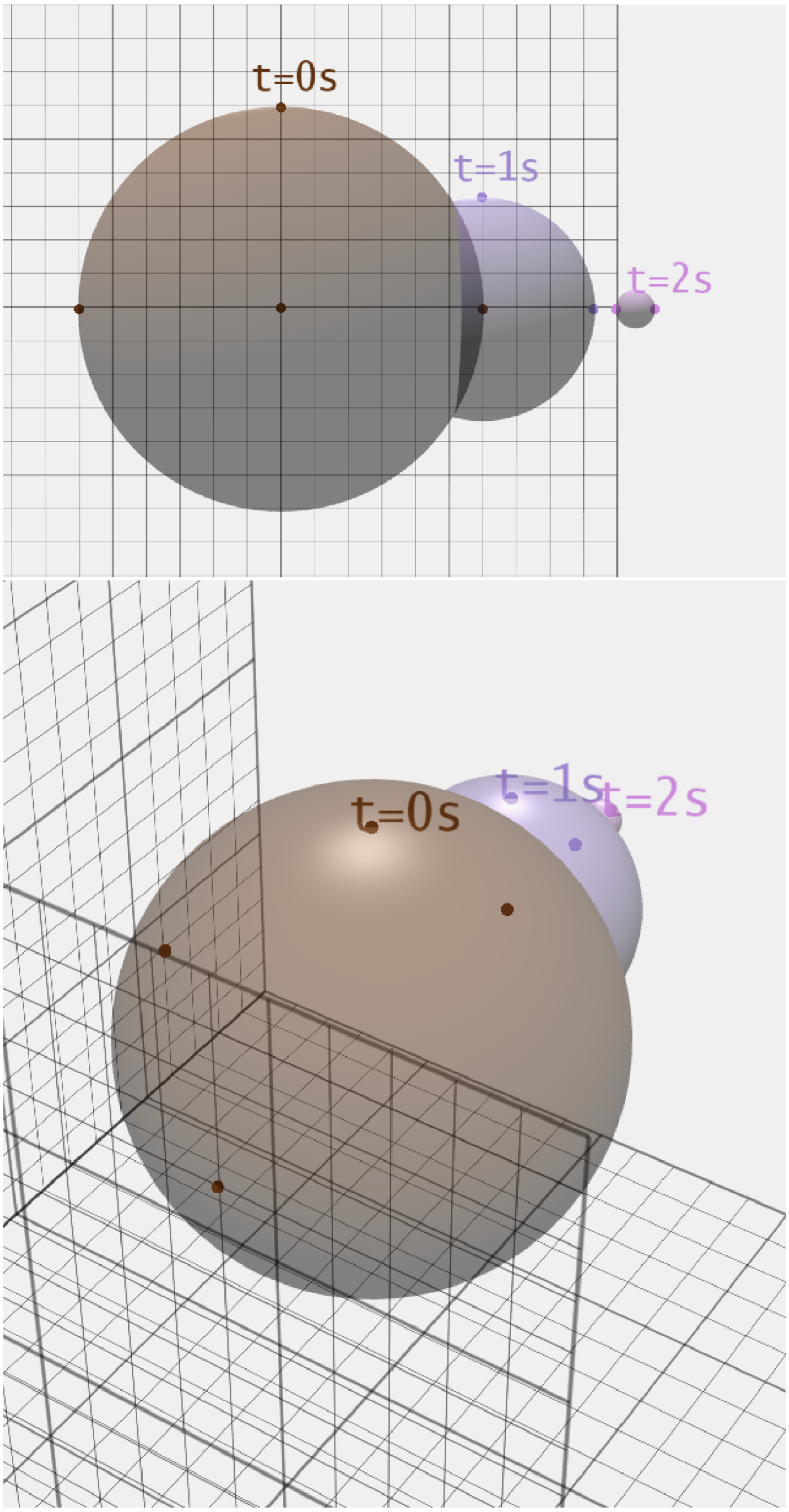


Fig. 5.16: Forces acting on soft body

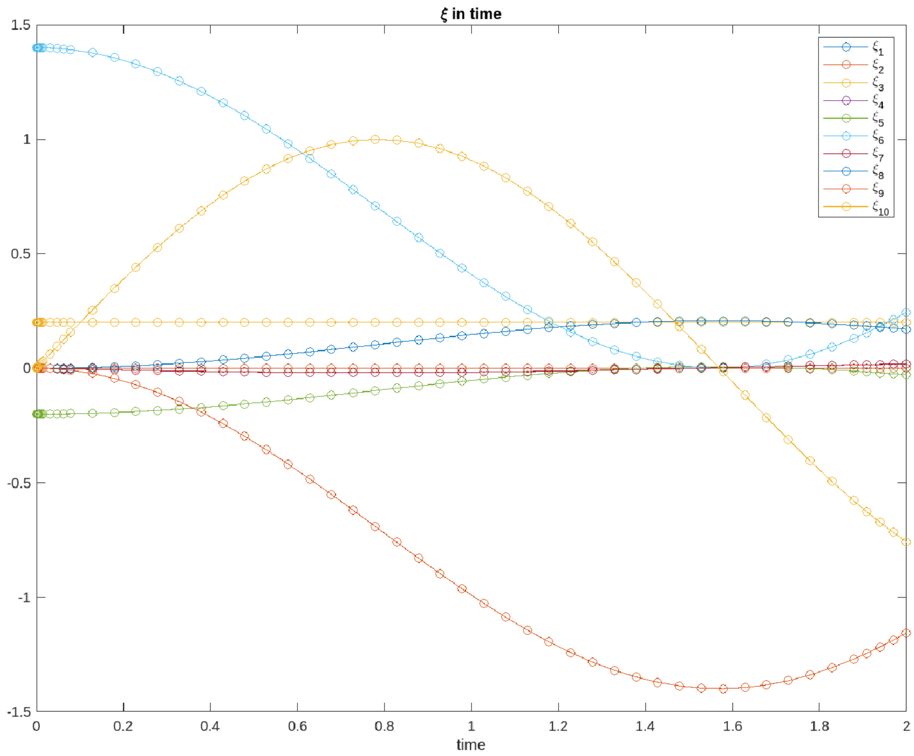


Fig. 5.17: Velocities of free rigid body motion

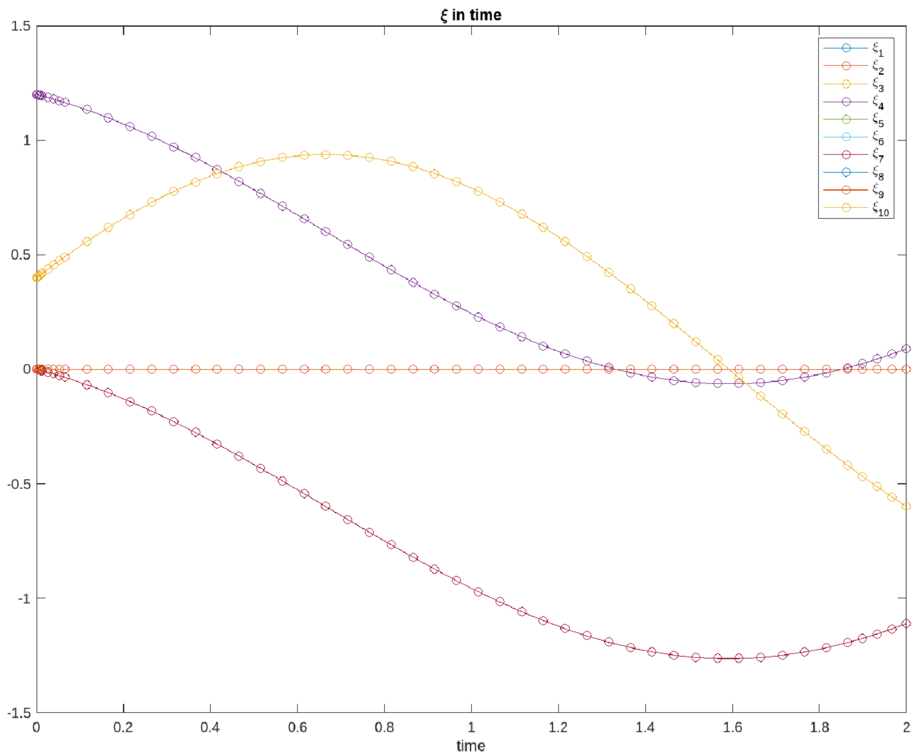


Fig. 5.18: Velocities during shrinking motion

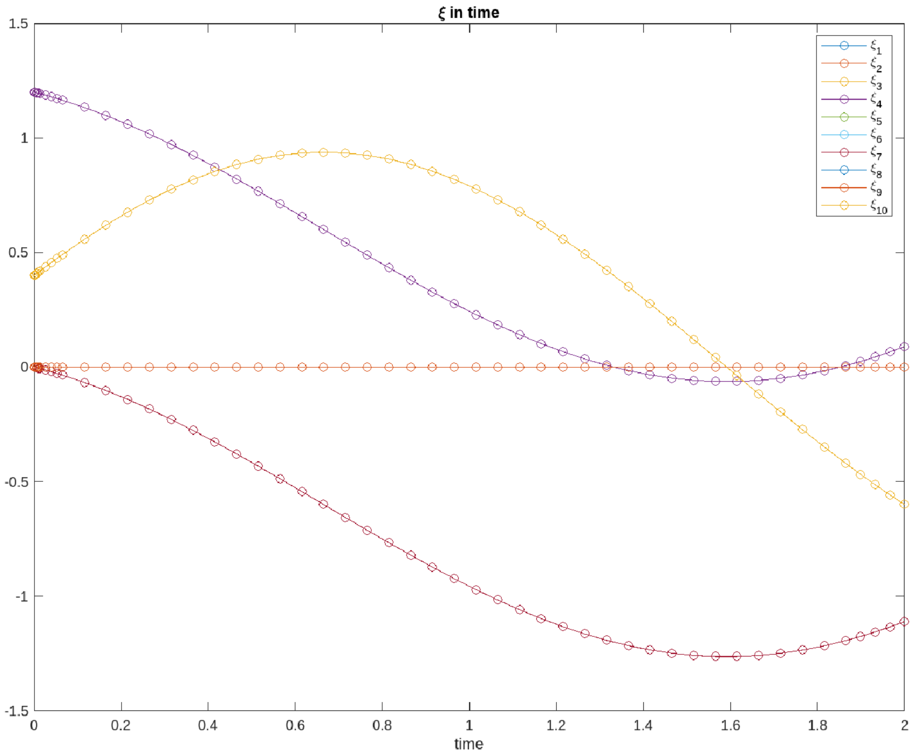


Fig. 5.19: Velocities during expansion

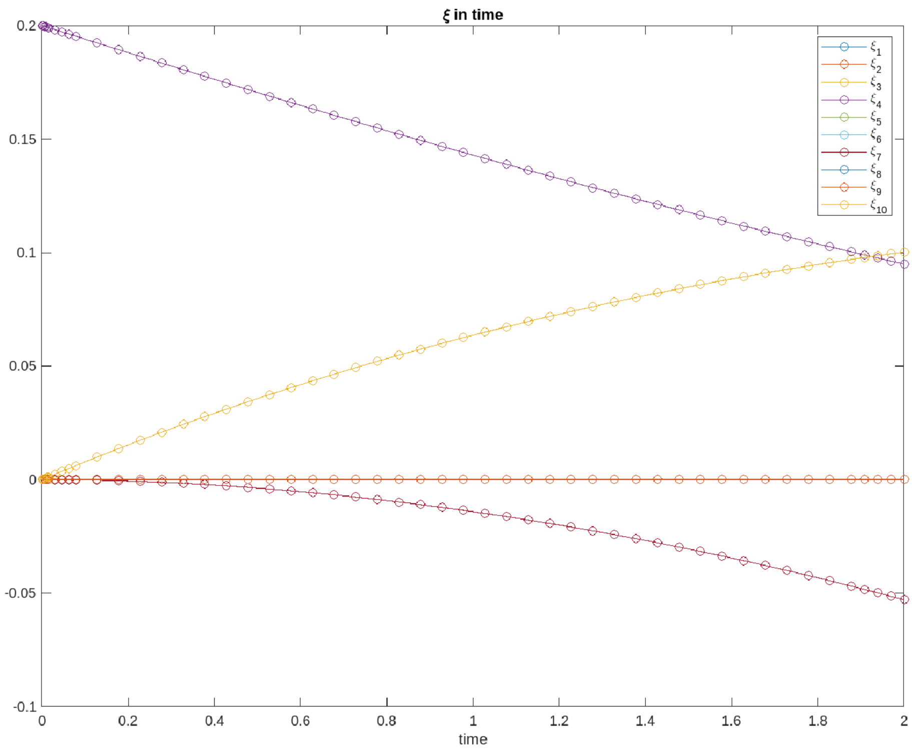


Fig. 5.20: Forces acting upon soft body

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